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F-HARMONIC MAPS IN KAHLER GEOMETRY

A DISSERTATION APPROVED FOR THE
DEPARTMENT OF MATHEMATICS

BY

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DEDICATION

to

My beloved grandparents

Khoa Van Nguyen and San Thi Ngo

My loving parents

Hau Phuc Nguyen and Thu Thi Nguyen

and their children

Huong

Huy

Hoàng

Hà

Hung

Hùng

Hòa

Huyên

Hang

Hiệp

Huân

and

Uncle Thi Nguyen

April and Caroline

Yen Nguyen

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Abstract

Let $u : M \to N$ be an F-harmonic map between Kahler manifolds of finite dimensions. Is $u \pm$ holomorphic? In the special case of a harmonic map, Y. T. Siu [50] gave an affirmative answer when the target manifold is of semi-strongly negative curvature. In other cases such as p-harmonic, exponentially harmonic maps etc., answers to the above question were less satisfactory. For the general case of F-harmonic maps, when the domain manifolds are complex space forms this thesis investigates the holomorphicity of F-harmonic maps and obtains Liouville-type theorems.

Chapter 1

Introduction

Details are all that mattter: God dwells there and you will never get to see Him if you don't struggle to get them right! **Stephen Gould**

The study of harmonic maps has been extensively developed in the frame work of differential geometry since its introduction in 1964 by J. Eells and J. H. Sampson [18]. In the early 1970's, A. Lichnerowicz [38] began to study harmonic maps in the setting of complex geometry. By the 80's, J. Jost and S. T. Yau [33] treated applications to nonpositively curved Kahler manifolds where the theory of harmonic maps had shown itself to be most successful. From its inception, extensions to the notions of p-harmonic [58], exponentially harmonic [14], f-harmonic [10][11], biharmonic and f-biharmonic maps [9] were continually introduced. Extensive research had been carried out and applied to broad areas in science and engineering including robot mechanics.

By the beginning of the present millennium, in an attempt to generalize all aspects of harmonic maps into a single concept, M. Ara ([1][2][3]) had introduced the notions of F-harmonic maps, F-stress energy tensor and studied the F-stability of these maps. This new concept unifies several varieties of harmonic maps such as p-harmonic maps, exponentially harmonic maps, minimal hypersurfaces, maximal spacelike hypersurfaces and steady compressible flows aside from the well-known classical harmonic maps [12].

This thesis investigates the role that F-harmonic maps play in Kahler geometry whose building blocks are the sixteen classes of almost Hermitian manifolds classified by A. Gray and L. M. Hervella [22]. In Chapter 1, we give local representations and the decomposition of the complexified differential of a C^∞ map in Hermitian geometry. In Chapter 2, we study F-harmonic maps from a different perspective: we give the notion of Ω-harmonic maps then derive all basic facts about F-harmonic maps via this definition. In Chapter 3, we explore the realm of Kahler geometry through concrete examples. In Chapter 4, we give applications of F-harmonic maps in the setting of Kahler geometry and prove several results yielding partial affirmative answers to the question posed in the abstract when the domain manifolds are complex space forms. We also obtain Liouville-type theorems. All manifolds in consideration are C^∞(or smooth), connected and of finite dimensions.

1.1 Main results

Theorem 2.15 : Let $u : (M^n, g) \to (N^k, h)$ be a stable F-harmonic map from a complete noncompact Riemannian manifold M into a complete Riemannian manifold N. Let ϕ be a smooth function on M. Then the following inequality holds:

$$0 \le \int_M F''(\tfrac{|du|^2}{2}) \{ \; |du|^2 \, |\nabla\phi|^2 + \phi^2 \, | \textstyle\sum_{i=1}^n B(\tilde{e}_i, \tilde{e}_i)|^2 \; \} \, dv_g$$
$$+ \int_M F'(\tfrac{|du|^2}{2}) \{ \, k|\nabla\phi|^2 + \phi^2 \textstyle\sum_{a=1}^k \sum_{i=1}^n (\, 2 \, |B(V_a, \tilde{e}_i)|^2$$
$$- < B(V_a, V_a), B(\tilde{e}_i, \tilde{e}_i) >) \} \, dv_g \, ,$$

where dv_g is the volume element of M and $\tilde{e}_i := du(e_i)$.

Theorem 2.15 generalizes Wei's theorem [58] .

Theorem 3.18 : Let (M^n, g, J) be an n-dimensional complete noncompact Kahler manifold. If at each point of M the sum of any q eigenvalues of the Ricci tensor is nonnegative then any 2-finite harmonic form of type $(0, q)$ or $(q, 0)$ is parallel. In addition, if M has infinite volume or the sums of any q eigenvalues of the Ricci tensor are all positive at some point of M then any such form vanishes.

Theorem 3.18 generalizes Greene and Wu's work [24] .

Theorem 4.14 : Let $u : (\mathbb{C}^n, g) \longrightarrow (N, h)$ be a C^∞ map into a Kahler manifold and $q < 0$ be a constant satisfying $2 - q = n$, where g is the standard metric on $\mathbb{C}^n$ and $n \ge 3$. Let $F : [0, \infty) \longrightarrow [0, \infty)$ be a strictly increasing C^2 function such that $F(t) \le 2tF'(t) < n \, F(t)$ for $t \in (0, \infty)$. If u is an F-harmonic map satisfying the above conditions then u is constant, provided u has slowly divergent energy.

Corollary 4.15 :Let $u : (\mathbb{C}^n, g) \longrightarrow (N^k, h)$ be a C^∞ map into a Kahler manifold and $q < 0$ be a constant satisfying $2 - q = n$, where g is the standard metric on $\mathbb{C}^n$ and $n \geq 3$. Let $F : [0, \infty) \longrightarrow [0, \infty)$ be a strictly increasing C^2 function such that

$$F(t) \leq 2tF'(t) < n\, F(t) \ , \quad \text{for } t \in (0, \infty) \ .$$

If u is an F-harmonic map satisfying the above conditions then u is constant, provided u has the following enegy growth

$$\int_{B(R)} F(\tfrac{|du|^2}{2})\, dv_g = o(R^\lambda) \quad \text{as} \quad R \to \infty \ .$$

Theorem 4.18 : For $n \geq 1$, let M^n be a complete simply connected, noncompact Kahler manifold of holomorphic sectional curvature HR^M which satisfies $-a^2 \leq HR^M \leq -b^2$, where a, b are some positive constants. Let N be any Kahler manifold and $F : [0, \infty) \longrightarrow [0, \infty)$ be a strictly increasing C^2 function such that

$$(n-1)b\, F(t) - 2ta\, F'(t) \geq 0 \ \text{ for } t \in (0, \infty) \ .$$

If $u : (M^n, g) \to (N^k, h)$ is an F-harmonic map with following growth condition

$$\int_{B(\rho)} F(\tfrac{|du|^2}{2})\, dv_g = o\,(\rho^\lambda) \quad as \quad \rho \to \infty \ , \quad \text{then u is constant.}$$

Corollary 4.19 : For $n \geq 1$, let M^n be a complete simply connected, noncompact Kahler manifold of holomorphic sectional curvature HR^M which satisfies $-a^2 \leq HR^M \leq -b^2$, where a, b are some positive constants. Let N be any Kahler manifold and $F : [0, \infty) \longrightarrow [0, \infty)$ be a strictly increasing C^2 function such that

$$(n-1)b\, F(t) - 2ta\, F'(t) \geq 0 \ \text{ for } t \in (0, \infty) \ .$$

If $u : (M^n, g) \to (N^k, h)$ is an F-harmonic map with slowly divergent F-energy then u is constant.

Corollary 4.20 : Any F-harmonic map with slowly divergent F-energy from the complex hyperbolic space $\mathbb{C}H^n$ to any Kahler manifold must be constant, provided the condition on the function F as in Theorem 4.22 is satisfied.

1.2 Differentiable maps in complex geometry

Let $f : (M^n, g, J) \to (N^k, h, J')$ be a C$^\infty$ map between almost Hermitian manifolds of dimensions n, k together with Riemannian metrics g, h and almost complex structures J, J', respectively. The complexified differential of f

$$df^C : TM^C \to TN^C$$

determines the partial differentials by compositions with inclusions of $TM^{1,0}$, $TM^{0,1}$ in TM^C and projections of TN^C onto $TN^{1,0}$, $TN^{0,1}$ as follows

$$df^C|TM^{1,0} = \partial f + \partial\bar{f} : TM^{1,0} \to TN^{1,0} \otimes TN^{0,1}$$

$$df^C|TM^{0,1} = \bar{\partial} f + \bar{\partial}\bar{f} : TM^{0,1} \to TN^{1,0} \otimes TN^{0,1}$$

$$df^C = df^C|TM^{1,0} + df^C|TM^{0,1} = \partial f + \partial\bar{f} + \bar{\partial} f + \bar{\partial}\bar{f}$$

Let $\{z^1, ..., z^n\}, \{w^1, ..., w^k\}$ be local complex coordinate systems in M, N, respectively. Then the partial differentials of f are represented in local coordinates by

$$\partial f : TM^{1,0} \to TN^{1,0}, \quad \partial f = \sum_{i,\alpha} f_i^\alpha \, dz^i \otimes \frac{\partial}{\partial w^\alpha} \in \Gamma(T^*M^{1,0} \otimes f^*TN^{1,0})$$

$$\partial\bar{f} : TM^{1,0} \to TN^{0,1}, \quad \partial\bar{f} = \sum_{i,\alpha} f_i^{\bar{\alpha}} \, dz^i \otimes \frac{\partial}{\partial \bar{w}^\alpha} \in \Gamma(T^*M^{1,0} \otimes f^*TN^{0,1})$$

$$\bar{\partial} f : TM^{0,1} \to TN^{1,0}, \quad \bar{\partial} f = \sum_{i,\alpha} f_{\bar{i}}^\alpha \, d\bar{z}^i \otimes \frac{\partial}{\partial w^\alpha} \in \Gamma(T^*M^{0,1} \otimes f^*TN^{1,0})$$

$$\bar{\partial}\bar{f} : TM^{0,1} \to TN^{0,1}, \quad \bar{\partial}\bar{f} = \sum_{i,\alpha} f_{\bar{i}}^{\bar{\alpha}} \, d\bar{z}^i \otimes \frac{\partial}{\partial \bar{w}^\alpha} \in \Gamma(T^*M^{0,1} \otimes f^*TN^{0,1})$$

where $i = 1, ..., n$, $\alpha = 1, ..., k$.

Henceforth, for convenience we denote df^C simply by df .

Denote by $< , >$ the J-invariant real inner product of various tensor bundles of M induced by g. The complex bilinear extension is also denoted by the same $< , >$. Define the Hermitian inner product $<< , >>$ by

$$<< u, v >> = < u, \bar{v} >$$

There always exists a local orthonomal Hermitian frame field on an almost Hermitian manifold. Let $\{e_j, Je_j\}_{j=1,...,n}$ be this local Hermitian frame field on M.

Then with respect to the Hermitian inner product $<<,>>$, we obtain the following corresponding $\pm$ holomorphic orthonormal frame fields

$$\{ \eta_j = \tfrac{1}{\sqrt{2}}(e_j - iJe_j) \}_{j=1,\ldots,n} \quad \text{and} \quad \{ \eta_{\bar{j}} = \tfrac{1}{\sqrt{2}}(e_j + iJe_j) \}_{j=1,\ldots,n}$$

spanning $TM^{1,0}$ and $TM^{0,1}$, respectively, such that

$$<< \eta_j , \eta_k >> = << \eta_{\bar{j}} , \eta_{\bar{k}} >> = \delta_{jk}$$

$$< e_i, e_j > = < Je_i, Je_j > = \delta_{ij}$$

$$< e_i, Je_j > = < Je_i, e_j > = 0 .$$

Similarly, we can choose a local Hermitian frame field $\{\tilde{e}_\alpha, J'\tilde{e}_\alpha\}_{\alpha=1,\ldots,k}$ on N with corresponding $\pm$ holomorphic orthonormal frame fields

$$\{ \tilde{\eta}_\alpha = \tfrac{1}{\sqrt{2}}(\tilde{e}_\alpha - iJ'\tilde{e}_\alpha) \}_{\alpha=1,\ldots,k} \quad \text{and} \quad \{ \tilde{\eta}_{\bar{\alpha}} = \tfrac{1}{\sqrt{2}}(\tilde{e}_\alpha + iJ'\tilde{e}_\alpha) \}_{\alpha=1,\ldots,k}$$

spanning $TN^{1,0}$ and $TN^{0,1}$, respectively.

The complexified differential df has the following local representation

$$df : TM^{1,0} \oplus TM^{0,1} \to TN^{1,0} \oplus TN^{0,1} \quad \text{defined by}$$

$$\begin{aligned} df(\eta_j + \eta_{\bar{j}}) &= df|TM^{1,0}(\eta_j) + df|TM^{0,1}(\eta_{\bar{j}}) \\ &= \tfrac{1}{2} [(df|TM^{1,0}(\eta_j))|TN^{1,0} + (df|TM^{1,0}(\eta_j))|TN^{0,1} \\ &\qquad + (df|TM^{0,1}(\eta_{\bar{j}}))|TN^{1,0} + (df|TM^{0,1}(\eta_{\bar{j}}))|TN^{0,1}] \\ &= \textstyle\sum_\alpha [f_j^\alpha \tilde{\eta}_\alpha + f_j^{\bar{\alpha}} \tilde{\eta}_{\bar{\alpha}} + f_{\bar{j}}^\alpha \tilde{\eta}_\alpha + f_{\bar{j}}^{\bar{\alpha}} \tilde{\eta}_{\bar{\alpha}}] \end{aligned}$$

where

$$df|TM^{1,0} : TM^{1,0} \to TN^{1,0} \oplus TN^{0,1} \quad \text{is given by}$$

$$\begin{aligned} df|TM^{1,0}(\eta_j) &= \tfrac{1}{2} (df|TM^{1,0}(\eta_j))|TN^{1,0} + \tfrac{1}{2} (df|TM^{1,0}(\eta_j))|TN^{0,1} \\ &= \partial f(\eta_j) + \partial \bar{f}(\eta_j) \\ &= \textstyle\sum_\alpha (f_j^\alpha \tilde{\eta}_\alpha + f_j^{\bar{\alpha}} \tilde{\eta}_{\bar{\alpha}}) \end{aligned}$$

and

$$df|TM^{0,1} : TM^{0,1} \to TN^{1,0} \oplus TN^{0,1} \quad \text{is given by}$$

$$df|TM^{0,1}(\eta_{\bar{j}}) = \tfrac{1}{2} (df|TM^{0,1}(\eta_{\bar{j}}))|TN^{1,0} + \tfrac{1}{2} (df|TM^{0,1}(\eta_{\bar{j}}))|TN^{0,1}$$

$$= \bar{\partial} f(\eta_{\bar{j}}) + \bar{\partial} \bar{f}(\eta_{\bar{j}})$$
$$= \sum_\alpha (f_{\bar{j}}^\alpha \tilde{\eta}_\alpha + f_{\bar{j}}^{\bar{\alpha}} \tilde{\eta}_{\bar{\alpha}})$$

Let $\{\theta^j, \theta^{\bar{j}}\}_{j=1,\ldots,n}$, $\{\tilde{\theta}^\alpha, \tilde{\theta}^{\bar{\alpha}}\}_{\alpha=1,\ldots,k}$ be the coframe fields in M, N dual to $\{\eta_j, \eta_{\bar{j}}\}$, $\{\tilde{\eta}_\alpha, \tilde{\eta}_{\bar{\alpha}}\}$. The complexified second fundamental form decomposes as

$$\begin{aligned} \nabla df^C &= (\nabla^{1.0} + \nabla^{0,1})\,(\partial + \bar{\partial})\,(f) \\ &= \nabla^{1,0}\,\partial f + \nabla^{1,0}\,\bar{\partial} f + \nabla^{0,1}\,\partial f + \nabla^{0,1}\,\bar{\partial} f \\ &= \nabla df^{\,2,0} + \nabla df^{\,1,1} + \nabla df^{\,0,2} \end{aligned}$$

where the middle two terms of the second equation are the (1,1)-parts of ∇df^C . The local representation of ∇df^C is given by

$$\begin{aligned} \nabla df^C = \textstyle\sum_{k,j,\alpha} [\ & f_{kj}^\alpha\, \theta^k \otimes \theta^j \otimes \tilde{\eta}_\alpha + f_{kj}^{\bar{\alpha}}\, \theta^k \otimes \theta^j \otimes \tilde{\eta}_{\bar{\alpha}} \\ & + f_{\bar{k}j}^\alpha\, \theta^{\bar{k}} \otimes \theta^j \otimes \tilde{\eta}_\alpha + f_{\bar{k}j}^{\bar{\alpha}}\, \theta^{\bar{k}} \otimes \theta^j \otimes \tilde{\eta}_{\bar{\alpha}} + f_{k\bar{j}}^\alpha\, \theta^k \otimes \theta^{\bar{j}} \otimes \tilde{\eta}_\alpha + f_{k\bar{j}}^{\bar{\alpha}}\, \theta^k \otimes \theta^{\bar{j}} \otimes \tilde{\eta}_{\bar{\alpha}} \\ & + f_{\bar{k}\bar{j}}^\alpha\, \theta^{\bar{k}} \otimes \theta^{\bar{j}} \otimes \tilde{\eta}_\alpha + f_{\bar{k}\bar{j}}^{\bar{\alpha}}\, \theta^{\bar{k}} \otimes \theta^{\bar{j}} \otimes \tilde{\eta}_{\bar{\alpha}}\ ] \end{aligned}$$

where $f_{kj}^\alpha = f_{jk}^\alpha$, $f_{k\bar{j}}^\alpha = f_{\bar{j}k}^\alpha$, $\overline{f_{kj}^\alpha} = f_{\bar{k}\bar{j}}^{\bar{\alpha}}$, $\overline{f_{k\bar{j}}^\alpha} = f_{\bar{k}j}^{\bar{\alpha}}$.

Next, we investigate the invariant form of the partial differentials.

Let $X \in \Gamma(TM^C)$. Then we can obtain $\pm$ holomorphic vector fields

$$Z = X - iJX \in \Gamma(TM^{1,0}) \quad , \quad \bar{Z} = X + iJX \in \Gamma(TM^{0,1})$$
$$i.e. \quad X = X^{1,0} + X^{0,1} = \tfrac{1}{2} Z + \tfrac{1}{2} \bar{Z}$$

and the values of the partial differentials on vector fields can be calculated as follows

$$\begin{aligned} \partial f(Z) &= \tfrac{1}{2}\,(df|TM^{1,0})(Z)\,|TN^{1,0} \\ &= \tfrac{1}{2}\,[df(X) - idf(JX)]\,|TN^{1,0} \\ &= \tfrac{1}{2}\,[df(X) - iJ'df(X) - idf(JX) - iJ'(-idf(JX))] \\ &= \tfrac{1}{2}\,[df(X) - iJ'df(X) - idf(JX) - J'(df(JX))] \end{aligned}$$

$$\partial \bar{f}(Z) = \tfrac{1}{2}\,(df|TM^{1,0})(Z)\,|TN^{0,1}$$

$$\begin{aligned}
&= \tfrac{1}{2}\,[df(X) - idf(JX)]\,|TN^{0,1} \\
&= \tfrac{1}{2}\,[df(X) + iJ'df(X) - idf(JX) + iJ'(-idf(JX))] \\
&= \tfrac{1}{2}\,[df(X) + iJ'df(X) - idf(JX) + J'(df(JX))]
\end{aligned}$$

$$\begin{aligned}
\bar{\partial} f(\bar{Z}) &= \tfrac{1}{2}\,(df|TM^{0,1})(\bar{Z})\,|TN^{1,0} \\
&= \tfrac{1}{2}\,[df(X) + idf(JX)]\,|TN^{1,0} \\
&= \tfrac{1}{2}\,[df(X) - iJ'df(X) + idf(JX) - iJ'(idf(JX))] \\
&= \tfrac{1}{2}\,[df(X) - iJ'df(X) + idf(JX) + J'(df(JX))]
\end{aligned}$$

$$\begin{aligned}
\bar{\partial} \bar{f}(\bar{Z}) &= \tfrac{1}{2}\,(df|TM^{0,1})(\bar{Z})\,|TN^{0,1} \\
&= \tfrac{1}{2}\,[df(X) + idf(JX)]\,|TN^{0,1} \\
&= \tfrac{1}{2}\,[df(X) + iJ'df(X) + idf(JX) + iJ'(idf(JX))] \\
&= \tfrac{1}{2}\,[df(X) + iJ'df(X) + idf(JX) - J'(df(JX))]
\end{aligned}$$

Note that since $X = \frac{1}{2}Z + \frac{1}{2}\bar{Z}$, we get

$$df(X) = \tfrac{1}{2}\, df(\, Z + \bar{Z}) = \tfrac{1}{2}\, [\, \partial f(Z) + \partial \bar{f}(Z) + \bar{\partial} f(\bar{Z}) + \bar{\partial}\bar{f}(\bar{Z})\,]\ .$$

Lemma 1.1. *Let $f : (M^n, g, J) \to (N^k, h, J')$ be a C^∞ map between almost Hermitian manifolds of dimensions n, k together with Riemannian metrics g, h and almost complex structures J, J', respectively. Then we have*

$$\tfrac{1}{2}\,|df|^2 = |\partial f|^2 + |\bar{\partial} f|^2$$

where $df = df^{\mathcal{C}}$ is the complexified differential of a smooth map between almost Hermitian manifolds.

Proof : Choose a local Hermitian orthonormal frame field $\{e_j, Je_j\}_{j=1,\ldots,n}$ in M with corresponding $\pm$ holomorphic orthonormal frame fields

$$\{\ \eta_j = \tfrac{1}{\sqrt{2}}(e_j - iJe_j)\ \}_{j=1,\dots,n} \quad \text{and} \quad \{\ \eta_{\bar{j}} = \tfrac{1}{\sqrt{2}}(e_j + iJe_j)\ \}_{j=1,\dots,n}$$

In local coordinates, the partial energy densities of f are defied as follows

$$\begin{aligned}
|\partial f|^2 &= \textstyle\sum_{j=1}^n \ << \partial f(\eta_j)\ ,\ \partial f(\eta_j) >> \\
&= \textstyle\sum_{j=1}^n \ < \partial f(\eta_j)\ ,\ \overline{\partial f(\eta_j)} > \\
&= \textstyle\sum_{j=1}^n \ < \partial f(\eta_j)\ ,\ \bar{\partial}\bar{f}(\eta_{\bar{j}}) > \\
&= \tfrac{1}{2}\ \textstyle\sum_{j=1}^n \ < \partial f(e_j - iJe_j)\ ,\ \bar{\partial}\bar{f}(e_j + iJe_j) > \\
&= \tfrac{1}{2}\ \textstyle\sum_{j=1}^n \ < df(e_j - iJe_j)\ |TN^{1,0}\ ,\ df(e_j + iJe_j)\ |TN^{0,1} > \\
&= \tfrac{1}{2}\ \textstyle\sum_{j=1}^n \ < \ \tfrac{1}{2}\ [\ df(e_j) - iJ'df(e_j) - idf(Je_j) - iJ'(-idf(Je_j))\ ]\ , \\
&\qquad\qquad \tfrac{1}{2}\ [\ df(e_j) + iJ'df(e_j) + idf(Je_j) + iJ'(idf(Je_j))\ ]\ > \\
&= \tfrac{1}{8}\ \textstyle\sum_{j=1}^n \ < \ df(e_j) - iJ'df(e_j) - idf(Je_j) - J'df(Je_j)\ , \\
&\qquad\qquad df(e_j) + iJ'df(e_j) + idf(Je_j) - J'df(Je_j)\ > \\
\\
&= \tfrac{1}{8}\ \textstyle\sum_{j=1}^n [< df(e_j), df(e_j) > - < iJ'df(e_j), df(e_j) > \\
&\qquad - < idf(Je_j), df(e_j) > - < J'df(Je_j), df(e_j) > \\
\\
&\qquad + < df(e_j), iJ'df(e_j) > - < iJ'df(e_j), iJ'df(e_j) > \\
&\qquad - < idf(Je_j), iJ'df(e_j) > - < J'df(Je_j), iJ'df(e_j) > \\
\\
&\qquad + < df(e_j), idf(Je_j) > - < iJ'df(e_j), idf(Je_j) > \\
&\qquad - < idf(Je_j), idf(Je_j) > - < J'df(Je_j), idf(Je_j) > \\
\\
&\qquad - < df(e_j), J'df(Je_j) > + < iJ'df(e_j), J'df(Je_j) > \\
&\qquad + < idf(Je_j), J'df(Je_j) > + < J'df(Je_j), J'df(Je_j) >] \\
\\
&= \tfrac{1}{4}\ \textstyle\sum_{j=1}^n [< df(e_j), df(e_j) > + < df(Je_j), df(Je_j) > \\
&\qquad\qquad +2 < df(Je_j), J'df(e_j) >]
\end{aligned}$$

$$
\begin{aligned}
|\bar\partial f|^2 = & \sum_{j=1}^{n} \ << \bar\partial f(\eta_{\bar j}) \ , \ \bar\partial f(\eta_{\bar j}) >> \\
= & \sum_{j=1}^{n} \ < \bar\partial f(\eta_{\bar j}) \ , \ \overline{\bar\partial f(\eta_{\bar j})} > \\
= & \sum_{j=1}^{n} \ < \bar\partial f(\eta_{\bar j}) \ , \ \partial \bar f(\eta_j) > \\
= & \ \tfrac{1}{2} \sum_{j=1}^{n} \ < \bar\partial f(e_j + iJe_j) \ , \ \partial \bar f(e_j - iJe_j) > \\
= & \ \tfrac{1}{2} \sum_{j=1}^{n} \ < df(e_j + iJe_j) \ |TN^{1,0} \ , \ df(e_j - iJe_j) \ |TN^{0,1} > \\
= & \ \tfrac{1}{2} \sum_{j=1}^{n} \ < \ \tfrac{1}{2} \ [\ df(e_j) - iJ'df(e_j) + idf(Je_j) - iJ'(idf(Je_j)) \ ] \ , \\
& \qquad \tfrac{1}{2} \ [\ df(e_j) + iJ'df(e_j) - idf(Je_j) + iJ'(-idf(Je_j)) \ ] \ > \\
= & \ \tfrac{1}{8} \sum_{j=1}^{n} \ < \ df(e_j) - iJ'df(e_j) + idf(Je_j) + J'df(Je_j) \ , \\
& \qquad df(e_j) + iJ'df(e_j) - idf(Je_j) + J'df(Je_j) \ >
\end{aligned}
$$

$$
\begin{aligned}
= & \ \tfrac{1}{8} \sum_{j=1}^{n} [< df(e_j), df(e_j) > - < iJ'df(e_j), df(e_j) > \\
& + < idf(Je_j), df(e_j) > + < J'df(Je_j), df(e_j) > \\
& \\
& + < df(e_j), iJ'df(e_j) > - < iJ'df(e_j), iJ'df(e_j) > \\
& + < idf(Je_j), iJ'df(e_j) > + < J'df(Je_j), iJ'df(e_j) > \\
& \\
& - < df(e_j), idf(Je_j) > + < iJ'df(e_j), idf(Je_j) > \\
& - < idf(Je_j), idf(Je_j) > - < J'df(Je_j), idf(Je_j) > \\
& \\
& + < df(e_j), J'df(Je_j) > - < iJ'df(e_j), J'df(Je_j) > \\
& + < idf(Je_j), J'df(Je_j) > + < J'df(Je_j), J'df(Je_j) >]
\end{aligned}
$$

$$
\begin{aligned}
= & \ \tfrac{1}{4} \sum_{j=1}^{n} [< df(e_j), df(e_j) > + < df(Je_j), df(Je_j) > \\
& \qquad -2 < df(Je_j), J'df(e_j) >]
\end{aligned}
$$

$$
\begin{aligned}
|\partial\bar{f}|^2 = & \sum_{j=1}^{n} << \partial\bar{f}(\eta_j) \, , \, \partial\bar{f}(\eta_j) >> \\
= & \sum_{j=1}^{n} < \partial\bar{f}(\eta_j) \, , \, \overline{\partial\bar{f}(\eta_j)} > \\
= & \sum_{j=1}^{n} < \partial\bar{f}(\eta_j) \, , \, \bar{\partial} f(\eta_{\bar{j}}) > \\
= & \frac{1}{2} \sum_{j=1}^{n} < \partial\bar{f}(e_j - iJe_j) \, , \, \bar{\partial} f(e_j + iJe_j) > \\
= & \frac{1}{2} \sum_{j=1}^{n} < df(e_j - iJe_j) \, |TN^{0,1} \, , \, df(e_j + iJe_j) \, |TN^{1,0} > \\
= & \frac{1}{2} \sum_{j=1}^{n} < \frac{1}{2} \left[df(e_j) + iJ'df(e_j) - idf(Je_j) + iJ'(-idf(Je_j)) \right] , \\
& \qquad \frac{1}{2} \left[df(e_j) - iJ'df(e_j) + idf(Je_j) - iJ'(idf(Je_j)) \right] > \\
= & \frac{1}{8} \sum_{j=1}^{n} < df(e_j) + iJ'df(e_j) - idf(Je_j) + J'df(Je_j) \, , \\
& \qquad df(e_j) - iJ'df(e_j) + idf(Je_j) + J'df(Je_j) > \\
\\
= & \frac{1}{8} \sum_{j=1}^{n} [< df(e_j), df(e_j) > + < iJ'df(e_j), df(e_j) > \\
& - < idf(Je_j), df(e_j) > + < J'df(Je_j), df(e_j) > \\
\\
& - < df(e_j), iJ'df(e_j) > - < iJ'df(e_j), iJ'df(e_j) > \\
& + < idf(Je_j), iJ'df(e_j) > - < J'df(Je_j), iJ'df(e_j) > \\
\\
& + < df(e_j), idf(Je_j) > + < iJ'df(e_j), idf(Je_j) > \\
& - < idf(Je_j), idf(Je_j) > + < J'df(Je_j), idf(Je_j) > \\
\\
& + < df(e_j), J'df(Je_j) > + < iJ'df(e_j), J'df(Je_j) > \\
& - < idf(Je_j), J'df(Je_j) > + < J'df(Je_j), J'df(Je_j) >] \\
\\
= & \frac{1}{4} \sum_{j=1}^{n} [< df(e_j), df(e_j) > + < df(Je_j), df(Je_j) > \\
& \qquad -2 < df(Je_j), J'df(e_j) >]
\end{aligned}
$$

$$
\begin{aligned}
|\bar\partial \bar f|^2 = & \sum_{j=1}^n << \bar\partial \bar f(\eta_{\bar j}) \, , \, \bar\partial \bar f(\eta_{\bar j}) >> \\
= & \sum_{j=1}^n < \bar\partial \bar f(\eta_{\bar j}) \, , \, \overline{\bar\partial \bar f(\eta_{\bar j})} > \\
= & \sum_{j=1}^n < \bar\partial \bar f(\eta_{\bar j}) \, , \, \partial f(\eta_j) > \\
= & \frac{1}{2} \sum_{j=1}^n < \bar\partial \bar f(e_j + iJe_j) \, , \, \partial f(e_j - iJe_j) > \\
= & \frac{1}{2} \sum_{j=1}^n < df(e_j + iJe_j) \, |TN^{0,1} \, , \, df(e_j - iJe_j) \, |TN^{1,0} > \\
= & \frac{1}{2} \sum_{j=1}^n < \frac{1}{2} [\, df(e_j) + iJ'df(e_j) + idf(Je_j) + iJ'(idf(Je_j)) \,] \, , \\
& \qquad \frac{1}{2} [\, df(e_j) - iJ'df(e_j) - idf(Je_j) - iJ'(-idf(Je_j)) \,] \; > \\
= & \frac{1}{8} \sum_{j=1}^n < \; df(e_j) + iJ'df(e_j) + idf(Je_j) - J'df(Je_j) \, , \\
& \qquad df(e_j) - iJ'df(e_j) - idf(Je_j) - J'df(Je_j) \; > \\
\\
= & \frac{1}{8} \sum_{j=1}^n [< df(e_j), df(e_j) > + < iJ'df(e_j), df(e_j) > \\
& + < idf(Je_j), df(e_j) > - < J'df(Je_j), df(e_j) > \\
\\
& - < df(e_j), iJ'df(e_j) > - < iJ'df(e_j), iJ'df(e_j) > \\
& - < idf(Je_j), iJ'df(e_j) > + < J'df(Je_j), iJ'df(e_j) > \\
\\
& - < df(e_j), idf(Je_j) > - < iJ'df(e_j), idf(Je_j) > \\
& - < idf(Je_j), idf(Je_j) > + < J'df(Je_j), idf(Je_j) > \\
\\
& - < df(e_j), J'df(Je_j) > - < iJ'df(e_j), J'df(Je_j) > \\
& - < idf(Je_j), J'df(Je_j) > + < J'df(Je_j), J'df(Je_j) >] \\
\\
= & \frac{1}{4} \sum_{j=1}^n [< df(e_j), df(e_j) > + < df(Je_j), df(Je_j) > \\
& \qquad +2 < df(Je_j), J'df(e_j) >] \, .
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
\frac{1}{2} \, |df|^2 \; = & \; \frac{1}{2} \sum_{j=i}^n [\, < df(e_j), df(e_j) > \; + \; < df(Je_j), df(Je_j) >] \\
= & \; |\partial f|^2 \; + \; |\bar\partial f|^2 \, . \qquad \Box
\end{aligned}
$$

1.3 Connections in the space of differential maps

Let $u : (M^n, g, J) \to (N^k, h, J')$ be a C$^\infty$ map between almost Hermitian manifolds of dimensions n, k together with Riemannian metrics g, h and almost complex structures J, J', respectively, i.e. let $u \in C^\infty(M, N)$.

Let ∇^M denote the Levi-Civita connection of M which induces a map

$$\nabla^M : \Gamma(TM) \to \Gamma(T^*M \otimes TM) .$$

The musical isomorphisms $\sharp$ and $\flat$ between TM and T^*M induces a dual connection ∇^* on T^*M as follows : for $X, Y \in \Gamma(TM),\ w \in \Gamma(T^*M)$,

$$\begin{aligned}
(\nabla^*_X w)(Y) &= (\nabla^M_X w^\sharp)^\flat(Y) \\
&= g_x(\nabla^M_X w^\sharp, Y) \\
&= X(g_x(w^\sharp, Y)) - g_x(w^\sharp, \nabla^M_X Y) \\
&= Xw(Y) - w(\nabla^M_X Y) .
\end{aligned}$$

Furthermore, the compatibility of ∇^M with g_{ij} induces the compatibility of ∇^* with the inverse metric g^{ij} on T^*M as follows : for $w, z \in \Gamma(T^*M)$

$$Xg^*(w, z) = g^*(\nabla^*_X w, z) + g^*(w, \nabla^*_X z)$$

Indeed,

$$\begin{aligned}
RHS &= g((\nabla^*_X w)^\sharp, z^\sharp) + g(w^\sharp, (\nabla^*_X z)^\sharp) \\
&= g(\nabla^M_X w^\sharp, z^\sharp) + g(w^\sharp, \nabla^M_X z^\sharp) \\
&= Xw(z^\sharp) = LHS .
\end{aligned}$$

Thus, ∇^* is a Riemannian connection.

Consider the induced vector bundle $u^*TN \to M$. At each point $x \in M$, the basis $\{ \frac{\partial}{\partial y^1}, ..., \frac{\partial}{\partial y^k} \}$ of $T_{u(x)}N$ gives rise to a basis $\{(\frac{\partial}{\partial y^1} \circ u)(x), ..., (\frac{\partial}{\partial y^k} \circ u)(x)\}$ for the fiber $T_{u(x)}N$ of u^*TN over x . Define a connection $\tilde{\nabla}$ in u^*TN induced by the Levi-Civita connection ∇^N on N by

$$(\tilde{\nabla}_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial y^\alpha} \circ u)(x) = \nabla^N_{du_x(\frac{\partial}{\partial x^i})} \frac{\partial}{\partial y^\alpha} .$$

Equivalently, in the invariant form

$$\tilde{\nabla}_X W = \nabla^N_{du(X)} W , \quad where \ X \in \Gamma(TM) , \ W \in \Gamma(u^* TN) .$$

If $h \in \Gamma(TN \otimes TN)$ is a metric in TN , then

$$(u^* h)_x = h_{u(x)}$$

defines a fiber metric in $u^* TN$.

The connections ∇^* in $T^* M$ and $\tilde{\nabla}$ in $u^* TN$ induce a connection ∇ on $T^* M \otimes u^* TN$ as follows :

$$\nabla_X (w \otimes W) = (\nabla^*_X w) \otimes W + w \otimes (\tilde{\nabla}_X W)$$

where $w \in T^* M$, $W \in u^* TN$, $X \in TM$. Thus, the differential of the map $u \in C^\infty(M, N)$ defines the C^∞ section $du \in \Gamma(T^* M \otimes u^* TN)$ and the covariant differential of du , $\nabla du \in \Gamma(T^* M \otimes T^* M \otimes u^* TN)$, defines a 2-form with value in the induced bundle, called *the second fundamental form* of the map u .

Lemma 1.2. *Let* $u \in C^\infty(M, N)$ *and* $X, Y \in \Gamma(TM)$. *Then,*

$$\nabla du(X, Y) = \tilde{\nabla}_X du(Y) - du(\nabla^M_X Y) .$$

Proof : For $w \in \Gamma(T^* M)$, $W \in \Gamma(u^* TN)$,

$$\begin{aligned}
&\tilde{\nabla}_X [(w \otimes W)(Y)] - (w \otimes W)(\nabla^M_X Y) \\
&\quad = \tilde{\nabla}_X [w(Y) \otimes W] - w(\nabla^M_X Y) \otimes W \\
&\quad = X w(Y) \otimes W + w(Y) \otimes \tilde{\nabla}_X W - w(\nabla^M_X Y) \otimes W \\
&\quad = (\nabla^*_X w)(Y) \otimes W + w(\nabla^M_X Y) \otimes W + w(Y) \otimes \tilde{\nabla}_X W - w(\nabla^M_X Y) \otimes W \\
&\quad = [(\nabla^*_X w) \otimes W + w \otimes (\tilde{\nabla}_X W)](Y) \\
&\quad = (\nabla_X (w \otimes W))(Y)
\end{aligned}$$

$$= \nabla(w \otimes W)(X, Y)$$

Since du is a special case of an arbitrary 1-form with value in the induced bundle u^*TN, the lemma follows at once . $\square$

Chapter 2

F-Harmonic maps

The true sign of intelligence is not knowledge but imagination.

A. Einstein

2.1 Ω-Harmonic maps

Let $u : (M^n, g) \to (N^k, h)$ be a C^∞ map between Riemannian manifolds of dimensions n, k and with Riemannian metrics g, h, respectively. We follow the notations in [8]. Let

$$\Omega : M \times N \times \mathbb{R} \to (0, \infty)$$
$$(x, y, t) \longmapsto \Omega(x, y, t)$$

be a positive function. For any compact domain D of M, the Ω- energy functional of u is defined by

$$E_\Omega(u; D) = \int_D \Omega(x, u(x), e(u)(x)) \, dv_g$$

where dv_g is the volume element and $e(u)$ is the energy density of u defined by

$$e(u) = \sum_{i=1}^n \frac{1}{2} h(du(e_i), du(e_i)) = \frac{1}{2} |du|^2 ,$$

where $|du|$ is the Hilbert-Schmidt norm of the differential du .

Here $\{e_i\}_{i=1}^n$ is an orthonormal frame field on M. A C^∞ map u is called Ω-harmonic if it is a critical point of the Ω-energy over any compact subset $D \subset M$.

The First Variation of the $\Omega -$ energy functional.

Denote $\partial_t = \frac{\partial}{\partial t}$, $\Omega' = \partial_t(\Omega)$, $\Omega'' = \partial_t(\partial_t(\Omega))$ and define

$$\Omega_u(x) = \Omega(x, u(x), e(u)(x))$$

$$\Omega'_u(x) = \Omega'(x, u(x), e(u)(x)) = \frac{\partial}{\partial t}\Omega(x, u(x), e(u)(x))$$

$$\Omega''_u(x) = \Omega''(x, u(x), e(u)(x)) = \frac{\partial^2}{\partial t^2}\Omega(x, u(x), e(u)(x))$$

Let $\{ u_t \}_{t \in (-\epsilon, \epsilon)}$ be a C$^\infty$ variation of u supported in D and denote the variation vector field of u by $V = \frac{\partial u_t}{\partial t}|_{t=0} = du_t(\frac{\partial}{\partial t})|_{t=0}$. Define

$$\phi : M \times (-\epsilon, \epsilon) \to N \text{ by}$$

$$\phi(x,t) = u_t(x) , \text{ where } u_o(x) = u(x) .$$

Let ∇^ϕ , $\tilde{\nabla}$ be the induced connections on ϕ^*TN and u^*TN . Then for any vector field X on M , considered as a vector field on M$\times(-\epsilon, \epsilon)$, we have

$$[\frac{\partial}{\partial t} , X] = 0 .$$

Let $x \in M$. Choose a local orthonornal frame field $\{e_i\}_{i=1,...,n}$ which is normal at x, i.e.

$$\nabla_{e_i} e_j|_x = 0 \quad \forall i, j = 1, ..., n.$$

Then , at x we have

$$\begin{aligned}
&\frac{d}{dt} E_\Omega (u_t; D)|_{t=0} \\
&\quad = \int_D \frac{\partial}{\partial t} \Omega(x, u_t(x), e(u_t)(x)) |_{t=0} \ dv_g \\
&\quad = \int_D [d\Omega (d\phi(\frac{\partial}{\partial t})) + d\Omega(\frac{\partial}{\partial t}(e(u_t)))] |_{t=0} \ dv_g \\
&\quad = \int_D [< (grad^N\Omega) \circ u , V > + \sum_{i=1}^n \Omega'_u < \tilde{\nabla}_{e_i} V , du(e_i) >] \ dv_g \\
&\quad = \int_D [< (grad^N\Omega) \circ u , V > + \sum e_i < V , \Omega'_u du(e_i) > \\
&\qquad - \sum < V , \tilde{\nabla}_{e_i}(\Omega'_u du(e_i)) >] dv_g
\end{aligned}$$

$$= \int_D \, [< \, (grad^N \Omega) \, \circ \, u \, , \, V \, > + \sum \, div \, (< \, V \, , \, \Omega'_u \, du(e_i) > \, e_i)$$
$$- \sum \, < \, V \, , \, e_i(\Omega'_u) \, du(e_i) > - \sum \, < \, V \, , \, \Omega'_u \, \tilde{\nabla}_{e_i} du(e_i) \, >] \, \, dv_g$$
$$= \int_D \, < \, (grad^N \Omega) \circ u \, , \, V \, > dv_g + \int_D \sum \, div \, (< \, V \, , \, \Omega'_u \, du(e_i) > \, e_i) dv_g$$
$$- \int_D \, < \, V \, , \, du(grad^M \, \Omega'_u) > dv_g \, - \, \int_D \, < \, V \, , \, \Omega'_u \, \tau(u) > dv_g$$
$$= \int_D < V \, , \, (grad^N \, \Omega) \, \circ \, u \, - \, du(grad^M \, \Omega'_u) \, - \, \Omega'_u \, \tau(u) > dv_g$$
$$= \, - \, \int_D \, < \, \tau_\Omega \, (u) \, , \, V \, > \quad dv_g$$

where in the second equality we have used the following fact

$$\begin{aligned} \frac{\partial}{\partial t}(e(u_t))|_{t=0} &= < \tilde{\nabla}_{\frac{\partial}{\partial t}} du_t(e_i), du_t(e_i) > |_{t=0} \\ &= < \tilde{\nabla}_{e_i} du_t(\tfrac{\partial}{\partial t}), du_t(e_i) > |_{t=0} \\ &= < \tilde{\nabla}_{e_i} V, du(e_i) > \end{aligned}$$

and in the last equality, the Ω - tension field is given by

$$\tau_\Omega(u) \; = \; - \, (grad^N \, \Omega \,) \, \circ \, u \, + \, du(\, grad^M \, \Omega'_u \,) \, + \, \Omega'_u \, \tau(u) \, ,$$

where $\tau(u)$ is the tension field of u given by

$$\begin{aligned} \tau(u) &= \; trace \, \nabla du \\ &= \; \textstyle\sum_{i=1}^n \; \nabla du(e_i, e_i) \\ &= \; \textstyle\sum_{i=1}^n \, [\, \tilde{\nabla}_{e_i} du(e_i) \, - \, du(\nabla^M_{e_i} e_i) \,] \, . \qquad \Box \end{aligned}$$

The Second Variation of the Ω – energy functional.

Let u : $(M^n, g) \, \to \, (N^k, h)$ be an Ω-harmonic map between Riemannian manifolds and $\{ \, u_{t,s} \, \}_{t,s \, \in \, (-\epsilon,\epsilon)}$ be a 2-parameter variation with compact support in D. Set

$$V \; = \; \frac{\partial u_{t,s}}{\partial t} \, |_{s,t=0} \qquad , \qquad W \; = \; \frac{\partial u_{t,s}}{\partial s} \, |_{s,t=0}$$

Define $\phi \; : \; M \times (- \, \epsilon, \epsilon) \times (- \, \epsilon, \epsilon) \; \longrightarrow \; N$ by

$$\phi(x,t,s) \; = \; u_{t,s}(x) \;\; , \;\; u_{o,o}(x) \; = \; u(x) \, .$$

For any vector field X on M, considered as a vector field on $M \times (-\epsilon, \epsilon) \times (-\epsilon, \epsilon)$ we have

$$[\,\partial_t\,,\,X\,]\;=\;[\,\partial_s\,,\,X\,]\;=\;[\,\partial_t\,,\,\partial_s\,]\;=\;0\,.$$

Let $x \in \ M$. Choose a local orthonornal frame field $\{e_i\}_{i=1,...,n}$ which is normal at x, i.e.

$$\nabla_{e_i} e_j|_x \;=\; 0 \quad \forall i, j = 1, ..., n.$$

Then , at x we have

$$\begin{aligned}
&\tfrac{\partial^2}{\partial s \partial t}\, E_\Omega\,(u_{t,s}\,;\,D)\,|_{s,t=0}\\
&\quad=\;\int_D\,\tfrac{\partial}{\partial s}\,[\,\tfrac{\partial}{\partial t}\,\Omega(x, u_{t,s}(x), e(u_{t,s})(x))\,]\,|_{s,t=0}\quad dv_g\\
&\quad=\;\int_D\,\tfrac{\partial}{\partial s}\,[\,d\Omega\,(d\phi(\partial_t))\;+\;d\Omega\,(\partial_t(e(u_{t,s})))\,]\,|_{s,t=0}\quad dv_g\\
&\quad=\int_D \tfrac{\partial}{\partial s}[<(grad^N\Omega)\circ u, d\phi(\partial_t)> + \textstyle\sum \Omega'_{u_{t,s}} < \nabla^\phi_{\partial_t} d\phi(e_i), d\phi(e_i)>]|_{s,t=0} dv_g\\
&\quad=\;\int_D\,[<\;\nabla^\phi_{\partial_s}(grad^N\,\Omega)\;\circ\;u\;,\;\;d\phi\,(\partial_t)\;>\\
&\qquad\qquad+\;<\;(grad^N\,\Omega)\;\circ\;u\;,\;\nabla^\phi_{\partial_s}\,d\phi\,(\partial_t)\;>\\
&\qquad\qquad+\;\textstyle\sum\;<\;\nabla^\phi_{\partial_s}\nabla^\phi_{\partial_t} d\phi(e_i)\;,\;d\phi(e_i)\;>\;\Omega'_{u_{t,s}}\\
&\qquad\qquad+\;\textstyle\sum\;<\;\nabla^\phi_{\partial_t}\,d\phi(e_i)\;,\;\nabla^\phi_{\partial_s} d\phi(e_i)\;>\;\Omega'_{u_{t,s}}\\
&\qquad\qquad+\;\textstyle\sum\;<\;\nabla^\phi_{\partial_t}\,d\phi(e_i)\;,\;d\phi(e_i)\;>\;\partial_s\,(\Omega'_{u_{t,s}})\,]\,|_{s,t=0}\quad dv_g
\end{aligned}$$

The following calculations of each term in the above integral are straightforward.

In the first term of the integral, since $[V, W] \;=\; 0$, we have

$$\begin{aligned}
&< V, \nabla_W\; grad\,\Omega > = \;\nabla_W < V, grad\,\Omega > \;-\; < \nabla_W V, grad\,\Omega >\\
&\qquad=\;\nabla_W\; d\Omega(V) - \;< \nabla_W V, grad\,\Omega >\\
&\qquad=\;\nabla_V\; d\Omega(W)\;+\;d\Omega([W, V]) - \;< \nabla_W V, grad\,\Omega >\\
&\qquad=\;\nabla_V < W, grad\,\Omega > \;-\; < \nabla_V W, grad\,\Omega >\\
&\qquad= < W, \nabla_V\; grad\,\Omega > . \quad \text{Thus}
\end{aligned}$$

$$< d\phi(\tfrac{\partial}{\partial t})\,,\;\nabla^\phi_{\partial_s}(grad^N\Omega)\;\circ\;u >\;|_{s,t=0}\;= < W\;,\;(\nabla^N_V grad^N\,\Omega)\;\circ\;u\;>$$

In the third term, the definition and properties of the curvature tensor yield

$$\begin{aligned}
&< \nabla^\phi_{\partial_s}\nabla^\phi_{\partial_t} d\phi(e_i)\;,\;d\phi(e_i)\;>\;\Omega'_{u_{t,s}}\;|_{s,t=0}\\
&\qquad= < \nabla^\phi_{\partial_s}\nabla^\phi_{e_i} d\phi(\partial_t)\;,\;d\phi(e_i)\;>\;\Omega'_{u_{t,s}}\;|_{s,t=0}\\
&\qquad=\;\Omega'_{u_{t,s}}\;< R^N(d\phi(\partial_s), d\phi(e_i))d\phi(\partial_t)\;,\;d\phi(e_i) > |_{s,t=0}
\end{aligned}$$

$$+\ \Omega'_{u_{t,s}}\ <\nabla^{\phi}_{e_i}\nabla^{\phi}_{\partial_s}\ d\phi(\partial_t)\ ,\ d\phi(e_i)>\ |_{s,t=0}$$

$$=\ \Omega'\ <R^N(W,d\phi(e_i))V\ ,\ d\phi(e_i)>$$

$$+\ \Omega'_{u_{t,s}}\ <\nabla^{\phi}_{e_i}\nabla^{\phi}_{\partial_s}\ d\phi(\partial_t)\ ,\ d\phi(e_i)>\ |_{s,t=0}$$

$$=\ -\,\Omega'\ <R^N(V,d\phi(e_i))d\phi(e_i),W>$$

$$+\,\Omega'_{u_{t,s}}\ <\nabla^{\phi}_{e_i}\nabla^{\phi}_{\partial_s}\ d\phi(\partial_t)\ ,\ d\phi(e_i)>\ |_{s,t=0}$$

$$=\ -\,\Omega'\ <R^N(V,d\phi(e_i))d\phi(e_i),W>$$

$$+\ e_i\ <\ \nabla^{\phi}_{\partial_s}\ d\phi(\partial_t)\ ,\ \Omega'_{u_{t,s}}d\phi(e_i)\ >\ |_{s,t=0}$$

$$-\ \ <\ \nabla^{\phi}_{\partial_s}\ d\phi(\partial_t)\ ,\ \nabla^{\phi}_{e_i}(\Omega'_{u_{t,s}}d\phi(e_i))\ >\ |_{s,t=0}$$

$$=\ -\,\Omega'\ <R^N(V,d\phi(e_i))d\phi(e_i),W>$$

$$+\ e_i\ <\ \nabla^{\phi}_{\partial_s}\ d\phi(\partial_t)\ ,\ \Omega'_{u_{t,s}}d\phi(e_i)\ >\ |_{s,t=0}$$

$$-\ \ <\ \nabla^{\phi}_{\partial_s}\ d\phi(\partial_t)\ ,\ \nabla^{\phi}_{e_i}(\Omega'_{u_{t,s}})d\phi(e_i)\ >\ |_{s,t=0}$$

$$-\ \ <\ \nabla^{\phi}_{\partial_s}\ d\phi(\partial_t)\ ,\ \Omega'_{u_{t,s}}\ \nabla^{\phi}_{e_i}d\phi(e_i)\ >\ |_{s,t=0}$$

$$=\ -\,\Omega'\ <R^N(V,d\phi(e_i))d\phi(e_i),W>$$

$$+\ div(<\ \nabla^{\phi}_{\partial_s}\ d\phi(\partial_t)\ ,\ \Omega'_{u_{t,s}}d\phi(e_i)\ >\ e_i)\ |_{s,t=0}$$

$$-\ \ <\ \nabla^{\phi}_{\partial_s}\ d\phi(\partial_t)\ ,\ d\phi\ (grad^M\ \Omega'_{u_{t,s}})\ >\ |_{s,t=0}$$

$$-\ \ <\ \nabla^{\phi}_{\partial_s}\ d\phi(\partial_t)\ ,\ \Omega'_{u_{t,s}}\ \tau(\phi)\ >\ |_{s,t=0}$$

In the fourth term, since

$$e_i\ <\Omega'\,\tilde{\nabla}_{e_i}d\phi(\partial_t)\ ,\ d\phi(\partial_s)\ >\ -\ <\ \tilde{\nabla}_{e_i}(\Omega'\,\tilde{\nabla}_{e_i}d\phi(\partial_t))\ ,\ d\phi(\partial_s)\ >$$

$$=<\ \Omega'\,\tilde{\nabla}_{e_i}d\phi(\partial_t)\ ,\ \tilde{\nabla}_{e_i}d\phi(\partial_s)\ >\ ,\quad \text{we get}$$

$$<\ \nabla^{\phi}_{\partial_t}d\phi(e_i)\ ,\ \nabla^{\phi}_{\partial_s}d\phi(e_i)\ >\ \Omega'$$

$$=<\ \Omega'\,\tilde{\nabla}_{e_i}d\phi(\partial_t)\ ,\ \tilde{\nabla}_{e_i}d\phi(\partial_s)\ >$$

$$=\ div(<\ \Omega'\,\tilde{\nabla}_{e_i}d\phi(\partial_t)\ ,\ d\phi(\partial_s)\ >\ e_i)-\ <\ \tilde{\nabla}_{e_i}(\Omega'\,\tilde{\nabla}_{e_i}d\phi(\partial_t))\ ,\ d\phi(\partial_s)\ >$$

In the fifth term, from the following equations

$$d\Omega'\ (\ d\phi(\partial_s)\ )\,|_{s,t=0}\ =<W,(grad^N\ \Omega')\ \circ\ u\ >,$$

$$d\Omega'\ (\ \partial_s(e(u_{t,s}))\ )\,|_{s,t=0}\ =\ \Omega''_u\ <\ \tilde{\nabla}_{e_i}W\ ,\ du(e_i)\ >,$$

$$d\Omega'\ (\ d\phi(\partial_s)\ )\ +\ d\Omega'\ (\ \partial_s(e(u_{t,s}))\ )\ =\ \partial_s\ (\ \Omega'(x,u_{t,s}(x),e(u_{t,s})(x))\ )$$

$$= \partial_s \, (\, \Omega'_{u_{t,s}} \,) \ ,$$

we obtain the following

$$< \ \nabla^{\phi}_{\partial_t} d\phi(e_i) \ , \ d\phi(e_i) \ > \ \partial_s \ (\Omega'_{u_{t,s}}) \ |_{s,t=0}$$
$$= < \tilde{\nabla} V, du >< W, (grad^N \, \Omega') \circ u > \ + \ < \tilde{\nabla} V, du > \ \Omega''_u < \tilde{\nabla}_{e_i} W, du(e_i) >$$
$$= < W, < \tilde{\nabla} V, du > (grad^N \, \Omega') \ \circ \ u >$$
$$+ \ e_i(< W, < \tilde{\nabla} V, du > \ \Omega''_u \, du(e_i) >)$$
$$- \ < W, \tilde{\nabla}_{e_i} [< \tilde{\nabla} V, du > \ \Omega''_u \, du(e_i) \,] \ >$$
$$= < W, < \tilde{\nabla} V, du > (grad^N \, \Omega') \ \circ \ u >$$
$$+ \ div(< W, < \tilde{\nabla} V, du > \ \Omega''_u \, du(e_i) > \ e_i)$$
$$- \ < W, \tilde{\nabla}_{e_i} [< \tilde{\nabla} V, du > \ \Omega''_u \, du(e_i) \,] \ >$$

Note that the second term in the integral combines with the last two negative terms in the third term of the integral to give the Ω-tension field with the negative sign which vanishes for an Ω-harmonic map .

By the divergence theorem, all the integrals involved with divergence vanish and we finally obtain

$$\frac{\partial^2}{\partial s \partial t} \, E_\Omega \, (u_{t,s} \ ; \ D) \ |_{s,t=0}$$
$$= \ \int_D \ [< (\nabla^N_V grad^N \, \Omega) \ \circ \ u \ , \ W >$$
$$- \ \Omega' \ \sum \ < R^N(V, d\phi(e_i)) \, d\phi(e_i) \ , \ W >$$
$$- \ \sum \ < \ \tilde{\nabla}_{e_i}(\Omega' \, \tilde{\nabla}_{e_i} V) \ , \ W \ >$$
$$+ \ << \tilde{\nabla} V, du > (grad^N \, \Omega') \ \circ \ u \ , \ W >$$
$$- \ \sum \ < \tilde{\nabla}_{e_i} \ [< \tilde{\nabla} V, du > \ \Omega''_u \, du(e_i)] \ , \ W \ >] \ dv_g$$
$$= \quad \int_D \ < \ J_{\Omega,u} \, (V), W \ > \ dv_g \ \ ,$$

where the Ω-Jacobi operator $J_{\Omega,u} \, (V) \ \in \ \Gamma(u^* TN \,)$ is given by

$$J_{\Omega,u} \, (V) \ = \ - \, \Omega'_u \, tr \, R^N(V, du) du \ - \ tr \, \tilde{\nabla} \, [\Omega'_u \, \tilde{\nabla} V] \ + \ (\ \nabla^N_V \ grad^N \, \Omega \ ) \circ u$$
$$+ \ < \tilde{\nabla} V, du > \ (\ grad^N \, \Omega' \ ) \circ u \ - \ tr \, \tilde{\nabla} \, [< \tilde{\nabla} V, du > \ \Omega''_u \, du] \ . \quad \square$$

The Stress – Energy Tensor of an Ω – harmonic map.

Let $u : (M^n, g) \to (N^k, h)$ be a C^∞ map between Riemannian manifolds of dimensions n, k and with Riemannian metrics g, h, respectively. Let

$$\Omega : M \times N \times \mathbb{R} \to (0, \infty)$$

$$(x, y, t) \longmapsto \Omega(x, y, t)$$

be a positive function. Then

$$\frac{d}{dt} E_\Omega (u; D)|_{t=0}$$
$$= \int_D \frac{\partial}{\partial t} (\Omega(x, u(x), e(u)(x)) \,|_{t=0}\, dv_g + \int_D \Omega(x, u(x), e(u)(x)) \frac{\partial}{\partial t}(dv_{g_t}) \,|_{t=0}$$
$$= \int_D \frac{\partial}{\partial t}(e(u)) \, \Omega'_u \, dv_g \; + \; \int_D \Omega_u \frac{\partial}{\partial t}(dv_{g_t}) \, ,$$

where

$$\frac{\partial}{\partial t}(e(u)) \; = \; -\frac{1}{2} < u^*h, \frac{\partial}{\partial t}\, g >_{\otimes^2 T^*M} \quad , \quad \frac{\partial}{\partial t}(dv_{g_t}) \; = \; \frac{1}{2} < g, \frac{\partial}{\partial t}\, g >_{\otimes^2 T^*M} \; dv_g$$

Thus , we obtain the following lemma.

Lemma 2.1. $\frac{d}{dt} E_\Omega (u; D)|_{t=0} \; = \; \frac{1}{2} \int_D < \Omega_u \, g \; - \; \Omega'_u \, u^*h \, , \, \frac{\partial}{\partial t}\, g \; > \; dv_g \, .$

Definition 2.2. The stress energy tensor of an Ω-harmonic map $u : M \to N$ is given by

$$S_\Omega (u) \; = \; \Omega_u \, g \; - \; \Omega'_u \, u^*h \, .$$

2.2 F-harmonic Maps

Let $u : (M^n, g) \to (N^k, h)$ be a C^∞ map betwenn Riemannian manifolds of dimensions n, k and with Riemannian metrics g, h. Let $F : [0, \infty) \to [0, \infty)$ be a strictly increasing C^2 function with $F(0) = 0$. To study F-harmonic maps we set

$$\Omega_u\ (x, u(x), e(u)(x)) \ = \ F(e(u)(x)) \ = \ F(\ \tfrac{1}{2}\ |du_x|^2\ )\ .$$

Then u is an F-harmonic map if for every compact subset $D \subset M$, u is a critical point of the F-energy functional

$$E_F(u) \ = \ \textstyle\int_D \ F(\ \tfrac{1}{2}|du|^2\ )\ dv_g\ ,$$

where $|du|$ is the Hilbert-Schmidt norm of the differential du . This is equivalent to saying that iff for any compactly supported variation $u_t : M \to N, -\epsilon < t < \epsilon$, with $u_o = u$, the following equation holds

$$\tfrac{\partial}{\partial t}\ E_F(u_t) \ = \ 0\ .$$

Let ∇^M , ∇^N be the Levi-Civita connections of M , N and $\tilde{\nabla}$ be the induced connection on u^*TN defined by

$$\tilde{\nabla}_X W \ = \ \nabla^N_{du(X)} W\ , \quad where\ \ X \in \Gamma(TM)\ ,\ W \in \Gamma(u^*TN)\ .$$

Choose a local orthonormal frame field $\{e_i\}$ on M. If we set $\nabla = \nabla^{T^*M\ \otimes\ u^*TN}$, then the F-tension field is given by

$$\begin{aligned}
\tau_F(u) \ &= \ d^*(F'(\tfrac{|du|^2}{2})du) \\
&= \ trace\ \nabla\ (F'(\tfrac{|du|^2}{2})\ du) \\
&= \ \textstyle\sum_{i=1}^n\ (\nabla\ (F'(\tfrac{|du|^2}{2})du))\ (e_i, e_i) \\
&= \ \textstyle\sum_{i=1}^n\ (\nabla_{e_i}\ (F'(\tfrac{|du|^2}{2})du))\ (e_i) \\
&= \ \textstyle\sum_{i=1}^n\ \tilde{\nabla}_{e_i}\ (F'(\tfrac{|du|^2}{2})du)(e_i) \ - \ \sum_{i=1}^n\ (F'(\tfrac{|du|^2}{2})du)(\nabla^M_{e_i} e_i) \\
&= \ \textstyle\sum_{i=1}^n\ (\tilde{\nabla}_{e_i} F'(\tfrac{|du|^2}{2}))du(e_i) \ + \ \sum_{i=1}^n\ F'(\tfrac{|du|^2}{2})\ \tilde{\nabla}_{e_i} du(e_i) \\
&\qquad\qquad - \ \textstyle\sum_{i=1}^n\ F'(\tfrac{|du|^2}{2})\ du(\nabla^M_{e_i} e_i) \\
&= \ \textstyle\sum_{i=1}^n\ < grad^M F'(\tfrac{|du|^2}{2}), e_i > du(e_i) \ + \ F'(\tfrac{|du|^2}{2})\ \tau(u)
\end{aligned}$$

$$= du(\ grad^M\ F'(\tfrac{|du|^2}{2}))\ +\ F'(\tfrac{|du|^2}{2})\,\tau(u)$$

where $\tau(u)\ =\ trace\,\nabla du\ =\ \sum\,[\ \tilde{\nabla}_{e_i}du(e_i)\ -\ du(\ \nabla^M_{e_i}e_i)\,]\,.$

Proposition 2.3. *(cf. [1]) The First Variation formula for the F-energy functional is*

given by

$$\tfrac{d}{dt}\,E_F(u_t\ ;\ D)\,|_{t=0}\ =\ -\ \int_D\ <\ V\ ,\ \tau_F(u)\ >\ dv_g\,,$$

where $V = \frac{\partial u_t}{\partial t}|_{t=0}$.

Proof : Let $\Omega\,(x,y,t)\ =\ F(t)$,

$$\Omega_u\,(x,u(x),e(u)(x))\ =\ F(e(u)(x))\ =\ F(\tfrac{|du|^2}{2})\ \text{and}\ \ \Omega'_u\ =\ F'(\tfrac{|du|^2}{2})\ .$$

Then from the last section,

$$\begin{aligned}
\tfrac{d}{dt}\,E_\Omega\,(u_t;D)|_{t=0}\ &=\ \int_D\ \tfrac{\partial}{\partial t}\,\Omega(x,u_t(x),e(u_t)(x)\,)\,|_{t=0}\ \ dv_g\\
&=\ \int_D\ [\ d\Omega\ (\ d\phi(\tfrac{\partial}{\partial t})\ +\ d\Omega(\ \tfrac{\partial}{\partial t}(e(u_t))\ )\,]\,|_{t=0}\ \ dv_g\\
&=\ \int_D\ [<\ (grad^N\Omega)\ \circ\ u\ ,\ V\ >\ +\ \textstyle\sum_{i=1}^n\ \Omega'_u\ <\tilde{\nabla}_{e_i}V\ ,\ du(e_i)>]\ \ dv_g\\
&=\ \int_D\ [\textstyle\sum\ e_i\ <V\ ,\ \Omega'_u\,du(e_i)>\ -\ \sum<V\ ,\ \tilde{\nabla}_{e_i}(\Omega'_u du(e_i))\ >]\ dv_g\\
&=\ \int_D\ [\textstyle\sum\ div\,(<\ V\ ,\ \Omega'_u\,du(e_i)>\ e_i)\ -\ \sum\ <\ V\ ,\ e_i(\Omega'_u)\,du(e_i)>\\
&\qquad\qquad -\ \textstyle\sum\ <\ V\ ,\ \Omega'_u\,\tilde{\nabla}_{e_i}du(e_i)\ >]\ \ dv_g\\
&=\ \int_D\textstyle\sum\ div\,(<\ V\ ,\ \Omega'_u\,du(e_i)>\ e_i)\,dv_g - \int_D\ <\ V\ ,\ du(grad^M\,\Omega'_u)>\ dv_g\\
&\qquad\qquad -\ \int_D\ <\ V\ ,\ \Omega'_u\,\tau(u)>\ dv_g\\
&=\ -\ \int_D\ <\ V\ ,\ du(grad^M\,\Omega'_u)\ +\ \Omega'_u\,\tau(u)>\ \ dv_g\\
&=\ -\ \int_D\ <V\ ,\ \tau_F(u)>\ dv_g\,.\quad\Box
\end{aligned}$$

Proposition 2.4. *(cf. [1]) The Second Variation formula for F-harmonic maps is given by*

$$\frac{\partial^2}{\partial s\partial t}\, E_F\,(u_{t,s}\,;\,D)\,|_{s,t=0}$$
$$= \int_D F''(\tfrac{|du|^2}{2})\, <\, \tilde\nabla V\,,\, du\, ><\, \tilde\nabla W\,,\, du\, >\, dv_g$$
$$+ \int_D F'(\tfrac{|du|^2}{2})\,[\,<\, \tilde\nabla V, \tilde\nabla W\, >\, -\, \sum_{i=1}^n\, h(\,R^N(V, du(e_i))du(e_i), W\,)\,]\, dv_g$$
$$= \int_D < J_{F,u}(V), W >\, dv_g\,,$$

where $<\,,\,>$ *denotes the inner product on* $T^*M\,\otimes\,u^*TN$ *, and the variational vector fields are*

$$V\,=\,\frac{\partial u_{t,s}}{\partial t}\,|_{s,t=0}\qquad,\qquad W\,=\,\frac{\partial u_{t,s}}{\partial s}\,|_{s,t=0}\,.$$

In particular, the F-Jacobi operator is given by

$$J_{F,u}\,(V)\,=\,-\,F'(\tfrac{|du|^2}{2})\,tr\,R^N(V,du)du\,-\,tr\,\tilde\nabla\,[F'(\tfrac{|du|^2}{2})\,\tilde\nabla V]$$
$$-\,tr\,\tilde\nabla\,[<\tilde\nabla V, du > F''(\tfrac{|du|^2}{2})du]\,.$$

Proof : Let $\Omega\,(x,y,t)\,=\,F(t)$. In particular, let

$\Omega_u\,(x,u(x),e(u)(x))\,=\,F(e(u)(x))\,=\,F(\frac{|du|^2}{2})$, then

$\Omega'_u\,=\,F'(\frac{|du|^2}{2})$ and $\Omega''_u\,=\,F''(\frac{|du|^2}{2})$.

We calculate

$$-\,<\,tr\,\tilde\nabla\,[\Omega'_u\,\tilde\nabla\,V]\,,\,W\,>$$
$$=\,-\sum\,<\,\tilde\nabla_{e_i}[\Omega'_u\,\tilde\nabla_{e_i}V]\,,\,W\,>$$
$$=\,-\sum\,e_i\,<\,\Omega'_u\,\tilde\nabla_{e_i}V\,,\,W\,>\,+\sum\,<\,\Omega'_u\,\tilde\nabla_{e_i}V\,,\,\tilde\nabla_{e_i}W\,>$$
$$=\,-\sum\,div\,(\,<\,\Omega'_u\,\tilde\nabla_{e_i}V\,,\,W\,>\,e_i\,)\,+\,\Omega'_u\,<\,\tilde\nabla V\,,\,\tilde\nabla W\,>$$

and

$$-\,<\,tr\,\tilde\nabla\,[<\,\tilde\nabla\,V, du\,>\,\Omega''_u\,du]\,,\,W\,>$$
$$=\,-\sum\,<\,\tilde\nabla_{e_i}\,(\,<\,\tilde\nabla V\,,\,du >\,\Omega''_u\,du(e_i))\,,\,W\,>$$
$$=\,-\sum\,e_i << \tilde\nabla V\,,\,du >\,\Omega''_u\,du(e_i)\,,\,W >$$
$$+\sum << \tilde\nabla V\,,\,du >\,\Omega''_u\,du(e_i)\,,\,\tilde\nabla_{e_i}W\,>$$

$$= - \sum div\,(<<\tilde{\nabla}V\,,\,du\,>\,\Omega''_u\,du(e_i)\,,\,W\,>\,e_i\,) + \Omega''_u\,<\,\tilde{\nabla}V\,,\,du\,><\,\tilde{\nabla}W\,,\,du\,>$$

Since Ω is now a function on M independent of N, it follows that

$$grad^N\,\Omega\;=\;grad^N\,\Omega'\;=\;0$$

and thus in this case, the $\Omega-$Jacobi operator simplifies to

$$J_{\Omega,u}\,(V)\;=\;-\,\Omega'_u\,tr\,R^N(V,du)du - tr\,\tilde{\nabla}\,[\Omega'_u\,\tilde{\nabla}V] - tr\,\tilde{\nabla}\,[<\tilde{\nabla}V,du>\;\Omega''_u\,du]$$

In particular, the F-Jacobi operator is given by

$$\begin{aligned} J_{F,u}\,(V)\;&=\;-\,F'(\tfrac{|du|^2}{2})\,tr\,R^N(V,du)du\;-\;tr\,\tilde{\nabla}\,[F'(\tfrac{|du|^2}{2})\,\tilde{\nabla}V] \\ &\quad - tr\tilde{\nabla}\,[<\tilde{\nabla}V,du>F''(\tfrac{|du|^2}{2})du] \\ &=\;-\,F'(\tfrac{|du|^2}{2})\,\sum\,R^N(V,du(e_i))du(e_i)\;-\;\sum\,\tilde{\nabla}_{e_i}\,[F'(\tfrac{|du|^2}{2})\,\tilde{\nabla}_{e_i}V] \\ &\quad -\;\sum\,\tilde{\nabla}_{e_i}\,[<\tilde{\nabla}V,du>F''(\tfrac{|du|^2}{2})du(e_i)] \end{aligned}$$

Taking inner product with W, integrating and using the divergence theorem yield

$$\begin{aligned} \int_D <J_{F,u}(V),W>dv_g \\ &= -\int_D F'(\tfrac{|du|^2}{2})<\sum_{i=1}^n R^N(V,du(e_i))du(e_i),W>dv_g \\ &\quad +\;\int_D F'(\tfrac{|du|^2}{2})\;<\tilde{\nabla}V,\tilde{\nabla}W>\;dv_g \\ &\quad +\;\int_D F''(\tfrac{|du|^2}{2})\;<\tilde{\nabla}V\,,\,du><\tilde{\nabla}W\,,\,du>\;dv_g\,.\quad\Box \end{aligned}$$

Set $\quad I(V,W)\;=\;\frac{\partial^2}{\partial s\partial t}\,|_{s,t=0}\;\;E_F(u_{s,t})\,.$

Definition 2.5. An F-harmonic map u is stable (or F-stable) if for any compactly supported vector field V along u , we have

$$I(V,V)\;\geq\;0\,,$$

i.e. if the eigenvalues of the F-Jacobi operator $J_{F,u}$ are all nonnegative.

Definition 2.6. The stress energy tensor of an F-harmonic map $u : M \to N$ is defined by

$$S_F\,(u) \;=\; F(e(u))\,g \;-\; F'(e(u))\,u^*h \;\;=\;\; F(\tfrac{|du|^2}{2})\,g \;-\; F'(\tfrac{|du|^2}{2})\,u^*h\,.$$

The following two propositions are well-known. However, the proofs vary among the authors. Here, we give our own version in order to fix notations for later use.

Proposition 2.7. $(\,div\; S_F(u))\,(X) \;=\; -\; <\; \tau_F(u), du(X) > .$

Proof : Choose a local orthonormal frame field $\{e_i\}$ normal at $p \in M$.

For $X \in T_pM,$ we have

$$
\begin{aligned}
&(\,div\; S_F(u))\,(X) \\
&= \textstyle\sum_{i=1}^n (\nabla_{e_i}\, S_F(u))(e_i, X) \\
&= \textstyle\sum_{i=1}^n \nabla_{e_i}\,(\,S_F(u)(e_i, X)) \;-\; \sum_{i=1}^n\; S_F(u)\,(e_i, \nabla_{e_i} X) \\
&= \textstyle\sum_{i=1}^n \nabla_{e_i}\,[F(\frac{|du|^2}{2}) < e_i, X > \;-\, F'(\frac{|du|^2}{2}) < du(e_i), du(X) >] \\
&\qquad - \textstyle\sum\; F(\frac{|du|^2}{2}) < e_i, \nabla_{e_i} X > \;+\; \sum F'(\frac{|du|^2}{2}) < du(e_i), du(\nabla_{e_i} X) > \\
&= \textstyle\sum_{i=1}^n (\,\nabla_{e_i}\; F'(\frac{|du|^2}{2})) < e_i, X > \;+\; \sum\; F(\frac{|du|^2}{2}) < e_i, \nabla_{e_i} X >] \\
&\qquad - \textstyle\sum (e_i F'(\frac{|du|^2}{2})) < du(e_i), du(X) - \sum\; F'(\frac{|du|^2}{2}) e_i < du(e_i), du(X) > \\
&\qquad - \textstyle\sum\; F(\frac{|du|^2}{2}) < e_i, \nabla_{e_i} X > \;+\; \sum F'(\frac{|du|^2}{2}) < du(e_i), du(\nabla_{e_i} X) > \\
&= \textstyle\sum_{i=1}^n\; F'(\frac{|du|^2}{2})\; \sum_{j=1}^n\; < \tilde{\nabla}_{e_i} du(e_j), du(e_j) >< e_i, X > \\
&- \textstyle\sum (e_i F'(\frac{|du|^2}{2})) < du(e_i), du(X) > - \sum\; F'(\frac{|du|^2}{2}) < \tilde{\nabla}_{e_i} du(e_i), du(X) > \\
&- \textstyle\sum\; F'(\frac{|du|^2}{2}) < du(e_i), \tilde{\nabla}_{e_i} du(X) > \;+\; \sum F'(\frac{|du|^2}{2}) < du(e_i), du(\nabla_{e_i} X) > \\
&= \textstyle\sum_{j=1}^n\; F'(\frac{|du|^2}{2})\;\; < \tilde{\nabla}_X du(e_j), du(e_j) > \\
&\qquad - \textstyle\sum\; < \nabla F'(\frac{|du|^2}{2}), e_i >< du(e_i), du(X) > \\
&\qquad - \textstyle\sum\; F'(\frac{|du|^2}{2}) < \tilde{\nabla}_{e_i} du(e_i) \;-\; du(\nabla_{e_i} e_i), du(X) > \\
&\qquad - \textstyle\sum\; F'(\frac{|du|^2}{2}) < du(e_i), \tilde{\nabla}_X du(e_i) >
\end{aligned}
$$

$$= - < du(\nabla F'(\tfrac{|du|^2}{2})), du(X) > - F'(\tfrac{|du|^2}{2}) < \tau(U), du(X) >$$
$$= - < du(\nabla F'(\tfrac{|du|^2}{2})) + F'(\tfrac{|du|^2}{2})\tau(u) , du(X) > . \quad \square$$

Corollary 2.8. *Any F-harmonic map satisfies the conservation law, i.e.*

$$div\ S_F(u) \equiv 0 .$$

Proof : This follows directly from Proposition 2.8.

Proposition 2.9.

$$div\ (F(\tfrac{|du|^2}{2})X) = \sum_{i=1}^n div(F'(\tfrac{|du|^2}{2}) < du(X), du(e_i) > e_i)$$
$$- < du(X), \tau_F(u) > + < S_F(u), \nabla\theta_X > .$$

Proof : For brevity we will use ∇ for gradient when the context is clear.

Choose a local orthonormal frame field $\{e_i\}$ on M . Then, for $X \in TM$,

$$div(F(\tfrac{|du|^2}{2})X)$$
$$= F(\tfrac{|du|^2}{2})divX + < \nabla F(\tfrac{|du|^2}{2}), X >$$
$$= F(\tfrac{|du|^2}{2})divX + \nabla_X F(\tfrac{|du|^2}{2})$$
$$= F(\tfrac{|du|^2}{2})divX + \sum F'(\tfrac{|du|^2}{2}) < \tilde{\nabla}_X du(e_i), du(e_i) >$$
$$= F(\tfrac{|du|^2}{2})divX + \sum F'(\tfrac{|du|^2}{2}) < (\tilde{\nabla}_X du)(e_i), du(e_i) >$$
$$= F(\tfrac{|du|^2}{2})divX + \sum F'(\tfrac{|du|^2}{2}) < (\tilde{\nabla}_{e_i} du)(X), du(e_i) >$$
$$= F(\tfrac{|du|^2}{2})divX + \sum F'(\tfrac{|du|^2}{2}) < \tilde{\nabla}_{e_i} du(X), du(e_i) >$$
$$- \sum F'(\tfrac{|du|^2}{2}) < du(\nabla_{e_i} X), du(e_i) >$$
$$= F(\tfrac{|du|^2}{2})divX + \sum < \tilde{\nabla}_{e_i} du(X), F'(\tfrac{|du|^2}{2})du(e_i) >$$
$$- \sum F'(\tfrac{|du|^2}{2}) < du(\nabla_{e_i} X), du(e_i) >$$
$$= F(\tfrac{|du|^2}{2})divX + \sum e_i < du(X), F'(\tfrac{|du|^2}{2})du(e_i) >$$
$$- \sum < du(X), e_i(F'(\tfrac{|du|^2}{2})du(e_i)) > - \sum F'(\tfrac{|du|^2}{2}) < du(\nabla_{e_i} X), du(e_i) >$$

$$= F(\tfrac{|du|^2}{2}) divX + \sum e_i \, (F'(\tfrac{|du|^2}{2}) < du(X), du(e_i) >)$$

$$- \sum < du(X), (e_i F'(\tfrac{|du|^2}{2})) du(e_i)) > - \sum < du(X), F'(\tfrac{|du|^2}{2}) \, \tilde{\nabla}_{e_i} du(e_i) >$$

$$- \sum F'(\tfrac{|du|^2}{2}) < du(\nabla_{e_i} X), du(e_i) >$$

$$= F(\tfrac{|du|^2}{2}) divX + \sum (e_i F'(\tfrac{|du|^2}{2})) < du(X), du(e_i) >$$

$$+ \sum F'(\tfrac{|du|^2}{2}) e_i < du(X), du(e_i) > - \sum < du(X), < \nabla F'(\tfrac{|du|^2}{2}), e_i > du(e_i) >$$

$$- \sum < du(X), F'(\tfrac{|du|^2}{2}) \, \tilde{\nabla}_{e_i} du(e_i) > - \sum F'(\tfrac{|du|^2}{2}) < du(\nabla_{e_i} X), du(e_i) >$$

$$= F(\tfrac{|du|^2}{2}) divX + \sum < \nabla F'(\tfrac{|du|^2}{2}), e_i > < du(X), du(e_i) >$$

$$+ \sum F'(\tfrac{|du|^2}{2}) < \nabla < du(X), du(e_i) >, e_i > - < du(X), du(\nabla F'(\tfrac{|du|^2}{2})) >$$

$$- \sum < du(X), F'(\tfrac{|du|^2}{2}) (\tilde{\nabla}_{e_i} du(e_i) - du(\nabla_{e_i} e_i)) >$$

$$- \sum F'(\tfrac{|du|^2}{2}) < du(\nabla_{e_i} X), du(e_i) >$$

$$= F(\tfrac{|du|^2}{2}) divX + \sum < \nabla F'(\tfrac{|du|^2}{2}) < du(X), du(e_i) >, e_i >$$

$$+ \sum F'(\tfrac{|du|^2}{2}) < \nabla < du(X), du(e_i) >, e_i >$$

$$- < du(X), du(\nabla F'(\tfrac{|du|^2}{2})) + F'(\tfrac{|du|^2}{2}) \tau(u) >$$

$$- \sum F'(\tfrac{|du|^2}{2}) < du(\nabla_{e_i} X), du(e_i) >$$

$$= F(\tfrac{|du|^2}{2}) divX + \sum < \nabla (F'(\tfrac{|du|^2}{2}) < du(X), du(e_i) >), e_i >$$

$$- < du(X), \tau_F(u) > - \sum F'(\tfrac{|du|^2}{2}) < du(\sum < \nabla_{e_i} X, e_j > e_j), du(e_i) >$$

$$= F(\tfrac{|du|^2}{2}) divX + \sum div(F'(\tfrac{|du|^2}{2}) < du(X), du(e_i) > e_i)$$

$$- < du(X), \tau_F(u) >$$

$$- \sum_{ij} F'(\tfrac{|du|^2}{2}) < du(e_j), du(e_i) > < \nabla_{e_i} X, e_j >$$

$$= \sum div \, (F'(\tfrac{|du|^2}{2}) < du(X), du(e_i) > e_i) - < du(X), \tau_F(u) >$$

$$+ \sum_{ij} [\, F(\tfrac{|du|^2}{2}) \, \delta_{ij} - F'(\tfrac{|du|^2}{2}) < du(e_j), du(e_i) >] < \nabla_{e_i} X, e_j >$$

$$= \sum div \, (F'(\tfrac{|du|^2}{2}) < du(X), du(e_i) > e_i) - < du(X), \tau_F(u) >$$

$$+ < S_F(u), \nabla \theta_X > . \quad \Box$$

Corollary 2.10. *If* $u:(M,g)\to(N,h)$ *is a* C^2 *F-harmonic map and* $D\subset\subset M$ *a* C^1 *compact domain with smooth hypersurface boundary* ∂D, *then*

$$\int_{\partial D} S_F(u)(X,\nu)\,dS_g \;=\; \int_D \;< S_F(u),\nabla\theta_X > dv_g + \int_D (div\, S_F(u))(X)\,dv_g$$

where θ_X *is the dual of* $X\in TM$ *and* ν *is the unit normal vector of* ∂D *.*

Proof : Applying Stokes' theorem to the preceding two propositions. □

Lemma 2.11. *(Weitzenböck Formula) [16] : For any p-form* $\sigma\in\mathcal{A}^p(E)$ *,*

$$\Delta\sigma \;=\; -\,trace\,\nabla^2\sigma \;+\; S(\sigma)$$

where $S_x\sigma(X_1,...X_p)=\sum_{ik}(-1)^k\,(R(e_i,X_k)\sigma)(e_i,X_1,...,\hat{X}_k,...,X_p)$ *, if* $p\geq 1$

$$= \;0\,,\; \textit{if}\;\; p=0\,.$$

Remark 2.12. Let $f:M\to N$ be a C^∞ map.

Then for $df\in\mathcal{A}^1(f^*TN):=\Gamma(T^*M\otimes f^*TN)$, we have

$$\begin{aligned} S_x df(X) \;&=\; -\sum (R(e_i,X)df)(e_i) \\ &=\; -\sum R(e_i,X)(df(e_i)) \;+\; \sum df(R(e_i,X)e_i) \\ &=\; -\sum R^N(df(e_i),df(X))df(e_i) \;+\; df(Ric^M X)\,,\ \text{and} \\ \Delta df(X) \;&=\; -\,trace\,\nabla^2 df(X) \;+ S(df)(X) \\ &=\; -\,\tilde\nabla^*\tilde\nabla df(X) \;-\; \sum R^N(df(e_i),df(X))df(e_i) \;+\; df(Ric^M X) \end{aligned}$$

Lemma 2.13. *(Böchner Formula for F-harmonic maps) [3] :*

$$\begin{aligned} \Delta F(\tfrac{|du|^2}{2}) \;&=\; F''(\tfrac{|du|^2}{2})\,|du|^2\,|\,\nabla|du|\,|^2 \;+\; F'(\tfrac{|du|^2}{2})\,[\,-<\Delta_H du,du> + |\nabla du|^2 \\ &\quad -\sum_{ij} < R^N(du(e_i),du(e_j))du(e_j),du(e_i)> + \sum_i < du(Ric^M e_i),du(e_i)>], \end{aligned}$$

where $\Delta_H \;=\; d\delta \;+\; \delta d$ *is the Hodge-Laplace operator on forms.*

Proof :

$$\Delta F(\tfrac{|du|^2}{2}) \;=\; \nabla\nabla F(\tfrac{|du|^2}{2})$$
$$= \;\nabla(F'(\tfrac{|du|^2}{2}) < \nabla du, du >)$$
$$= \;F''(\tfrac{|du|^2}{2}) \;< \nabla du, du >^2 \;+ F'(\tfrac{|du|^2}{2}) < \nabla\nabla du, du > \;+ F'(\tfrac{|du|^2}{2})\;|\nabla du|^2$$
$$= \;F''(\tfrac{|du|^2}{2})\;|du|^2\;|\nabla|du|\;|^2 \;+\; F'(\tfrac{|du|^2}{2})\;\{ - < \Delta_H du, du > \;+ |\nabla du|^2$$
$$- \sum_{ij} < R^N(du(e_i), du(e_j))du(e_j), du(e_i) > + \sum < du(Ric^M e_i), du(e_i) >\}. \;\Box$$

Proposition 2.14. *Let* (N, h) *be a Riemannian manifold and* $F : [0, \infty) \to [0, \infty)$ *be a strictly increasing* C^2 *function. If* $u : S^2 \to (N, h)$ *is an F-harmonic map from the unit 2-sphere, then the following equality holds*

$$trace\; I(du(W), du(W))$$
$$= - \int_{S^2} F'(\tfrac{|du|^2}{2})|\tau(u)|^2 dv_g + \int_{S^2} trace < du(\nabla_W\; grad F'(\tfrac{|du|^2}{2})), du(W) > dv_g$$
$$+ \int_{S^2}\; trace\; (< \tilde{\nabla} du(W), du >)^2\; F''(\tfrac{|du|^2}{2})\; dv_g\,,$$

where I *is the index form and the vector field* W *is the orthogonal projection of any parallel vector field in* $\mathbb{R}^3$*.*

Proof : Consider the isometric embedding of S^2 in $\mathbb{R}^3$.

Let $p \in S^2$ and $a \in \mathbb{R}^3$. Define $\phi(p) = < a, p >$, $\forall p \in S^2$, and set

$$W \;=\; grad\,\phi\,.$$

Let ∇ and $\bar{\nabla}$ be the Levi-Civita connections on S^2 and $\mathbb{R}^3$ with respect to the standard flat metric, respectively. Choose an orthonormal frame field $\{e_i\}_{i=1,2}$ in S^2 normal at $p \in S^2$, i. e.

$$\nabla_{e_i} e_j|_p = 0\,.$$

Then,
$$W \;=\; \sum_{i=1}^{2}\; e_i(\phi)\; e_i$$
$$= \;\sum_{i=1}^{2}\; e_i < a, p > \; e_i$$
$$= \;\sum_{i=1}^{2}\; < a, \bar{\nabla}_{e_i} p > \; e_i$$

$$= \sum_{i=1}^{2} < a, e_i > \ e_i$$

Thus W is the orthogonal projection of a parallel vector field in $\mathbb{R}^3$ onto S^2 and it is easy to see that $\nabla_{e_i} W|_p = 0$, $\forall i = 1, 2.$

Then for any vector field $X \in \Gamma(TS^2)$, we obtain

$$\nabla_X W \ = \ -\phi X \quad \text{and}$$

$$\text{trace } \nabla^2 W \ = \ -W \ .$$

To see this, let p be a point on S^2 with the above local orthonormal frame field $\{e_i\}$. Then at p, we have

$$\begin{aligned}
\nabla_X W = \ & (\bar{\nabla}_X W)^T \\
= \ & \bar{\nabla}_X W \ - \ (\bar{\nabla}_X W)^\perp \\
= \ & \bar{\nabla}_X (\sum < a, e_i > e_i) - \ < \bar{\nabla}_X (\sum < a, e_i > e_i), p > p \\
= \ & \sum < a, \bar{\nabla}_X e_i > e_i \ + \ \sum < a, e_i > (\bar{\nabla}_X e_i) \\
& \quad - \ < \sum < a, \bar{\nabla}_X e_i > e_i, p > p \ - \ \sum < a, e_i >< \bar{\nabla}_X e_i, p > p \\
= \ & \sum < a, \bar{\nabla}_X e_i > e_i + \sum < a, e_i > \ [\, (\nabla_X e_i) \ + \ (\bar{\nabla}_X e_i)^\perp \,] \\
& \quad - \ \sum < a, e_i >< \bar{\nabla}_X e_i, p > p \\
= \ & \sum < a, \bar{\nabla}_X e_i > e_i \\
= \ & \sum < a, \nabla_X e_i + (\bar{\nabla}_X e_i)^\perp > e_i \\
= \ & \sum < a, < \bar{\nabla}_X e_i, p > p > e_i \\
= \ & - \ \sum < a, < e_i, \bar{\nabla}_X p > p > e_i \quad , \qquad \text{since} < e_i, p >= 0 \\
= \ & - \ < a, p > \sum < e_i, \bar{\nabla}_X p > e_i \\
= \ & - \ < a, p > \sum < e_i, X > e_i \\
= \ & - \ < a, p > X \\
= \ & - \, \phi \, \text{X} \ ,
\end{aligned}$$

and

$$\begin{aligned}
\text{trace } \nabla^2 W \ = \ & \sum_{i=1}^{2} \nabla_{e_i} \nabla_{e_i} W \\
= \ & \sum \ \nabla_{e_i} (-\phi \, e_i)
\end{aligned}$$

$$= -\sum e_i(\phi)e_i$$

$$= -\sum < \nabla\phi, e_i > e_i$$

$$= -\nabla\phi$$

$$= -W\ .$$

Let $du(W) \in \Gamma(u^*TN)$ be a vector field along the map u where $W \in \Gamma(TS^2)$ is a vector field on S^2 defined as above. We write $F' = F'(\frac{|du|^2}{2})$ and consider the trace of the index form

$$trace\ I(du(W), du(W))$$

$$= \int_{S^2} trace < J_{F,u}(du(W)), du(W) > \ dv_g\ ,$$

where the F-Jacobi operator is defined by

$$J_{F,u}(du(W)) = -F'\sum R^N(du(W), du(e_i))du(e_i) - \sum\tilde{\nabla}_{e_i}[F'\tilde{\nabla}_{e_i}(du(W))]$$
$$- \sum\ \tilde{\nabla}_{e_i}\ [< \tilde{\nabla}(du(W)), du > F''\ du(e_i)]$$

Henceforth, all calculations are carried out locally at p $\in S^2$.

$$\sum_{i=1}^2 \tilde{\nabla}_{e_i}\ [F'\ \ \tilde{\nabla}_{e_i}(du(W)]$$

$$= \sum(\tilde{\nabla}_{e_i}F')(\tilde{\nabla}_{e_i}(du(W))) + \sum F'\tilde{\nabla}_{e_i}\tilde{\nabla}_{e_i}du(W)$$

$$= \tilde{\nabla}_{gradF'}du(W) + F'\sum\tilde{\nabla}_{e_i}\tilde{\nabla}_{e_i}du(W)$$

$$= \tilde{\nabla}_W du(gradF') + du([gradF', W]) + F'\sum\tilde{\nabla}_{e_i}\tilde{\nabla}_W du(e_i)$$
$$+F'\sum\tilde{\nabla}_{e_i}du([e_i, W])\ ,$$

since $\nabla_X du(Y) = \nabla_Y du(X) + du([X,Y])$

$$= \tilde{\nabla}_W du(gradF') + du(\nabla_{gradF'}W) - du(\nabla_W gradF')$$
$$+F'\sum[R^N(du(e_i), du(W))du(e_i) + \tilde{\nabla}_W\tilde{\nabla}_{e_i}du(e_i) + \tilde{\nabla}_{[e_i,W]}du(e_i)]$$
$$+F'\sum\tilde{\nabla}_{e_i}du([e_i, W])$$

$$= \tilde{\nabla}_W du(gradF') - du(\nabla_W gradF') + F'\sum R^N(du(e_i), du(W))du(e_i)$$
$$+F'\sum\tilde{\nabla}_W\tilde{\nabla}_{e_i}du(e_i) + F'\sum\tilde{\nabla}_{[e_i,W]}du(e_i)$$
$$+ F'\sum\tilde{\nabla}_{[e_i,W]}du(e_i) + F'\sum du([e_i,[e_i,W]])\ ,$$

where we use

$$\nabla_{gradF'}W = \nabla_{\sum <gradF',e_i> e_i}W = \sum < gradF', e_i > \nabla_{e_i}W = 0$$

and $[e_i, W] = \nabla_{e_i}W - \nabla_W e_i = 0$,

$= \sum \tilde{\nabla}_W du(gradF')$ - $du(\nabla_W gradF')$ + $F' \sum R^N(du(e_i), du(W))du(e_i)$

$+F'\tilde{\nabla}_W[\tau(u) + \sum du(\nabla_{e_i}e_i)]$ + $F' \sum du(\nabla_{e_i}[e_i, W] - \nabla_{[e_i,W]}e_i)$,

here we use

$$\tau(u) = \sum [\ \tilde{\nabla}_{e_i}du(e_i) - du(\nabla_{e_i}e_i)\ ]$$

$= \sum \tilde{\nabla}_W du(gradF')$ - $du(\nabla_W gradF')$ + $F' \sum R^N(du(e_i), du(W))du(e_i)$

$+F'\tilde{\nabla}_W\tau(u)$ + $F' \sum \tilde{\nabla}_W du(\nabla_{e_i}e_i)$ + $F' \sum du(\nabla_{e_i}\nabla_{e_i}W)$

$-F' \sum du(\nabla_{e_i}\nabla_W e_i)$

$= \sum \tilde{\nabla}_W du(gradF')$ - $du(\nabla_W gradF')$ - $F' \sum R^N(du(W), du(e_i))du(e_i)$

$+\tilde{\nabla}_W(F'\tau(u))$ - $(\nabla_W F')\tau(u)$ + $F' \sum(\tilde{\nabla}_W du)(\nabla_{e_i}e_i)$

$+F' \sum du(\nabla_W\nabla_{e_i}e_i)$ + $F' \sum du(\nabla_{e_i}\nabla_{e_i}W)$ - $F' \sum du(\nabla_{e_i}\nabla_W e_i)$

= - $du(\nabla_W gradF')$ - $F' \sum R^N(du(W), du(e_i))du(e_i)$ - $(\nabla_W F')\tau(u)$

$+F' \sum du[\nabla_W\nabla_{e_i}e_i - \nabla_{e_i}\nabla_W e_i]$ + $F' \sum du(\nabla_{e_i}\nabla_{e_i}W)$, here

$\tau_F(u) = du(gradF') + F'\ \tau(u) = 0$ and $\nabla_{e_i}e_i = 0$,

= - $du(\nabla_W gradF')$ - $F' \sum R^N(du(W), du(e_i))du(e_i)$ - $(\nabla_W F')\tau(u)$

$+F'du(\sum R^M(W, e_i)e_i)$ + $F'du(\sum(\nabla_{e_i}\nabla_{e_i}W)$

= - $du(\nabla_W gradF')$ - $F' \sum R^N(du(W), du(e_i))du(e_i)$ - $(\nabla_W F')\tau(u)$

$+F'du(Ricci\ W)$ + $F'du(trace\ \nabla^2 W)$,

here we note that on the unit n-sphere S^n

$$Ricci^{S^n}(W)\ =\ (n-1)\,W$$

and since trace $\nabla^2 W = -W$, the last two terms vanish

= - $du(\nabla_W gradF')$ - $F' \sum R^N(du(W), du(e_i))du(e_i)$ - $(\nabla_W F')\tau(u)$.

The following calculations are straightforward :

$$
\begin{aligned}
&trace\, du(\ W(F'(\tfrac{|du|^2}{2}))\, W\ ) \\
&\quad = \ trace\, du\ (\ \textstyle\sum_{ij} < a, e_i >\ e_i(F'(\tfrac{|du|^2}{2}))\ < a, e_j > e_j) \\
&\quad = \ trace\, du\ (\ \textstyle\sum_{ij}\ < grad\ F'(\tfrac{|du|^2}{2}), e_i > e_j\ < a, e_i >< a, e_j >\ ) \\
&\quad = \quad du\ (\ \textstyle\sum_{i}\ < grad\ F'(\tfrac{|du|^2}{2}), e_i > e_i\ < a, e_i >^2\ ) \\
&\quad = \quad du\ (\ grad\ F'(\tfrac{|du|^2}{2})\ ) \\
&trace\ <\ W(F'(\tfrac{|du|^2}{2}))\ \tau(u), du(W)\ > \\
&\quad = \ trace\ <\ \tau(u), du(W(F'(\tfrac{|du|^2}{2}))W) > \\
&\quad =<\ \tau(u), du(\ grad\ F'(\tfrac{|du|^2}{2})\ )\ > \\
&\quad =<\ \tau(u), -\ F'(\tfrac{|du|^2}{2})\ \tau(u)\ >, \quad \text{by the F-harmonicity condition .}
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
&trace < J_{F,u}(du(W)), du(W) > \\
&\quad = trace\, [-F' < \textstyle\sum R^N(du(W), du(e_i))du(e_i), du(W) > + < du(\nabla_W gradF'), du(W) > \\
&\qquad + F' < \textstyle\sum R^N(du(W), du(e_i))du(e_i), du(W) > + \ < (\nabla_W F')\tau(u), du(W) > \\
&\qquad\qquad - \textstyle\sum\ <\ \tilde{\nabla}_{e_i}\, (< \tilde{\nabla}(du(W)), du > F''du(e_i))\ ,\ du(W)\ >] \\
&\quad = \ trace\, [<\ W(F')\, \tau(u)\ ,\ du(W)\ > + <\ du(\nabla_W\ grad\ F')\ ,\ du(W)\ > \\
&\qquad\qquad - \textstyle\sum\ e_i\ (\ <\ (< \tilde{\nabla} du(W), du > F''du(e_i))\ ,\ du(W)\ >) \\
&\qquad\qquad + \textstyle\sum\ <\ (< \tilde{\nabla} du(W), du > F''du(e_i))\ ,\ \tilde{\nabla}_{e_i} du(W)\ >] \\
&\quad = \ -\ F'\ |\tau(u)|^2\ +\ trace\ <\ du(\nabla_W\ grad\ F')\ ,\ du(W)\ > \\
&\qquad\qquad - trace \textstyle\sum\ div\ (<< \tilde{\nabla} du(W), du > F''du(e_i), du(W) > e_i) \\
&\qquad\qquad + trace\ (< \tilde{\nabla} du(W), du >)^2\ F''
\end{aligned}
$$

By the divergence theorem, the integral of the third term in the last equality vanishes and the lemma follows by integration. □

Theorem 2.15. *Let* $u : (M^n, g) \to (N^k, h)$ *be a stable F-harmonic map from a complete noncompact Riemannian manifold M into a complete Riemannian manifold N. Let* ϕ *be a smooth function on M. Then the following inequality holds:*

$$\begin{aligned} 0 \;\le\; & \int_M F''(\tfrac{|du|^2}{2}) \{ \; |du|^2 \, |\nabla \phi|^2 \; + \; \phi^2 \, | \textstyle\sum_{i=1}^n B(\tilde{e}_i, \tilde{e}_i)|^2 \; \} \, dv_g \\ & + \int_M F'(\tfrac{|du|^2}{2}) \{ \; k|\nabla \phi|^2 \; + \; \phi^2 \textstyle\sum_{a=1}^k \sum_{i=1}^n (\, 2 \, |B(V_a, \tilde{e}_i)|^2 \\ & \qquad - \; < B(V_a, V_a), B(\tilde{e}_i, \tilde{e}_i) >) \} \, dv_g \, , \end{aligned}$$

where dv_g *is the volume element of M and* $\tilde{e}_i := du(e_i)$.

Proof : The Nash's embedding theorem says that we can isometrically embed N^k into $\mathbb{R}^r$ for some r. Let $\{V_a\}_{a=1}^r$ be an orthonomal basis in $\mathbb{R}^r$ where

$V_1^T, ..., V_k^T \;=\; V_1, ..., V_k$ are tangent to N and

$V_{k+1}^\perp, ..., V_r^\perp \;=\; V_{k+1}, ..., V_r$ are normal to N.

Denote $f_t^{\phi \; V_a^T}$ the flow generated by V_a^T and apply the second variation formula with

$$u_t = f_t^{\phi \; V_a^T} \circ u \quad \text{and} \quad u_o = u \, ,$$

then we sum over $a = 1, ..., r$ with $s = t$.

$$\begin{aligned} & \textstyle\sum_{a=1}^r \tfrac{d^2}{dt^2} \, E_F(f_t^{\phi \; V_a^T} \circ u)|_{t=0} \\ & \quad = \textstyle\sum_{a=1}^r \int_M [\, F''(\tfrac{|du|^2}{2}) \, (\, \sum_{i=1}^n \; < \tilde{\nabla}_{e_i} \, \phi V_a^T, \tilde{e}_i >)^2 \\ & \qquad\quad + F'(\tfrac{|du|^2}{2}) \, \textstyle\sum_{i=1}^n \{ \, |\tilde{\nabla}_{e_i} \phi V_a^T|^2 - \; < R^N(\phi V_a^T, \tilde{e}_i)\tilde{e}_i, \phi V_a^T >) \, \}] \, dv_g \end{aligned}$$

Denote $\bar{\nabla}$ the Riemannian connection in $\mathbb{R}^r$. Since V_a is parallel in $\mathbb{R}^r$, we get

$$\begin{aligned} \tilde{\nabla}_{e_i} V_a^T \; &= \; \nabla^N_{\tilde{e}_i} \, V_a^T \\ &= \; (\bar{\nabla}_{\tilde{e}_i} \, V_a^T)^T \\ &= \; (\bar{\nabla}_{\tilde{e}_i} \, [\, V_a \; - \; V_a^\perp])^T \\ &= \; - \, (\bar{\nabla}_{\tilde{e}_i} \, V_a^\perp)^T \\ &= \; A_{\, V_a^\perp}(\tilde{e}_i) \\ \tilde{\nabla}_{e_i} \phi \, V_a^T \; &= \; (e_i \phi) \, V_a^T \; + \; \phi \, \tilde{\nabla}_{e_i} V_a^T \end{aligned}$$

Thus,

$$\sum_{a=1}^{r} \big(\sum_{i=1}^{n} < \tilde{\nabla}_{e_i} \phi\, V_a^T, \tilde{e}_i > \big)^2$$
$$= \sum_{a=1}^{r} \big(\sum_{i=1}^{n} < (e_i\, \phi)\, V_a^T, \tilde{e}_i > + \phi < \tilde{\nabla}_{e_i} V_a^T, \tilde{e}_i > \big)^2$$
$$= \sum_{a=1}^{r} \big(\sum_{i=1}^{n} < (e_i\, \phi)\, V_a^T, \tilde{e}_i > \big)^2 + \phi^2 \sum_{a=1}^{r} \big(\sum_{i=1}^{n} < A_{V_a^\perp} (\tilde{e}_i), \tilde{e}_i > \big)^2$$
$$= \sum_{a=1}^{r} < V_a^T, du(\nabla\phi) >^2 + \phi^2 \sum_{a=1}^{r} \big(\sum_{i=1}^{n} < B(\tilde{e}_i, \tilde{e}_i), V_a^\perp > \big)^2$$
$$= \sum_{a=1}^{k} < V_a, du(\nabla\phi) >^2 + \phi^2 \sum_{a=k+1}^{r} < \sum_{i=1}^{n} B(\tilde{e}_i, \tilde{e}_i), V_a >^2$$
$$= |\, du(\nabla\phi)\,|^2 + \phi^2\, |\, \sum_{i=1}^{n} B(\tilde{e}_i, \tilde{e}_i)\,|^2$$
$$= |\, du\,|^2\, |\, \nabla\phi\,|^2 + \phi^2\, |\, \sum_{i=1}^{n} B(\tilde{e}_i, \tilde{e}_i)\,|^2 ,$$

since we can choose a local orthonornal frame field $e_1, ..., e_n$ such that e_1 is the unit vector field in the direction of the gradient vector field $\nabla\phi$, it follows that

$$|\, du(\nabla\phi)\,|^2 = |\, du(|\nabla\phi|\, e_1)\,|^2 = |du(e_1)|^2\, |\nabla\phi|^2 = |du|^2\, |\nabla\phi|^2 .$$

Next,

$$\sum_{a=1}^{r} \sum_{i=1}^{n} |\, \tilde{\nabla}_{e_i} \phi\, V_a^T\,|^2$$
$$= \sum_{a=1}^{r} \sum_{i=1}^{n} |\, (e_i\phi)\, V_a^T + \phi\, \tilde{\nabla}_{e_i} V_a^T|^2$$
$$= \sum_{a=1}^{r} \sum_{i=1}^{n} \{ (e_i\phi)^2\, |V_a^T|^2 + 2(e_i\phi)\phi < V_a^T, \tilde{\nabla}_{e_i} V_a^T > + \phi^2 |\tilde{\nabla}_{e_i} V_a^T|^2 \}$$
$$= k\, |\nabla\phi|^2 + \sum_{a=1}^{r} \sum_{i=1}^{n} \{ 2(e_i\phi)\phi < V_a^T, A_{V_a^\perp} (\tilde{e}_i) > + \phi^2 |\tilde{\nabla}_{e_i} V_a^T|^2 \}$$
$$= k\, |\nabla\phi|^2 + \sum_{a=1}^{r} \sum_{i=1}^{n} \{ \phi^2\, |\, A_{V_a^\perp} (\tilde{e}_i)|^2 \}$$

By the Gauss curvature equation, we obtain

$$\sum_{a=1}^{r} \sum_{i=1}^{n} \{ |\, A_{V_a^\perp} (\tilde{e}_i)|^2 - < R^N(V_a^T, \tilde{e}_i)\tilde{e}_i, V_a^T > \}$$
$$= \sum_{a=1}^{k} \sum_{i=1}^{n} \big(2\, |\, B(V_a, \tilde{e}_i)\,|^2 - < B(V_a, V_a), B(\tilde{e}_i, \tilde{e}_i) > \big) . \qquad \square$$

Chapter 3

Kahler geometry

Le génie est la longue patience.

French

The Kahler structures were introduced with the following motivation : given any Hermitian metric on a complex manifold (M, h), the fundamental 2-form ω can be expressed in local holomorphic coordinates as follows

$$\omega \ = \ i \sum h_{\alpha\bar{\beta}} \, dz^\alpha \wedge d\bar{z}^\beta \quad , \quad h_{\alpha\bar{\beta}} \ = \ h(\tfrac{\partial}{\partial z^\alpha}, \tfrac{\partial}{\partial \bar{z}^\beta})$$

The Kahler condition $d\omega \ = \ 0$ is equivalent to the local existence of some function u such that

$$h_{\alpha\bar{\beta}} \ = \ \tfrac{\partial^2 u}{\partial z^\alpha \partial \bar{z}^\beta}$$

i.e. the whole metric tensor is defined by a unique function! This remarkable property of the metric allows one to obtain simple explicit expressions for the Ricci and curvature tensors and a long list of miracles then occurs.

There is another remarkable property of Kahler metrics : every point x in a Riemannian manifold has a local coordinate system $\{x^i\}$ such that the metric osculates to the Euclidean metric to the order 2 at x. These special coordinate systems are the normal coordinates around each point. On a Hermitian manifold, the existence of normal holomorphic coordinates around each point is equivalent to the Kahler condition, i.e. the metric is Kahler. Applications of Kahler manifolds have been widely researched in differential geometry, complex analysis, algebraic geometry, and theoretical physics.

In section 3.1, we gather a few relevant facts about Kahler geomtry. In section 3.2, we calculated the holomorphic sectional curvature of the complex hyperbolic space $\mathbb{C}H^n$. Then we give a detail description of the complex projective space $\mathbb{C}P^n$. Along the way, we show that when n = 1, $\mathbb{C}P^1$ behaves exactly like the unit sphere S^2. A list of other well-known Kahler manifolds are also given.

3.1 Kahler manifolds

An almost complex manifold M is a real manifold with a field J of endomorphisms of TM such that $J^2 = -I$. This operator J can be extended linearly to an operator, also denoted by J, on the complexified tangent bundle TM^C with fiber $T_xM \otimes \mathbb{C}$ at x in M, which induces a decomposotion

$$TM^C = TM^{1,0} \oplus TM^{0,1}$$

of the bundles of the eigenspaces of J on TM^C associated to the eigenvalues i, - i and further induces a dual decomposition of the complexified cotangent bundles

$$T^*M^C = T^*M^{1,0} \oplus T^*M^{0,1}$$

A Hermitian metric on an almost complex manifold M is a Riemannian metric g satisfying

$g(JX, JY) = g(X, Y)$, $\forall X, Y \in \Gamma(TM)$. Then it follows that

$$g(X, JY) = -g(JX, Y)$$
$$g(X, JX) = 0$$

An almost complex manifold with a Hermitian metric is called almost Hermitian.

Proposition 3.1. *[61] Every almost complex manifold admits a Hermitian metric provided it is paracompact.*

Proof : Since it is paracompact, we can take a Riemannian metric h and set

$$g(X, Y) = h(X, Y) + h(JX, JY). \quad \text{Then}$$
$$\begin{aligned} g(JX, JY) &= h(JX, JY) + h(J^2X, J^2Y) \\ &= h(JX, JY) + h(X, Y) \\ &= g(X, Y). \quad \square \end{aligned}$$

The Hermitian metric g on an almost Hermitian manifold M extends to a complex bilinear form on TM^C and thus induces on $TM^{1,0}$ the Hermitian form associated to $X, Y \in T_xM^{1,0}$ the number $g(X, \bar{Y})$.

The almost complex structure J of an n-dimensional manifold M is integrable if locally there exists coordinates $z^i = x^i + y^i$, $1 \leq j \leq n$, for which

$$J(\tfrac{\partial}{\partial x^j}) = \tfrac{\partial}{\partial y^j} \quad \text{and}$$
$$J(\tfrac{\partial}{\partial y^j}) = -\tfrac{\partial}{\partial x^j} \quad \text{for all } 1 \leq j \leq n .$$

Theorem 3.2. *[42] Any integrable almost complex structure is induced by a complex structure.*

Proposition 3.3. *[36] An almost complex structure is integrable iff the Lie bracket of vector fields preserves* $TM^{0,1}$ *, i.e.* $[TM^{0,1} , TM^{0,1}] \subset TM^{0,1}$.

To every almost complex structure J, we can associate a (2,1)-tensor N^J, the Nijenhuis tensor defined by

$N^J(X,Y) = [X,Y] - [JX,JY] + J([JX,Y] + [X,JY])\ ,\ \forall\, X,Y \in \Gamma(TM).$

Proposition 3.4. *[32] Let J be an almost complex structure on a real 2n-dimensional manifold M. Then J is a complex structure iff $N^J = 0$.*

Remark 3.5. Every almost complex manifold is necessarily of even dimension. To see this, let (M^n, J) be an almost complex manifold of complex dimension n. Its almost complex structure J_p acts on the tangent space. Choose a real basis of vector fields. The $J^\nu_\mu(p)$ are real, where $J_p = J^\nu_\mu(p)\,\frac{\partial}{\partial x^\nu} \otimes dx^\mu$. It follows that

$$[Det(J)]^2 = Det(J^2) = Det(-I) = (-1)^n\ .$$

Since $[Det(J)]$ is real, $[Det(J)]^2$ is positive, hence n must be even.

Remark 3.6. [6] For $n \neq 2, 6,\ S^n$ does not admit any almost complex structure.

The *Kahler form* on an almost Hermitian manifold (M, g, J) is the 2-form

$$w(X,Y) = g(X,JY)$$

It is easy to check that

$$w(X,Y) = w(JX,JY) \quad . \quad \text{Indeed,}$$

$$w(JX,JY) = g(JX,J^2Y) = -\,g(JX,Y) = g(X,JY) = w(X,Y)$$

Definition 3.7. An almost Hermitian manifold (M, g, J) is almost Kahler if the Kahler form is closed , i.e. $dw = 0$. Furthermore, if J is induced by a complex structure then almost complex, almost Hermitian, and almost Kahler manifolds are called complex, Hermitian, and Kahler manifolds.

Let (M^n, g) be a Kahler manifold and R be its curvature tensor. Then we have

$$\begin{aligned} R(X,Y)J &= JR(X,Y) \quad \text{, [36] p.145} \\ R(JX,JY) &= R(X,Y) \\ R(JX,Y) &= -\,R(X,JY) \\ Ric(JX,JY) &= Ric(X,Y) \;=\; \tfrac{1}{2}\, trace(JR(X,JY)) \\ (\nabla_Z Ric)(X,Y) &= (\nabla_X Ric)(Y,Z) \;+\; (\nabla_{JY} Ric)(JX,Z) \end{aligned}$$

Proposition 3.8. *[36] Let (M^n, g, J) be a Kahler manifold. For $n \geq 2$, if M is of constant sectional curvature then M is flat.*

Proof : $R(X,Y)Z \;=\; c\,[\,g(Y,Z)X \;-\; g(X,Z)Y\,]$ for $X, Y, Z \in \Gamma(TM)$

Since $R(JX,JY) \;=\; R(X,Y)$, we get

$$\begin{aligned} R(X,Y)Y &= c\,[\;g(Y,Y)X \;-\; g(X,Y)Y\,] \\ &= c\,[\,g(Y,Z)X \;-\; g(X,Z)Y\,] \\ &= R(JX,JY)Y \qquad \text{which implies} \end{aligned}$$

$$(2n \;-\; 1)\,c\,X \;=\; c\,X$$

Since $n \geq 2$, we have

$$(2n \;-\; 2)\,c \;=\; 0 \quad \Longrightarrow \quad c \;=\; 0. \quad \square$$

In view of this proposition, the notion of constant sectional curvature for Kahler manifolds is no longer essential. Thus, the notion of constant holomorphic sectional curvature in Kahler geometry is the analog of sectional curvature in the Riemannian case.

The *holomorphic sectional curvature* $HR(v, Jv)$ for a unit tangent vector v in a Kahler manifold is the sectional curvature of the plane generated by $\{v, Jv\}$. If $HR(v, Jv)$ does not depend on v, then M is of constant holomorphic sectional curvature. A complex manifold with constant holomorphic sectional curvature is a complex space form which must be locally isometric to one of the following complete, simply connected Kahler manifolds [65] : $\mathbb{C}^n$, $\mathbb{C}P^n(4k^2)$, $\mathbb{C}H^n(-4k^2)$, where $-4k^2$ means that the sectional curvature lies in $[\,-4k^2,\ -k^2\,]$ and likewise for $4k^2$. For these spaces, sectional curvature of the planes spanned by orthonormal vectors u, v is

$$g(R(u,v)v,u) \ = \ \tfrac{1}{4}\, HR\,[1 \ + 3(g(u,Jv))^2\,]\,.$$

We also have the following relation :

holomorphic sectional curvature $\subset$ holomorphic bisectional curvature $\subset$ sectional curvature

Proposition 3.9. *[32] A Kahler manifold of constant holomorphic sectional curvature is an Einstein manifold .*

Remark 3.10. [21] If the holomorphic bisectional curvature is positive (negative) , then so is the Ricci tensor $\sum_{i=1}^{n} R(X_i, JX_i, X, JY)$, where $\{X_1, ..., X_n, JX_1, ..., JX_n\}$ is an orthonormal basis of T_pM .

Remark 3.11. [25] The Fubini-Study metric on $\mathbb{C}P^n$ has positive bisectional curvature.

3.2 Examples of Kahler manifolds

Example 1. **The complex hyperbolic space** $\mathbb{C}H^n$.

Lemma 3.12. *The holomorphic sectional curvature of the complex hyperbolic space* $\mathbb{C}H^n$ *is negatively quarter-pinched.*

Proof : It is well-known [28] that we can define the complex hyperbolic space $\mathbb{C}H^n$ as

$$\mathbb{C}H^n \;\cong\; SU(n,1)\;/\;S(U(n)\times\, U(1))$$

Let $p = \mathbb{C}$ be the first coordinate axis. The isotropy group is given by $S(U(n)\times U(1))$ which are the matrices in $U(n)\times U(1)$ of determinant 1. This group is naturally isomorphic to $U(n)$ via the map

$$B \;\longmapsto\; \begin{pmatrix} B & 0 \\ 0 & -\,trace\;B \end{pmatrix}$$

The involution that makes $\mathbb{C}H^n$ symmetric is given by the conjugation by

$$S \;=\; \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix}.$$

The canonical decomposition of the Lie algebra is

$$su(n,1) \;=\; u(n) \oplus\; u(1)\; \oplus\; \mathbf{m}, \quad \text{where}$$

$su(n,1)$={ $\begin{pmatrix} z_1 & z_2 \\ z_2^* & z_3 \end{pmatrix}$ | z_1, z_3 skew-Hermitian of order n,1 ; $tr\; z_1{+}z_3 = 0$; $and\; z_2\; arbitrary$}

$$=\left\{\begin{pmatrix} A & z \\ z* & -traceA \end{pmatrix} \mid z = \begin{bmatrix} z^1 \\ : \\ : \\ z^n \end{bmatrix} \in \mathbb{C}^n \ , \ A = -A^*\right\}$$

The inclusion of $u(n)$ in $su(n,1)$ is given by $B \longmapsto \begin{pmatrix} B & 0 \\ 0 & -\ trace\ B \end{pmatrix}$.

Thus we can write elements of $su(n,1)$ as

$$\begin{pmatrix} A & z \\ z* & -traceA \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & -traceA \end{pmatrix} + \begin{pmatrix} 0 & z \\ z* & 0 \end{pmatrix}$$

where we identify the Lie subalgebra $\mathbf{m}$ with $\mathbb{C}^n$ via

$$\mathbf{m} = \left\{\begin{pmatrix} 0 & z \\ z* & 0 \end{pmatrix} \mid \mathbf{z} \in \mathbb{C}^{\mathbf{n}}\right\} \cong \mathbb{C}^{\mathbf{n}}.$$

We can use the following standard inner product

$$< A, B > = -\frac{1}{2}\, trace(AB) = \frac{1}{2}\, trace(AB^*)\ .$$

$$Let \quad z = \begin{bmatrix} z^1 \\ : \\ : \\ z^n \end{bmatrix} \ , \quad w = \begin{bmatrix} w^1 \\ : \\ : \\ w^n \end{bmatrix} \in \mathbb{C}^n\ .$$ We calculate

$$< \begin{pmatrix} 0 & z \\ z* & 0 \end{pmatrix} , \begin{pmatrix} 0 & w \\ w* & 0 \end{pmatrix} > = \frac{1}{2} trace(\begin{pmatrix} 0 & z \\ z* & 0 \end{pmatrix} \begin{pmatrix} 0 & w \\ w* & 0 \end{pmatrix})$$

$$= \frac{1}{2} trace \begin{pmatrix} zw^* & 0 \\ 0 & z^*w \end{pmatrix}$$

$$= \frac{1}{2}(trace(zw^*) \ + \ z^*w)$$

$$= \frac{1}{2}(\ w^*z \ + \ z^*w \ )$$

$$= Re < z, w >$$

where $< z, w >$ is the standard Hermitian inner product on $\mathbb{C}^n$ which is conjugate linear in the second variable. Note that

$$z^*w \ = \ (\bar{z}^1 \ ... \ \bar{z}^n) \begin{pmatrix} w^1 \\ : \\ : \\ w^n \end{pmatrix} \ = \ \sum \bar{z}^k w^k \ = < w, z >$$

$$w^*z \ = \ (\bar{w}^1 \ ... \ \bar{w}^n) \begin{pmatrix} z^1 \\ : \\ : \\ z^n \end{pmatrix} \ = \ \sum \bar{w}^k z^k \ = < z, w >$$

which implies $< z, w > = \overline{< w, z >}$.

The next step is to calculate the Lie bracket on $\mathbf{m}$: let $p \in \mathbb{C}H^n$, then at p we have

$$[z, w] = [\begin{pmatrix} 0 & z \\ z* & 0 \end{pmatrix} \ , \ \begin{pmatrix} 0 & w \\ w* & 0 \end{pmatrix}]$$

$$= \begin{pmatrix} 0 & z \\ z* & 0 \end{pmatrix} \begin{pmatrix} 0 & w \\ w* & 0 \end{pmatrix} - \begin{pmatrix} 0 & w \\ w* & 0 \end{pmatrix} \begin{pmatrix} 0 & z \\ z* & 0 \end{pmatrix}$$

$$= \begin{pmatrix} zw^* - wz^* & 0 \\ 0 & z^*w - w^*z \end{pmatrix}$$

From Corollary 6.3.5 (p.295 [32]) and Lemma 3.2 (p.243 [47]) , since $\mathbb{C}H^n$ is a symmetric space, we get the following for its curvature tensor R.

$R(z,w)w = [w,[z,w]]$

$$= [\begin{pmatrix} 0 & w \\ w* & 0 \end{pmatrix}, \begin{pmatrix} zw^* - wz^* & 0 \\ 0 & z^*w - w^*z \end{pmatrix}]$$

$$= \begin{pmatrix} 0 & w \\ w* & 0 \end{pmatrix} \begin{pmatrix} zw^* - wz^* & 0 \\ 0 & z^*w - w^*z \end{pmatrix} - \begin{pmatrix} zw^* - wz^* & 0 \\ 0 & z^*w - w^*z \end{pmatrix} \begin{pmatrix} 0 & w \\ w* & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & w(z^*w - w^*z) - (zw^* - wz^*)w \\ w^*(zw^* - wz^*) - (z^*w - w^*z)w^* & 0 \end{pmatrix}$$

We observe the following.

$[w^*(zw^* - wz^*) - (z^*w - w^*z)w^*]^*$

$= (zw^* - wz^*)^*w - w(z^*w - w^*z)^*$

$= (wz^* - zw^*)w - w(w^*z - z^*w)$

$= w(z^*w - w^*z) - (zw^* - wz^*)w$

Therefore the identification $\mathbf{m} \cong \mathbb{C}^\mathbf{n}$ yields

$$R(z,w)w = w(z^*w - w^*z) - (zw^* - wz^*)w$$

To compute the sectional curvature, we choose an orthonormal basis $\{z, w\}$ of a plane where $|z|^2 = |w|^2 = 1$ and $Re < z, w >= 0$. Then the sectional curvature of the plane spanned by $\{z, w\}$ is given by

$$sect(z,w) = < R(z,w)w, z >$$

$$= < w(z^*w - w^*z) - (zw^* - wz^*)w \ , \ z >$$

$$= z^*w(z^*w - w^*z) - z^*(zw^* - wz^*)w$$

$$= z^*wz^*w - z^*ww^*z - z^*zw^*w + z^*wz^*w$$

$$= 2\, z^*wz^*w - z^*ww^*z - 1$$

$$= 2\, Re^2 < w,z > - 2\, Im^2 \ < w,z >$$

$$+ 4\, i\, Re < w,z > Im < w,z > \ - Re^2 < w,z > - Im^2 < w,z > -1$$

$$= Re^2 < w,z > - 3\, Im^2 \ < w,z >$$

$$+ 4\, iRe < w,z > Im < w,z > - 1$$

$$= - 3\, Im^2 < w,z > - 1 \quad , \quad since\ Re < w,z > = 0$$

Thus it is easy to see that

if $< z, w >= 0$ then sect is equal to -1 , and

if $w = i\, z$ then sect is equal to -4 .

Since $0 \le |Im < w,z > | \le 1$, it folows that all other sectional curvatures lie between $[-4, -1]$, i.e. $\mathbb{C}H^n$ is negatively quarter-pinched. $\square$

Remark 3.13. Let D^n be the open unit ball in $\mathbb{C}^n$ defined by

$D^n = \{ (z^1, ..., z^n) \mid \sum z^\alpha \bar{z}^\alpha < 1 \}$. Set

$$w = 4\, i\, \partial\bar{\partial}\, (1 - \sum z^\alpha \bar{z}^\alpha) .$$

Then the associated metric g is

$$ds^2 = 4\, \frac{(1-\sum z^\alpha \bar{z}^\alpha)(\sum dz^\alpha d\bar{z}^\alpha) + (\sum \bar{z}^\alpha dz^\alpha)(\sum z^\alpha d\bar{z}^\alpha)}{(1-\sum z^\alpha \bar{z}^\alpha)^2}$$

It is well-known that the complex hyperbolic space $\mathbb{C}H^n$ can be identified with D^n [36] and thus its Kahler metric is this metric.

Example 2. **The complex projective space** $\mathbb{C}P^n$.

Consider the complex vector space $\mathbb{C}^{n+1}$. A complex linear subspace of complex dimension1 in $\mathbb{C}^{n+1}$ is a complex line. Define

$$\begin{aligned} \mathbb{C}P^n \; &:= \; \text{the space of all complex lines in } \; \mathbb{C}^{n+1} \\ &:= \; (\, \mathbb{C}^{n+1} - \{0\} \,) \,/\, \mathbb{C}^* \;\; \text{, where } \mathbb{C}^* \text{ acts by multiplication on } \mathbb{C}^{n+1} \\ &:= \; (\, \mathbb{C}^{n+1} - \{0\} \,) \,/ \, \sim \;\; \text{, where } \; z \sim \; w \; \text{ iff } \; \exists\, \lambda \in \mathbb{C}^* = \mathbb{C} - \{0\} \end{aligned}$$

such that $w = \lambda z$. Two points of $\mathbb{C}^{n+1} - \{0\}$ are equivalent iff they are complex linearly dependent, i.e. they lie on the same line. Only the origin [0,...,0] does not define a point in $\mathbb{C}P^n$. Denote [z] = the equivalence class of z. Write

$$z \; = \; (z^0, ..., z^n) \in \mathbb{C}^{n+1} \; .$$

The standard open covering of $\mathbb{C}P^n$ is given by the n+1 open subsets

$$\begin{aligned} U_i \; &= \; \{ \; [z] = [z^0, ..., z^n] \mid z^i \neq 0 \; \} \; \subset \; \mathbb{C}P^n \\ &= \text{the space of all lines not contained in the complex hyperplane } \{z^i = 0\} \end{aligned}$$

If $\mathbb{C}P^n$ is endowed with the quotient topology via

$$\pi \; : \; \mathbb{C}^{n+1} \backslash \{0\} \; \longrightarrow \; (\, \mathbb{C}^{n+1} \backslash \{0\} \,) \,/\, \mathbb{C}^*$$

then the U_i 's are indeed open and we obtain a bijection

$$\phi_i \;\; : \;\; U_i \longrightarrow \mathbb{C}^n$$

$$\phi_i \left([z^0, ..., z^n] \right) \; = \; \left(\tfrac{z^0}{z^i}, ..., \tfrac{\hat{z}^i}{z^i}, ..., \tfrac{z^n}{z^i}\right) \; = \; (w^1, ..., w^n) \in \mathbb{C}^n$$

Thus, $\mathbb{C}P^n$ becomes a C^∞ manifold since the transition maps are diffeomorphisms

$$\phi_j \circ \phi_i^{-1} \; : \; \phi_i(U_i \cap U_j) \; = \; \{ z = (z^1, ..., z^n) \in \mathbb{C}^n \mid z^j \neq 0 \} \; \longrightarrow \; \phi_j(U_i \cap U_j)$$

$$\phi_j \circ \phi_i^{-1}(z^1, .., z^n) = \phi_j(\, [z^1, .., z^i, 1, z^{i+1}, .., z^n]) = \left(\tfrac{z^1}{z^j}, .., \tfrac{z^i}{z^j}, \tfrac{1}{z^j}, \tfrac{z^{i+1}}{z^j}, .., \tfrac{\hat{z}^j}{z^j}, .., \tfrac{z^n}{z^j}\right)$$

They are also holomorphic : indeed, write $z^k = x^k + iy^k$, then for

$$\tfrac{\partial}{\partial z^k} \; = \; \tfrac{1}{2}\left(\tfrac{\partial}{\partial x^k} \; - \; i\tfrac{\partial}{\partial y^k}\right) \quad \text{and} \quad \tfrac{\partial}{\partial \bar{z}^k} \; = \; \tfrac{1}{2}\left(\tfrac{\partial}{\partial x^k} \; + \; i\tfrac{\partial}{\partial y^k}\right) \quad \text{we have}$$

$$\tfrac{\partial}{\partial \bar{z}^k} \;\; \phi_j \circ \phi_i^{-1}(z^1, ..., z^n) \; = \; 0 \quad , \quad for \; k = 1, ..., n.$$

This shows that $\mathbb{C}P^n$ is a complex manifold [32] .

Consider the (n+1)-tuple

$$(z^0, ..., z^n)$$

satisfying the restriction that not all z^j vanish identically; as *homogeneous coordinates* $[z] = [z^0, ..., z^n]$. These are not coordinates in the usual sense because a point in an n-dimensional manifold here is described by (n+1) complex numbers. The coordinates are defined only up to muliplication with an arbitrary nonvanishing complex number λ

$$[z^0, ..., z^n] \; = \; [\lambda z^0, ..., \lambda z^n]$$

This fact is expressed by the adjective "homogeneous". The coordinates $(z^1, ..., z^n)$ defined by the charts ϕ_i are Euclidean coordinates.

The vector space structure of $\mathbb{C}^{n+1}$ induces an analogous structure on $\mathbb{C}P^n$ by homogenization : each linear inclusion $\mathbb{C}^{k+1} \subset \mathbb{C}^{n+1}$ induces an inclusion $\mathbb{C}P^k \subset \mathbb{C}P^n$. The image of such an iclusion is called a linear subspace. The image of a hyperplane in $\mathbb{C}^{n+1}$ is again called a hyperplane and the image of a 2-dimesional space $\mathbb{C}^2$ is a line.

Instead of considering $\mathbb{C}P^n$ as a quotient of $\mathbb{C}^{n+1} - \{0\}$ we may also view it as a compactification of $\mathbb{C}^n$. We say that the hyperplane H at infinity is added to $\mathbb{C}^n$: the inclusion

$\mathbb{C}^n \longrightarrow \mathbb{C}P^n$ is given by

$(z^1, ..., z^n) \longmapsto [1, z^1, ..., z^n]$:= H := a hyperplane $\mathbb{C}P^{n-1}$

Thus, we have a disjoint union of complex Euclidean spaces :

(*) $\mathbb{C}P^n \; = \; \mathbb{C}^n \cup \mathbb{C}P^{n-1} \; = \; \mathbb{C}^n \cup \mathbb{C}^{n-1} \cup ... \cup \mathbb{C}^0$

Topologically,

$\mathbb{C}P^n$ = the union of (n+1) cells of real dimension 0, 2, ..., 2n .

By the Mayer-Vietoris sequence, we may easily compute the cohomology of $\mathbb{C}P^n$ from (*) . In order to represent $\mathbb{C}P^n$ as the union of 2 open sets as required for the application of this sequence, we put

$$U \;=\; \mathbb{C}^n$$

$$V \;=\; \{z \in \mathbb{C}^n \mid \; ||z||^2 \;=\; z^j \bar{z}^j > 1\} \,\cup\, \mathbb{C}P^{n-1}$$

Then V has $\mathbb{C}P^{n-1}$ as a deformation retract, i.e.

$$r_t : V \longrightarrow V \;,\; r_t(z) \;=\; tz \quad \text{for } z \in \mathbb{C}^n \quad \text{where } t \text{ runs from 1 to } \infty \;,$$

$$r_t(w) \;=\; w \quad \text{for } w \in \mathbb{C}P^{n-1} \;,$$

and $U \cap V$ is homotopically equivalent to the unit sphere S^{2n-1} of $\mathbb{C}^n$.

It follows from (*) that $\mathbb{C}P^1$ is diffeomorphic to S^2 [32]. Indeed, recall that S^2 may be described via stereographic projection from the north and south poles by 2 charts with image $\mathbb{C}$ and the transition map

$$z \longmapsto \tfrac{1}{z}$$

which is actually the transition map

$$[1, z] \longmapsto [\tfrac{1}{z}, 1] \quad \text{of } \mathbb{C}P^1 \;.$$

To introduce a metric on $\mathbb{C}P^n$, let

$$\pi \;:\; \mathbb{C}^{n+1} - \{0\} \;\longrightarrow\; \mathbb{C}P^n \qquad \text{be the standard projection}$$

and consider the holomorphic map

$$Z \;:\; U \subset \mathbb{C}P^n \;\longrightarrow\; \mathbb{C}^{n+1} - \{0\} \qquad \text{which is a lift of } Id_{\,\mathbb{C}P^n}$$

i.e. a holomorphic map with $\pi \circ Z \;=\; Id_{\,\mathbb{C}P^n}$. We put

$$w \;=\; \tfrac{i}{2}\, \partial\bar{\partial}\, log\, ||Z||^2 \quad \text{and denote} \quad \partial = \tfrac{\partial}{\partial Z^j}\, dZ^j \;,\; \bar{\partial} = \tfrac{\partial}{\partial \bar{Z}^k}\, d\bar{Z}^k \;.$$

If $Z' \;:\; U \longrightarrow \mathbb{C}^{n+1} - \{0\}$ is another lift, we have

$$Z' \;=\; \phi\, Z \;\;, \text{ where } \phi \text{ is a nowhere vanishing holomorphic function.}$$

Since $\bar{\partial}\, log\, \phi \,=\, 0 \,=\, \partial\, log\, \bar{\phi}$ with ϕ being holomorphic and nowhere vanishing, we get

$$\begin{aligned}\tfrac{i}{2}\,\partial\bar{\partial}\, log\, \|Z'\|^2 \;&=\; \tfrac{i}{2}\,\partial\bar{\partial}\,(\, log\, \|Z\|^2 \,+\, log\,\phi \,+\, log\,\bar{\phi})\\ &=\; w \,+\, \tfrac{i}{2}\,(\,\partial\bar{\partial}\, log\,\phi \,-\, \bar{\partial}\partial\, log\,\bar{\phi}\,)\\ &=\; w\,.\end{aligned}$$

Thus, w does not depend on the choice of charts and defines a 2-form on $\mathbb{C}P^n$.

Next, we want to represent w in local coordinates : let

$$U_0 \;=\; \{\, [Z^0, ..., Z^n] \mid Z^0 \neq 0\,\}$$

$Z \;=\; (1, z^1, ..., z^n)$ which is a lift of π over U_0 , since $z^i \,=\, \frac{Z^i}{Z^0}\ on\ U_o$.

Then,

$$\begin{aligned} w \;&=\; \tfrac{i}{2}\,\partial\bar{\partial}\, log\,(1 + z^j\bar{z}^j)\\ &=\; \tfrac{i}{2}\,\partial\,(\,\tfrac{z^j d\bar{z}^j}{1+z^k\bar{z}^k}\,)\\ &=\; \tfrac{i}{2}\,[\,\tfrac{dz^j\wedge d\bar{z}^j}{1+z^k\bar{z}^k} \,-\, \tfrac{z^k\bar{z}^j\ dz^j\wedge d\bar{z}^k}{(1+z^l\bar{z}^l)^2}\,]\end{aligned}$$

At [1,0,...,0] , we get

$$\begin{aligned} w \;&=\; \tfrac{i}{2}\, dz^j\wedge d\bar{z}^j\\ &=\; \tfrac{i}{2}\,[\,(dx^j + idy^j)\wedge(dx^j - idy^j)\,]\\ &=\; \tfrac{i}{2}\,[\, idy^j\wedge dx^j \,-\, idx^j\wedge dy^j\,]\\ &=\; \tfrac{i}{2}\,(\,-2i(dx^j\wedge dy^j))\\ &=\; dx^j\wedge dy^j\,.\end{aligned}$$

Thus, w is positive definite at the point [1,0,...,0] . Since w is invariant under the operation of U(n+1) on $\mathbb{C}P^n$, it is positive definite everywhere. We generalize the object w above in the following.

Let M be a complex manifold with local coordinates $z = (z^1, ..., z^n)$. A Hermitian metric on M is given by an expression of the form

$$h_{j\bar{k}}(z)\, dz^j \otimes d\bar{z}^k$$

where $h_{j\bar{k}}(z)$ depends smoothly on z and is positive definite and Hermitian for every z . The expression

$$\frac{i}{2}\, h_{j\bar{k}}(z)\, dz^j \wedge d\bar{z}^k$$

is called the Kahler form of the Hermitian metric.

A hermitian metric $h_{j\bar{k}}(z)\, dz^j \otimes d\bar{z}^k$ is called a Kahler metric, if $\forall z \in M$, $\exists$ a neigborhood U of z and a function u : U $\longrightarrow \mathbb{R}$ such that

$$\frac{i}{2}\, h_{j\bar{k}}(z)\, dz^j \wedge d\bar{z}^k \;=\; \partial\bar{\partial}u.$$

Then $\partial\bar{\partial}u$ is called the Kahler form. The 2-form w above defines a Kahler metric on $\mathbb{C}P^n$ called the Fubini-Study metric [36], which has many special properties.

To obtain the Fubini-Study metric, we consider the homogeneous coordinate system $\{z^0, z^1, ..., z^n\}$. For every j , let U_j be an open subset of $\mathbb{C}P^n$ defined by $z^j \neq 0$. Set

$$t_j^k \;=\; \frac{z^k}{z^j} \;\;,\;\; j,k \;=\; 0,1,...,n$$

On each U_j , take $\{\; t_j^0, ..., \hat{t}_j^i, ..., t_j^n \;\}$ as a local coordinate system and consider the function

$$f_j \;=\; \sum_{i=0}^n t_j^i\, \bar{t}_j^i \;=\; \sum_{i=0}^n (t_k^i\, \bar{t}_k^i)\, t_j^k\, \bar{t}_j^k \;=\; f_k\, t_j^k\, \bar{t}_j^k \quad on \;\; U_j \cap U_k$$. Then

$$log\; f_j \;=\; log\; f_k \;+\; log\; t_j^k \;+\; \overline{log t}_j^k\; .$$

Since t_j^k is holomorphic in $U_j \cap U_k$, we have

$$\bar{\partial}\, log\; t_j^k \;=\; 0 \quad , \quad \partial\, \overline{log t}_j^k \;=\; \bar{\partial}\, \overline{log t}_j^k \;=\; 0$$

From $\partial\bar{\partial} \;=\; -\,\bar{\partial}\partial$, we obtain on $U_j \cap U_k$

$$\partial\bar{\partial}\, log f_j \;=\; \partial\bar{\partial}\, log f_k$$

On each U_j , setting

$$w \;=\; -4\, i\, \partial\bar{\partial}\, log f_j$$

gives a globally defined closed (1,1)-form w on $\mathbb{C}P^n$.

On the other hand,

$$f_0 \;=\; \sum_{j=0}^n t_0^j\, \bar{t}_0^j \;=\; 1 \;+\; \sum_{\alpha=1}^n t^\alpha\, \bar{t}^\alpha$$

$w \; = \; - \, 4 \, i \, \sum_{\alpha,\beta=1}^{n} \frac{\partial^2 log f_0}{\partial t^\alpha \; \partial t^\beta} \, dt^\alpha \wedge d\bar{t}^\beta \;\;, \;\; where \; t^\alpha \; = \; t_0^\alpha \;, \; \alpha = 1, ..., n$. Thus

$w \; = \; - \, 4 \, i \, \frac{\sum dt^\alpha \wedge d\bar{t}^\alpha \; + \; \sum t^\alpha \bar{t}^\alpha \sum dt^\alpha \wedge d\bar{t}^\alpha \; - \; \sum \bar{t}^\alpha dt^\alpha \wedge \sum t^\alpha d\bar{t}^\alpha}{(\; 1 \; + \; \sum t^\alpha \bar{t}^\alpha)^2}$

The metric tensor g associated with this Kahler form w is indeed the Fubini-Study metric given by

$$ds^2 \; = \; 4 \; \frac{(1+\sum t^\alpha \bar{t}^\alpha)(\sum dt^\alpha d\bar{t}^\alpha) \; - \; (\sum \bar{t}^\alpha dt^\alpha)(\sum t^\alpha d\bar{t}^\alpha)}{(\; 1+\sum t^\alpha \bar{t}^\alpha)^2} \; .$$

Example 3. The complex Euclidean space $\mathbb{C}^n$ with metric $ds^2 = \sum_{j=1}^{n} \; dz^j d\bar{z}^j$.

The fundamental 2-form w is given by $\quad w \; = \; - \, i \, \sum_{j=1}^{n} \; dz^j \wedge d\bar{z}^j$

which is clearly closed and so the metric defines a Kahler structure on $\mathbb{C}^n$. Thus, $\mathbb{C}^n$ is a complete, simply connected flat Kahler manifold.

Example 4. Any complex 1-dimensional manifold Σ i.e. any Riemann surface is automatically a Kahler manifold since dw is a 3-form and therefore vanishes on the real 2-dimensional manifold Σ . Any complex submanifold N of a Kahler manifold M is a Kahler manifold. In particular, all complex projective manifolds, i.e. those that admit a holomorphic embedding into some complex projective space, are Kahler manifolds.

3.3 Noncompact Kahler manifolds

Definition 3.14. [57] A differential form ω satisfying

$$\liminf_{r\to\infty} \frac{1}{r^2} \int_{B(x_0,r)} |w|^p \, dv_g < \infty ,$$

for some real number p , for some point $x_0 \in$ M is said to be 2-balanced. In particular, every L^p form is 2-balanced. In particular, a differential form ω satisfying the above inquality is 2-finite and every L^p form is 2-finite ([56]). More importantly, 2-finite implies 2-balanced.

Remark 3.15. [55] Every 2-balanced, q> 0, holomorphic function f : M $\to$ $\mathbb{C}$ on a complete noncompact Kahler manifold is constant.

Remark 3.16. The case f being 2-finite, q> 0 and the case f being 2-moderate, q $>$ 0 with F $\in \mathcal{F}$ being nondecreasing are due to Karp [34].

Theorem 3.17. *[57] Let M be a complete noncompact manifold of nonnegative Ricci curvature. Then every harmonic 1-form or harmonic (n-1) form on M satisfying*

$$\liminf_{r\to\infty} \frac{1}{r^2} \int_{B(x_0,r)} |w|^p \, dv_g < \infty , \quad p > 2$$

for some $x_0 \in M$ is parallel. If the Ricci curvature is positive at a point, then every harmonic 1-form or harmonic (n-1) form satisfying the above condition vanishes identically. Furthermore, for p $>$ 1 , every L^p harmonic 1-form or harmonic (n-1) form on M vanishes.

Theorem 3.18. *Let (M^n, g, J) be a complete noncompact Kahler manifold. If at each point of M the sum of any q eigenvalues of the Ricci tensor is nonnegative then any 2-finite harmonic form of type (0,q) or (q,0) is parallel. In addition, if M has infinite volume or the sums of any q eigenvalues of the Ricci tensor are all positive at some point of M then any such form vanishes.*

Proof : Let x$\in$ M and α be a 2-finite, $p > 1$, harmonic form of type $(0, q)$, where $0 < q < n$. Choose an orthonormal frame field $\{V_1, ..., V_n, \bar{V}_1, ..., \bar{V}_n\}$ and its dual orthonormal coframe field $\{w^1, ..., w^n, \bar{w}^1, ..., \bar{w}^n\}$ whereV_i are complex vector fields of type (1,0). Since the calculation is local and does not depend on the choice of frames, we can choose these frames to be normal at x. Furthermore, at everypoint of a Kahler manifold, there exists a local complex coordinate system which is normal at the given point [36], i.e. let $z = (z^1, ..., z^n)$ be the local complex coordinate system normal at a given point x $\in M$. Then we have

$$z(x) = 0$$

$$g_{i\bar{j}}(x) = \delta_{ij}$$

$$dg_{i\bar{j}}(x) = 0 \quad i.e. \quad DV_i = Dw^i = 0 \,,$$

where D is the Levi-Civita connection on M.

The complex structure J on M induces the following decompositions :

$$d \; = \partial + \bar{\partial} \qquad \text{where} \qquad \partial = \sum w^i \wedge D_{V_i} \quad , \quad \bar{\partial} = \sum \bar{w}^i \wedge D_{\bar{V}_i}$$

$$d^* = \partial^* + \bar{\partial}^* \qquad\qquad \partial^* = -\sum \iota(V_i) D_{\bar{V}_i} \quad , \quad \bar{\partial}^* = -\sum \iota(\bar{V}_i) D_{V_i}$$

and $\iota(V_i)$ is the interior multiplication (i.e. contraction) with the vector V_i .

The complex Laplacian is then given by two equivalent formulas via conjugation :

$$\Box_{\partial} \;=\; \partial\partial^* + \partial^*\partial \;=\; \overline{\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}} \;=\; \bar{\Box}_{\bar{\partial}}$$

Thus, we get

$$\begin{aligned}
\partial\partial^* &= \sum w^i \wedge D_{V_i}(-\sum \iota(V_j)D_{\bar{V}_j}) \;=\; \sum_{ij} w^i \wedge \iota(V_j)D_{V_i}D_{\bar{V}_j} \\
\partial^*\partial &= -\sum \iota(V_j)D_{\bar{V}_j}(\sum w^i \wedge D_{V_i}) \;=\; -\sum_{ij} \iota(V_j)[w^i \wedge D_{\bar{V}_j}D_{V_i}] \\
&= -\sum_{ij} \iota(V_j)w^i \wedge D_{\bar{V}_j}D_{V_i} \;+\; \sum_{ij} w^i \wedge \iota(V_j)D_{\bar{V}_j}D_{V_i}
\end{aligned}$$

Since $\sum R_{V_i\bar{V}_i} = -\sum D_{V_i}D_{\bar{V}_i} + \sum D_{\bar{V}_iV_i}$, we obtain

$$\begin{aligned}
\Box_\partial &= -\sum_i D_{\bar{V}_i}D_{V_i} \;+\; \sum_{ij} w^i \wedge \iota(V_j)R_{V_i\bar{V}_j} \\
&= -\sum_i D_{V_i}D_{\bar{V}_i} \;-\; \sum_i R_{V_i\bar{V}_i} \;+\; \sum_{ij} w^i \wedge \iota(V_j)R_{V_i\bar{V}_j}
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\bar{\partial}\bar{\partial}^* &= \sum \bar{w}^i \wedge D_{\bar{V}_i}(-\sum \iota(\bar{V}_j)D_{V_j}) \;=\; -\sum_{ij} \bar{w}^i \wedge \iota(\bar{V}_j)D_{\bar{V}_i}D_{V_j} \\
\bar{\partial}^*\bar{\partial} &= -\sum \iota(\bar{V}_j)D_{V_j}(\sum \bar{w}^i \wedge D_{\bar{V}_i}) \\
&= -\sum_{ij} \iota(\bar{V}_j)[D_{V_j}\bar{w}^i \wedge D_{\bar{V}_i} \;+\; \bar{w}^i \wedge D_{V_j}D_{\bar{V}_i}] \\
&= -\sum D_{V_i}D_{\bar{V}_i} \;+\; \sum \bar{w}^i \wedge \iota(\bar{V}_j)D_{V_j}D_{\bar{V}i}
\end{aligned}$$

and

$$\Box_{\bar{\partial}} = -\sum D_{V_i}D_{\bar{V}_i} \;-\; \sum \bar{w}^i \wedge \iota(\bar{V}_j)R_{V_j\bar{V}_i}$$

To show that $\Box_\partial = \bar{\Box}_{\bar{\partial}}$, we a pply conjugation :

$$\begin{aligned}
\bar{\Box}_{\bar{\partial}} &= \overline{-\sum D_{V_i}D_{\bar{V}_i} \;-\; \sum \bar{w}^i \wedge \iota(\bar{V}_j)R_{V_j\bar{V}_i}} \\
&= -\sum_i D_{\bar{V}_i}D_{V_i} \;-\; \sum_{ij} w^i \wedge \iota(V_j)R_{\bar{V}_jV_i} \\
&= -\sum_i D_{V_i}D_{\bar{V}_i} \;-\; \sum_i R_{V_i\bar{V}_i} \;+\; \sum_{ij} w^i \wedge \iota(V_j)R_{V_i\bar{V}_j} \\
&= \Box_\partial
\end{aligned}$$

here we have used the skew- symmetric property and the definition of the curvature tensor.

__**Remark 1**__ : Let f = $< \alpha, \alpha >$, where α is a smooth 1-form on M.

We want to show that $f^{\frac{1}{2}}$ is subharmonic, i.e. to show that

$$\Box f^{\frac{1}{2}} \geq 0 .$$

However, the function $f^{\frac{1}{2}}$ may not be C$^\infty$ at the zeros of the differential form α and thus we will show that $\Box(f+\epsilon)^{\frac{1}{2}} \geq 0$ instead. Once we have shown that the function $(f+\epsilon)^{\frac{1}{2}}$ is subharmonic for each $\epsilon > 0$ then we can conclude that $f^{\frac{1}{2}}$ is also subharmonic since $f^{\frac{1}{2}}$ is the limit of subharmonic functions $(f+\epsilon)^{\frac{1}{2}}$ uniformly on compact sets.

__**Remark 2**__ : If a C^∞ positive function h is subharmonic then so is h^p for $p \geq 1$. Indeed,

$$\Box h^p = div(\nabla h^p) = div(ph^{p-1}\nabla h) = ph^{p-1}\Box h + p(p-1)h^{p-2}|\nabla h|^2 \geq 0.$$

We compute

$$\begin{aligned}
&\Box(<\alpha,\alpha>+\epsilon)^{\frac{1}{2}} \\
&= \textstyle\sum_{i=1}^n V_i\bar{V}_i(<\alpha,\alpha>+\epsilon)^{\frac{1}{2}} \\
&= \textstyle\sum_{i=1}^n V_i\,[\,\frac{1}{2}(<\alpha,\alpha>+\epsilon)^{\frac{-1}{2}}(<D_{\bar{V}_i}\alpha,\alpha>+<\alpha,D_{V_i}\alpha>)] \\
&= \textstyle\sum (\frac{-1}{4})(<\alpha,\alpha>+\epsilon)^{\frac{-3}{2}}(<D_{V_i}\alpha,\alpha>+<\alpha,D_{\bar{V}_i}\alpha>)(<D_{\bar{V}_i}\alpha,\alpha> \\
&+ <\alpha,D_{V_i}\alpha>) + \textstyle\sum \frac{1}{2}(<\alpha,\alpha>+\epsilon)^{\frac{-1}{2}}[<D_{V_i}D_{\bar{V}_i}\alpha,\alpha>+<D_{\bar{V}_i}\alpha,D_{\bar{V}_i}\alpha> \\
&+<D_{V_i}\alpha,D_{V_i}\alpha>+<\alpha,D_{\bar{V}_i}D_{V_i}\alpha>] \\
&= \textstyle\sum (\frac{1}{4})(<\alpha,\alpha>+\epsilon)^{\frac{-3}{2}}\,[-<D_{V_i}\alpha,\alpha><D_{\bar{V}_i}\alpha,\alpha> \\
&-<\alpha,D_{V_i}\alpha><\alpha,D_{\bar{V}_i}\alpha>-<\alpha,D_{\bar{V}_i}\alpha><D_{\bar{V}_i}\alpha,\alpha> \\
&-<D_{V_i}\alpha,\alpha><\alpha,D_{V_i}\alpha> +2\,(<\alpha,\alpha>+\epsilon)(\,||D_{\bar{V}_i}\alpha||^2+||D_{V_i}\alpha||^2\,)] \\
&+\textstyle\sum \frac{1}{2}(<\alpha,\alpha>+\epsilon)^{\frac{-1}{2}}(<D_{V_i}D_{\bar{V}_i}\alpha,\alpha>+<\alpha,D_{\bar{V}_i}D_{V_i}\alpha>)
\end{aligned}$$

Thus, if the expression inside the square brackets in the last equality is nonnegative then

$$\begin{aligned}
&\Box(<\alpha,\alpha>+\epsilon)^{\frac{1}{2}} \\
&\geq \ \tfrac{1}{2}(<\alpha,\alpha>+\epsilon)^{\frac{-1}{2}}(\textstyle\sum <D_{V_i}D_{\bar{V}_i}\alpha,\alpha>+\sum <\alpha,D_{\bar{V}_i}D_{V_i}\alpha>)
\end{aligned}$$

Next we will show that this expression inside the square brackets is indeed nonnegative. Recall that for a complex number z = x + iy , we have

$$z\bar{z} = |z|^2 \quad where \;\; |z| = \sqrt{x^2+y^2}$$

and $\quad Re\ z \leq |Re\ z| \leq |z|.$

Applying these properties of complex numbers yields

$$< \alpha, D_{\bar{V}_i}\alpha >< D_{\bar{V}_i}\alpha, \alpha > \; = \; < \alpha, D_{\bar{V}_i}\alpha > \overline{< \alpha, D_{\bar{V}_i}\alpha >} \; = \; | < \alpha, D_{\bar{V}_i}\alpha > |^2$$

$$< D_{V_i}\alpha, \alpha >< \alpha, D_{V_i}\alpha > \; = \; \; | < D_{V_i}\alpha, \alpha > |^2$$

Furthermore,

$$- < D_{V_i}\alpha, \alpha >< D_{\bar{V}_i}\alpha, \alpha > - < \alpha, D_{V_i}\alpha >< \alpha, D_{\bar{V}_i}\alpha > + ||\alpha||^2(||D_{\bar{V}_i}\alpha||^2 + ||D_{V_i}\alpha||^2)$$

$$= - < D_{V_i}\alpha, \alpha >< D_{\bar{V}_i}\alpha, \alpha > - \overline{< D_{V_i}\alpha, \alpha >< D_{\bar{V}_i}\alpha, \alpha >} + \; ||\alpha||^2(||D_{\bar{V}_i}\alpha||^2 + ||D_{V_i}\alpha||^2)$$

$$= -2Re < D_{V_i}\alpha, \alpha >< D_{\bar{V}_i}\alpha, \alpha > + ||\alpha||^2(||D_{\bar{V}_i}\alpha||^2 + ||D_{V_i}\alpha||^2)$$

$$\geq \;\; | < D_{V_i}\alpha, \alpha > | \; | < D_{\bar{V}_i}\alpha, \alpha > | + \; ||\alpha||^2(||D_{\bar{V}_i}\alpha||^2 + ||D_{V_i}\alpha||^2)$$

$$\geq \;\; -2 \, ||D_{V_i}\alpha|| \; ||\alpha|| \; ||D_{\bar{V}_i}\alpha|| \; ||\alpha|| + ||\alpha||^2(||D_{\bar{V}_i}\alpha||^2 + ||D_{V_i}\alpha||^2)$$

$$= \;\; ||\alpha||^2(||D_{\bar{V}_i}\alpha||^2 + ||D_{V_i}\alpha||^2) - 2 \; ||D_{\bar{V}_i}\alpha|| \; ||D_{V_i}\alpha||)$$

$$= \;\; ||\alpha||^2(||D_{\bar{V}_i}\alpha|| - ||D_{V_i}\alpha||)^2$$

$$\geq \;\; 0$$

where we have used the Cauchy-Schwarz inequality

$$- \, | < v, w > |^2 \; + \; ||v||^2 \; ||w||^2 \; \geq \;\; 0$$

to obtain the following inequalities

$$- \, | < D_{V_i}\alpha, \alpha > |^2 \; + \; ||D_{V_i}\alpha||^2 \; ||\alpha||^2 \;\; \geq \;\; 0$$

$$- \, | < \alpha, D_{\bar{V}_i}\alpha > |^2 \; + \; ||\alpha||^2 \; ||D_{\bar{V}_i}\alpha||^2 \;\; \geq \;\; 0$$

Thus, we can rewrite the expression inside the square brackets as follows

$$[- < D_{V_i}\alpha, \alpha >< D_{\bar{V}_i}\alpha, \alpha > - < \alpha, D_{V_i}\alpha >< \alpha, D_{\bar{V}_i}\alpha > + ||\alpha||^2(||D_{\bar{V}_i}\alpha||^2 + ||D_{V_i}\alpha||^2)$$

$$- < \alpha, D_{\bar{V}_i}\alpha >< D_{\bar{V}_i}\alpha, \alpha > - < D_{V_i}\alpha, \alpha >< \alpha, D_{V_i}\alpha > + ||\alpha||^2 \; ||D_{\bar{V}_i}\alpha||^2$$

$+\,||D_{V_i}\alpha||^2\;||\alpha||^2 + 2\epsilon\;(||D_{\bar{V}_i}\alpha||^2 + ||D_{V_i}\alpha||^2)] \quad \geq \quad 0$

Hence, we have proved that

$$\Box(<\alpha,\alpha> +\epsilon)^{\frac{1}{2}} \;\geq\; \tfrac{1}{2}(<\alpha,\alpha> +\epsilon)^{\frac{-1}{2}}(<\sum D_{V_i}D_{\bar{V}_i}\alpha,\alpha> + <\alpha,\sum D_{\bar{V}_i}D_{V_i}\alpha>)$$

The next step is to show that the right hand side of this inequality is nonnegative.

Since $R_{V_i\bar{V}_j}\alpha$ preserves type and $\iota(V_j)\alpha = 0$, for al form α of type (0,q), we get

$$\textstyle\sum_{ij} w^i \wedge \iota(V_j)R_{V_i\bar{V}_j}\alpha \;=\; 0$$

Thus, from previous calcululation and the above observation we obtain

$$\Box\alpha \;=\; -\sum D_{V_i}D_{\bar{V}_i}\alpha \;-\; \sum R_{V_i\bar{V}_i}\alpha$$

which implies

$$\sum D_{V_i}D_{\bar{V}_i}\alpha \;=\; -\,\Box\alpha \;-\; \sum R_{V_i\bar{V}_i}\alpha$$

$$\sum D_{\bar{V}_i}D_{V_i}\alpha \;=\; -\,\Box\alpha\;, \qquad \text{since} \quad R_{V_i\bar{V}_i} \;=\; -\,D_{V_i}D_{\bar{V}_i} \;+\; D_{\bar{V}_i}D_{V_i}.$$

If α is harmonic then

$$<\sum D_{V_i}D_{\bar{V}_i}\alpha,\alpha> \;+\; <\alpha,\sum D_{\bar{V}_i}D_{V_i}\alpha> \;=\; -\; <\sum R_{V_i\bar{V}_i}\alpha,\alpha>$$

Recall that the pointwise Hermitian inner product $<,>$ on forms is defined as folllows :

for the multi-index I = $(i_1, ..., i_q)$ such that $i_1 < ... < i_q$, we write

$$w^I \;=\; w^{i_1} \;\wedge\; ... \wedge\; w^{i_q}$$

$$\bar{w}^J \;=\; \bar{w}^{j_1} \;\wedge ... \;\wedge\; \bar{w}^{j_q}$$

By the property of complex inner product [36],

$$< w^I, w^J > = \;0\; = <\bar{w}^I,\bar{w}^J> \quad and \quad < w^I,\bar{w}^J> = \;\delta^{IJ}$$

Extend $<,>$ to act on forms and define the corresponding norm as follows :

$$<\phi,\bar{\phi}> = \;||\phi||^2 \;\;,\;\; \phi \;\in\; \mathcal{A}^*$$

We claim that $\mathcal{R} = \sum R_{V_i\bar{V}_i}$ is a Hermitian operator. To see this, let ξ, η be covectors of type (0,q) and write

$$\xi = \sum_{|I|=q} \xi_I \, \bar{w}^I \ , \ \eta = \sum_{|J|=q} \eta_J \, \bar{w}^J$$

Then,

$$< \xi, \bar{\eta} > = \sum_{|I|=q} \xi_I \, \eta_I$$

Since $R_{V_i\bar{V}_i} f = f \, R_{V_i\bar{V}_i} 1 = 0$ for any function f, it follows that

$$0 = R_{V_i\bar{V}_i} < \xi, \bar{\eta} > = < R_{V_i\bar{V}_i}\xi, \bar{\eta} > + < \xi, R_{V_i\bar{V}_i}\bar{\eta} >$$

which implies

$$< R_{V_i\bar{V}_i}\xi, \bar{\eta} > = - < \xi, R_{V_i\bar{V}_i}\bar{\eta} > = < \xi, R_{\bar{V}_i V_i}\bar{\eta} > = < \xi, \overline{R_{V_i\bar{V}_i}\eta} >$$

This show that $\mathcal{R}$ is a Hermitian operator and hence it can be diagonalized by some orthonormal basis of eigenvectors of type (0,1), say $\{\bar{W}_1, ..., \bar{W}_n\}$ relative to which the eigenvalues λ_i are real, i.e.

$$\mathcal{R}(\bar{W}_i) = \lambda_i \bar{W}_i$$

Let $\{\theta^1, ..., \theta^n, \bar{\theta}^1, ..., \bar{\theta}^n\}$ be the coframe field dual to

$$\{W_1, ..., W_n, \bar{W}_1, ..., \bar{W}_n\}.$$

Then duality gives

$$0 = \mathcal{R}(\bar{\theta}^j(\bar{W}_j)) = \mathcal{R}(\bar{\theta}^j)\bar{W}_j + \bar{\theta}^j \mathcal{R}(\bar{W}_j) = \mathcal{R}(\bar{\theta}^j)\bar{W}_j + \lambda_j \, \bar{\theta}^j \, \bar{W}_j$$

which gives

$$\mathcal{R}(\bar{\theta}^j) = - \lambda_j \, \bar{\theta}^j$$

For the (0,q) - form $\alpha = \sum_{|I|=q} \alpha_J \, \bar{\theta}^J$, we have

$$\begin{aligned} \mathcal{R}(\bar{\theta}^{j_1} \wedge ... \wedge \bar{\theta}^{j_q}) &= \sum_{k=1}^q \bar{\theta}^{j_1} \wedge ... \wedge \mathcal{R}(\bar{\theta}^{j_k}) \wedge ... \wedge \bar{\theta}^{j_q} \\ &= - (\lambda_{j_1} + ... + \lambda_{j_q}) \, \bar{\theta}^J \quad \text{and thus} \\ \mathcal{R}(\alpha) &= - \sum_{|J|=q} \alpha_J \, (\textstyle\sum_{k=1}^q \lambda_{j_k}) \, \bar{\theta}^J \end{aligned}$$

At a given point x $\in$ M, if $\sum_{k=1}^{q} \lambda_{j_k} \geq 0$ then for all $j_1 < ... < j_q$, it follows that

$$- < \mathcal{R}\alpha, \alpha > = < \sum_{|J|=q} (\sum_{k=1}^{q} \lambda_{j_k})\, \bar{\theta}^J \;,\; \sum_{|I|=q} \alpha_I\, \bar{\theta}^J >$$
$$= \sum_J \alpha_J^2 \,(\sum_{k=1}^{q} \lambda_{j_k})$$
$$\geq 0$$

We have proved that

$\Box(< \alpha, \alpha > + \epsilon)^{\frac{1}{2}} \geq 0$ and hence $< \alpha, \alpha >^{\frac{1}{2}}$ is subharmonic .

Moreover, if $\alpha \neq 0$ *and* $\sum_{k=1}^{q} \lambda_{j_k} > 0$ *then* $\Box(< \alpha, \alpha > + \epsilon)^{\frac{1}{2}} > 0.$

To prove the first assertion we apply Yau's Theorem 1 in [64] as follows:

if $\int_M < \alpha, \alpha >^{\frac{p}{2}} < \infty$ for some $p,\ 1 < p < \infty,$ then

$< \alpha, \alpha > \equiv$ constant for $1 < p < \infty$.

It follows that

$$0 = \Box < \alpha, \alpha > = \sum V_i \bar{V}_i < \alpha, \alpha > = \sum V_i \,[< D_{\bar{V}_i}\alpha, \alpha > + < \alpha, D_{V_i}\alpha >]$$
$$= \sum(< D_{V_i} D_{\bar{V}_i}\alpha, \alpha > + < D_{\bar{V}_i}\alpha, D_{\bar{V}_i}\alpha > + < D_{V_i}\alpha, D_{V_i}\alpha >$$
$$+ < \alpha, D_{V_i} D_{\bar{V}_i}\alpha >)$$
$$= \sum (< D_{V_i} D_{\bar{V}_i}\alpha, \alpha > + < \alpha, D_{V_i} D_{\bar{V}_i}\alpha >) + \sum (||D_{\bar{V}_i}\alpha||^2 + ||D_{V_i}\alpha||^2)$$

which means $D\alpha = 0$ or α is parallel.

To prove the second assertion we observe that if the volume is infinite then since α is 2-finite $< \alpha, \alpha >$ must be zero. Thus $\alpha \equiv 0$.

Now, regardless of the volume of M, if

$$< D_{V_i} D_{\bar{V}_i}\alpha, \alpha > + < \alpha, D_{V_i} D_{\bar{V}_i}\alpha > > 0$$

then $\alpha = 0$ at any point where $\sum_{k=1}^{q} \ \lambda_{j_k} \ > \ 0$ for all $j_1 < ... < j_q$.

Finally,we observe that the calculations for forms of type (q,0) would follow readily on the same line from those for forms of type (0,q) because conjugation is an isometry and conjugate of forms of type (0,q) are forms of type (q,0). $\square$

Chapter 4

Applications

The beauty of Mathematics lies in its complexity : indeed, one must be willing and able to traverve a rabid river of intricate logics and abstruse abstractions and to trek through rugged mountains and deep gorges of convoluted calculi to arrive at a surreal valley of truth ; and whence, the beauty of an ordinary mathematical writing transcends, permeates and enlightens one's soul to the greatest of all possible human intellectual satisfaction. **H. T. Nguyen**

4.1 F-Harmonic maps from a complete Kahler manifold with a pole

Let M be a complete simply-connected Kahler manifold. A pole is a point $x_o \in M$ such that the exponential map

$$exp \; : \; T_{x_o}M \longrightarrow M \quad \text{is a diffeomorphism.}$$

The radial curvature K of a manifold with a pole is the restriction of the holomorphic sectional curvature to all the radial planes which contain the unit vector

$$\partial_r(x) \ = \ \frac{\partial}{\partial r}(x)$$

in T_xM tangent to the unique geodesic joining x_o to x and pointing away from x_o. Denote $r(x) = dist(x_o, x)$ the distance from x_o and define the tensor

$$g - dr \otimes dr \ = \ \begin{cases} 0 & on\ the\ radial\ direction\ \partial_r \\ g & on\ the\ orthogonal\ complement\ \partial_r^{\perp}. \end{cases}$$

Lemma 4.1. *[23]* **(Hessian Comparison Theorems in Riemannian Geometry)**

Let (M, g) be a complete Riemannian manifold with a pole x_o with its radial curvature K_r .

(i) If $-a^2 \le K_r \le -b^2$, $a, b > 0$, *then*

$$b\, coth(br)\, [g - dr \otimes dr] \ \le \ Hess(r) \ \le \ a\, coth(ar)\, [g - dr \otimes dr]\, .$$

(ii) If $K_r \ = \ 0$, *then*

$$\frac{1}{r}\, [g - dr \otimes dr] \ = \ Hess(r)\, .$$

(iii) If $\frac{-A}{(1+r^2)^{1+\epsilon}} \le K_r \le \frac{B}{(1+r^2)^{1+\epsilon}}$ *where* $\epsilon > 0$, $A \ge 0$, $0 \le B < 2\epsilon$, *then*

$$\frac{1 - \frac{B}{2\epsilon}}{r}\, [g - dr \otimes dr] \ \le \ Hess(r) \ \le \ \frac{e^{\frac{A}{2\epsilon}}}{r}\, [g - dr \otimes dr]\, .$$

(iv) If $-Ar^{2q} \ \le \ K_r \ \le -Br^{2q}$ *where* $A \ge B > 0$ *and* $q > 0$, *then*

$$B_o r^q \ [g - dr \otimes dr] \ \le \ Hess(r) \ \le \ (\sqrt{A}\, coth\, \sqrt{A})\, r^q\, [g - dr \otimes dr]$$

for $r \ge 1$, *where* $B_o = \ min\{1, -\frac{q+1}{2} \ + \ (B + (\frac{q+1}{2}^2)^{\frac{1}{2}}\}$.

Definition 4.2. The F-degree d_F is defined as

$$d_F \ = \ sup_{\ t \ge 0} \frac{t\ F'(t)}{F(t)}\ .$$

Lemma 4.3. *[13] Let (M, g) be a complete Riemannian manifold with a pole* x_o *. If there exist positive functions* $h_1(r),\ h_2(r)$ *on* $M - \{x_o\}$ *such that*

(i) $h_1(r)\,[g - dr \otimes dr] \ \leq\ Hess(r)\ \leq\ h_2(r)\,[g - dr \otimes dr]$ *and*

(ii) $1\ \leq\ r\,h_2(r)$ *,*

then for $X = r\,\partial_r = r\,\nabla r$ *, we have*

$$< S_F(u)\,, \nabla\theta_X > \ \geq\ F(\tfrac{|du|^2}{2})\,[\,1 + (n-1)rh_1(r)\ -\ 2p\,d_F\,rh_2(r)\,]\,.$$

Theorem 4.4. *[13] Let (M, g) be a complete Riemannian manifold with a pole* x_o *and* $E \to M$ *a Riemannian vector bundle. Let* $\omega \in \mathcal{A}^p(E)$ *be a differential p-form with value in the vector bundle E. Assume that the radial curvature* K_r *satisfies one of the following three conditions:*

(i) $-a^2 \leq K_r \leq -b^2$ *where* $a, b > 0$ *and* $(n-1)b - 2\,p\,a\,d_F \geq 0$

(ii) $K_r = 0$ *where* $n - 2\,p\,d_F > 0$

(iii) $\frac{-A}{(1+r^2)^{1+\epsilon}} \leq K_r \leq \frac{B}{(1+r^2)^{1+\epsilon}}$ *where* $\epsilon > 0,\ \ A \geq 0,\ \ 0 < B < 2\epsilon,$ *and* $n - (n-1)\frac{B}{2\epsilon} - 2\,p\,e^{\frac{A}{2\epsilon}}\,d_F$ *.*

If ω *satisfies an F-conservation law, then for any* $0\ <\ R_1\ \leq\ R_2$ *,*

$$\frac{1}{R_2^\lambda}\int_{B(R_2)} F(\tfrac{|du|^2}{2})\,dv_g\ \ \geq\ \ \frac{1}{R_1^\lambda}\int_{B(R_1)} F(\tfrac{|du|^2}{2})\,dv_g\,,$$

where

$$\lambda\ =\ \begin{cases} n - 2p\frac{a}{b}d_F & if\ \ K_r\ satisfies\ (i) \\ n - 2pd_F & if\ \ K_r\ satisfies\ (ii) \\ n - (n-1)\frac{B}{2\epsilon}\ -\ 2pe^{\frac{A}{2\epsilon}}d_F & if\ \ K_r\ satisfies\ (iii), \end{cases}$$

Proposition 4.5. *(cf. [13]) Let (M, g) be a complete Riemannian manifold with a pole* x_o *. Assume that the radial curvature* K_r *satisfies one of the following three* *conditions:*

(i) $\quad -a^2 \leq K_r \leq -b^2 \quad$ *where* $a, b > 0$ *and* $(n-1)b - 2\,p\,a\,d_F \geq 0$

(ii) $\quad K_r = 0 \quad$ *where* $n - 2\,p\,d_F > 0$

(iii) $\frac{-A}{(1+r^2)^{1+\epsilon}} \leq K_r \leq \frac{B}{(1+r^2)^{1+\epsilon}}$ *where* $\epsilon > 0,\ \ A \geq 0,\ \ 0 < B < 2\epsilon,$ *and*

$$n - (n-1)\tfrac{B}{2\epsilon} - 2\,p\,e^{\frac{A}{2\epsilon}}\,d_F\ .$$

Let $F : [0, \infty) \longrightarrow [0, \infty)$ *be a strictly increasing* C^∞ *function such that* $F(0) = 0$. *If* $u : (M^n, g) \to (N^k, h)$ *is an F-harmonic map, then for any* $0 < R_1 \leq R_2$,

$$\frac{1}{R_2^\lambda} \int_{B(R_2)} F(\tfrac{|du|^2}{2})\, dv_g \ \geq\ \frac{1}{R_1^\lambda} \int_{B(R_1)} F(\tfrac{|du|^2}{2})\, dv_g\ ,$$

where

$$\lambda = \begin{cases} n - 2p\frac{a}{b}d_F & if\ \ K_r\ satisfies\ (i) \\ n - 2pd_F & if\ \ K_r\ satisfies\ (ii) \\ n - (n-1)\frac{B}{2\epsilon} - 2pe^{\frac{A}{2\epsilon}}d_F & if\ \ K_r\ satisfies\ (iii), \end{cases}$$

Proof : By Corollary 2.9, the differential du of an F-harmonic map u, considered as a 1-form with value in the induced bundle, satisfies an F-conservation law, i.e. $S_F(u)$ is divergence-free. Thus, Lemma 4.1, Lemma 4.3 and Theorem 4.4 yield the monotonicity formula for an F-harmonic map. □

Theorem 4.6. *(cf. [13]) Let* (M^n, g) *be a complete Kahler manifold with a pole* x_o *and* (N^k, h) *be any Kahler manifold. Assume that the radial curvature* K_r *satisfies one of the following three conditions:*

(i) $\quad -a^2 \leq K_r \leq -b^2 \quad$ *where* $a, b > 0$ *and* $(n-1)b - 2\,p\,a\,d_F \geq 0$

(ii) $\quad K_r = 0 \quad$ *where* $n - 2\,p\,d_F > 0$

(iii) $\frac{-A}{(1+r^2)^{1+\epsilon}} \leq K_r \leq \frac{B}{(1+r^2)^{1+\epsilon}}$ *where* $\epsilon > 0,\ \ A \geq 0,\ \ 0 < B < 2\epsilon,$ *and*

$$n - (n-1)\tfrac{B}{2\epsilon} - 2\,p\,e^{\frac{A}{2\epsilon}}\,d_F\ .$$

Let $F : [0, \infty) \longrightarrow [0, \infty)$ *be a strictly increasing* C^∞ *function such that* $F(0) = 0$*. If* $u : (M^n, g) \to (N^k, h)$ *is an F-harmonic map with the growth condition*

$$\int_{B(\rho)} F(\frac{|du|^2}{2})\, dv_g \;=\; o\,(\rho^\lambda) \quad as \quad \rho \to \infty\,,$$

where

$$\lambda \;=\; \begin{cases} n - 2p\frac{a}{b}d_F & if \;\; K_r \; satisfies \; (i) \\ n - 2pd_F & if \;\; K_r \; satisfies \; (ii) \\ n - (n-1)\frac{B}{2\epsilon} \;-\; 2pe^{\frac{A}{2\epsilon}}d_F & if \;\; K_r \; satisfies \; (iii), \end{cases}$$

then u is constant.

Proof : By Proposition 4.6, for any $0 < R_1 \leq R_2$, we obtain the monotonicity formula for F-harmonic maps

$$\frac{1}{R_2^\lambda} \int_{B(R_2)} F(\frac{|du|^2}{2})\, dv_g \;\geq\; \frac{1}{R_1^\lambda} \int_{B(R_1)} F(\frac{|du|^2}{2})\, dv_g\,,$$

where

$$\lambda \;=\; \begin{cases} n - 2p\frac{a}{b}d_F & if \;\; K_r \; satisfies \; (i) \\ n - 2pd_F & if \;\; K_r \; satisfies \; (ii) \\ n - (n-1)\frac{B}{2\epsilon} \;-\; 2pe^{\frac{A}{2\epsilon}}d_F & if \;\; K_r \; satisfies \; (iii). \end{cases}$$

By assumption on growth, for all $R \geq R_1 > 0$, we have

$$0 \;=\; \lim_{R\to\infty} \frac{1}{R} \int_{B(R)} F(\frac{|du|^2}{2})\, dv_g \;\geq\; \frac{1}{R_1} \int_{B(R_1)} F(\frac{|du|^2}{2})\, dv_g\,.$$

This limit is zero because F is a non-negative smooth function. Furthermore, the integral on the right-hand side of the inequality is also non-negative which implies that $F(\frac{|du|^2}{2}) = 0$. Thus, $|du| = 0$ and hence u is constant . $\Box$

Remark 4.7. (cf. [13]) Since $S_{F,u}$ is divergence-free, du satisfies an F-conservation law. Thus, we can always obtain a monotonicity formula for an F-harmonic map regardless how the radial curvature varies, provided we have the Hessian comparison estimates with bounds and some positive constant

$$c \;\leq\; 1 + (n-1) r h_1(r) \;-\; 2p\, d_F\, r h_2(r)$$

as in Lemma 4.3.

Lemma 4.8. *[13] Let* (M^n, g) *be a complete Kahler manifold with a pole* x_o*. If* M *has constant holomorphic sectional curvature* $-a^2$, $a \geq 0$, *where* $n-1-2pd_F \geq 0$ *when* $a \neq 0$, *and* $n - 2pd_F > 0$ *when* $a = 0$, *then*

$$\frac{1}{R_1^{n-2pd_F}} \int_{B(R_1)} F(\tfrac{|du|^2}{2})\, dv_g \;\leq\; \frac{1}{R_2^{n-2pd_F}} \int_{B(R_2)} F(\tfrac{|du|^2}{2})\, dv_g$$

for $R_2 \geq R_1 > 0$.

Proposition 4.9. *(cf. [13]) Let (M, g) be a complete Riemannian manifold with a pole* x_o*. Assume that the radial curvature* K_r *satisfies the following condition:*

(iv) $-Ar^{2q} \;\leq\; K_r \;\leq\; -Br^{2q}$ *where* $A \geq B > 0$ *and* $q > 0$.

Denote $\delta := (n-1)B_o \;-\; 2pd_F \sqrt{A}\, coth\, \sqrt{A} \geq 0$ *where*

$$B_o = \;min\{1, -\tfrac{q+1}{2} \;+\; (B + (\tfrac{q+1}{2})^2)^{\frac{1}{2}}\} .$$

If $\int_{\partial B(1)} [\, F(\frac{|du|^2}{2}) \;-\; F'(\frac{|du|^2}{2}) < i_{\frac{\partial}{\partial r}} du, i_{\frac{\partial}{\partial r}} du >]\, ds \;\geq\; 0$ *then*

$$\frac{1}{R_1^{1+\delta}} \int_{B(R_1)-B(1)} F(\tfrac{|du|^2}{2})\, dv_g \;\leq\; \frac{1}{R_2^{1+\delta}} \int_{B(R_2)-B(1)} F(\tfrac{|du|^2}{2})\, dv_g$$

for any $R_2 \geq R_1 \geq 1$.

Proof : Take $X = r\nabla r$. By Lemmas 4.2 and 4.4, we have

$$< S_F(u), \nabla\theta_X > \;\geq\; F(\tfrac{|du|^2}{2})\, (1 + \delta r^{q+1})$$

and

$$S_F(u)(X, \tfrac{\partial}{\partial r}) \;=\; F(\tfrac{|du|^2}{2}) \;-\; F'(\tfrac{|du|^2}{2}) \;< i_{\frac{\partial}{\partial r}} du, i_{\frac{\partial}{\partial r}} du > \quad \text{on} \quad \partial B(1) ,$$

$$S_F(u)(X, \tfrac{\partial}{\partial r}) \;=\; RF(\tfrac{|du|^2}{2}) \;-\; RF'(\tfrac{|du|^2}{2}) \;< i_{\frac{\partial}{\partial r}} du, i_{\frac{\partial}{\partial r}} du > \quad \text{on} \quad \partial B(R)$$

By Corollary 2.9, $S_F(u)$ is divergence-free. Thus, by Corollary 2.11 we have

$$\int_{\partial D} S_F(u)(X,\nu)\, dS_g \;=\; \int_D \; < S_F(u), \nabla\theta_X > dv_g \,.$$

It follows that

$$\begin{aligned} R \int_{\partial B(R)} [\, F(\tfrac{|du|^2}{2}) \;-\; F'(\tfrac{|du|^2}{2}) < \, i_{\frac{\partial}{\partial r}} du, i_{\frac{\partial}{\partial r}} du \; >]\, ds \\ - \int_{\partial B(1)} [\, F(\tfrac{|du|^2}{2}) \;-\; F'(\tfrac{|du|^2}{2}) < \, i_{\frac{\partial}{\partial r}} du, i_{\frac{\partial}{\partial r}} du \; >]\, ds \\ \geq \; \int_{B(R)-B(1)} (1+\delta r^{q+1})\, F(\tfrac{|du|^2}{2})\, dv_g \,. \end{aligned}$$

Therefore, if $\int_{\partial B(1)} [\, F(\frac{|du|^2}{2}) \;-\; F'(\frac{|du|^2}{2}) < \, i_{\frac{\partial}{\partial r}} du, i_{\frac{\partial}{\partial r}} du \; >]\, ds \;\geq\; 0$ then

$$R \int_{\partial B(R)} F(\tfrac{|du|^2}{2})\, ds \;\;\geq\;\; (1+\delta) \int_{B(R)-B(1)} F(\tfrac{|du|^2}{2})\, dv_g$$

for any $R > 1$.

The coarea formula gives

$$\frac{d\int_{B(R)-B(1)} F(\frac{|du|^2}{2})\, dv_g}{\int_{B(R)-B(1)} F(\frac{|du|^2}{2})\, dv_g} \;\;\geq\;\; \frac{1+\delta}{R}\, dR$$

for a.e. $R \geq 1$.

Integrating over $[R_1, R_2]$ proves the proposition. $\square$

Theorem 4.10. *(cf. [13]) Let (M^n, g) be a complete Kahler manifold with a pole x_o and (N^k, h) be any Kahler manifold. Assume that the radial curvature K_r satisfies the following condition:*

(iv) $-\,Ar^{2q} \;\leq\; K_r \;\;\leq\; -\,Br^{2q}$ *where* $A \geq B > 0$ *and* $q > 0$.

Denote $\delta \;:=\; (n-1)B_o \;-\; 2pd_F\,\sqrt{A}\,coth\,\sqrt{A} \geq 0$ *where*

$$B_o = \;\; min\{1, -\tfrac{q+1}{2} \;+\; (B + (\tfrac{q+1}{2})^2)^{\frac{1}{2}}\} \,.$$

Let $F : [0,\infty) \longrightarrow [0,\infty)$ *be a strictly increasing* C^∞ *function such that* $F(0) = 0$*. If* $u : (M^n, g) \to (N^k, h)$ *is an F-harmonic map with the growth condition*

$$\int_{B(\rho)} F(\tfrac{|du|^2}{2})\, dv_g \;=\; o\,(\rho^{1+\delta}) \quad as \quad \rho \to \infty\, ,$$

then u is constant on $M \setminus B(1)$ *.*

Proof : This follows immediately from Proposition 4.9 . □

Definition 4.11. (cf. [30]) For a smooth map $u : M \to N$, the F-energy is slowly divergent if there exists a positive function $\phi(r)$ on M satisfying

$$\int_{R_1}^{\infty} \frac{1}{r\phi(r)}\, dr \;=\; \infty\, ,$$

for some $R_1 > 0$, such that

$$\lim_{R\to\infty} \int_{B(R)} \frac{F(\frac{|du|^2}{2})}{\phi(r(x))}\, dv_g \;<\; \infty\, ,$$

where $r(x)$ is the distance function from a fixed point $x_o \in M$ and $B(R)$ is the geodesic ball of radius R centered at x_o .

Theorem 4.12. *[13] Suppose* du *has slowly divergent F-energy. Then*

(i) For any $\lambda > 0,\ \lim_{r\to\infty} \frac{\phi(r)}{r^\lambda} \neq \infty$ *.*

(ii) If $\lim_{r\to\infty} \frac{\phi(r)}{r^\lambda}$ *exists for some* $\lambda > 0$ *, then*

$$\int_{B(R)} F(\tfrac{|du|^2}{2})\, dv_g \;=\; o(R^\lambda) \quad as \quad R \to \infty\, .$$

Remark 4.13. In light of this theorem, the growth of order λ and of order $(1+\delta)$ are weaker than the slowly divergent growth and finite growth.

4.2 F-harmonic maps from a complex space form

An n-dimensional complex manifold of constant holomorphic sectional curvature is called a complex space form and it must be locally isometric to one of the following complete simply connected universal covering spaces [65] : $\mathbb{C}P^n(4k^2)$, $\mathbb{C}^n$, $\mathbb{C}H^n(-4k^2)$, where $-4k^2$ means that the holomorphic sectional curvature of the complex hyperbolic space $\mathbb{C}H^n$ lies in $[-4k^2, -k^2]$ and $4k^2$ means that the holomorphic sectional curvature of the complex projective space $\mathbb{C}P^n$ lies in $[k^2, 4k^2]$.

Theorem 4.14. *Let $u : (\mathbb{C}^n, g) \longrightarrow (N^k, h)$ be a C^∞ map into a Kahler manifold and $q < 0$ be a constant satisfying $2 - q = n$, where g is the standard metric on $\mathbb{C}^n$ and $n \geq 3$. Let $F : [0, \infty) \longrightarrow [0, \infty)$ be a strictly increasing C^2 function such that*

$$F(t) \ \leq \ 2tF'(t) \ < \ n\, F(t) \ \ , \ \ \textit{for} \ \ t \in (0, \infty)\,.$$

If u is an F-harmonic map satisfying the above conditions then u is constant, provided u has slowly divergent enegy.

Proof : We apply the method used in [40]. Let x_o be a point in $\mathbb{C}^n$ and $B(R)$ an open geodesic ball with radius R and center x_o . Let $r = r(x)$ be the distance from x_o and $\frac{\partial}{\partial r}$ the unit radial vector field pointing away from x_o . Let $\{U_t\}_{t\in\mathbb{R}^+}$ be a 1-parameter family of C^∞ maps

$$U_t : \mathbb{C}^n \longrightarrow N \ : \ U_t(x) \ = \ u(tx) \ , \ x \in \mathbb{C}^n \ . \ \text{Set}$$

$$(*) \qquad E(R,t) \ = \ \int_{B(R)} F\,(\tfrac{|dU_t|^2}{2})\, dv_g \ \ ,$$

where dv_g is the volume element.

Applying Green's theorem yields

$$\frac{\partial}{\partial t}\, E(R,t)\,|_{t=1} \ = \ \int_{B(R)} F'(\tfrac{|dU_t|^2}{2}) \ < dU_t, \tfrac{d}{dt}(dU_t) > |_{t=1} \ dv_g$$

$$= \int_{B(R)} < F'(\tfrac{|du|^2}{2})du, \tilde{\nabla}(du(r\tfrac{\partial}{\partial r})) > \; dv_g$$
$$= \int_{B(R)} < d^* \, (F'(\tfrac{|du|^2}{2})du), du(r\tfrac{\partial}{\partial r}) > \; dv_g$$
$$+ R \int_{\partial B(R)} F'(\tfrac{|du|^2}{2}) < du(\tfrac{\partial}{\partial \nu}), du(r\tfrac{\partial}{\partial r}) > \; dS_g \, ,$$

where $\frac{\partial}{\partial \nu}$ is the unit normal and dS_g is the volume element with respect to the induced Kahler metric on $\partial B(R)$.

By the F-harmonic condition $d^*(F'(\frac{|du|^2}{2})du) = 0$, we obtain

$$\frac{\partial}{\partial t} E(R,t)\,|_{t=1} \geq 0 \; .$$

On the other hand, reparametrizing the integral $(*)$ gives

$$E(R,t) = t^{-n} \int_{B(tR)} F(\tfrac{1}{2}\, t^2\, h_{kl}(u(x))\, u_i^k(x)\, u_i^l(x) \,)\, dx \, .$$

By a direct calculation, we have

$$\frac{\partial}{\partial t} E(R,t) = (-n)\, t^{-n-1} \int_{B(tR)} F(\tfrac{1}{2}\, t^2\, h_{kl}(u(x))\, u_i^k(x)\, u_i^l(x) \,)\, dx$$
$$+ t^{-n} \int_{\partial B(tR)} R(Rt)^{n-2}\, F(\tfrac{1}{2}\, t^2\, h_{kl}(u(x))\, u_i^k(x)\, u_i^l(x) \,)\, dS_g$$
$$+ t^{-n+1} \int_{B(tR)} F'(\tfrac{1}{2}\, t^2\, h_{kl}(u(x))\, u_i^k(x)\, u_i^l(x) \,)\, h_{kl}(u(x))\, u_i^k(x)\, u_i^l(x)\, dx.$$

If we assume that $F(t) \leq 2tF'(t) < n\, F(t)$, then

$$F(\tfrac{|du|^2}{2}) \leq F'(\tfrac{|du|^2}{2})\, |du|^2$$
$$- F'(\tfrac{|du|^2}{2})\, |du|^2 \leq - F(\tfrac{|du|^2}{2})$$

Thus,

$$\tfrac{n}{n-1}\, F(\tfrac{|du|^2}{2}) - \tfrac{2}{n-1}\, F'(\tfrac{|du|^2}{2})\, |du|^2 \leq \tfrac{n}{n-1}\, F(\tfrac{|du|^2}{2}) - \tfrac{1}{n-1}\, F'(\tfrac{|du|^2}{2})\, |du|^2$$
$$= F(\tfrac{|du|^2}{2}) \, .$$

The assumption $2 - q = n$ implies that $1 - q < n$. Thus,

$$\tfrac{1-n}{n-1} - \tfrac{q}{n-1} < 0 \quad \text{, i.e.} \quad -1 < \tfrac{q}{n-1} \, .$$

Note that $\quad |du|^2 = \frac{1}{2}\, h_{kl}(u(x))\, u_i^k(x)\, u_i^l(x)$.

So after reparametrization, we have

$$\begin{aligned}
&\frac{\partial}{\partial t}\, E(R,t)\,|_{t=1} \\
&= \int_{B(R)} [\, -n\, F(\tfrac{|du|^2}{2}) + 2\, F'(\tfrac{|du|^2}{2})\, |du|^2\,]\, dx + R \int_{\partial B(R)} R^{\,n-2}\, F(\tfrac{|du|^2}{2})\, dS_g \\
&= -\int_{B(R)} [\, n\, F(\tfrac{|du|^2}{2}) - 2\, F'(\tfrac{|du|^2}{2})\, |du|^2\,]\, dx + R \int_{\partial B(R)} R^{\,n-2}\, F(\tfrac{|du|^2}{2})\, dS_g \\
&< \frac{q}{n-1} \int_{B(R)} [\, n\, F(\tfrac{|du|^2}{2}) - 2\, F'(\tfrac{|du|^2}{2})\, |du|^2\,]\, dx + R \int_{\partial B(R)} R^{\,n-2}\, F(\tfrac{|du|^2}{2})\, dS_g \\
&\leq q \int_{B(R)} F(\tfrac{|du|^2}{2})\, dx \;+\; R \int_{\partial B(R)} R^{\,n-2}\, F(\tfrac{|du|^2}{2})\, dS_g \\
&= q\, E(R,1) \;+\; R\, \frac{d}{dR}\, E(R,1)\, .
\end{aligned}$$

Then $\quad q\, E(R,1) \;+\; R\, \frac{d}{dR}\, E(R,1) \;\geq\; 0$,

and thus for all $R > 0$, we have

$$\begin{aligned}
\frac{d}{dR}\, [\, R^q\, E(R,1)\,] &= \frac{d}{dR}(R^q)\, E(R,1) \;+\; R^q\, \frac{d}{dR}\, E(R,1) \\
&= q\, R^{q-1}\, E(R,1) \;+\; R^q\, \frac{d}{dR}\, E(R,1) \\
&= R^{q-1}\, [q\, E(R,1) \;+\; R\, \frac{d}{dR}\, E(R,1)] \\
&\geq 0\, .
\end{aligned}$$

So $R^q\, E(R,1)$ is a non-decreasing function of R.

If u is not constant then there exists a point $x \in \mathbb{C}^n$ such that at this point $|du|^2 \neq 0$, and so there exists some $R_o > 0$ and $C > 0$, such that

$$\int_{B(R_o)} F\, (\tfrac{|du|^2}{2})\, dv_g \;\;\geq\;\; C\, .$$

Since $R^q\, E(R,1)$ is a non-decreasing function of R, for all $R \geq R_o$, we get

$$R^q\, E(R,1) \;\geq\; R_o^q\, E(R_o,1) \quad , \text{ i.e.}$$

$$E(R,1) \;\geq\; (\tfrac{R_o}{R})^q\, E(R_o,1)\, . \text{ Thus,}$$

$$\int_{B(R)} F(\tfrac{|du|^2}{2}) dv_g \;\;\geq\;\; (\tfrac{R_o}{R})^q \int_{B(R_o)} F(\tfrac{|du|^2}{2})\, dv_g \;\;\geq\;\; C\, (\tfrac{R_o}{R})^q\, .$$

Furthermore, we also have

$$
\begin{aligned}
R \, \tfrac{d}{dR} \, E(R,1) \;&\ge\; - \, q \, E(R,1) \quad , \text{ i. e.} \\
\int_{\partial B(R)} F(\tfrac{|du|^2}{2}) \, dv_g \;&\ge\; - \, \tfrac{q}{R^{n-1}} \int_{B(R)} F(\tfrac{|du|^2}{2}) \, dv_g \\
&\ge\; - \, q \, C \, R_o^q \, (\tfrac{1}{R^{n-1+q}}) \\
&=\; - \, q \, C \, R_o^q \, (\tfrac{1}{R}) \quad , \quad \text{since } \; 2 - q = n \, .
\end{aligned}
$$

Hence, by the coarea formula and the definition of slow divergence we obtain

$$
\begin{aligned}
\lim_{R \to \infty} \int_{B(R)} \frac{F(\frac{|du|^2}{2})(x)}{\phi(r(x))} \, v_g \;&=\; \int_0^\infty \frac{dR}{\phi(R)} \int_{\partial B(R)} F(\tfrac{|du|^2}{2}) v_g \\
&\ge\; - \, q \, C \, R_o^q \int_0^\infty \frac{dR}{R \, \phi(R)} \\
&\ge\; - \, q \, C \, R_o^q \int_{R_o}^\infty \frac{dR}{R \, \phi(R)} \\
&\ge\; \infty \, ,
\end{aligned}
$$

a contradiction to the slowly divergent condition of the F-energy. $\square$

Corollary 4.15. *Let* $u : (\mathbb{C}^n, g) \longrightarrow (N^k, h)$ *be a* C^∞ *map into a Kahler manifold and* $q < 0$ *be a constant satisfying* $2 - q = n$ *, where* g *is the standard metric on* $\mathbb{C}^n$ *and* $n \ge 3$ *. Let* $F : [0, \infty) \longrightarrow [0, \infty)$ *be a strictly increasing* C^2 *function such that*

$$F(t) \;\le\; 2tF'(t) \;<\; n \, F(t) \quad , \quad \textit{for} \;\; t \in (0, \infty) \, .$$

If u *is an F-harmonic map satisfying the above conditions then* u *is constant, provided* u *has the following enegy growth*

$$\int_{B(R)} F(\tfrac{|du|^2}{2}) \, dv_g \;=\; o(R^\lambda) \quad \textit{as} \quad R \to \infty \, .$$

Proof : Proceeding as in the theorem yields

$$q \, E(R,1) \;+\; R \, \tfrac{d}{dR} \, E(R,1) \;\ge\; 0 \quad .$$

Thus, we have

$$
\begin{aligned}
q\,E(R,1) \;&\geq\; -\,R\,\tfrac{d}{dR}\,E(R,1)\\
\int_{R_1}^{R_2}\,\tfrac{q}{R}\,dR \;&\geq\; -\,\int_{R_1}^{R_2}\,\tfrac{dE(R,1)}{E(R,1)}\\
q\,[\,lnR_2 - lnR_1\,] \;&\geq\; -\,ln\,E(R_2,1) \;+\; ln\,E(R_1,1)\\
\tfrac{R_2^q}{R_1^q} \;&\geq\; \tfrac{E(R_1,1)}{E(R_2,1)}
\end{aligned}
$$

which yields a monotonicity formula

$$\tfrac{1}{R_1^{-q}}\,\int_{B(R_1)}\,F(\tfrac{|du|^2}{2})\,dv_g \;\;\leq\;\; \tfrac{1}{R_2^{-q}}\,\int_{B(R_2)}\,F(\tfrac{|du|^2}{2})\,dv_g\;.$$

By assumption on growth, for all $R \geq R_1 > 0$ and since $q < 0$, we have

$$0 \;=\; \lim_{R\to\infty}\,\tfrac{1}{R^{-q}}\,\int_{B(R)} F(\tfrac{|du|^2}{2})\,dv_g \;\;\geq\;\; \tfrac{1}{R_1^{-q}}\,\int_{B(R_1)} F(\tfrac{|du|^2}{2})\,dv_g\;.$$

This limit is zero because F is a non-negative smooth function. Furthermore, the integral on the right-hand side of the inequality is also non-negative which implies that $F(\frac{|du|^2}{2}) = 0$. Thus,

$|du|^2 \;= 0$ and hence, u is constant. □

Remark 4.16. J. Wan proved that " *Any harmonic map from* $\mathbb{C}^n$ *to any Kahler manifold is* $\pm$ *holomorphic under an assumption of energy density* [53] . "

Remark 4.17. H. C. J. Sealey : " *For* $n \geq 2$ *, any holomorphic map of finite energy from* $\mathbb{C}^n$ *to any Kahler manifold is constant* [49] . "

Theorem 4.18. *For* $n \geq 1$ *, let* M^n *be a complete simply connected, noncompact Kahler manifold of holomorphic sectional curvature* HR^M *which satisfies*

$-a^2 \leq HR^M \leq -b^2$, *where a, b are some positive constants. Let N be any Kahler manifold and* $F : [0, \infty) \longrightarrow [0, \infty)$ *be a strictly increasing* C^2 *function such that*

$$(n-1)b\, F(t) \;-\; 2ta\, F'(t) \;\geq\; 0 \;\; for\; t \in (0, \infty)\,.$$

If $u : (M^n, g) \to (N^k, h)$ *is an F-harmonic map with following growth condition*

$$\int_{B(\rho)} F(\tfrac{|du|^2}{2})\, dv_g \;=\; o\,(\rho^\lambda) \quad as \quad \rho \to \infty\,,$$

then u is constant.

Proof : We use the same technique as in [40]. Let x_o be a point in M.

Take $X = r\frac{\partial}{\partial r} \in T_{x_o}M$, where $r = r(x)$ is the distance from x_o and $\frac{\partial}{\partial r}$ is the unit radial vector field pointing away from x_o . By Corollary 2.11 and the definition of the stress energy tensor, we have

$$\begin{aligned}
&\int_{B(R)} (div\; S_F(u))(X)\, dv_g \;+\; \int_{B(R)} \;< S_F(u), \nabla X >\; dv_g \\
&\quad = \int_{\partial B(R)} F(\tfrac{|du|^2}{2}) g(X, \nu)\, dv_g \;-\; \int_{\partial B(R)} F(\tfrac{|du|^2}{2})\, h(du(X), du(\nu))\, dv_g \\
&\quad = R \int_{\partial B(R)} F(\tfrac{|du|^2}{2})\, dv_g \;-\; R \int_{\partial B(R)} F(\tfrac{|du|^2}{2})\, h(du(\tfrac{\partial}{\partial r}), du(\tfrac{\partial}{\partial r}))\, dv_g \\
&\quad \leq R \int_{\partial B(R)} F(\tfrac{|du|^2}{2})\, dv_g \;\;.
\end{aligned}$$

Choose a local Hermitian orthonormal frame field $\{e_1, ..., e_{n-1}, e_n = \frac{\partial}{\partial r}\}$ on M . The following local calculations are straightforward

$$\begin{aligned}
&\nabla_{\frac{\partial}{\partial r}} X \;=\; \tfrac{\partial}{\partial r} \\
&\nabla_{e_i} X \;\;=\; r\, \nabla_{e_i} \tfrac{\partial}{\partial r} \;=\; r\, Hess(r)(e_i, e_j) e_j \\
&div\; X \;=\; 1 \;+\; r\, Hess(r)(e_i, e_i)\;, \;\; 1 \leq i \leq n-1\;,
\end{aligned}$$

where $Hess(r)(e_i, e_j) \;=\; \nabla_{e_j}\, \nabla_{e_i} r \;-\; (\nabla_{e_j} e_i)\, r$.

Thus,

$F(\frac{|du|^2}{2})\, h(du(e_i), du(e_j))\, g(\nabla_{e_i} X, e_j)$

$$= F(\tfrac{|du|^2}{2}) \, [\, rHess(r)(e_i, e_j) \, h(du(e_i), du(e_j)) \; + \; h(du(\tfrac{\partial}{\partial r}), du(\tfrac{\partial}{\partial r})) \,] \; .$$

The Hessian comparison theorem [62] yields

$$
\begin{aligned}
&< S_F(u), \nabla X > \\
&= F(\tfrac{|du|^2}{2}) \; div \; X \; - \; F'(\tfrac{|du|^2}{2}) \; h(du(e_i), du(e_j)) \; g(\nabla_{e_i} X, e_j) \\
&= F(\tfrac{|du|^2}{2})(1 + rHess(r)(e_i, e_j)) - F'(\tfrac{|du|^2}{2})[\, |du(\tfrac{\partial}{\partial r})|^2 + rHess(r)(e_i, e_j) h(du(e_i), du(e_j))]. \\
&\geq F(\tfrac{|du|^2}{2})[1 + (n-1)(br)coth(br)] - F'(\tfrac{|du|^2}{2})[\, |du(\tfrac{\partial}{\partial r})|^2 + (ar)coth(ar) h(du(e_i), du(e_i))] \\
&\geq F(\tfrac{|du|^2}{2})[1 + (n-1)(br)coth(br)] \\
&\qquad - F'(\tfrac{|du|^2}{2})[(ar)coth(ar)|du(\tfrac{\partial}{\partial r})|^2 + (ar)coth(ar) h(du(e_i), du(e_i))] \\
&= F(\tfrac{|du|^2}{2}) \, [1 + (n-1)(br)coth(br)] \; - \; F'(\tfrac{|du|^2}{2}) \, (ar)coth(ar)|du|^2 \\
&\geq F(\tfrac{|du|^2}{2}) + rcoth(br) \, [\, (n-1)bF(\tfrac{|du|^2}{2}) \; - \; a|du|^2 F'(\tfrac{|du|^2}{2}) \,] \\
&\geq F(\tfrac{|du|^2}{2}) \quad ,
\end{aligned}
$$

because $(n-1)bF(t) - 2taF'(\frac{|du|^2}{2}) \; \geq \; 0$.

Since F-harmonic maps are divergence-free, it follows that

$$R \int_{\partial B(R)} F(\tfrac{|du|^2}{2}) \, dv_g \; \geq \; \int_{B(R)} F(\tfrac{|du|^2}{2}) \, dv_g \; .$$

Following Dong and Wei [13], a monotonicity formula could be obtained as follows: from the proof of Theorem 4.9 since F-harmonic maps are divergence-free, we have

$$R \int_{\partial B(R)} F(\tfrac{|du|^2}{2}) \, dv_g \; \geq \; \int_{B(R)} F(\tfrac{|du|^2}{2}) \, dv_g \quad , \quad \text{i.e.}$$

$$\frac{\int_{\partial B(R)} F(\frac{|du|^2}{2}) \, dv_g}{\int_{B(R)} F(\frac{|du|^2}{2}) \, dv_g} \; \geq \; \frac{1}{R}$$

The coarea formula

$$\frac{d}{dR} \int_{B(R)} F(\tfrac{|du|^2}{2}) \, dv_g \; = \; \int_{\partial B(R)} F(\tfrac{|du|^2}{2}) \, dv_g$$

gives

$$\frac{\frac{d}{dR}\ \int_{B(R)}\ F(\frac{|du|^2}{2})\ dv_g}{\int_{\partial B(R)}\ F(\frac{|du|^2}{2})\ dv_g}\ \geq\ \frac{1}{R}\quad \text{for a. e. } R > 0\ .$$

Integrating over $[R_1, R_2]$, $R_1 > 0$, yields the following monotonicity formula

$$\frac{1}{R_2}\ \int_{B(R_2)}\ F(\frac{|du|^2}{2})\ dv_g\ \geq\ \frac{1}{R_1}\ \int_{B(R_1)}\ F(\frac{|du|^2}{2})\ dv_g\ .$$

By assumption on growth and the above monotonicity formula, for all $R \geq R_1 > 0$, we have

$$0\ =\ \lim_{R\to\infty}\ \frac{1}{R}\ \int_{B(R)} F(\frac{|du|^2}{2})\ dv_g\ \ \geq\ \ \frac{1}{R_1}\ \int_{B(R_1)} F(\frac{|du|^2}{2})\ dv_g\ .$$

This limit is zero because F is a non-negative smooth function. Furthermore, the integral on the right-hand side of the inequality is also non-negative which implies that $F(\frac{|du|^2}{2}) = 0$. Thus,

$$|du|^2\ =\ 0 \text{ and hence, u is constant .} \qquad \square$$

Corollary 4.19. *For $n \geq 1$, let M^n be a complete simply connected, noncompact Kahler manifold of holomorphic sectional curvature HR^M which satisfies $-a^2 \leq HR^M \leq -b^2$, where a, b are some positive constants. Let N be any Kahler manifold and $F : [0, \infty) \longrightarrow [0, \infty)$ be a strictly increasing C^2 function such that*

$$(n-1)b\ F(t)\ -\ 2ta\ F'(t)\ \geq\ 0\ \ \text{for } t \in (0, \infty)\ .$$

If $u : (M^n, g) \to (N^k, h)$ is an F-harmonic map with slowly divergent F-energy then u is constant.

Proof : Dong and Wei proved in [13] that the slowly divergent growth implies the following growth condition

$$\int_{B(\rho)}\ F(\frac{|du|^2}{2})\ dv_g\ =\ o\ (\rho^\lambda)\quad as\quad \rho \to \infty\ .$$

So the corollary is a consequence of this result.

However, a direct proof can be obtained as follows: proceeding as in the theorem, since F-harmonic maps are divergence-free, it follows that

$$R \int_{\partial B(R)} F(\tfrac{|du|^2}{2})\, dv_g \;\geq\; \int_{B(R)} F(\tfrac{|du|^2}{2})\, dv_g \,.$$

If u is not constant then there is a point $x \in M$ such that $|du|^2 \neq 0$. This means that there exists some $R_o > 0$ and some positive constant C_o such that for all $R > R_o$,

$$\int_{B(R)} F(\tfrac{|du|^2}{2})\, dv_g \;\geq\; C_o \;.$$

Thus,

$$\int_{\partial B(R)} F(\tfrac{|du|^2}{2})\, dv_g \;\geq\; \tfrac{C_o}{R} \,.$$

The slowly divergent condition then implies that,

$$\begin{aligned} \lim_{R\to\infty} \int_{B(R)} \frac{F(\frac{|du|^2}{2})(x)}{\phi(r(x))}\, dv_g \;&=\; \int_0^\infty \frac{dR}{\phi(R)} \int_{\partial B(R)} F(\tfrac{|du|^2}{2})\, dv_g \\ &\geq\; C_o \int_0^\infty \frac{dR}{R\,\phi(R)} \\ &\geq\; C_o \int_{R_o}^\infty \frac{dR}{R\,\phi(R)} \;=\; \infty \,. \end{aligned}$$

This contradicts the assumption that the F-energy of u is slowly divergent. □

Corollary 4.20. *Any F-harmonic map with slowly divergent F-energy from the complex hyperbolic space $\mathbb{C}H^n$ to any Kahler manifold must be constant, provided the condition on the function F as in Theorem 4.22 is satisfied.*

Proof : Since $\mathbb{C}H^n$ is negatively quarter-pinched, the corollary follows immediately from the theorem. □

Remark 4.21. H. C. J. Sealey proved that : *For* $n \geq 2$ *, the complex hyperbolic spcace* $\mathbb{C}H^n$ *supports no nonconstant harmonic maps of finite energy. In particular, any nonconstant holomorphic map from* $\mathbb{C}H^n$ *to a Kahler manifold must have infinite energy* [49].

Bibliography

[1] M. Ara, *Geometry of F-harmonic maps*. Kodai Math. J. **22** (1999), 243-263.

[2] M. Ara, *Stability of F-harmonic maps into pinched manifolds*, Hiroshima Math. J. **31** (2000), 171-181.

[3] M. Ara, *Instability and nonexistence theorems for F-harmonic maps*, Illinois J. Math. **45** (2001), 675-680.

[4] P. Baird, *Stress-energy tensors and the Lichnerowicz Laplacian*. J. Geom. Phys. **58**, (2008), 1329-1342.

[5] W. Ballmann, *Lectures on Kahler Manifolds*. European Mathematical Society, 2006.

[6] A. Borel and J. P. Serre, *Determination des p-puissances reduites de Steenrod dans la cohomologie des groupes classiques*, Applications, C. R. Acad. Sci. Paris 233 (1951), 680-682. (MR 13, 319.)

[7] J. Cheeger and D. G. Ebin, *Comparison Theorems in Riemannian Geometry*. New York: Noth-Holland/Elsevier, 1975.

[8] A. H. Cherif and M. Djaa, *Geometry of Energy and Bienergy Variations between Riemannian Manifolds*, Kyungpook Math. J. 55(2015), 715-730.

[9] Y. J. Chiang, *f-biharmonic Maps between Riemannian Manifolds*, Department of Mathematics, University of Mary Washington Fredericksburg, VA 22401, USA (2012).

[10] N. Course, *f-harmonic maps*, Thesis, University of Warwick, Coventry, CV4 7Al, UK (2004)

[11] N. Course, *f-harmonic maps which map the boundary of the domain to one point in the target*, New York Journal of Mathematics, **13** (2007), 423-435.

[12] Y. Dong and H. Lin and G. Yang, *Liouville theorems for F-harmonic maps and their applications*, arXiv:1111.1882v1[math.DG] 8 Nov 2011. Results. Math. 69 no. 1-2, (2016), 105-127.

[13] Y. Dong and S. W. Wei, *On vanishing theorems for vector bundle value p-forms and their applications*, Comm. Math. Phys. **304** (2011), 329-368.

[14] J. Eells, *p-Harmonic and Exponentially Harmonic Maps*, lectures given at Leeds University, June 1993.

[15] J. Eells and L. Lemaire, *Selected Topics in Harmonic Maps*. American Mathematical Society, Regional Conference Series in Mathematics, no. **50**, 1983.

[16] J. Eells and L. Lemaire, *A report on harmonic maps*, C.B.M.S. **50** A. M. S. 1978.

[17] J. Eells and L. Lemaire, *Another report on harmonic maps*, Bull. London Math. Soc. **20(5)** (1988), 385-524.

[18] J. Eells and J. H. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. **86** (1964), 106-160.

[19] H. Federer, *Geometric measure theorey*, Springer-Verlag, New York, 1969.

[20] M. Gaffney, *A special Stokes' theorem for complete Riemannian manifolds*, Ann. of Math. **60** (1954), 140-145.

[21] S. I. Goldberg and S. Kobayashi, *Holomorphic Bisectional Curvature*, J. Diff. Geom. **1** (1967), 225-233.

[22] A. Gray and L. H. Hervella, *The sixteen classes of almost Hermitian manifolds and their linear invariants*, Ann. Mat. Pura Appl., **123** (1980), 35-58.

[23] R. E. Greene and H. Wu, *Function theory on manifolds which possess a pole*. Lecture Notes in Math. **699**, Berlin, Heidleberg-New York: Springer-Verlag. (1979).

[24] R. E. Greene and H. Wu, *Harmonic forms on noncompact Riemannian and Kahler manifolds*. The Michigan Mathematical Journal **28**, Issue 1 (1981), 63-81.

[25] R. E. Greene and H. Wu, *On Kahler manifolds of positive bisectional curvature and a theorem of Hartogs*, Abhandlungen aus dem Mathematischen Seminar der Universitt Hamburg, 1978, Vol.47(1), pp.171-185 [Peer Reviewed Journal]

[26] K. Grove, H. Karcher and E. A. Ruh, *Group actions and curvature*, Inv. Math. **23** (1974), 31-48.

[27] K. Grove and H. Karcher and E. A. Ruh, *Jacobi fields and Finsler metrics on compact Lie groups with an application to differential pinching problem*, Math. Ann. **21** (1974), 7-21.

[28] S. Helgason, *Differential Geometry, Lie Groups and Symmetric Spaces*. New York-London: Academic Press, 1962.

[29] R. Howard and S. W. Wei, *Non-existence of stable harmonic maps to and from certain homogeneous spaces and submanifolds of Euclidean space*, Trans. Amer. Math. Soc. **294** (1986), 319-331.

[30] H. S. Hu, *A nonexistence theorem for harmonic maps with slowly divergent energy*, Chinese Ann. of Math. Ser. B **5** (4), (1984), 737-740.

[31] D. Huybrechts, *Complex geometry: an introduction*. Springer-Verlag, Berlin, New York, Universitext, 1995.

[32] J. Jost, *Riemannian Geometry and Geometric Analysis*. Berlin-Heidelberg: Springer-Verlag, 2011.

[33] J. Jost and S. T. Yau, *Harmonic mappings and Kahler Manifolds*, Math. Ann. **262**, (1983), 145-166.

[34] L. Karp, *Subharmonic functions on real and complex manifolds*, Math. Z. **179** (1982), no 4, 535-554.

[35] M. Kassi, *A Liouville theorem for F-harmonic maps with finite F-energy*. Electron. J. Differ. Equ. **15**, (2006), 1-9.

[36] S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, vols. I and II, New York: Wiley-Interscience, 1963.

[37] P. F. Leung, *On the stability of harmonic maps*, J. London Math. Soc. **949** (1982) 122-129.

[38] A. Lichnerowicz, *Applications harmonique et varietes Kahleriennes*, Symp. Math. III, Bologna (1970), 341-402.

[39] J. C. Liu, *Liouville theorems of stable F-harmonic maps for compact convex hypersurfaces*, Hiroshima Math. J. **36** (2006), 221-234.

[40] J. C. Liu, *Liouville-type theorems for F-harmonic maps on noncompact manifolds*, Kodai math. J. **28** (2005), 483-493.

[41] A. Moroianu, *Lectures on Kahler Geometry*. Cambridge University Press, New York, 2007.

[42] A. Newlander and L. Nirenberg, *Complex analytic coordinates in almost complex manifolds*, Ann. of Math., **65** (1957), 391-404.

[43] Y. Ohnita, *On pluriharmonicity of stable harmonic maps*, J. London Math. Soc. (2) **35** (1987), 563-568.

[44] Y. Ohnita and S. Udagawa, *Complex-analyticity of pluriharmonic maps and their constructions*, Lecture Notes in Math. 1468. Springer-Verlag. (1991), 371-407.

[45] Y. Ohnita and S. Udagawa, *Stability, complex-analyticity and constancy of pluriharmonic maps from compact Kahler manifolds*, Math. Z. **205**, (1990), 629-644.

[46] T. Okayasu, *Pinching and nonexistence of stable harmonic maps*, Tohoku Math. J. **40** (1988), 213-220.

[47] P. Petersen, *Riemannian Geometry*, Second Edition. Graduate Text in Mathematics, **171**. Springer, New York, 2006.

[48] J. H. Sampson, *Applications of harmonic maps to Kahler geometry*, Contemp Math. **49** (1986), 125-134.

[49] H. C. J. Sealey, *Some conditions ensuring the vanishing of harmonic differential forms with applications to harmonic maps and Yang-Mills theory*, Math. Proc. Camb. Phil. Soc. **91** (1982), 441-452.

[50] Y. T. Siu, *The complex-analyticity of harmonic maps and the strong rigidity of compact Kahler manifolds*, Ann. Math. **112** (1980), 73-111.

[51] S. T. Siu and S.T. Yau, *Compact Kahler manifolds of positive bisectional curvature*, Invent. Math. **59** (1980), 189-204.

[52] S. Udagawa, *Pluriharmonic maps and minimal immersions of Kahler manifolds*, J. London Math. Soc. (2) **37** (1988), 375-384.

[53] J. Wan, *Harmonic Maps From* $\mathbb{C}^n$ *To Kahler Manifolds*, Pacific Journal of Math. vol 275. No. **1**, (2015), 183-189.

[54] F. W. Warner, *Foundations of Differentiable Manifolds and Lie Groups*, New York: Springer-Verlag, 1983.

[55] S. W. Wei, *p-Harmonic Geometry And Related Topics*, Bull. Transilv. Univ. Brasov Ser. III 1(50), (2008), 415-453.

[56] S. W. Wei, J. Li, L. Wu, *Generalizations of the uniformization theorem and Bochner' method in p-harmonic geometry*, Commun. Math. Anal. 2008, Conference 1, 46-68.

[57] S. W. Wei and L. Wu, *Vanishing Theorems for 2-Banlanced Harmonic Forms*, Global J. of Pure and Applied Math **2** (2015), 745-753.

[58] S. W. Wei and C. M. Yau, *Regularity of p-energy minimizing maps and p-superstrongly unstable indices.* Geom. Analysis **4**, (2) 1994, 247-272.

[59] J. Wolf, *Spaces of Constant Curvature*, Wilmington: Publish or Perish, 1984.

[60] H. H. Wu, *The Bochner Technique in Differential Geometry*, Math. Report vol. 3, part 2, London: Harwood Academic Publishers, 1988.

[61] K. Yano and M. Kon, *Structures on Manifolds*. World Scientific Publishing Co. Pte. Ltd., 1984.

[62] Yuanlong Xin, *Geometry of Harmonic Maps*. Birkhauser, Boston, 1996.

[63] Y. L. Xin, *Some result on stable harmonic maps*. Duke Math. J. **47** (1980), no. 3, 609-613.

[64] S. T. Yau, *Some function-theoretic properties of complete Riemannian manifolds and their applications to geometry*. Indiana Univ. Math. J. **25** (1976) no. 7, 659-670.

[65] F. Zheng, *Complex Differential Geometry*. Studies In Advanced Mathematics. vol 18. AMS/IP. 2002.