THE UNIVERSITY OF OKLAHOMA

GRADUATE COLLEGE

UNPACKING MATHEMATICAL CONTENT THROUGH PROBLEM SOLVING

A Dissertation

SUBMITTED TO THE GRADUATE FACULTY

in partial fulfillment of the requirements for the

degree of

Doctor of Philosophy

By

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Norman, Oklahoma
2002

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UNPACKING MATHEMATICAL CONTENT THROUGH PROBLEM SOLVING

A Dissertation APPROVED FOR THE DEPARTMENT OF

INSTRUCTIONAL LEADERSHIP AND ACADEMIC CURRICULUM

BY

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ABSTRACT

Current calls for reform of preservice teacher education include suggested changes in the mathematics content courses required for elementary education programs. One suggestion is the inclusion of more complex problem solving activities. The context of this study was a homogeneous group of preservice teachers in a two-course sequence of college mathematics content courses. I investigated the use of complex problem solving as a vehicle for these preservice teachers to unpack previously learned mathematics, reconstruct their understandings, and connect mathematical concepts for deeper understanding. With the support of a mathematical classroom community, collaboration, and reflection, students reported gaining deeper understandings of the mathematics, a change in their beliefs about the nature of mathematics and themselves as mathematicians, and a significant decrease in negative emotional baggage.

As I considered what preservice teachers bring with them to the college mathematics content classroom, and how using problem solving to unpack mathematical content could deepen mathematical knowing, I found that frustration played a large role in the process. Choosing tasks that perturb student thinking and bring them to a level of frustration that provokes facilitative anxiety seemed to best encourage students to engage in mathematical collaboration, communication, and reflection during the problem solving process. Students reported frustration as the most common experience when starting and working on problems. However, in their reflective essays at the end of the semester, students had coupled their feelings of frustration with their feelings of accomplishment, satisfaction, and higher levels of mathematical confidence.

UNPACKING MATHEMATICAL CONTENT THROUGH PROBLEM SOLVING

CHAPTER ONE

INTRODUCTION

> You cannot teach a man anything; you can only help him find it within himself.
> Galileo

Mathematics Education

The National Council of Teachers of Mathematics (NCTM, 2000), the National Research Council (NRC, 1989, 1996), and other groups such as the Glenn Commission (National Commission on Mathematics and Science Teaching, [NCMST], 2000) note that societal needs for mathematics are rising with the level of technology, and that past expectations of basic computational ability are not sufficient for survival in today's society (see also Carpenter & Lehrer, 1999; Hiebert et al., 1997). Battista (1999) writes, "In the Information Age and the web era, obtaining the facts is not the problem; analyzing and making sense of them is" (p. 428). Many groups are advocating higher standards for mathematics education at all levels of education (American Mathematical Association of Two-Year Colleges [AMATYC], 1995; Conference Board of the Mathematical Sciences [CBMS], 2001; Kirkpatrick, Swafford, & Findell, 2001; Leitzel, 1991; Mathematical Sciences Education Board [MSEB], 1991, 1993; National Council of Teachers of Mathematics, 2000). However, there are many questions about how to implement reform in mathematics education, who should make the decisions, and how the effects should be evaluated. Many mathematics educators feel that the change must be a systemwide change, and that a paradigm shift is required.

A New Paradigm

Along with greater expectations for mathematics education, a new paradigm of mathematics instruction is leading us away from the traditional transmission mode where known mathematics are transmitted to students in a linear fashion. The new paradigm views the very nature of mathematics differently, as an active and creative discipline, and with the understanding that all knowledge must be actively constructed by the student. Schifter & Fosnot (1993) explain the implications for mathematics education under this new paradigm:

> If the creation of the conceptual networks that constitute each individual's map of reality – including her mathematical understanding – is the product of constructive and interpretive activity, then it follows that no matter how lucidly and patiently teachers explain to their students, they cannot understand for their students. Though this is a truism, it nonetheless has profound implications for pedagogical theory and practice: Once one accepts that the learner must herself actively explore mathematical concepts in order to build the necessary structures of understanding, it then follows that teaching mathematics must be reconceived as the provision of meaningful problems designed to encourage and facilitate the constructive process. In effect, the mathematics classroom becomes a problem solving environment in which developing an approach to thinking about mathematical issues, including the ability to pose questions for oneself, and building the confidence necessary to approach new problems are valued more highly than memorizing algorithms and using them to get right answers. (p. 9)

This new paradigm is just beginning to emerge in mathematics education. Many inservice teachers are not familiar with or have not had any experience under this new paradigm. If this worldview is to expand and take force in our school systems, it needs to be introduced and modeled in our college and pre-college courses (Tosey, 2002). One of the most important populations to reach with the new paradigm is preservice teachers. They will be the ones who introduce and model this paradigm for their students in the next generation.

Preservice Teachers

College students who are preparing to teach in the elementary schools are a population that often struggles in college mathematics classes. They must take sufficient mathematics to meet degree requirements, and learn enough about mathematics to be successful in their future careers. However, the preparation of elementary school teachers often falls far short of preparing them with the knowledge skills or dispositions they need (CBMS, 2001; Kirkpatrick et al., 2001). Changes in the mathematics content preparation of elementary school teachers are being proposed in response to higher expectations. Some propose adding on more classes, others feel that the existing classes must change. The reform of teacher preparation in post-secondary institutions is central to providing quality mathematics education for all students. However, exactly what mathematical content should be acquired and when, where, and how, are still unresolved questions in current research (MSEB, 2001; NRC, 1996; National Science Foundation [NSF], 1996).

One reform suggestion is to incorporate more problem solving activities in the mathematics classroom in an effort to increase mathematical understanding (NCTM, 2000; NRC, 1996). However, little research has involved looking at how this might be applied to the college mathematics classroom (Lampert, 1990). The Preparing Elementary Teachers to Teach Mathematics Project (PETTM) observed preservice teachers in college mathematics classes at Indiana University for three years (LeBlanc, Lester & Kroll, 1992). Rather than teaching *about* problem solving or teaching *for* problem solving, this program promotes teaching *via* problem solving. In this view, problem solving is not a topic to be taught, but a context for learning all mathematics. My research study tood this view by investigating the use of problem solving activities in a college mathematics content course for preservice elementary teachers.

Organization of the Dissertation

Chapter One explains the purpose of the study and rationale for this research. Chapter Two provides background information and a literature review. Chapter Three includes a description of the research setting, population, methodology, questions and research criteria. An overview of the data is provided in Chapters Four through Six, with the discussion of findings, conclusions, and implications in Chapter Seven.

Purpose of the Study

There are numerous research studies on problem solving in the mathematics classroom with children. However, similar studies with college students is limited, especially research involving preservice teachers and problem solving in college mathematics content courses. There is a need for qualitative research to investigate what problem solving activities might look like in the college mathematics classroom. A combination of problem solving activities with periodic reflection in a mathematical community (Lappan & Even, 1989) may offer valuable insights in improving preservice teacher mathematics education.

Another unanswered question in the research is what conceptions of mathematics and mathematics teaching and learning do preservice teachers bring to teacher education (Thompson, 1992), along with deeply rooted beliefs and emotions that can affect mathematical understanding.

As I have studied in mathematics education and taught college level mathematics content courses, I have come to envision a model of learning based on the metaphor of the "edge of chaos," a precept of systems thinking. In the seventh chapter I present and explain how I think this model can represent how students come to mathematical understanding through unpacking content and reconstructing for greater understanding and connections.

This dissertation is organized around the following focus questions:

1. What do preservice teachers bring with them to a college mathematics content course?

2. How may a problem-rich learning environment enable preservice teachers to unpack the mathematics content they have already acquired?

3. How does unpacking the content through problem solving affect mathematical understandings?

CHAPTER TWO

BACKGROUND

> *What is not fully understood is not possessed.*
> *Johann Goethe*

Preservice Teachers

Preservice elementary teachers can be viewed as a subgroup of the college population with some common characteristics. Consisting of more than 90% white females, they are more often than not non-traditional students (Fisher, 1992; Green & Weaver, 1998; Su, 1993). Historically, they tend to dislike and avoid mathematics (Ball, 1990a; CBMS, 2001; National Center for Research on Teacher Education [NCRTE], 1991), many have not had a mathematics class in more than three years (Ball, 1990a), and they bring misconceptions and negative attitudes toward mathematics to the college mathematics classroom (Philippou & Christou, 1998).

Preservice elementary teachers score lower on standardized achievement tests, exhibit higher levels of mathematical anxiety, and possess more negative attitudes toward mathematics than the general college population (Fisher, 1992; Kelly & Tomhave, 1985; Rech, Hartzell, & Stephens, 1993). Their mathematical understanding is often colored by their emotional responses to the subject. They are more apprehensive about teaching mathematics than other subjects. Many hope to teach at lower grade levels so their lack of content knowledge will not have as great an impact (Ball, 1988a, 1988b).

Preservice elementary teachers often believe, speak, and act as if they have non-mathematical minds (Powell & Frankenstein, 1997). Their own school training probably focused on an algorithmic approach to mathematics that was unlikely to contribute to meaningful understanding (Ball, 1990a; Civil, 1990; Thompson, 1992). Thus they tend to approach their mathematics studies from a dualistic point of view, seeing mathematics as an absolute and arbitrary collection of facts and rule-bound procedures to be learned by memorization (CBMS, 2001; NCRTE, 1991; Schifter & Fosnot, 1993). This dualism propels an inclination to immediately search for prescribed algorithms and memorized procedures (Ball, 1990; Benbow, 1993; Schoenfeld, 1989).

Preservice elementary teachers often feel mathematical success depends on having an innate mathematical mind (Ball, 1990a; Beatty, 2001; Belenky, Clinchy, Goldberger & Tarule, 1996; Buerk, 2000; Frank, 1990; Wheatley, Blumsack & Jakubowski, 1995; Wilcox, Schram, Lappan & Lanier, 1991). They see mathematics differently from the other disciplines they study. For example, Benbow (1993) reports that preservice teachers see mathematics as more dichotomized into completely right or wrong answers than either English or Social Studies. These beliefs about mathematics, culturally embedded by twelve or more years of experience, are the main obstacle for reform of teacher education (Ball, 1988a; Civil, 1990).

Preservice elementary teachers have low levels of autonomy in the mathematics classroom, exhibiting a dependence on external authority (teacher, textbook, peer) for constant verification of the one right method and one right answer (Ball, 1990a; Civil, 1990; LeBlanc et al., 1992; Wilson & Ball, 1991). They feel that they cannot serve as their own source of verification and that mathematical thinking, processes, and solutions must be validated externally (Dupree, 1999). They often expect the instructor to "tell" them the mathematics and "show" them what and how to do it (Chazan & Ball, 1995; Wilcox, et al., 1991). They see little value in engaging in mathematical discourse about multiple processes and possible solutions when they view mathematics as simply a right or wrong answer (Ball, 1989; Civil, 1990). Thus they have little propensity to look for alternative solutions or to verify their solutions using other methods (Ma, 1999).

There is very little research on this particular student population in college mathematics content courses. Research in the past fifteen years has been overwhelmingly focused on preservice teachers' pedagogical knowledge and beliefs rather than on their content knowledge and understanding (Ball, Lubienski, & Mewborn, 2001; NCRTE, 1991). Recently, there is an emerging research base about teaching and learning with understanding, and how students themselves construct meaning in mathematics (Carpenter & Lehrer, 1999). However, there is a need for more research in the specific area of preservice teachers and the college mathematics content classroom.

Sufficient Mathematics or Different Mathematics

It may seem that prospective elementary school teachers learn sufficient basic skills during their own K-12 schooling to teach elementary mathematics. However, the general public and policy makers continue to emphasize that most elementary school teachers don't know enough about mathematics to teach it well (CBMS, 2001; Kirkpatrick et al., 2001). Many states have increased the number of required mathematics courses for high school graduation and for elementary teacher certification. However, taking more mathematics classes may not be the answer, if the kind of mathematics needed for teaching is different from what is taught in traditional college mathematics courses (CBMS, 2001; Featherstone, Smith, Beasley, Corbin & Shank, 1995; Heaton, 1994; Kirkpatrick et al., 2001; McDiarmid, 1992). There is a need for greater understanding rather than simply "adding on" mathematical knowledge through additional courses. Unless students learn with understanding, additional mathematics knowledge is unlikely to be useful outside the college classroom (Carpenter & Lehrer, 1999). It is like using a map to find your way, if it is someone else's map, you still have to figure it out. But if you explored the territory and made the map from your experiences, then you might better understand and apply the map in new and diverse situations (Cohen & Stewart, 1994).

Mathematicians, mathematics educators, researchers, and policymakers recognize the different nature of the mathematics necessary for K-12 teaching, and how intellectually rich elementary mathematics can be (American Mathematical Association/Mathematical Association of America [AMA/MAA], 2001; Ball, 1992; CBMS, 2001; Interstate New Teacher Assessment and Support Consortium [INTASC], 1992; Kirkpatrick et al., 2001; Lampert & Ball, 1999; Lappan & Even, 1989; McDiarmid, 1992; NCTM, 2000; Usiskin, 2002). Ball & Bass (2000) suggest that preservice teachers need to learn, know, and understand mathematics in a different way than mathematicians and scientists. They maintain a distinction between "knowing how to do math and knowing it in ways that enable use in practice" (p. 94).

The National Center for Research on Teacher Education (NCRTE, 1991) distinguishes between knowledge *of* mathematics and knowledge *about* mathematics. Knowledge *of* mathematics refers to concepts, procedures, and the connections among them. Knowledge *about* mathematics refers to subject matter knowledge *for teaching*, a subject-specific pedagogical content knowledge (Shulman, 1986). Wilson (1989) describes this pedagogical content knowledge as

> understandings and beliefs about the range of alternatives for teaching a particular piece of subject matter to particular students in particular schools, as well as knowledge and beliefs about the ways in which students learn the content in question. This knowledge also enables teachers to generate instructional representations that are justifiable on the basis of the

> discipline itself, on theories of teaching and learning, on knowledge of the interests and prior knowledge of students, and on educational goals and objectives. (p. 1)

Pedagogical content knowledge involves a way of thinking, reasoning, and solving problems that enables a teacher to analyze textbook examples, create assignments, respond to student questions, judge answers on assessments, offer alternate methods, choose manipulatives, and compare multiple representations. A teacher needs to know more than just the correct answer. A teacher needs to understand the underlying meanings and connections in the mathematics (Ball, 1990c; CBMS, 2001; Lampert & Ball, 1999; Ball & Bass, 2000; Kirkpatrick et al., 2001; NCRTE, 1991; Wilson, 1989). In a similar way, mathematical reasoning is as fundamental to learning and knowing mathematics as text comprehension is to reading (Ball & Bass, in press).

Ma (1999) describes the need for depth, breadth, and thoroughness in the mathematical knowledge of elementary teachers by comparing a profound understanding of fundamental mathematics to a taxi driver's knowledge of the city. The driver can flexibly and adaptively arrive at many significant places in a wide variety of ways from multiple perspectives. Lewis (2001) compares a woodsman walking through a familiar wooded area to a tourist who would soon be lost if he left the main trail. Lampert (1990) speaks of the jagged and uncertain cross-country terrain in contrast to walking the well-worn path. In the same way, preservice teachers need to know mathematics from multiple angles in order to teach with understanding and handle any questions or misconceptions that arise. In this way a teacher may securely and safely wander off the main lesson path to follow up on interesting questions or investigate student errors.

Developing understanding involves more than just connecting prior and subsequent knowledge. It involves the creation of rich, integrated knowledge structures, with multiple paths of retrieval and elaborate relationships. Structured knowledge is less likely to be forgotten and more likely to be in a format that makes it useful across multiple applications (Carpenter & Lehrer, 1999).

Learned Mathematics

Preservice teachers bring a store of "learned" mathematics with them to the college mathematics classroom. Consisting of bits and pieces of disconnected information learned by rote, it is often not useful or coherent (Featherstone et al., 1995). Frequently, this mathematical knowledge is tacitly rather than explicitly understood. Students can go through the mathematical motions to successfully complete exercises, but may not understand what they are doing, why it works, when it is appropriate to use, or if their answer makes sense (Ball, 1991, 1998). Caine, Caine & Crowell (1994) speak of surface knowledge that has no real meaning for the learner, as it is simply memorized for the test and almost invariably forgotten. There are few connections with other knowledge or social and emotional issues. Whitehead (1929) calls it "inert" knowledge, and suggests that it is the greatest threat to education. In addition, this store of "learned" mathematics is accompanied by attitudes, inclinations, and habits of thinking that affect the way students learn and incorporate new ideas (Ball, 1988a; Wilcox et al., 1999).

Langer (1997) describes mindless learning as accepting transmitted information unconditionally, memorizing out of context, and operating from a single perspective. Mindlessness leads to overlearned skills and students who respond as if operating on autopilot (division of fractions means flip and multiply). When confronted with a problem that they feel they should be able to do, they attempt to "retrieve" the proper mathematical bit that will take them quickly to the correct solution. They spend more time trying to "remember" how to do it, rather than trying to make sense of the problem, and when they can't remember how, they feel they can't do it (Ball, 1988). When a formula or "trick," is retrieved and applied, they are so happy to find a solution that they don't stop to think about why it works or if the solution even makes sense (Civil, 1990; LeBlanc et al., 1992). Often this initial rote learning interferes with subsequent development of meaningful learning (Ball 1990c; Pesek & Kirshner, 2000; Rasch, Finch & Williams, 1992).

Learning basic skills so well that they are a reflexive action precludes understanding and being able to apply those skills in new and different situations. Even the basics should be thoughtfully considered and applied with judgment and critical thinking. Mindful learning involves connecting new ideas, openness to new information, and awareness of more than one perspective (Langer, 1997). So how can preservice students revisit these overlearned skills and begin to use them mindfully? Civil (1990) suggests that students need to become inquisitive about the elementary mathematics that they have been using for years. They need the opportunity of reconnecting with their own mathematical capacities (MSEB, 2001). Doll (1989) used the term "unpacking" to describe how young students picked apart problems and reconstructed them into new problems. Caine et al. (1994) suggested unpacking of fundamental concepts to seek deeper meaning. Ball & Bass (2000) refer to this as unpacking the content, a decompression or deconstruction of basic mathematical concepts and skills in order to reexamine and reconstruct with understanding.

Unpacking the Content

Preservice teachers work with mathematical content for students in its growing state. It seems they have to work backward from their own mature, compressed understanding of the content, to unpack the constituent elements of even the most basic mathematical notions (Ball & Bass, 2000; Kirkpatrick et al., 2001). Unpacking the content allows teachers to examine the undergirdings and interconnections of mathematical algorithms, definitions, and properties, in such a way that they can later identify when children are applying and understanding them correctly, or where there are mistakes and misconceptions.

This unpacking also serves as a conduit for preservice teachers to begin developing a personal relationship with mathematics (Ball, 2002). "Learning is a process of continually restructuring prior knowledge, not just adding to it" (MSEB, 1993). Preservice teachers need to work their way through "the disequilibrium, disorientation, and confusion of unsettling that which they had always done, and done well" (Schifter & Fosnot, 1993, p. xi). Instructors can also use unpacking as an assessment tool. Ball (1989) writes

> First, educators must judge what prior learnings can contribute to future growth and which may impede it. This implies a need to examine what learners bring – what they already know, believe, assume, and are inclined to do. Educators must also have a vision of where learners are headed and what ideas, beliefs, attitudes, and dispositions are likely to prove useful for moving in the direction. Second, educators must be able to construct the conditions for experiences which can foster future growth . . . past experiences can also be reinterpreted and reconstructed, given new lenses, new assumptions, new ideas [to help] students reinterpret their past experiences with mathematics and redirect their future experiences with it. (p. 5)

Problem Solving

Many suggest (see Cobb, Yackel & Wood, 1993; Confrey, 1990; Davis, Maher & Noddings, 1990; Duckworth, 1987; Hiebert et al., 1996; NCTM, 2000; Schifter &

Fosnot, 1993; Wheatley, 1991) that providing a problem-rich learning environment will encourage students to revisit their learned mathematics and expand their understanding. Problem solving is a complex activity that involves more than merely recalling facts and well-learned procedures. It is not a mathematical topic to be relegated to the first or last chapter of the textbook. It is a way of thinking and an approach to solving all mathematical situations (LeBlanc et al., 1992). It is through problem solving activities that mathematical concepts acquire their meaning (Smith, 1997). Preservice teachers especially need to become aware of and develop their own abilities as mathematical reasoners. They need multiple opportunities to develop, follow, and critique their own and others' mathematical arguments, to make inventions; to make sense and assign meanings; and to interact mathematically in a way that will serve to develop their mathematical understanding (Bastable, 2001; Murray, Olivier, & Human, 1998).

Choosing tasks

Choosing a good task is difficult. Most story problems are exercises rather than actual problems. A task is problematic only if the solution method is not known in advance (Charles & Lester, 1982; Cobb & Wood, 1988; Hiebert et al., 1997; Murray et al., 1998; NCTM, 2000; Schoenfeld, 1985; Wheatley, 1991). Schoenfeld (2001) states, "In the real world, problems do not come neatly packaged with methods of solution attached; our job is to figure out how to approach them" (p. 53). Doll (1993) suggests that the mathematics curriculum should be selected with four criteria: Richness, Recursion, Relation, and Rigor. Choosing mathematical tasks with Richness provides a full-bodied, complex scenario that affords exploration of multiple and important mathematical ideas. Recursion offers patterns that can be identified in iterative explorations. Relation provides connections between and among mathematical ideas. Rigor requires verification and validation of solutions.

Good tasks also include collaborative sense making, developing strategies, collecting and recording data, and reflection on processes and solutions (Remillard, 1990). Reflection is an integral part of a good problem, from beginning to finish. Reflection can be mental, verbal, or written; it may be applied individually, in a group, or with a whole class. Doll (1993) emphasizes that the process of reflection should be critical, public, and communal for classrooms. Reflective writing helps students by being therapeutic, deepening their content learning, improving their problem-solving skills, and changing their perception of mathematics (LeBlanc et al., 1992). Reflection is also the culminating portion of the problem, where the mathematical results are reinterpreted in terms of the initial problem situation (Romberg & Kaput, 1999). New aspects of understanding come from the reflection, not the solution (Russell, 1999). Looking back may be the most important part of problem solving, because it is what is learned after the problem is solved that really counts (Wilson, Fernandez & Hadaway, 1993).

Making Mistakes

Problem solving legitimately involves some false starts and blind alleys (MSEB, 1993). It is the nature of a genuine problem to provide a challenge at the beginning. If a student looks at a problem and knows exactly what to do, then it is not a genuine problem but simply an exercise. A genuine problem gives no obvious clue about how to start the problem solving process. Students often employ trial and error to test possible methods. This often leads to mistakes, but what is important is what the student does with those mistakes. Mistakes should be looked on as an opportunity to begin limiting the possibilities. Mistakes should not be covered up or ignored, but used as a constructive learning opportunity (Heibert, et al., 1997).

Seeking Understanding

Heibert et al. (1997) state, "Knowing mathematics, *really knowing it*, means understanding it. . . . Understanding is crucial because things learned with understanding can be used flexibly, adapted to new situations, and used to learn new things. Things learned with understanding are the most useful things to know in a changing and unpredictable world" (p. 1; emphasis in original). Developing mathematical understanding involves more than simply adding to the knowledge pool, more than just connecting new knowledge to prior knowledge. It requires the creation of rich, integrated knowledge structures. These knowledge structures are less susceptible to forgetting because there are multiple paths to retrieving it (Carpenter & Lehrer, 1999). Good problem solving tasks can build such structures and paths.

Mathematical Community

Hiebert et al. (1997) speak of five dimensions that frame a classroom. These include the nature of the task, the role of the teacher, the social culture of the classroom, the available mathematical tools, and the accessibility of the mathematics for every student. The social culture of the classroom has a significant effect on the learning of the students. It is how students relate to and interact with each other, and what the expectations are for the mathematics they will be doing. Also important is the valuing of ideas, the autonomy of the students, and an appreciation for making mistakes.

Learning is increased in a setting that allows students to share and compare their thoughts. A classroom community that promotes student collaboration can provide a rich and supportive environment for students to take risks and explore and justify their ideas. In such a classroom, students respectfully listen to each other, but also make value judgments about the thoughts that are shared, with critical and nonjudgmental questioning of others' conjectures (Bransford, Brown & Cocking, 2000; Cassel, 2002; Cobb, Yackel & Wood, 1992; Mewborn, 1999; NCTM, 1989). Collaboration involves interactive communication, not students working side by side and then checking their answers together. In this way, there is a sharing of authority and contributions are equally valued (Mewborn, 1999).

Prior experiences in mathematics classrooms, along with society's attitudes (negative social norms) toward mathematics, combine to convey strong messages that work against a genuine community environment. A community is involved in a common process of inquiry, but the habits and preconceptions formed in the culture of the traditional mathematics classroom are often highly resistant to reform to community (Schifter & Fosnot, 1993).

Sociomathematical norms

It is important for mathematics students to move from an ingrained emphasis on speed and accuracy to an emphasis on reasoning, and from mechanical memorization to valuing conceptual understanding (Wilson & Ball, 1991). Social norms are often introduced as expectations of the teacher relating to students' actions, but sociomathematical norms must grow out of the students' involvement with the teacher, each other, and appropriate mathematical activities. Sociomathematical norms are involved with deciding what is appropriate, sufficient, and different in mathematical discussions, procedures and solutions (see Figure 1).

In the process of negotiating social and sociomathematical norms, students construct and reconstruct personal beliefs and values that help them become increasingly autonomous in mathematics. Social norms benefit students by developing social autonomy, while sociomathematical norms benefit students by developing intellectual autonomy (Yackel & Cobb, 1996). These sociomathematical norms become an intrinsic aspect of the mathematical culture of the classroom and serve to heighten the intellectual autonomy of the students.

NORMS	SOCIAL NORMS	SOCIOMATHEMATICAL NORMS
Mathematical sophistication	•Expected to contribute to mathematical discussions	•What is an appropriate and valuable contribution •What counts as mathematically sophisticated, efficient, elegant
Mathematical explanation	•Expected to explain solutions and ways of thinking •Justification is mathematical rather than status-based (I'm older; I'm smarter)	•What is an acceptable mathematical explanation •What is acceptable argument or justification
Mathematical difference	•Expected to offer different ways of thinking, methods, representations, and solutions	•What counts as mathematically different •Identify and judge similarities and differences among various solutions
Mathematical communication	•Expected to listen and try to understand others' ideas and solutions •Expected to ask questions, challenge ideas, and reflect	•Become intellectually autonomous •How does it make sense •What is taken-as-shared basis for communication

Figure 1: Social and sociomathematical norms (compiled from Yackel & Cobb, 1996).

Autonomy

Many times students perceive their mathematical knowledge as something that someone in authority has told them or shown them. They see it as someone else's knowledge (teacher, textbook, old dead mathematicians) that they are just practicing (Carpenter & Lehrer, 1999). As long as students seek approval from an external authority, their own thoughts and beliefs will remain hidden (Schifter & Fosnot, 1993). Ball (1998) writes

> If the teacher is the source of validation and answers, this communicates to students that mathematical truth is something one has, not something one establishes with knowledge and reason. The answers are in books or stored in the heads of those with more education. Either of these alternatives, while perhaps efficient, misrepresents what it means to learn or to know mathematics, making it seem that the task is to acquire and store knowledge in one's head. Neither do these approaches help students acquire the skills and understanding needed to judge the validity of their own ideas and results – to be "independent learners" or to be "empowered," part of the rhetoric of education today. (p. 95)

One possible outcome of a collaborative community is an increase in student autonomy. A supportive teacher and a safe classroom environment that promote a sharing of both authority and responsibility can encourage intellectual autonomy. Kamii (1985/2000) defines autonomy as the "ability to be self-governing in the intellectual realm, the ability to make judgments for one-self, searches and questions for one-self (not just accepting with checking), puts things in proper relationships" (p. 57). Intellectual autonomy occurs when students are encouraged to take responsibility for their knowledge construction in conjunction with other class members (Wheatley et al., 1995).

When a student relies on external authority (teacher, textbook, peer) to verify a method or solution, their only justification is "somebody told me." A less dependent learner may justify it because "it gets me the correct answer." An autonomous student with high internal authority can justify methods and solutions by saying, "I can deduce it from other things I know," as they have convinced themselves. They are then in a position to convince others with appropriate arguments (Faux & Mason, 2001). A student can justify a mathematical claim by dogmatic assertion, by force of authority, or by reasoning through a sequence of steps, each justified and universally persuasive from a base of common knowledge (Ball & Bass, 2001).

Current Related Research

Unfortunately, little research has examined what such a problem-rich learning environment might look like in the college mathematics classroom. Duckworth (1987) researched problem solving with inservice elementary teachers. The teachers showed insights for understanding and the nature of mathematics, and took more mathematical initiatives with the children they were teaching. Schram et al. (1988), piloted a sequence of innovative mathematics courses for preservice teachers in the Teacher Education and Learning to Teach Study (TELTS). Their results showed a change in students' perceptions of the nature of mathematics, the nature of a mathematics classroom, and their understanding of how mathematics is learned. A followup report (NCRTE, 1991) found that only the preservice teacher education courses that were focused on mathematical content and understanding (rather than methods) showed these change in attitudes and understanding.

Civil (1990) researched preservice teachers in a mathematics content course. She found that students were confused but willing to try to make sense of mathematics. Their beliefs about doing mathematics the traditional school way often interfered with the exploratory nature of the tasks in the class. She concluded that content courses for preservice teachers should challenge these beliefs, present them with tasks likely to create cognitive conflict, and provide opportunities for peer communication. LeBlanc et al. (1992) were involved in PETTM, a four-year study of preservice teachers in mathematics courses with a problem solving focus and use of reflective writing. They found that when students are encouraged to be reflective about their problem solving, their reflective ability improves, their content understanding increases, their motivation expands, and they begin to look for and make connections between mathematical concepts.

Lester & Mau (1993) taught preservice teachers in a new mathematics content course that emphasized problem solving in cooperative groups. Results included a change from transmission teaching to development of personal autonomy and increased confidence and determination. Santos (1993) wrote about preservice teachers' problem solving with fractions. She found her students broadened their mathematical understanding and enhanced their metacognitive awareness of themselves as learners. Austin (1996) incorporated problem solving in a class with inservice teachers. These teachers reported finding their own voices, recognizing the mathematician within, and gaining confidence mathematically.

Perrine's (2001) research focused on preservice teachers and a problem-solving mathematics course. She found a statistically significant gain in proportional reasoning over the semester and through winter break as students seemed to understand and retain more information when involved in problem solving activities. There was also an accompanying gain in change of attitude toward mathematics. Ball (1998a; 1988b; 1990a; 2002; Ball & Bass, 2000; Ball et al., 2001) is currently researching problem solving with preservice and inservice elementary teachers. I had an opportunity to observe some of her research in action at a workshop in January 2002. As I watched twenty preservice teachers struggle with seemingly simple fraction scenarios, I was amazed to see the students delve into their preconceptions and misconceptions to begin making connections between "learned" definitions and skills and among new and different representations. I realized that this was what I had been trying to accomplish with my own preservice students, and this experience helped crystallize in my mind the direction I wanted to take in my teaching and my research.

Goal of this Study

As I thought about how to accomplish my newly defined goal, I realized that I needed to organize my thoughts about what I want to accomplish with my own students in my college mathematics classes. I want to

- help students develop a positive relationship with mathematics
- provide a problem-rich environment to begin making sense of the mathematics
- negotiate sociomathematical norms
- develop a community of mathematical discourse (Ball, 1988a, 1993) that allows sharing of responsibility and authority

With these goals in mind, I investigated the use of problem solving activities with periodic reflections and an emphasis on community (Lappan & Even, 1989).

CHAPTER THREE

RESEARCH QUESTIONS

> When a mind is stretched by a new idea, it never goes back to its original dimensions.
> Unknown

Focus Questions

Given the typical preservice elementary teacher, and the need for learning a different kind of mathematics to prepare them for teaching, it seems that they need to be able to unpack, revisit, and reconstruct the learned mathematics content they bring with them to the college mathematics classroom. What mathematics do they bring with them, and what do they consciously believe and subconsciously assume about it? How can we provide preservice teachers with the time and space to explore mathematics in such a way that already "learned" content can be unpacked, revisited, and reconstructed with purposeful understanding? How can we provide the opportunity for them to construct relationships between and among mathematical ideas in a college mathematics classroom? With many concerns about how I can best serve my preservice students in the mathematics classroom, I am focusing my research on the following questions:

1. What do preservice teachers bring with them to a college mathematics content course?

2. How may a problem-rich learning environment enable preservice teachers to unpack the mathematics content they have already acquired?

3. How does unpacking the content through problem solving affect mathematical understandings?

These are important questions to look at in the current sweep of reform in mathematics education and teacher preparation. We need to start where preservice teachers *are* in their mathematics understanding, and provide opportunities for them to grow and gain the understanding they need to successfully teach mathematics in their future classrooms. Much of the content they bring to the college mathematics classroom may not be in a form that can be used in practice.

There is very little research published in the area of mathematics content for preservice teachers. This study will be a valuable addition to the research base concerning preservice teachers learning mathematics content for understanding. The information gleaned in this study may help improve the teacher preparation process and inform and direct future studies in this area.

Research Setting

I teach a two-semester sequence of mathematics content courses specifically designed for students majoring in Elementary, Early Childhood, and Special education. With two sections of each course, I have approximately 140 students total per semester. Almost all students in the first course take the second course the following semester. A few students in the second course are new to me: those who have transferred in, tested out, or are returning to school. Most students are sophomores or juniors, unless they have transferred, changed majors, or are returning to school.

The first course, MATH 2213: *Mathematical Systems*, involves the real number system, operations, properties, patterns, sequences, and functions. The second course, MATH 3213: *Data Analysis and Geometric Systems*, offers concepts from probability, statistics, and informal geometry. Prerequisites include two semesters of freshman level mathematics, usually Math for Critical Thinking and College Algebra, although some students have taken a calculus course. Both of my classes have a course packet of readings, handouts, and activities, which replace the traditional textbook. Students are encouraged to attend class regularly as the focus is on personal interaction with teacher, peers, and problem solving activities.

Since both courses consist of relatively basic mathematics, my emphasis is not on calculation or speed, but on understanding, relating, extending and connecting general mathematical ideas. Many simple mathematical algorithms are familiar to students through their own schooling and decades of use. However, when we begin exploring the basis of these calculational tools and familiar algorithms in class, most students have no idea why they are used or where they came from. Why do we flip and multiply when dividing fractions? Why do we count the total decimal places when multiplying decimals? These are questions they have never thought about. They follow these rules because they have been taught to do so, and have (for the most part) been successful achieving answers with these algorithms, without really thinking about them.

Establishing a Mathematical Community

As each semester begins, I tried to model my classroom as a community, a safe and supportive environment where we can explore mathematics together. All were encouraged to participate with equal value and authority, with the teacher as a facilitator. Class decisions were generally made by consensus, and all were expected to contribute to discussions and share their ideas. I encouraged and expected students to work together in pairs or small groups. I allowed students to self-select their groups and honor the choice to not be a part of a structured group. Some of the groups remained steady throughout the semester and some changed membership due to non-attenders or two groups working together on a certain problem. Some students who were absent the day a Problem of the Week (POW) was introduced were only able to work on it alone at home. Some students involved various others (friends, roommates, family) outside the classroom.

At first, I had to deal with students' inclination to try to guess the expected response instead of relying on and developing their own understandings. I wanted my students to explain and justify their solutions, try to make sense of others' explanations, indicate agreement and disagreement, and think about and offer alternative methods and solutions. However, students must be willing to negotiate this process (Cobb, 2000) as we seek for equal authority and shared responsibility for learning and understanding. I began by refusing to judge methods or verify solutions, instead inquiring where their thinking was going and how they could convince themselves and their peers with their reasoning. Probing questions about the applicability of their pattern or conjecture in similar or different situations allowed students to check the viability of their own conjectures (Wheatley et al., 1995).

At first, many of the students were uncomfortable and frustrated with my actions, which are so different from their previous experiences in mathematics class. Soon students seemed to appreciate the freedom to explore alternative methods, to share their thinking, and to risk examining their misconceptions and errors in reasoning (Hiebert et al., 1997). They began to feel the sense of a mathematical community that is supportive and enlightening. They also reported appreciating group work for the first time or to a greater extent than before.

Establishing a Problem-Rich Mathematics Environment

A problem-rich environment was part of community development and a key to establishing classroom norms that value discussion, explanation of reasoning, problem solving and challenging each other's ideas. Tasks were designed for potential learning opportunities, more time was spent engaging in complex problems than in lecture and drill, and the emphasis was placed on process rather than right answers. Time and space were provided for students to work on problems individually, in small groups or large groups, during class time. The problems were open-ended and predisposed to multiple methods and solutions. Students were expected to continue to pursue these problems over an extended period during the semester. They were encouraged to use whatever resources they felt would help them solve the problem rather than expecting them to have engaged in problem solving to practice particular skills.

One approach to problem solving activities was to help students problematize the mathematics they thought they already knew. A good task was a genuine problem for students, one for which they did not have a handy memorized rule to apply. Within the frame of the task was the opportunity to explore mathematical ideas and come up with reasonable methods for reaching solutions (Hiebert et al., 1997). Appropriate tasks made the subject problematic for students (students saw the task as an interesting problem), began connecting where students already *are*, and engaged students in thinking about important mathematics.

Revisiting mathematical concepts in a way that offered possible contradictions or exposed poor assumptions and "thin understanding" (Ball, 1990b, 1990c) was an attempt to "rock the boat" of their learned mathematics. Both the students and I were continually surprised by the levels of complexity within even such simple-seeming mathematical concepts as even and odd numbers or prime factors.

Research Design

I chose an emergent design for my research, as described in the work of Cobb (2000) and Steffe & Thompson (2000). This theoretical stance was underpinned by teacher as researcher, and the classroom teaching experiment, a dynamic research plan directed toward understanding the progress students make over an extended time. "The main goal is to develop descriptions of existing situations, or conjectures about possible situations" (Kelly & Lesh, 2000, p. 363).

The social perspective drew on ethnomethodology (Cobb, 2000). It focused on mathematical activity in the social context of the classroom, and the reflexivity between students' constructive activities and their participation in social processes. I documented the microculture established by the classroom community and developed theoretical constructs that were used to make sense of what is going on in the classroom. I focused on the conceptual reorganizations that students made while engaged in mathematical activities and interaction with their peers. This methodology considered the reflexivity between theory and practice (Cobb, 2000), where practice generated theory and theory informed practice, and cycled again as practice validated theory and theory suggested practice.

The teaching experiment was an extension of my past exploratory teaching with previous students in these classes. From my experiences and readings, I developed some ideas of what may happen during a semester of problem solving activities with my students. However, since my students were human beings that are self-organizing and self-regulating, I also anticipated some independent contributions and some surprises

(Steffe & Thompson, 2000).

In an emergent design teaching experiment, I as teacher-as-researcher did my best to "forget" the expected results and tried to keep an open mind during subsequent teaching episodes. I was then free to adapt to the constraints and possible new paths encountered while interacting with the students. I carefully chose the ideas and questions presented to the class in order to stimulate interaction, new ideas and further questions, which were analyzed and followed up during subsequent teaching episodes. As always, the students' anticipated and unanticipated language and actions steered me in a new or different direction. Wholly unexpected possibilities opened from a surprise question or comment by a student or group of students. I was continually and simultaneously listening and making choices of which interactions to pursue and which to redirect by further questioning. Thus the research goal structure was continually being modified to fit with the students' mathematical activity (Steffe & Thompson, 2000).

Research Population

I taught one and then both of the mathematics content courses for the past four years. My research population consisted of students from the last four semesters (not including summers) with most students taking the courses consecutively. Demographic data gathered by survey (see Appendix A) from the 129 participating students in the most recent semester are offered here as a representative sample of the research population. One student declined and two students were unable to continue participation in the study. Their information was not recorded for analysis.

Ninety-three percent of the students were female, a gender imbalance that is consistent with other research on preservice teacher populations (Fisher, 1992; Green &

Weaver, 1998; Su, 1993) and my experience with these classes for the past four years. Student race/ethnicity was 89% Caucasian and 6% Native American, with 5% reporting Black, Hispanic, Asian, or mixed heritage. The relatively high percentage of Native American students is reflected in both the state and university populations. About 18% of the students described themselves as non-traditional students. The majority of students were sophomores and juniors, with six freshmen and 25 seniors.

Data Collection Methods

In a teaching experiment, the collection of data is accomplished over an extended period of time in a classroom situation. Multiple data sources are used to examine the questions from multiple perspectives and to provide multiple confirmations of interpretation by the teacher-researcher. I collected data in five primary, interdependent ways: short response papers, narrative journals, reflective essays, interviews, and a final questionnaire. I also recorded my own observations of students during class time in a teacher journal. A short survey collected demographic information to compare the population of preservice teachers to the characteristics of other research populations. Below I describe each of the six data sources.

Short Response Papers

The first data collection method was a series of short response papers (402 total) collected only in the first course, as students and I began our relationships. Students in the second course were familiar to me and knew what I expected of them. These short response papers served as immediate feedback by class members after their initial collaborative work on posed problems. These papers are not expected to be finished papers with a well-thought out analysis, but an immediate debriefing of a specific classroom interaction, a short reflective communication for each student to begin contemplating his or her own mathematical thinking.

For example, while discussing a specific mathematical principle, several students seemed to have breakthrough insights that electrified the atmosphere of the classroom. Just before the end of the class period, the students were asked to stop working and record their thinking and feelings about the problem. The papers were immediately turned in for me to read, respond with written comments, and be returned the next class period. My written comments varied according to the student response, sometimes including questions about a specific phrase or section, sometimes acknowledging a particularly strong emotion, or encouraging more sharing of thoughts in the future. Many of the comments were further questions, encouraging the students to think deeper in this area.

The short response papers provided a baseline of student thinking and encouraged initial communication with me as a new teacher at the beginning of the study. Students continued to provide quick glimpses into their own thinking and feeling throughout the problem solving activities. The short response papers provided a "spot check" of student involvement and understanding, facilitated student reflection, and were used to begin identification of the mathematical understanding that students brought with them to my classroom. The response papers also provided information about students' weaknesses and misunderstandings. These data were used to help answer my first research question, *What mathematics do preservice teachers bring with them to a college mathematics content course?*

Narrative Journaling

The second data collection method was based on a series of problem solving activities given over a six-week period of the semester. These problem solving activities are informally known by my students as POWs. These problems have been adapted and refined over the last four semesters to challenge previous mathematical learning, extend mathematical understandings, encourage students to handle diverse complex situations, and to explore and make connections among mathematical ideas. These problems are open-ended, with the opportunity to use multiple methods and arrive at multiple solutions (see Appendix B for sample POWs).

Five or six POWs were presented in each course. The problem contexts were carefully chosen to be interesting and provocative to the students, and to allow them enough time and creative space to revisit basic mathematical concepts in hopes of unpacking the content and deepening their understanding. Each problem was coordinated with a particular course of study. POWs presented in the *Mathematical Systems* class centered around basic mathematical concepts such as even and odd numbers, prime factors, and patterns. POWs in the *Data Analysis and Geometric Systems* class encompassed various geometric concepts such as measurement and dimensionality. The emphasis is on the process of arriving at a solution, not on the correct answer to the problem. The process is designed to be a collaborative effort in a reflective setting. Each student should be sharing their ideas with the group, appraising others' ideas, and defending and corroborating solutions. The goal is to provide a problem-rich learning environment to allow them to unpack the mathematics content they have already "learned."

Normally a full class period was given to working on each POW. Students were encouraged to work in pairs or small groups, and expected to continue working on the POWs outside of class on their own time. During class, I floated around, listening to group discussions and observing what students were trying and recording. At times I queried a group about their thinking, or posed a probing question to try to deepen their investigations. Much of the unpacking occurred through these conversations as students sought to justify their answers to one another.

Students were required to keep a narrative journal of their attempts on each POW, whether their methods were fruitful, what their thinking processes were, and the reasoning behind their solution(s). I emphasize that I do not expect or desire a technical description of their procedures, but want a glimpse into what they are thinking, since I can't see inside their heads. I told them that the narrative journals were as important as the problem solving effort, as it served as a reflective process, and encouraged them to make connections and develop relationships with other mathematical concepts and problems. In the journals, students were to include any sketches, charts, diagrams, or tables they used, and to describe any manipulatives or other representations used. They also described how working with a peer or group of peers affected their interaction with this problem. A grading rubric was given to the students so they had an idea of my expectations for their narrative journals and reflective essays (see Appendix C).

Each student chose four POWs to be included in a final course portfolio, including the narrative journals and an overall reflective essay. This assignment was graded and formed a major portion (20%) of the students' final grades in the class. I collected 418 portfolios from 320 students over the course of four semesters. The majority of students are very willing to share their thoughts and actions in dealing with the POWs. Most students were motivated to engage in these problem solving activities because they realized that in just a short while they will be teaching mathematics in their own elementary classrooms. They realized they needed to be able to communicate mathematically with their future students, and seemed to realize that participating in the problem solving activities and written reflections might be a way of facilitating their ability to communicate mathematically.

The narrative journals provided information to answer two of my research questions, *What mathematics do preservice teachers bring with them to a college mathematics course?* and *How may a problem-rich learning environment allow preservice teachers to unpack the mathematics content they have already acquired?*

Reflective Essays

The third data source consisted of 231 reflective essays included as part of the portfolio in both courses during the last two semesters. It was designed to let the students reflect on what they experienced and learned in the course. They were encouraged to comment on the readings, class discussions, POWs and other problem solving activities, and their interactions with others in the class. In the description of the portfolio assignment, I gave a short list of writing prompts for students who weren't sure what to write about in their reflective essay. Some of the suggested questions to discuss included:

How did the POWs help you in your mathematical understanding?

How has your mathematical understanding changed during this course?

What do you know and understand NOW that you didn't get before?

The reflective essay was included as part of the portfolio assignment as an overarching description of the students' experiences, interactions, and mathematical growth throughout the semester. This data source specifically addressed my third research question, *How does unpacking content affect their mathematical understandings?*

Interviews

Although the POW portfolio was a major source of data in my research study and a large part of answering my research questions, I expected there to be gaps in answering my research questions and also that new areas might be opened to explore. Therefore, the preliminary analysis of general trends and unexpected ideas in the POW portfolios were used to design questions for followup group interviews. I selected three groups of students, based on their portfolios and observed classroom interactions. I chose a group that seemed to function well in facilitating learning from each course, and a group which did not seem to facilitate learning for their group members. I interviewed each group with questions about their interactions, understandings, and emotions while working together on the POWs. I also "checked" my interpretations of their portfolios and my class observations with them to verify that I was not misunderstanding or reading anything into them that did not belong. This was a check for research bias.

The beginning questions for these interviews were developed from the preliminary data analysis. However, I expected the interview questions to begin to diverge in response to the replies and interactions of those being interviewed. The group interviews lasted about an hour each, and were audio taped and transcribed. Some of the initial interviews allowed adjustments and additions in subsequent interview questions as gaps were filled and new ideas identified for exploration. Data gathered in the interview process provided answers to all three research questions, but especially the third question, *How does unpacking content affect their mathematical understandings?*

Questionnaire

The data collected in the portfolios and interviews were used to design a final set of four questions that I included on the final exam. Two of the questions were about a specific mathematical topic and two questions were about mathematical understanding (see Figure 2). The final questionnaires were another attempt to fill any gaps and refine my analyses in order to answer my research questions.

Math 2213 Final Questionnaire

1. What do you now understand about the mathematical concepts of **even** and **odd** that you did not realize before this class?

2. Describe the numbers **zero** and **one** and the role(s) they play in the system of real numbers.

3. What is the most significant change that happened to your mathematical understanding this semester?

4. How did working on the Problems of the Week help you to deepen your understanding of mathematics?

Math 3213 Final Questionnaire

1. Which geometrical concept(s) have you gained a deeper understanding of in this class?

2. What is the most significant change that happened in your mathematical understanding this semester?

3. How did working on the Problems of the Week help you to deepen your understanding of mathematics?

4. What has been the greatest help to your mathematical understanding over the two semesters of classes (Math 2213 and 3213)?

Figure 2: Final questionnaires for Math 2213 and Math 3213.

Teacher Journal

The last data collection method was a teacher journal. A teaching journal is designed to provide a detailed, near-immediate description of what has occurred in class from the teacher's perspective. After several class periods during the semester, I "debriefed" myself by recording my observations, thoughts, and impressions of students and group interactions. It also provided a place for me to "think out loud" about what I should have or could have done in a given situation. It provided a reflective place for me to deal with frustrations and confusions, and to make notations of actions and words that I was especially excited about and pleased to hear from my students' interactions. Another part of my teacher journal included audio taped observations of groups working on a problem of the week. These tapes were transcribed and added to my teacher journal.

The teacher journal was written on my personal computer and indexed by course and date. It provided a way to look back and identify general trends, insights that I have had, and to record questions for further study and analysis. The data were used to refine future directions of the study. Since this journal was a record of my perspective of what I observed and heard in the classroom, it was important that I used the additional sources of data to verify my interpretations. My journal was a constant reminder to me to strive to remain open-minded and unbiased in my interpretations and impressions. It was also a way to revisit my research questions and help me realign my data collection to provide responses to my research questions.

Data Analysis

The data collected in this research study was qualitative and collected throughout each semester. The fourth semester began with collecting the short response papers and ended with the final questionnaires. The design of the teaching experiment lent itself to continuous analysis, beginning with the first interactions in the classroom. The goal was to bring as much order as possible to the data so that conclusions could be drawn based upon the criteria selected for analysis. In general, I was looking for trends when I grouped and summarized the data. An ongoing mental analysis provided insights that fed back into the teaching process, producing an iterative discovery cycle. This recursive analysis began to shape the results in a fluid way. At points I had to "freeze frame" this cycle and retrospectively look back at where we started, where we are, and where we were going along the way.

The primary analysis technique was reading the five major data collections. The first reading was exploratory to get an overall idea of what the data were offering. Subsequent readings allowed me to begin identifying and categorizing general trends, similarities and differences. I also identified and extracted examples to be used to support each theme. Subsequent readings of data records were used specifically to check whether they supported or matched the indexed categories. Each student chose a four-digit index number to be their identification in the records. The individual data record was annotated and then recorded in the proper category using the index number for identification and correlation. This was a way of associating related pieces of data in order to easily refer back to the original data record if necessary.

Each POW was designed with multiple mathematical concepts to explore. However, students came to each POW with different experiences, different attitudes, and different levels of motivation and persistence, hence, my expectations were met, not met, or exceeded to varying degrees. I tried to determine an explanation in each case. In addition, as I "saw" connections and interactions in the data, I needed to determine if the students actually made those connections or was it just my interpretation of what I was reading or hearing.

The first major milestone was the secondary analysis of the short response papers, teacher journal, narrative journals, and reflective essays using the indexing system. The first goal of the secondary analysis was to identify and interpret answers to my research questions. Another goal was to define divergent areas that need to be validated and questions that still need to be addressed. The analysis also helped determine which working groups and individual students were interviewed. The results were refined into questions for the followup interviews and the final questionnaire.

The third analysis round was done after the interviews, and helped refine the final questionnaire. A preliminary summary of the data was organized and checked to see if all research questions had been sufficiently answered. The final analysis was a last review of all collected data, including the final questionnaire data. The focus was on validating results found in previous analyses and determining extracts and examples from the data records to be included in the dissertation itself.

Criteria Standards

Replicability of data is not relevant or even desirable in the context of a teaching experiment, since one of the theoretical bases of this type of research is that initial conditions and interim changes may lead to different endpoints. Instead there is a need to develop ways of analyzing innovations tested in teaching experiments that will be commensurable in different classrooms based on context and meaning (Cobb, 2000). When research participants are "complex, dynamic, self regulating, and continually adapting systems, and when a basic assumption is made that a single situation will be interpreted differently by different subjects, then replication cannot refer to simplistic notions about doing the same things again under the same conditions" (Kelly & Lesh, 2000, p. 362).

In this case, the relevant criteria would be the generalizability and trustworthiness of the analysis. What is understood from a given case may be relevant when interpreting other cases. This may guide future research and development of classroom activities. Trustworthiness is a consideration to the extent to which the analysis of longitudinal analysis is systematic and thorough. A strength of these criteria is the prolonged engagement with the students. However, emergent design is a classroom-based research paradigm and is not intended to suggest results that are applicable on a larger political level with practices outside the classroom (Cobb, 2000).

Credibility in qualitative research is based on prolonged engagement and persistent observation of participants, and triangulation of the results. Checking written conclusions with the participants provides a final check of credibility. I checked dependability as the consistency of my research process to indicate the stability of my inquiry processes over the time of the research study. I confirmed the quality of the results of my research by supporting my findings with reference to literature and findings by other authors that confirm my interpretations (Guba & Lincoln, 1989).

CHAPTER FOUR

WHAT STUDENTS BRING

> The illiterate of the 21st century will not be those who cannot read and write, but those who cannot learn, unlearn, and relearn.
> Alvin Toffler

Research Findings

In this chapter I will present my research data, gleaned from short response papers, narrative journals, reflective essays, interviews, final questionnaire, and my teacher journal. As I read and reread the students' writings, several common themes emerged. Many students described their previous experiences with mathematics, expressing common fears, attitudes and beliefs, and discussing issues of time, power, and autonomy. I have chosen to organize these thematic data within the overlapping spheres of my research questions, and have included appropriate excerpts from student writings and other data sources.

The data were read and analyzed with the intention of addressing the following three research questions:

> 1. What do preservice teachers bring with them to a college mathematics content course?
>
> 2. How may a problem-rich learning environment enable preservice teachers to unpack the mathematics content they have already acquired?
>
> 3. How does unpacking mathematics content through problem solving affect mathematical understandings?

What Students Bring to Preservice Mathematics Classes

The first research question asks what students bring with them to a college mathematics content course. The action verb *bring* serves as a metaphor (Ball, 1988a) for students who do not arrive empty-headed in the college mathematics classroom. With more than 2,000 hours in an "apprenticeship of observation" (Lortie, 1975), spread over fourteen or more years of mathematics education, students come with already formed knowledge, beliefs, and emotions concerning mathematics. Mathematical understanding emerges from the interaction of students' knowledge, beliefs, and emotions (Ball, 1991; 1998). I must consider these three facets of mathematical experiences as I consider what students bring with them to the college mathematics classroom.

Mathematical Knowledge

Mathematical knowledge includes understanding of topics, procedures, concepts, and the relationships among them. My students have already studied the majority of the topics found in the mathematics courses I teach. They have many years of arithmetic experience and all but a very few older students took a geometry course in high school. Most have taken two prerequisite college mathematics courses (usually critical thinking and college algebra). A few students have taken calculus. It is expected that this prerequisite content knowledge is brought to the classes I teach, but the reality falls short of these expectations. I found that, although students brought considerable mathematical knowledge with them (10 to 15 years' worth), this knowledge appears to be faded, false, fragmented, and only understood on a tacit level.

Faded knowledge. Some content knowledge has been forgotten or is dusty from years of non-use. I recorded in my journal hearing a student say to her group, "I thought I understood it in like second grade, but now I'm not sure." Several students made similar comments in their short response papers:

> *... those old skills that I thought I would never have to think about again.*
>
> *Sometimes it is su[r]prising that I do not remember a lot of these concepts, just by name recognition.*
>
> *I may have a vague remembrance of the concept, but I never completely and full remember what it means.*
>
> *... I have forgotten almost every thing [sic] that I learned back in my early childhood days.*
>
> *I did not know that I had forgotten or gotten rusty on my basic math concepts ...*
>
> *I'm sure I was taught prime numbers and factorization in elementary school, however, that has been such a long time ago, I do not remember any of it.*
>
> *I don't really remember how to find the LCM. I know I did it about 10 years ago ...*

Ball & Bass (in press) refer to previous mathematical understandings that are no longer accessible to students as "faded knowledge," which matches the student descriptions students given above. Even excellent mathematics students do not always have a fresh recall of all the mathematical concepts they have encountered. Students also referred to their faded knowledge in their portfolio writings:

> *I am amazed at how much can be forgotten over a few years if not put into practice regularly.*
>
> *I came into the class vaguely recalling certain formulas and properties ...*
>
> *Over the years, so many of the concepts ... have been lost in the cobwebs of my brain.*
>
> *... the "dusty" math side of my brain.*

. . . information that was long ago pushed to the back of my mind.

Sometimes I feel that because I am 22 years old I am expected to "know" this stuff. When in all actuality, I retain very little if any of the information I am given in a math course.

I was amazed when I got into this class how much I did not know for having made mostly A's in all my math classes.

Fifty-one of the 129 students used eighteen different verbs in their narrative journals and reflective essays to describe the retrieval of this faded knowledge. Each verb began with the prefix *re-*, which means to do the action again (see Figure 3). The most commonly used verbs were refresh, relearn, and review. One student wrote in her reflective essay, "This class refreshes all of the concepts that I learned from

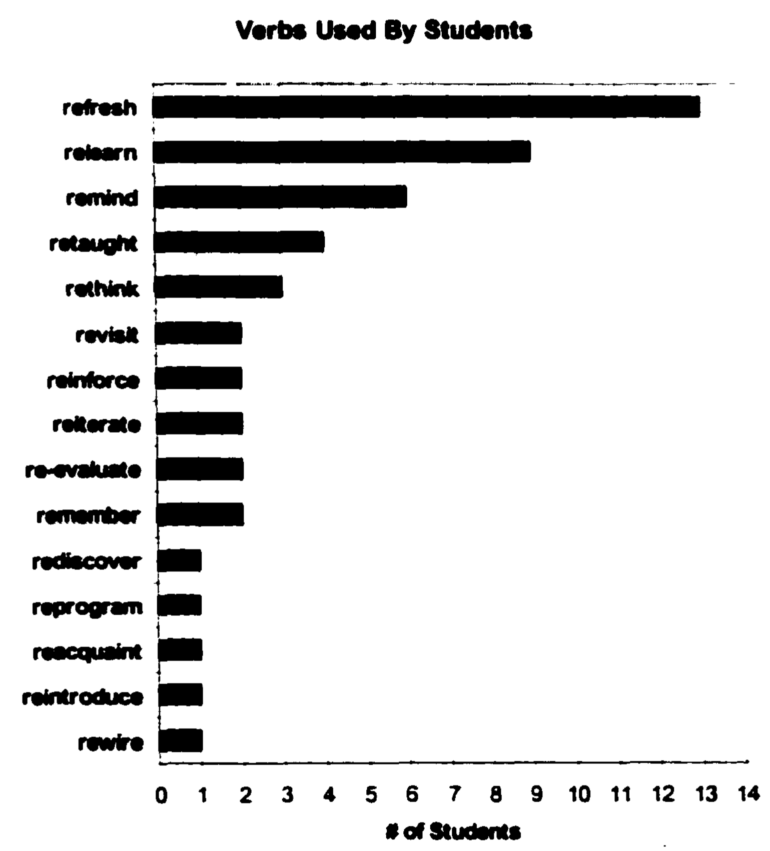

Figure 3: Fifty-one of 129 students used verbs in their writings indicating revisiting of previously learned material.

elementary school all of the way through high school, and it, also, reinforces and reteaches the concepts that I did not totally understand the first time I was taught them." The revisiting of this previously learned knowledge served to unpack the mathematical knowledge students brought with them to my mathematics classes. Students were surprised by their need to do this. One student wrote, "What I never considered is that I would also need to relearn elementary mathematical concepts that I assumed I understood." Part of the reason that these concepts may need to be relearned is due to the fragmentary nature of the information students bring to the mathematics classroom. Students had some experience with certain mathematical concepts and procedures, but this knowledge was often so fragmented as to prevent broad understanding or application in other instances.

False knowledge. Sometimes the mathematical knowledge students bring with them is incorrect. One student wrote in her reflective essay, "before taking this class, I thought that a prime number did not have any factors." Another wrote in a short response paper, "0 cannot be divided by 2." Yet another example is recorded in my teacher journal. During a class discussion of prime numbers, two students flatly stated that one was prime because that's what they had been taught. Many students were willing to accept their vociferous announcement as truth, based on an absent external authority. As I offered multiple scenarios to try to perturb their thinking, they either repeated their injunction, or blatantly excused such observed exceptions for the number one. For instance, on a list of factors for the numbers one through twenty-five, most students listed the number one as having just one factor. However, some students listed the factor one twice and then insisted it was a prime number because "it had two factors." When I asked if we could double list factors for other numbers, they countered that you didn't need to do that for other numbers, just for one.

Later the same class period, we were discussing prime factorization. I pointedly asked why we didn't include the factor one in every prime factorization if it was a prime number. A student suggested that the one is "understood" in every prime factorization, it just isn't written. Some students fell back on external authority or previous experience and were unwilling to be perturbed to reconsider their erroneous information. However, one student did write in her narrative journal that she finally understood and was fully convinced that one was not a prime number, for it did not fall into the pattern of primes in the 100 Cards POW. She was able to resolve her dilemma about the number one and prime numbers by making connections among concepts reinforced by the patterns in this problem.

Fragmented knowledge. The knowledge students bring to college mathematics classes often consists of disconnected fragments of memorized terms, facts, formulas or procedures. Students often retain only partial understanding of a topic, missing some key components or only having a shallow understanding of the breadth of the concept. For example, students admitted in their reflective essays:

> *I'm a little embarrassed to admit that before this class, I really didn't understand prime factorization.*
>
> *This course is based on some prior knowledge and I lacked in some of the areas we covered.*
>
> *Doing problems of the week helped me see where I stood at on a math level. I surprisingly didn't know very much.*

The 100 Cards POW (see Appendix B) involved various turnings of numbered cards that resulted in a pattern of cards facing up and down. One of the solutions involved recognizing that the perfect square numbers were facing down due to being turned an odd number of times (once for each factor of the perfect square number). Of the 137 students including this POW in their portfolio, 24 students specifically mentioned in their narrative journals that this solution was the set of perfect square numbers *plus the one card*. In addition, 41 students left the number one off their list of perfect squares. Even if this was a result of their marking method (several students ignored the initial turn and began recording on the second student), they did not notice that the number one <u>should</u> have been on the list of perfect squares. Overall, about half the students did not associate the number one with the list of perfect squares. Their knowledge of this concept was not complete enough to recognize the omission.

Another student wrote in her narrative journal that she was unable to follow her group's discussion of the Earth POW (see Appendix B). She was confused by their explanation that the increase in the circumference would always be 2π units. She wrote in her journal that she thought that the answer should have been a *number* value instead of a *formula* like 2π in the equation. Seventeen students struggled with adding that one extra foot *all the way around* the Earth, some adding an extra foot for each foot of circumference at the equator, and some adding one foot to the radius on *both* sides. Although they could easily remember and apply the circumference formula, they had no understanding of what the formula represented or what the function of 2π was.

Along with knowledge of topics, procedures, and concepts, there must be a relationship among these three. However, students often compartmentalize their mathematical knowledge into distinct sections such as subtraction, factorization, or algebra, without realizing that it is the interconnectivity that provides the power of mathematics. This isolation of knowledge prevents the development of connections between multiple concepts and procedures. When the goal is to attain the correct answer quickly instead of developing long-term understanding, then relationships of learning built over time don't seem to matter. "I memorized what I had to pass the test, in no way understanding the theory behind the formulas, and then forg[o]t when the test was over," one student wrote in her reflective essay. In the short response papers for the GCF/LCM POW, six students wrote how surprised they were that prime numbers had anything to do with the least common multiple and greatest common factor. In their earlier schooling they had never made these connections.

A few students wrote about the relationships and connections that they found between and among mathematical concepts while working on the POWs:

> *I think the more links you have to other sources [of] knowledge, the more likely you are to remember that information.*
>
> *I know the basic formulas for circles, but this problem required an actualization of how they all relate together.*
>
> *I never used previous knowledge or made any type of connections between problems and other tools I was given.*
>
> *The Problems of the Week helped me to learn in that they showed me that math is joined in so many ways to each other. These problems showed me that math concepts are to be used with other math concepts instead of thinking that today we are working on algebra but not subtraction. They showed me that to figure out some problems I am going to have to use one or many different concepts to find an answer.*

As the semester progressed, students began to construct connections and relationships between topics, which affected the depth of their understanding. Students began to expand their tacit knowledge into a more explicit knowledge.

Tacit knowledge. Knowledge brought by students seemed to exist only on a tacit or technical level. Consisting of a collection of memorized rules, formulas, and procedures, tacit knowledge allows students to go through the mathematical motions, but they cannot explain what, how, or why they do what they do (Ball, 1991). One student wrote in her essay, "If someone had asked me how to do something before taking this course, I would have just solved the problem for him or her . . ." Another explained in her short response paper, "Although I had written it down, it seemed "useless" because I couldn't explain my explanations." Students may have the technical prowess to perform the mathematical calculations, but they are unable to explain how, when, and why concepts are used, or whether their solution even makes sense.

I recorded in my teacher journal that during the first class presentations (second week of the course) students complained that they understood the concepts they were presenting, but how difficult it was to explain it to their partners and the class as a whole. Other comments were included in student portfolios:

> *I was not taught to "think" about math.*
>
> *I don't think I could actually explain how base 10 works; it's just something that I know and have been programmed to use day to day.*
>
> *Since I never really had to think about the math concepts in detail, I had a weak comprehension of how they really worked. I could apply concepts, but not understand why I was applying them.*
>
> *. . . I have always just gone through the motions and solved the problem without ever thinking about how I was solving it and why.*
>
> *I would recognize the answer if I saw it but I could not put [it] into my own words. . . . I could get the right answers to problems but when it came to explaining it on paper [it] was difficult.*

In the Base 5 POW (see Appendix B), for example, students were asked to think about identifying even and odd numbers in base five. In base ten, students usually determine even and odd numbers by visually checking the last digit of the number. This very technical process does not hold for the base five system. One student wrote in her short response paper, "Although I had a clear understanding of even and odds, I realized they are so embedded in my mind that I don't ever have to think of what it really means to be even or odd." Students wrote in their narrative journals that they had a great deal of difficulty separating their thinking about even and odd in base five from this traditional base ten property. It appears that this one easy trick overshadowed all other thinking about odd and even numbers.

Because of this difficulty, I chose to include a question about even and odd in the final questionnaire. When asked what they knew now about even and odd numbers that they didn't know before, almost a third of the 129 students wrote that they now knew the actual definitions and various other properties of such numbers. Six students admitted that they only knew how to list odd and even numbers before. Five students reported that they had learned nothing new about these concepts.

Students seem to only tacitly understand many mathematical concepts. There is a need for students, especially preservice elementary teachers, to revisit mathematical content with an opportunity to expand and deepen their understanding to a level that will facilitate their future teaching careers. Preservice teachers will need to be able to understand mathematically at a level that will enable them to verbally explain and give examples, and to follow and deal with any misconceptions or alternate procedures of their future students. They will also need a variety of facts and terminology that are not intrinsic to the mathematics but are socially and culturally determined. They will need to model and explain this type of knowledge that Piaget referred to as social knowledge (Kamii, 1985/2000).

Social knowledge. Social knowledge is information that is socially determined and hence must be transmitted through the culture. An example is the number of days in each month which was a critical piece of information needed for the Palindrome Date POW (see Appendix B). Some students remembered how many days but others described using memorized devices such as a poem, song, or the famous "knuckle trick," to determine the number of days in each month. Students who had no available associated social knowledge used calendars as a resource. One student even commented in her short response paper that the problem assumed a Christian calendar, and that other cultures would have different notation and hence different palindrome dates.

Two students commented in their narrative journals:

> *. . . I did have previous outside-of-class knowledge that I brought to class with me.*
>
> *I am constantly able to use knowledge from outside the course within this course.*

The students in my classes brought many types of mathematical knowledge with them to the college classroom, but this knowledge is sometimes faded, false, fragmented, and only tacitly understood. Overall, their mathematical content knowledge can be considered deficient and insufficient both personally and for teaching. Such mathematical knowledge does not form a strong foundation for increased understanding and embracing new mathematical concepts. In addition, such mathematical knowledge is accompanied by mathematical beliefs and emotions that may further encumber additional learning.

Mathematical Beliefs

The students in my classes brought beliefs about the nature of mathematics and their own mathematical abilities to the classroom. The majority saw mathematics as dualistic and power laden, with a dependence on external authority for verification. They believed that answers should be found quickly and easily. A student wrote in her short response paper, "I thought that I would have to answer quickly and that I should be able to work it out in my head." These types of beliefs have a strong influence on both their mathematical performance and their mathematical understandings. They also implicate ideas about the nature of mathematics, that they need to come to answers quickly and that there is only one acceptable or correct way to do a problem, and that mathematics is reserved for those who are "gifted" or special, not for them. Those who have this special insight, they come to believe, should provide guidance and authority for determining correct approaches and answers to mathematics problems.

Nature of mathematics. My students seemed to view mathematics differently from other disciplines they studied. In their reflective essays, students compared mathematics to other fields of study:

> *Arithmetic will never be as easy to learn as spelling is.*
>
> *I have always enjoyed creative subjects such as English . . . I have never felt that freedom in math.*
>
> *Children are encouraged to express themselves through art, music or dramatic play, but when it comes to math, they are often tied down to one system of working through problems.*
>
> *In some subjects, such as English and Science, we are taught to explore the possibilities and use our imagination. In math, however, this is not the case.*

There seems to be no room for creativity or imagination when the only goal is a quick and correct solution. In their portfolios, some of the students described mathematics as "cut and dry," "not only difficult, but also boring," "a series of formulas," "an accumulation of irrelevant concepts and theorems," or "an absolute science." Several students described it as "black and white," as one student writing in her reflective essay confessed: "I have been raised knowing that math is black and white. The world around math is full of gray, but when you move into the world of math there is no in-between." Other comments from student writings about the nature of mathematics included:

> *I was taught to believe there are no words in math.*
>
> *Math has always been a number thing to me, so when you add the words in, it gets all messed up.*
>
> *All of my previous teachings had led me to believe that math was entirely concrete and numbers and letters should never be mixed.*
>
> *. . . all things math[e]matical are formulas.*

One of the major differences students saw between mathematics and other disciplines was the application of the "one right method" in achieving the "one right answer."

<u>One right way</u>. More than half of the students made reference in their writings to their previous experiences in mathematics at the elementary, secondary, and college levels. The most common description centered around their perception of the one right way to do mathematics. For example, students wrote in their essays:

> *I have been conditioned throughout my other math classes that there is one correct method or formula for each problem, and that each problem should be worked according to that method or formula.*
>
> *I was convinced that there really was only one right answer and only one right way to find the answer.*
>
> *Throughout my previous mathematical education, the existence of one method to follow and one correct answer to discover was always conveyed as the most important component to the comprehension of mathematics.*
>
> *As a student you are taught many things and how to do them, but you are never taught why to do them that way or even if there are other options.*

Students felt they were not allowed to determine their own methods but were always required to reproduce the modeled version. They were often penalized if another method was used, even if the solution was correct. Students were not encouraged or even allowed to question why a problem is done a certain way or a certain formula is applicable. One student who did question why reported being told by teachers, "that is just the way it is" or "there is no answer" or "you will learn that later when you are older." There is an impression that the canon must be transmitted in toto without any questioning. However, some students seemed to resent the perceived power and authority of the teacher and textbook. Students' comments included:

> *. . . they wanted it done their way, and if it was not done that way, they would count it wrong, and not even care that the answer was right.*
>
> *All math classes that I have taken, the teacher has you learn one way in finding the right answer, and when it came to test time you needed to remember how to do that problem like the teacher or it was counted wrong.*
>
> *There was only one correct way of getting to the correct solution, and if you missed either part, you were totally wrong.*
>
> *I was taught that if you did not do it a certain way (the teacher's way) then it was wrong regardless of the end [product].*
>
> *. . . I became frustrated with my elementary experiences with math. I cannot help but think about what my understanding could have been . . .*
>
> *As I worked on these problems, I was forced to really think about why I have been accepting things my teachers told me ten years ago.*
>
> *I guess it really bothered me that the teacher had that much power . . .*

These last two student comments indicated the concern some students had with the authority and control of the teacher and the sometimes oppressive atmosphere of the mathematics classroom. In addition to students' perception of the nature of mathematics as a discipline and the nature of the mathematics classroom, students also bring perceptions of themselves as mathematical beings. Perceptions of themselves as mathematical learners directly linked to their beliefs about mathematics and their successes and confidence with mathematics.

Self perception. My students revealed a variety of different perceptions about themselves as mathematics learners, as suggested by their descriptions of their relationships with mathematics. Many students felt successful in their mathematical careers. Others felt incompetent. They shared their beliefs about themselves as mathematicians in their journals, essays, and interviews. A student commented in an interview, "I think I really have never had a grasp on like some of those things, like math, I'm just not a very good math person." Others shared their beliefs about themselves as mathematicians in their journals and essays:

> *I have always been good at math. I enjoy it more than any other subject and have taken some pretty tough math classes . . .*
>
> *Math has always been in my blood. As far back as I can remember I have excelled at math.*
>
> *It just frustrates me so much because I have always understood what is going on in math until now. Math has been my best subject since first grade and now that I have reached a math class that deals with elementary concepts, I am lost.*
>
> *Due to all of the negative association I had with math, I have always viewed myself as not having much of an aptitude for math.*

In my experience as a mathematics educator, there seems to be a balance of students having good and bad experiences with mathematics. A few students seem to be on each extreme, either very happy and successful with their mathematical experiences or highly anxious and debilitated by their previous experiences. A few of the older, nontraditional students have not had a mathematics course in many years, or had only two mathematics classes in high school. On the other hand, there always seem to be a few students who have changed their major to education after being in a program that required them to take several calculus courses. Students' mathematical experiences seem to reach across the board. However, one commonality seems to be the degree to which students seek external verification of their mathematical work.

External authority. My students showed an inordinate urge to validate their answers. Whether they were choosing a process or finding a solution, they wanted verification from an external authority. They seemed to have little faith in themselves or their methods, even when they thought they were right. On the first day of class, I presented students with a task concerning four women crossing a bridge in pairs. As I recorded in my journal, students were appalled when I refused to give or verify the solution at the end of class. Others expected me to announce the solution the next class period. They were frustrated when I refused to do so, asking them to justify their own answers. Several recorded the novelty of this point of view in their short response papers:

> *This was very different from my past classes, where my questions would have been answered by the back of the book, my teacher, or another student's help.*
>
> *. . . I could not seem to get the right answer right away. This made me furious and [the teacher] would not tell me if I was right or wrong.*
>
> *My first reaction was that they problems of the week were pointless if I could not check my answer.*
>
> *The instructor was there only to observe, not to give us any tips or pats on the back for a right answer. This was frustrating at times, because we had all grown accustomed to going to the teacher directly when we questioned our techniques. It was hard to accept what we thought was true and run with it.*

During group work on the 100 Cards POW, a student was audio taped saying, "Sure wish there was an answer book to this." Later in the class period when her group came to a tentative conclusion, they sought for verification. Here is part of the conversation:

A: *Should we call her over and ask her? Will she tell us?*
B: *(emphatically) No!*

This negative response indicates that the student realized that I would not validate their answer, and that they would need to justify it for themselves. The need for external verification seems to stem from the perception that mathematics is a field with known answers, with a goal of matching the answer in the back of the book, or the teacher's answer key. When mathematics is perceived as determined by an external authority, then students feel that they cannot know for themselves whether or not a solution is correct. This perception is contrary to the experiences of real life mathematics, where solutions for new questions have no way of being externally verified, but must stand on internal and socially negotiated verification.

The last portion of what students bring with them to the mathematics classroom is an abundance of mathematical emotions. Although emotions are tightly entwined with beliefs, I discuss them separately here.

Mathematical Emotions

Students bring many emotions about mathematics to the college mathematics classroom. These emotions are feelings and attitudes such as fear, frustration, enjoyment, and satisfaction. Many of the negative emotions can be considered oppressive baggage, that is, emotions that are overwhelming and paralyzing. These emotions also affect students' mathematical beliefs and mathematical knowledge. This oppressive baggage often interferes with learning and understanding (Brubaker, 1994).

Negative emotions. Most of the students described having negative emotions about mathematics at some time or another. One student represented several students' emotional relationship with mathematics when she wrote in her reflective essay, "I have a love hate relationship with math." Students chose strong words in their essays to describe how they felt about mathematics, from dislike to avoidance and hatred. Other negative emotions cited were fear, anger, frustration, dread, and anxiety. Three students described their negative feelings:

> *Breaking loose of so many years of bad math baggage will be difficult . . .*
>
> *Due to all of the negative association I had with math, I have always viewed myself as not having much of an aptitude for math.*
>
> *On the fourth shape we were supposed to find the volume. I was so scared because I could not for the life of me remember what the formula to area was. Man did I feel stupid. Wasn't I supposed to already know this information? How did I get so far in math when I could not even remember a crucial formula?*

The most common emotion cited by students was frustration, as reported in their short response papers, narrative journals, and reflective essays. Students wrote that they were frustrated by the POWs, the class, the teacher, their groups, and themselves. During the group interview, a student described her frustration, "I think it was frustrating to me because I've always been good at math and these problem of the weeks [sic] made me actually face concepts I never had to face before." A student wrote in her short response paper, "These problems are always frustrating at first . . ." Other students wrote in their portfolios:

> *The problems of the week have become an extra source of frustration for me.*
>
> *I have journeyed from curiosity, to hopelessness, to potential, to frustration.*
>
> *I found [the POWs] very difficult and frustrating.*

Some of the other words and phrases described in student writings included:

angry	discouraged	gave me the shivers
furious	felt dumb	seemed impossible
panicked	overwhelmed	seemed pointless
mind boggled	had a sinking feeling	drove me crazy
freaked out	fried my brain	hatred
cringed on seeing	hurt my brain	irritated
tensed up	completely thrown off	confused
stressed out	baffled	gives me a headache

Part of the emotional feelings cited were related to a fear of appearing stupid in front of others in the class. Voicing an opinion, risking a comment, or sharing thoughts can be a very emotional act for students. One of their greatest fears seems to be a public display in which they are determined to be wrong.

Fear of being wrong. Many of the students in my classes expressed an inordinate fear of being wrong in private or public. This fear kept them from trying new things, from risking their opinions and defending their methods. Students seemed to second guess themselves frequently. About 15% of the students specifically addressed their fear of being wrong in their journals and essays:

> *I was terrified of being wrong and humiliated in front of my classmates.*
>
> *I am finding more and more though, it is not the math that I have feared so much, but instead I have been afraid of being wrong and not doing the problem the right way in other math classes.*
>
> *. . . I realize that there was also a fear instilled in me very early – a fear of that one little mistake that would render the rest of my efforts futile.*
>
> *We all had the same difficulty: none of us had the confidence to try different things. We were afraid of being wrong . . .*

One of the ways of dealing with this fear was the openness of the classroom community, the emphasis on communication in groups and whole class, and a technique of the class verbalizing "I agree" or "I disagree" when someone has explained his or her thinking. This technique offered a safe and supportive way of affirming or respectfully disagreeing with the mathematical thinking being offered. However, not all of the emotions students described were negative feelings.

Other emotions. Affective variables such as motivation, engagement, and perseverance have their own effects on student learning. Students wrote:

> *I would never try and gain an understanding of what was taking place while solving a problem, I just wanted to learn a formula and plug numbers in and come up with the right answer every time. I did not want to have to put any effort into it, I just wanted the right answer as quickly and easily as possible.*
>
> *In the beginning, it was hard for me to motivate myself to attempt them [POWs].*
>
> *I never thought that I would be this motivated to complete a problem of the week.*
>
> *Usually, if I have a problem like these, I give up and wait for someone to tell me the answer.*
>
> *In all of my previous experiences with mathematics, my goal had been to learn the method and obtain the correct answer. I knew that if I could obtain the correct answer I would receive a high grade. This goal is what motivated me in mathematics.*

Summary

Preservice teachers bring a plethora of baggage with them into the college mathematics classroom. This baggage includes previous knowledge, beliefs about mathematics, perceptions about themselves as mathematicians, and emotions that may help or hinder their mathematical understanding. Much of this baggage is oppressive, students are not empowered, and sometimes the baggage even holds them back from attaining further learning and understanding. Students need to examine this baggage with the intent of evaluating what they have, what is missing or too shallow, what is valuable, and how it all connects. The next step is to reorganize the baggage in a way that it can be accessed easily and interconnected for deeper understanding. The next chapter explores how a problem-rich learning environment may enable students to unpack the mathematical baggage they bring to the college mathematics classroom.

CHAPTER FIVE

UNPACKING THROUGH PROBLEM SOLVING

> **We shall not cease from exploration, and the end of all our exploring will be to arrive where we started and know the place for the first time.**
> T. S. Eliot

Chapter Four explored the first research question concerning the mathematical baggage preservice teachers bring with them to the college mathematics classroom. This baggage consists of mathematical knowledge, accompanied by beliefs and emotions about mathematics and about themselves as mathematical beings. Some of this baggage is oppressive, and may interfere with additional learning and prevent the deeper development of mathematical understanding. In this chapter I explore how students may begin to unpack, revisit, and reconstruct their previous learning in a way that encourages their future efforts at increasing their understanding of mathematics. This begins with my second research question.

Unpacking Through Problem Solving

The second question asks, "How may a problem-rich learning environment allow preservice teachers to unpack the mathematics content they have already acquired?" With content knowledge that is faded, false, fragmented, and tacit, the POWs offered students an opportunity to engage in problem solving activities that had the possibility of perturbing their thinking. Students had opportunities to practice starting a problem, brainstorm multiple possible methods, take the risk of being wrong, and create representations that might help their understanding. Communication between and among groups was encouraged, and there was a long-term period available to continue to return and revisit each problem.

The POWs were a new and different way of approaching mathematics for the majority of the students. A POW was assigned each week for a six-week portion of the semester. An initial class period was dedicated to each POW, enabling students to work in groups for their initial involvement in each problem. Students were expected to continue working on the problem on their own time and at their own pace. At the end of the six-week period, the students were given an additional two weeks to arrange four POWs in a student portfolio. Most students included four POWs they were successful in solving, but some students chose to include a POW that they had persevered in engaging but had not totally solved to their own satisfaction.

Each POW was aligned with a particular topic in class but offered multiple mathematical concepts and procedures to explore as well as multiple solution approaches. POWs also offered a wide range of entry levels to accommodate various student abilities and levels of engagement. For instance, the 100 Cards POW provided an opportunity to explore even and odd numbers, factors and multiples, prime and perfect square numbers, and patterns. The Earth POW offered the opportunity to deal with the concepts of radius, circumference, ratio, π as an irrational number, approximations, measurement conversion, and estimation. Not every student delved into each of these concepts, but most worked with more than one.

The POWs provided an opportunity for students to engage in mathematical thinking. Each problem is designed to elicit questions as students seek to explore and discover the relationships among multiple mathematical concepts and procedures. POWs also seemed to provoke perturbations and invoke engagement in revisiting the mathematics they thought they already knew. Perturbation was one of the emergent themes in the research.

Perturbation

As complex problems, POWs seemed to offer opportunities for cognitive and emotive conflict. Although I was aware of a few points that I expected to trip up students, there were many more areas that seemed to perturb students' thinking than I expected. Most of the POWs seemed to have points where students would get stuck, and where they had to really unpack and examine the concepts in order to resolve the conflicts. For instance, the Palindrome Date POW appeared to present an infinite amount of possible dates to check until students began to recognize and apply the constraints of the calendar. After an exploratory lesson on radius, diameter, and circumference, students still struggled with the idea of adding one foot *all the way around* the Earth. In another task, students found a conflict in trying to combine fractions of a lifetime and whole years. The Cube POW (see Appendix B) involved 216 volume units and 216 surface area units, leading some students to assume the block was hollow. One student wrote in her narrative journal:

> *. . . there must be 216 painted blocks on the outside of the cube. But I knew that was totally wrong since there [were] 216 blocks in the entire cube and I was only counting the outside blocks. Finally I realized that I was counting all of the edge blocks more than once.*

Perturbation was one of the major emergent themes that took me by surprise. Having explored several of the POWs myself, I still did not foresee some of the conflicting situations that students found. As an example, I share an episode that occurred at the chalkboard after the class had worked on the String POW (see Appendix

B) in the hallway for the entire class period. I had approached the group to find out what they were so engrossed in discussing after class time was over. They were describing two points of view about a five-sided star drawn as a non-simple curve. My journal entry describes what happened:

> *[Two students drew] five-pointed stars – one with crossing lines (like string) and one just a perimeter drawing. They mark and number the sides and lines in both examples. They turned to me to see which way was the correct way. At this point I asked the group what the difference was between the two stars. We pointed and discussed and decided that one was a polygon (perimeter) and the other was not, was a non-simple closed curve. So both ways of thinking were correct, just using a different model.*

This group of students had been perturbed in their thinking about naming the figure created by passing a string to every fifth (of twelve) students standing in a circle until the string returned to the initial person. I had assumed that students would just call it a "star" shape and go on. However, students were perturbed in trying to find a name for such a different figure, and were finding it difficult with their previous background in polygons. The perturbation seemed to arise from their personal conviction that their thinking was correct, yet the other group was just as sure of their defended solution. I was proud of my students and how they engaged the cognitive conflicts they ran up against.

A student wrote about perturbation arising from the POWs in her reflective essay:

> *It initially shook my understanding of mathematics and my confidence in my ability to solve mathematics problems. However, in the end it has given me a deeper understanding of mathematics. . . . I had to step outside of my comfort zone of "normal" math problems.*

Another student explained, "sometimes you have to . . . become uncomfortable with the situation to find an answer." Both of these students realized that perturbation was not to be avoided but an integral part of the problem solving. Students seemed to be perturbed and experience frustration at the beginning of the problem, during the problem solving process, and sometimes even after the solution had been identified. However, most students continued to engage with the problem through these episodes of frustration. Further discussion of these episodes of frustration may be found in Chapter Seven.

Engagement

Problematization of the task context seemed to be an individual thing for the students in my classes. Not all students allowed themselves to be perturbed and engaged fully in a problem solving activity. A problem that caught the students' interest in the beginning usually continued to engage them long enough to begin unpacking the mathematical content of the context. This initial motivation was often followed by the intrigue of the solution(s). A student wrote in her short response paper that POWs were "much more captivating than working generic math problems." Two other students commented in their reflective essays:

> *All of these problems created very intriguing relationships and patterns and evoked my further interest . . .*
>
> *I found that this choosing of the tasks makes a huge difference in the interest and involvement of the students. Worthwhile and interesting activities peak [sic] student interests.*

Students who are intrigued by a complex problem may begin to challenge their beliefs and mathematical disposition by examining and confronting their fundamental beliefs and assumptions about mathematics, and what it means to know mathematics. Some of their underlying assumptions can be brought to light, explored, and named. Naming assumptions is one of the strongest ways to begin changing false or harmful beliefs. One example described in my teacher journal was when students were considering the fractional area of a picture (see Figure 4) on the overhead projector. The first suggestions were fractions representing the colored portion, as this is what is modeled in textbooks. Then students decided that there were two complementary fractions to each interpretation, one for the colored portion and one for the non-colored portion. Other interpretations of the picture were shared, and students were fascinated by the multitude of fractions represented in just one picture.

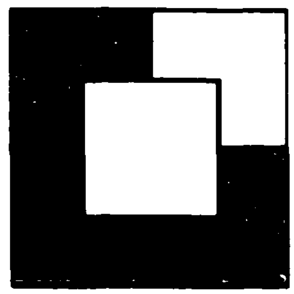

Figure 4: Fraction problem.

Sometimes assumptions are constraints that limit the deepening of mathematical understanding and furthering of mathematical explorations. Students wrote in their portfolios:

> *I was limiting myself to fractions where the denominator was that of the original ones in the [magic] square.*
>
> *I assumed much too quickly that the pattern revolved around the 3/8 and 1/4 that were already in the square.*
>
> *We assumed that we would need to work through massive amounts of guess and checks so we began jotting down random dates and then flipping them backwards to form the palindrome.*
>
> *I just assumed that because it was math, it would be impossible to solve.*

Other times students can examine their assumptions and begin to consciously consider whether they are valid and decide how to deal with them. Three students wrote in their narrative journals:

> *I think this problem . . . forced me to let go of all my "set in stone" ideas about math.*
>
> *. . . I welcome the opportunity to break out of my preconceived notions about mathematics.*
>
> *I had to test every previous assumption or notion that I had and then be able to explain why I decided on what I did.*

Another facet of student engagement was the involvement of others in the POWs. The first day of Math 2213 I gave the students the Bridge task.[1] A student wrote in her short response paper that she planned to try to stump her boyfriend with the problem. Students tended to "share" these problems more often than they would a worksheet or standard word problem. Students wrote of roommates who laughed at their struggles with a problem and then a week later were cursing the same problem as they struggled together in the early hours of the morning. One student took her POW "homework" along on a babysitting job and ended up with the parents involved in the process they observed. Another student reported that her colleagues at work were fascinated by the problems and always asked if there was a new one to work on during lunchtime. These problems appear to be more engaging and intriguing to students (and others) than the standard worksheets and word problems. Instead of giving up when they get stuck, students tended to return again and again to these problems.

---

[1] Four women need to cross a rickety bridge. The crossing takes one woman twenty-five minutes, another twenty minutes, the third ten minutes and the last five minutes. Only two women can be on the bridge at one time, and the pair must have the flashlight with them as they cross. How can the women cross the bridge in pairs, with one returning the light, in one hour?

Revisiting Concepts

The POWs provided an opportunity for students to revisit concepts that they thought they understood or had never understood in previous classes. Two students indicated in their short response papers, "As an elementary student, I never understood prime numbers and factors very well," and "I'm just now understanding things I learned years ago." The story context of the POWs provided an incentive to explore and discuss mathematical ideas in a way that brought to light misconceptions and fragmented understanding. Students wrote in their reflective essays:

> *I began to analyze concepts that I had never before questioned and started reexamining the way in which I had always defined certain mathematical components.*
>
> *. . . it made me take basic principles that I took for granted, and rethink why they are the way they are.*
>
> *You would think that going back to basic concepts would be easy. I thought surely that I knew the basics. I was wrong.*
>
> *My experiences . . . with the POW's have opened my eyes to how wrong I was to assume that I had a higher understanding of math. Since I never really had to think about the math concepts in detail, I had a weak comprehension of how they really worked.*
>
> *I now understand that in order to fully grasp a math concept that you must understand the underlying theory behind each math problem.*

My students are beginning to evaluate their mathematical understandings with their future teaching needs in mind. A student wrote, ". . . we want to learn and make sense of everything we are doing in this class because we are going to have to use it again in our own classroom." Students seem to be realizing that being able to do a problem and get a correct answer is not the only skill they need when they begin teaching their own students. They are beginning to realize that mathematics is all around them and involved in every part of their life. This real life mathematics can be a very different situation from the traditional exercises worked in most classrooms. Learning mathematics for real life problems entails more time and more of a creative space than is generally allowed for standard exercises.

Time and Space

POWs provided time and space for dealing with these more complex problems. Real life problems are usually dealt with over a period of time, allowing a search for resources, taking a break and coming back to the problem, and working with other people. The POWs were more likely to offer a similar experience than doing a worksheet. Students wrote about putting away problems for a day or weekend and when approaching them again, having more success. One student wrote, "Sometimes you just need to look at it for a couple of days before you can come to what might be the right answer." Another mentioned in her short response paper that she would "definitely be thinking about this outside of class, and I won't be able to get it out of my head for a while."

Students discovered more about themselves and mathematics by having these "thinking spaces" (Ball, 1990c). It allowed them the extra time and space to develop their thoughts and to be sure of their understanding. In a group interview, one student commented that she had gotten stuck on a problem and asked a friend who was not in the class for help. However, she phrased her question as, "Will you think about this with me?" rather than a request for finding an answer. Having time and space along with a change in attitude allowed this student to continue to engage the problem with a friend.

Other students commented about time and space in their portfolios:

The POW's allowed room for me to experiment, and determine my own *learning fashion.*

I had never worked on a problem this long before and had actually understood what I was doing.

I personally have never spent this much time on a single problem . . .

. . . I would rather understand something and it take longer than not understand and be done in a short time and have had no idea what I just did.

It seems that the POWs were personally motivating to many students. Instead of waiting for class to start or just talking, students would spend the time before class started working together on the problems and what they had accomplished since they last met. Several times I had to announce that time for class was up, they were so involved in the activity that they had lost track of time. Several students mentioned having similar experiences while working on the problems at home.

One student remarked in a group interview that a POW "haunted" her; another wrote that it "called my name." Three students wrote in their first response papers, "it will stay on my mind and annoy me until I figure it out for myself," "it will bug me all day," and "it will probably eat at me until I obtain an answer." Another student wrote in her narrative journal, "It bothered me when I couldn't figure it out and before it didn't [bother me]." Some students gave examples of thinking about the POWs after class, out of class, and even during other classes. Students explained that they shared these "brainteasers" with parents, siblings, friends, roommates, neighbors, wives and husbands, boyfriends and fiances, fellow workers and even employers. Some of the comments from portfolios included:

I was sitting in my Literature class watching a movie and all of [a] sudden it hit me, I quickly wrote down the [palindrome] date and then tried to focus on Literature class again.

> *POWs usually leave an impression on me. I continue thinking about these problems at random times.*
>
> *I usually don't enjoy doing work in the classroom but when doing these assignments I would not even look at the clock once.*
>
> *I would spend hours of my own time and not even realize it [on] one problem.*
>
> *. . . when I did begin working on each problem, it was hard to make myself stop. . . . When I had to find my own method, I rarely checked the clock or counted how many problems I had left to do.*
>
> *One night, around midnight, I started working on the problems and looked at the clock and it was 3:00 AM in the morning!!!! It was so surprising to me how these problems made time fly.*

POWs enabled students to unpack the mathematics they had encountered throughout their own schooling. POWs were designed to perturb students' thinking and encourage them to revisit mathematical ideas in an effort to resolve the conflict. POWs engaged and motivated students more than worksheets or exercises. Students were able to explore multiple methods and representations on their way to deeper understanding. The use of POWS provided a problem-rich learning environment that enabled students to unpack, revisit, and reconstruct mathematics they had previously encountered.

Summary

Chapter Four emphasized and explored what preservice teachers bring with them to a college mathematics classroom. The mathematical knowledge, beliefs, and emotions that students bring have an effect on their ability to participate in learning activities that lead to greater mathematical understanding. Chapter Five specifically addresses the problem solving through POWs as an opportunity for students to unpack and extend their previous mathematics learning. Through POWs, students were engaged over periods of time and space, working with classmates, friends, co-workers, and significant others. The POWs provided opportunities for students to focus on and wrestle with problematic aspects of problems, experiencing frustration but having the opportunity to work through their frustrations to the point where perturbation was a motivational rather than debilitating aspect of the problem situation. Without the perturbation as well as the spacing to work on the problems, students would not have had the opportunity to unpack their mathematics and extend their understandings. The quality of the POWs was significant as was the classroom culture that encouraged prolonged engagement and conversation about the problems.

Chapter Six will address the third and final research question, how unpacking mathematical content affects students' mathematical understandings. Students who have brought mathematical content knowledge to the college mathematics classroom, participated in problem solving activities in order to unpack, revisit, and reconstruct this knowledge, found that their understanding was affected in many aspects. These aspects are discussed in the next chapter.

CHAPTER SIX

UNPACKING FOR UNDERSTANDING

> We don't receive wisdom; we must discover it for ourselves
> after a journey that no one can take for us or spare us.
> Marcel Proust

The third and last question asks, "How does unpacking content affect mathematical understandings?" Students made multiple references in their narrative journals, reflective essays, and on the final questionnaire to ways that working on the POWs helped their mathematical understanding. Unpacking previously learned mathematical content through problem solving encouraged reflection, deepened mathematical understanding, and had a positive effect on their beliefs and emotions about mathematics and themselves. One student seemed to sum it up when she wrote, "I think that the POWs were given to us so that we can try to dig deeply for information that we might already know, but apply it in a new way."

Unpacking Encourages Reflection

Part of the unpacking process is the mental, verbal, and written act of reflection. Students remarked that they were "thinking about their thinking" as they metacognitively reflected on their own thought processes while problem solving. Students described their mental reflections:

> *Because the Problems of the Week requires a lot of thought, I was able to see just how my mind works to find the answers.*
>
> *. . . the problems really made me think about them in new ways and examine the way I think when I do math.*
>
> *[The POWs] have helped me understand more about the way I think in*

math.

> *. . . the neat thing about it is that I was actually having to think about what I was doing and think about that thinking . . . first we have to be able to think about our own thoughts in order to defend them.*

This internal dialogue was accompanied by verbal reflection within and among the groups. Students wrote:

> *Doing group work allowed me to verbalize my thought processes in order to better understand the particular problem and how I was going about solving it.*

> *I find that talking things out amongst others who are presenting different viewpoints helps me to better understand the material being covered. It provides an open forum for ideas and different viewpoints. It also allows for a free flow of ideas in an environment that is condusive [sic] to promoting an open-minded evaluation of unique viewpoints and even opposing ideas.*

The portfolios provided an impetus for written reflection, which students reported as a valuable activity. But even the short response papers offered an initial reflective process. One student wrote in her short response to the 100 Cards POW that she thought it was <u>not</u> a lesson in primes. Then she drew a star and at the bottom of the page her starred footnote read, "Oh, I just realized that this may be a lesson in primes after all." Just jotting down a few quick thoughts was enough for her to see something new. Students also reported that they often caught errors or noticed something new while writing their reflective essays. Comments from reflective essays included:

> *For some reason, simply writing what I was thinking was immensely beneficial.*

> *Wow! I did not realize how much I have learned through these experiences until I began to write them down. (I guess that is why you have us write a reflective paper).*

> *A key part to this problem for me was actually writing everything out to make a complete visual picture.*

All in all though having to write about the problems really made me think about them in new ways and examine the way I think when I do math.

The idea of writing my thought processes down caused me to be fully aware of exactly what was going on in my head.

As I wrote out my responses for each problem it was amazing how much more I would see that I had not seen before I started to try and explain my methods and reasoning.

Often students were amazed at how much they have been able to do themselves. The reflective practices of unpacking enabled them to see clearly what they were accomplishing. Some of the students wrote:

I did not realize how much I had learned in this class without knowing it. Often times in class I feel as if I am not being productive until I write my POW response. Then I can see what concepts I relearned or learned for the first time.

I did not figure this out until I started writing this . . . when you . . . have to write about something a lot can be revealed to you.

I did not realize how much I have learned in this course so far until writing this reflective essay.

. . . it was very interesting to see all of the journaling that proved how much I did, learned, and described.

This type of reflection enabled students to see a difference between simply "learning" and completely understanding mathematics. A student explained:

I have learned that the act of learning just hits the surface of education. It entails the teaching of a concept and processing the information, while understanding, although it coincides with learning, is something totally different. . . . It requires a person to retain the information clearly after they learn it.

Unpacking Deepens Understanding

The final questionnaire was given only during the fourth semester and specifically asked 129 students how the POWs increased their mathematical understanding and to identify the most significant change in their mathematical understanding. Student responses could be grouped into general categories, with some students responding in more than one category (see Figures 5 and 6). The most common responses were the way the POWs challenged their thinking and expanded their understanding by making

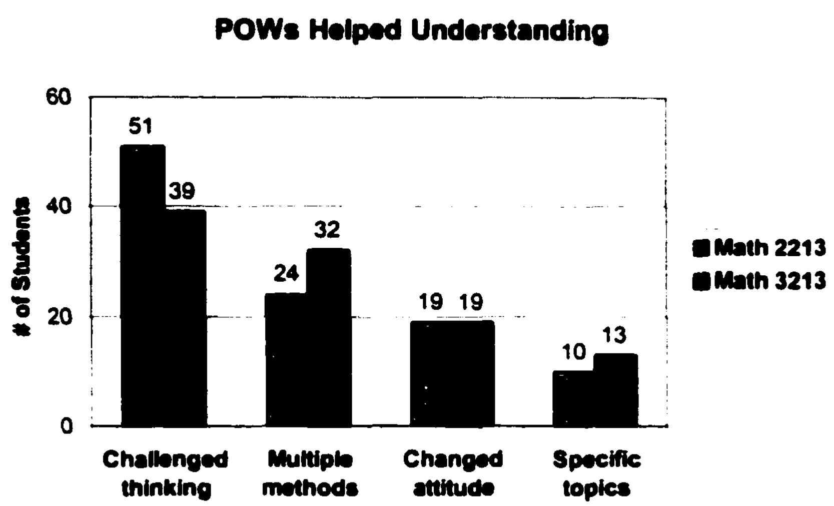

Figure 5: 129 students reported four major ways that POWs helped deepen mathematical understanding.

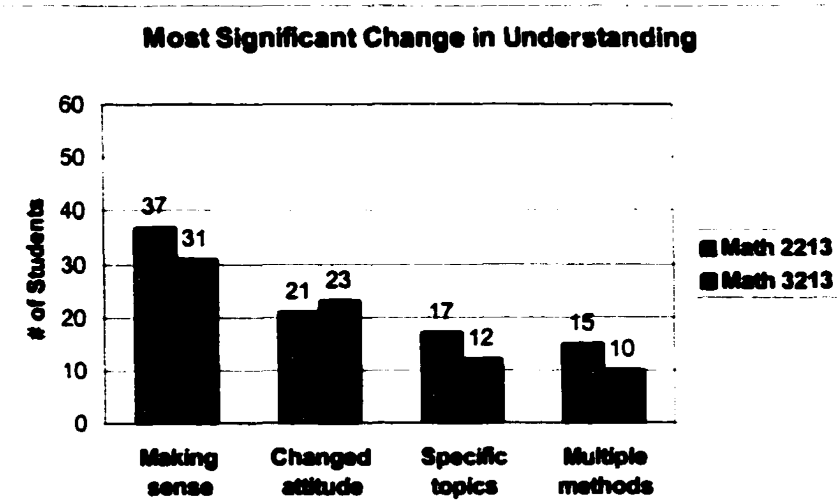

Figure 6: 129 students reported their most significant changes in mathematical understanding.

sense of the mathematics. Students also responded that unpacking through the POWs helped to change their attitude about mathematics and their abilities. Two other categories of responses involved the value of learning multiple methods and perspectives and help with understanding specific mathematical topics such as fractions. These four categories were also what students reported being the most significant change in their mathematical understanding.

Unpacking for understanding. On the final questionnaire, 90 of the 129 students responded that the POWs helped their mathematical understanding the most by challenging their thinking. More than half of the students described the most significant change in their understanding as making sense of the mathematics. The most commonly used phrase was "thinking outside the box." Students were able to stretch their thinking, understand and connect new concepts, and figure out the 'whys' of what was going on. Student comments included:

> *I truly benefited [sic] from this problem of the week because it not only made me learn more about mathematics but also enabled me to gain a new depth to this knowledge.*

> *If you can extend your thinking and ask questions about the concept, ponder other applications, and know how to go about investigating your new ideas because you have a firm grasp on the original concept, that to me means you have truly learned the mathematical concept.*

Explicit understanding. Students wrote that they now understood more than just how to do the mathematics. They could now explain their thinking, understand other people's perspectives, and felt ready to teach these particular concepts. They began to realize that there is more to mathematics than just finding an answer. Their understanding became much more explicit. Student comments from essays included:

> *Although I learned this concept in the years of my elementary education, I didn't ever really take the time to understand the true definition and*

characteristics present within this mathematical process until discussing it this semester.

It is one thing to be familiar with a term or concept but it is another thing entirely to be able to convey your knowledge to another person.

My experiences . . . with the POW's have opened my eyes to how wrong I was to assume that I had a higher understanding of math. Since I never really had to think about the math concepts in detail, I had a weak comprehension of how they really worked. I could apply concepts, but not understand why I was applying them. . . . While I was solving the problems of the week, I could see methods that would get me to the answer that I was not used to applying.

I think that this project is awesome and I will always look back on it and see it as when I really began to UNDERSTAND math.

I noticed that I made more progress with these problems and more understanding of the situations than I have had in any other math class.

Even though most of the concepts presented in this class were not extremely difficult, I learned them more thoroughly than I would have if they had been taught in an ordinary classroom fashion because of the way that they were presented.

As students' understanding developed they began to reason mathematically and defend their solutions. Asking for verification from the teacher quickly dwindled as students struggled to find their own justification. Two students wrote in their portfolios:

Anyone could figure out the answer to the card question, but it's not until you relate it to the factorization worksheet that you have proof as to why your answers are correct.

The Problems of the Week challenged me to find my own way of answering a problem. They also taught me how to defend my answer so that the answer that I got was justified.

Some students even exceeded all my expectations and were able to expand their intellectual curiosity (Mewborn, 1999) to ask questions about what else might be possible.

Intellectual curiosity. Traditional transmission methods limit students' mathematical activity to looking for the expected right answer, and when it is attained and externally verified, they are satisfied. There is no additional thinking, exploring, or need for understanding, because the goal (the correct answer) has been accomplished. When students truly begin to understand a mathematical concept and its relationship and connection to other concepts, then a creative space begins to emerge. Several students proposed conjectures or alternative situations outside the envelope of the problem being presented. One student explained, "There are rules that have to be followed, but there is creative space within the rules."

A student investigating which cards end up face down in the 100 Cards POW found the pattern of the perfect squares and then wondered whether the perfect cubes also had a pattern. She investigated and found none. Another student had finished the Other Bases POW and was working on an extension project. She brought me her work and asked if I had considered a rational base as the answer, since base six was too small and base seven was too big. She thought the solution must lay somewhere in between, maybe base 6.8 or so. I certainly had never encountered this idea before!

In another example, while exploring the parameters for a fraction to be written as a terminating decimal, students had observed that the allowable denominators of two and five were factors of ten. A student immediately asked if the denominator would be restricted to only five in the base five system. I was totally amazed that she was able to consider this application in other bases and connect it to previous class discussions.

A student explained it better than I can when she wrote, "Math is not about the answers but the ideas. Answers signal the end to a thought, but ideas are only the beginning of an endless road to limitless potential." Other examples of intellectual curiosity from the narrative journals included:

> *. . . I started to wonder if other cultures could use a different definition of even and odd. What if the definition was if the number is divisible into groups of three evenly? Then if you didn't have a whole group of three it would be an odd number. That would change entirely what people think about even and odd numbers.*

> *. . . if you change just one of the numbers given to you, then there is a new answer. . . . An interesting thing to find out would be how many magic squares could be made with a specific denominator.*

Multiple representations. POWs offered an opportunity to explore and apply multiple representations as part of the unpacking process. Most students reported trying several representations before finding one (or more) that seemed to enable their comprehension. Often students were able to use manipulatives or models during class time, but they also used various representations outside of class. Students represented the problem mentally, physically with manipulatives, on paper with diagrams and symbols, and also used written and spoken speech such as metaphors to help their understanding. A student wrote in her reflective essay, "These methods may include but are not limited to sketches, charts, graphs, or talking the problem through and visualizing the possible solutions."

The majority of students began the Earth POW with a sketch or picture on the POW sheet. When the solution did not even come close to matching their initial estimates, many students went looking for a physical model that they could explore hands-on. In their narrative journals they described using common, everyday circular items such as a clock, ball, pop bottle, cup lid, tomato, or a belt around their own waist. A few students explored similar problems involving smaller numbers that were more intuitive than the millions of feet involved with the equator. One student drew the two circumferences and then snipped and straightened them in order to compare lengths. This representation allowed this student to confirm that the difference would indeed be on the smaller side of the initial gut estimate.

The Cube POW also allowed students to choose multiple representations (see Appendix D). The majority of students began with a sketch of a cube on their paper, a two-dimensional representation that only showed three of the six faces. Other students drew nets or separated all six faces. When students could not visualize the other sides of the cube using the sketch, they made paper models or physical models of blocks and Legos. Some students used common household items such as a box, bank, lunch box, Rubik's cube, or sugar cubes. Some students visualized the cube with "layers like a cake" or imagined Velcro that allowed the outside layer to be pulled off. One student found herself working on the problem in a waiting room. She noticed that both the floor and ceiling were covered by six tiles in length and six tiles in width. She envisioned herself inside the cube, and used this representation to find her answers.

Although students discovered and explored multiple representations while working with the POWs, there was still a hierarchical perception of their value. One group, who was far ahead of the rest during the 100 Cards POW (see Appendix D for sample representations), told me that they had the answers but that they had to use the "kindergarten method" to find them. When I asked them what method they had used, they showed me how they counted factors by flipping their palms up and down. This technique answered several questions, since the orientation of their palm also answered whether the card was up or down. However, they perceived this representation as cheating or immature, although it was actually the strongest and most efficient representation I observed for this problem, as it took the students directly to a solution along a valid and verifiable path.

Understanding through metaphor. Students seeking for understanding often found a metaphor within the context of the problem that enabled them to crystallize and explain their thinking. In the Earth POW, students were asked how much more rope would be needed to "float" above the equator compared to tied around the Earth at the equator. When students arrived at solutions that were extremely contrary to their initial estimates, they were perturbed with their answers. In seeking to verify and understand their solutions, some employed the use of metaphoric reasoning. One student explained that the height difference of mountains and valleys is insignificant compared to the whole Earth; another compared one light year to the distance of the entire universe. A third student envisioned placing a penny at each end of a football field. Each of these metaphors gives a hint of their attempt to understand why only six more units of measure were required to "float" the rope one unit above the equator when they expected the answer to be a very large amount.

Another example of metaphor use was given by a student working on the Magic Square POW. In answering the question about more than one solution, she thought about a puzzle, where every piece has its place, and they can't be placed elsewhere or the entire picture would be distorted. She felt that the fractions in the solution had their own places and could not be moved without distorting the magic square. Another student visualized the Cube POW by thinking of an orange, with layers that could be peeled away. Each of these metaphors enabled students to connect new situations with prior experiences in a way that supported their new understanding.

Making connections. Students were able to discover, create, and analyze mathematical connections within the context of the POWs. As they unpacked and revisited mathematical concepts, they were able to see the relationships between and among the ideas. A student working on the 100 Cards POW described it as the "biggest eye opener" and that she would never have made the connection between an odd number of factors and the perfect squares otherwise. Other students wrote in their reflective essays and narrative journals:

> *. . . the answers are embedded within the relationships between the concepts presented in the problem.*
>
> *I think the more links you have to other sources [of] knowledge, the more likely you are to remember that information.*
>
> *I think that [it] is astonishing how we go through school and learn all these concepts but never get a deep understanding about how they are all connected together.*
>
> *. . . I was required to look at the big picture and not just a small portion.*
>
> *. . . it was the final question of why I got the answers that I did that caused me to ponder and deeply analyze the relationships present in mathematics . . .*
>
> *The Problems of the Week helped me to learn in that they showed me that math is joined in so many ways to each other. These problems showed me that math concepts are to be used with other math concepts instead of thinking that today we are working on algebra but not subtraction. They showed me that to figure out some problems I am going to have to use one or many different concepts to find an answer.*
>
> *Math is not, here is a concept, and then here is another, but if we look deeply into the concepts, addition, subtraction, and etc are all combined and used differently to form other formulas and theories.*
>
> *In this problem we dealt with circumference, radius, and difference. We already understood these things separately, but through this problem I saw a relationship between the three.*

Unpacking through problem solving deepens the mathematical understanding of students by making it more explicit. It also helps to develop their ability to reason and verify their solutions, increases their intellectual curiosity, allows them to explore multiple representations, and to make connections between and among mathematical concepts. But unpacking also allows for change in other areas, such a mathematical beliefs and emotions.

Unpacking Affects Mathematical Beliefs

Almost 30% of the students wrote that the POWs influenced their change of attitude toward mathematics. One third of the students reported that this change of attitude was the most significant result of their deeper understanding. They looked at the nature of mathematics differently than before.

Nature of mathematics. Many students wrote of the changes in their beliefs about the nature of mathematics. Where formerly students considered mathematics to be dualistic, absolute, and computational, they now see it in a broader view. They began to realize that even "simple" mathematics could be rich and complex. They found through their own experience that mathematics is an active, growing part of their lives. They felt that they could learn a lot from mistakes. This change in their perception of the nature of mathematics is reflected in the following comments:

> *I always thought that in math, there was only one way of doing things. What I understand now is that I can truly be creative and mathematical at the same time!*
>
> *This new way of looking at math found me seeing this long-dreaded subject as a much more creative subject than I had seen it to be before.*
>
> *These problems have expressed a creative side that I never thought math could contain. . . . I was excited to find a creative side to mathematics.*
>
> *Lately I have realized that math is discovery. I never really thought I would think originally and logically at the same time.*
>
> *This course has enlightened me to see math with new eyes, math as a subject with creative possibilities.*
>
> *Math has become exciting, like solving a mystery instead of simply acting out methods to derive solutions.*

I have realized that math is a process and not just a series of formulas.

I did not think there was another side to mathematical concepts except formulas and concepts . . . I began to open my eyes and realize that there was a whole other side of math that I had never known.

Mathematics is not just a one dimensional world of numerical values. It is a limitless world of relationships and reason.

Mathematics is an endless world of possibilities, ideas, and opportunities to learn.

. . . I feel as if there has been this whole other world of math that I have been missing out on for the last 19 years.

Rather than a structured, formal set up, math has turned out to be just as debatable as any other subject area.

Other's perspectives. Much of the reason to view mathematics differently stemmed from the realization that people see mathematics in a different ways. Working in groups helped students realize that other people don't always see things the same way that they do. On the final questionnaire, 56 of 129 students mentioned that the most important part of POWs helping their understanding was the realization that there are multiple methods of solution and multiple perspectives to the problem. Multiple methods was the second most common answer on this final question. Twenty-five students chose this reason as their most significant change in understanding, explaining that it expanded their world views. Although 19% may not seem a significant portion of the responses, students reiterated this result in their narrative journals and reflective essays as an important effect of the POWs.

Student comments about working with others in their group included:

. . . usually you are taught or understand one way so you think everyone else things that way too.

I heard what other people were doing to solve problems and they were different than mine but we were all getting the same answer.

I learned that there is not always one way to do something and that everyone does not always see things in my way.

. . . we found different ways to do the same things that found the same solutions. Then we would talk about how we each came up with our specific answer and taught each other a new way to work out each problem.

Working with people gives you a different perspective on things that you may never have thought of if you were working alone.

I have noticed when I work in a pair or group I expand my knowledge a lot more than if I was working individually or just sitting in class.

When I would become frustrated and think I had exhausted every possible method of solving a problem of the week, my partner or partners would come up with another possible method.

Ownership. Students were amazed by the many methods available to solve the problems. This was a great contrast to their previous experiences in mathematics classes described in their reflective essays. As they became more familiar with alternate methods, they began to realize that some methods were more efficient than others. However, sometimes they were more comfortable working their way even though it might take longer. Two students wrote in their short response papers:

That may be a long way around the problem but it was the best way I found that I was able to understand.

This is probably one of the longest ways, but it seems to be working for me.

Students wrote in their reflective essays:

I am sure the methods I used to arrive at the answers for these problems may not have been the most efficient, but they worked for me.

With the POWs, you would actually have to think about it for a while and come up with a plan. Sometimes, many ideas would pop into my head at once and I would have to think about which one I thought would work best.

. . . sometimes you must think beyond the scope of the problem and imag[ine] the alternate possibilities there are and the various routes you have to getting to them.

Now I know that if I get by myself, and just think things through, the way it makes sense to me, I will eventually end up in the right place.

I worked hard at finding the best possible way for me to understand what I was doing and not just go through the motions as I had always done before.

. . . exploring new ways and finding my own personal understanding of a concept, not the teacher's understanding.

The POWs allowed room for me to experiment, and determine my own learning fashion.

I had to look at problems from different perspectives and correlate these ideas with my own. This help[ed] to understand my ideas better and take ownership of them.

As students identified the perspectives of others, and realized that multiple methods and perspectives were of value, they confronted the myth of the one right way, and began to develop ownership of their personal mathematical understanding. They realized that although their method might not be the most efficient or the easiest for others to understand, it made sense to them and thus was a valuable possession. The freedom to engage in the mathematics "their own way" was a tremendous boost to their self-confidence and understanding.

Intellectual autonomy. Students learned about themselves and their mathematical abilities. They discovered that they could be in charge of their own mathematics, that they could be the authority in deciding which process to use and whether a solution was viable. This burgeoning intellectual autonomy was very valuable to students. Students wrote in their portfolios:

The POW portfolio was the most helpful tool for me in making these discoveries about myself . . .

However, for me, this was also the most challenging part of the course: the freedom to learn how I wanted. I have never had any class, especially a math class, that allowed me to do my own thing. Needless to say, at first, I was a little discouraged. I was so used to being told what to do, or what steps to take, that when it became solely my decision, I did not know what to do. I eventually learned that I had to rely on my own ability and to trust myself that I knew what to do.

I was so relieved that I did not have to ask you to help me. I really wanted to try to do it on my own.

Many times when I don't understand something, I rely on a source to know the answer. POWs on the other hand, did not allow me to do this. POWs really taught me to rely only on myself to get the job done.

The most valuable thing that I gained in this course was the ability to trust myself . . . I found that I had to rely mostly on my own abilities to solve problems . . . I do not need anyone to tell me that I am right . . . I know that I am capable of arriving at the correct answer without the assistance of anyone. I learned this because of the POWs.

This was very different from my past classes, where my questions would have been answered by the back of the book, my teacher, or another student's help. That is the most valuable thing I gained in the class: self-reliance when solving mathematical problems. It is quite a challenge to be confident in your work when you cannot check or compare your answers.

Unpacking through problem solving not only increases mathematical understanding, but has a way of bringing to light the mathematical beliefs students may not know that they are harboring. One student described such a belief in her short response paper. She indicated that she had the thought embedded in her brain that if the teacher looks at your paper without making a comment, then your work must be wrong. She had to remind herself that in this class, both right and wrong answers are shared, and the validation is up to the students individually and as a class.

As students confront these beliefs, they are able to distinguish the validity of their perceptions about the nature of mathematics as a discipline, to open their eyes to the differing perspectives of other people, and begin to feel an ownership of their mathematics. This increases their intellectual autonomy which can also have an effect on their mathematical emotions.

Unpacking Affects Mathematical Emotions

Many of the negative emotions students reported feeling about mathematics and mathematics classes seemed to change over the course of the semester. Students reported less anxiety and fear, a change in their feelings about mathematics, more confidence and a feeling of empowerment.

Less anxiety and fear. Students reported that working with the POWs helped to lessen their anxiety and fear. They also learned to deal with these feelings better. Some of them wrote in their reflective essays:

> *Now after working through some of these POWs . . . I realize that sometimes you get all worked up about nothing.*
>
> *I have expanded my abilities and lessened my fears. Through this class I have come to understand and even appreciate my fears about math.*
>
> *It's funny how a simple math problem can help overcome math anxiety.*
>
> *. . . the POWs were a step in helping me overcome my math fears.*
>
> *I really feel like I overcame some fears by working through that problem.*
>
> *I learned not to fear it so much and to not give up so easily . . .*
>
> *The POWs helped me to understand where my stress with math was focused so that I could work past it.*
>
> *This class has help lifted the stress math put on me and allows me the freedom to experience math in a new way.*

As their fear and anxiety lessened, they began to appreciate some of the negative emotions and realized they could be not only be controlled but could be put to use. One of the changes involved looking at error in a new light.

Valuing Error. As they lost some of their fear of being wrong, students came to see that making errors and being wrong was not always a bad thing, it was just something that happened in the process of getting somewhere on the problem. A student wrote about the benefits of failure in her reflective essay: "When I get a problem wrong, I take more time on that situation and dissect my procedure methods.". Students commented that they learned more from their own errors than they did imitating a given method. The thinking and rethinking required to determine why an answer is wrong was conducive to understanding and increased the appreciation for and connections with the concept. It also affected their intellectual autonomy, as they were able to state, "I changed my mind" instead of "I was wrong." They can then explain why and how their thinking changed as a result of their own understanding and the support of the community.

Students wrote about learning from their mistakes in their reflective essays:

> *Learning from your mistakes is much more beneficial than being told how to do something.*
>
> *I think that it is not only important to understand what you did to make a problem right, but also what you did that made it wrong.*
>
> *I think that sometimes the best learning is the learning that occurs when mistakes are made and then fixed. I know that for me, if I make a mistake and instead of being reprimanded for it I am able to understand and overcome or correct the mistake, then I am more likely to understand how or why the mistake was made.*
>
> *In high school, I felt that getting a wrong answer was failing. Now I understand that getting a wrong answer is so much more beneficial in the long run. When I get a problem wrong, I take more time on that situation and dissect my procedure methods. After I realize how and why I came to the wrong conclusion, I leave that problem with a fuller understanding of all the workings to that solution.*
>
> *So whatever I may come up with in these problems is not wrong. It is just my way of thinking and figuring out certain problems. It might be the long way or the short way, but it works.*

> *I have learned that sometimes you learn about the places you travel to better by getting lost, than arriving at the destination with no problems. If you make mistakes you learn valuable lessons that you might never know if you never messed up.*

Students also explored the differences when they appeared to have a wrong answer, discovering that sometimes there was more than one right answer, or more than one way of interpreting the question. It continued to remind them that there are multiple methods and even multiple answers to mathematical problems. One student wrote in her reflective essay, "I learned that although my way might not have been correct, it seemed a logical attempt, and that trying is the best I can do." Other students wrote:

> *I realized that I had done them differently than others. I did not do them wrong just different.*

> *Making a mistake like this, was somewhat beneficial. It made me realize that there could be another answer to the problem than one way.*

> *When my partners and I were finished drawing our pictures mine was completely different [from] theirs. This worried me, but then I realized that I had drawn mine from a completely different viewpoint. That made me think of a lesson that in math there are always many different ways to look at problems and come out with the same results.*

Sometimes students even understood that making mistakes meant that they really were exploring the situation, seeking for a better strategy, or for a method that was understandable to them. Luth (2000) explains that students make errors because they are actively testing their hypotheses and working out the rules, not necessarily because they don't understand the mathematics. One student reflects this explanation as she writes in her reflective essay:

> *I was not afraid to attempt something and be wrong. I knew that I was not wrong; I just had not found the best strategy for me. I think that approaching the problems from this angle helped me very much in finding my answers. I did not get as stressed out or worried about finding the right answer, I saw finding a good strategy for me as the most important aspect of the problem.*

This student experienced less stress and worry as she was able to take the time and engage the problems in the way that made the most sense to her. The realization that exploration and risk taking are valuable and worthwhile seemed to parallel a change of emotion. The change of emotion then underscored the belief that errors were not a negative thing but could be a valued step in the journey to a solution to the problem.

Change of emotions. Students reported significant changes of emotion toward mathematics, the mathematics classroom, and their own abilities, beliefs, and emotions. They reported less fear and anxiety and more confidence in themselves. They found they could enjoy mathematics when it was more than just numbers and calculations. Even as early as the short response papers, students made such comments as, "We finished fairly quickly, which helped boost my POW confidence." Some of their comments included:

> *. . . for the first time in my life I enjoyed going to "math" class, doing math!*
>
> *I enjoy math more because I feel that I have a clue as to what I am talking about.*
>
> *I actually started to care about learning math.*
>
> *I learned to explore concepts that I have never really understood and have always avoided.*
>
> *I don't feel like such a failure in math . . .*

Confidence and self-efficacy. Forty students specifically addressed their increased confidence in doing mathematics over the course of the problem solving activities. Self-efficacy in dealing with specific mathematical concepts such as fractions, word problems, and geometry have given a substantial boost to students' beliefs that they can both *do* and *understand* mathematics. My students commented in their reflective essays and narrative journals:

> *. . . the lessons I learned and the confidence I gained were worth more than a right answer.*
>
> *However, this time [Math 3213] I felt much more confident about devising my own methods and techniques in order to discover solutions.*
>
> *. . . it seems crazy that a math problem could have such an effect on my esteem. But where I once felt incompetent, I now feel more competent and daring. It just amazed me so much that I could come up with ideas on how to solve a math problem without a step-by-step guide of how to do it.*
>
> *I felt quite accomplished when I figured out the problem and I went back to check my work and it was still right. This problem boosted my mathematical self-esteem . . .*
>
> *The satisfaction that I would get after I would complete a problem was kind of funny. I would feel like I had done something for the class but really it was only helping myself.*
>
> *. . . while working this POW I was feeling very confident in my reasoning and usage of the process of elimination. . . . The most valuable thing that I gained was more self-confidence when thinking critically.*
>
> *Coming into this class I already had a high mathematical confidence, though now after half a semester I have achieved a whole new level of confidence.*
>
> *I feel that these problems were what I needed to gain the confidence needed to survive my "math journey." I am much more confident with my work and with expressing my work in a creative way.*
>
> *. . . I also gained confidence in myself, solving what at first seemed like impossible math brainteasers.*
>
> *. . . now my confidence comes from exploring new ways and finding my own personal understanding of a concept, not the teacher's understanding.*

Empowerment. Autonomous learners understand and do their own thinking, stand up for their own beliefs, and take responsibility for their own learning. Their fear of failure transforms to realizing they can learn from their mistakes. They are willing to take risks, and understand it that frustration is part of the problem solving process. They recognize and confront their initial assumptions, look for other resources and connections to previous and outside knowledge, and continue to engage in mathematical thinking at other times and places than just mathematics class. A student wrote, "[The teacher] opened a window slightly and then allowed me to open it up all the way and climb through." This student took that small opening and was able to enlarge and complete the rest of the action herself. Other student portfolio comments about empowerment included:

> *I felt like I was in charge, and I could do anything I wanted or needed to do in order to solve the problem . . . The POW's allowed room for me to experiment, and determine my own learning fashion.*
>
> *I felt so proud when I came up with the answer on my own, and this gave me the confidence and encouragement that I needed and helped me prove to myself that I was capable of doing things on my own, and that I could trust myself and my abilities.*
>
> *I really can do things that I thought I couldn't do before. I've always doubted myself and my math ability up until this point.*
>
> *I feel empowered to continue on in my quest for mathematical knowledge . . .*
>
> *I feel like I am actually participating in and contributing to my own education.*
>
> *I was so used to being told what to do, or what steps to take, that when it became solely my decision, I did not know what to do. I eventually learned that I had to rely on my own ability and to trust myself that I knew what to do . . . I think that by giving us the freedom to do what we wanted, we had to learn how to learn, each in our own way.*

Not every student felt like the Problems of the Week helped their mathematical understanding. It is important to note that what seemed to work so well for most of the class failed to be helpful for a few students.

POWs Were Not Helpful for All

Seven of the 129 students stated in their reflective essays that the POWs did not help their mathematical understanding at all; in fact, a few indicated that it may have made it worse because of their confusion and frustration. These students were perturbed by the POWs, but did not seem able to hurdle the frustration level they encountered as did others in the class. The process of unpacking did not seem to occur, and so these problems were just that, an unsolvable problem for them. Three of these students wrote in their reflective essays:

> *So far, I have only been able to finish one POW and I actually feel very stupid. . . . To be honest, so far I have not really learned anything new about math through the POWs because I don't understand all of them, except the first one. . . . It just frustrates me so much because I have always understood what is going on in math until now. Math has been my best subject since first grade and now that I have reached a math class that deals with elementary concepts, I am lost.*

> *The problem of the weeks really didn't help me as much as I thought that it would. I just thought of it as something extra to do that took up more of my time. To me none of these problems helped me to understand geometric shapes any more than I already do.*

> *To tell you the truth I don't feel that all of the POWs that you gave us helped my understanding of math. . . . overall I feel that I didn't get anything more than a little more practice with numbers.*

One of the students reported feeling this way in her first reflective essay for the first course. However, after the second course, she wrote that she enjoyed the POWs, that they increased her self-confidence, and that she felt like she understood some problems for the first time. Several of the other students who wrote negative comments about the POWs were all in the first course this semester. One of them did not pass and had to retake the first course. Two of them passed the first course but chose not to enroll in the second course in subsequent semesters. The other three successfully passed both courses. It is difficult to determine what the problem was for these students, as attendance, previous experiences, levels of anxiety, and other factors all have a part. Another possibility is the student may have chosen to opt out, which is always a choice in an open community. Students were observed to opt out in their choice of POWs to include in their portfolios, their choice of attending class, and their choice of working in a group.

Further research is needed to determine what didn't work for these students, why it didn't work, and what possible additions and alterations may be more successful. Overall, the great majority of students remarked that working on the POWs significantly affected their mathematical understandings in a positive way. They felt that their understanding grew in a more explicit manner, that they were able to make connections and consider multiple perspectives. Their understanding of the nature of mathematics changed and some students were able to successfully reason and justify their answers.

Summary

With the mathematics that students bring with them to the college mathematics classroom falling short of expectations, I endeavored to provide a problem-rich learning environment where students could begin to unpack and revisit previous mathematical understandings. Most students were successful in extending their understanding of many concepts and procedures to more explicit and connected levels. In the next chapter I will discuss these findings and other emergent data, and offer a model based on chaos theory that helps explain why students experience open-ended opportunities to learn mathematics so differently.

CHAPTER SEVEN

RESEARCH ANALYSIS & DISCUSSION

> It must never be forgotten that education is not a process of packing articles in a trunk.
> Alfred North Whitehead

Overview of the Study

This chapter begins with an overview of the study, introduces a few of the emergent themes in my research, describes learning on the edge of chaos, and finishes with some implications of my research. With the assumptions of a need for reform in preservice mathematical education, and a different kind of mathematics needed by preservice teachers than generally offered in the traditional college mathematics class, I embarked on this research study to explore a different approach to the teaching of mathematics. My research context involved a homogeneous group of preservice teachers immersed in a mathematical community with a problem-rich learning environment and negotiated sociomathematical norms. With the expectation of group collaboration and communication, I set out a program involving complex problem scenarios, coupled with time, space, and reflection. The three questions that focused my research were:

> 1. What do preservice teachers bring with them to a college mathematics content course?
>
> 2. How may a problem-rich learning environment enable preservice teachers to unpack the mathematics content they have already "learned?"
>
> 3. How does unpacking the content affect preservice teachers' mathematical understandings?

Deficient and Insufficient Mathematics

My research found that although students bring thirteen to fifteen years of mathematical experiences with them to the college mathematics classroom, this knowledge was often faded, false, fragmented, and tacitly understood. Overall, it tended to be deficient and insufficient for both personal and future teaching purposes. This content knowledge was accompanied by as many years of mathematical beliefs and emotions that were often oppressive baggage that interfered with understanding mathematics and connecting added knowledge.

Interaction with complex problem scenarios such as the POWs in a supportive and non-threatening environment appears to enable students to unpack their previously learned mathematics and revisit concepts with the intention of examining, sorting, valuing, and reconstructing them into a more useful configuration. The POWs seemed to be able to perturb student thinking even more than I had anticipated. Levels of perturbation, initial anxiety followed by engagement, active problem solving, and ultimately revisiting their solution strategies as they often became frustrated with or motivated by the need to understand why their solutions worked. This offered an insight into the continuum of frustration that transcends students from inactivity to actively engaging in their own mathematical thinking. The classroom culture and relationships they developed in class were important for their willingness to take risks and go beyond initial levels of frustration with the problems. The level of engagement for most students was very high, and they used this motivation to revisit concepts that they thought they had long understood. With extra time and thinking spaces, students were able to unpack and reconstruct many mathematical concepts and procedures.

Alfred North Whitehead (1929) wrote, "Knowledge does not keep any better than fish (p. 147)." But you can keep a fish – for awhile – if it is packed in ice, away from air, heat, and light. However, one cannot keep fish packed forever. To really enjoy it, one must unpack it and ingest it! This is similar to mathematical knowledge. It cannot keep forever, packed away from understanding, application, and connections. The "aimless accumulation of inert and unutilized knowledge" (Whitehead, 1929, p. 58) must be aired and treated as fresh fish. Whitehead suggests that mathematical knowledge needs to come to students "just drawn out of the sea and with the freshness of its immediate importance (p. 147)." Cached knowledge needs to be unpacked, inventoried, evaluated, and put to use. Only through this process can previously learned mathematics be available and used for its maximum impact.

Students reported that unpacking mathematical content through problem solving activities affected their mathematical understanding in multiple ways. Unpacking allowed opportunity for reflection, encouraged more explicit understanding, affected their beliefs about the nature of mathematics, changed their perceptions about themselves as mathematical beings, and influenced their emotions surrounding mathematics.

Reflection

Unpacking their mathematics encouraged reflection at multiple levels, mentally, verbally, and written, at the beginning, throughout, and after the problem was solved. One student wrote in her journal about her experience analyzing a POW instead of just answering a POW. "I think that analyzing is reflecting on what you have just done in the math problem and answering questions about the how's and why's of your methods and answers. This requires a lot of thinking, communicating, and verbalizing about math."

Articulation is a public form of reflection (Carpenter & Lehrer, 1999), and when students are challenged to articulate their own ideas, they learn about their own powers of thought (Schifter & Fosnot, 1993). Students' developing ability to reflect on their own thinking and actions is evidence of the emerging nature of their mathematical understanding (Carpenter & Lehrer, 1999).

Written reflection through journaling and reflective essays was a valuable portion of the unpacking done through the POWs. Although at first students were reluctant to "write" in a mathematics course, they soon realized the value and implications of reflecting their thoughts and understanding through mental, verbal, and written forms. A student explained about writing her narrative journal, "A key part to this problem for me was actually writing everything out to make a complete visual picture." Students felt that the more practice they had explaining the more deeply and explicitly they understood the concepts involved.

Explicit Understanding

Unpacking through problem solving extended students' mathematical content understanding to a more explicit level. Many students referred to their new and deeper understanding of mathematical topics they had previously had difficulty with, such as fractions, word problems, and geometry. Other students found that many mathematical topics were related, and that deeper understanding also included making connections among the different topics, such as prime factors and lowest common multiple.

In some cases, more explicit understanding encouraged intellectual curiosity that took students beyond the expected into the realm of possibilities. A few students began to think about what might happen if one or more parameters of the problem were changed, and the effects it might have on the solution. They were exploring ideas like rational number base systems and corollaries for standard theorems (see Intellectual curiosity in Chapter Six).

Unpacking content provided an opportunity to explore and engage in multiple representations. Students were able to compare the efficiency and appropriateness of different ways of modeling problem scenarios. With multiple concepts involved in each POW, there was an opportunity to begin making connections and drawing relationships between and among mathematical topics, concepts, and procedures. As their understanding became more explicit, their ability to communicate their mathematical thinking increased. They found they could explain what they were thinking and doing to others and in written form. These more explicit understandings were supported by and related to the accompanying changes in students' beliefs and emotions about mathematics.

Beliefs

Unpacking also had an effect on students' beliefs about the nature of mathematics. They were able to use their current experiences with mathematics to validate or negate their previous perceptions of the discipline. Students were surprised and intrigued to discover that many other students had different perspectives than they did. Working in groups and sharing ideas and questions was a valuable experience in learning about how others think and interact with mathematics. This will be a valuable asset in dealing with their future students, as many of them noted in their writings. Students were also able to start claiming ownership of their mathematics as they constructed and engineered their own understanding, connections, and perceptions.

Another strong outcome of unpacking was an increase in autonomy. If the teacher or text is the ultimate source of truth, when students move on and those sources are no longer immediately available, then verification cannot be achieved. Unpacking helped students discover verification and validation within their own thinking. Students were able to share in the authority and responsibility of learning and understanding, and developed ways of validating and verifying their methods solutions themselves, instead of relying on external sources.

Students who had not been successful in past mathematics courses began to feel a sense of accomplishment in even their simplest efforts. A student wrote, "I've always doubted myself and my math ability up until this point." Students who were confident and always performed well in mathematics classes were beginning to expand their understandings and were able to extend their thoughts beyond the technical level. One such student wrote in her reflective essay, "Coming into this class I already had a high mathematical confidence, though now after half a semester I have achieved a whole new level of confidence."

Emotions

Unpacking had a very strong effect on the negative emotions that many students had about mathematics. They reported a decrease in their anxiety and fears. There is less to fear from mathematics when seen as more than just numbers and calculations. Students began to identify, confront, and learn to deal with their various negative emotions. Their attitudes toward mathematics changed as their perceptions altered. They began to enjoy mathematics as they viewed it as a more creative and interesting discipline. There was a parallel upswing in confidence and mathematical self-efficacy, resulting in an increase in empowerment as they began to feel in charge of their own mathematical learning. Lester & Mau (1993) reported similar results in their study with problem solving and preservice teachers. "Students amazed themselves, and that amazement turned to confidence and an increased determination for the remainder of the semester (p. 10)."

Students who do not have to conform to one restrictive method or answer to an oppressive authority may find that unpacking through problem solving encourages reflection, deeper understanding, a change in beliefs and emotions about mathematics and themselves, and an increase in confidence and autonomy. Involvement in such experiences can be empowering to students. The act of unpacking fosters significant changes in students' attitudes about what mathematics should be and what it means to do, understand, and teach mathematics. This observation was an emerging theme in my research.

Emerging Themes

There were three additional themes that emerged from the data. Students were very outspoken about their previous experiences in mathematics classrooms. They almost exclusively wrote about experiencing and believing that mathematics could only be done one way. Another emergent theme was the amount of frustration encountered at multiple stages of the problem solving process, and the surprising coupling of this frustration with a sense of accomplishment and satisfaction when the solution was found.

One Way Baggage

One of the principal emerging themes involved the "one way" baggage that students reported bringing with them to the college mathematics classroom. Forty-seven students spent extensive time in their reflective essays discussing their previous experiences with mathematics. They were very specific that there was always and only one way: one right method to achieve the one right answer. Students shared in their essays:

> *I am very used to having the instructor tell me which formula to use and on which problems to use it.*
>
> *. . . coming to a conclusion that appeased a math instructor or a textbook publisher was the ultimate accomplishment.*
>
> *I always thought before that a teacher should teach something one way and all the students should do it her way.*

This oppressive baggage colored students' beliefs about the nature of mathematics and their perceptions about themselves as mathematicians. If there is only one way, then the teacher tells it, the students memorize it, and re-tell it on the test. Once the test is scored, there is no further purpose for that information, so it is either forgotten or stored in some mental toolkit. Some of the comments shared in the reflective essays included:

> *I memorized what I had to pass the test, in no way understanding the theory behind the formulas, and then forgot when the test was over*
>
> *. . . all of my previous math courses have taught me to rely heavily on rote training through memorization of formula after formula.*
>
> *After over ten years of being told to plug certain numbers into certain places to create specific results it has imbedded [sic] in me a one track mind.*

One student remarked in her reflective essay about dealing with circumference and area of a circle. "I always just followed the formula and tried to memorize it for the test. Now, I don't have to just memorize it for the test, because I know when to apply it."

Knowledge does not fade when there are connections and understanding about when and why to apply it.

As students engaged in collaborative problem solving with genuine tasks, accompanied by communication and reflection, this oppressive baggage began to disturb students. It was no longer just the way things had been, it became a vexing experience that students somehow felt should not have happened. Students described these experiences with strong words in their reflective essays:

> *I believe that too often in mathematics classrooms rules and formulas are given without allowing the students to see how they were derived and why they work.*

> *I have been taught to see things the way the teacher wants them, whether I understand the steps or not. I think that is exactly why so many people hate math.*

> *It seems that the teacher was so busy teaching algebra that she didn't care if we actually understood algebra.*

> *But I now see that the teachers simply were not open to different ways of thinking.*

> *I am now seeing the terrible consequences of not being allowed to explore math and the concepts behind it. It is a lot harder to go back to the beginnings of math as an adult, than it would have been as a child.*

> *I wish that I had had a math teacher who would have been more worried about whether or not the concepts were understood and not just about the right answer.*

> *If all people do not learn the same way how can there only be one "right" way to do a problem? There can't.*

> *I am curious at how a similar age group could have been taught so many different approaches to the same problem and yet were all told that there is only one way to come to a solution.*

As students realized that mathematics was not as restrictive and constrained as they had believed, their personal mathematical goals began to change. It was no longer sufficient to just get the right answer, get a high score, and pass the class. They wanted and expected more than that, as explained in their reflective essays:

> *I need explanation, something I can touch and see a difference in. Not just doing it to do it. I need to know what I am doing, what I am trying to get at, why I am doing it and is there a significant purpose to it . . .*
>
> *Hopefully by the end of this class I will be able to claim all my math as my own.*
>
> *. . . I often caught myself thinking about how exciting it would be to reach out to my young students some day with such exciting and alive mathematical reasoning. Now that's a new thought. Math being alive for me.*
>
> *I dreaded learning and teaching math. Now it is the subject that I am most excited to teach.*
>
> *. . . this course has made me understand what my math goals should consist of. From now on, instead of doing math to just get the answer, I need to actually understand what math is all about.*

Wanting more and different experiences in the mathematics classroom, many students were determined to avail themselves of the opportunity to really learn and understand the mathematics this time around. They realized that their career goals would be best served by engaging in mathematical problem solving that would increase their understanding to an explicit level. They may not have been successful in explaining their thinking or their methods, or understanding others' perspectives and explanations in the past, but they knew this would be important in their future careers. I believe that this motivation helped them deal with their confusion, frustrations, and anxieties.

<u>Frustration</u>

Another emerging theme was the amount of frustration encountered while involved in problem solving activities. Frustration was the most common descriptor in the short response papers, and continued to be mentioned in the narrative journals for various POWs. However, when it came to the reflective essays, the tenor of the writing about frustration changed. In almost every case, when frustration was mentioned in the essays (and there were less occurrences there), it was accompanied by an explanation of accomplishment and satisfaction. It seems that over the course of the semester, students were able to funnel their feelings of frustration into feelings of success. Here are a few of the student remarks on the corresponding emotions of frustration and accomplishment:

> *I feel like I've gained a lot, personally and mentally from performing these problems. Even though at times I felt like giving up, my curiosity and determination pulled me through and in the end I felt a huge since [sic] of accomplishment for giving them my full attention.*

> *These problems are always frustrating at first, but then when you figure out the answer it is a very rewarding feeling.*

> *Not only do you experience the frustration of the confused student, you also get to experience the JOY and pleasure of figuring out the problem, which seemed so difficult to you a few moments before that.*

> *Though this class has been frustrating at times, I still get a rewarding feeling when I finally get the answer. Likewise, the frustration of this class has led me to a deeper understanding of simple mathematical concepts, and helped me to see simple math problems with a new and broader perspective.*

> *Overall I must admit I thoroughly enjoyed the problems of the week, although they were extremely challenging. I would have never said that at the beginning when they were first given, because they were very frustrating. I received great satisfaction once I was able to solve the problem and completely understand it.*

Although I was surprised by this emerging theme of frustration leading to accomplishment, it is not uncommon. Frustration and confusion with problem solving often come before the satisfaction of accomplishment. Schifter & Fosnot (1993) explain that

> being challenged, encountering novelty, confronting one's

> misconceptions–in short, building new and stronger understandings–typically involves bewilderment and frustration. However, if mathematics students can learn to recognize that the discomfort they experience is part of the process, they can also learn to tolerate it. By encouraging students to monitor their own learning, teachers can help them achieve greater control over that process. (p. 11)

It seemed that students were frustrated at different points during the problem solving activity. Most students were frustrated trying to start the problem. Then students seemed to "get stuck" at certain points along the way. Sometimes students were frustrated with their solutions, and had to dig deeper to understand why the answer was unexpected or counterintuitive.

Getting started. Many students wrote of the difficulty of starting the problems. Students would read and reread the problem and wonder where in the world to start. Several students used the word "overwhelming" to describe their feelings at first sight of the problem. These problems did not follow the format of a textbook chapter or involve a single procedure that was to be routinely practiced. These problems took some thinking and looking for possible methods. This was an entirely new experience for the majority of the students. At the beginning of the tasks, their short response papers indicated the challenge of starting:

> *Whenever I first read the problem of the week I was thinking, "how in the world am I going to do this?"*
>
> *I cannot just look at this problem and know where to start.*
>
> *We had a hard time trying to figure out a way to begin . . .*
>
> *At first I had no clue how to even approach this problem.*

And similar comments were found at the end of the process, in their reflective essays:

> *I think that [the] most challenging thing about these problems was figuring out a way to start the problem. I[t] always took me several different ways to decide how I was going to go about getting an answer.*

> *The most challenging part of the POWs was finding a starting point. I would read each problem and just sit wondering where to begin. I had to learn to try different things until I came across something that made sense.*
>
> *I would spend twenty minutes just staring at the worksheet or my scrap paper trying to decide the "right" way to do the problem.*

The beginning is not the only place students felt frustration. Sometimes they got bogged down in the middle of the problem solving process. This "getting stuck" may be caused by confusion turned up by the unpacking process, a lack of confidence that allows them to second guess themselves, feeling that they have already tried all possibilities, and unvoiced assumptions that block their progress.

Getting stuck. Problem solving often involves getting stuck at various points of the process. There may be false starts and blind alleys (MSEB, 1993). Students who are unpacking mathematical content that they thought they previously understood may get stuck as they pick apart and reconstruct the concepts involved. When the revisiting concludes in a deeper understanding or an "aha!" experience, students can get back on track with the problem solving.

Sometimes students have such a lack of confidence in their mathematical abilities that they cannot trust themselves to choose a correct path or trust their solution. Part of this is their fear of being wrong, both publicly and personally. A student wrote about, "a fear of that one little mistake that would render the rest of my efforts futile." It is difficult to move on if you think the possibility of error will lead you far from the correct path in a long and useless journey. These students tend to go beyond a healthy critical check of their thinking into second guessing themselves, trapped in a spiral of negatively questioning their efforts and abilities. As students worked collaboratively over time with the POWs, their confidence level increased, and they were able to begin breaking out of this destructive cycle.

A common point of frustration was the feeling that they had tried all possible combinations and were not successful. Some students even stated that the problem could not have an answer because they had tried everything and nothing worked. Two examples were the Bridge task and Magic Square POWs. Students felt they had tried all possible crossing combinations for the ladies on the bridge. Actually, most tried only a handful (not all 108 combinations) that they felt would give them the best chance at an answer. Some students also reported trying all possible fractions for the Magic Square POW when in reality they only tried all possible eighths, without realizing that other fractions were possible. Students explained their frustration in their short response papers:

> *I became frustrated because I was not getting the answer. I had become blocked and was not seeing any other possibilities than the ones I was trying.*
>
> *I got stuck in the same situation every time . . .*
>
> *Once I try all the ways that I <u>think</u> there are to solve it, my mind goes blank.*
>
> *I started to get frustrated because every way that I tried I kept coming up with 65 minutes.*

Sometimes students are laboring under unconscious assumptions that serve as a roadblock to further progress on a problem. Again, with the Bridge task, students kept trying combinations of crossings that exceeded the allotted hour's time. A strength to note here is that students were able to routinely exclude the majority of possible combinations that obviously were excessive. But their constrained choices usually did not include the shortest combination because of unspoken assumptions of who was able to carry the lamp and who should pair with whom. In their short response papers, students wrote:

We didn't think outside of what "had" to be true . . .

I got stuck in that idea but I finally realized . . .

[Then I] broke my "mind mold" and I discovered the answer!

Students working on the Palindrome Date POW often assumed that the most "recent" date would fall into the past few years or at least the last century. They were amazed when the most recent palindrome dates seemed to fall in the fourteenth century. Some students were able to verify this seeming contradiction by logical explorations, others stated that they would continue to attack the problem because there had to be a more recent date than the 1300's. These embedded assumptions stalled the students and sometimes led them down the wrong path.

Often students turned to an internal dialogue (a form of reflection) to identify the obstacle(s) to their progress (EDC, 2000). Usually with continued collaboration, communication, reflection, and deeper unpacking, students were able to work through or find a way around the sticking points and continue on their path to a solution. However, even smooth sailing at the beginning and during a problem does not discount the possibility of cognitive conflict when a solution is found.

<u>Cognitive conflict</u>. With the Earth POW, some students were frustrated at the end of the problem solving process. When the solution was counterintuitive to all expectations, the cognitive conflict brought a good deal of frustration at a point when they felt they should "be done." When a solution was far out of the expected zone, they stopped and worried that they had made an error. Most students just reported the answer and were done. But some students allowed themselves to be perturbed enough that they checked their work to confirm its correctness, and then confronted their feelings that the

answer couldn't be the right one! This is where multiple representations and metaphors enabled those students to find an understanding that resolves the cognitive conflict. Frustration over wanting to understand why a solution works or why an answer is correct was, for many of the students, a new kind of frustration with mathematics; one that lead to more mathematics rather than limiting their mathematics.

Here are a few comments from student journals:

> *. . . I just assumed that in order to add a foot all the way around would require a large amount of extra rope.*
>
> *I was so baffled by the result that I repeated the problem several times, but each time I did it, I got the same answer. I decided to try the same problem with a miniature soccer ball that I have . . . I got the same answer. Next I made up imaginary spheres with different circumferences. . . Each time I did the problem I came up with about 6.283 [units]. That's when it dawned on me that the answer I kept getting was equal to two times pi. . . . It was very surprising to me that the size of the sphere in question . . . is completely irrelevant.*
>
> *I was fairly certain that I had calculated correctly, but I did not understand . . . Thinking for a few minutes made me realize why this was correct. The earth's circumference is so huge. Adding 2 feet to the diameter is not going to change the circumference hardly at all. It's like putting a penny at either end of a football field; the length will not change enough to notice a difference.*

Frustration can occur at the beginning, in the middle, and at the end of the problem solving process. At each point of frustration, students may give up or continue to engage by unpacking, reflecting, communicating, or considering other perspectives. Frustration tends to occur when students are perturbed in their thinking, causing them to reorganize their thinking, actions, and understandings. Doll (1993) asks an important question, "Under what conditions, then, does perturbation become a positive factor in the self-organizing process?" (p. 164).

Analogies to Frustration

Here are two analogies to show that frustration can be a helpful and necessary part of problem solving and mathematical understanding. Schifter & Fosnot (1993) make an analogy of a boat sailing on calm waters. Not until the water gets churned up a bit does the sailor need to do any work. During turbulence the sailor must work to keep an even keel. However, the sailor should avoid rough water that puts the boat in danger. Schifter & Fosnot continue

> We try in our interventions to engender cognitive conflict where we see faulty logic or to stimulate further thinking where we see an opportunity to open up a new area of exploration . . . In any case, interventions are chosen not so much to help participants get right answers as to help them construct more powerful concepts. (p. 28)

Perturbation is an important part of a problem-rich learning environment. If the task at hand can be solved quickly and simply, then it is just an exercise rather than a problem. Tasks which "rock the boat" or "shake them up" offer an excellent situation for unpacking and reexamining learned mathematics.

I would also like to compare frustration to a low grade fever. Such a fever is an indication that the body is fighting an unstable condition such as a virus or infection. Medical doctors often tell parents not to treat a low grade fever, as it is the best mechanism the body has to fight back to stable health. However, if the fever rises high enough, it becomes a danger to the body. I compare a low grade fever to facilitative anxiety, just enough frustration to be helpful but not harmful in a learning situation. If the frustration level is too low, it may not perturb a student's thinking enough for unpacking to occur. However, when the level of frustration gets high enough to be numbing or mind-paralyzing, then it is a threat to the student, and the anxiety becomes debilitative rather than facilitative. It is important to keep the challenge within the appropriate zone of frustration to encourage critical thinking skills and seeking for understanding without harm or permanent damage to confidence or ability.

Not understanding is one of the most frustrating and ultimately defeating experiences a student can have. Understanding, on the other hand, is one of the most intellectually satisfying experiences (Heibert, et al., 1997). Since frustration seems to be an integral part of complex problem solving, the goal must be to bridge between these two feelings in an effort to move forward through the frustration and reach the satisfaction of accomplishment. One way to do this is constraining the frustration within a range that facilitates rather than debilitates.

Facilitative Anxiety

The notion of facilitative anxiety suggests that some apprehension over a task is a positive factor, tending to keep one poised, alert, and just slightly off balance. Enjoyment occurs at the boundary where challenges are balanced with the ability to act (Csikszentmihalyi, 1990; Brown, 1994). Doll (1993) writes that it is the perturbations, errors, mistakes, confusions that make up the disequilibrium that helps a student reorganize "with more insight and on a higher level than previously attained" (p. 83). He adds that the disequilibrium must be structurally disturbing to prompt reorganization.

Facilitative anxiety is characterized by the confidence that one can succeed if one makes the effort. Students with facilitative anxiety show a high degree of organization in their thoughts and actions. They are able to mediate stressful situations in a flexible way. On the other hand, students with debilitative anxiety have a high degree of disorganization of thoughts and actions, respond rigidly, experience confusion, emotional upset, a feeling of helplessness, and their minds "go blank" (Hartman, 1997).

The level of anxiety that is facilitative lies in a range between too much and too little. Too much anxiety and students will dismiss the activity as impossible or meaningless, leading to disruption without learning. Too little anxiety will not produce any reorganization in response to new information. There must be a dynamic tension between challenge and comfort (Bransford, Brown & Cocking, 2000; Bruner, 1973; Doll, 1989; Cleveland, 1994; Romberg & Kaput, 1999). Nothing changes in the midst of stability, it is too comfortable and placid. But too much discomfort does not lend itself to learning, either. Learning must occur in the space where comfort and challenge begin to coexist.

Some of my students exhibited a facilitative level of anxiety in their unpacking and problem solving. They wrote

> *I have learned the importance of letting a challenge motivate me, instead of frustrate me. . . . I believe that the challenge has pushed us to go above you personal level of learning.*
>
> *It seemed like every question on the Problems of the Week had millions of things to be considered and just as I thought I had figured it all out, one more confusing thing was brought to my attention. I had to stretch outside of my comfort zone and question things that I had never even thought about before.*
>
> *This usually leaves me feeling even more frustrated and less confident in my thinking. . . . On the other hand, just when I feel like there is no way that I can understand, I come out with a better sense of the math concept at hand.*

These students were able to keep their anxiety within a facilitative range that enabled them to achieve more learning and understanding. I believe this range of facilitative anxiety is similar to the concept of the "edge of chaos" encountered in complexity and systems theory.

Complexity and Learning

Complex adaptive systems are networks of "agents" acting in parallel. Each agent constantly acts and reacts with other agents in the system, producing an interactive environment. Control is highly dispersed, and behavior arises from competition and cooperation between the agents at many levels of organization. The complexity arises from the interactions, and does not exist in the agents themselves (Waldrop, 1992). Hartwell (1995) calls the process of individual human learning "the most dynamic process, the most emergent reality, in the universe" (p. 13). I want to follow up on this idea of learning as a complex system and classrooms as learning networks.

Prigogine & Stengers (1984) developed a complex systems model with three stages – a process of equilibrium, disequilibrium, and re-equilibrium. Their model can also be related to Bruner's notion of disruption, Dewey's view of experience, Piaget's disequilibrium theory (Doll, 1986; 1993), and Vygotsky's zone of proximal development. All of these theories deal with three phases and the learner's iterative movement between them.

Tosey (2002) discusses how we can apply complexity to learning. First, an open complex system is unpredictable, and we can only anticipate what might happen and attend to what does happen in practice. Students come with "initial conditions" that affect their learning ability, and their interactions with each other and with mathematics. Carefully chosen tasks can perturb thinking and bring students to a critical point of reorganization. Even insignificant-seeming interactions can be followed by a qualitative leap in understanding. The breakup of previously held ideas and the sudden creation of new ideas (aha!) is hypothesized to take the form of a phase transition (Kelso, 1999).

The significant moment is the "aha!" of learning, the sudden comprehension or understanding of a novel relationship (Trygestad, 1997). Of course, along the way there are also the "oh, no!" moments, when students stumble on a conflict leading to perturbation. However, this is a critical part of the process of reorganization. Doll (1993) writes, "A system self-organizes only when there is a perturbation, problem, or disturbance – when the system is unsettled and needs to resettle, to continue functioning (p. 163)." He also suggests that the curriculum be filled with enough ambiguity, challenge, and perturbation to invite the learner to enter into dialogue with both the curriculum and others. Therefore the teacher must create enough cognitive dissonance to motivate students to reorganize prior knowledge (Tyrgestad, 1997). Such teachers will be comfortable with ambiguity, unknown outcomes, provide space for novelty and play, and encourage risks and new behaviors. Such a teacher will help contain anxiety through support and encouragement, facilitating joint reflections, and engaging in dialogue (Oekerman, 1997).

Educators already recognize that they cannot control or determine learning. Students are essentially self-organizing and the most valuable kind of learning is emergent and constructed. If we focus on learning as a product that can be engineered, we limit the educational experiences of our students. Instead, we create conditions under which learning is likely to emerge. This necessarily means working at the edge of chaos (Tosey, 2002).

<u>Edge of Chaos</u>

Systems which are the most adaptable, creative, and alive, are operating in the zone between stability and instability. It is not a border line but another phase state between the states of order and disorder. When the system (student) is operating in the narrow zone between order and chaos, there is a space for novelty, new ideas, unexpected directions of activity flow, where risks are taken and new behavior is tried out. There are new and surprising outcomes that emerge from creative activities (Waldrop, 1992; Gell-Mann, 1994; Oekerman, 1997), where small changes can produce big effects (Lipmanowicz, Petzinger, Hutchens, Lewin, & Lindberg, 2002). The area is where the system is both solid enough to store information, yet fluid enough to transmit it (Waldrop, 1992), is called the "edge of chaos."

We have all heard phrases like "take it to the edge," "pushing the edge of the envelope," and "over the edge". The phrase "edge of chaos" was coined by Doyne Farmer, of the Santa Fe Institute and first used in print in 1988 (Waldrop, 1992). Thus this concept is fairly recent and is evolving from a science to a cultural metaphor with underlying themes of control, creativity, and subtlety (Briggs & Peat, 1999). The process is essentially the same in cognition as in biology. The agents are individual minds and the feedback comes from teachers, the environment, and direct experience. Learning at the edge can provide conditions that maximize learning (Waldrop, 1992). I propose a learning model based on the edge of chaos in an attempt to represent the use of problem solving, facilitative anxiety, and unpacking.

Edge of Chaos Model

In the middle of the model there is an amorphous region of stability which represents the prior mathematical learning of a student, what they often identify as their "comfort zone." Surrounding this zone is the space of not-yet-known and unknowable mathematics. Students definitely would refer to this as their "discomfort zone." Between these two seemingly complementary regions is a third zone where stability and instability are in a coexistent flux. It is this region which is called the "edge of chaos" and which represents the target zone of optimal learning (see Figure 7). A magical moment when the flux of perception shifts and the chaos begins to self-organize is known as the aha! moment (Briggs & Peat, 1999). When prior learning is activated and

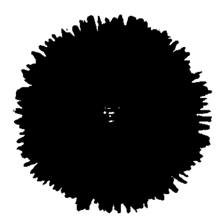

Figure 7: The edge of chaos.

challenged, the imbalance or cognitive dissonance, causes cognitive development. Cognitive development often occurs in spurts at points of disequilibrium, at the level of difficulty allowing the learner to understand new information independently (Trygestad, 1997).

This model could also include the emotive or affective domains, especially as we think of facilitative anxiety, and trying to keep the students on edge with just enough anxiety to perturb their thinking and engage their interest. Any emotion that is too extreme or far from the comfort zone of a student may throw them into chaos and prevent learning. Little or no emotion involved does not provide the motivation, perseverance, or stamina for tackling a complex problem, and the student is in a stagnant position in the middle of the model.

Tasks From the Zone

A task should build on what students know, produce cognitive conflict, and create some curiosity and motivation to resolve the conflict (Romberg & Kaput, 1999). There is a challenge to design tasks with the flexibility to challenge a student's upper reaches of understanding, provide a window on mathematical thinking, and give a sense of accomplishment (MSEB, 1993). For example, choosing Task B in Figure 8 (below), far inside the comfort zone, will probably not allow perturbation and unpacking. Choosing

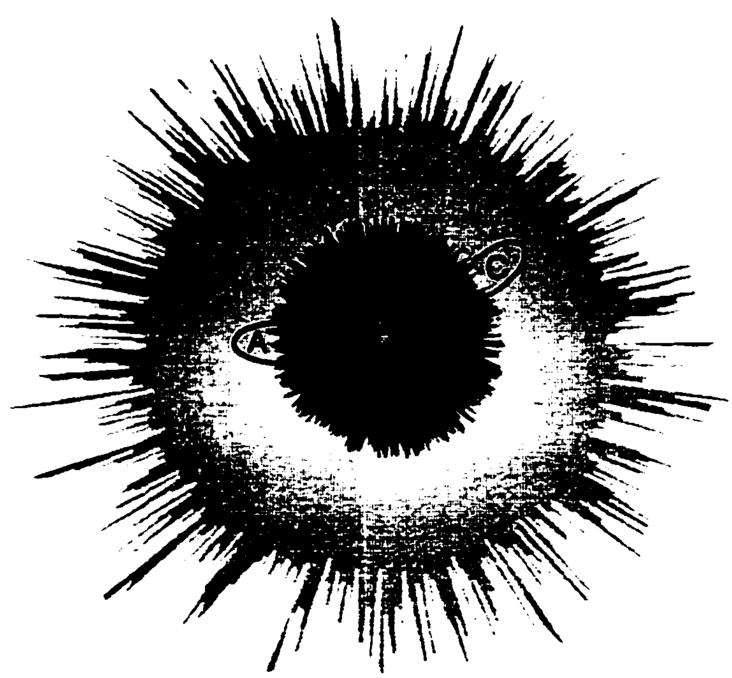

Figure 8: Choosing tasks on the edge of chaos.

Task C which is outside the comfort zone entirely will be too overwhelming to students and probably lead them to give up or obtain the answer by other methods than with understanding. A task should be chosen with roots in the comfort zone but extend student engagement and understanding into the edge of chaos and beyond. The teacher must be willing to give up control, to recognize that development, growth, and understanding are not instantaneous and continuous, but come in punctuated spurts. Thus time is a key factor – students must have the opportunity to reflect, to try alternatives, and to disagree. They need to explain why they did what they did and to explore other methodologies (Doll, 1986).

Tasks themselves have a depth or dimension as they intersect with the space of the psycho-emotive comfort zone. Tasks rich in problem solving potential, as described above, are more likely to intersect the learning space in ways that perturb the student without leaving the student in a cold sweat. Choosing a task that will propel students to the edge is very important. Since each student has his or her own unique comfort zone, each student's edge is a different shape and volume. A task needs to be designed to cross multiple edges of chaos, and provide multiple entry levels and extensions for students who engage at a higher level of understanding. Superimposing comfort zones for all students in the class can give an idea of how broad the task needs to be to offer each student enough comfort to begin and enough challenge to unpack.

All models are imperfect but can be very useful for understanding. In fact it is the imperfections of the model that are critical for guiding future work, seeing what we don't know and where to explore next (New England Complex Systems Institute [NECSI], 2000). This model may be extended to include other fields of learning, or across time for individual students as the edge of chaos expands outward to include a larger comfort zone. However, we must be careful when extending to check our assumptions and ask critical questions about how the model fits what we observe happening in the classrooms and with students actions and understandings.

Research Implications

My research findings about what preservice teachers bring with them to a college mathematics content course are similar to other research findings. An important point is to note that the content knowledge, beliefs, and emotions that students bring are so closely intertwined that they cannot be treated separately. It is the interactions of these components that steer the course of understanding mathematics. The changes in understanding were accompanied by parallel changes in mathematical beliefs and emotions, and vice versa. Deeper understanding cannot emerge without the accompanying effects on beliefs and emotions about mathematics as a discipline and the student as a mathematical being. Any research questions or findings must take this into account when being analyzed or discussed.

My research suggests that using POWs in a safe mathematical community, supported by sociomathematical norms and reflection, helped students to unpack and reconstruct their previous learning of mathematics. Perturbation, engagement, and frustration are important facets supporting the unpacking of mathematical content. Without frustration at multiple points in the problem solving process, students may not engage deeply enough to begin or continue unpacking. Reflection is also an important part of problem solving, where students can think about their thinking and their interaction with the mathematics and each other. Reflection is a way of evaluating and weighing methods, solutions, and implications within and without the context of the problem.

One very important question is if unpacking through problem solving will "do any good in the long run." Mary Kennedy asserts that the history of American education is a "history of reform efforts, most of which have left teaching unchanged" (Kennedy, 1991, p. 3). Research has shown that changes observed during a semester often do not transfer to other contexts, that new teachers generally revert to traditional modes of teaching within three years, and that the educational system cannot sustain reformed mathematics pedagogy. Philosophical changes are short lived, internal and external pressures are heavy, and the sheer mass of familiar daily realities pulls teachers back into the old safe routines (Schifter & Fosnot, 1993). Even with two semesters of problem solving, and three subsequent semesters of mathematics methods classes taught along the same philosophy, we are trying to counter the effects of up to sixteen years of traditional methods (Schifter & Fosnot, 1993).

I want my students to be agents of change, but they are products of the very system that needs to be changed. I believe that what most students experienced in my classrooms <u>has</u> changed their outlook on mathematics and themselves as mathematicians. I feel that there <u>will</u> be a long term effect on their teaching. Having glimpsed another worldview, I think the memory (and feelings) of that world will begin to impinge on their perceived realities. One student wrote in her reflective essay, "I have changed my ideas about how mathematics should be taught and learned. . . . I believe that there is a fairly big gap in what the nature of math is and should be and what is practiced in the classroom. . . . mathematics is a form of reasoning. This includes making sense of things, thinking in a logical manner, and making inferences and conclusions. This is what math in school should be about."

William James (1904) said, "Things which we are quite unable definitely to recall have nevertheless impressed themselves, in some way, upon the structure of the mind. We are different for having once learned them. The resistances in our systems of brain-paths are altered." So I think that the experiences of two semesters, followed by additional methods classes under the new paradigm, will have a long term influence on both the thinking and the actions of my preservice students. Further research is indicated with this population.

Need for More Research

There is a need for more research to examine my findings in terms of other courses, other fields, and other student populations. For instance, I am not sure of the role of frustration with children. Frustration may be an integral part of unpacking for adult students who have had extensive mathematical experiences, but may not play as large role with young students encountering mathematical concepts for the first time. If frustration plays a similar role with children, then it needs to be further examined in the current trend of helping children "feel good" about themselves and "making math fun and easy."

There is a need for more research with students who did not feel that POWs were helpful for their mathematical understanding. The questions may be why POWs don't work for them, or what other variables are affecting the success of unpacking through problem solving. It may be that students who do not attend regularly, or do not participate in group activities, have not been able to engage enough to show a change. Students may choose to opt out rather than spend the time and effort needed to engage the mathematics. Perhaps the anxiety level is so crippling that students are not in a position to be helped in these situations. An accompanying question about the role of group dynamics and whether homogeneous or heterogeneous groups work best could also be pursued with these students.

Another question I would like to pursue in upcoming semesters is that of extending the unpacking process through more whole class discussion. My research methods for this study focused mainly on small group and personal unpacking and reflection. I would like to incorporate more time in the classroom (beyond one class period per POW) to have students discuss their interim methods and findings with the whole class. Perhaps this would help those who are not currently capable of engaging, or those whose anxiety levels are too high to get all they could from the problem solving activities. Maybe more sharing of thoughts and processes may lead to more engagement <u>after</u> the problem is solved, in reflecting and connecting with what has been learned.

A suggested follow-up study would involve a case study of a few students to explore a student's engagement in problem solving activities such as the POWs change over time and exposure to multiple tasks, or how a student's frustration episodes change from early to later encounters with problem solving situations like POWs. This may extend the findings of this study in a way that may inform the field of mathematics education.

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Appendix A:

Demographic Survey

Math 2213/3213 Information Sheet

Index number _________

Class (please circle one) *Fr* *So* *Jr* *Sr*

Please check one blank in each of the following questions –

Gender _____ *Female* _____ *Male*

Ethnicity (please describe) ____

Traditional student ___________
(Generally under age 25, full-time, unmarried, no children)

Non-traditional student ___________
(Generally over age 25, returning or part-time; married, children)

Grade(s) I would like or be willing to teach (please circle all that apply)

Pre-K K 1 2 3 4 5 6 7 8

Grades I would not like or be willing to teach (please circle all that apply)

Pre-K K 1 2 3 4 5 6 7 8

Appendix B:

Sample POWs

(Problems of the Week have been adapted from multiple sources. I have refined these problems to specifically meet the curriculum and students needs)

Math 2213

Palindrome Date POW

Recently we had a day with an extraordinary date. Tuesday, 2 October 2001, has an interesting pattern when written as month, day, and year. If you record the date in a six-digit format (like filling in scantron bubbles) it appears like this:

10022001

This number is a palindrome, a number which is the same written forwards or backwards. Your mission, should you choose to accept it, is to find the two ***most recent calendar dates*** that were also palindrome dates.

QUESTIONS:

What process(es) did you use to identify these dates?

How can you be sure that these are the most recent dates?

EXTENSION:

Use your process(es) to find the ***next five future*** palindrome dates.

Math 2213

Base 5 POW

You are an anthropologist, and you discover a primitive race of Martians with one arm, one hand and five fingers. You are interested in their arithmetic, which is modeled on base five.

QUESTIONS:

The place values for base 10 are units, tens, hundreds, thousands, etc. What are the place values for the first four positions in the base 5 system?

______ ______ ______ ______

List the first 30 base five numbers.

Complete the base five addition table for the first twelve numbers.

EXTENSION:

Look at your list of the first 30 base five numbers and identify which are odd and even.

In our base ten, we often identify even and odd numbers by checking the last digit. Does this property work with base five numbers – can they be identified as even or odd by just looking at the last digit?

Math 2213

Magic square POW

A magic square is an arrangement of numbers in a grid so that each row, column, and main diagonal (corner to corner) add up to a certain number.

The following magic square contains fractions instead of the traditional whole numbers. The first two fractions have been placed to get you started. Place a fraction in each remaining square so that the sum of each row, column, and main diagonal equals 1.

3/8		
		1/4

EXTENSION:

Is the arrangement you found the only one? WHY or WHY NOT?

Math 2213

100 Cards POW

One hundred cards are numbered 1 through 100 on one side, lying number side up, on a long table. One hundred students are sent to the table, one by one, with a specific task for each student to do.

The first student turns every card over.
Then the second student turns over the second, fourth, sixth card, etc.
Then the third student turns over the third, sixth, ninth card, etc.
Then the fourth student turns over the fourth, eighth card, etc.

And so forth, until all one hundred students have completed their respective tasks.

QUESTIONS:

Which numbered cards are still facing number down at the end of the process? WHY? What do we call these numbers?

Which cards were only turned over exactly twice? WHY? What do we call these numbers?

Which numbered cards where turned over the most times? WHY?

EXTENSION:

Can this problem be generalized to 1000 cards? 5000 cards? Why or why not?

Math 3213

String POW

Begin with a group of 12 students standing in a circle, with another student as a helper in the middle. Have the first student hold the end of the string. Have the helper give the string to every third student **until the string gets back to the first student** to form a closed curve.

What geometric shape was made? Record your results on the chart.

Predict what shape will be made if the process is repeated with every fourth student holding the string. Wrap the string and check your answer.

Predict what shape will be made if the process is repeated with every fifth student. Wrap the string and check your answer.

Predict what shape will be made if the process is repeated with every sixth student holding the string. Maybe you are getting tired of looping string. How could you model this activity on paper? Finish filling out the chart.

QUESTIONS:

What is determining which shape will appear?
How can you predict what the shape may be?
What patterns show up on your chart?

What method could you use to determine the shape of looping string on every ***nth*** students' finger?

EXTENSION:

What would happen if there were 11 or 13 students in the circle?
What shapes would appear? Why?

Math 3213

Earth POW

Imagine you tied a rope tightly around the Earth at the equator. The circumference of the Earth is approximately 25,000 miles.

Exactly how long is your rope in feet?

Now imagine we want the rope to "float" in the air one **foot** above the Earth all the way around the equator. How many **more** feet of rope do you think you would need?

My guess ___________________

Calculate how many more feet of rope you would need.

How does your answer compare to your estimate? Why?

Now imagine we want the rope to "float" in the air one **mile** above the Earth all the way around the equator. How much **more** rope do you think you would need?

My guess ____________________

Calculate how many more miles of rope you would need.

What do you notice about your answers to the two scenarios of floating one foot and floating one mile? WHY does this happen???

Math 3213

Cube POW

A wooden cube is made up of 6x6x6 = 216 little blocks glued together. Then the outside surface of the wooden cube is painted red.

QUESTIONS:

How many of the original 216 cubes are painted on –

No sides?

One side?

Two sides?

Three sides?

More than three sides?

EXTENSION:

If we remove an outside layer one block thick from the original cube, how many blocks will be left? What is the volume of this smaller cube?

If we remove another outside layer one block thick from the smaller cube, how many blocks will be left? What is the volume of this smaller cube?

If we remove another outside layer one block thick from this second smaller cube, how many blocks will be left? What is the volume of this second smaller cube?

Appendix C:

Portfolio Grading Rubric

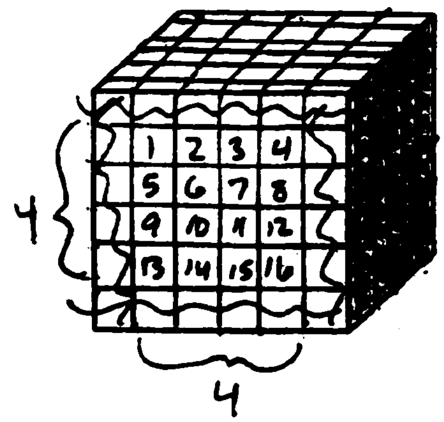
Portfolio Grading Rubric

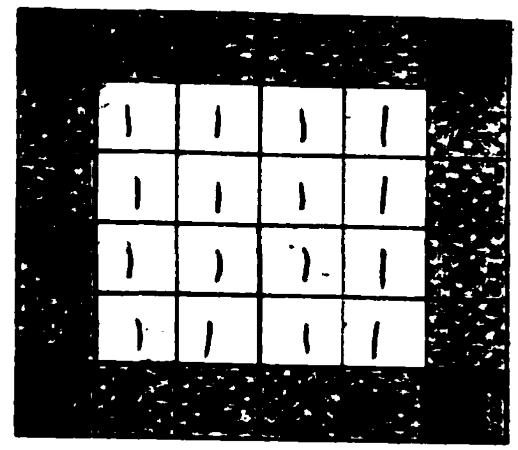
	Excellent	Fair	Poor
Format	exceptionally organized/neat excellent grammar/spelling stapled by POW, name/title on each two copies in a paper pocket folder permission form included	organized, neat good grammar/spelling other folder	disorganized torn edges/loose pages poor spelling/grammar unstapled/no folder one copy/no name/no title
Completeness	extensive explanations attempts multiple methods/solutions examples/counterexamples sketches/diagrams/charts/lists/tables checked solutions	some explanation only gives solution technical details only	no explanation no diagrams, etc no solution
Effort	four POWs full mathematical effort all extension questions creative, original thinking uses appropriate resources	moderate mathematical effort traditional thinking	minimal mathematical effort missing POW(s) unanswered questions
Connections	connects to personal experiences connects to other mathematics concepts connects to classroom experiences	mentions w/o explanation	no connections mentioned
Journaling	fully describes reasoning fully explains solution several pages each POW evidence of trial/error describes context of learning	some reasoning described some explanation of solution 1-2 pages each POW only shows successes	no reasoning described only solution offered less than one page each POW
Reflective essay	3+ pages of reflection explores personal math progress reflects on personal math goals	1-2 pages of reflection	less than 1 page of reflection

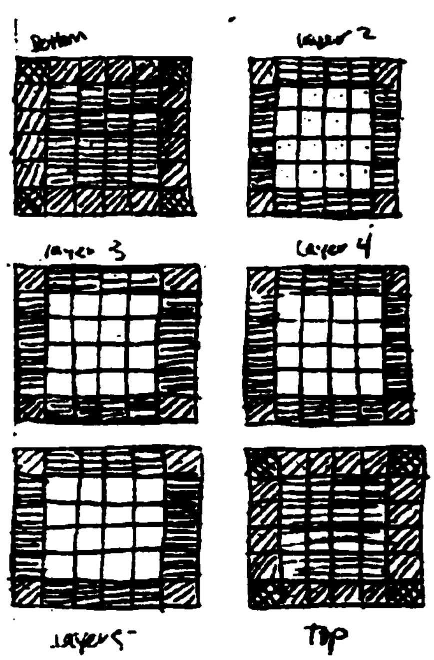
Appendix D:

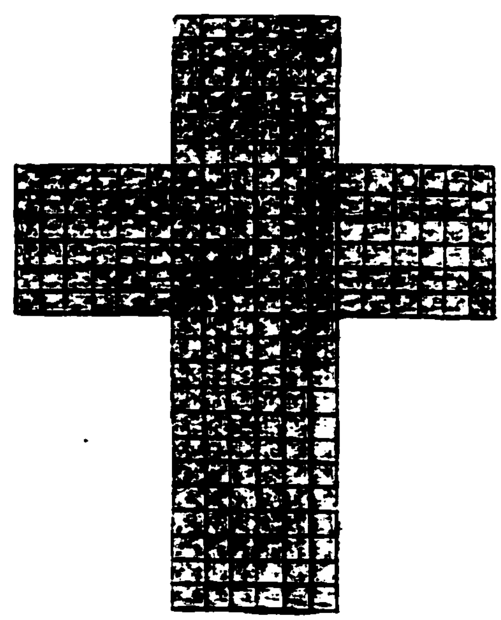
Multiple Representations

Multiple Representations for the Cube POW









Multiple Representations for the 100 Cards POW