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BENJIMAN CLOYES THARP
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REPRESENTATIONS OF THE MARKED BRAUER ALGEBRA

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BY

Dr. Jonathan Kujawa, Chair

Dr. Max Forester

Dr. Patrick Livingood

Dr. Murad Özaydın

Dr. Tomasz Przebinda

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Abstract

The marked Brauer algebra is a generalization of the diagrammatic Brauer algebra which diagrammatizes Moon's centralizer algebra for the type $\mathfrak{p}$ Lie superalgebra. We prove that the marked Brauer algebra is a standard based algebra in the sense of Du-Rui and determine the circumstances under which the algebra is quasi-hereditary. In particular, we observe that the category of modules for the marked Brauer algebra is a highest weight category. Standard modules are constructed and the simple modules are classified using the general framework of standard based algebras. We describe the induction and restriction of standard modules and study weights of these modules which arise from the action of a collection of Jucys-Murphy type elements in the marked Brauer algebra. Finally, we introduce arc diagrams for the marked Brauer algebra, which provide a combinatorial criterion for determining decomposition multiplicities of standard modules.

Chapter 1

Introduction

1.1 Background

Schur-Weyl duality has featured prominently in the study of representation theory since its introduction by Issai Schur in the early 1900s. In its original form, it says the following. Let V be a finite-dimensional complex vector space and consider the n-fold tensor product of V with itself, $V^{\otimes n} := V \otimes \cdots \otimes V$ (n factors of V). The general linear group $GL(V)$ acts diagonally on $V^{\otimes n}$ on the left, meaning if $g \in GL(V)$, then g acts simultaneously on each tensorand:

$$g.\left(v_1 \otimes \cdots \otimes v_n\right) := (g.v_1) \otimes \cdots \otimes (g.v_n)\,.$$

At the same time, the symmetric group S_n acts on $V^{\otimes n}$ on the right by place permutations of the tensorands, so if $\sigma \in S_n$, then

$$\left(v_1 \otimes \cdots \otimes v_n\right).\sigma := v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}.$$

These two actions commute with each other and, in fact, generate each other's full centralizer in $\mathrm{End}_{\mathbb{C}}\left(V^{\otimes n}\right)$. As a corollary, when $\dim V \geq n$ we obtain a decomposition of $V^{\otimes n}$ as a $GL(V) \times S_n$-module into a direct sum of modules of the form $L_\lambda \otimes S^\lambda$, where S^λ is the Specht module corresponding to the partition λ of n and L_λ is the simple $GL(V)$-module with highest weight λ. Since the set of Specht modules $\{S^\lambda : \lambda \text{ is a partition of } n\}$ is precisely the complete set of nonisomorphic simple S_n-modules, this decomposition gives a correspondence between simple S_n-modules and certain simple modules for $GL(V)$. Moreover, this correspondence can be promoted to a functor between module categories.

The symmetric group S_n is a nice example of an algebraic object which can be described diagrammatically: the simple transposition $s_i = (i, i+1)$ can be depicted using the diagram below.

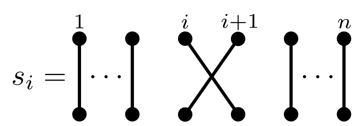


Figure 1

Multiplication in the symmetric group then corresponds to first vertically stacking diagrams and then simplifying to create a new diagram which has the same connected components as the stack. For example, in S_3 we have $(123) = (12)(23)$, which looks like

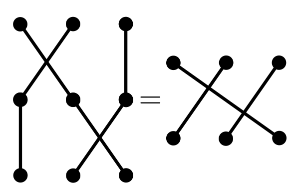


Figure 2

if we pass to the diagrams. In this way, we can obtain a diagrammatic version of S_n, in which a permutation σ corresponds to the diagram which has node i in the bottom row connected by an edge to node $\sigma(i)$ in the top row. More generally, mathematical objects which have a diagrammatic description allow us to reduce potentially complicated abstract calculations to concrete pictorial manipulations.

There are many natural generalizations of classical Schur-Weyl duality obtained by replacing $GL(V)$ by some other Lie theoretic object. One such generalization was discovered by Richard Brauer in 1937. If we fix a nondegenerate symmetric or skew-symmetric bilinear form $(,)$ on V, then the group of isometries of V which preserve this form,

$$G = \{g \in GL(V) : (gu, gv) = (u, v)\, \forall u, v \in V\}$$

is the orthogonal group $O(V)$ or the symplectic group $Sp(V)$, respectively. Being a subgroup of $GL(V)$, G acts diagonally on $V^{\otimes n}$ on the left and this action commutes with the right action of the symmetric group described above. However, since G is smaller than $GL(V)$, the symmetric group algebra is too small to generate the full centralizer of the action of G in $\mathrm{End}_{\mathbb{C}}\left(V^{\otimes n}\right)$. Brauer described this centralizer algebra, now called the Brauer algebra, $B_n\left(\delta_G\right)$ where $\delta_{O(V)} = \dim V$ and $\delta_{Sp(V)} = -\dim V$, as a diagram algebra having generators $e_1, s_1, \ldots, e_{n-1}, s_{n-1}$ depicted by

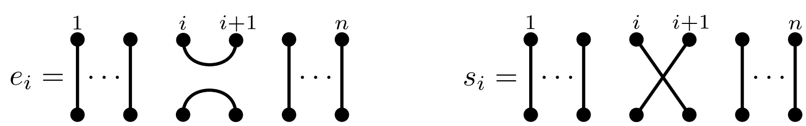


Figure 3

subject to a manageably small list of relations (see Theorem 2.1 for details). The interpretation of s_i is as described above for the symmetric group, while the action of e_i involves the bilinear form in a concrete way. Multiplication in the Brauer algebra is again by vertical stacking, except we could now potentially have connected components which are isolated in the middle of the stacked diagrams. These connected components must be removed in order to recover a new Brauer diagram, which, by definition, is comprised of all connected components involving the topmost and bottommost rows of vertices. To account for possible lost information, we scale the simplified diagram by δ_G^k, where k is the number of connected components isolated in the middle of the stack. A detailed example of multiplication in $B_n\left(\delta_G\right)$ is given in the next chapter. Using $B_n(\delta_G)$, we now recover the full results of Schur-Weyl duality with $O(V)$ or $Sp(V)$ replacing $GL(V)$.

The Brauer algebra has consistently been studied since its introduction. The work of Graham-Lehrer [21] , Cox-De Visscher-Martin [14], Shalile [40], and others has shown that this algebra has a rich representation theory and interesting structure. In particular, the definition of $B_n(\delta)$ makes sense for arbitrary $\delta \in \mathbb{C}$. For generic δ, the Brauer algebra is semisimple. More generally, Graham-Lehrer showed that $B_n(\delta)$ is a cellular algebra, so it has a certain distinguished basis with a compatible anti-automorphism as well as a collection of cell modules. These modules are easy to describe and yet collectively contain much representation-theoretic information. More recently, Shalile has studied the representation theory of the Brauer algebra using weights in a manner similar to the Okounkov-Vershik approach to the representation theory of the symmetric group.

Other authors have continued to search for the phenomenon of Schur-Weyl duality for other Lie-theoretic objects besides the three classical groups mentioned above. One particularly pleasing example involves the orthosymplectic Lie superalgebra. Let W be a finite-dimensional superspace over $\mathbb{C}$; that is, $W = W_{\overline{0}} \oplus W_{\overline{1}}$ is a $\mathbb{Z}_2$-graded vector space, where $W_{\overline{0}}$ and $W_{\overline{1}}$ are called the even and odd parts of W, respectively. We call $w \in W$ homogeneous of degree $\overline{a} \in \mathbb{Z}_2$ if $w \in W_{\overline{a}}$ and denote by $\overline{w} \in \mathbb{Z}_2$ the degree of w. Note that the n-fold tensor power $W^{\otimes n}$ and $\mathfrak{gl}(W) := \mathrm{End}_{\mathbb{C}}(W)$ naturally inherit a $\mathbb{Z}_2$-grading from that of W. A Lie superalgebra is a superspace $\mathfrak{g} = \mathfrak{g}_{\overline{0}} \oplus \mathfrak{g}_{\overline{1}}$ with a super-bracket operation $[\cdot,\cdot] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ which satisfies graded versions of the usual Lie algebra axioms. For example, $\mathfrak{gl}(W)$ is naturally a Lie superalgebra under the super-commutator $[x, y] := xy - (-1)^{\overline{x}\cdot\overline{y}}\, yx$, for homogeneous $x, y \in \mathfrak{gl}(W)$.

Fix a nondegenerate even super-symmetric bilinear form $B = (\cdot,\cdot)$ on W. By definition, an even bilinear form on W satisfies $(W_{\overline{a}}, W_{\overline{b}}) = 0$ when $\overline{a} \neq \overline{b}$. A bilinear form on W is called super-symmetric if $(x,y) = (-1)^{\overline{x}\cdot\overline{y}}(y,x)$ for homogeneous elements $x, y \in W$. Such a form restricts to a symmetric bilinear form on $W_{\overline{0}}$ and a skew-symmetric bilinear form on $W_{\overline{1}}$. The Lie subsuperalgebra of $\mathfrak{gl}(W)$ which preserves this form is the orthosymplectic Lie superalgebra, $\mathfrak{osp}(W)$. If we then let the Brauer algebra $B_n(\delta_{\mathfrak{osp}})$, where $\delta_{\mathfrak{osp}}$ is the super-dimension $\dim W_{\overline{0}} - \dim W_{\overline{1}}$ of W, act on $W^{\otimes n}$ by signed versions of the actions described above, we again recover the conclusions of Schur-Weyl duality (see [18]). Since the bilinear form is a mixture of symmetric and skew-symmetric, this result conceptually unifies the results for $O(V)$ and $Sp(V)$ described above and gives some indication why both $\pm \dim V$ appear as parameters for these cases.

If we now take B to be a nondegenerate odd super-symmetric bilinear form, so $(W_{\overline{a}}, W_{\overline{b}}) = 0$ when $\overline{a} = \overline{b}$, then $\dim W_{\overline{0}} = \dim W_{\overline{1}}$ and the Lie subsuperalgebra of $\mathfrak{gl}(W)$ which preserves B is the type $\mathfrak{p}$ Lie superalgebra. Since B is nondegenerate, we can choose a basis $\{w_1, \ldots, w_{2r}\}$ for W, where $\{w_1, \ldots, w_r\}$ is a basis for $W_{\overline{0}}$ and $\{w_{r+1}, \ldots, w_{2r}\}$ is a basis for $W_{\overline{1}}$, with $(w_j, w_{r+i}) = (w_{r+i}, w_j) = \delta_{ij}$ and $(w_i, w_j) = (w_{r+i}, w_{r+j}) = 0$. Then, we find that a matrix realization of $\mathfrak{p}(r)$ consists of block matrices of the form

$$\begin{pmatrix} X & Y \\ Z & -X^{\top} \end{pmatrix}$$

with X any $r \times r$ matrix, Y a symmetric $r \times r$ matrix, and Z a skew-symmetric $r \times r$ matrix. In 2003, Moon [33] gave a presentation by generators and relations for the centralizer algebra of $\mathfrak{p}(r)$ in $\mathrm{End}_{\mathbb{C}}(W^{\otimes n})$ for $r \geq n$ and observed some similarities between this algebra and the Brauer algebra $B_n(0)$, but was unable to prove a definite link between these two algebras.

The diagrammatic marked Brauer algebra $B_n(\delta, \varepsilon)$, where $\varepsilon \in \{\pm 1\}$ and $\delta = 0$ when $\varepsilon = -1$, generalizes the ordinary Brauer algebra, in the sense that $B_n(\delta, 1)$ is isomorphic to the Brauer algebra, and provides a diagrammatic realization of

Moon's algebra when $\varepsilon = -1$. This algebra will be carefully defined in the next section. Using this algebra, we [29] were able to unify the results described above for $\mathfrak{osp}(W)$ and $\mathfrak{p}(r)$ by varying the parity of the bilinear form B:

Theorem. *Let $\mathfrak{g}$ be the Lie subsuperalgebra of $\mathfrak{gl}(W)$ which preserves B. Set $\varepsilon = (-1)^{\overline{B}}$, where $\overline{B}$ is $\overline{0}$ when B is even and $\overline{1}$ when B is odd, and $\delta = \dim W_{\overline{0}} - \dim W_{\overline{1}}$. Then $End_{\mathfrak{g}}\left(W^{\otimes n}\right) \cong B_n\left(\delta, \varepsilon\right)$ when $\dim W_{\overline{0}}$ or $\dim W_{\overline{1}}$ is sufficiently large compared to n.*

1.2 Summary of Results

We now turn to a study of the marked Brauer algebra, $B_n\left(0, -1\right)$, and its representation theory. The next chapter begins with a careful definition of $B_n\left(\delta, \varepsilon\right)$ and a study of some of its properties. In particular, we show that $B_n\left(0, -1\right)$ has a standard basis in the sense of [17] which is related to the cellular basis for the Brauer algebra $B_n(\delta)$. As a result, we find that $B_n\left(0, -1\right)$ is a quasi-hereditary algebra when n is odd, and $B_n\left(0, -1\right)$ modulo a nil ideal is quasi-hereditary when n is even. This observation justifies the introduction of standard modules for $B_n\left(0, -1\right)$ in Chapter 3. In Section 2.3, we introduce a version of Jucys-Murphy elements for $B_n\left(0, -1\right)$, which we later use to define weights of standard modules. These weights will form the basis of much of our later work on decomposition multiplicities.

In Section 3.1, we introduce standard modules for $B_n\left(0, -1\right)$ as they are defined in the general theory of standard based algebras. These modules are labeled by elements of a poset Λ and are easy to construct. Moreover, these standard modules coincide with the standard modules from the theory of quasi-hereditary algebras, are indecomposable, and have a unique simple quotient. In fact, every simple $B_n\left(0, -1\right)$-module is the quotient of some standard module, so a subset of the elements of Λ label the simple $B_n\left(0, -1\right)$-modules. We study the standard modules for the remain-der of this work. In particular, we obtain decomposition theorems for the restriction of a standard module to the subalgebras $\mathbb{C}S_n$ and $B_{n-1}(0,-1)$ of $B_n(0,-1)$ as well as for the induction of a standard module to $B_{n+1}(0,-1)$. We obtain refinements of the induction and restriction functors by considering the generalized eigenspaces for the Jucys-Murphy elements.

In the final chapter, we introduce combinatorial tools which will be used to determine the composition factors of the standard modules. We define the decomposition matrix for $B_n(0,-1)$ and show that it is a block matrix with four blocks. Two of the blocks are the decomposition matrices for the symmetric group algebra $\mathbb{C}S_n$ and the algebra $B_{n-2}(0,-1)$, while a third block is all zeros. We then restrict our attention to determining the decomposition multiplicities which appear in the final block. We define weights of standard modules and depict them graphically using weight diagrams. There is a procedure to decorate weight diagrams with arcs, and we prove a link between the resulting arc diagrams and the decomposition multiplicities for standard modules. As a corollary, we find that all decomposition multiplicities must be 0 or 1. More generally, we conjecture that these arc diagrams completely determine the nonzero decomposition multiplicities for standard modules.

Chapter 2

The Marked Brauer Algebra

Fix a parameter $\delta \in \mathbb{C}$. A *Brauer n-diagram* is a graph having two rows of n vertices each with the top row labeled $\{1, \ldots, n\}$ and the bottom row labeled $\{1', \ldots, n'\}$ where each vertex is connected to exactly one other vertex by an edge. Edges connecting two vertices in the top row will be called *cups* and edges connecting two vertices in the bottom row will be called *caps*, while edges connecting a vertex in the top row to a vertex in the bottom row will be called *through strings*. Ordered pairs will be used to denote edges in a Brauer diagram, so (i, j) with $i < j$ denotes a cup, (i', j') with $i < j$ a cap, and (i, j') a through string. Given two Brauer n-diagrams D_1 and D_2, we can form their product $D_1 D_2$ as follows:

1. Stack D_1 on top of D_2, identifying the vertices in the bottom row of D_1 with those in the top row of D_2.

2. Create a new n-diagram consisting of all connected components of the stacked diagrams except for those which lie entirely in the middle.

3. Scale the resulting diagram by $\delta^{c(D_1,D_2)}$, where $c(D_1, D_2)$ is the number of connected components of the stacked diagrams isolated in the middle.

This multiplication is associative and makes the complex vector space spanned by the collection of all Brauer n-diagrams into an algebra, $B_n(\delta)$. For example, in $B_7(\delta_G)$ we have

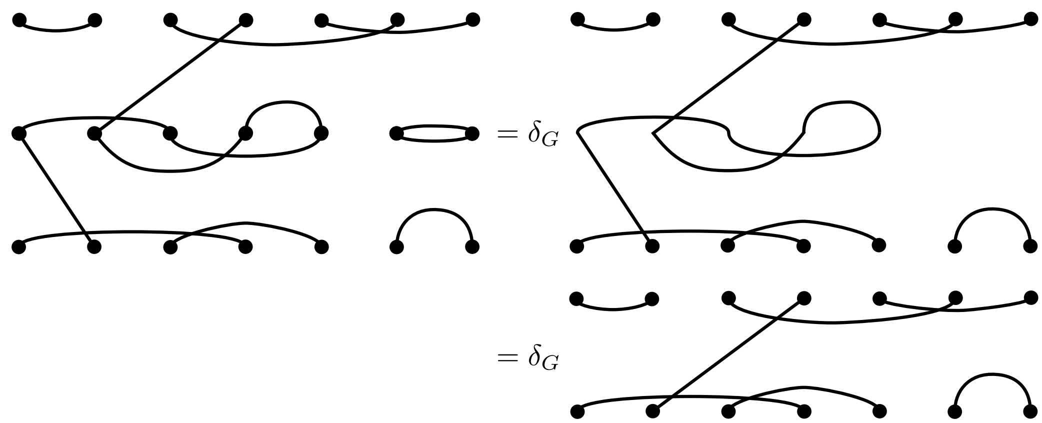


Figure 4

since there is one connected component isolated in the middle of the stacked diagrams.

A *marked Brauer n-diagram* is obtained from an ordinary Brauer n-diagram by marking the cups with beads and the caps with arrows in such a way that no two markings lie on the same horizontal line. To this end, we will imagine the space between the rows of a marked Brauer diagram as being divided into $2n-2$ distinct lateral bands called *levels*. Levels are numbered $1, \ldots n-1, (n-1)', \ldots, 1'$ from top to bottom and each may contain at most one marking. For example, a marked 5-diagram is shown below with levels indicated.

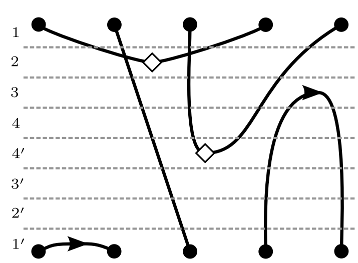


Figure 5

We call two markings *adjacent* if the vertical space between the markings contains no other markings. For example, in the marked diagram below, markings 1 and 2, 2 and 4, and 3 and 4 are the only pairs of adjacent markings.

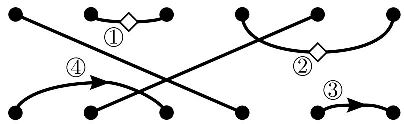


Figure 6

Two marked Brauer diagrams are considered equivalent if one can be obtained from the other by a regular isotopy which preserves the relative heights of markings. This means the regular isotopy may move markings around, but it must preserve adjacency and non-adjacency. For example, the two marked diagrams below are equivalent.

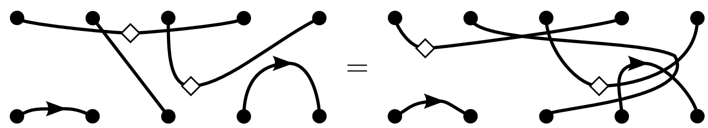


Figure 7

Fix an additional parameter $\varepsilon \in \{\pm 1\}$. The *marked Brauer algebra* $B_n(\delta, \varepsilon)$ is the associative, unital $\mathbb{C}$-algebra spanned by the collection of marked Brauer n-diagrams with the multiplication described below. In addition to planar isotopy, two local relations generate equivalent diagrams in $B_n(\delta, \varepsilon)$. The first says that one can reverse an arrow on a cap if the resulting diagram is scaled by a factor of ε:

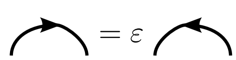


Figure 8

This gives one reason why we require $\varepsilon^2 = 1$, since twice reversing an arrow results in the original diagram, now scaled by ε^2. The second local relation says that a factor of ε must be introduced when two adjacent markings are moved past one another. Pictorially, this looks like

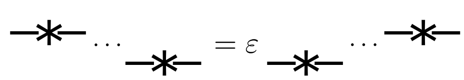


Figure 9

where the stars can be either type of marking. The example below shows this relation in use.

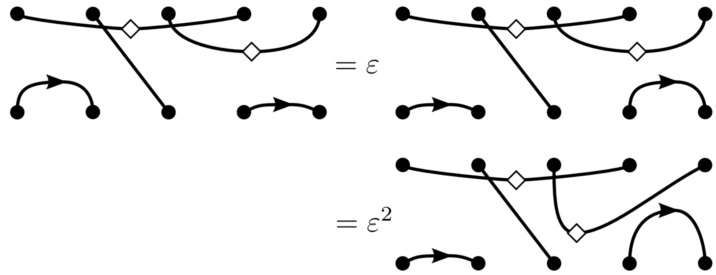


Figure 10

Multiplying two marked Brauer diagrams starts with the procedure outlined above for the ordinary Brauer algebra, resulting in a new diagram which is scaled by a power of δ and potentially contains edges which contain multiple beads and arrows at various levels. By possibly shifting markings between levels (at the cost of factors of ε as described above), we may cancel adjacent beads and arrows which appear on the same edge according to the following local simplification rules:

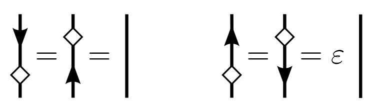


Figure 11

For example, consider the following product of marked Brauer diagrams in $B_7(\delta, \varepsilon)$:

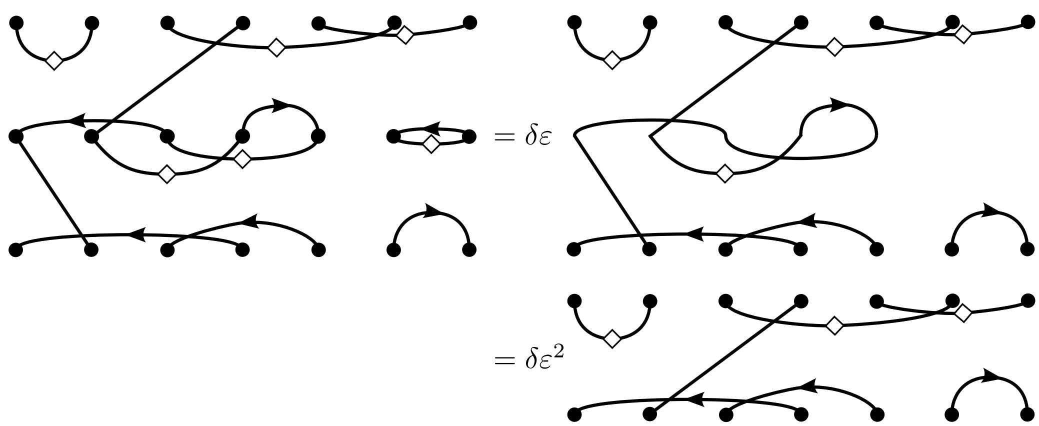


Figure 12

The set of all marked Brauer diagrams is in general much too large to form a basis for $B_n(\delta, \varepsilon)$: any two marked Brauer diagrams having the same connected components can be transformed into one another at the cost of factors of ε using the local simplification rules. We call a marked Brauer n-diagram *standard* if each cup (i,j) (respectively, cap (i',j')) sits in level i (resp. i') and all arrows point right. This defines an ordering on all the cups and caps in a marked diagram where

1. All cups are before all caps.

2. Cups are in increasing order by their left vertices from left to right.

3. Caps are in decreasing order by their left vertices from left to right.

The figure below is a standard marked Brauer 7-diagram.

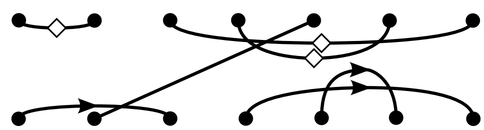


Figure 13

The following result, proven in [29], shows that the marked Brauer algebra has nice choices of basis and generating set. Observe that the generators are simply marked analogs of those for the ordinary Brauer algebra, which are depicted in the Introduction.

Theorem 2.1. *The set of standard marked Brauer n-diagrams provides a basis for $B_n(\delta,\varepsilon)$, so* $\dim B_n(\delta,\varepsilon) = (2n-1)!!$. *Moreover, $B_n(\delta,\varepsilon)$ is generated by the diagrams*

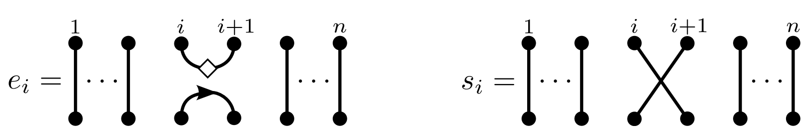


Figure 14

for $i = 1,\ldots,n-1$ subject to the following relations for all i,j with $j \neq i, i\pm 1$:

$$
\begin{array}{cc}
s_i^2 = 1 & e_i^2 = \delta e_i \\
e_i s_i = \varepsilon e_i & s_i e_i = e_i \\
s_i s_j = s_j s_i & e_i e_j = e_j e_i \\
s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} & e_i e_{i+1} e_i = \varepsilon e_i \\
s_i e_j = e_j s_i & e_{i+1} e_i e_{i+1} = \varepsilon e_{i+1} \\
s_i e_{i+1} e_i = \varepsilon s_{i+1} e_i & e_{i+1} e_i s_{i+1} = \varepsilon e_{i+1} s_i
\end{array}
$$

From these relations, we observe that $B_n(\delta, 1)$ is isomorphic to the unmarked Brauer algebra $B_n(\delta)$, while $B_n(0, -1)$ is isomorphic to Moon's algebra A_n from [33]. Moreover, we see that the elements $s_1, \ldots, s_{n-1}$ generate a subalgebra of $B_n(\delta, \varepsilon)$ which is isomorphic to the symmetric group algebra $\mathbb{C}S_n$. We also note that the subalgebra generated by $e_1, \ldots, e_{n-1}$ strongly resembles the Temperley-Lieb algebra, $TL_n(\delta)$, except for the presence of factors of ε in some of the relations. This subalgebra will be further investigated in Section 2.2.

The relations also imply the necessity for $\delta = 0$ when $\varepsilon = -1$: the equation

$$\delta e_i = e_i^2 = e_i\left(s_i e_i\right) = \left(e_i s_i\right) e_i = \left(\varepsilon e_i\right) e_i = \varepsilon \delta e_i$$

is valid for any δ when $\varepsilon = 1$, but requires $\delta = 0$ when $\varepsilon = -1$ since the characteristic of our ground field is 0.

We can obtain a slightly smaller generating set for $B_n(\delta, \varepsilon)$, which will be useful for simplifying proofs in later sections. To avoid confusing the notation for the edges of a diagram with that for transpositions in $\mathbb{C}S_n \subset B_n(\delta, \varepsilon)$, let $s_{i,j}$ denote the transposition (ij), where $i < j$. Then, for each $i = 2, \ldots, n-1$, we can write $e_i = s_{1,i} s_{2,i+1} e_1 s_{2,i+1} s_{1,i}$, as depicted below.

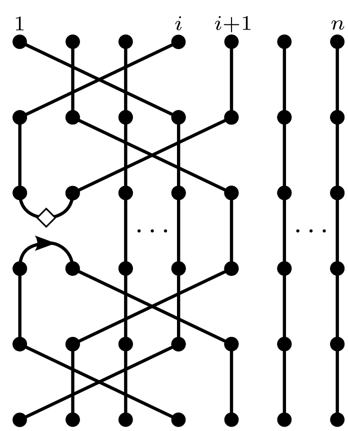


Figure 15

This proves the following result.

Lemma 2.2. *$B_n(\delta, \varepsilon)$ is generated as an algebra by any generating set for $\mathbb{C}S_n$ together with the element e_1. In particular, $B_n(\delta, \varepsilon)$ is generated by $e_1, s_1, s_2, \ldots, s_{n-1}$.*

Let $n = m + 2k$. Let I_n^m be the subspace of $B_n(\delta, \varepsilon)$ generated by the collection of all diagrams having m or fewer through strings. This is in fact a two-sided ideal in $B_n(\delta, \varepsilon)$ since multiplying any two marked Brauer diagrams can never increase the number of through strings. This gives a filtration of $B_n(\delta, \varepsilon)$ by two-sided ideals: $0 \subseteq I_n^t \subseteq \cdots \subseteq I_n^{n-2} \subseteq I_n^n = B_n(\delta, \varepsilon)$, where $t = 0$ if n is even and $t = 1$ if n is odd. Let B_n^m be the quotient algebra I_n^m / I_n^{m-2}, which is spanned by all diagrams having exactly m through strings. In this quotient, any product which results in a diagram having fewer than m through strings is zero. Note that $B_n^n = I_n^n / I_n^{n-2}$ is isomorphic to the symmetric group algebra $\mathbb{C}S_n$. We then have a vector space isomorphism,

$$B_n(\delta, \varepsilon) \cong B_n^n \oplus B_n^{n-2} \oplus \cdots \oplus B_n^t \cong \mathbb{C}S_n \oplus B_n^{n-2} \oplus \cdots \oplus B_n^t$$

where $t = 0$ if n is even and $t = 1$ if n is odd.

In what follows, we will often simply write B_n for $B_n(\delta, \varepsilon)$, specifying δ and ε only when confusion might arise. Since most of our work will focus on $B_n(0, -1)$, we will refer to factors of ε as *signs*.

2.1 Semisimplicity

Certainly, $B_1(0, -1)$ is semisimple since this algebra can be thought of as the group algebra of the symmetric group on one letter, $\mathbb{C}S_1$. This is, in fact, the only value of n for which $B_n(0, -1)$ is semisimple, as Moon proved in [33]. We can be prove this fact immediately for all even n (see below) using a simplified version of Moon's proof. We offer a different proof of the full result at the end of Section 4.3.

Proposition 2.3. *$B_n(0, -1)$ is not a semisimple algebra when n is even.*

Proof. It suffices to exhibit a nonzero nil ideal of $B_n(0,-1)$, since such an ideal is contained in the Jacobson radical of the algebra. The ideal I_n^0 consisting of all elements in $B_n(0,-1)$ having no through strings is a nonzero nil ideal since every element must square to zero, as a product of two diagrams having no through strings must necessarily have a connected component in the middle. Hence, $J\left(B_{2n}(0,-1)\right) \neq 0$ so $B_n(0,-1)$ is not semisimple. □

It is possible, and advantageous, to directly analyze the failure of $B_3(0,-1)$ to be semisimple. Let $x := (1+s_1+e_1)\cdot e_2$. It is straightforward to check, using the relations in Theorem 2.1, that the two-sided ideal $\langle x\rangle = \{x, xs_1, xs_1s_2\}$ and that each of these three elements squares to 0. Hence, $\langle x\rangle$ is a nil ideal of B_3, so $B_3(0,-1)$ is not semisimple.

2.2 The (Marked) Temperley-Lieb Algebra

It is well known that the Brauer Algebra $B_n(\delta)$ contains the Temperley-Lieb algebra $TL_n(\delta)$ as the unital subalgebra consisting of all planar Brauer diagrams (that is, those diagrams which can be drawn so that their edges do not intersect). To avoid confusing marked and unmarked e_i's, let $E_1,\ldots,E_{n-1} \in B_r(\delta)$ denote the unmarked generators of $B_n(\delta)$. It is well-known that $E_1,\ldots,E_{n-1}$ generate $TL_n(\delta)$, subject to the relations $E_i^2 = \delta E_i$, $E_iE_j = E_jE_i$ when $|i-j|>1$, and $E_iE_{i\pm1}E_i = E_i$ for all possible i,j.

Using our marked diagrams and the multiplication of marked diagrams described above, we may define a marked Temperley-Lieb algebra $TL_n(\delta,\varepsilon)$, where $\varepsilon \in \{\pm1\}$ to be the algebra generated by the marked diagrams $e_1,\ldots,e_{n-1}$ subject to $e_i^2 = \delta e_i$, $e_ie_j = e_je_i$ when $|i-j|>1$, and $e_ie_{i\pm1}e_i = \varepsilon e_i$ for all possible i,j. In [7], Brundan and Ellis also studied this object in the context of supercategories. By definition, we see that $TL_n(\delta,\varepsilon)$ is a subalgebra of $B_n(\delta,\varepsilon)$ and $TL_n(\delta,1) \cong TL_n(\delta)$ by forgetting the markings. Interestingly, unlike in $B_n(\delta, \varepsilon)$ we can have $\varepsilon = -1$ *and* $\delta \neq 0$ in $TL_n(\delta, \varepsilon)$ since this algebra contains no "twist" maps. It is straightforward to check that $TL_2(\delta, -1) \cong TL_2(\delta)$ for all δ, but $TL_n(\delta, -1)$ is in general a different object from $TL_n(\delta)$ when $n > 2$. However, we have the following result when $\delta = 0$.

Proposition 2.4. *The marked Temperley-Lieb algebra $TL_n(0, \varepsilon)$ is isomorphic to the ordinary Temperley-Lieb algebra $TL_n(0)$.*

Proof. Define $\varphi : \mathrm{TL}_n(0) \to \mathrm{TL}_n(0, \varepsilon)$ on the generators by sending E_i to $\varepsilon^i e_i$. The map is clearly a bijection of vector spaces. Moreover, the map is a well-defined algebra homomorphism since for all i

$$\varphi(E_i)\varphi(E_i) = \left(\varepsilon^i e_i\right)\left(\varepsilon^i e_i\right) = 0 = \varphi(0) = \varphi\left(E_i^2\right)$$

and

$$\varphi(E_i)\varphi(E_{i\pm1})\varphi(E_i) = \left(\varepsilon^i e_i\right)\left(\varepsilon^{i\pm1} e_{i\pm1}\right)\left(\varepsilon^i e_i\right) = \varepsilon^{i\pm1} \cdot \varepsilon e_i = \varepsilon^i e_i = \varphi(E_i)$$

while if $|i - j| > 1$, then

$$\varphi(E_i)\varphi(E_j) = \left(\varepsilon^i e_i\right)\left(\varepsilon^j e_j\right) = \left(\varepsilon^j e_j\right)\left(\varepsilon^i e_i\right) = \varphi(E_j)\varphi(E_i)$$

Thus, the map is in fact an isomorphism of algebras. $\square$

2.3 Jucys-Murphy Elements for $B_n(0, -1)$

Suppose $i < j$. Recall $s_{i,j}$ is the transposition (ij), so $s_{i,j}$ is the Brauer diagram having a single crossing of the i^{th} and j^{th} strands and all other vertices connected by vertical strands.

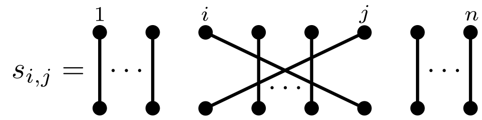


Figure 16

Let $e_{i,j}$ be the standard marked Brauer diagram having a single cup (i, j), a single cap (i', j'), and all other vertices connected by vertical strands.

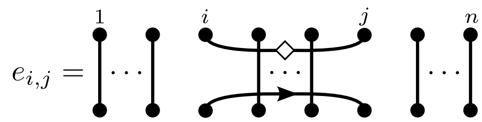


Figure 17

Inspired by Nazarov in [34] and Shalile in [40], we define the following n elements in B_n and call them *Jucys-Murphy elements*: $X_1 = 0$ and, for $k = 2, \ldots, n$, let $X_k = \sum_{j=1}^{k-1} (s_{j,k} + e_{j,k})$.

Lemma 2.5. *The following commutation relations hold when* $|j - k| > 1$*:*

$$s_i s_{j,k} = \begin{cases} s_{j,k} s_i & \text{if } i \neq j-1, j, k-1, k \\ s_{j-1,k} s_{j-1} & \text{if } i = j-1 \\ s_{j+1,k} s_j & \text{if } i = j \\ s_{j,k-1} s_{k-1} & \text{if } i = k-1 \\ s_{j,k+1} s_k & \text{if } i = k \end{cases}$$

and

$$s_i e_{j,k} = \begin{cases} e_{j,k} s_i & \text{if } i \neq j-1, j, k-1, k \\ e_{j-1,k} s_{j-1} & \text{if } i = j-1 \\ e_{j+1,k} s_j & \text{if } i = j \\ e_{j,k-1} s_{k-1} & \text{if } i = k-1 \\ e_{j,k+1} s_k & \text{if } i = k \end{cases}$$

Proof. These are all verified by direct calculation. When $|j - k| = 1$, we use the relations in Theorem 2.1, so assume $|j - k| > 1$. To see that s_i commutes with $s_{j,k}$ and

$e_{j,k}$ when $i \neq j-1, j, k-1, k$ one observes that s_i braids the i^{th} and $(i+1)^{\text{st}}$ strands which are connected to vertical strands in both $s_{j,k}$ and $e_{j,k}$. For the others, first observe that $s_{j,k} = s_{k-1} \cdots s_{j+1} s_j s_{j+1} \cdots s_{k-1}$ and $e_{j,k} = s_{k-1} \cdots s_{j+1} e_j s_{j+1} \cdots s_{k-1}$. Then,

$$\begin{aligned}
s_{j-1} s_{j,k} &= s_{j-1} \left(s_{k-1} \cdots s_{j+1} s_j s_{j+1} \cdots s_{k-1} \right) \\
&= s_{k-1} \cdots s_{j+1} \left(s_{j-1} s_j \right) s_{j+1} \cdots s_{k-1} \\
&= s_{k-1} \cdots s_{j+1} \left(s_j s_{j-1} s_j s_{j-1} \right) s_{j+1} \cdots s_{k-1} \\
&= \left(s_{k-1} \cdots s_{j+1} s_j s_{j-1} s_j s_{j+1} \cdots s_{k-1} \right) s_{j-1} \\
&= s_{j-1,k} s_{j-1}
\end{aligned}$$

$$\begin{aligned}
s_j s_{j,k} &= s_j \left(s_{k-1} \cdots s_{j+1} s_j s_{j+1} \cdots s_{k-1} \right) \\
&= s_{k-1} \cdots s_{j+2} \left(s_j s_{j+1} s_j \right) s_{j+1} \cdots s_{k-1} \\
&= s_{k-1} \cdots s_{j+2} \left(s_{j+1} s_j s_{j+1} \right) s_{j+1} \cdots s_{k-1} \\
&= \left(s_{k-1} \cdots s_{j+2} s_{j+1} s_{j+2} \cdots s_{k-1} \right) s_j \\
&= s_{j+1,k} s_j
\end{aligned}$$

$$\begin{aligned}
s_{k-1} s_{j,k} &= s_{k-1} \left(s_{k-1} \cdots s_{j+1} s_j s_{j+1} \cdots s_{k-1} \right) \\
&= \left(s_{k-2} \cdots s_{j+1} s_j s_{j+1} \cdots s_{k-2} \right) s_{k-1} \\
&= s_{j,k-1} s_{k-1}
\end{aligned}$$

$$\begin{aligned}
s_k s_{j,k} &= s_k \left(s_{k-1} \cdots s_{j+1} s_j s_{j+1} \cdots s_{k-1} \right) \\
&= \left(s_k s_{k-1} \cdots s_{j+1} s_j s_{j+1} \cdots s_{k-1} s_k \right) s_k \\
&= s_{j,k+1} s_k
\end{aligned}$$

Now, using the relations in Theorem 2.1 we have

$$\begin{aligned} s_{j-1}e_j &= s_{j-1}\left(e_j s_{j-1}\right)s_{j-1} \\ &= s_{j-1}\left(-e_j e_{j-1} s_j\right)s_{j-1} \\ &= \left(-s_{j-1}e_j e_{j-1}\right)s_j s_{j-1} \\ &= s_j e_{j-1} s_j s_{j-1} \end{aligned}$$

and

$$\begin{aligned} s_{j+1}e_j &= -s_j e_{j+1} e_j \\ &= s_j\left(-e_{j+1}e_j s_{j+1}\right)s_{j+1} \\ &= s_j\left(e_{j+1}s_j\right)s_{j+1} \end{aligned}$$

so

$$\begin{aligned} s_{j-1}e_{j,k} &= s_{j-1}\left(s_{k-1}\cdots s_{j+1}e_j s_{j+1}\cdots s_{k-1}\right) \\ &= s_{k-1}\cdots s_{j+1}\left(s_{j-1}e_j\right)s_{j+1}\cdots s_{k-1} \\ &= s_{k-1}\cdots s_{j+1}\left(s_j e_{j-1} s_j s_{j-1}\right)s_{j+1}\cdots s_{k-1} \\ &= \left(s_{k-1}\cdots s_{j+1}s_j e_{j-1} s_j s_{j+1}\cdots s_{k-1}\right)s_{j-1} \\ &= e_{j-1,k}s_{j-1} \end{aligned}$$

$$\begin{aligned} s_j e_{j,k} &= s_j\left(s_{k-1}\cdots s_{j+1}e_j s_{j+1}\cdots s_{k-1}\right) \\ &= s_{k-1}\cdots s_{j+2}s_j\left(s_{j+1}e_j\right)s_{j+1}\cdots s_{k-1} \\ &= s_{k-1}\cdots s_{j+2}s_j\left(s_j e_{j+1} s_j s_{j+1}\right)s_{j+1}\cdots s_{k-1} \\ &= \left(s_{k-1}\cdots s_{j+2}e_{j+1}s_{j+2}\cdots s_{k-1}\right)s_j \\ &= e_{j+1,k}s_j \end{aligned}$$

$$\begin{aligned} s_{k-1}e_{j,k} &= s_{k-1}\left(s_{k-1}\cdots s_{j+1}e_j s_{j+1}\cdots s_{k-1}\right) \\ &= \left(s_{k-2}\cdots s_{j+1}e_j s_{j+1}\cdots s_{k-2}\right)s_{k-1} \\ &= e_{j,k-1}s_{k-1} \end{aligned}$$

$$\begin{aligned} s_k e_{j,k} &= s_k\left(s_{k-1}\cdots s_{j+1}e_j s_{j+1}\cdots s_{k-1}\right) \\ &= \left(s_k\cdots s_{j+1}e_j s_{j+1}\cdots s_k\right)s_k \\ &= e_{j,k+1}s_k \end{aligned}$$

which completes the proof. □

Lemma 2.6. *When $i \neq j-1, j, k-1, k$, we have $e_i s_{j,k} = s_{j,k} e_i$ and $e_i e_{j,k} = e_{j,k} e_i$. In the remaining cases, the following commutation relations hold when $|j-k| > 1$:*

$$e_i s_{j,k} = \begin{cases} -e_{j-1}e_{j-1,k} & \text{if } i = j-1 \\ -e_j e_{j+1,k} & \text{if } i = j \\ e_{k-1}e_{j,k-1} & \text{if } i = k-1 \\ e_k e_{j,k+1} & \text{if } i = k \end{cases} \quad \text{and } s_{j,k}e_i = \begin{cases} -e_{j-1,k}e_{j-1} & \text{if } i = j-1 \\ -e_{j+1,k}e_j & \text{if } i = j \\ e_{j,k-1}e_{k-1} & \text{if } i = k-1 \\ e_{j,k+1}e_k & \text{if } i = k \end{cases}$$

Proof. By direct calculation, as follows. When $|j-k| = 1$, we use the relations in Theorem 2.1, so assume $|j-k| > 1$. Since e_i involves only the i^{th} and $(i+1)^{\text{st}}$ strands which are connected to vertical lines in $s_{j,k}$ and $e_{j,k}$ when $i \neq j-1, j, k-1, k$, we easily see that e_i commutes with $s_{j,k}$ and $e_{j,k}$. Using the relations in Theorem 2.1 we have

$$\begin{aligned} e_{j-1} &= -e_{j-1}e_j e_{j-1} \\ &= \left(e_{j-1}s_{j-1}\right)e_j e_{j-1} \\ &= e_{j-1}\left(s_{j-1}e_j e_{j-1}\right) \\ &= e_{j-1}\left(-s_j e_{j-1}\right) \end{aligned}$$

$$
\begin{aligned}
e_j e_{j+1} &= e_j\left(e_{j+1}s_j\right)s_j \\
&= e_j\left(-e_{j+1}e_j s_{j+1}\right)s_j \\
&= \left(-e_j e_{j+1} e_j\right)s_j s_j s_{j+1} s_j \\
&= \left(e_j s_j\right)\left(s_j s_{j+1} s_j\right) \\
&= -e_j s_{j+1} s_j s_{j+1}
\end{aligned}
$$

Hence,

$$
\begin{aligned}
e_{j-1}s_{j,k} &= e_{j-1}\left(s_{k-1}\cdots s_{j+1}s_j s_{j+1}\cdots s_{k-1}\right) \\
&= s_{k-1}\cdots s_{j+1}e_{j-1}s_j s_{j+1}\cdots s_{k-1} \\
&= s_{k-1}\cdots s_{j+1}\left(-e_{j-1}s_j e_{j-1}\right)s_j s_{j+1}\cdots s_{k-1} \\
&= -e_{j-1}\left(s_{k-1}\cdots s_{j+1}s_j e_{j-1}s_j s_{j+1}\cdots s_{k-1}\right) \\
&= -e_{j-1}e_{j-1,k}
\end{aligned}
$$

$$
\begin{aligned}
e_j s_{j,k} &= e_j\left(s_{k-1}\cdots s_{j+1}s_j s_{j+1}\cdots s_{k-1}\right) \\
&= s_{k-1}\cdots s_{j+2}\left(e_j s_{j+1}s_j s_{j+1}\right)s_{j+2}\cdots s_{k-1} \\
&= s_{k-1}\cdots s_{j+2}\left(-e_j e_{j+1}\right)s_{j+2}\cdots s_{k-1} \\
&= -e_j\left(s_{k-1}\cdots s_{j+2}e_{j+1}s_{j+2}\cdots s_{k-1}\right) \\
&= -e_j e_{j+1,k}
\end{aligned}
$$

$$
\begin{aligned}
s_{j,k}e_{j-1} &= \left(s_{k-1}\cdots s_{j+1}s_js_{j+1}\cdots s_{k-1}\right)e_{j-1} \\
&= s_{k-1}\cdots s_{j+1}s_je_{j-1}s_{j+1}\cdots s_{k-1} \\
&= s_{k-1}\cdots s_{j+1}s_j\left(-e_{j-1}s_je_{j-1}\right)s_{j+1}\cdots s_{k-1} \\
&= -\left(s_{k-1}\cdots s_{j+1}s_je_{j-1}s_js_{j+1}\cdots s_{k-1}\right)e_{j-1} \\
&= -e_{j-1,k}e_{j-1}
\end{aligned}
$$

$$
\begin{aligned}
s_{j,k}e_j &= \left(s_{k-1}\cdots s_{j+1}s_js_{j+1}\cdots s_{k-1}\right)e_j \\
&= s_{k-1}\cdots s_{j+1}s_j\left(s_{j+1}e_j\right)s_{j+2}\cdots s_{k-1} \\
&= s_{k-1}\cdots s_{j+1}s_j\left(-s_je_{j+1}e_j\right)s_{j+2}\cdots s_{k-1} \\
&= -\left(s_{k-1}\cdots s_{j+2}e_{j+1}s_{j+2}\cdots s_{k-1}\right)e_j \\
&= -e_{j+1,k}e_j
\end{aligned}
$$

The remaining four formulas are most easily verified by drawing the diagrams.

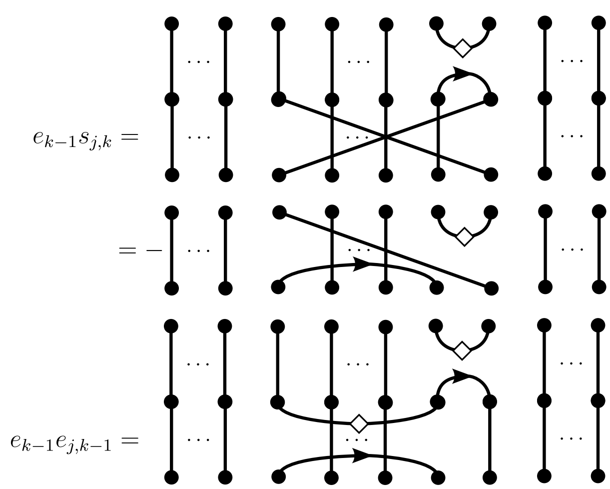


Figure 18

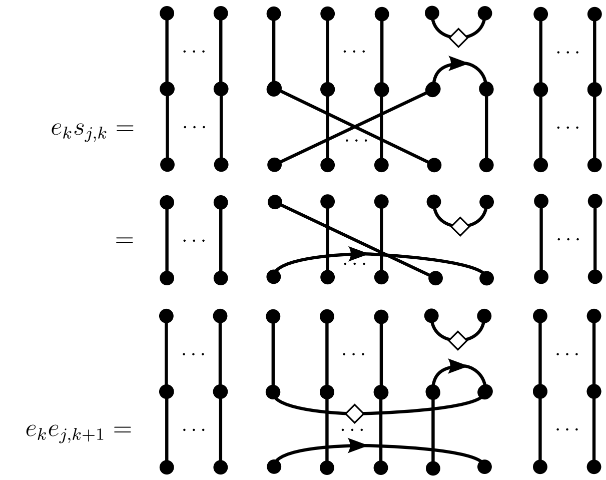


Figure 19

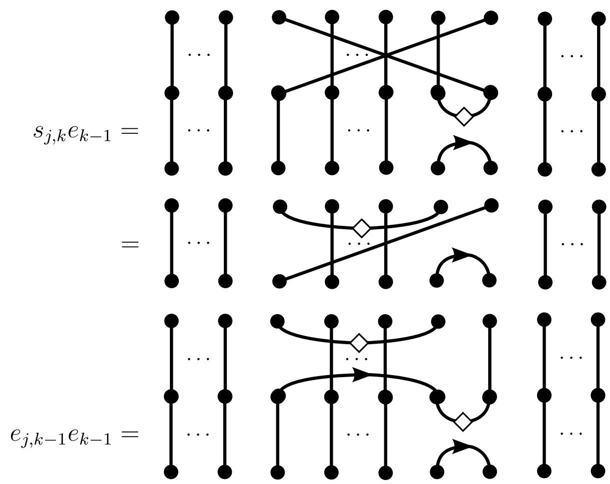


Figure 20

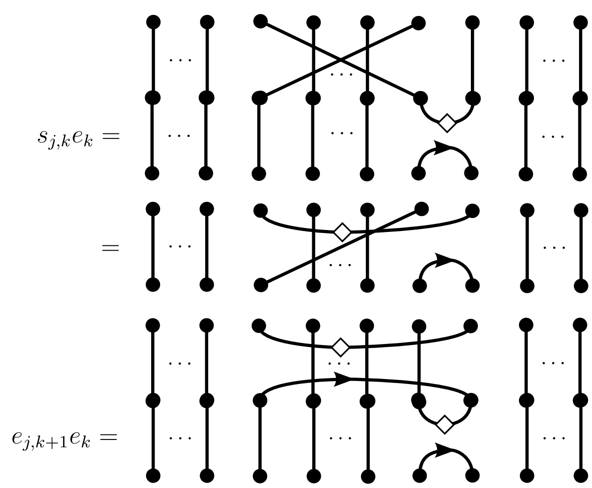


Figure 21

This completes the proof. $\square$

Proposition 2.7. *The following relations hold for the Jucys-Murphy elements:*

1. $s_i X_k = X_k s_i$ *and* $e_i X_k = X_k e_i$ *if* $i \neq k-1, k$,
2. $s_k X_k - X_{k+1} s_k = e_k - 1$ *and* $s_k X_{k+1} - X_k s_k = e_k + 1$,
3. $e_k \left(X_k - X_{k+1} \right) = e_k$ *and* $\left(X_k - X_{k+1} \right) e_k = -e_k$.

In particular, X_k *commutes with the subalgebra* $B_{k-1} \subseteq B_k$*. Hence,* $\langle X_1, \ldots, X_n \rangle$ *is a commutative subalgebra of* B_n*.*

Proof. If $i \neq k-1, k$, we have, by Lemmas 2.5 and 2.6,

$$\begin{aligned}
s_i X_k &= \sum_{j=1}^{k-1} (s_i s_{j,k} + s_i e_{j,k}) \\
&= \sum_{j \neq i, i+1} (s_i s_{j,k} + s_i e_{j,k}) + s_i s_{i,k} + s_i e_{i,k} + s_i s_{i+1,k} + s_i e_{i+1,k} \\
&= \sum_{j \neq i, i+1} (s_{j,k} s_i + e_{j,k} s_i) + s_{i+1,k} s_i + e_{i+1,k} s_i + s_{i,k} s_i + e_{i,k} s_i \\
&= \sum_{j=1}^{k-1} (s_{j,k} s_i + e_{j,k} s_i) \\
&= X_k s_i
\end{aligned}$$

and

$$\begin{aligned}
e_i X_k &= \sum_{j=1}^{k-1} (e_i s_{j,k} + e_i e_{j,k}) \\
&= \sum_{j \neq i, i+1} (e_i s_{j,k} + e_i e_{j,k}) + e_i s_{i,k} + e_i e_{i,k} + e_i s_{i+1,k} + e_i e_{i+1,k} \\
&= \sum_{j \neq i, i+1} (e_i s_{j,k} + e_i e_{j,k}) - e_i e_{i+1,k} + e_i e_{i,k} - e_i e_{i,k} + e_i e_{i+1,k} \\
&= \sum_{j \neq i, i+1} (e_i s_{j,k} + e_i e_{j,k}) + 0 \\
&= \sum_{j \neq i, i+1} (e_i s_{j,k} + e_i e_{j,k}) - e_{i+1,k} e_i + e_{i,k} e_i - e_{i,k} e_i + e_{i+1,k} e_i \\
&= \sum_{j \neq i, i+1} (s_{j,k} e_i + e_{j,k} e_i) + s_{i,k} e_i + e_{i,k} e_i + s_{i+1,k} e_i + e_{i+1,k} e_i
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{k-1} (s_{j,k}e_i + e_{j,k}e_i) \\
&= X_k e_i
\end{aligned}
$$

Since $B_{k-1} \subseteq B_k$ is generated by $\{e_1, s_1, \ldots, e_{k-2}, s_{k-2}\}$, this shows that X_k commutes with B_{k-1}. In fact, this shows that any two Jucys-Murphy elements commute: given X_j and X_k with $j < k$, we have $X_j \in B_{k-1}$ so the two elements commute by the previous observation. Now,

$$
\begin{aligned}
s_k X_k &= \sum_{j=1}^{k-1} (s_k s_{j,k} + s_k e_{j,k}) \\
&= \sum_{j=1}^{k-1} (s_{j,k+1} s_k + e_{j,k+1} s_k) + s_{k,k+1} s_k - 1 + e_{k,k+1} s_k + e_k \\
&= \sum_{j=1}^{k} (s_{j,k+1} s_k + e_{j,k+1} s_k) - 1 + e_k \\
&= X_{k+1} s_k + e_k - 1
\end{aligned}
$$

and so

$$
\begin{aligned}
s_k X_{k+1} &= X_k s_k - s_k e_k s_k + s_k s_k \\
&= X_k s_k + e_k + 1
\end{aligned}
$$

Finally, we have

$$
\begin{aligned}
e_k X_k &= \sum_{j=1}^{k-1} (e_k s_{j,k} + e_k e_{j,k}) \\
&= \sum_{j=1}^{k-1} (e_k e_{j,k+1} + e_k s_{j,k+1}) + e_k e_{k,k+1} + e_k s_{k,k+1} - e_k s_{k,k+1} \\
&= \sum_{j=1}^{k} (e_k s_{j,k+1} + e_k e_{j,k+1}) + e_k \\
&= e_k X_{k+1} + e_k
\end{aligned}
$$

and

$$
\begin{aligned}
X_k e_k &= \sum_{j=1}^{k-1} (s_{j,k} e_k + e_{j,k} e_k) \\
&= \sum_{j=1}^{k-1} (e_{j,k+1} e_k + s_{j,k+1} e_k) + e_{k,k+1} e_k + s_{k,k+1} e_k - s_{k,k+1} e_k \\
&= \sum_{j=1}^{k} (s_{j,k+1} e_k + e_{j,k+1} e_k) - e_k \\
&= X_{k+1} e_k - e_k
\end{aligned}
$$

which completes the proof. □

2.4 Center

By considering an arbitrary linear combination of basis vectors and systematically forcing relations among its coefficients, we find that the center of $B_2(0,-1)$ is trivial while the element

$$z = (s_1 - e_1)(s_2 + e_2) + (s_2 + e_2)(s_1 - e_1)$$

is the unique nonidentity generator of the center of $B_3(0,-1)$. Unfortunately, this method becomes too labor intensive to yield results in $B_n(0,-1)$ with $n > 3$ because of the rapid growth of the dimension of the algebra. The computer algebra system GAP was used to confirm that the center of $B_4(0,-1)$ is trivial, however the program failed to yield results for $B_5(0,-1)$.

In [12], Coulembier asserts that the element

$$\Theta_n := \prod_{2 \le i < j \le n} \left[1 - (X_i - X_j)^2\right]$$

where $\Theta_2 := 0$, is central in $B_n(0,-1)$. It is straightforward to check that $\Theta_3 = z - 2$.

Observe that

$$\begin{aligned}\Theta_n &= \left(\prod_{2\le i<j\le n-1}\left[1-(X_i-X_j)^2\right]\right)\cdot\prod_{2\le i<n}\left[1-(X_i-X_n)^2\right]\\ &= \Theta_{n-1}\cdot\prod_{2\le i<n}\left[1-(X_i-X_n)^2\right]\end{aligned}$$

Since the center of $B_4(0,-1)$ is trivial, we may conclude $\Theta_4 \in \mathbb{C}$. In fact, a calculation in GAP shows that $\Theta_4 = 0$ and, from the recursive formula above, it follows that $\Theta_n = 0$ for all $n \geq 4$.

Unfortunately, this fact does not tell us whether the center of $B_n(0,-1)$ is trivial for $n \geq 5$. However, this is believed to be true because of the connection between $B_n(0,-1)$ and the type $\mathfrak{p}$ Lie superalgebra, whose universal enveloping algebra has trivial center (see [20] for more details).

2.5 A Standard Basis for $B_n(0,-1)$

Let $(\Lambda, \leq)$ be a partially ordered set. Following [2], a subset $\Phi \subseteq \Lambda$ will be called an ideal of Λ if $\lambda \leq \mu$ implies $\lambda \in \Phi$ for any $\mu \in \Phi$ and $\lambda \in \Lambda$. We will call $\Phi \subseteq \Lambda$ a coideal is $\Lambda \backslash \Phi$ is an ideal. A *standard full-based algebra* on a poset Λ is a triple $(A, \mathscr{B}, \Lambda)$ where A is a $\mathbb{C}$-algebra such that

1. A has a basis $\mathscr{B}$, called a defining base, which is fibred over Λ, i.e. $\mathscr{B} = \bigsqcup_{\lambda\in\Lambda} \mathscr{B}^\lambda$.

2. For each $\lambda \in \Lambda$ there are index sets $I(\lambda)$ and $J(\lambda)$ such that

$$\mathscr{B}^\lambda = \left\{a_{ij}^\lambda : i \in I(\lambda), j \in J(\lambda)\right\}$$

3. For any $a \in A$ and $a_{ij}^{\lambda} \in \mathscr{B}$, we have

$$\begin{aligned} a \cdot a_{ij}^{\lambda} &\equiv \sum_{i' \in I(\lambda)} f_{i'}^{\lambda}(a,i) a_{i'j}^{\lambda} \bmod \left(A^{>\lambda}\right), \\ a_{ij}^{\lambda} \cdot a &\equiv \sum_{j' \in J(\lambda)} f_{j'}^{\lambda}(j,a) a_{ij'}^{\lambda} \bmod \left(A^{>\lambda}\right), \end{aligned}$$

where the coefficients $f_{i'}^{\lambda}(a,i)$ and $f_{j'}^{\lambda}(j,a)$ are independent of j and i, respectively.

4. For each $\lambda \in \Lambda$, there is a nonzero bilinear form β_λ on the $\mathbb{C}$-span of $\mathscr{B}^\lambda \times \mathscr{B}^\lambda$ such that $\beta_\lambda \left(a_{ij}^{\lambda}, a_{kl}^{\lambda}\right)$ is the coefficient in front of a_{kj}^{λ} in the product $a_{kl}^{\lambda} a_{ij}^{\lambda}$.

When an associative algebra has properties 1-3, it is called a *standard based algebra*. While the bilinear form in property 4 can always be defined for any standard based algebra, it need not always be nonzero for all $\lambda \in \Lambda$. When it is nonzero, we call the algebra *full*. Notice that being 'standard based' is almost the same as being cellular except for one key difference: a cellular algebra must also possess an involution (an anti-homomorphism which squares to the identity) which is compatible with the labeling of basis elements. In particular, a cellular basis is also a standard base, if we forget the involution. Although the ordinary Brauer algebras, $B_n(\delta, 1)$ are known to be cellular, the marked Brauer algebras $B_n(0, -1)$ do not possess the required involutions (as discussed in [29]) so are not cellular. They are, however standard based algebras, as we discuss below.

Recall that a *partition* of m, denoted $\lambda \vdash m$, is a weakly decreasing sequence of nonnegative integers, $(\lambda_1, \lambda_2, \dots)$ with $\sum\limits_{i \geq 1} \lambda_i = m$. Write $|\lambda|$ for $\sum\limits_{i \geq 1} \lambda_i$. We visualize partitions using *Young diagrams*, which are arrays of boxes having λ_i boxes in the i^{th} row, for $i \geq 1$. For example, the partition $(3, 2, 2, 1, 0, \dots)$ of size 8 is depicted by the Young diagram below.

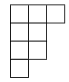


Figure 22

Let Λ_m be the set of all partitions of size m. Given $\lambda, \mu \in \Lambda_m$, the *dominance order* on Λ_m says $\lambda \trianglelefteq \mu$ if $\sum_{i=1}^{k} \lambda_i \leq \sum_{i=1}^{k} \mu_i$ for all $k \geq 1$. When $m \leq 5$, the dominance order is a total order. The dominance order on partitions of size 6 can be depicted as follows (increasing from left to right).

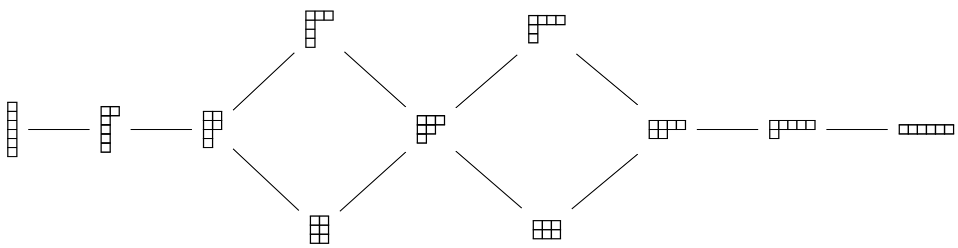


Figure 23

Let $\Lambda(n) = \Lambda_n \sqcup \Lambda_{n-2} \sqcup \cdots \sqcup \Lambda_t$, where $t = 0$ if n is even and $t = 1$ if n is odd. We define a partial order on $\Lambda(n)$ by declaring $\lambda \geq \mu$ if $|\lambda| \leq |\mu|$ or if $|\lambda| = |\mu|$ and $\lambda \trianglelefteq \mu$. Again, this is a total order for $n \leq 5$. For example, on $\Lambda(5)$, we have the order below.

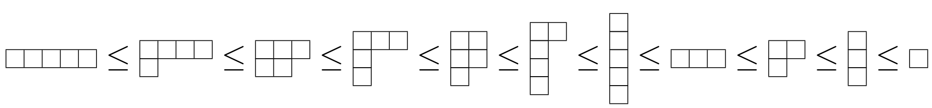


Figure 24

Recall that a *Young tableau* of shape $\lambda \vdash m$ is a filling of the boxes of the Young diagram for λ with the numbers $1, \ldots, m$. We call such a filling *standard* if the entries in each row and column are in strictly increasing order. For example, a standard tableau of shape $(3, 2, 2, 1, 0, \ldots)$ is given below.

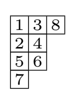


Figure 25

For each $\lambda \in \Lambda(n)$, let $\mathcal{T}^\lambda$ denote the set of standard Young Tableaux of shape λ.

Let $\mathcal{P}_n^m$ be the collection of all unmarked half-diagrams having n total vertices, m of which are free (not connected to any other vertex) and the remaining $2k$ are connected in pairs. Note that we may draw the arcs as cups or caps since we need only record connected components. For example, the following are elements of $\mathcal{P}_7^3$:

$x =$  $y =$ 

Figure 26

We may use the identification $B_n(0,-1) \cong B_n^n \oplus B_n^{n-2} \oplus \cdots \oplus B_n^t$ to obtain a basis for $B_n(0,-1)$ by choosing a basis for each summand B_n^m. In particular, finding a standard base for each summand will then give a standard base for $B_n(0,-1)$. Given $x, y \in \mathcal{P}_n^m$, let $D_{x,y} \in B_n^m$ be the standard marked Brauer diagram having top determined by $x \in \mathcal{P}_n^m$, bottom determined by $y \in \mathcal{P}_n^m$, and all through strings connected so that they do not cross. For example, with the above choice of x and y in $\mathcal{P}_7^3$, we have

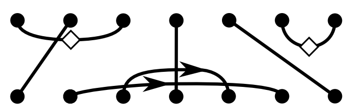
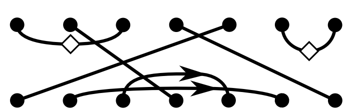
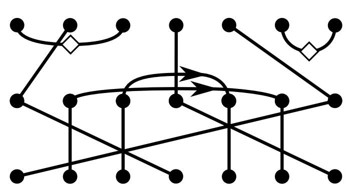
$D_{x,y} =$ 

Figure 27

For any standard $D \in B_n^m$, we can write $D = D_{x,y}\sigma$ for appropriate choices of $x, y \in \mathcal{P}_n^m$, where $\sigma \in S_n$ records the crossings of the through strings in D. For example, in B_7^3, we find

$D =$ 

$=$ 

$= D_{x,y}\sigma$

Figure 28

If D is not in standard form, then we simply have $D = \pm D_{x,y}\sigma$, where the sign depends on the number of cups and caps of D which are not in standard order. With $J_n^m = \mathbb{C}\{D_{x,y} : x, y \in \mathcal{P}_n^m\}$, we may then uniquely identify a standard diagram $D = D_{x,y}\sigma \in B_n^m$ with $D_{x,y} \otimes \sigma \in J_n^m \otimes \mathbb{C}S_m$. Since the collection of standard diagrams in B_n^m is a basis for B_n^m, this identification gives an isomorphism of vector spaces $B_n^m \cong J_n^m \otimes \mathbb{C}S_m$.

Recall that any two $S, T \in \mathcal{T}^\lambda$ correspond uniquely to an element $w \in S_m$ under the *Robinson-Schensted-Knuth correspondence* (see [28] for details). For example, the RSK correspondence yields the following associations in S_3.

S	T	w
$\ytableaushort{123}$	$\ytableaushort{123}$	1
$\ytableaushort{13,2}$	$\ytableaushort{13,2}$	(12)
$\ytableaushort{12,3}$	$\ytableaushort{12,3}$	(23)
$\ytableaushort{13,2}$	$\ytableaushort{12,3}$	(123)
$\ytableaushort{12,3}$	$\ytableaushort{13,2}$	(132)
$\ytableaushort{1,2,3}$	$\ytableaushort{1,2,3}$	(13)

Table 1

Given $S, T \in \mathcal{T}^\lambda$, let $C_{S,T}^\lambda$ be the *Kazhdan-Lusztig basis element*

$$C_{S,T}^\lambda = C_w = \sum_{v \preceq w} (-1)^{l(v)+l(w)} \overline{P_{v,w}}(1) v$$

in $\mathbb{C}S_m$, where $P_{v,w} \in \mathbb{Z}[q, q^{-1}]$ is the *Kazhdan-Lusztig polynomial* corresponding to $v, w \in S_m$, and the sum is taken over those v which are smaller than w in the *Bruhat order*. See [5, 27] for further details about these elements. It is well-known (see [21, 32]) that $\left\{C_{S,T}^\lambda : S, T \in \mathcal{T}^\lambda, \lambda \in \Lambda_m\right\}$ forms a cellular basis for $\mathbb{C}S_m$, so this collection is a standard base for $\mathbb{C}S_m$ by forgetting the involution. For example, this standard base for $\mathbb{C}S_3$ is

$$\begin{aligned}
C_1 &= 1 \\
C_{(12)} &= -1+(12) \\
C_{(23)} &= -1+(23) \\
C_{(123)} &= 1-(12)-(23)+(123) \\
C_{(132)} &= 1-(12)-(23)+(132) \\
C_{(13)} &= -1+(12)+(23)-(123)-(132)+(13)
\end{aligned}$$

For each $\lambda \in \Lambda_m$, take $I(\lambda) = \mathcal{P}_n^m \times \mathcal{T}^\lambda$. Given $(x,S),(y,T) \in I(\lambda)$, define $a^\lambda_{(x,S),(y,T)}$ to be the element $D_{x,y}C^\lambda_{S,T}$. Note that $a^\lambda_{(x,S),(y,T)}$ corresponds uniquely to the element $D_{x,y} \otimes C^\lambda_{S,T}$ in the vector space $J_n^m \otimes \mathbb{C}S_m$. So, we may take

$$\left\{a^\lambda_{(x,S),(y,T)} : (x,S),(y,T) \in I(\lambda), \lambda \in \Lambda_m\right\}$$

as a basis for B_n^m.

Theorem 2.8. *$B_n(0,-1)$ is a standard based algebra. The algebra is full when n is odd. When n is even, the algebra $B_n(0,-1)/I_n^0$ is full.*

Proof. With

$$\mathscr{B}^\lambda = \left\{a^\lambda_{(x,S),(y,T)} : (x,S),(y,T) \in I(\lambda)\right\}$$

and $J(\lambda) = I(\lambda)$ for all $\lambda \in \Lambda(n)$, we then have that $\mathscr{B} = \bigsqcup_{\lambda\in\Lambda} \mathscr{B}^\lambda$ is a basis for $B_n(0,-1)$, satisfying properties 1 and 2. Let $\lambda \in \Lambda_m$. Given a standard diagram $D \in B_n(0,-1)$ and $a^\lambda_{(x,S),(y,T)} \in \mathscr{B}^\lambda$, we have the following possibilities for the product $D \cdot a^\lambda_{(x,S),(y,T)}$:

1. If there is a connected component in the middle row when D is stacked on top of $D_{x,y}$, then $D \cdot a^\lambda_{(x,S),(y,T)} = 0$.

2. If D has fewer than $|\lambda|$ through strings, then the diagram $D \cdot D_{x,y}$ has more cups/caps than $D_{x,y}$, so the product $D \cdot a^{\lambda}_{(x,S),(y,T)}$ represents a higher term with respect to the partial order on $\Lambda(n)$. Hence, $D \cdot a^{\lambda}_{(x,S),(y,T)} \equiv 0 \bmod \left(B_n^{>\lambda}\right)$.

3. If D has at least $|\lambda|$ through strings, then $D_{x,y}$ has at least as many cups/caps as D, so some of the caps of D may interact with the cups of $D_{x,y}$ in order to form additional cups. Note that the caps of y play no role in the simplifications - only the cups/caps of D and the cups of x are used. So, we can write $D \cdot D_{x,y} = (-1)^{q(D,x)} D_{x',y}\sigma$, for some $x' \in \mathcal{P}_n^{|\lambda|}$ and $\sigma \in S_{|\lambda|}$, hence $D \cdot a^{\lambda}_{(x,S),(y,T)} = (-1)^{q(D,x)} D_{x',y}\sigma C^{\lambda}_{S,T}$. Now, $\sigma C^{\lambda}_{S,T}$ is a linear combination of other $C^{\lambda}_{U,T}$ (modulo higher terms) with coefficients depending only on S, U, and σ (which in turn depends on D and x), so the product $D \cdot a^{\lambda}_{(x,S),(y,T)}$ is then a linear combination of $a^{\lambda}_{(x',U),(y,T)}$ (modulo higher terms) with coefficients which depend on D, (x,S), and (x',U). In particular, we observe that the coefficients in this linear combination are independent of $(y,T) \in I(\lambda)$.

Thus $D \cdot a^{\lambda}_{(x,S),(y,T)}$ satisfies property 3, so it follows that $a \cdot a^{\lambda}_{(x,S),(y,T)}$ also satisfies this property for all $a \in B_n(0,-1)$ since a can be written as a linear combination of standard marked Brauer diagrams. Similarly, one shows that $a^{\lambda}_{(x,S),(y,T)} \cdot D$ satisfies property 3, so $B_n(0,-1)$ is a standard based algebra.

It remains to check that the bilinear form defined as in property 4 is nonzero on each Λ_m with $m > 0$. Note that the diagrams in B_n/I_n^0 always have at least one through string. Let $b_{n,m} \in \mathcal{P}_n^m$ have the first m vertices free followed by k successive cups, as shown below.

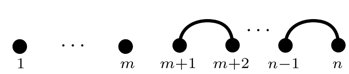


Figure 29

Similarly, let $b'_{n,m} \in \mathcal{P}_n^m$ have the first $m-1$ vertices free, followed by k successive cups with the last vertex free. Then, the product $D_{b_{n,m},b'_{n,m}} \cdot D_{b_{n,m},b_{n,m}}$,

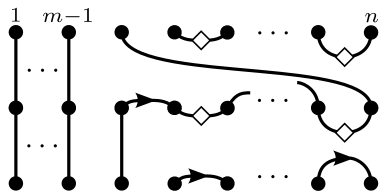


Figure 30

is equal to $D_{b_{n,m},b_{n,m}}$ since the markings on the through string (m, m') cancel left-to-right with no signs. Choose $S, T, U, V \in \mathcal{T}^\lambda$ such that the coefficient in front of $C^\lambda_{S,V}$ in the product $C^\lambda_{S,T}C^\lambda_{U,V}$ is nonzero. Since we view S_m as the group of permutations of $\{1, \dots m\}$, it follows that $C^\lambda_{S,T}D_{b_{n,m},b_{n,m}} = D_{b_{n,m},b_{n,m}}C^\lambda_{S,T}$ since all of the through strings in $D_{b_{n,m},b_{n,m}}$ are vertical lines connecting the first m vertices in both rows. Then, $\beta_\lambda\left(a^\lambda_{(b_{n,m},U),(b_{n,m},V)}, a^\lambda_{(b_{n,m},S),\left(b'_{n,m},T\right)}\right)$ is the coefficient in front of $a^\lambda_{(b_{n,m},S),(b_{n,m},V)} = D_{b'_{n,m},b_{n,m}}C^\lambda_{S,V}$ in the product

$$\begin{aligned} a^\lambda_{(b_{n,m},S),\left(b'_{n,m},T\right)} a^\lambda_{(b_{n,m},U),(b_{n,m},V)} &= \left(D_{b_{n,m},b'_{n,m}}C^\lambda_{S,T}\right)\left(D_{b_{n,m},b_{n,m}}C^\lambda_{U,V}\right) \\ &= D_{b_{n,m},b'_{n,m}}D_{b_{n,m},b_{n,m}}C^\lambda_{S,T}C^\lambda_{U,V} \\ &= D_{b_{n,m},b_{n,m}}C^\lambda_{S,T}C^\lambda_{U,V} \end{aligned}$$

which is nonzero. Hence, $\beta_\lambda \neq 0$. □

Theorem 2.9. *B_n/I^0_n is split quasi-hereditary. Hence, the category of finite dimensional B_n/I^0_n-modules is a highest weight category.*

Proof. This is a direct consequence of Theorems 3.2.1 and 3.1.2 in [17] since B_n/I^0_n is standard full-based. □

Note that when n is odd, we have $I^0_n = 0$ and hence $B_n = B_n/I^0_n$ is quasi-hereditary in this case. When n is even, we have already remarked in Proposition 2.3 that I^0_n is a nil ideal since the product of any two of its elements is zero. The quasi-hereditary algebra B_n/I^0_n thus consists of all marked Brauer diagrams which have at least two throughstrings, where the product of any two diagrams which simplifies to have no throughstrings is zero.

Since the Brauer algebra, $B_n(\delta)$, is cellular, its cellular basis also serves as a standard base. By ignoring the signs which appear in the calculations above, we note that the previous two results hold for $B_n(\delta, 1) \cong B_n(\delta)$. In particular, when $\delta = 0$, we find the algebra $B_n(0, 1)/I_n^0$ is quasi-hereditary. It is interesting to note that it appears no one has previously observed this fact. Indeed, those authors who have studied the Brauer algebra within the context of quasi-hereditary algebras have treated separately the case when n is even and $\delta = 0$. Perhaps this observation could be used to treat all Brauer algebras simultaneously using the methods of [14].

Chapter 3

Representations of $B_n(0,-1)$

3.1 Standard Modules for $B_n(0,-1)$

Assume $n = m + 2k$ and $\lambda \vdash m$. Recall $b_{n,m} \in \mathcal{P}_n^m$ is the half-diagram consisting of m free vertices followed by k caps, as depicted in Figure 29.

Let T_λ denote the *canonical* standard Young tableau of shape λ having entries $1, \dots \lambda_1$ in the first row, $\lambda_1 + 1, \dots, \lambda_1 + \lambda_2$ in the second row, and so on, where $\lambda = (\lambda_1, \dots, \lambda_t)$. For example, if $\lambda = (3, 2, 2, 1, 0, \dots)$, then

$$T_\lambda = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} 6 & 7 \\ \cline{1-2} 8 \\ \cline{1-1} \end{array}$$

Figure 31

We define the *standard module* $\Delta_n(\lambda)$ as in [17] to be the left $B_n(0,-1)$-module with basis consisting of all elements

$$a^\lambda_{(x,S)} := a^\lambda_{(x,S),(b_{n,m},T_\lambda)} = D_{x,b_{n,m}} C^\lambda_{S,T_\lambda}$$

where $(x, S) \in I(\lambda)$. The action of $B_n(0,-1)$ is then given by

$$a \cdot a^{\lambda}_{(x,S)} \equiv \sum_{(x',T)\in I(\lambda)} f^{\lambda}_{(x',T)}\left(a,(x,S)\right) a^{\lambda}_{(x',T)} \bmod \left(B_n^{>\lambda}\right)$$

where $a \in B_n(0,-1)$, $(x,S) \in I(\lambda)$, and the coefficients $f^{\lambda}_{(x',T)}\left(a,(x,S)\right)$ are those in property 3 of the definition of a standard based algebra given in Section 2.5. We now give an alternate description of these standard modules, which is considerably easier to work with.

Let S^{λ} be the *Specht module* for S_m corresponding to the partition λ. Recall $\left\{S^{\lambda} : \lambda \vdash m\right\}$ is a complete set of non-isomorphic simple modules for the symmetric group. The single-parameter set $\left\{C^{\lambda}_{S,T_{\lambda}} : S \in \mathcal{T}^{\lambda}\right\}$ of Kazhdan-Lusztig basis elements for $\mathbb{C}S_m$ provides a basis for S^{λ} when we compute the action modulo terms labeled by partitions which are higher in the partial order on Λ_m. For example, using the Kazhdan-Lusztig basis described previously for $m = 3$, we find that $S^{\square\square\square} \cong \mathbb{C}\{C_1\}$ since

$$(12).C_1 = 1 + (-1 + (12)) \equiv C_1$$

$$(23).C_1 = 1 + (-1 + (23)) \equiv C_1$$

modulo the higher terms $C_{(12)}$ and $C_{(23)}$, respectively.

Let $\overline{J_n^m}$ be the subspace of all diagrams in J_n^m having fixed bottom $b_{n,m}$. Thus, $\overline{J_n^m} = \left\{D_{x,b_{n,m}} : x \in \mathcal{P}_n^m\right\}$. We then have a vector space isomorphism $\Delta_n(\lambda) \cong \overline{J_n^m} \otimes S^{\lambda}$ via the correspondence $a^{\lambda}_{(x,S)} \leftrightarrow D_{x,b_{n,m}} \otimes C^{\lambda}_{S,T_{\lambda}}$. The function $f_n^m : \mathcal{P}_n^m \to \overline{J_n^m}$ given by $x \mapsto D_{x,b_{n,m}}$ provides a one-to-one correspondence between $\mathcal{P}_n^m$ and $\overline{J_n^m}$ and will be heavily used in future sections.

Let $\overline{B_n^m}$ be the space having as its basis the collection of all standard diagrams in B_n^m with fixed bottom $b_{n,m}$. This is a left B_n-submodule of B_n^m since multiplying on the left by any (marked) Brauer diagram will yield zero or a new diagram having bottom $b_{n,m}$. This submodule is also a right S_m-module, where we view $S_m \subseteq S_n$ as permutations of the set $\{1, \ldots, m\}$, since stacking such a permutation below a diagram with bottom $b_{n,m}$ only permutes the first m vertices and does not affect the last k caps. Given any diagram $D \in \overline{B_n^m}$, we can find a unique $x \in \mathcal{P}_n^m$ which has the same cups and free vertices as D and a unique permutation $\sigma \in S_m$ such that $D = \pm f_n^m(x)\sigma$. So, the collection of diagrams $\{f_n^m(x)\sigma : x \in \mathcal{P}_n^m, \sigma \in S_m\}$ gives a basis for $\overline{B_n^m}$. Moreover, the identification $f_n^m(x)\sigma \leftrightarrow f_n^m(x) \otimes \sigma$ gives a vector space isomorphism $\overline{B_n^m} \cong \overline{J_n^m} \otimes \mathbb{C}S_m$.

Proposition 3.1. *Let $n = m + 2k$ and $\lambda \vdash m$. Let $\{v_1, \ldots, v_{d_\lambda}\}$ be a basis for the Specht module S^λ. Then $\Delta_n(\lambda) \cong \overline{B_n^m} \otimes_{\mathbb{C}S_m} S^\lambda$ as a left B_n-module and has as a basis $\{f_n^m(x) \otimes v_i : x \in \mathcal{P}_n^m, 1 \leq i \leq d_\lambda\}$, so $\dim \Delta_n(\lambda) = \binom{n}{m} \cdot (2k-1)!! \cdot d_\lambda$.*

Proof. As vector spaces, we have

$$\begin{aligned} \overline{B_n^m} \otimes_{\mathbb{C}S_m} S^\lambda &\cong \left(\overline{J_n^m} \otimes \mathbb{C}S_m\right) \otimes_{\mathbb{C}S_m} S^\lambda \\ &\cong \overline{J_n^m} \otimes S^\lambda \\ &\cong \Delta_n(\lambda) \end{aligned}$$

Since the image of f_n^m serves as a basis for $\overline{J_n^m} \subseteq \overline{B_n^m}$, this shows that the claimed set is indeed a basis for $\overline{B_n^m} \otimes_{\mathbb{C}S_m} S^\lambda$. The dimension formula now follows since $\left|\overline{J_n^m}\right| = |\mathcal{P}_n^m| = \binom{n}{m} \cdot (2k-1)!!$, as there are $\binom{n}{m}$ ways to choose the m free vertices for a diagram followed by $(2k-1)!!$ ways to connected the remaining $2k$ vertices into pairs.

It remains to check that the vector space isomorphism above is actually as left B_n-modules. Let $\varphi : \Delta_n(\lambda) \to \overline{B_n^m} \otimes_{\mathbb{C}S_m} S^\lambda$ be the linear map which sends a standard basis element $a_{(x,S)}^\lambda = D_{x,b_{n,m}} C_{S,T_\lambda}^\lambda$ to $D_{x,b_{n,m}} \otimes C_{S,T_\lambda}^\lambda$. Given $D \otimes v \in \overline{B_n^m} \otimes_{\mathbb{C}S_m} S^\lambda$ with D a standard diagram, we can write $D = D_{x,b_{n,m}}\sigma$ for some $x \in \mathcal{P}_n^m$ and

$\sigma \in S_m$, so that

$$\begin{aligned} D \otimes v &= f_n^m(x) \otimes \sigma v \\ &= \sum_{S \in \mathcal{T}^\lambda} \alpha_S D_{x,b_{n,m}} \otimes C^\lambda_{S,T_\lambda} \end{aligned}$$

It is straightforward to check that the linear map $\psi : \overline{B_n^m} \otimes_{\mathbb{C}S_m} S^\lambda \to \Delta_n(\lambda)$ which sends $D \otimes v$ to $\sum_{S \in \mathcal{T}^\lambda} \alpha_S D_{x,b_{n,m}} C^\lambda_{S,T_\lambda}$ gives a well-defined inverse to φ. To see that φ commutes with the action of B_n, it suffices to take $D' \in B_n$ to be a standard diagram. If $D' D_{x,b_{n,m}} = 0$, then we certainly have $\varphi\left(D' a^\lambda_{(x,S)}\right) = D'\varphi\left(a^\lambda_{(x,S)}\right)$, as both sides are 0. So, suppose $D' D_{x,b_{n,m}} \neq 0$. We can find $q \in \mathbb{Z}_{\geq 0}$, $y \in \mathcal{P}_n^m$, and $\sigma \in S_m$ so that $D' D_{x,b_{n,m}} = (-1)^q D_{y,b_{n,m}} \sigma$, and hence

$$\begin{aligned} D' a^\lambda_{(x,S)} &= (-1)^q D_{y,b_{n,m}} \sigma C^\lambda_{S,T_\lambda} \\ &= (-1)^q \sum_{S' \in \mathcal{T}^\lambda} \alpha_{S'} D_{y,b_{n,m}} C^\lambda_{S',T_\lambda} \end{aligned}$$

Applying φ, we find

$$\begin{aligned} \varphi\left(D' a^\lambda_{(x,S)}\right) &= (-1)^q \sum_{S' \in \mathcal{T}^\lambda} \alpha_{S'} D_{y,b_{n,m}} \otimes C^\lambda_{S',T_\lambda} \\ &= (-1)^q D_{y,b_{n,m}} \otimes \sigma C^\lambda_{S,T_\lambda} \\ &= (-1)^q D_{y,b_{n,m}} \sigma \otimes C^\lambda_{S,T_\lambda} \\ &= D' D_{x,b_{n,m}} \otimes C^\lambda_{S,T_\lambda} \\ &= D'\varphi\left(a^\lambda_{(x,S)}\right) \end{aligned}$$

Hence, the isomorphism is as B_n-modules. □

To close this section, we note that one can similarly define standard right B_n-modules, $\Delta_n^{\mathrm{op}}(\lambda)$, following the discussion in [17]. These modules have $\mathbb{C}$-basis $b^\lambda_{(x,S)} := a^\lambda_{(b_{n,m},T_\lambda),(x,S)}$, where $(x,S) \in I(\lambda)$, with B_n action given by the second formula of property 3 in the definition of a standard based algebra found in Section 2.5. Although we will only use these modules in the next section to define a bilinear form on standard modules, these modules also give rise to the *costandard modules*, $\nabla_n(\lambda)$, whose study is reserved for future work.

3.2 Properties of Standard Modules

We now collect some information about the standard modules from the general theory of standard based algebras found in [17]. Given $\lambda \in \Lambda_m$, let β_λ^Δ be the bilinear form on $\Delta_n(\lambda) \times \Delta_n^{\mathrm{op}}(\lambda)$ which sends $\left(a^\lambda_{(x,S)}, b^\lambda_{(y,T)}\right)$ to the coefficient in front of $a_{(b_{n,m},T_\lambda),(b_{n,m},T_\lambda)}$ in the product $a_{(b_{n,m},T_\lambda),(y,T)} \cdot a_{(x,S),(b_{n,m},T_\lambda)}$. Note that this is simply the restriction of the bilinear form β defined in Section 2.5 to the subspace of $B_n^m \times B_n^m$ spanned by $\left\{a^\lambda_{(x,S),(b_{n,m},T_\lambda)} : (x,S) \in I(\lambda)\right\} \times \left\{a^\lambda_{(b_{n,m},T_\lambda),(y,T)} : (y,T) \in I(\lambda)\right\}$. In particular, this means $\beta_\lambda^\Delta \neq 0$ if and only if $\beta_\lambda \neq 0$.

Theorem 3.2. *Let $\Lambda'(n) = \left\{\lambda \in \Lambda(n) : \beta_\lambda^\Delta \neq 0\right\}$ and let $\lambda \in \Lambda'(n)$. The following results about standard modules hold.*

1. *$Hom_{B_n}(\Delta_n(\lambda), \Delta_n(\mu)) = \begin{cases} 0 & \text{unless } \lambda \leq \mu \\ \mathbb{C} & \text{if } \mu = \lambda \end{cases}$. Hence, each standard module is indecomposable.*

2. *$\Delta_n(\lambda)$ is a cyclic B_n-module.*

3. *$rad(\Delta_n(\lambda))$ is the radical of the bilinear form β_λ^Δ.*

4. *$L_n(\lambda) := \Delta_n(\lambda)/rad(\Delta_n(\lambda))$ is absolutely irreducible and occurs exactly once as a composition factor of $\Delta_n(\lambda)$.*

5. *If $L_n(\mu)$ is a composition factor of $\Delta_n(\lambda)$, then $\mu \leq \lambda$.*

6. *$\{L_n(\lambda) : \lambda \in \Lambda'(n)\}$ is a complete set of all non-isomorphic simple B_n-modules. Hence, when n is odd, the simple modules of B_n are parametrized by the set of partitions of $n, n-2, \ldots, , 1$, and, when n is even, the simple modules of B_n are parametrized by the set of partitions of $n, n-2, \ldots, 2$.*

7. *The standard modules agree with their counterparts in the highest weight category B_n-***mod.**

8. *Let $\Delta = \{M \in Ob\,(B_n - \textbf{mod}) : M \cong \Delta_n(\mu)$ for some $\mu \in \Lambda(n)\}$. The projective cover $P_n(\lambda)$ of $L_n(\lambda)$ has a Δ-filtration. If $(P_n(\lambda) : \Delta_n(\mu))$ denotes the number of sections isomorphic to $\Delta_n(\mu)$ in a Δ-filtration of $P_n(\lambda)$, then $(P_n(\lambda) : \Delta_n(\lambda)) = 1$, while $(P_n(\lambda) : \Delta_n(\mu)) = 0$ implies $\lambda \leq \mu$.*

9. *Weak BGG Reciprocity: $(P_n(\lambda) : \Delta_n(\mu))$ is equal to the number of times $L_n(\lambda)$ occurs as a composition factor in $\nabla_n(\mu)$.*

10. *Strong BGG Reciprocity: $(P_n(\lambda) : \Delta_n(\mu))$ is equal to the number of times $L_n(\lambda')$ occurs as a composition factor in $\Delta_n(\mu')$, where λ' is the conjugate partition to λ.*

Proof. Properties 1-8 follow immediately from Corollaries 2.3.3, 2.3.4,3.2.2, Proposition 2.4.4, and Theorem 2.4.1 in [17] since B_n/I_n^0 is standard full-based. Property 9 follows from Theorem 3.1 in [25], while Property 10 follows from Theorem 3 in [12]. □

The labeling of simple modules in Property 6 above was previously obtained in [29] without the use of standard modules.

Corollary 3.3. *If $\lambda \in \Lambda'(n)$, then rad$\,(\Delta_n(\lambda))$ is the unique maximal submodule of $\Delta_n(\lambda)$, and hence $L_n(\lambda)$ is the unique simple quotient of $\Delta_n(\lambda)$.*

Proof. If not, then there is a maximal submodule M which is distinct from the radical of $(\Delta_n(\lambda))$. By definition of the radical, we must have rad$\,(\Delta_n(\lambda)) \leq M$ so by the correspondence theorem $M/\mathrm{rad}\,(\Delta_n(\lambda))$ is then a submodule of $\Delta_n(\lambda)/\mathrm{rad}\,(\Delta_n(\lambda)) = L_n(\lambda)$. But, $L_n(\lambda)$ is absolutely irreducible by Theorem 3.2(4), so $M = \mathrm{rad}\,(\Delta_n(\lambda))$ or $M = \Delta_n(\lambda)$. $\square$

3.3 Standard Modules Labeled by $\emptyset$

In this section, we begin a study of the standard module $\Delta_n(\emptyset)$, with n even. This standard module comes from the ideal I_n^0 of B_n, which is precisely the portion of B_n which causes the algebra to fail to be quasi-hereditary. Because of this, our later methods for determining composition factors will not apply. We begin with a lemma.

Lemma 3.4. *When n is even, there is a surjective B_n-module homomorphism $\Delta_n(\square\square) \to \Delta_n(\emptyset)$. This map is an isomorphism if and only if $n = 2$.*

Proof. Say $S^{\square\square} = \mathbb{C}v$ while $S^{\emptyset} = \mathbb{C}w$. Let $\varphi : \Delta_n(\square\square) \to \Delta_n(\emptyset)$be the map given by $f_n^2(x) \otimes v \mapsto f_n^2(x)e_1 \otimes w$. Since the first two vertices of the bottom row of $f_n^2(x)$ are connected to through strings, the diagram $f_n^2(x)e_1$ will have no through strings, as required. This map is a homomorphism of B_n-modules since if $D \in B_n$ then

$$\begin{aligned} \varphi\left(D.f_n^2(x) \otimes v\right) &= \left(D.f_n^2(x)\right) e_1 \otimes w \\ &= D.\left(f_n^2(x)e_1\right) \otimes w \\ &= D.\varphi\left(f_n^2(x) \otimes v\right) \end{aligned}$$

by the associativity of diagram multiplication.

Given $f_n^0(x) \otimes w \in \Delta_n(\emptyset)$ where $x \in \mathcal{P}_n^0$ has cups $(a_1, b_1), \ldots \left(a_{n/2}, b_{n/2}\right)$ with $a_1 < a_2 < \cdots < a_{n/2}$, let $x_1 \in \mathcal{P}_n^2$ be the half-diagram with cups $(a_2, b_2), \ldots \left(a_{n/2}, b_{n/2}\right)$. Then, $f_n^2(x_1)$ will have through strings $(a_1, 1')$ and $(a_2, 2')$ so that $\varphi\left(f_n^2(x_1) \otimes v\right) = f_n^2(x_1)\, e_1 \otimes w = f_n^0(x) \otimes w$. Note that no sign is required since the cap of e_1 is lower than the caps of $f_n^2(x_1)$ as required since it is the leftmost cap in $f_n^2(x_1)\,e_1$ while the cup of e_1 must be raised past all the cups and caps of $f_n^2(x_1)$ at the cost of $2\cdot\binom{n-2}{2}$ signs. This shows that φ is surjective.

Observe that

$$\begin{aligned}
\dim \Delta_n(\square\square) &= \binom{n}{2}\cdot\left(2\left(\frac{n-2}{2}\right)-1\right)!! \\
&= \frac{n!}{2!\cdot(n-2)!}(n-3)!! \\
&= \frac{n!}{2\cdot(n-2)!!} \\
&= \frac{n}{2}\cdot(n-1)!! \\
&\geq (n-1)!! \\
&= \dim \Delta_n(\emptyset)
\end{aligned}$$

so this map will be injective (and hence an isomorphism) if and only if $n=2$. $\square$

Corollary 3.5. *When n is even, $L_n(\emptyset)\cong L_n(\square\square)$.*

Proof. Composing the surjective homomorphism $\Delta_n(\square\square)\to\Delta_n(\emptyset)$ described above with the quotient map $\Delta_n(\emptyset)\to L_n(\emptyset)$, we find that $L_n(\emptyset)$ is a simple quotient of $\Delta_n(\square\square)$. But, $L_n(\square\square)$ is the unique simple quotient of $\Delta_n(\square\square)$, so the two simple modules must be isomorphic. $\square$

It should be possible and interesting to explicitly describe the kernel of the map in Lemma 3.4 following the arguments presented in [16] and [24], however this calculation is reserved for a future project since it is not needed in the present work.

3.4 Eigenvalues of Jucys-Murphy Elements on Standard Modules

The table below summarizes the eigenvalues of the Jucys-Murphy elements which occur on each standard module for $B_n(0,-1)$, $n = 1, 2, 3, 4, 5$. The exponents are the multiplicities with which these eigenvalues occur. The X_1 column is omitted since $X_1 = 0$. These results were obtained by direct calculations using the computer algebra system MAXIMA and provided invaluable evidence to motivate many of the results which appear in later sections.

Standard Module	X_2	X_3	X_4
$\Delta_2\left(\emptyset\right)$	1^1	-	-
$\Delta_2\left(\ydiagram{2}\right)$	1^1	-	-
$\Delta_2\left(\ydiagram{1,1}\right)$	-1^1	-	-
$\Delta_3\left(\ydiagram{1}\right)$	$1^2, -1^1$	$2^1, 0^2$	-
$\Delta_3\left(\ydiagram{3}\right)$	1^1	2^1	-
$\Delta_3\left(\ydiagram{2,1}\right)$	$1^1, -1^1$	$1^1, -1^1$	-
$\Delta_3\left(\ydiagram{1,1,1}\right)$	-1^1	-2^1	-
$\Delta_4\left(\emptyset\right)$	$1^2, -1^1$	$2^1, 0^2$	1^3
$\Delta_4\left(\ydiagram{2}\right)$	$1^4, -1^2$	$2^2, 1^1, 0^2, -1^1$	$3^1, 1^3, 0^2$
$\Delta_4\left(\ydiagram{1,1}\right)$	$1^3, -1^3$	$2^1, 1^1, 0^2, -1^1 - 2^1$	$2^2, -1^4$
$\Delta_4\left(\ydiagram{4}\right)$	1^1	2^1	3^1
$\Delta_4\left(\ydiagram{3,1}\right)$	$1^2, -1^1$	$2^1, 1^1, -1^1$	$2^2, -1^1$
$\Delta_4\left(\ydiagram{2,2}\right)$	$1^1, -1^1$	$1^1, -1^1$	0^2
$\Delta_4\left(\ydiagram{2,1,1}\right)$	$1^1, -1^2$	$1^1, -1^1, -2^1$	$1^1, -2^2$
$\Delta_4\left(\ydiagram{1,1,1,1}\right)$	-1^1	-2^1	-3^1

Table 2

Standard Module	X_2	X_3	X_4	X_5
$\Delta_5\left(\ydiagram{1}\right)$	$1^9, -1^6$	$2^4, 1^2, 0^6, -1^2, -2^1$	$3^1, 2^2, 1^6, 0^2, -1^4$	$2^6, 0^9$
$\Delta_5\left(\ydiagram{3}\right)$	$1^7, -1^3$	$2^4, 1^2, 0^2, -1^2$	$3^2, 2^2, 1^3, 0^2, -1^1$	$4^1, 2^6, 0^3$
$\Delta_5\left(\ydiagram{2,1}\right)$	$1^{11}, -1^9$	$2^4, 1^5, 0^4, -1^5, -2^2$	$3^1, 2^4, 1^4, 0^4, -1^5, -2^2$	$3^3, 1^8, -1^9$
$\Delta_5\left(\ydiagram{1,1,1}\right)$	$1^4, 1^6$	$2^1, 1^2, 0^2, -1^2, -2^3$	$2^2, 1^1, -1^4, -2^2, -3^1$	$2^3, -2^7$
$\Delta_5\left(\ydiagram{5}\right)$	1^1	2^1	3^1	4^1
$\Delta_5\left(\ydiagram{4,1}\right)$	$1^3, -1^1$	$2^2, 1^1, -1^1$	$3^1, 2^2, -1^1$	$3^3, -1^1$
$\Delta_5\left(\ydiagram{3,2}\right)$	$1^3, -1^2$	$2^1, 1^2, -1^2$	$2^2, 0^2, -1^1$	$2^2, 0^3$
$\Delta_5\left(\ydiagram{3,1,1}\right)$	$1^3, -1^3$	$2^1, 1^2, -1^2, -2^1$	$2^2, 1^1, -1^1, -2^2$	$2^3, -2^3$
$\Delta_5\left(\ydiagram{2,2,1}\right)$	$1^2, -1^3$	$1^2, -1^2, -2^1$	$1^1, 0^2, -2^2$	$0^3, -2^2$
$\Delta_5\left(\ydiagram{2,1,1,1}\right)$	$1^1, -1^3$	$1^1, -1^1, -2^2$	$1^1, -2^2, -3^1$	$1^1, -3^3$
$\Delta_5\left(\ydiagram{1,1,1,1,1}\right)$	-1^1	-2^1	-3^1	-4^1

Table 3

Chapter 4

Restriction and Induction in $B_n(0, -1)$

4.1 Restriction of Standard Modules to $\mathbb{C}S_n$

We first recall some definitions which we will need in order to state the main result of this section. A partition η is a *hook* if $\eta = (n-k, 1, 1, \ldots, 1)$, where k-many 1's appear, for some $0 \leq k \leq n-1$. The integer $n-k$ is called the *width* of the hook, while $k+1$ is its *depth*. For example, the hook $(3, 1, 1, 1)$ has depth 4 and width 3, as we can easily see by looking at its Young diagram:

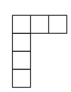


Figure 32

Within the Young diagram for a partition, the *hook about the box* (i, j) is the collection of boxes $\{(i, k) : k \geq j\} \cup \{(l, j) : l > i\}$. For example, in the partition $(5, 4, 2, 2, 1)$, the hook about $(2, 2)$ is labeled using blank and filled circles while the hook about $(3, 2)$ is labeled using filled circles.

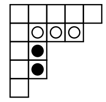


Figure 33

We will call a hook about the box (i, i) in a partition a *diagonal hook*. So, the hook about $(2, 2)$ depicted in the example above is a diagonal hook.

Given two partitions λ and μ, we write $\mu \subseteq \lambda$ if $\mu_i \leq \lambda_i$ for all i. Visually, this means that the Young diagram for μ fits entirely inside of that for λ when the diagrams are superimposed with top and left edges aligned. For example, the partition $\mu = (3, 2, 2, 1)$ is contained in the partition $\lambda = (5, 4, 2, 2, 1)$, as we can see in the picture below, in which the boxes for μ are shaded.

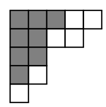


Figure 34

When $\mu \subseteq \lambda$, the *skew diagram* λ/μ is obtained by removing the boxes of μ from the Young diagram for λ; if $\mu \nsubseteq \lambda$, then λ/μ is defined to be $\emptyset$. For the two partitions shown above, λ/μ looks like

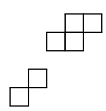


Figure 35

As with Young diagrams, we may fill the boxes of a skew diagram with positive integers. Given a tuple $\nu = (\nu_1, \ldots, v_t)$ of nonnegative integers, we say a filling of λ/μ having ν_i-many instances of the positive integer i is of *weight* ν. Certainly, such a filling exists if and only if $\sum_{i=1}^{t} \nu_i = |\lambda| - |\mu|$. Call a filling *semistandard* if the the numbers weakly increase from left to right in each row and strictly increase from top to bottom in each column, and call the skew diagram which is filled in this way a *semistandard tableau of shape λ/μ and weight ν*. For example, the $\nu = (4,1,1)$-weight filling of λ/μ shown below is semistandard.

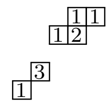


Figure 36

We associate to a semistandard tableau T of shape λ/μ and weight ν a word $w(T)$ in the alphabet $\{1, 2, \ldots, |\nu|\}$ obtained by listing the entries of each box of T starting at the top right and moving right-to-left and top-to-bottom. In the example above, the associated word is 112131. Call $w(T) = w_1 w_2 \cdots w_r$ a *lattice permutation* if, for all $j = 2, 3, \ldots, |\nu|$, the number of j's in the subword $w_1 w_2 \cdots w_i$ is less than or equal to the number of $j-1$'s in the same subword for all $i = 1, 2, \ldots, r$. In particular, if $\nu = (\nu_1, \ldots, \nu_t)$, this condition forces $\nu_j \leq \nu_{j-1}$ for all $j = 2, 3, \ldots, t$, so $\nu \vdash (|\lambda| - |\mu|)$ when $w(T)$ is a lattice permutation. The word associated to our example above is a lattice permutation, however the similar word 113121 (which also comes from a semistandard tableau of the shape above) is not a lattice permutation since the number of 3's in the subword 113 is greater than the number of 2's in this subword.

We call T a *Littlewood-Richardson tableau* if its associated word is a lattice permutation. So, the semistandard tableau shown above is a Littlewood-Richardson tableau. The Littlewood-Richardson coefficient $c^{\lambda}_{\mu\nu}$ is the number of Littlewood-Richardson tableau of shape λ/μ and weight ν. For example, let $\lambda = \ydiagram{4,1}$, $\mu = \square$, and $\nu = \ydiagram{3,1}$. Then, $\lambda/\mu = \ydiagram{1+3,1}$, so there are two semistandard tableau of shape λ/μ and weight ν, namely $\ytableaushort{\none111,2}$ and $\ytableaushort{\none112,1}$. The associated words are 1112 and 2111, respectively, of which only 1112 is a lattice permutation. Hence, $c^{\lambda}_{\mu\nu} = 1$.

Because the symmetric group algebra is semisimple over $\mathbb{C}$, the restriction of

$\Delta_n(\mu)$ to $\mathbb{C}S_n$ will decompose as a direct sum of Specht modules,

$$\mathrm{res}^{B_n}_{\mathbb{C}S_n} \Delta_n(\mu) = \bigoplus_{\lambda} d_{\mu,\lambda} S^{\lambda}$$

with some multiplicities $d_{\mu,\lambda}$. We now compute these multiplicities in terms of the Littlewood-Richardson coefficients.

Theorem 4.1. *Suppose $\mu \vdash m$, where $n = m+2k$. The multiplicity of S^{λ}, $\lambda \vdash n$, in $res^{B_n}_{\mathbb{C}S_n} \Delta_n(\mu)$ is $\sum_{\nu \in X} c^{\lambda}_{\mu\nu}$, where X is the set of partitions of $2k$ whose diagonal hooks have depth one less than width and $c^{\lambda}_{\mu\nu}$ are the Littlewood-Richardson coefficients.*

Before giving the proof, which mimics a similar proof by Hanlon and Wales in [23] for the ordinary Brauer algebra, we consider the following example with $n = 5$ and $\mu = \square$. Among the partitions of $4 = n - |\mu|$,

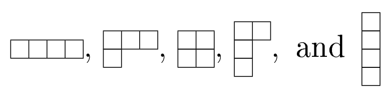


Figure 37

only $\nu = \yng(3,1)$ satisfies the diagonal hook condition, since the width of this hook is three, while its depth is two. We compute the Littlewood-Richardson coefficients $c^{\lambda}_{\mu\nu}$ for the various $\lambda \vdash 5$. Our calculation above showed $c^{\lambda}_{\mu\nu} = 1$ for $\lambda = \yng(4,1)$. If λ is one of

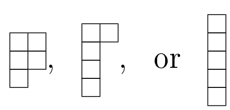


Figure 38

then $c^{\lambda}_{\mu\nu} = 0$ since any filling of the skew shapes,

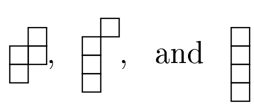


Figure 39

by weight ν must place at least two 1's in a column, which is not a semistandard filling. For $\lambda = \ydiagram{5}$, we note that $\ytableaushort{1112}$ is the only semistandard filling of weight ν, and the associated word, 2111, is not a lattice permutation. So, $c^{\lambda}_{\mu\nu} = 0$ in this case. Finally, for $\lambda = \ydiagram{3,2}$ or $\lambda = \ydiagram{3,1,1}$, we have precisely one semistandard ν-weight filling of λ/μ, $\ytableaushort{\none11,12}$ and $\ytableaushort{\none11,1,2}$, respectively, and both associated words are lattice permutations. These calculations show

$$\operatorname{res}^{B_n}_{\mathbb{C}S_n} \Delta_n(\mu) = S^{\ydiagram{4,1}} \oplus S^{\ydiagram{3,2}} \oplus S^{\ydiagram{3,1,1}}$$

which has been empirically verified by explicit computation using the computer algebra system MAXIMA.

Proof. Note that $\overline{B^0_{2k}}$ is naturally an S_{2k}-module, so that $\overline{B^0_{2k}} \otimes S^{\mu}$ is a module for the Young subgroup $S_{2k} \times S_m$ of S_n. We first claim that

$$\operatorname{res}^{B_n}_{\mathbb{C}S_n} \Delta_n(\mu) \cong \operatorname{ind}^{S_n}_{S_{2k} \times S_m} \overline{B^0_{2k}} \otimes S^{\mu}$$

as S_n-modules. We have

$$\begin{aligned} \dim \operatorname{ind}^{S_n}_{S_{2k} \times S_m} \overline{B^0_{2k}} \otimes S^{\mu} &= \frac{n!}{(2k)! \cdot m!} \cdot \dim \overline{B^0_{2k}} \otimes S^{\mu} \\ &= \binom{n}{m} \cdot (2k-1)!! \cdot \dim S^{\mu} \\ &= \dim \Delta_n(\mu) \end{aligned}$$

so it suffices to exhibit a surjective S_n-module homomorphism

$$\operatorname{ind}^{S_n}_{S_{2k} \times S_m} \overline{B^0_{2k}} \otimes S^{\mu} \to \operatorname{res}^{B_n}_{\mathbb{C}S_n} \Delta_n(\mu)$$

Noting that $\overline{B^0_{2k}} = \mathbb{C}S_{2k}.f^0_{2k}(b_{2k,0})$, we define $\varphi : \operatorname{ind}^{S_n}_{S_{2k} \times S_m} \overline{B^0_{2k}} \otimes S^{\mu} \to \operatorname{res}^{B_n}_{\mathbb{C}S_n} \Delta_n(\mu)$ to be the map given by $\sigma \otimes f^0_{2k}(b_{2k,0}) \otimes v \mapsto \sigma f^m_n \left(\overline{b_{n,m}}\right) \otimes v$, where $\overline{b_{n,m}} \in \mathcal{P}^m_n$ is the half-diagram with cups $(1,2), \ldots, (2k-1, 2k)$ and free vertices $2k+1, \ldots 2k+m$.

Note that $(\sigma, \tau) \in S_{2k} \times S_m \leq S_n$ acts diagonally on $\overline{B_{2k}^0} \otimes S^\mu$, while, inside B_n, the element (σ, τ) is the horizontal juxtaposition of the diagrams σ and τ, so the left action of $S_{2k} \times S_m$ on $\mathrm{res}_{\mathbb{C}S_n}^{B_n} \Delta_n(\mu)$ mimics the diagonal action on the element $f_n^m\left(\overline{b_{n,m}}\right) \otimes v$: the copy of τ acting on the last m places of $f_n^m\left(\overline{b_{n,m}}\right)$ can slide though the diagram and subsequently move across the tensor product to act on S^μ.

Given a standard diagram $f_n^m(x) \in \overline{B_n^m}$ we can find $\sigma_x \in S_n$ so that $f_n^m(x) = (-1)^q \sigma_x f_n^m\left(\overline{b_{n,m}}\right)$, where q depends on the number of cups of $f_n^m\left(\overline{b_{n,m}}\right)$ which are moved out of the standard order by σ_x. Then,

$$\begin{aligned} \varphi\left((-1)^q \sigma_x \otimes f_{2k}^0\left(b_{2k,0}\right) \otimes v\right) &= (-1)^q \sigma_x f_n^m\left(\overline{b_{n,m}}\right) \otimes v \\ &= f_n^m(x) \otimes v \end{aligned}$$

so the map is surjective. Finally, for $\tau \in S_n$ we have

$$\begin{aligned} \varphi\left(\tau.\sigma \otimes f_{2k}^0\left(b_{2k,0}\right) \otimes v\right) &= \varphi\left((\tau\sigma) \otimes f_{2k}^0\left(b_{2k,0}\right) \otimes v\right) \\ &= (\tau\sigma) f_n^m\left(\overline{b_{n,m}}\right) \otimes v \\ &= \tau.\sigma f_n^m\left(\overline{b_{n,m}}\right) \otimes v \\ &= \tau.\varphi\left(\sigma \otimes f_{2k}^0\left(b_{2k,0}\right) \otimes v\right) \end{aligned}$$

so the map is an isomorphism of of S_n-modules as desired.

Recall the hyperoctahedral group $H_k = \mathbb{Z}_2 \wr S_k$, which embeds into S_{2k} as the centralizer of the element $(12)(34)\cdots(2k-1,2k)$. An element $(\mathbf{a}, \sigma) \in H_k$ acts on $f_{2k}^0\left(b_{2k,0}\right)$ as follows. If the k-tuple $\mathbf{a}$ has a 1 in the i^{th} place and 0 elsewhere, then $\mathbf{a}$ acts by interchanging the endpoints of the i^{th} cup of $f_{2k}^0\left(b_{2k,0}\right)$ while fixing everything else. This action is trivial since interchanging the vertices of a cup does not affect the cup. The permutation $\sigma \in S_k$ acts by permuting the k cups of $f_{2k}^0\left(b_{2k,0}\right)$. This means that $\mathbb{C}\left\{f_{2k}^0\left(b_{2k,0}\right)\right\}$ is isomorphic, as an H_k-module, to the sign representation, sgn_k, for H_k since the even permutations will fix $f^0_{2k}\left(b_{2k,0}\right)$ while odd permutations will act by -1.

We claim $\overline{B^0_{2k}} \cong \mathrm{ind}^{S_{2k}}_{H_k}\mathbb{C}\left\{f^0_{2k}\left(b_{2k,0}\right)\right\}$ as S_{2k}-modules. We have

$$\dim \mathrm{ind}^{S_{2k}}_{H_k}\mathbb{C}\left\{f^0_{2k}\left(b_{2k,0}\right)\right\} = \frac{(2k)!}{(2k)!!}\cdot 1 = (2k-1)!! = \dim \overline{B^0_{2k}}$$

so we need only show the map $\psi : \mathrm{ind}^{S_{2k}}_{H_k}\mathbb{C}\left\{f^0_{2k}\left(b_{2k,0}\right)\right\} \to \overline{B^0_{2k}}$ given by $\sigma \otimes f^0_{2k}\left(b_{2k,0}\right) \mapsto \sigma f^0_{2k}\left(b_{2k,0}\right)$ is a surjective S_{2k}-module homomorphism. That ψ is an S_{2k}-module map is clear since

$$\begin{aligned}
\psi\left(\tau.\sigma \otimes f^0_{2k}\left(b_{2k,0}\right)\right) &= \psi\left((\tau\sigma) \otimes f^0_{2k}\left(b_{2k,0}\right)\right) \\
&= (\tau\sigma)\, f^0_{2k}\left(b_{2k,0}\right) \\
&= \tau.\sigma f^0_{2k}\left(b_{2k,0}\right) \\
&= \tau.\psi\left(\sigma \otimes f^0_{2k}\left(b_{2k,0}\right)\right)
\end{aligned}$$

for all $\tau \in S_{2k}$.

Given $f^0_{2k}(x) \in \overline{B^0_{2k}}$, choose $\sigma_x \in S_{2k}$ so that $f^0_{2k}(x) = (-1)^q \sigma_x f^0_{2k}\left(b_{2k,0}\right)$, where q depends on the number of cups of $f^0_{2k}\left(b_{2k,0}\right)$ which are moved out of the standard order by σ_x. Then,

$$\begin{aligned}
\psi\left((-1)^q \sigma_x \otimes f^0_{2k}\left(b_{2k,0}\right)\right) &= (-1)^q \sigma_x f^0_{2k}\left(b_{2k,0}\right) \\
&= f^0_{2k}(x)
\end{aligned}$$

so the map is an isomorphism of S_{2k}-modules as desired. Hence, as S_{2k}-modules, we have

$$
\begin{aligned}
\overline{B_{2k}^{0}} &\cong \operatorname{ind}_{H_k}^{S_{2k}} \mathbb{C}\left\{ f_{2k}^{0}\left(b_{2k,0}\right)\right\} \\
&\cong \operatorname{ind}_{H_k}^{S_{2k}} \operatorname{sgn}_k \\
&\cong \bigoplus_{\nu \in X} S^{\nu}
\end{aligned}
$$

where X is the set of partitions of $2k$ whose diagonal hooks have depth one less than width (see Theorem 2.15 in [19] for a proof of the last isomorphism).

Combining the results above, we now have $\operatorname{res}_{\mathbb{C}S_n}^{B_n} \Delta_n(\mu) \cong \bigoplus_{\nu \in X} \operatorname{ind}_{S_{2k}\times S_m}^{S_n} S^{\nu} \otimes S^{\mu}$. By the Littlewood-Richardson Rule, for each fixed ν and μ,

$$
\operatorname{ind}_{S_{2k}\times S_m}^{S_n} S^{\nu} \otimes S^{\mu} \cong \bigoplus_{\lambda \vdash n} c_{\mu\nu}^{\lambda} S^{\lambda}
$$

where $c_{\mu\nu}^{\lambda}$ is the Littlewood-Richardson coefficient. Hence, the multiplicity of S^{λ} in $\operatorname{res}_{\mathbb{C}S_n}^{B_n} \Delta_n(\mu)$ is given by $\sum_{\nu \in X} c_{\mu\nu}^{\lambda}$, as claimed. □

4.2 Restriction of Standard Modules to $B_{n-1}(0,-1)$

There is a natural choice of inclusion $B_{n-1} \hookrightarrow B_n$ given by appending an extra vertical strand to the right of a diagram $D \in B_{n-1}$:

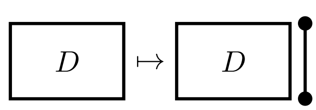


Figure 40

We can therefore consider the restriction of any B_n-module to the subalgebra B_{n-1} which is the image of this map.

The work below will require a significant amount of calculation with marked diagrams. In order to track the signs that result from rearranging the cups and caps in a diagram, we introduce the following function. Let $\operatorname{mrk}(D, l)$ denote the number of markings in D which lie strictly above level l. This means we always have $\mathrm{mrk}(D,1)=0$. For example, in the diagram

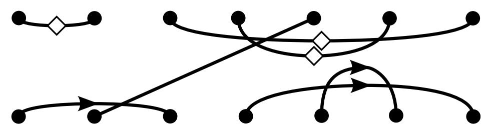


Figure 41

we have $\mathrm{mrk}(D,2)=1$ since the bead on the cup $(1,2)$ sits in level 1, $\mathrm{mrk}(D,4)=2$ since there are two beads above level 4, and $\mathrm{mrk}(D,4')=4$ since there are three beads and one arrow above level $4'$.

Consider the Young diagram for a partition $\lambda \vdash m$. We call a box b *removable* from λ if removing b results in a Young diagram for a partition of $m-1$. Thus, b is removable if there are no boxes to the right of b in the same row as b and there are no boxes immediately below b. For example, the removable boxes in the partition below are shaded.

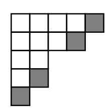


Figure 42

When $\mu \vdash m-1$ can be obtained from λ by removing a removable box of λ, we write $\mu \lhd \lambda$. A box b' is called *addable* to λ if adding b' results in a new partition of size $m+1$. For example, all addable boxes to the partition $(5,4,2,2,1)$ are shaded in the diagram below.

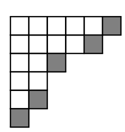


Figure 43

When $\nu \vdash m+1$ can be obtained from λ by adding an addable box to λ, we write $\nu \rhd \lambda$.

To simplify notation, we will write $\mathrm{res}^n_{n-1}M$ for the restriction of a B_n-module M to the subalgebra B_{n-1}.

Theorem 4.2. *Let $n = m + 2k$ with $k > 0$ and $\lambda \vdash m$. We have the following short exact sequence of B_{n-1}-modules:*

$$0 \to \bigoplus_{\mu \lhd \lambda} \Delta_{n-1}(\mu) \to res^n_{n-1}\Delta_n(\lambda) \to \bigoplus_{\nu \rhd \lambda} \Delta_{n-1}(\nu) \to 0$$

Proof. We follow the proof given in [16] for the ordinary Brauer algebra, adjusting for signs as necessary. To simplify notation, let $V := \mathrm{res}^n_{n-1}\Delta_n(\lambda)$. Let $\{v_1, \ldots, v_d\}$ be a basis for S^λ. Let A be the set of $x \in \mathcal{P}^m_n$ which have vertex n free and let $B = \mathcal{P}^m_n \backslash A$, so B consists of half-diagrams which have vertex n connected to some other vertex. Recall there is a function $f^m_n : \mathcal{P}^m_n \to \overline{B^m_n}$ given by $x \mapsto D_{x,b_{n,m}}$. That is, the image of x is the standard marked Brauer diagram having top x, bottom $b_{n,m}$, and through strings connected to the remaining vertices so that they do not cross.

Let W be the subspace of V spanned by the set $\{f^m_n(a) \otimes v_i : a \in A, 1 \leq i \leq d\}$. To see that this subspace is B_{n-1}-invariant, we first show that W is invariant under the action of each $\sigma \in S_{n-1}$. For any $a \in A$, we have $\sigma.f^m_n(a) = \pm\sigma_t f^m_n(\sigma.a)$, where σ_t is the part of σ which only interacts with the through strings of $f^m_n(a)$. The part of σ which interacts with the cups of $f^m_n(a)$ may generate a sign, but this scalar factor will not affect whether or not $\sigma.f^m_n(a)$ lies in the subspace W. Note that $\sigma.a$ again lies in A since σ fixes n. Since the m through strings of $f^m_n(a)$ are connected to the first m vertices in the bottom row so that they do not cross, we can push σ_t through $f^m_n(a)$ so that $\sigma.f^m_n(a) = (-1)^q f^m_n(a')\sigma'$ where q is 0 or 1 as discussed above and $\sigma' \in S_m$ rearranges the through strings in the same way as σ_t. In particular, σ' must fix m since (n, m') is a through string of $f^m_n(a)$ which is fixed by σ, so in fact $\sigma' \in S_{m-1}$. Hence, $\sigma.\left(f^m_n(a) \otimes v_i\right) = (-1)^q f^m_n(\sigma.a)\sigma' \otimes v_i = (-1)^q f^m_n(\sigma.a) \otimes \sigma' v_i$ which again lies in W since $\sigma' v_i \in S^\lambda$. Now, B_{n-1} is generated by S_{n-1} together with e_1 so we need only check that W is invariant under the action of e_1. There are five cases to consider.

1. If vertices 1 and 2 are both connected to through strings in $f_n^m(a)$, then $e_1.f_n^m(a) = 0$ in the quotient space $\overline{B_n^m}$ since the resulting diagram has an additional cup.

2. If $(1, 2)$ is a cup in $f_n^m(a)$, then $e_1.f_n^m(a) = 0$.

3. Suppose vertex 1 is connected to the through string $(1, 1')$ while vertex 2 is connected to a cup $(2, l)$ in $f_n^m(a)$. Note l must be less than n since vertex n must be free. There cannot be any additional cups in $f_n^m(a)$ above $(2, l)$, so $e_1.f_n^m(a)$ has a new cup $(1, 2)$, which is above all previous cups, and a new through string $(l, 1')$. All other through strings and cups of $f_n^m(a)$ are unchanged and no signs are incurred. So, the action of e_1 simply interchanges the roles of vertices 1 and l; that is $e_1.f_n^m(a) = (1, l)f_n^m(a)$. Since $(1, l) \in S_{n-1}$, the considerations above then show that $e_1.\left(f_n^m(a) \otimes v_i\right) \in W$.

4. Suppose vertex 1 is connected to a cup $(1, l)$ while vertex 2 is connected to the through string $(2, 1')$ in $f_n^m(a)$. Again, there cannot be any additional cups in $f_n^m(a)$ above $(1, l)$, so $e_1.f_n^m(a)$ has a new cup $(1, 2)$ which is above all other cups, a new through string $(l, 1')$, and all other cups and through strings unchanged. The resulting diagram is scaled by a sign from canceling the bead of $(1, l)$ in $f_n^m(a)$ with the adjacent arrow of $(1', 2')$ in e_1. So, $e_1.f_n^m(a) = -(2, l)f_n^m(a)$ since the action of e_1 interchanges the roles of vertices 2 and l. Thus, $e_1.\left(f_n^m(a) \otimes v_i\right) = -(2, l)f_n^m(a) \otimes v_i \in W$ since $(2, l) \in S_{n-1}$.

5. Suppose vertices 1 and 2 are connected to distinct cups $(1, l)$ and $(2, j)$ in $f_n^m(a)$. The diagram $e_1.f_n^m(a)$, has new a new cup connecting vertices l and j and a new cup $(1, 2)$. The resulting simplified diagram is scaled by a sign from canceling the bead on cup $(1, l)$ in $f_n^m(a)$ with the arrow on cap $(1', 2')$

in e_1 as well as a factor of $(-1)^{\mathrm{mrk}(f_n^m(a),l)-1}$ if $l < j$ or $(-1)^{\mathrm{mrk}(f_n^m(a),j)-1}$ if $l > j$ from moving the cup (l, j) or (j, l), respectively, from level 2 to level l or j, respectively. So, the diagram $e_1.f_n^m(a)$ is a scalar multiple of the diagram $(2, l)f_n^m(a)$ since the action of e_1 incurs a sign and interchanges the roles of vertices 2 and l. Since $(2, l) \in S_{n-1}$, we have $e_1.\left(f_n^m(a) \otimes v_i\right) \in W$.

Thus, W is a B_{n-1}-module.

We now construct a B_{n-1}-module isomorphism between W and

$$\overline{B_{n-1}^{m-1}} \otimes_{\mathbb{C}S_{m-1}} \mathrm{res}_{S_{m-1}}^{S_m} S^\lambda$$

This will give the desired result since it is well-known that $\mathrm{res}_{S_{m-1}}^{S_m} S^\lambda \cong \bigoplus_{\mu \lhd \lambda} S^\mu$, so

$$\begin{aligned} \overline{B_{n-1}^{m-1}} \otimes_{\mathbb{C}S_{m-1}} \mathrm{res}_{S_{m-1}}^{S_m} S^\lambda &\cong \overline{B_{n-1}^{m-1}} \otimes_{\mathbb{C}S_{m-1}} \left(\bigoplus_{\mu \lhd \lambda} S^\mu \right) \\ &\cong \bigoplus_{\mu \lhd \lambda} \overline{B_{n-1}^{m-1}} \otimes_{\mathbb{C}S_{m-1}} S^\mu \\ &= \bigoplus_{\mu \lhd \lambda} \Delta_{n-1}(\mu) \end{aligned}$$

Let $\varphi_1 : A \to \mathcal{P}_{n-1}^{m-1}$ be the map which removes the vertex n from each $a \in A$. Since vertex n is free in each $a \in A$, $\varphi_1(a)$ has $n-1$ total vertices with $m-1$ free. This map is readily seen to be a bijection of sets since any given half-diagram in $\mathcal{P}_{n-1}^{m-1}$ corresponds uniquely to half-diagram in A by adding or removing a free vertex in position n. Let $\varphi : W \to \overline{B_{n-1}^{m-1}} \otimes_{\mathbb{C}S_{m-1}} \mathrm{res}_{S_{m-1}}^{S_m} S^\lambda$ be the map $f_n^m(a) \otimes v_i \mapsto f_{n-1}^{m-1}(\varphi_1(a)) \otimes v_i$. This map is injective (since φ_1 and the f maps are injective) and the image is a basis for $\overline{B_{n-1}^{m-1}} \otimes_{\mathbb{C}S_{m-1}} \mathrm{res}_{S_{m-1}}^{S_m} S^\lambda$ since φ_1 is surjective, so φ is a vector space isomorphism. Plainly, the map φ takes a basis element $f_n^m(a) \otimes v_i$ of W to a basis element of $\overline{B_{n-1}^{m-1}} \otimes_{\mathbb{C}S_{m-1}} \mathrm{res}_{S_{m-1}}^{S_m} S^\lambda$ by deleting the through string (n, m') and the vertices n and m' in $f_n^m(a)$ and leaving everything else unchanged.

We must show that φ commutes with the actions of $\sigma \in S_{n-1}$ and e_1. To see that $\sigma\varphi = \varphi\sigma$, we first observe that $\sigma.\varphi_1(a) = \varphi_1(\sigma.a)$ for all $a \in A$ since σ fixes vertex n and φ_1 only affects vertex n. Note that since the diagrams $f_n^m(a)$ and $f_{n-1}^{m-1}(\varphi_1(a))$ have the same cups, then the diagrams $\sigma.f_n^m(a)$ and $\sigma.f_{n-1}^{m-1}(\varphi_1(a))$ again have the same cups and both will be scaled by the same power of -1. Also, since σ fixes n we have $\sigma.f_n^m(a) = (-1)^q\sigma_t f_n^m(\sigma.a)$ and $\sigma.f_{n-1}^{m-1}(\varphi_1(a)) = (-1)^q\sigma_t f_{n-1}^{m-1}(\sigma.\varphi_1(a))$ for the same permutation $\sigma_t \in S_{n-1}$ which only affects the through strings of the diagrams. Arguing as above, the permutation σ_t can be pushed through the diagrams $f_n^m(\sigma.a)$ and $f_{n-1}^{m-1}(\sigma.\varphi_1(a))$ to obtain a new permutation $\sigma' \in S_{m-1}$ which acts on these diagrams on the right. Hence, we have

$$\begin{aligned}
\varphi\left(\sigma.f_n^m(a) \otimes v\right) &= \varphi\left((-1)^q f_n^m(\sigma.a)\sigma' \otimes v\right) \\
&= (-1)^q \varphi\left(f_n^m(\sigma.a) \otimes \sigma' v\right) \\
&= (-1)^q f_{n-1}^{m-1}\left(\varphi_1(\sigma.a)\right) \otimes \sigma' v \\
&= (-1)^q f_{n-1}^{m-1}\left(\sigma.\varphi_1(a)\right)\sigma' \otimes v \\
&= \sigma.\left(f_{n-1}^{m-1}\left(\varphi_1(a)\right) \otimes v\right) \\
&= \sigma.\varphi\left(f_n^m(a) \otimes v\right)
\end{aligned}$$

So, φ is an S_{n-1}-map. Now, to check that $e_1\varphi = \varphi e_1$,we must again consider five cases. In each case, it suffices to check that $e_1.f_n^m(a)$ and $e_1.f_{n-1}^{m-1}(\varphi_1(a))$ agree, up to the presence of the through string (n, m') since the map φ will delete this edge and its adjacent vertices.

1. If vertices 1 and 2 are both connected to through strings in $f_n^m(a)$, then the same is true in $f_{n-1}^{m-1}(\varphi_1(a))$. So, $e_1.f_n^m(a) = 0$ and $e_1.f_{n-1}^{m-1}(\varphi_1(a)) = 0$ in each respective quotient space.

2. If $(1, 2)$ is a cup in a, then $e_1.f_n^m(a) = 0$ and $e_1.f_{n-1}^{m-1}(\varphi_1(a)) = 0$.

3. Suppose $(1,1')$ and $(2,l)$, $l < n$, appear in both diagrams. Then, as in the case analysis above, $e_1.f_n^m(a) = (1,l)f_n^m(a)$ and

$$e_1.f_{n-1}^{m-1}(\varphi_1(a)) = (1,l)f_{n-1}^{m-1}(\varphi_1(a))$$

so the action of e_1 is by the permutation $(1,l) \in S_{n-1}$. We just showed that φ is an S_{n-1}-map, so this shows φ commutes with the action of e_1 in this case.

4. Suppose $(1,l)$, $l < n$, and $(2,1')$ appear in both diagrams. Then, canceling the bead of $(1,l)$ in each diagram with the arrow of $(1',2')$ in e_1 introduces a sign. Simplifying the resulting diagram introduces no further signs. So, $e_1.f_n^m(a) = -(2,l)f_n^m(a)$ and $e_1.f_{n-1}^{m-1}(\varphi_1(a)) = -(2,l)f_{n-1}^{m-1}(\varphi_1(a))$, so φ commutes with the action of e_1 in this case.

5. Suppose $(1,l)$ and $(2,j)$ appear in both diagrams, where $l, j < n$. Then, a sign is introduced for canceling the bead of $(1,l)$ in each diagram with the arrow of $(1',2')$ in e_1. This simplification results in a new cup (l,j) if $l < j$ or (j,l) if $l > j$ which sits at depth 2 in each diagram. Additional signs will be introduced when this cup is moved to its proper depth in the resulting diagram. However, we now see that the action of e_1 on either diagram is simply that of the permutation $(2,l) \in S_{n-1}$. The considerations above show that φ commutes with the action of e_1 in this case.

Thus, φ is a B_{n-1}-module isomorphism.

We now consider the quotient space V/W. A basis for this space consists of all elements of the form $f_n^m(b) \otimes v_i$ where $b \in B$. Let $\psi_1 : B \to \mathcal{P}_{n-1}^{m+1}$ be the map which takes $b \in B$ to the half-diagram with vertex n and the edge adjacent to it removed so that the resulting half-diagram has one additional free vertex and one fewer arc. Given $b' \in \mathcal{P}_{n-1}^{m+1}$, we can add a vertex n to b' and connect this vertex to any of the $m+1$ free vertices in b' with an arc to obtain a diagram in B. Hence, ψ_1 is an

$(m+1)$-to-1 surjective map. Moreover, ψ_1 commutes with the action of S_{n-1} since ψ_1 does not change any of the first $n-1$ vertices of $b \in B$.

Suppose (l,n) is the cup connected to vertex n in $b \in B$. Let $\psi_1^* : B \to \overline{B_{n-1}^{m+1}}$ be the map $b \mapsto (-1)^{\mathrm{mrk}(f_n^m(b),l)} f_{n-1}^{m+1}(\psi_1(b))$. Note that given any $b' \in \mathcal{P}_{n-1}^{m+1}$ and any $1 \leq j \leq m+1$, we can create $b \in B$ from b' by adding a vertex n and connecting n to the j^{th} free vertex, call it l, in b' by an arc. Then, $\psi_1^*(b) = (-1)^{\mathrm{mrk}(f_n^m(b),l)} f_{n-1}^{m+1}(b')$, so ψ_1^* is surjective.

Let τ_i denote the transposition $(i, m+1)$ so that $\tau_{m+1} = 1$. Recall that

$$\mathrm{ind}_{S_m}^{S_{m+1}} S^\lambda \cong \bigoplus_{i=1}^{m+1} \mathbb{C}\tau_i \otimes_{S_m} S^\lambda$$

and the action of $\sigma \in S_{m+1}$ on $\mathrm{ind}_{S_m}^{S_{m+1}} S^\lambda$ is given by

$$\sigma.\left(\tau_i \otimes v_j\right) = \tau_{\sigma^{-1}(i)} \left(\tau_{\sigma^{-1}(i)}\sigma\tau_i\right) \otimes v_j = \tau_{\sigma^{-1}(i)} \otimes \left(\tau_{\sigma^{-1}(i)}\sigma\tau_i\right) v_j$$

since $\tau_{\sigma^{-1}(i)}\sigma\tau_i$ fixes $m+1$, so can be viewed as a permutation in S_m.

Suppose (l,n) is the cup connected to vertex n in $b \in B$ and that there are $j-1$ free vertices in b to the left of l so that $\psi_1^*(b)$ has (l,j') among its through strings. Let $\psi : V/W \to \overline{B_{n-1}^{m+1}} \otimes_{\mathbb{C}S_{m+1}} \mathrm{ind}_{S_m}^{S_{m+1}} S^\lambda$ be given by $f_n^m(b) \otimes v \mapsto \psi_1^*(b) \otimes (\sigma_{j,m+1} \otimes v)$, where $\sigma_{j,m+1} = (j, m+1, m, m-1, \ldots, j+1) \in S_{m+1}$ is the permutation which sends j to $m+1$ and shifts every integer k strictly between j and $m+1$ down by one. Note that $\psi_1^*(b) \otimes (\sigma_{j,m+1} \otimes v) = \psi_1^*(b)\sigma_{j,m+1} \otimes (1 \otimes v)$, since the $\sigma_{j,m+1}$ can move across the leftmost tensor product, and the diagram $\psi_1^*(b)\sigma_{j,m+1} = (-1)^{\mathrm{mrk}(f_n^m(b),l)} f_{n-1}^{m+1}(b')\,\sigma_{j,m+1}$ has top $\psi_1(b)$ and bottom $b_{n-1,m+1}$ with vertex l connected to $(m+1)'$ and all other through strings drawn so that they do not cross each other. Once we show that this map is an isomorphism of B_{n-1}-modules, we will have the desired result since it is well-known that $\mathrm{ind}_{S_m}^{S_{m+1}} S^\lambda \cong \bigoplus_{\nu \rhd \lambda} S^\nu$ so

$$\begin{aligned}
\overline{B_{n-1}^{m+1}} \otimes_{\mathbb{C}S_{m+1}} \mathrm{ind}_{S_m}^{S_{m+1}} S^\lambda &\cong \overline{B_{n-1}^{m+1}} \otimes_{\mathbb{C}S_{m+1}} \left(\bigoplus_{\nu \rhd \lambda} S^\nu \right) \\
&\cong \bigoplus_{\nu \rhd \lambda} \overline{B_{n-1}^{m+1}} \otimes_{\mathbb{C}S_{m+1}} S^\nu \\
&= \bigoplus_{\nu \rhd \lambda} \Delta_{n-1}(\nu)
\end{aligned}$$

We will first show this map is a vector space isomorphism. Now, $\dim V/W = |B| \cdot \dim S^\lambda$ and since ψ_1 is an $m+1$-to-1 map, we have $\dim \overline{B_{n-1}^{m+1}} \otimes_{\mathbb{C}S_{m+1}} \mathrm{ind}_{S_m}^{S_{m+1}} S^\lambda = \frac{|B|}{m+1} \cdot \left((m+1) \cdot \dim S^\lambda\right)$, so these two spaces have the same dimension. Hence, it suffices to check that ψ is surjective. Given a basis element $f_{n-1}^{m+1}(b') \otimes (\tau_i \otimes v_j) \in \overline{B_{n-1}^{m+1}} \otimes_{\mathbb{C}S_{m+1}} \mathrm{ind}_{S_m}^{S_{m+1}} S^\lambda$, choose $b \in B$ as above with cup (l, n) having $i-1$ free vertices to the left of l, so that $\psi_1^*(b) = (-1)^{\mathrm{mrk}(f_n^m(b),l)} f_{n-1}^{m+1}(b')$. Then,

$$\begin{aligned}
\psi\left((-1)^{\mathrm{mrk}(f_n^m(b),l)} f_n^m(b) \otimes \left(\sigma_{i,m+1}^{-1} \tau_i v_j\right)\right) &= f_{n-1}^{m+1}(b')\, \sigma_{i,m+1} \otimes \left(1 \otimes \left(\sigma_{i,m+1}^{-1} \tau_i v_j\right)\right) \\
&= f_{n-1}^{m+1}(b') \otimes \sigma_{i,m+1}. \left(\tau_{m+1} \otimes \sigma_{i,m+1}^{-1} \tau_i v_j\right) \\
&= f_{n-1}^{m+1}(b') \otimes \\
&\qquad \left(\tau_i \otimes (\tau_i \sigma_{i,m+1} \tau_{m+1})\, \sigma_{i,m+1}^{-1} \tau_i v_j\right) \\
&= f_{n-1}^{m+1}(b') \otimes (\tau_i \otimes v_j)
\end{aligned}$$

so the map is surjective and, hence, is an isomorphism of vector spaces. It remains to check that ψ is a map of B_{n-1}-modules.

We first show ψ commutes with the action of S_{n-1}. That is, for $b \in B$ with cup (l, n) where there are $j-1$ free vertices before l, we must compare $\sigma.f_n^m(b)$ to $\sigma.f_{n-1}^{m+1}(\psi_1(b))\, \sigma_{j,m+1}$, where $\sigma_{j,m+1} = (j, m+1, m, \ldots, j+1)$ as above. It suffices to take σ to be a simple transposition s_i, with $1 \le i \le n-2$. There are three cases to consider.

1. Suppose $i \neq l-1, l$, so that s_i does not affect the cup (l, n). There are four subcases.

   (a) Suppose i and $i+1$ are free in b. Then, both are connected to through strings, (i, q') and $(i+1, (q+1)')$ where $q \leq m-1$, so the crossing of s_i can be pulled through $f_n^m(b)$ giving $s_i f_n^m(b) = f_n^m(b) s_q$. In particular, acting by s_i does not generate any signs. We then have

$$\begin{aligned} \psi\left(s_i f_n^m(b) \otimes v\right) &= \psi\left(f_n^m(b) \otimes (s_q v)\right) \\ &= (-1)^{\operatorname{mrk}(f_n^m(b),l)} f_{n-1}^{m+1}\left(\psi_1(b)\right) \sigma_{j,m+1} \otimes \left(1 \otimes (s_q v)\right) \\ &= (-1)^{\operatorname{mrk}(f_n^m(b),l)} f_{n-1}^{m+1}\left(\psi_1(b)\right) \sigma_{j,m+1} s_q \otimes (1 \otimes v) \end{aligned}$$

   Note that if $i+1 < l$, then $q+1 \leq j-1$ (since there are $j-1$ free vertices before l) and if $i > l$, then $j \leq q \leq m-1$. Hence,

$$\sigma_{j,m+1} s_q = \begin{cases} s_q \sigma_{j,m+1} & \text{if } q \leq j-2 \\ s_{q+1} \sigma_{j,m+1} & \text{if } q \geq j \end{cases}$$

   since $\sigma_{j,m+1}$ shifts the integers strictly between j and $m+1$ down by one. Moreover, if $i+1 < l$, then $f_{n-1}^{m+1}\left(\psi_1(b)\right)$ has through strings (i, q') and $(i+1, (q+1)')$, while if $i > l$, then $f_{n-1}^{m+1}\left(\psi_1(b)\right)$ has through strings $(i, (q+1)')$ and $(i+1, (q+2)')$ since l is the j^{th} free vertex in $\psi_1(b)$, so the through strings to the left of l have their bottom endpoints shifted up by one compared to $f_n^m(b)$. Thus, by pulling the transposition s_q or s_{q+1} up through $f_{n-1}^{m+1}\left(\psi_1(b)\right)$, we see that $f_{n-1}^{m+1}\left(\psi_1(b)\right) \sigma_{j,m+1} s_q = s_i f_{n-1}^{m+1}\left(\psi_1(b)\right)$, which shows $\psi\left(s_i f_n^m(b) \otimes v\right) = s_i \psi\left(f_n^m(b) \otimes v\right)$ in this case.

   (b) Suppose i is free in b while $i+1$ is not. Say (i, q') and $(i+1, r)$ or $(r, i+1)$ are edges in $f_n^m(b)$. The effect of s_i on $f_n^m(b)$ is to interchange the top endpoint of a through string and the adjacent endpoint of a cup, so no sign will be generated (since no markings will pass each other) and $s_i f_n^m(b) = f_n^m(s_i b)$ since both diagrams have through string (i, q') and cup (i, r) or (r, i). Hence, $(-1)^{\mathrm{mrk}(f_n^m(s_i b), l)} = (-1)^{\mathrm{mrk}(f_n^m(b), l)}$. Each of these statements holds for $f_{n-1}^{m+1}(\psi_1(b))$ as well, so we have $s_i f_{n-1}^{m+1}(\psi_1(b)) = f_{n-1}^{m+1}(s_i \psi_1(b)) = f_{n-1}^{m+1}(\psi_1(s_i b))$ since ψ_1 commutes with the action of S_{n-1} by the discussion above. Hence,

$$\begin{aligned}
\psi\left(s_i f_n^m(b) \otimes v\right) &= \psi\left(f_n^m(s_i b) \otimes v\right) \\
&= (-1)^{\mathrm{mrk}(f_n^m(s_i b), l)} f_{n-1}^{m+1}\left(\psi_1(s_i b)\right) \sigma_{j,m+1} \otimes (1 \otimes v) \\
&= (-1)^{\mathrm{mrk}(f_n^m(b), l)} s_i f_{n-1}^{m+1}\left(\psi_1(b)\right) \sigma_{j,m+1} \otimes (1 \otimes v) \\
&= s_i \psi\left(f_n^m(b) \otimes v\right)
\end{aligned}$$

(c) Suppose $i+1$ is free in b while i is not. The proof of this case is just that for case (b) with the roles of i and $i+1$ interchanged, so is not repeated.

(d) Suppose neither i nor $i+1$ is free in b. Then, both vertices are connected to cups in $f_n^m(b)$. If at least one of i or $i+1$ is connected to the right endpoint of their respective cup, then the action of s_i will not introduce any signs (since no markings will need to change levels) and we have $s_i f_n^m(b) = f_n^m(s_i b)$. If both i and $i+1$ are connected to the left endpoints of their respective cups, then acting by s_i on $f_n^m(b)$ reverses the order of these two cups so that $s_i f_n^m(b) = -f_n^m(s_i b)$. Note that in either case, we have $(-1)^{\mathrm{mrk}(f_n^m(s_i b), l)} = (-1)^{\mathrm{mrk}(f_n^m(b), l)}$ since the cups connected to i and $i+1$ are either both before or both after l. Moreover, since the cups connected to i and $i+1$ must also appear in the same locations of $f_{n-1}^{m+1}(\psi_1(b))$ as they do in $f_n^m(b)$, both of these statements hold for the action of s_i on this diagram as well. Hence, we have

$$\begin{aligned}
\psi\left(s_i f_n^m(b) \otimes v\right) &= (-1)^q \psi\left(f_n^m(s_i b) \otimes v\right) \\
&= (-1)^{q+\mathrm{mrk}(f_n^m(s_i b),l)} f_{n-1}^{m+1}\left(\psi_1(s_i b)\right) \sigma_{j,m+1} \otimes (1 \otimes v) \\
&= (-1)^{q+\mathrm{mrk}(f_n^m(b),l)+q} s_i f_{n-1}^{m+1}\left(\psi_1(b)\right) \sigma_{j,m+1} \otimes (1 \otimes v) \\
&= s_i \psi\left(f_n^m(b) \otimes v\right)
\end{aligned}$$

where $q = 0$ if at least one of i or $i+1$ is connected to the right endpoint of their respective cup and $q = 1$ if both i and $i+1$ are connected to the left endpoints of their respective cups.

2. Suppose $i = l - 1$. There are two subcases to consider.

   (a) Suppose i is free in b. Then, acting by s_i on $f_n^m(b)$ does not induce a sign or introduce new crossings among the through strings, so $s_i f_n^m(b) = f_n^m(s_i b)$. Observe that $(-1)^{\mathrm{mrk}(f_n^m(s_i b),i)} = (-1)^{\mathrm{mrk}(f_n^m(b),l)}$ since acting by s_i does not affect any of the cups above l and i is free in b. Since $s_i b$ has cup (i, n) with $j-2$ free vertices before i we have

$$\begin{aligned}
\psi\left(s_i f_n^m(b) \otimes v\right) &= \psi\left(f_n^m(s_i b) \otimes v\right) \\
&= (-1)^{\mathrm{mrk}(f_n^m(s_i b),i)} f_{n-1}^{m+1}\left(\psi_1(s_i b)\right) \sigma_{j-1,m+1} \otimes (1 \otimes v) \\
&= (-1)^{\mathrm{mrk}(f_n^m(b),l)} f_{n-1}^{m+1}\left(s_i \psi_1(b)\right) \sigma_{j-1,m+1} \otimes (1 \otimes v)
\end{aligned}$$

   Now, i is the $(j-1)^{\text{st}}$ free vertex in $s_i\psi_1(b)$, so $f_{n-1}^{m+1}\left(s_i\psi_1(b)\right)\sigma_{j-1,m+1}$ has $(i, (m+1)')$ and $(l, (j-1)')$ among its through strings since l must then be the j^{th} free vertex in $s_i\psi_1(b)$ and the $\sigma_{j-1,m+1}$ shifts the bottom endpoint connected to l down by one. But, $f_{n-1}^{m+1}\left(\psi_1(b)\right)\sigma_{j,m+1}$ has $(l, (m+1)')$ and $(i, (j-1)')$ among its through strings, so acting by s_i on this diagram changes these through strings into $(i,(m+1)')$ and $(l,(j-1)')$. Since the action of s_i preserves each edge except those connected to i and $i+1$, we thus conclude that $f_{n-1}^{m+1}(s_i\psi_1(b))\,\sigma_{j-1,m+1} = s_i f_{n-1}^{m+1}(\psi_1(b))\,\sigma_{j,m+1}$ since the two through strings affected by s_i coincide in both diagrams. Hence, we have

$$\begin{aligned}\psi\left(s_i f_n^m(b)\otimes v\right) &= (-1)^{\mathrm{mrk}(f_n^m(b),l)} s_i f_{n-1}^{m+1}(\psi_1(b))\,\sigma_{j,m+1}\otimes(1\otimes v)\\ &= s_i\psi\left(f_n^m(b)\otimes v\right)\end{aligned}$$

as required.

(b) Suppose i is not free in b. Note that acting by s_i preserves the number of free vertices in b which lie to the left of l. If i is the right endpoint of a cup in b, then acting by s_i on $f_n^m(b)$ yields two new cups, (i,n) and one whose right endpoint is l, with no new sign, so $s_i f_n^m(b) = f_n^m(s_i b)$. Also, in this case we have $(-1)^{\mathrm{mrk}(f_n^m(s_i b),i)} = (-1)^{\mathrm{mrk}(f_n^m(b),l)}$ since the number of cups above i and l is the same in either diagram as these vertices are adjacent and no cup begins at i. If i is the left endpoint of a cup in b, then acting by s_i on $f_n^m(b)$ yields two new cups, (i,n) and one whose left endpoint is l, as well as a sign since the two new cups are out of order, so $s_i f_n^m(b) = -f_n^m(s_i b)$. Also, in this case we have $(-1)^{\mathrm{mrk}(f_n^m(s_i b),i)} = (-1)^{\mathrm{mrk}(f_n^m(b),l)-1}$ since the cup starting at i is above l in $f_n^m(b)$. Hence, we have

$$\begin{aligned}\psi\left(s_i f_n^m(b)\otimes v\right) &= (-1)^q\psi\left(f_n^m(s_i b)\otimes v\right)\\ &= (-1)^{q+\mathrm{mrk}(f_n^m(s_i b),i)} f_{n-1}^{m+1}(\psi_1(s_i b))\,\sigma_{j,m+1}\otimes(1\otimes v)\\ &= (-1)^{q+\mathrm{mrk}(f_n^m(b),l)-q} f_{n-1}^{m+1}(s_i\psi_1(b))\,\sigma_{j,m+1}\otimes(1\otimes v)\\ &= (-1)^{\mathrm{mrk}(f_n^m(b),l)} f_{n-1}^{m+1}(s_i\psi_1(b))\,\sigma_{j,m+1}\otimes(1\otimes v)\end{aligned}$$

where $q = 0$ if i is the right endpoint of a cup in b and $q = 1$ if i is the left endpoint of a cup in b. Now, $f_{n-1}^{m+1}(s_i\psi_1(b))\,\sigma_{j,m+1}$ has $(i,(m+1)')$ among its through strings and also has a cup containing l, which is also the case for $s_i f_{n-1}^{m+1}(\psi_1(b))\,\sigma_{j,m+1}$ since s_i changes the through string $(l,(m+1)')$ into $(i,(m+1)')$ and changes the cup containing i into a cup containing l. Note that no signs are introduced when analyzing the action of s_i on either of these diagrams since s_i moves the endpoint of the cup to an adjacent free vertex. We conclude that $f_{n-1}^{m+1}(s_i\psi_1(b))\,\sigma_{j,m+1} = s_i f_{n-1}^{m+1}(\psi_1(b))\,\sigma_{j,m+1}$ since the two edges affected by s_i agree. Hence, we have

$$\begin{aligned}\psi\left(s_i f_n^m(b)\otimes v\right) &= (-1)^{\mathrm{mrk}(f_n^m(b),l)} s_i f_{n-1}^{m+1}(\psi_1(b))\,\sigma_{j,m+1}\otimes(1\otimes v)\\ &= s_i\psi\left(f_n^m(b)\otimes v\right)\end{aligned}$$

3. Suppose $i = l$. There are two subcases to consider.

   (a) Suppose $i+1$ is free in b. Acting by s_i on $f_n^m(b)$ does not induce a sign or introduce new crossings among the through strings, so $s_i f_n^m(b) = f_n^m(s_i b)$. Observe that $(-1)^{\mathrm{mrk}(f_n^m(s_i b),i+1)} = (-1)^{\mathrm{mrk}(f_n^m(b),l)}$ since the number of cups above l equals that above $i+1$, since these vertices are adjacent and $i+1$ is free in b. Since $s_i b$ has cup $(i+1,n)$ with j free vertices to the left of $i+1$, we have

$$\begin{aligned}\psi\left(s_i f_n^m(b)\otimes v\right) &= \psi\left(f_n^m(s_i b)\otimes v\right)\\ &= (-1)^{\mathrm{mrk}(f_n^m(s_i b),i+1)} f_{n-1}^{m+1}(\psi_1(s_i b))\,\sigma_{j+1,m+1}\otimes(1\otimes v)\\ &= (-1)^{\mathrm{mrk}(f_n^m(b),l)} f_{n-1}^{m+1}(s_i\psi_1(b))\,\sigma_{j+1,m+1}\otimes(1\otimes v)\end{aligned}$$

   Now, $i+1$ is the $(j+1)^{\mathrm{st}}$ free vertex in $s_i\psi_1(b)$, so $f_{n-1}^{m+1}(s_i\psi_1(b))\,\sigma_{j+1,m+1}$

has $(i+1,(m+1)')$ and (l,j') among its through strings since l must then be the j^{th} free vertex in $s_i\psi_1(b)$. But, $f_{n-1}^{m+1}(\psi_1(b))\,\sigma_{j,m+1}$ has $(l,(m+1)')$ and $(i+1,j')$ among its through strings, so acting by s_i on this diagram changes these through strings into $(i+1,(m+1)')$ and (l,j'). Since the action of s_i preserves each edge except those connected to i and $i+1$, we thus conclude that $f_{n-1}^{m+1}(s_i\psi_1(b))\,\sigma_{j+1,m+1} = s_i f_{n-1}^{m+1}(\psi_1(b))\,\sigma_{j,m+1}$ since the two through strings affected by s_i coincide in both diagrams. Hence, we have

$$\begin{aligned}\psi\left(s_i f_n^m(b)\otimes v\right) &= (-1)^{\mathrm{mrk}(f_n^m(b),l)} s_i f_{n-1}^{m+1}(\psi_1(b))\,\sigma_{j,m+1}\otimes(1\otimes v)\\ &= s_i\psi\left(f_n^m(b)\otimes v\right)\end{aligned}$$

as required.

(b) Suppose $i+1$ is not free in b. Note that acting by s_i preserves the number of free vertices in b which lie to the left of l, and hence to the left of $i+1$. If $i+1$ is the right endpoint of a cup in b, then acting by s_i on $f_n^m(b)$ yields two new cups, $(i+1,n)$ and one whose right endpoint is l, with no new sign since no markings are interchanged, so $s_i f_n^m(b) = f_n^m(s_i b)$. Also, in this case we have $(-1)^{\mathrm{mrk}(f_n^m(s_i b),i+1)} = (-1)^{\mathrm{mrk}(f_n^m(b),l)}$ since the number of cups above $i+1$ and l is the same in either diagram as these vertices are adjacent and $i+1$ is not the left endpoint of a cup. On the other hand, if $i+1$ is the left endpoint of a cup in b, then acting by s_i on $f_n^m(b)$ yields two new cups, $(i+1,n)$ and one whose left endpoint is l, as well as a sign since the two new cups are out of order, so $s_i f_n^m(b) = -f_n^m(s_i b)$. Also, in this case we have $(-1)^{\mathrm{mrk}(f_n^m(s_i b),i)} = (-1)^{\mathrm{mrk}(f_n^m(b),l)+1}$ since the cup starting at l is moved by s_i above $i+1$ in $f_n^m(b)$. Hence, we have

$$\begin{aligned}
\psi\left(s_i f_n^m(b) \otimes v\right) &= (-1)^q \psi\left(f_n^m(s_i b) \otimes v\right) \\
&= (-1)^{q+\mathrm{mrk}(f_n^m(s_i b), i+1)} f_{n-1}^{m+1}\left(\psi_1(s_i b)\right) \sigma_{j,m+1} \otimes (1 \otimes v) \\
&= (-1)^{q+\mathrm{mrk}(f_n^m(b), l)+q} f_{n-1}^{m+1}\left(s_i \psi_1(b)\right) \sigma_{j,m+1} \otimes (1 \otimes v) \\
&= (-1)^{\mathrm{mrk}(f_n^m(b), l)} f_{n-1}^{m+1}\left(s_i \psi_1(b)\right) \sigma_{j,m+1} \otimes (1 \otimes v)
\end{aligned}$$

where $q = 0$ if $i+1$ is the right endpoint of a cup in b and $q = 1$ if $i+1$ is the left endpoint of a cup in b. Now, $f_{n-1}^{m+1}\left(s_i \psi_1(b)\right) \sigma_{j,m+1}$ has $(i+1, (m+1)')$ among its through strings and also has a cup containing l, which is also the case for $s_i f_{n-1}^{m+1}\left(\psi_1(b)\right) \sigma_{j,m+1}$ since s_i changes the through string $(l, (m+1)')$ into $(i+1, (m+1)')$ and changes the cup containing $i+1$ into a cup containing l whose other endpoint is unchanged. Note that no signs are introduced when analyzing the action of s_i on either of these diagrams since s_i moves the endpoint of the cup to an adjacent free vertex in b. We conclude that $f_{n-1}^{m+1}\left(s_i \psi_1(b)\right) \sigma_{j,m+1} = s_i f_{n-1}^{m+1}\left(\psi_1(b)\right) \sigma_{j,m+1}$ since the two edges affected by s_i agree. Hence, we have

$$\begin{aligned}
\psi\left(s_i f_n^m(b) \otimes v\right) &= (-1)^{\mathrm{mrk}(f_n^m(b), l)} s_i f_{n-1}^{m+1}\left(\psi_1(b)\right) \sigma_{j,m+1} \otimes (1 \otimes v) \\
&= s_i \psi\left(f_n^m(b) \otimes v\right)
\end{aligned}$$

Hence, ψ is an S_{n-1}-map. Finally, we show that ψ commutes with the action of e_1. There are three overall cases to consider.

1. Suppose $l \neq 1, 2$. There are five subcases.

   (a) Suppose vertices 1 and 2 are both free in b. Then, acting by e_1 on the diagrams $f_n^m(b)$ and $f_{n-1}^{m+1}\left(\psi_1(b)\right)$ produces diagrams which have fewer through strings, so $e_1 f_n^m(b) = 0$ in $\overline{B_n^m}$ and $e_1 f_{n-1}^{m+1}(\psi_1(b)) = 0$ in $\overline{B_{n-1}^{m+1}}$. Hence, $\psi(e_1 f_n^m(b) \otimes v) = 0 = e_1 \psi(f_n^m(b) \otimes v)$.

(b) Suppose $(1,2)$ is a cup in b. Then, this is also a cup in $\psi_1(b)$, so $e_1 f_n^m(b) = 0 = e_1 f_{n-1}^{m+1}(\psi_1(b))$. Hence, $\psi(e_1 f_n^m(b) \otimes v) = 0 = e_1 \psi(f_n^m(b) \otimes v)$.

(c) Suppose $(1,q)$ is a cup while 2 is free in b. Since there are no other free vertices to the left of 2, we must then have that $(2,1')$ is a through string in $f_n^m(b)$. Acting by e_1 will turn the cup $(1,q)$ into part of the through string $(q,1')$ with a sign (since the arrow points away from the bead) and will introduce a new cup, $(1,2)$, which is already above all other cups so no sign is generated. Hence, the action of e_1 on $f_n^m(b)$ is by the transposition $(2,q) \in S_{n-1}$, scaled by a sign. Moreover, both of the edges $(1,q)$ and $(2,1')$ are also present in $f_{n-1}^{m+1}(\psi_1(b))$, so our analysis above applies to this diagram as well. Since we know that ψ is an S_{n-1}-map, we then get that ψ commutes with e_1:

$$\begin{aligned}
\psi(e_1 f_n^m(b) \otimes v) &= -\psi((2,q) f_n^m(b) \otimes v) \\
&= -(2,q)\psi(f_n^m(b) \otimes v) \\
&= -(-1)^{\mathrm{mrk}(f_n^m(b),l)}(2,q) f_{n-1}^{m+1}(\psi_1(b))\, \sigma_{j,m+1} \otimes (1 \otimes v) \\
&= (-1)^{\mathrm{mrk}(f_n^m(b),l)} e_1 f_{n-1}^{m+1}(\psi_1(b))\, \sigma_{j,m+1} \otimes (1 \otimes v) \\
&= e_1 \psi(f_n^m(b) \otimes v)
\end{aligned}$$

(d) Suppose $(2,q)$ is a cup while 1 is free in b. Then, $(1,1')$ is a through string in $f_n^m(b)$ since 1 is the leftmost vertex in the top row. Acting by e_1 will turn the cap of e_1 and the cup $(2,q)$ into a through string $(q,1')$ with no sign (since the arrow points into the bead) and will introduce a new cup $(1,2)$ which is already above all other cups so no sign is generated. Thus, the effect of acting by e_1 is to act by the transposition $(1,q) \in$

S_{n-1}. Moreover, both of the edges $(2,q)$ and $(1,1')$ are also present in $f_{n-1}^{m+1}(\psi_1(b))$, so our analysis above applies to this diagram as well. Since we know that ψ is an S_{n-1}-map, we then get that ψ commutes with e_1:

$$\begin{aligned}
\psi\left(e_1 f_n^m(b) \otimes v\right) &= \psi\left((1,q) f_n^m(b) \otimes v\right) \\
&= (1,q)\psi\left(f_n^m(b) \otimes v\right) \\
&= (-1)^{\mathrm{mrk}(f_n^m(b),l)}(1,q) f_{n-1}^{m+1}\left(\psi_1(b)\right)\sigma_{j,m+1} \otimes (1 \otimes v) \\
&= (-1)^{\mathrm{mrk}(f_n^m(b),l)} e_1 f_{n-1}^{m+1}\left(\psi_1(b)\right)\sigma_{j,m+1} \otimes (1 \otimes v) \\
&= e_1\psi\left(f_n^m(b) \otimes v\right)
\end{aligned}$$

(e) Suppose $(1,q)$ and $(2,r)$ are cups in b. Then, acting by e_1 on $f_n^m(b)$ produces two new cups $(1,2)$ and (q,r) (or (r,q) if $r<q$) as well as two signs: the first sign comes from canceling the arrow on the cap of e_1 with the bead on $(1,q)$, while the second sign $(-1)^{\mathrm{mrk}(f_n^m(b),x)-\mathrm{mrk}(f_n^m(b),3)} = (-1)^{\mathrm{mrk}(f_n^m(b),x)-2}$ comes from moving the bead at level 2 down to level $x=q$ if $q<r$ or $x=r$ if $r<q$. Hence, the total sign from the action of e_1 is $(-1)^{\mathrm{mrk}(f_n^m(b),x)-1}$, where $x=q$ if $q<r$ or $x=r$ if $r<q$. Ignoring the markings and signs, we see that e_1 changes the edges of $f_n^m(b)$ by the transposition $(2,q)\in S_{n-1}$. But, acting by $(2,q)$ on $f_n^m(b)$ introduces a sign $(-1)^{\mathrm{mrk}(f_n^m(b),x)-\mathrm{mrk}(f_n^m(b),3)} = (-1)^{\mathrm{mrk}(f_n^m(b),x)-2}$, where $x=q$ if $q<r$ or $x=r$ if $r<q$, since the bead at level 2 must be moved to level x, so we see that e_1 acts on $f_n^m(b)$ by $-(2,q)\in\mathbb{C}S_{n-1}$. Since both of the cups $(1,q)$ and $(2,r)$ are also present in $f_{n-1}^{m+1}(\psi_1(b))$, the same analysis

applies here as well. Hence, we have

$$\begin{aligned}
\psi\left(e_1 f_n^m(b) \otimes v\right) &= -\psi\left((2,q) f_n^m(b) \otimes v\right) \\
&= -(2,q)\psi\left(f_n^m(b) \otimes v\right) \\
&= -(-1)^{\mathrm{mrk}(f_n^m(b),l)}(2,q) f_{n-1}^{m+1}\left(\psi_1(b)\right)\sigma_{j,m+1} \otimes (1 \otimes v) \\
&= (-1)^{\mathrm{mrk}(f_n^m(b),l)} e_1 f_{n-1}^{m+1}\left(\psi_1(b)\right)\sigma_{j,m+1} \otimes (1 \otimes v) \\
&= e_1\psi\left(f_n^m(b) \otimes v\right)
\end{aligned}$$

2. Suppose $l = 1$. There are two subcases.

   (a) Suppose vertex 2 is free in b. Then, $(2, 1')$ is a through string in $f_n^m(b)$ since there are no free vertices to the left of 2. Acting by e_1 yields a new through string $(n, 1')$, so $e_1 f_n^m(b) \in W$ hence $e_1 f_n^m(b) = 0$ in V/W. On the other hand, $f_{n-1}^{m+1}(\psi_1(b))$ has $(1, (m+1)')$ and $(2, 1')$ among its through strings so acting by e_1 yields a diagram with fewer through strings, hence $e_1 f_{n-1}^{m+1}(\psi_1(b)) = 0$ in $\overline{B_{n-1}^{m+1}}$. Thus, we have $\psi\left(e_1 f_n^m(b) \otimes v\right) = 0 = e_1\psi\left(f_n^m(b) \otimes v\right)$.

   (b) Suppose $(2, q)$ is a cup in b. Acting by e_1 on $f_n^m(b)$ then introduces two new cups $(1, 2)$ and (q, n) as well as two signs: the first sign comes from canceling the bead on $(1, n)$ with the arrow on the cap of e_1, while the second $(-1)^{\mathrm{mrk}(f_n^m(b),q)-\mathrm{mrk}(f_n^m(b),3)} = (-1)^{\mathrm{mrk}(f_n^m(b),q)-2}$ comes from moving the bead at level 2 down to level q. Ignoring the markings, we see that e_1 changes the edges of $f_n^m(b)$ by the transposition $(1, q) \in S_{n-1}$. But, acting by $(1, q)$ on $f_n^m(b)$ introduces two signs: one sign to interchange the beads at levels 1 and 2 and $(-1)^{\mathrm{mrk}(f_n^m(b),q)-\mathrm{mrk}(f_n^m(b),3)} = (-1)^{\mathrm{mrk}(f_n^m(b),q)-2}$ to move the bead on the cup (q, n) from its new position at level 2 (moved from level 1 at the cost of the first sign) to level q. Hence, e_1 acts on

$f_n^m(b)$ by $(1,q) \in S_{n-1}$, so we have

$$\begin{aligned}
\psi\left(e_1 f_n^m(b) \otimes v\right) &= \psi\left((1,q) f_n^m(b) \otimes v\right) \\
&= (-1)^{\mathrm{mrk}(f_n^m(b),1)} (1,q) f_{n-1}^{m+1}\left(\psi_1(b)\right) \sigma_{j,m+1} \otimes (1 \otimes v) \\
&= (1,q) f_{n-1}^{m+1}\left(\psi_1(b)\right) \sigma_{j,m+1}
\end{aligned}$$

Now, $f_{n-1}^{m+1}\left(\psi_1(b)\right)$ has $(1,(m+1)')$ and $(2,q)$ among its edges, so acting by $(1,q)$ on this diagram transforms the cup $(2,q)$ into $(1,2)$ with no sign since the cup $(2,q)$ is highest in $f_{n-1}^{m+1}\left(\psi_1(b)\right)$ and changes the through string $(1,(m+1)')$ into $(q,(m+1)')$. But, this is also the effect of acting by e_1 on $f_{n-1}^{m+1}\left(\psi_1(b)\right)$ since the arrow of the cap of e_1 and the bead of $(2,q)$ cancel without sign creating the through string $(q,(m+1)')$ and the cup $(1,2)$ from e_1 descends into $f_{n-1}^{m+1}\left(\psi_1(b)\right)$ since it is the highest cup. Thus,

$$\begin{aligned}
\psi\left(e_1 f_n^m(b) \otimes v\right) &= (1,q) f_{n-1}^{m+1}\left(\psi_1(b)\right) \sigma_{j,m+1} \otimes (1 \otimes v) \\
&= e_1 f_{n-1}^{m+1}\left(\psi_1(b)\right) \sigma_{j,m+1} \otimes (1 \otimes v) \\
&= e_1 (-1)^{\mathrm{mrk}(f_n^m(b),1)} f_{n-1}^{m+1}\left(\psi_1(b)\right) \sigma_{j,m+1} \otimes (1 \otimes v) \\
&= e_1 \psi\left(f_n^m(b)\right)
\end{aligned}$$

3. Suppose $l = 2$. Then, $s_1 f_n^m(b)$ is back in case 2, so we know

$$\psi\left(e_1 s_1 f_n^m(b) \otimes v\right) = e_1 \psi\left(s_1 f_n^m(b) \otimes v\right)$$

But, $e_1 s_1 = -e_1$, so we have

$$
\begin{aligned}
\psi\left(e_1 f_n^m(b)\otimes v\right) &= \psi\left(e_1 s_1^2 f_n^m(b)\otimes v\right)\\
&= -\psi\left(e_1 s_1 f_n^m(b)\otimes v\right)\\
&= -e_1\psi\left(s_1 f_n^m(b)\otimes v\right)\\
&= e_1 s_1\psi\left(s_1 f_n^m(b)\otimes v\right)\\
&= e_1\psi\left(f_n^m(b)\otimes v\right)
\end{aligned}
$$

Hence, ψ commutes with the action of e_1, so is a map of B_{n-1}-modules, as required.

□

Recall from the proof above the set A consisting of those of $x \in \mathcal{P}_n^m$ which have vertex n free. In what follows, we show that some of the eigenvalues of the Jucys-Murphy element X_n on $\Delta_n(\lambda)$ uniquely characterize the standard B_{n-1}-modules which appear as submodules in $\mathrm{res}_{n-1}^n \Delta_n(\lambda)$.

Lemma 4.3. *For all $a \in A$ and $j < n$, we have*

1. $s_{j,n} f_n^m(a) = \begin{cases} f_n^m(a)(i,m) & \text{if } (j,i') \text{ is an edge in } f_n^m(a) \\ -e_{i,n} f_n^m(a) & \text{if } (i,j) \text{ or } (j,i) \text{ is an edge in } f_n^m(a) \end{cases}$

2. $e_{j,n} f_n^m(a) = \begin{cases} 0 & \text{if } (j,i') \text{ is an edge in } f_n^m(a) \\ -s_{i,n} f_n^m(a) & \text{if } (i,j) \text{ or } (j,i) \text{ is an edge in } f_n^m(a) \end{cases}$

Proof. By direct calculation. First suppose (j,i') is a through string in $f_n^m(a)$. Then, the transposition $s_{j,n}$ can be pulled down through the through strings (j,i') and (n,m'), yielding the transposition (i,m) acting on the right of $f_n^m(a)$. Also in this case, the diagram $e_{j,n} f_n^m(a)$ has fewer than m through strings, so the product is 0 in $\overline{B_n^m}$.

Now, suppose (i, j) is a cup in $f_n^m(a)$. Then, acting by $s_{j,n}$ on $f_n^m(a)$ changes (n, m') into (j, m') and (i, j) into (i, n) and does not introduce a sign since only the right endpoint of the cup is altered. Ignoring markings, acting by $e_{i,n}$ also effects these changes to the edges of $f_n^m(a)$. We must check the sign generated by the action of $e_{i,n}$. First, the cap of $e_{i,n}$ must descend down to level i of $f_n^m(a)$ at the cost of $(-1)^{\mathrm{mrk}(f_n^m(a),i)}$. Canceling the bead of (i, j) with the arrow of the cap of $e_{i,n}$ yields an additional sign. Finally, the cup (i, n) of $e_{i,n}$ must be pulled down to level i of $f_n^m(a)$, at the cost of an additional $(-1)^{\mathrm{mrk}(f_n^m(a),i)}$. Hence, the total sign from the action of e_1 is -1, so $s_{j,n} f_n^m(a) = -e_{i,n} f_n^m(a)$.

Finally, suppose (j, i) is a cup in $f_n^m(a)$. Acting by $s_{j,n}$ on $f_n^m(a)$ changes (n, m') into (j, m') and (j, i) into (i, n), and putting the resulting diagram back into standard form requires a sign $(-1)^{\mathrm{mrk}(f_n^m(a),i)-\mathrm{mrk}(f_n^m(a),j+1)}$ to move the bead of (i, n) from level j to level i. Ignoring the markings, the action of $e_{i,n}$ also produces these changes to $f_n^m(a)$, so we need only check the signs. First, the cap of $e_{i,n}$ must descend down to level i of $f_n^m(a)$ at the cost of $(-1)^{\mathrm{mrk}(f_n^m(a),i)}$. Canceling the bead of (j, i) and the arrow of the cap of $e_{i,n}$ introduces another sign. Finally, the cup (i, n) of $e_{i,n}$ must be pulled down to level i of $f_n^m(a)$ at the cost of an additional $(-1)^{\mathrm{mrk}(f_n^m(a),i)}$. Hence, the total sign from the action of e_1 is -1, so $s_{j,n} f_n^m(a) = -e_{i,n} f_n^m(a)$. $\square$

Recall that the *content* of the box in the i^{th} row and j^{th} column of a partition is $j - i$. This means contents are constant on diagonals, with 0 down the main diagonal, positive numbers down each superdiagonal, and negative numbers down each subdiagonal. For example, the partition $(5, 4, 2, 2, 1)$ and the contents of its boxes are shown below.

0	1	2	3	4
-1	0	1	2	
-2	-1			
-3	-2			
-4				

Figure 44

For T a standard tableau of shape λ, denote by $c_T(i)$ the content of the box containing i in the standard tableau T. So, for example, in the standard tableau

$$\begin{array}{|c|c|c|c|c|}
\hline
1 & 2 & 5 & 6 & 8 \\
\hline
3 & 4 & 7 & 11 \\
\cline{1-4}
9 & 10 \\
\cline{1-2}
12 & 14 \\
\cline{1-2}
13 \\
\cline{1-1}
\end{array}$$

Figure 45

we have $c_T(1) = 0$, $c_T(2) = 1$, $c_T(3) = -1$, and so on.

Recall also the *Jucys-Murphy elements* for $\mathbb{C}S_m$: $x_1 = 0$ and, for $k = 2, \ldots, m$, $x_k = \sum_{j=2}^{k-1} s_{j,k}$. The collection $\{x_1, \ldots, x_m\}$ generates a commutative subalgebra of $\mathbb{C}S_m$ and the sum $x_1 + \cdots + x_m$ is central in $\mathbb{C}S_m$. The *Young basis* $\left\{v_T : T \in \mathcal{T}^\lambda\right\}$ for the Specht module S^λ, sometimes also called the Murphy basis or Gelfand-Tsetlin basis, has the property that x_i acts on v_T by the scalar $c_T(i)$, for all $i = 2, \ldots, m$. Hence, for $T \in \mathcal{T}^\lambda$, the sum of the Jucys-Murphy elements acts on v_T by the sum of the contents of the partition λ. See [35] for details about this basis.

Proposition 4.4. *Let $\lambda \vdash m$ where $n = m + 2k$. Let $\left\{v_T : T \in \mathcal{T}^\lambda\right\}$ be the Young basis for S^λ. Then, for each $a \in A$, the element $f_n^m(a) \otimes v_T$ in $\Delta_n(\lambda)$ is an eigenvector for X_n of eigenvalue $c_T(m)$.*

Proof. Let $\{(i_1, j_1), \ldots, (i_k, j_k) : i_l < j_l, \forall l = 1, \ldots, k\}$ be the set of cups in a and let $\{(t_1, 1'), \ldots, (t_{m-1}, (m-1)'), (n, m')\}$ be the set of through strings in $f_n^m(a)$. By the previous lemma, $s_{i_l,n} + e_{j_l,n}$, $s_{j_l,n} + e_{i_l,n}$, and $e_{t_r,n}$ all act by 0 on $f_n^m(a)$ for $1 \leq l \leq k$ and $1 \leq r \leq m-1$, while each $s_{t_r,n}$ acts as the transposition (r, m) on the right. Hence, we have

$$
\begin{aligned}
X_n f_n^m(a) &= \sum_{l=1}^{n-1} (s_{l,n} + e_{l,n})\, f_n^m(a) \\
&= \sum_{c=1}^{k} (s_{i_c,n} + s_{j_c,n} + e_{i_c,n} + e_{j_c,n})\, f_n^m(a) + \sum_{r=1}^{m-1} (s_{t_r,n} + e_{t_r,n})\, f_n^m(a) \\
&= \sum_{r=1}^{m-1} f_n^m(a)(r,m) \\
&= f_n^m(a) x_m
\end{aligned}
$$

where x_m denotes the m^{th} Jucys-Murphy element for the symmetric group. Thus,

$$
\begin{aligned}
X_n f_n^m(a) \otimes v_T &= f_n^m(a) x_m \otimes v_T \\
&= f_n^m(a) \otimes x_m v_T \\
&= f_n^m(a) \otimes c_T(m) v_T
\end{aligned}
$$

so X_n acts on $f_n^m(a) \otimes v_T$ by the scalar $c_T(m)$. □

Recall from Theorem 4.2 that $\mathrm{res}_{n-1}^n \Delta_n(\lambda)$ has a submodule $W \cong \bigoplus_{\mu \lhd \lambda} \Delta_{n-1}(\mu)$.

Corollary 4.5. *Suppose $\lambda \vdash m$ and $\mu \lhd \lambda$. Then, X_n acts on the summand $\Delta_{n-1}(\mu)$ of the submodule W of $res_{n-1}^n \Delta_n(\lambda)$ by scalar multiplication by the content of the box removed from λ to form μ.*

Proof. Since X_n commutes with $B_{n-1} \subseteq B_n$, it preserves the decomposition of W as a direct sum of B_{n-1}-submodules. Now, each summand $\Delta_{n-1}(\mu)$ of W is indecomposable as a B_{n-1}-module, so X_n must act on $\Delta_{n-1}(\mu)$ by a single eigenvalue: if we had a decomposition of $\Delta_{n-1}(\mu)$ into eigenspaces for X_n, then each one would again be a B_{n-1}-submodule of $\Delta_{n-1}(\mu)$ since X_n and B_{n-1} commute. By the previous proposition, X_n acts on a vector $f_n^m(a) \otimes v_T \in \Delta_{n-1}(\mu)$ by the eigenvalue $c_T(m)$, which is the content of the box containing m in the standard tableau T of shape

λ. Since a standard tableau for $\mu \vdash m-1$ does not have a box labeled by m and $\mu \subset \lambda$, $c_T(m)$ must be the content of the box removed from λ to form μ. $\square$

Suppose $\lambda \vdash m$ and $\mu \lhd \lambda$. Since λ has a unique removable box of a given content, the eigenvalue of X_n on W uniquely characterizes those vectors $f_n^m(a) \otimes v_T$ which appear in the summand $\Delta_{n-1}(\lambda)$ of W. Let $\mathcal{T}^\lambda(\mu)$ be the set of standard tableaux of shape λ for which the box labeled m is λ/μ. Note that removing the box containing m from a tableau $T \in \mathcal{T}^\lambda(\mu)$ results in a standard tableau of shape μ and, in fact, we can recover every standard tableau of shape μ from some $T \in \mathcal{T}^\lambda(\mu)$ in this way. By the previous results, if $T \in \mathcal{T}^\lambda(\mu)$, then X_n acts on $f_n^m(a) \otimes v_T$ by the eigenvalue $c_T(m)$ for all $a \in A$. Hence, the set $\left\{f_n^m(a) \otimes v_T : a \in A, T \in \mathcal{T}^\lambda(\mu)\right\}$ forms a basis for the (isomorphic copy of the) summand $\Delta_{n-1}(\mu)$ of W.

We now show that X_n acts on each summand of the quotient

$$\operatorname{res}^n_{n-1}\Delta_n(\lambda)/W \cong \bigoplus_{\nu \rhd \lambda} \Delta_{n-1}(\nu)$$

by an eigenvalue determined by the partitions λ and $\nu \rhd \lambda$. Recall from the proof of Theorem 4.2 the set $B = \mathcal{P}_n^m \backslash A$, which consists of those half-diagrams which have vertex n connected to some other vertex. We begin with a technical lemma to discover how the individual summands of the Jucys-Murphy element X_n act on a diagram $f_n^m(b)$, for $b \in B$.

Lemma 4.6. *For $b \in B$ with cup (l, n), we have*

1. $$s_{j,n} f_n^m(b) = \begin{cases} f_n^m(b) & \text{if } j = l \\ -e_{i,n} f_n^m(b) & \text{if } (j,i) \text{ or } (i,j) \text{ is a cup in } b \\ 0 \text{ in } V/W & \text{if } j \text{ is free in } b \end{cases}$$

2. $e_{j,n} f_n^m(b) = \begin{cases} 0 & \textit{if } j = l \\ -s_{i,n} f_n^m(b) & \textit{if } (j,i) \textit{ or } (i,j) \textit{ is a cup in } b \\ (jl) f_n^m(b) & \textit{if } j \textit{ is free in } b \end{cases}$

Proof. Since $f_n^m(b)$ has cup (l,n), we have $e_{l,n} f_n^m(b) = 0$ and $s_{l,n} f_n^m(b) = f_n^m(b)$. Suppose j is free in b. Say (j,q') is the through string in $f_n^m(b)$ connected to vertex j. Then, $s_{j,n} f_n^m(b) = 0$ in V/W since the resulting diagram has a through string connected to vertex n, so lies in W. Ignoring markings for the moment, both $e_{j,n} f(b)$ and $(jl) f_n^m(b)$ have a new cup (j,n) and a new through string (l,q'). We consider two cases to check that the signs match.

1. Suppose $j < l$. Then, to compute the sign for the action of $e_{j,n}$, we move the cup (l,n) up through $f_n^m(b)$ at the cost of $(-1)^{\mathrm{mrk}(f(b),l)}$, cancel the bead of (l,n) with the arrow of the cap of $e_{j,n}$ with no sign, and then lower the cup (j,n) back into $f_n^m(b)$ at the cost of $(-1)^{\mathrm{mrk}(f_n^m(b),j)}$. So, the total sign is $(-1)^{\mathrm{mrk}(f_n^m(b),l)+\mathrm{mrk}(f_n^m(b),j)}$. Since acting by (jl) stretches the left endpoint of the cup (l,n) to vertex j and there is no cup in $f_n^m(b)$ at level j, the sign involved is $(-1)^{\mathrm{mrk}(f_n^m(b),l)-\mathrm{mrk}(f_n^m(b),j)}$, which matches our calculation for the action of $e_{j,n}$ (mod 2). Hence, $e_{j,n} f_n^m(b) = (jl) f_n^m(b)$ in this case.

2. Suppose $j > l$. To compute the sign for the action of $e_{j,n}$, we move the cup (l,n) up through $f_n^m(b)$ at the cost of $(-1)^{\mathrm{mrk}(f_n^m(b),l)}$, cancel the bead of (l,n) with the arrow of the cap of $e_{j,n}$ with no sign, and then lower the cup (j,n) back into $f_n^m(b)$ at the cost of $(-1)^{\mathrm{mrk}(f_n^m(b),j)-1}$ since the cup (l,n) is no longer part of the diagram. So, the total sign is $(-1)^{\mathrm{mrk}(f_n^m(b),l)+\mathrm{mrk}(f_n^m(b),j)-1}$. Since acting by (jl) stretches the left endpoint of the cup (l,n) to vertex j and there is no cup in $f_n^m(b)$ at level j, the sign involved is $(-1)^{\mathrm{mrk}(f_n^m(b),j)-\mathrm{mrk}(f_n^m(b),l+1)} = (-1)^{\mathrm{mrk}(f_n^m(b),j)-\mathrm{mrk}(f_n^m(b),l)-1}$, which matches our calculation for the action of $e_{j,n}$ (mod 2). Hence, $e_{j,n} f_n^m(b) = (jl) f_n^m(b)$ in this case.

Now, suppose (j,i) is a cup in b. Ignoring markings, acting by $s_{j,n}$ on $f_n^m(b)$ changes the cup (j,i) into (i,n) and the cup (l,n) into (j,l) if $j<l$ or (l,j) if $j>l$. Also ignoring markings, acting by $e_{i,n}$ on $f_n^m(b)$ introduces a new cup (i,n) and combines the cups (j,i) and (l,n) with the cap of $e_{i,n}$ to form the cup (j,l) if $j<l$ or (l,j) if $j>l$. Since the diagrams resulting from these two actions agree, we need only check the signs. There are three cases to consider.

1. Suppose $j<i<l$. To compute the sign for the action of $s_{j,n}$ on $f_n^m(b)$, we move the newly formed cup (i,n) from its starting point at level j down to level i at the cost of $(-1)^{\mathrm{mrk}(f_n^m(b),i)-\mathrm{mrk}(f_n^m(b),j+1)}$ and then move the cup (j,l) from level l up to level j at the cost of $(-1)^{\mathrm{mrk}(f_n^m(b),l)-\mathrm{mrk}(f_n^m(b),j+1)+1}$, where the extra sign comes from the fact that the bead on (j,l) must pass the bead on (i,n). Hence, the total sign for the action of $s_{j,n}$ is $(-1)^{\mathrm{mrk}(f_n^m(b),i)+\mathrm{mrk}(f_n^m(b),l)+1}$. Now, to compute the sign for the action of $e_{i,n}$, we first move the cup (l,n) up through $f_n^m(b)$ at the cost of $(-1)^{\mathrm{mrk}(f_n^m(b),l)}$, cancel the bead of this cup with the arrow in $e_{i,n}$ with no sign, and then lower the cup (i,n) of $e_{i,n}$ down into $f_n^m(b)$ at the cost of $(-1)^{\mathrm{mrk}(f_n^m(b),i)}$. Hence, the total sign for the action of $e_{i,n}$ is $(-1)^{\mathrm{mrk}(f_n^m(b),i)+\mathrm{mrk}(f_n^m(b),l)}$, so $s_{j,n}f_n^m(b) = -e_{i,n}f_n^m(b)$, as desired.

2. Suppose $j<l<i$. To compute the sign for the action of $s_{j,n}$ on $f_n^m(b)$, we first move the newly formed cup (i,n) from its starting point at level j down to level i at the cost of $(-1)^{\mathrm{mrk}(f_n^m(b),i)-\mathrm{mrk}(f_n^m(b),j+1)}$. Note that the bead on this cup passes the bead on (j,l) during this move and the sign for that passage is counted. Now, move the cup (j,l) from level l up to level j at the cost of $(-1)^{\mathrm{mrk}(f_n^m(b),l)-1-\mathrm{mrk}(f_n^m(b),j+1)-1}$, where the missing signs appear since there is no longer a cup at level j. Hence, the total sign for the action of $s_{j,n}$ is $(-1)^{\mathrm{mrk}(f_n^m(b),i)+\mathrm{mrk}(f_n^m(b),l)}$. Now, to compute the sign for the action of $e_{i,n}$, we first move the cup (l,n) up through $f_n^m(b)$ at the cost of $(-1)^{\mathrm{mrk}(f_n^m(b),l)}$, cancel the bead of this cup with the arrow in $e_{i,n}$ with no sign, and then lower

the cup of $e_{i,n}$ down into $f_n^m(b)$ at the cost of $(-1)^{\mathrm{mrk}(f_n^m(b),i)-1}$ since there is no longer a cup at level l. Hence, the total sign for the action of $e_{i,n}$ is $(-1)^{\mathrm{mrk}(f_n^m(b),i)+\mathrm{mrk}(f_n^m(b),l)-1}$, so $s_{j,n}f_n^m(b) = -e_{i,n}f_n^m(b)$.

3. Suppose $l < j$. To compute the sign for the action of $s_{j,n}$, we move the cup (i,n) from level j down to level i at the cost of $(-1)^{\mathrm{mrk}(f_n^m(b),i)-\mathrm{mrk}(f_n^m(b),j+1)}$. Note that the cup (l,j) is already in its proper position since only its right endpoint was changed, so no other signs are generated. Now, to compute the sign for the action of $e_{i,n}$, move the cup (j,i) up through $f_n^m(b)$ at the cost of $(-1)^{\mathrm{mrk}(f_n^m(b),j)}$, cancel the bead of (j,i) with the arrow of $e_{i,n}$ at the cost of another sign, and then move the cup of $e_{i,n}$ down into $f_n^m(b)$ at the cost of an additional $(-1)^{\mathrm{mrk}(f_n^m(b),i)-1}$, where the missing sign is due to the fact that there is no longer a cup at level j. Hence, the total sign for the action of $e_{i,n}$ is $(-1)^{\mathrm{mrk}(f_n^m(b),j)+\mathrm{mrk}(f_n^m(b),i)}$, and since $\mathrm{mrk}\left(f_n^m(b),j+1\right) = \mathrm{mrk}\left(f_n^m(b),j\right)+1$ we have that $s_{j,n}f_n^m(b) = -e_{i,n}f_n^m(b)$.

Finally, suppose (i,j) is a cup in b. Then, $(ij)f_n^m(b) = f_n^m(b)$ since the transposition fixes the cup (i,j). Also, note that $s_{j,n} = (ij)s_{i,n}(ij)$ and $e_{i,n} = (ij)e_{j,n}(ij)$. Then, by the case above,

$$\begin{aligned} s_{j,n}f_n^m(b) &= (ij)s_{i,n}(ij)f_n^m(b) \\ &= (ij)s_{i,n}f_n^m(b) \\ &= -(ij)e_{j,n}f_n^m(b) \\ &= -(ij)e_{j,n}(ij)f_n^m(b) \\ &= -e_{i,n}f_n^m(b) \end{aligned}$$

This completes the proof. □

Lemma 4.7. *For $b \in B$, we have*

$$X_n . f_n^m(b) = \left(1 + \sum_{r=1}^{m} (t_r l)\right) f_n^m(b)$$

in $res_{n-1}^n \Delta_n(\lambda)/W$, where $t_1, \ldots, t_m$ are the free vertices in b.

Proof. Let $\{(i_1, j_1), \ldots, (i_{k-1}, j_{k-1}), (l, n) : i_l < j_l, \forall l = 1, \ldots, k-1\}$ be the set of cups in b and let $\{(t_1, 1'), \ldots, (t_m, m')\}$ be the set of through strings in $f_n^m(b)$. By the previous lemma, $s_{t_r,n}$, $e_{l,n}$, $s_{i_s,n} + e_{j_s,n}$, and $s_{j_s,n} + e_{i_s,n}$ all act by 0 on $f_n^m(b)$ for all $r = 1, \ldots, m$ and $s = 1, \ldots, k-1$, while $e_{t_r,n}$ acts by the transposition $(t_r l)$ on $f_n^m(b)$. Hence,

$$\begin{aligned} X_n f_n^m(b) &= \sum_{t=1}^{n-1} (s_{t,n} + e_{t,n}) f_n^m(b) \\ &= \sum_{c=1}^{k-1} (s_{i_c,n} + s_{j_c,n} + e_{i_c,n} + e_{j_c,n}) f_n^m(b) + (s_{l,n} + e_{l,n}) f_n^m(b) \\ &\quad + \sum_{r=1}^{m} (s_{t_r,n} + e_{t_r,n}) f_n^m(b) \\ &= f_n^m(b) + \sum_{r=1}^{m} (t_r l) f_n^m(b) \end{aligned}$$

as claimed. □

Proposition 4.8. *Let $\lambda \vdash m$ and suppose $\nu \rhd \lambda$. Let $\{v_T : T \in \mathcal{T}^\nu\}$ be the Young basis for S^ν, viewed as a subspace of $ind_{S_m}^{S_{m+1}} S^\lambda$. Then,*

$$\psi^{-1}\left(f_{n-1}^{m+1}(x) \otimes v_T\right) \in res_{n-1}^n \Delta_n(\lambda)/W$$

is an eigenvector for X_n of eigenvalue $1 + c_T(\nu/\lambda)$ for all $t \in \mathcal{T}^\nu$, $x \in \mathcal{P}_{n-1}^{m+1}$.

Proof. It suffices to prove this for $x = b_{n-1,m+1}$ since we can always find $\sigma_x \in S_{n-1}$

such that $f_{n-1}^{m+1}(x) = (-1)^{q_x}\sigma_x f_{n-1}^{m+1}(b_{n-1,m+1})$ so if

$$X_n.\psi^{-1}\left(f_{n-1}^{m+1}(b_{n-1,m+1}) \otimes v_T\right) = (1 + c_T(\nu/\lambda)) \cdot \psi^{-1}\left(f_{n-1}^{m+1}(b_{n-1,m+1}) \otimes v_T\right)$$

then since ψ^{-1} is a B_{n-1}-module homomorphism and X_n commutes with B_{n-1}, we have

$$\begin{aligned}
X_n.\psi^{-1}\left(f_{n-1}^{m+1}(x) \otimes v_T\right) &= X_n.\psi^{-1}\left((-1)^{q_x}\sigma_x f_{n-1}^{m+1}(b_{n-1,m+1}) \otimes v_T\right) \\
&= (-1)^{q_x}\sigma_x X_n \psi^{-1}\left(f_{n-1}^{m+1}(b_{n-1,m+1}) \otimes v_T\right) \\
&= (-1)^{q}\sigma_x (1 + c_T(\nu/\lambda)) \cdot \psi^{-1}\left(f_{n-1}^{m+1}(b_{n-1,m+1}) \otimes v_T\right) \\
&= (1 + c_T(\nu/\lambda)) \cdot \psi^{-1}\left((-1)^{q}\sigma_x f_{n-1}^{m+1}(b_{n-1,m+1}) \otimes v_T\right) \\
&= (1 + c_T(\nu/\lambda)) \cdot \psi^{-1}\left(f_{n-1}^{m+1}(x) \otimes v_T\right)
\end{aligned}$$

Recall $\tau_i = (i, m+1)$. The set of through strings in $D := f_{n-1}^{m+1}(b_{n-1,m+1})$ is $\{(1,1'), \ldots, (m+1, (m+1)')\}$. We can write a Young basis vector v_T in the S^ν component of $\mathrm{ind}_{S_m}^{S_{m+1}} S^\lambda$ in the basis $\left\{\tau_i \otimes v_S : 1 \le i \le m+1, S \in \mathcal{T}^\lambda\right\}$ of $\mathrm{ind}_{S_m}^{S_{m+1}} S^\lambda$ as $v_T = \sum_{i=1}^{m+1} \sum_{S \in \mathcal{T}^\lambda} \alpha_{i,S}\tau_i \otimes v_S$. By a calculation in the proof of Theorem 4.2, we then have $\psi^{-1}(D \otimes (\tau_i \otimes v_S)) = f_n^m(b_i) \otimes \sigma_{i,m+1}^{-1}\tau_i v_S$ in $\mathrm{res}_{n-1}^n \Delta_n(\lambda)/W$, where b_i is obtained from $b_{n-1,m+1}$ by adding a vertex n and cup (i,n). Note that no sign is required since (i,n) is the highest cup in $f_n^m(b_i)$. Moreover, $f_n^m(b_i)$ has through strings $\{(1,1'), \ldots, (i-1, (i-1)'), (i+1, i'), \ldots, (m+1, m')\}$. Hence, by the previous lemma

$$X_n.f_n^m(b_i) = \left(1 + \sum_{\substack{r=1,\\ r\neq i}}^{m+1} (ri)\right) f_n^m(b_i)$$

But, the transposition (ii) acts as identity on $f_n^m(b_i)$ so we can actually write $X_n.f_n^m(b_i) = \left(\sum_{r=1}^{m+1}(ri)\right) f_n^m(b_i)$. Since each transposition (ri) in this sum lies in

S_{n-1}, we then have

$$
\begin{aligned}
X_n.\psi^{-1}\left(D\otimes(\tau_i\otimes v_S)\right) &= X_n.f_n^m(b_i)\otimes\sigma_{i,m+1}^{-1}\tau_i v_S \\
&= \left(\sum_{r=1}^{m+1}(ri)\right) f_n^m(b_i)\otimes\sigma_{i,m+1}^{-1}\tau_i v_S \\
&= \left(\sum_{r=1}^{m+1}(ri)\right)\psi^{-1}\left(D\otimes(\tau_i\otimes v_S)\right) \\
&= \psi^{-1}\left(\left(\sum_{r=1}^{m+1}(ri)\right)D\otimes(\tau_i\otimes v_S)\right) \\
&= \psi^{-1}\left(D\left(\sum_{r=1}^{m+1}(ri)\right)\otimes(\tau_i\otimes v_S)\right)
\end{aligned}
$$

since ψ^{-1} is a B_{n-1}-module homomorphism and the through strings in D are straight vertical lines. Now, $\left(\sum_{r=1}^{m+1}(ri)\right)$ is an element of $\mathbb{C}S_{m+1}$ so can pass across the first tensor product yielding

$$
\begin{aligned}
\left(\sum_{r=1}^{m+1}(ri)\right)\tau_i &= (ii)\tau_i+\tau_i\left(\tau_i\left(\sum_{\substack{r=1,\\ r\neq i}}^{m+1}(ri)\right)\tau_i\right) \\
&= \tau_i+\tau_i\left(\sum_{\substack{r=1,\\ r\neq i}}^{m}(r,m+1)+(i,m+1)\right) \\
&= \tau_i\left(1+x_{m+1}\right)
\end{aligned}
$$

in the middle tensor factor, where x_{m+1} is the $(m+1)^{\text{st}}$ Jucys-Murphy element for the symmetric group algebra $\mathbb{C}S_{m+1}$. If we let t_k denote the sum of the first k Jucys-Murphy elements in $\mathbb{C}S_{m+1}$, then we have $x_{m+1}=t_{m+1}-t_m$. It is well known that t_{m+1} is central in $\mathbb{C}S_{m+1}$ so

$$
\begin{aligned}
\tau_i x_{m+1} &= \tau_i\left(t_{m+1}-t_m\right) \\
&= t_{m+1}\tau_i-\tau_i t_m
\end{aligned}
$$

and hence

$$\begin{aligned} X_n.\psi^{-1}\left(D\otimes(\tau_i\otimes v_S)\right) &= \psi^{-1}\left(D\otimes\left((\tau_i+t_{m+1}\tau_i-\tau_i t_m)\otimes v_S\right)\right) \\ &= \psi^{-1}\left(D\otimes(\tau_i\otimes v_S)\right)+\psi^{-1}\left(D\otimes t_{m+1}\left(\tau_i\otimes v_S\right)\right) \\ &\qquad -\psi^{-1}\left(D\otimes(\tau_i\otimes t_m v_S)\right) \end{aligned}$$

since $t_m \in \mathbb{C}S_m$. But t_m acts on each Young basis element $v_S \in S^\lambda$ by a single scalar value, namely the sum of the contents of all boxes in λ. If we denote this number by c_λ, then we have

$$\begin{aligned} X_n.\psi^{-1}\left(D\otimes(\tau_i\otimes v_S)\right) &= \psi^{-1}\left(D\otimes(\tau_i\otimes v_S)\right)+\psi^{-1}\left(D\otimes t_{m+1}\left(\tau_i\otimes v_S\right)\right) \\ &\qquad -c_\lambda\psi^{-1}\left(D\otimes(\tau_i\otimes v_S)\right) \end{aligned}$$

Since this calculation holds for all $i=1,\ldots,m+1$ and all $S\in\mathcal{T}^\lambda$, we thus have

$$\begin{aligned} X_n.\psi^{-1}\left(D\otimes v_T\right) &= \sum_{i=1}^{m+1}\sum_{S\in\mathcal{T}^\lambda}\alpha_{i,S}X_n.\psi^{-1}\left(D\otimes(\tau_i\otimes v_S)\right) \\ &= \sum_{i=1}^{m+1}\sum_{S\in\mathcal{T}^\lambda}\alpha_{i,S}\left[\psi^{-1}\left(D\otimes(\tau_i\otimes v_S)\right)+\psi^{-1}\left(D\otimes t_{m+1}\left(\tau_i\otimes v_S\right)\right)\right. \\ &\qquad \left.-c_\lambda\psi^{-1}\left(D\otimes(\tau_i\otimes v_S)\right)\right] \\ &= \psi^{-1}\left(D\otimes v_T\right)+\psi^{-1}\left(D\otimes t_{m+1}v_T\right)-c_\lambda\psi^{-1}\left(D\otimes v_T\right) \\ &= \left(1+c_\nu-c_\lambda\right)\psi^{-1}\left(D\otimes v_T\right) \end{aligned}$$

since t_{m+1} acts on the Young basis element v_T by c_ν, the sum of the contents of the boxes in ν. Finally, observe that since $\nu \triangleright \lambda$ we have $c_\nu - c_\lambda = c\left(\nu/\lambda\right)$, the content of the box added to λ to form ν, which proves the claim. □

Corollary 4.9. *Let $\lambda \vdash m$ and suppose $\nu \triangleright \lambda$. Then, X_n acts on the summand $\Delta_{n-1}(\nu)$ of the quotient $res^n_{n-1}\Delta_n(\lambda)/W$ by scalar multiplication by one plus the* *content of the box added to λ to form ν.*

Proof. Arguing as above for W, we know that X_n must act on each indecomposable summand $\Delta_{n-1}(\nu)$ of $\mathrm{res}^n_{n-1}\Delta_n(\lambda)/W$ by a single eigenvalue. Since $\Delta_{n-1}(\nu)$ has as its basis the collection of all vectors of the form $f^{m+1}_{n-1}(x) \otimes v_T$ with $x \in \mathcal{P}^{m+1}_{n-1}$ and $T \in \mathcal{T}^\nu$, the previous proposition tells us that X_n acts by the eigenvalue $1 + c\,(\nu/\lambda)$ on the image of each such vector inside $\mathrm{res}^n_{n-1}\Delta_n(\lambda)/W$. □

4.3 Localization and Globalization Functors

When $n \geq 3$, let e be the idempotent shown below.

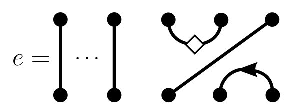


Figure 46

Note that eB_ne is again an algebra with identity element e.

Lemma 4.10. *When $n \geq 3$, $eB_ne \cong B_{n-2}$ as algebras.*

Proof. First note that $A := \mathbb{C}\{D \in B_n : D \text{ has edges } (n-2, n-1), ((n-1)', n')\}$ is a subalgebra of B_n since the product of any two such diagrams again has the desired cup and cap. We claim $eB_ne = A$. Given $D \in B_n$, we have that either $eDe = 0$ or eDe has $(n-2, n-1)$ and $((n-1)', n')$ among its edges, which shows $eB_ne \subseteq A$. Conversely, if D is a standard diagram in A then we claim $D = -eDe$. Certainly, eDe has $(n-2, n-1)$ and $((n-1)', n')$ among its edges. Since the edges of e connecting vertices in the set $V = \{1, 1', \ldots, n-3, (n-3)'\}$ are straight vertical lines, any edges of D which connect two vertices in V will again be present in eDe. So, we need only check that eDe and D have the same edges connected to vertices n and $(n-2)'$. Note that vertex n (in the top e) is connected first to $n-2$ in D, and then back to vertex n in D via the cup $(n-2, n-1)$ of D and cap $((n-1)', n')$ of the top e. So, if (x, n) is an edge in D, then it will also be an edge of eDe. Similarly, we see that vertex $(n-2)'$ in the bottom e connects via a through string and adjacent cup-cap pair to vertex $(n-2)'$ in D and so if $(y,(n-2)')$ is an edge of D, then this will also be an edge of eDe. It remains to justify that no signs are introduced by the two multiplications of D by e. First, observe that the pair of signs from simplifying the adjacent markings cancel one another. Second, to make eDe nearer to a standard diagram, the remaining cups in D must be pushed up past the cup in the top e, while the remaining caps in D must be pulled below the cap in the bottom e. Since D has the same number of cups as caps, these two operations generate the same sign twice. Finally, to fully standardize eDe, the arrow on the bottom cap of e must be reversed at the cost of a sign. Thus, we have that $eB_ne = A$, as desired.

We have a natural algebra inclusion $i : B_{n-2} \hookrightarrow B_n$ which pads two vertical strands on the right of each diagram in B_{n-2}. Let $\psi : B_{n-2} \to eB_ne$ be the linear map which first includes $D \in B_{n-2}$ into B_n and then multiplies on both the left and right by e. Diagrammatically, $\psi(D) = e \cdot i(D) \cdot e$ looks like

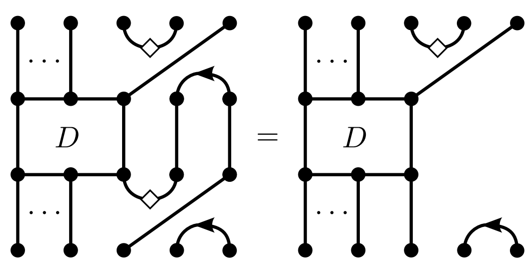


Figure 47

since canceling the bead and arrow which appear on the strand connected to $(n-2)'$ incurs no sign as the arrow on the cap is pulled down past the even number of markings in D and the arrow points into the bead. Looking at the rightmost diagram above, we observe that ψ is bijective so it remains to verify that ψ is an algebra homomorphism. But this also follows diagrammatically, since

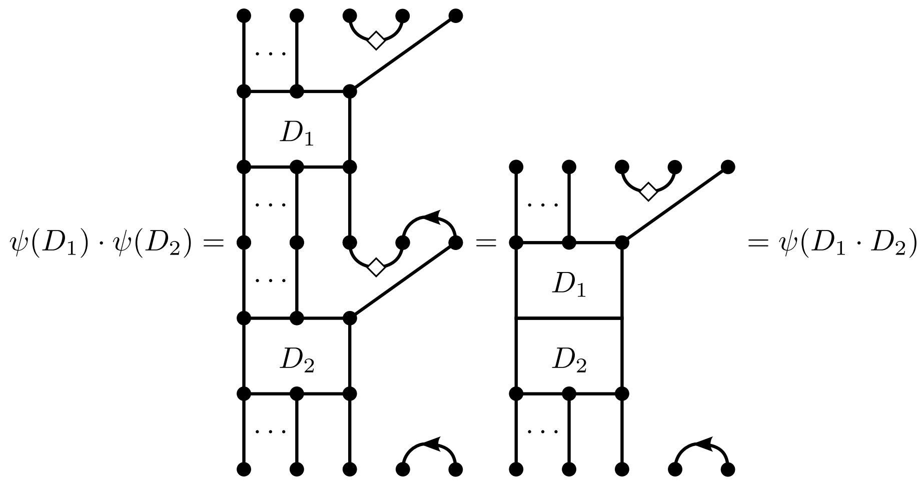


Figure 48

Hence ψ gives an algebra isomorphism $B_{n-2} \cong eB_ne$ as desired. $\square$

Using e, we define the *localization* and *globalization functors*, respectively, as in [22]: $F_n : B_n$-**mod** $\to B_{n-2}$-**mod** by $M \mapsto eM$ and $G_n : B_{n-2}$-**mod** $\to B_n$-**mod** by $M \mapsto B_ne \otimes_{B_{n-2}} M$. These functors are called Schur functors in some references. The following proposition records Green's results from Section 6.2 of [22], translated into our notation.

Proposition 4.11. *The following results hold for the functors F_n and G_n.*

1. *F_n is an exact functor. So, if $0 \to M_1 \to M_2 \to M_3 \to 0$ is an exact sequence of B_n-modules, then $0 \to eM_1 \to eM_2 \to eM_3 \to 0$ is an exact sequence of B_{n-2}-modules.*

2. *If M is a simple B_n-module, then $F_n(M)$ is either 0 or a simple B_{n-2}-module.*

3. *$F_n \circ G_n$ is (isomorphic to) the identity functor on $B_{n-2} -$ **mod**.*

4. *$\{eL_n(\lambda) : \lambda \in \Lambda'(n), eL_n(\lambda) \neq 0\}$ is a complete irredundant set of simple B_{n-2}-modules.*

We now calculate the effect of these functors on standard modules and, in the case of F_n, on simple modules.

Lemma 4.12. *If* $n = m + 2k$ *and* $\lambda \vdash m$, *then*

$$F_n\left(\Delta_n(\lambda)\right) \cong \begin{cases} \Delta_{n-2}(\lambda) & \text{if } m < n \\ 0 & \text{if } m = n \end{cases}$$

as B_{n-2}*-modules. Hence,* $F_n\left(\Delta_n(\lambda)\right) = 0$ *if and only if* $\lambda \vdash n$.

Proof. Certainly, if $\lambda \vdash n$, then $e\Delta_n(\lambda) = 0$ since $e\overline{B_n^n} = 0$. So, suppose $\lambda \vdash m < n$. Note that each element of $e\Delta_n(\lambda)$ can be written as a linear combination of elements of the form $D \otimes v$ with $v \in S^\lambda$ and D a diagram in $\overline{B_n^m}$ which has $(n-2, n-1)$ among its edges. On the other hand, given such an element $D \otimes v$ we see that it also lies in $e\Delta_n(\lambda)$ since

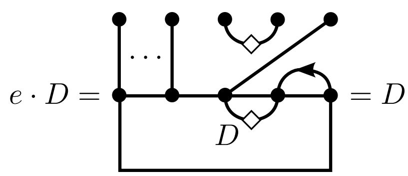


Figure 49

as canceling the cup in D with the cap in e incurs the same sign as lowering the cap in e down to standard position. So, $e\Delta_n(\lambda)$ is precisely the span of the collection of elements of the form $D \otimes v$ with $v \in S^\lambda$ and D a diagram in $\overline{B_n^m}$ which has $(n-2, n-1)$ among its edges. Using this identification, we have a map $\varphi : \Delta_{n-2}(\lambda) \to e\Delta_n(\lambda)$ given by $D \otimes v \mapsto e \cdot i(D) \otimes v$, where $i : B_{n-2} \hookrightarrow B_n$ is the usual inclusion which pads two vertical lines on the right of a diagram.

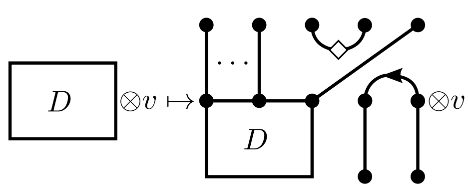


Figure 50

Diagrammatically, this map is readily seen to be injective and surjective since the image is uniquely determined by the diagram $D \in \overline{B_{n-2}^m}$. Moreover, this map is an isomorphism of left $eB_ne \cong B_{n-2}$-modules since

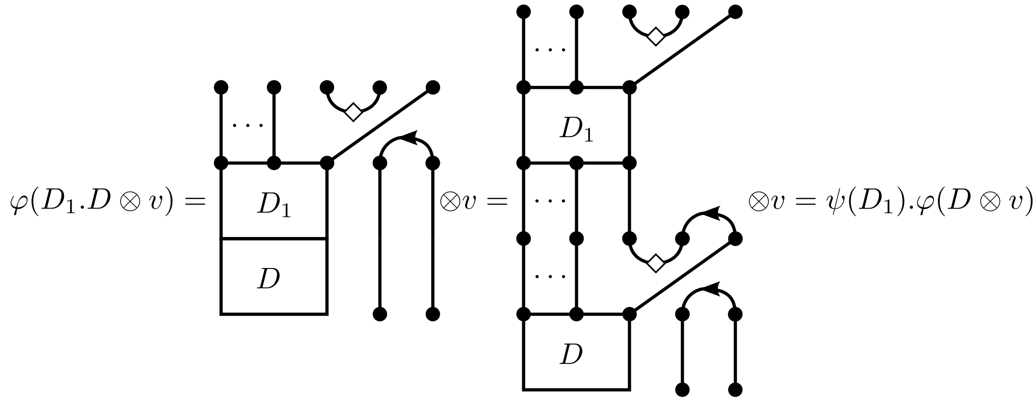


Figure 51

where $\psi : B_{n-2} \to eB_ne$ is the algebra isomorphism from Lemma 4.10. Hence, we have $F_n(\Delta_n(\lambda)) = e\Delta_n(\lambda) \cong \Delta_{n-2}(\lambda)$, as desired. $\square$

Lemma 4.13. *If $n = m + 2k$ and $\lambda \vdash m$, then*

$$F_n(L_n(\lambda)) \cong \begin{cases} L_{n-2}(\lambda) & \text{if } m < n \\ 0 & \text{if } m = n \end{cases}$$

as B_{n-2}-modules. Hence, $F_n(L_n(\lambda)) = 0$ if and only if $\lambda \vdash n$.

Proof. If $\lambda \vdash n$, then $F_n(\Delta_n(\lambda)) = 0$ by the previous lemma and so $F_n(L_n(\lambda)) = 0$ as well. Now, suppose $\lambda \vdash m < n$. By Proposition 4.11(2), $F_n(L_n(\lambda))$ is either 0 or a simple B_{n-2}-module. By Theorem 3.2(6) and Proposition 4.11(4), the two sets $\{L_{n-2}(\mu) : \mu \in \Lambda'(n-2)\}$ and $\{eL_n(\gamma) : \gamma \in \Lambda'(n), eL_n(\gamma) \neq 0\}$ both describe the complete set of non-isomorphic simple B_{n-2}-modules. Since $\Lambda'(n) = \Lambda_n \sqcup \Lambda'(n-2)$ and $F_n(L_n(\tau)) = 0$ when $\tau \in \Lambda_n$, it follows that $F_n(L_n(\lambda)) \neq 0$. Now, $F_n(L_n(\lambda)) \cong L_{n-2}(\mu)$ for some $\mu \in \Lambda'(n-2)$ and $F_n(\Delta_n(\lambda)) = \Delta_{n-2}(\lambda)$, so applying the exact functor F_n to the short exact sequence

$$0 \to \operatorname{rad}(\Delta_n(\lambda)) \to \Delta_n(\lambda) \to L_n(\lambda) \to 0$$

of B_n-modules from Theorem 3.2(4) yields another short exact sequence

$$0 \to F_n\left(\mathrm{rad}\left(\Delta_n(\lambda)\right)\right) \to \Delta_{n-2}(\lambda) \to L_{n-2}(\mu) \to 0$$

of B_{n-2}-modules. But, this says $L_{n-2}(\mu)$ is a simple quotient of $\Delta_{n-2}(\lambda)$, which has $L_{n-2}(\lambda)$ as its unique simple quotient, so it follows that $\mu = \lambda$. $\square$

Let $E_{n,m}$ be the idempotent in $\overline{B_n^m}$ depicted below, which has $k = \frac{1}{2}(n-m)$ cups and caps.

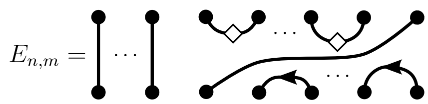


Figure 52

Note that $E_{n,n}$ is the identity, while $E_{n,n-2}$ is the idempotent e introduced at the beginning of this section.

Lemma 4.14. *Suppose $\lambda \vdash m$. Then, $G_n\left(\Delta_{n-2}\left(\lambda\right)\right) \cong \Delta_n\left(\lambda\right)$.*

Proof. Recall $\Delta_{n-2}(\lambda) = \overline{B_{n-2}^m} \otimes_{\mathbb{C}S_m} S^\lambda$, so

$$G_n\left(\Delta_{n-2}\left(\lambda\right)\right) = B_n e \otimes_{B_{n-2}} \left(\overline{B_{n-2}^m} \otimes_{\mathbb{C}S_m} S^\lambda\right)$$

We claim $B_n e \otimes_{B_{n-2}} \overline{B_{n-2}^m} \cong \overline{B_n^m}$ as $(B_n, \mathbb{C}S_m)$-bimodules. Let φ be the map which carries $D_1 e \otimes D_2 \in B_n e \otimes_{B_{n-2}} \overline{B_{n-2}^m}$ to $D_1 e \cdot i\left(D_2\right) \cdot e$. Then, $\varphi\left(D_1 e \otimes D_2\right)$ is given diagrammatically by

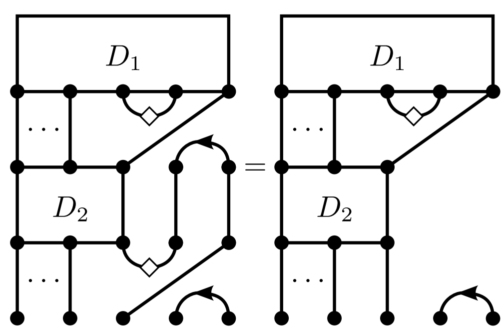
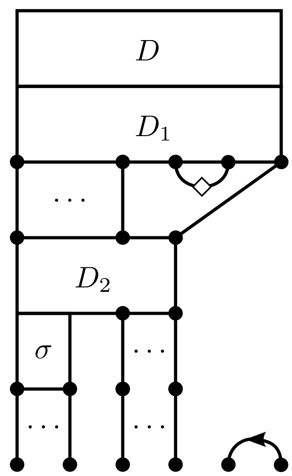


Figure 53

so we see that, after simplifying, $\varphi : B_n e \otimes_{B_{n-2}} \overline{B_{n-2}^m} \to \overline{B_n^m}$. This depiction of the map easily shows that φ is a map of $(B_n, \mathbb{C}S_m)$-bimodules since if $D \in B_n$ and $\sigma \in S_m$ we have

$\varphi(D.D_1e \otimes D_2.\sigma) =$



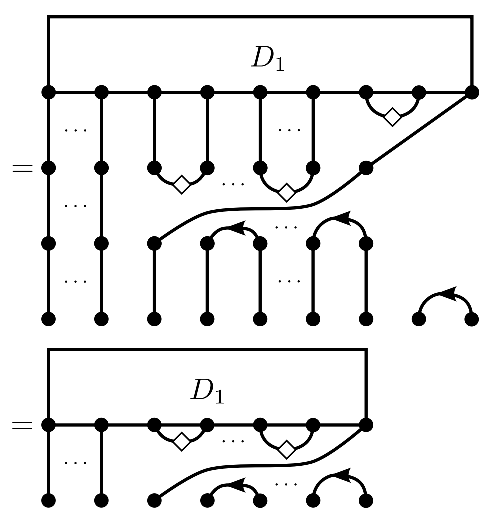
$= D.\varphi(D_1e \otimes D_2).i(\sigma)$

Figure 54

since $\sigma \in S_m$ and e commute.

We first show that φ is surjective. Observe that

$\varphi(D_1e \otimes E_{n-2,m}) =$



$= D_1E_{n,m}$

Figure 55

for any choice of $D_1 \in B_n$ since simplifying the bottom three diagrams introduces the following signs: $(-1)^{k-1}$ from raising the cups of $E_{n-2,m}$ past the cup from the topmost e, and $(-1)^{k-1}$ from the caps of $E_{n-2,m}$ being lowered past the cap of the bottommost e. To obtain any diagram $D \in \overline{B_n^m}$, we now simply choose $D_1 \in B_n$ so that it rearranges the cups and through strings of the copy of $E_{n,m}$ which appears in the rightmost diagram above to form D, scaling by a sign if necessary. This shows that φ is surjective.

To see that φ is injective as well, it suffices to show that $B_n e \otimes_{B_{n-2}} \overline{B^m_{n-2}}$ is spanned by at most $\dim \overline{B^m_n}$-many elements. Let $D_1 e \otimes D_2 \in B_n e \otimes_{B_{n-2}} \overline{B^m_{n-2}}$. We can find $\sigma \in S_{n-2}$ so that $D_2 = \pm \sigma E_{n-2,m}$. The σ can move across the tensor product if we include it into B_n, and we find

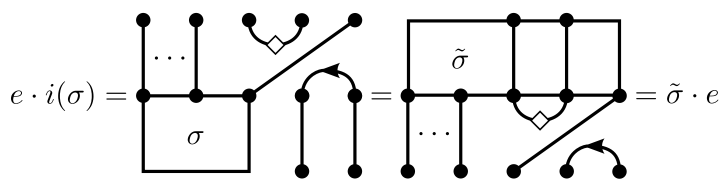


Figure 56

where $\tilde{\sigma}$ is obtained from σ by inserting two vertical lines between vertices $n-3$ and $n-2$. Note that this operation simply comes from a different embedding of B_{n-2} into B_n, so the diagram $\tilde{\sigma}$ is uniquely determined by σ. Hence, we have $D_1 e \otimes D_2 = \pm D_1 \tilde{\sigma} e \otimes E_{n-2,m}$. Now, $E_{n-2,m} \in B_{n-2}$, so this element can also move across the tensor product by including into B_n and we see that

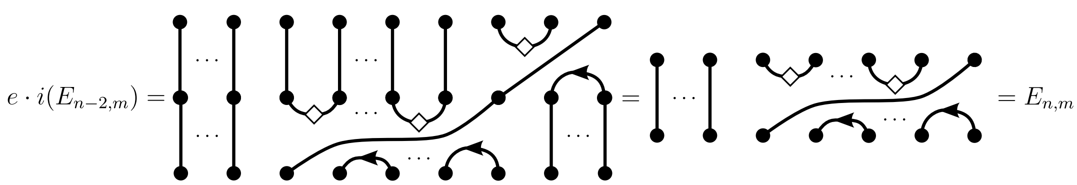


Figure 57

Hence,

$$
\begin{aligned}
D_1 e \otimes D_2 &= \pm D_1 \tilde{\sigma} e \otimes E_{n-2,m} \\
&= \pm D_1 \tilde{\sigma} e \otimes E_{n-2,m} E_{n-2,m} \\
&= \pm D_1 \tilde{\sigma} e \cdot i\left(E_{n-2,m}\right) \otimes E_{n-2,m} \\
&= \pm D_1 \tilde{\sigma} E_{n,m} \otimes E_{n-2,m}
\end{aligned}
$$

This shows that every element in $B_n e \otimes_{B_{n-2}} \overline{B^m_{n-2}}$ belongs to the span of elements of the form $D_1\tilde{\sigma}E_{n,m} \otimes E_{n-2,m}$. But, $D_1\tilde{\sigma}E_{n,m} \in \overline{B^m_n}$ since it has fixed bottom $b_{n,m}$ and the second tensorand is fixed, so there are $\dim \overline{B^m_n}$-many spanning elements of this form. Hence, the map φ is in fact an isomorphism of bimodules. □

As an application of the properties derived above for these functors, we now return to the study of semisimplicity of $B_n(0, -1)$.

Lemma 4.15. *If B_n is semisimple, then B_{n-2} is semisimple.*

Proof. Suppose B_n is semisimple. Given a nonzero finite-dimensional B_{n-2}-module, M, the B_n-module $G_n(M)$ is then semisimple, so $G_n(M) \cong \bigoplus_{i=1}^{t} L_n(\lambda_i)$ as B_n-modules. Then,

$$M \cong F_n \circ G_n(M) \cong \bigoplus_{i=1}^{t} F_n L_n(\lambda_i)$$

as B_{n-2}-modules, where $F_n L_n(\lambda_i) \cong L_{n-2}(\lambda_i)$ or $F_n L_n(\lambda_i) = 0$ for each i, by Lemma 4.13. As $M \neq 0$, we must have $F_n L_n(\lambda_i) \neq 0$ for some i, so M is semisimple as a B_{n-2}-module. Since a finite-dimensional algebra is semisimple if and only if all of its finitely generated modules are semisimple, this shows that B_{n-2} is semisimple. □

Theorem 4.16. *B_n is semisimple if and only if $n = 1$.*

Proof. We previously showed in Proposition 2.3 that B_n is not semisimple when n is even, so suppose n is odd. In the discussion following Proposition 2.3, we showed that B_3 is not semisimple. The lemma above now implies B_n cannot be semisimple when $n > 3$. □

4.4 Induction of Standard Modules to $B_{n+1}(0, -1)$

In this section, we use the restriction and globalization functors to study the induction functor, $\operatorname{ind}_n^{n+1} -$. By definition, the induction of B_n-module M up to B_{n+1} is $\operatorname{ind}_n^{n+1} M = B_{n+1} \otimes_{B_n} M$. We begin with a lemma.

Lemma 4.17. *$B_n e \cong B_{n-1}$ as (B_{n-1}, B_{n-2})-bimodules.*

Proof. Let V be the subspace of B_n spanned by those diagrams having $((n-1)', n')$ among their edges. We first claim that $B_n e = V$ as (B_{n-1}, B_{n-2})-bimodules. Certainly, each diagram in $B_n e$ has $((n-1)', n')$ as one of its edges, so $B_n e \subseteq V$. Let D be a standard diagram in V. We claim $D = De$. Both diagrams have the same top since multiplication on the right does not affect the top row. The vertical edges of e preserve any connections to vertices $\{1', \ldots, (n-3)'\}$, while the cup of e combines with the cap $((n-1)', n')$ in D to create another vertical line attached to $(n-2)'$, preserving the connection to this vertex. Canceling the bead and arrow in this step introduces a sign for each of the caps in D. Finally, the multiplication on the right by e reintroduces the cap $((n-1)', n')$ and moving this cap into standard position introduces a sign for each cup in D except $((n-1)', n')$ plus one more sign to flip the arrow of the cap from e. It follows that $V \subseteq B_n e$. That $B_n e = V$ as (B_{n-1}, B_{n-2})-bimodules now also follows since $D \in V$ can be written as De.

We can realize a map $\varphi : B_{n-1} \to B_n e$ by first including $D \in B_{n-1}$ into B_n in the usual way and then multiplying on the right by e_{n-1} since this gives a cap in the desired location. Diagrammatically, $\varphi(D)$ looks like

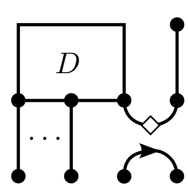


Figure 58

Visualizing φ in this way, we observe that the edges of $\varphi(D)$ other than $((n-1)', n')$ are uniquely determined by the edges of D. Hence, it is immediate that φ is a bijective map. It is also now easy to see that φ is a (B_{n-1}, B_{n-2})-bimodule homomorphism since if $D_1 \in B_{n-1}$ then

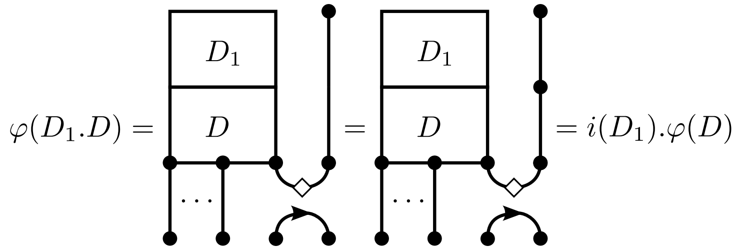


Figure 59

and if $D_2 \in B_{n-2}$ then

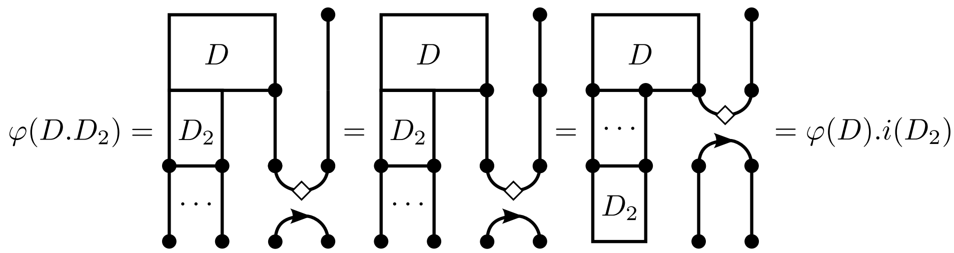


Figure 60

where $i : B_{n-r} \hookrightarrow B_n$ denotes the usual embedding which pads r vertical lines on the right of each diagram in B_{n-r}. Observe that these two action commute with each other, so the map is an isomorphism as bimodules. $\square$

Proposition 4.18. *If $|\lambda| \leq n$, then $ind_n^{n+1}\Delta_n(\lambda) \cong res_{n+1}^{n+2}\Delta_{n+2}(\lambda)$ as B_{n+1}-modules.*

Proof. By Lemmas 4.14 and 4.17 we have

$$\begin{aligned}
\mathrm{ind}_n^{n+1}\Delta_n(\lambda) &= B_{n+1} \otimes_{B_n} \Delta_n(\lambda) \\
&\cong \mathrm{res}_{n+1}^{n+2} B_{n+2}e \otimes_{B_n} \Delta_n(\lambda) \\
&= \mathrm{res}_{n+1}^{n+2} G_{n+2}(\Delta_n(\lambda)) \\
&\cong \mathrm{res}_{n+1}^{n+2} \Delta_{n+2}(\lambda)
\end{aligned}$$

as desired. $\square$

Corollary 4.19. *Let $n = m + 2k$ and $\lambda \vdash m$. We have the following short exact* *sequence of B_{n+1}-modules:*

$$0 \to \bigoplus_{\mu \lhd \lambda} \Delta_{n+1}(\mu) \to ind_n^{n+1} \Delta_n(\lambda) \to \bigoplus_{\nu \rhd \lambda} \Delta_{n+1}(\nu) \to 0$$

Proof. Theorem 4.2 gives the short exact sequence

$$0 \to \bigoplus_{\mu \lhd \lambda} \Delta_{n+1}(\mu) \to \mathrm{res}_{n+1}^{n+2} \Delta_{n+2}(\lambda) \to \bigoplus_{\nu \rhd \lambda} \Delta_{n+1}(\nu) \to 0$$

of B_{n+1}-modules. The result now follows since $\mathrm{res}_{n+1}^{n+2}\Delta_{n+2}(\lambda) \cong \mathrm{ind}_n^{n+1}\Delta_n(\lambda)$ by the previous proposition. □

The previous two results say we may view $\mathrm{ind}_n^{n+1}\Delta_n(\lambda)$ as the restriction of a B_{n+2}-module down to B_{n+1}. In particular, we can analyze the eigenspaces of the Jucys-Murphy element X_{n+2} acting on $\mathrm{ind}_n^{n+1}\Delta_n(\lambda)$, as in Section 4.2. From these results, we know X_{n+2} acts on each summand $\Delta_{n+1}(\gamma)$, with $\gamma \lhd \lambda$ or $\gamma \rhd \lambda$, by a single eigenvalue. By applying Corollaries 4.5 and 4.9 to the short exact sequence in the proof above, we obtain the following result.

Corollary 4.20. *Let $n = m + 2k$ and $\lambda \vdash m$. Suppose $\mu \lhd \lambda \lhd \nu$. Referring to the short exact sequence in the previous corollary, the eigenvalue of X_{n+2} on $\Delta_{n+1}(\mu)$ is the content of the box λ/μ, while the eigenvalue of X_{n+2} on $\Delta_{n+1}(\nu)$ is one plus the content of the box ν/λ.*

If $n > 0$ and $\gamma \in \Lambda(n-1)$, then we have $\Delta_{n+1}(\gamma) = G\Delta_{n-1}(\gamma)$. Also, when $|\lambda| < n$, both induction and restriction yield two-step filtrations by standard modules. So, with the same notation as in the corollary above, this result says that when $|\lambda| < n$, the eigenvalues of X_{n+2} acting on $\Delta_{n+1}(\mu)$ and $\Delta_{n+1}(\nu)$ coincide with those of X_n acting on $\Delta_{n-1}(\mu)$ and $\Delta_{n-1}(\nu)$, respectively. This can also be seen explicitly by observing that $X_{n+2}e = eX_n$ and using the definition of the functor G along with the fact that e is an idempotent.

4.5 Adjointness of Induction and Restriction

Proposition 4.21. *$\left(ind_{n-1}^{n}, res_{n-1}^{n}\right)$ is an adjoint pair of functors. That is, there is an isomorphism $Hom_{B_n}\left(ind_{n-1}^{n}M, N\right) \cong Hom_{B_{n-1}}\left(M, res_{n-1}^{n}N\right)$ which is natural in both variables.*

Proof. Let $\alpha : \mathrm{Hom}_{B_n}\left(\mathrm{ind}_{n-1}^{n}M, N\right) \to \mathrm{Hom}_{B_{n-1}}\left(M, \mathrm{res}_{n-1}^{n}N\right)$ by $\alpha(\varphi)(m) = \varphi(1\otimes m)$. Note that $\alpha(\varphi)$ is a B_{n-1}-module homomorphism: given $x \in B_{n-1}$, we have

$$\begin{aligned} x.\alpha(\varphi)(m) &= x.\varphi(1 \otimes m) \\ &= \varphi(x \otimes m) \\ &= \varphi(1 \otimes x.m) \\ &= \alpha(\varphi)(x.m) \end{aligned}$$

Let $\beta : \mathrm{Hom}_{B_{n-1}}\left(M, \mathrm{res}_{n-1}^{n}N\right) \to \mathrm{Hom}_{B_n}\left(\mathrm{ind}_{n-1}^{n}M, N\right)$ by $\beta(\psi)(D \otimes m) = D\psi(m)$. Note that $\beta(\psi)$ is well-defined since ψ is a B_{n-1}-module homomorphism and the tensor product is over B_{n-1}. The map $\beta(\psi)$ is a B_n-module homomorphism: given $x \in B_n$,

$$\begin{aligned} x.\beta(\psi)(D \otimes m) &= x.D\psi(m) \\ &= (xD)\psi(m) \\ &= \beta(\psi)(xD \otimes m) \\ &= \beta(\psi)(x.(D \otimes m)) \end{aligned}$$

These maps are mutual inverses:

$$\begin{aligned}(\alpha \circ \beta)(\psi)(m) &= \alpha\left(\beta(\psi)\right)(m) \\ &= \beta(\psi)(1 \otimes m) \\ &= 1 \cdot \psi(m) \\ &= \psi(m)\end{aligned}$$

so $\alpha \circ \beta$ is the identity map on $\mathrm{Hom}_{B_{n-1}}\left(M, \mathrm{res}_{n-1}^{n} N\right)$ and

$$\begin{aligned}(\beta \circ \alpha)(\varphi)(D \otimes m) &= \beta\left(\alpha(\varphi)\right)(D \otimes m) \\ &= D\alpha(\varphi)(m) \\ &= D\varphi(1 \otimes m) \\ &= \varphi(D \otimes m)\end{aligned}$$

so $\beta \circ \alpha$ is the identity map on $\mathrm{Hom}_{B_n}\left(\mathrm{ind}_{n-1}^{n} M, N\right)$. Hence, $\mathrm{Hom}_{B_n}\left(\mathrm{ind}_{n-1}^{n} M, N\right) \cong \mathrm{Hom}_{B_{n-1}}\left(M, \mathrm{res}_{n-1}^{n} N\right)$.

It remains to check that this isomorphism is natural in both variables; we will check the naturality of α. First, given a B_{n-1}-module homomorphism $f : M \to M'$, the diagram

$$\begin{tikzcd} \mathrm{Hom}_{B_n}\left(\mathrm{ind}_{n-1}^{n} M', N\right) \arrow[r, "\alpha"] \arrow[d, "(\mathbf{1}_{B_n} \otimes f)^*"'] & \mathrm{Hom}_{B_{n-1}}\left(M', \mathrm{res}_{n-1}^{n} N\right) \arrow[d, "f^*"] \\ \mathrm{Hom}_{B_n}\left(\mathrm{ind}_{n-1}^{n} M, N\right) \arrow[r, "\alpha"'] & \mathrm{Hom}_{B_{n-1}}\left(M, \mathrm{res}_{n-1}^{n} N\right) \end{tikzcd}$$

commutes since

$$\begin{aligned} (f^* \circ \alpha)(\varphi)(m) &= f^*(\alpha(\varphi))(m) \\ &= \alpha(\varphi)(f(m)) \\ &= \varphi(1 \otimes f(m)) \end{aligned}$$

while

$$\begin{aligned} (\alpha \circ ((\mathbf{1}_{B_n} \otimes f)^*))(\varphi)(m) &= \alpha((\mathbf{1}_{B_n} \otimes f)^*(\varphi))(m) \\ &= ((\mathbf{1}_{B_n} \otimes f)^*(\varphi))(1 \otimes m) \\ &= \varphi(1 \otimes f(m)) \end{aligned}$$

Similarly, given a B_n-module homomorphism $g : N \to N'$, the diagram

$$\begin{tikzcd} \operatorname{Hom}_{B_n}\left(\operatorname{ind}_{n-1}^n M, N\right) \arrow[r, "\alpha"] \arrow[d, "g_*"'] & \operatorname{Hom}_{B_{n-1}}\left(M, \operatorname{res}_{n-1}^n N\right) \arrow[d, "g_*"] \\ \operatorname{Hom}_{B_n}\left(\operatorname{ind}_{n-1}^n M, N'\right) \arrow[r, "\alpha"'] & \operatorname{Hom}_{B_{n-1}}\left(M, \operatorname{res}_{n-1}^n N'\right) \end{tikzcd}$$

commutes since

$$\begin{aligned} (g_* \circ \alpha)(\varphi)(m) &= g_*(\alpha(\varphi))(m) \\ &= g \circ (\alpha(\varphi))(m) \\ &= g \circ \varphi(1 \otimes m) \\ &= \alpha(g \circ \varphi)(m) \\ &= \alpha(g_*(\varphi))(m) \\ &= (\alpha \circ g_*)(\varphi)(m) \end{aligned}$$

Hence, the induction and restriction functors are adjoint. □

4.6 $i-\mathbf{ind}_{n-1}^{n}$ and $i-\mathbf{res}_{n-1}^{n}$

In this section, we introduce refinements of the induction and restriction functors defined in previous sections.

Consider the commutative subalgebra $A = \langle X_n \rangle$ of B_n generated by the Jucys-Murphy element X_n. Since B_n is finite-dimensional, this subalgebra can be thought of as a quotient of the polynomial ring in the variable X_n. We may decompose A into generalized eigenspaces, $A = \bigoplus_{i \in \mathbb{C}} A_i$, for the action of X_n on A. Since A is commutative and $X_n \in A$, this decomposition is as both left and right A-modules. Projection onto A_i, denoted $P_{i,n}$, is an idempotent in $\mathrm{Hom}_A(A, A)$. We have an isomorphism $\mathrm{Hom}_A(A, A) \cong A$ as algebras since the usual A-module isomorphism, $\varphi \mapsto \varphi(1)$, also respects the ring structure:

$$\varphi \circ \psi \mapsto \varphi \circ \psi(1) = \varphi(\psi(1)) = \varphi(1) \cdot \psi(1)$$

So, we may identify $P_{i,n}$ with an idempotent in A, which can then be written as a polynomial in the variable X_n. Note that $\{P_{i,n} : i \in \mathbb{C}\}$ is a collection of mutually orthogonal idempotents, since no nonzero vector can be a generalized eigenvector for two distinct generalized eigenvalues. Furthermore, the sum $\sum_{i \in \mathbb{C}} P_{i,n}$ is the identity map on A, so, since A is a unital subalgebra of B_n, we have $\sum_{i \in \mathbb{C}} P_{i,n} = 1$ in B_n.

Given a B_n-module M, we define $i - \mathrm{res}_{n-1}^{n} M$ to be $P_{i,n}M$. This space is a left B_{n-1}-module since $P_{i,n}$, being a polynomial in X_n, commutes with the subalgebra B_{n-1} of B_n. The decomposition $1 = \sum_{i \in \mathbb{C}} P_{i,n}$ then provides a decomposition $\mathrm{res}_{n-1}^{n} = \bigoplus_{i \in \mathbb{C}} i - \mathrm{res}_{n-1}^{n}$ of functors B_n-mod $\to B_{n-1}$-mod. Moreover, the space $P_{i,n}M$ is the generalized i-eigenspace for X_n acting on the left of M, so the generalized eigenspace decomposition $\bigoplus_{i \in \mathbb{C}} P_{i,n}M$ for X_n acting on M is then a decomposition as left B_{n-1}-modules. In particular, for standard modules, we obtain refined short exact sequences

$$0 \to \Delta_{n-1}(\lambda - \varepsilon) \to i - \mathrm{res}_{n-1}^{n} \Delta_n(\lambda) \to \Delta_{n-1}(\lambda + \varepsilon') \to 0$$

where ε is a removable box of λ with content i and ε' is an addable box of λ with content $i-1$.

Now, given a B_{n-1}-module N, we define $i - \mathrm{ind}_{n-1}^{n} N$ to be the left B_n-module $B_n P_{i,n} \otimes_{B_{n-1}} N$, so we again obtain a decomposition $\mathrm{ind}_{n-1}^{n} = \bigoplus_{i \in \mathbb{C}} i - \mathrm{ind}_{n-1}^{n}$ of functors B_{n-1}-mod $\to B_n$-mod. Unfortunately, we do not immediately obtain refined short exact sequences for the i-induction of standard modules. This is essentially due to the fact that our previous results used eigenvalues for X_{n+1} instead of X_n. However, we do find that i-induction and i-restriction are adjoint.

Proposition 4.22. $\left(i - ind_{n-1}^{n}, i - res_{n-1}^{n}\right)$ *is an adjoint pair of functors.*

Proof. Let M be a B_{n-1}-module and N a B_n-module. Define

$$\alpha : \mathrm{Hom}_{B_n}\left(i - \mathrm{ind}_{n-1}^{n} M, N\right) \to \mathrm{Hom}_{B_{n-1}}\left(M, i - \mathrm{res}_{n-1}^{n} N\right)$$

by $\alpha(\varphi)(m) = P_{i,n}\varphi(P_{i,n} \otimes m)$. As φ is a B_n-module homomorphism, we have

$$\begin{aligned} \alpha(\varphi)(m) &= \varphi(P_{i,n} \cdot P_{i,n} \otimes m) \\ &= \varphi(P_{i,n} \otimes m) \end{aligned}$$

This observation makes it clear that $\alpha(\varphi)$ is a B_{n-1}-module homomorphism.

Define

$$\beta : \mathrm{Hom}_{B_{n-1}}\left(M, i - \mathrm{res}_{n-1}^{n} N\right) \to \mathrm{Hom}_{B_n}\left(i - \mathrm{ind}_{n-1}^{n} M, N\right)$$

by $\beta(\psi)(DP_{i,n} \otimes m) = DP_{i,n}\psi(m)$. Since the image of ψ lies in $P_{i,n}N$, we have

$\psi = P_{i,n}\psi$, so that $\beta(\psi)(DP_{i,n} \otimes m) = D\psi(m)$. Note that β is a map of B_n-modules.

Checking that these two maps are mutual inverses is now simple:

$$\alpha \circ \beta(\psi)(m) = \beta(\psi)(P_{i,n} \otimes m) = \psi(m)$$

while

$$\begin{aligned} \beta \circ \alpha(\varphi)(DP_{i,n} \otimes m) &= D\alpha(\varphi)(m) \\ &= D\varphi(P_{i,n} \otimes m) \\ &= \varphi(DP_{i,n} \otimes m) \end{aligned}$$

Hence, the two functors are adjoint. □

In order to obtain a refined induction functor which behaves compatibly with eigenvalues for a certain Jucys-Murphy element, we might instead define

$$i - \underline{\mathrm{ind}}_{n-1}^{n} N = i - \mathrm{res}_{n}^{n+1} \circ G(N)$$

using our original definition for the induction functor. We again obtain a decomposition $\mathrm{ind}_{n-1}^{n} = \bigoplus_{i \in \mathbb{C}} i - \underline{\mathrm{ind}}_{n-1}^{n}$ as well as short exact sequences refined by eigenvalues for X_{n+1} (see Corollary 4.19):

$$0 \to \Delta_n(\lambda - \varepsilon) \to i - \underline{\mathrm{ind}}_{n-1}^{n} \Delta_n(\lambda) \to \Delta_n(\lambda + \varepsilon') \to 0$$

where ε is a removable box of λ with content i and ε' is an addable box of λ with content $i-1$. Unfortunately, $i - \underline{\mathrm{ind}}_{n-1}^{n}$ is not in general adjoint to $i - \mathrm{res}_{n-1}^{n}$. We will make use of the adjointness of $i - \mathrm{ind}_{n-1}^{n}$ and $i - \mathrm{res}_{n-1}^{n}$ in later work.

Chapter 5

Decomposition Multiplicities of Standard Modules

5.1 The Decomposition Matrix for $B_n(0, -1)$

By direct calculation using the computer algebra system MAXIMA to simplify the matrix calculations, we arrive at the following decompositions of standard modules for $B_n(0, -1)$, $n = 3, 4, 5$. Here, the notation $M = \begin{matrix} S \\ U \end{matrix}$ means U is a submodule of M and $M/U \cong S$.

$L_3\left(\ydiagram{1}\right)$ $\Delta_3\left(\ydiagram{1}\right) = L_3\left(\ydiagram{3}\right)$

$L_4\left(\ydiagram{2}\right)$ $\Delta_4\left(\ydiagram{2}\right) = L_4\left(\ydiagram{4}\right) \oplus L_4\left(\ydiagram{2,2}\right)$	$L_4\left(\ydiagram{1,1}\right)$ $\Delta_4\left(\ydiagram{1,1}\right) = L_4\left(\ydiagram{3,1}\right)$
$L_5\left(\ydiagram{1}\right)$ $L_5\left(\ydiagram{3}\right)$ $\Delta_5\left(\ydiagram{1}\right) = L_5\left(\ydiagram{3,2}\right)$	$L_5\left(\ydiagram{3}\right)$ $\Delta_5\left(\ydiagram{3}\right) = L_5\left(\ydiagram{5}\right) \oplus L_5\left(\ydiagram{3,2}\right)$
$L_5\left(\ydiagram{1,1,1}\right)$ $\Delta_5\left(\ydiagram{1,1,1}\right) = L_5\left(\ydiagram{3,1,1}\right)$	$L_5\left(\ydiagram{2,1}\right)$ $\Delta_5\left(\ydiagram{2,1}\right) = L_5\left(\ydiagram{4,1}\right)$

Table 4

Recall the *lexicographic order* on partitions of a fixed size given by $\lambda <_l \mu$ if, for some i, we have $\lambda_j = \mu_j$ for all $j < i$ and $\lambda_i < \mu_i$. This is a total order on each set Λ_m. For example, when $m = 6$, we have

$$\ydiagram{1,1,1,1,1,1} <_l \ydiagram{2,1,1,1,1} <_l \ydiagram{2,2,1,1} <_l \ydiagram{2,2,2} <_l \ydiagram{3,1,1,1} <_l \ydiagram{3,2,1} <_l \ydiagram{3,3} <_l \ydiagram{4,1,1} <_l \ydiagram{4,2} <_l \ydiagram{5,1} <_l \ydiagram{6}$$

Figure 61

We may define a total order on the set $\Lambda(n)$ by setting $\lambda > \mu$ if $|\lambda| < |\mu|$ or if $|\lambda| = |\mu|$ and $\lambda <_l \mu$. This total order agrees with our partial order on $\Lambda(n)$ for $n \leq 5$. For example, on $\Lambda(5)$, we have

$$\ydiagram{5} < \ydiagram{4,1} < \ydiagram{3,2} < \ydiagram{3,1,1} < \ydiagram{2,2,1} < \ydiagram{2,1,1,1} < \ydiagram{1,1,1,1,1} < \ydiagram{3} < \ydiagram{2,1} < \ydiagram{1,1,1} < \ydiagram{1}$$

Figure 62

Denote by $[\Delta_n(\lambda) : L_n(\mu)]$ the number of times the simple B_n-module $L_n(\mu)$

appears as a composition factor of the standard module $\Delta_n(\lambda)$ and let

$$\begin{aligned} D_n : &= (D_{\lambda,\mu})_{\lambda,\mu\in\Lambda(n)} \\ &= ([\Delta_n(\lambda) : L_n(\mu)])_{\lambda,\mu\in\Lambda(n)} \end{aligned}$$

be the decomposition matrix for B_n, whose rows and columns are ordered according to the total order on $\Lambda(n)$ described above. In terms of the decomposition matrix, the decomposition data above tell us, for instance,

$$D_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

Observe that the decomposition matrix for B_3 sits inside the matrix D_5 as the submatrix formed from the last four rows and columns:

$$D_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

The lemma below shows that this observation holds in general.

Lemma 5.1. *Suppose $\lambda, \mu \in \Lambda$ with $|\lambda|, |\mu| < n$. Then,*

$$[\Delta_n(\lambda) : L_n(\mu)] = [\Delta_{n-2}(\lambda) : L_{n-2}(\mu)]$$

Proof. Let $0 = M_0 \lneq M_1 \lneq \cdots \lneq M_t = \Delta_n(\lambda)$ be a composition series for $\Delta_n(\lambda)$. By Lemma 4.12, $0 = eM_0 \leq eM_1 \leq \cdots \leq eM_t = \Delta_{n-2}(\lambda)$ is a chain of submodules (although not necessarily a composition series) of $\Delta_{n-2}(\lambda)$. Suppose $L_n(\mu) \cong M_{r+1}/M_r$ for some r. Then by Lemma 4.13, $L_{n-2}(\mu) \cong F_n(M_{r+1}/M_r) \cong eM_{r+1}/eM_r$, so any refinement of the chain of submodules above for $\Delta_{n-2}(\lambda)$ will include the step $eM_r \leq eM_{r+1}$. That is, $L_{n-2}(\mu)$ is a composition factor of $\Delta_{n-2}(\lambda)$. Since this argument applies to each instance of $L_n(\mu)$ as a composition factor of $\Delta_n(\lambda)$ without change, the equality of decomposition numbers now follows. $\square$

Using the results obtained in previous sections, we can now prove that the decomposition matrix for the marked Brauer algebra is a block matrix, three of whose blocks are easily described. We will provide a roadmap to determine the contents of the third block later in Section 5.4.

Theorem 5.2. *The decomposition matrix D_n has the following block form*

$$\begin{pmatrix} I_{p(n)} & 0 \\ * & D_{n-2} \end{pmatrix}$$

where $p(n)$ is the number of partitions of n and D_{n-2} is the decomposition matrix for B_{n-2}. Moreover, this matrix is lower-triangular and has all diagonal entries equal to 1.

Proof. If $|\lambda| = |\mu| = n$, then the standard modules are simple and are isomorphic to the Specht modules for S_n. Hence, $[\Delta_n(\lambda) : L_n(\mu)] = \left[S^\lambda : S^\mu\right] = \delta_{\lambda\mu}$, so the submatrix of D_n whose rows and columns are labeled by partitions of n is the identity matrix of size $p(n)$, the number of partitions of size n. By Theorem 3.2(4), $[\Delta_n(\lambda) : L_n(\lambda)] = 1$, so $D_{\lambda\lambda} = 1$ for all $\lambda \in \Lambda(n)$. Also, by Theorem 3.2(5), $[\Delta_n(\lambda) : L_n(\mu)] = 0$ when $\mu > \lambda$, so D_n is lower-triangular. Finally, if $|\lambda|, |\mu| < n$, then Lemma 5.1 shows that $D_{\lambda\mu} = [\Delta_{n-2}(\lambda) : L_{n-2}(\mu)]$, so the entries of this submatrix come from the decomposition matrix for B_{n-2}. $\square$

It remains to compute the decomposition numbers which appear in the lower left block. In the following sections, we introduce tools which we will use to tackle this problem.

5.2 Weights

We follow Shalile's approach in [40] for the ordinary Brauer algebra for definitions and general inspiration for this section. Recall from Theorem 4.2 and its corollaries that we have a short exact sequence

$$0 \to \bigoplus_{\mu \lhd \lambda} \Delta_{n-1}(\mu) \to \operatorname{res}^n_{n-1} \Delta_n(\lambda) \to \bigoplus_{\nu \lhd \lambda} \Delta_{n-1}(\nu) \to 0$$

of B_{n-1}-modules where the Jucys-Murphy element X_n acts on each standard module $\Delta_{n-1}(\gamma)$ appearing in this exact sequence by

$$\begin{cases} c(\lambda/\gamma) & \text{if } \gamma \lhd \lambda \\ c(\gamma/\lambda) + 1 & \text{if } \gamma \rhd \lambda \end{cases}$$

where $c(\varepsilon)$ denotes the content of the box ε. Since there is only one addable or removable box of a given content in λ, each of these direct sums is multiplicity-free. Also, we see that the spectrum of X_n is entirely determined by the boxes which can be added to or removed from λ. Now, we can take each $\Delta_{n-1}(\gamma)$ which appears in this exact sequence and restrict it to B_{n-2} to obtain a new short exact sequence whose standard modules are labeled by partitions obtained from γ by adding or removing a box and on which X_{n-1} acts by eigenvalues determined by the contents of these boxes. We can continue this process until we restrict to B_1 in the final step. Thus, we can associate to $\Delta_n(\lambda)$ a Bratteli diagram whose vertices at level $i = 0, 1, \ldots, n-1$ are labeled by the partitions which label standard modules appearing in the short exact sequence for the restriction of $\Delta_n(\lambda)$ to B_{n-i} and there is an edge connecting α in level i to β in level $i+1$ if $\Delta_{n-i-1}(\beta)$ appears in the short exact sequence for the restriction of $\Delta_{n-i}(\alpha)$ to B_{n-i-1}. We label such an edge by the eigenvalue for X_{n-i} acting on $\Delta_{n-i-1}(\beta)$. For example, the Bratteli diagram associated to $\Delta_4(\square\square)$ appears below.

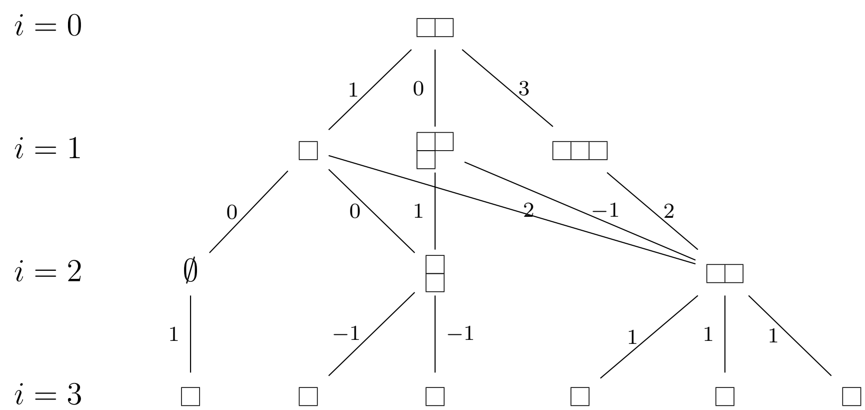


Figure 63

We call a path from the bottom of this diagram to the top an *up-down tableaux*. That is, an up-down tableaux t of length n for a partition $\lambda \vdash m$, where $n = m + 2k$, is a sequence of partitions $t = (t_1, \ldots, t_n) \in \Lambda(n)^n$ beginning with $t_1 = \square$ and ending with $t_n = \lambda$ such that t_i and t_{i+1} differ by precisely one box for all $i = 1, \ldots, n-1$. We will say that t_{i+1} is obtained from t_i by adding or removing a box at step i of t. Denote by $UD_n(\lambda)$ the set of all up-down tableaux of length n for λ. For example, the path second from left in the Bratteli diagram above corresponds to the up-down tableaux $(\square, \square\square, \square\square\square, \square\square)$.

Proposition 5.3. *For each $\lambda \in \Lambda(n)$, the standard module $\Delta_n(\lambda)$ has a basis labeled by the elements of $UD_n(\lambda)$. That is,* $\dim \Delta_n(\lambda) = |UD_n(\lambda)|$.

Proof. By induction on n, the base case $n = 0$ being trivial since $\Delta_0(\emptyset)$ is one-dimensional and $UD_0(\emptyset)$ consists of one element. When $n > 0$, we have by Theorem 4.2 the short exact sequence

$$0 \to \bigoplus_{\mu \lhd \lambda} \Delta_{n-1}(\mu) \to \operatorname{res}^n_{n-1} \Delta_n(\lambda) \to \bigoplus_{\nu \rhd \lambda} \Delta_{n-1}(\nu) \to 0$$

and, by induction, we have bases for the standard modules $\Delta_{n-1}(\mu)$ and $\Delta_{n-1}(\nu)$ labeled by $UD_{n-1}(\mu)$ and $UD_{n-1}(\nu)$, respectively. We then obtain a basis for $\Delta_n(\lambda)$ which is labeled by $\bigcup\limits_{\mu \lhd \lambda \lhd \nu} (UD_{n-1}(\mu) \cup UD_{n-1}(\nu))$ by taking the union of all the bases for $\Delta_{n-1}(\mu)$ along with preimages (under the quotient map) of all the bases for $\Delta_{n-2}(\nu)$ for all possible μ and ν with $\mu \lhd \lambda \lhd \nu$. But since $\mu \lhd \lambda \lhd \nu$, each of the up-down tableaux of length $n-1$ in $UD_{n-1}(\mu) \cup UD_{n-1}(\nu)$ can be completed to an up-down tableaux of length n for λ by adding or removing the appropriate box from μ or ν, respectively. Moreover, every such up-down tableaux of length n for λ can be obtained in this way since a box must either be added to or removed from λ in the last step. Hence, the basis for $\Delta_n(\lambda)$ is in fact labeled by $UD_n(\lambda)$. $\square$

We define the *weight* of an up-down tableaux $t \in UD_n(\lambda)$ to be

$$\operatorname{wt}(t) = (0, X_2(t_1), \ldots, X_n(t_{n-1})) \in \mathbb{Z}^n$$

where $X_{i+1}(t_i)$ is defined to be the eigenvalue of X_{i+1} on $\Delta_i(t_i)$ in the restriction of $\Delta_n(\lambda)$ to B_i for all $i = 1, \ldots, n-1$. That is,

$$X_{i+1}(t_i) := \begin{cases} c\left(t_{i+1}/t_i\right) & \text{if } t_i \lhd t_{i+1} \\ c\left(t_i/t_{i+1}\right) & \text{if } t_i \rhd t_{i+1} \end{cases}$$

when $i = 1, \ldots, n-1$. Notice the first entry is always 0 since $X_1 = 0$. For example, the weight of the up-down tableau given above is $(0, 1, 2, 3)$. We will say $\lambda, \mu \in \Lambda(n)$ have a *common weight of length* n if there exist $t \in UD_n(\lambda)$ and $u \in UD_n(\mu)$ with $\mathrm{wt}(t) = \mathrm{wt}(u)$. For example, $(\square, \square\square, \square\square\square, \square\square\square\square)$ has the same weight, $(0, 1, 2, 3)$, as the up-down tableaux considered previously.

It will often be useful to speak about both an up-down tableaux and its weight at the same time. To this end, we will often depict an up-down tableaux as a graph with one row of vertices labeled by the tableaux $t_1, \ldots, t_n$ with edges connecting t_i to t_{i+1} labeled by the weights $X_{i+1}(t_i)$ for all $i = 1, \ldots, n-1$. We will also call such graphs the up-down tableaux of length n for λ. For example, the first up-down tableaux given above can also be written as $(\square \stackrel{1}{-} \square\square \stackrel{2}{-} \square\square\square \stackrel{3}{-} \square\square)$.

Lemma 5.4. *If $\mu \vdash n$ and $\lambda \in \Lambda(n)$ have a common weight of length n, then $\lambda \subseteq \mu$.*

Proof. Let $t \in UD_n(\mu)$ and $u \in UD_n(\lambda)$ with $\mathrm{wt}(t) = \mathrm{wt}(u)$. Note that since $|\mu| = n$, a box must be added at each step of t. Proceed by induction on n to show $\lambda = u_n \subseteq t_n = \mu$. When $n = 1$, this is trivial since $\lambda = u_1 = t_1 = \mu$. If $n > 1$, then consider the sub-up-down tableaux $t' = (t_1, \ldots t_{n-1})$ and $u' = (u_1, \ldots, u_{n-1})$ of t and u, respectively. Then, $\mathrm{wt}(t') = \mathrm{wt}(u')$, so $u_{n-1} \subseteq t_{n-1}$ by induction. Now, a box ε of content c is added to t_{n-1} to form t_n, while a box of content c is added to u_{n-1} or a box of content $c-1$ is removed from u_{n-1} to form u_n. If a box is removed from u_{n-1}, then $u_n \subseteq u_{n-1} \subseteq t_{n-1} \subseteq t_n$. So, suppose a box ε' of content c is added to u_{n-1} to form u_n. Note that since $t_n = t_{n-1} + \varepsilon$, the space immediately below ε must be empty in t_{n-1} and hence in u_{n-1}. Since $u_{n-1} \subseteq t_{n-1}$, the box ε' must either already lie in t_{n-1}, in which case we would have $u_n \subseteq t_n$, or be addable to t_{n-1}, so that $\varepsilon' = \varepsilon$ since there is only one addable box to t_{n-1} of content c. Hence, $\lambda = u_n \subseteq t_n = \mu$ in either case. $\square$

Lemma 5.5. *If $\lambda, \mu \vdash n$ have a common weight of length n, then $\lambda = \mu$.*

Proof. Note that boxes can only be added in each step of the up-down tableaux t and u since $\lambda, \mu \vdash n$. Suppose $\lambda \neq \mu$. Then, for any pair of up-down tableaux $t \in UD_n(\lambda)$, $u \in UD_n(\mu)$ we can find a smallest integer k such that $t_k \neq u_k$. Then, the box added to $t_{k-1} = u_{k-1}$ to form t_k must be different from the box added to form u_k. Since each addable box to a partition has a unique content, the k^{th} entry of $\text{wt}(t)$ differs from that of $\text{wt}(u)$. $\square$

5.3 Weight Spaces

We wish to relate the weights of $\lambda \in \Lambda(n)$, which come from the eigenvalues of X_i on $\Delta_i(\gamma)$ where γ is obtained from λ by $n-i$ steps of adding/removing boxes, to the eigenvalues of X_i acting on $\Delta_n(\lambda)$ for $i = 1, \ldots, n$.

Proposition 5.6. *For each $\lambda \in \Lambda(n)$, $\Delta_n(\lambda)$ has a generalized eigenspace decomposition $\Delta_n(\lambda) = \bigoplus_{w \in \mathbb{C}^n} \Delta_n(\lambda)_w$, where for each $w = (w_1, \ldots w_n) \in \mathbb{C}^n$*

$$\Delta_n(\lambda)_w = \{v \in \Delta_n(\lambda) : \exists r \in \mathbb{Z}_{>0} \text{ such that } (X_i - w_i)^r v = 0, \forall i = 1, \ldots, n\}$$

and only finitely many $\Delta_n(\lambda)_w$ are nonzero.

Proof. We can find such a generalized eigenspace decomposition

$$\Delta_n(\lambda) = \bigoplus_{j=1}^{r_n} \Delta_n(\lambda)_{\alpha_{n,j}}$$

using X_n, where $\alpha_{n,1}, \ldots, \alpha_{n,r_n}$ are the distinct eigenvalues of X_n acting on $\Delta_n(\lambda)$. Each X_i then preserves each generalized eigenspace $\Delta_n(\lambda)_{\alpha_{n,j}}$ since X_i and X_n commute. For each fixed j, we may then find a generalized eigenspace decomposition $\Delta_n(\lambda)_{\alpha_{n,j}} = \bigoplus_{i=1}^{r_{n-1}} \left(\Delta_n(\lambda)_{\alpha_{n,j}}\right)_{\alpha^j_{n-1,i}}$ using X_{n-1}, where $\alpha^j_{n-1,1}, \ldots, \alpha^j_{n-1,r_{n-1}}$ are the distinct eigenvalues of X_{n-1} acting on $\Delta_n(\lambda)_{\alpha_{n,j}}$. Continue in this way to refine the decomposition so that each summand is a simultaneous generalized eigenspace for all the X_i. $\square$

Lemma 5.7. *Let $H = \langle X_1, \ldots, X_n \rangle$ be the subalgebra of B_n generated by its Jucys-Murphy elements. Given $\lambda \in \Lambda$, let $M := \bigoplus_{\mu \lhd \lambda} \Delta_{n-1}(\mu)$ denote the B_{n-1}-submodule of $V := res^n_{n-1}\Delta_n(\lambda)$ guaranteed by Theorem 4.2. Then,*

1. *M is an H-submodule of V.*

2. *There are generalized eigenspace decompositions for $V = \bigoplus_{w \in \mathbb{C}^n} V_w$ and $M = \bigoplus_{w \in \mathbb{C}^n} M_w$ for the action of $X_1, \ldots, X_n$ such that $V/M \cong \bigoplus_{w \in \mathbb{C}^n} V_w/M_w$ as an H-module.*

3. *If $v \in V/M$ is a common generalized eigenvector for $X_1, \ldots, X_n$, then $V_w \neq 0$. That is, V contains a common generalized eigenvector for $X_1, \ldots, X_n$.*

Proof. The first statement is immediate since M is a B_{n-1}-module and X_n acts on each summand of M by a single eigenvalue according to Theorem 4.2. The second statement is proven in complete generality in Lemma 2.1 of [40], so the proof applies to our setting as well.

For the last statement, let $w = (\lambda_1, \ldots, \lambda_n)$. Under the isomorphism in the second statement, the image of v is a nonzero vector in V_w/M_w, so, V_w is nontrivial. $\square$

Lemma 5.8. *Let $w = (w_1, \ldots w_n) \in \mathbb{C}^n$. Then,* $\dim \Delta_n(\lambda)_w$ *is precisely the number of up-down tableau of length n for λ with weight w. In particular, this means $\Delta_n(\lambda)_w \neq 0$ if and only if there is an up-down tableau of length n for λ of weight w.*

Proof. By induction on n. When $n = 1$, we have exactly one up-down tableau of length one, $(\square)$, with weight (0). The standard module $\Delta_1(\square)$ is one-dimensional and is the 0-eigenspace for X_1. So, the claim is true in this case.

Suppose $n > 1$. We have a short exact sequence of B_{n-1}-modules,

$$0 \to \Delta_{n-1}(\lambda - \varepsilon) \to w_n - \mathrm{res}_{n-1}^n \Delta_n(\lambda) \to \Delta_{n-1}(\lambda + \varepsilon') \to 0$$

where ε is a removable box of λ with content w_n and ε' is an addable box of λ with content $w_n - 1$. By induction, $\dim \Delta_{n-1}(\lambda - \varepsilon)_{(w_1,\ldots,w_{n-1})}$ and $\dim \Delta_{n-1}(\lambda + \varepsilon')_{(w_1,\ldots,w_{n-1})}$ are precisely the number of up-down tableau of length $n-1$ for $\lambda - \varepsilon$ and $\lambda + \varepsilon'$, respectively, with weight $(w_1, \ldots, w_{n-1})$. Each of these up-down tableau can be completed to an up-down tableau of length n for λ with weight w by adding the box ε or removing the box ε'. Since

$$\dim \Delta_n(\lambda)_w = \dim \Delta_{n-1}(\lambda - \varepsilon)_{(w_1,\ldots,w_{n-1})} + \dim \Delta_{n-1}(\lambda + \varepsilon')_{(w_1,\ldots,w_{n-1})}$$

the claim now follows. □

We will call a common generalized eigenvector for $X_1, \ldots X_n$ a *weight vector* and the corresponding tuple $(\lambda_1, \ldots, \lambda_n)$ of eigenvalues for $X_1, \ldots, X_n$ a *weight*.

Proposition 5.9. *If $[\Delta_n(\lambda) : L_n(\mu)] \neq 0$, then there are weight vectors $v_1 \in \Delta_n(\lambda)$ and $v_2 \in \Delta_n(\mu)$ with the same weight.*

Proof. Since $[\Delta_n(\lambda) : L_n(\mu)] \neq 0$, in a composition series $0 = M_0 \subset M_1 \subset \cdots \subset M_r = \Delta_n(\lambda)$ for $\Delta_n(\lambda)$ we have that $M_s/M_{s-1} \cong L_n(\mu)$ for some $1 \leq s \leq r$. So, we may obtain a nonzero B_n-module homomorphism $\varphi : \Delta_n(\mu) \to \Delta_n(\lambda)/M_{s-1}$ via the composition

$$\Delta_n(\mu) \twoheadrightarrow L_n(\mu) \xrightarrow{\cong} M_s/M_{s-1} \hookrightarrow \Delta_n(\lambda)/M_{s-1}$$

Given a weight vector $v_2 \in \Delta_n(\mu)$ of weight $w \in \mathbb{C}^n$, the vector $\varphi(v_2)$ must then be a weight vector in $\Delta_n(\lambda)/M_{s-1}$ of weight w since φ is a B_n-module homomorphism. Then, $\Delta_n(\lambda)/M_{s-1}$ contains a weight vector of weight w, so by Lemma 5.7(3), $\Delta_n(\lambda)$ must contain a weight vector v_1 which is also of weight w. □

This shows that λ and μ have a common weight when $[\Delta_n(\lambda) : L_n(\mu)] \neq 0$. We wish to prove that the opposite is also true, however, this is a more challenging task. The combinatorics introduced in the next section will be used to tackle this problem.

5.4 Arc Diagrams

Associate to each partition $\lambda = (\lambda_1, \lambda_2, \dots) \in \Lambda(n)$ a *weight diagram* $d(\lambda)$ which consists of a number line with black dots at each number in

$$\mathbf{c}(\lambda) := \{\lambda_i - i : i \in \mathbb{Z}_{>0}\}$$

So, if λ has t parts then

$$\mathbf{c}(\lambda) = \mathbb{Z}_{<-t} \cup \{\lambda_1 - 1, \dots, \lambda_t - t\}$$

That is, the weight diagram has a black dot above the content of the rightmost box in each row of λ. For example,

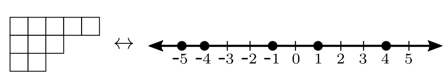
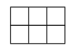


Figure 64

This correspondence is one-to-one: given a weight diagram with dots at $c_1 > c_2 > \cdots > c_t$, this weight diagram describes the partition $(c_1+1, c_2+2, \dots, c_t+t, 0, \dots)$. Note that adding a box to λ corresponds to moving a dot to the right one space, provided the space next to the dot is blank. Similarly, removing a box from λ corresponds to moving a dot to the left one space, provided this space is blank.

Inspired by similar constructions for the type $\mathfrak{p}$ Lie superalgebra found in [1] and conversations with Jonathan Comes, we create the *arc diagram* $\overline{d(\lambda)}$ for λ, starting with the weight diagram, as follows. Let $f_\lambda : \mathbb{Z} \to \{0,1\}$ be the indicator function for the set $\mathbb{Z}\backslash\mathbf{c}(\lambda)$, so $f_\lambda(i)$ is 0 when $d(\lambda)$ has a dot at i and is 1 when $d(\lambda)$ has no dot at i. Let $g_\lambda : \mathbb{Z} \to \{\pm 1\}$ be given by $g_\lambda(i) = (-1)^{f_\lambda(i)+1}$. For $i < j$, let $r_\lambda(i,j) = \sum_{a=i+1}^{j} g_\lambda(a)$. Note that $r_\lambda(i,i) = 0$ for all i and for $i < j < k$, we have $r_\lambda(i,k) = r_\lambda(i,j) + r_\lambda(j,k)$. Also, for $i > \lambda_1 - 1$, we have $r_\lambda(i,j) > 0$ for all $j > i$. For $i \notin \mathbf{c}(\lambda)$, let $R_\lambda(i) = \{j > i : r_\lambda(i,j) = 0 \text{ and } r_\lambda(1,a) \geq 0, \forall i < a < j\}$. For each $i \notin \mathbf{c}(\lambda)$, draw an arc in $d(\lambda)$ connecting i to $j \in R_\lambda(i)$. The resulting diagram is the arc diagram $\overline{d(\lambda)}$.

For example, the weight diagram for $\lambda =$  is

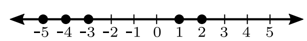


Figure 65

so we have $g_\lambda(i) = 1$ when $i \in \{-2,-1,0,3,4,\dots\}$, while $g_\lambda(i) = -1$ when $i \in \{\dots,-4,-3,1,2\}$. Observe that $r_\lambda(i,j) > 0$ for all $3 \leq i < j$, so we can have no arcs which start to right of 2. So, we need only consider when $i \in \{-2,-1,0\}$. The table below summarizes the values of $r_\lambda(i,j)$.

j	$r_\lambda(-2,j)$	$r_\lambda(-1,j)$	$r_\lambda(0,j)$
-1	1	0	-
0	2	1	0
1	1	0	-1
2	0	-1	-2
> 2	$j-2$	$j-3$	$j-4$

Table 5

Based on these values, we conclude $R_\lambda(-2) = \{2\}$, $R_\lambda(-1) = \{1\}$, and $R_\lambda(i) = \emptyset$ otherwise. Hence, the arc diagram for λ is

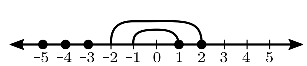


Figure 66

Lemma 5.10. *For $i \notin \mathbf{c}(\lambda)$ and $j \in R_\lambda(i)$, we have $j \in \mathbf{c}(\lambda)$. That is, $R_\lambda(i) \subseteq \mathbf{c}(\lambda)$.*

Proof. Suppose $j \in R_\lambda(i)$ but $j \notin \mathbf{c}(\lambda)$. Then, $g_\lambda(j) = 1$, so

$$0 = r_\lambda(i, j) = \sum_{a=i+1}^{j-1} g_\lambda(a) + g_\lambda(j) \geq 0 + 1 = 1$$

which is a contradiction. $\square$

Lemma 5.11. *For any $a < b \notin \mathbf{c}(\lambda)$, $c \in R_\lambda(a)$, and $d \in R_\lambda(b)$, we have either $b > c$ or $c > d$. That is, two arcs can only intersect at a common source which has no dot.*

Proof. Suppose $b < c$ and $c \leq d$, so $d(\lambda)$ is as pictured below.

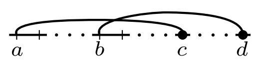


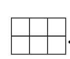
Figure 67

Since $c \in R_\lambda(a)$ and $a \leq b-1 < c$, we have $r_\lambda(a, b-1) \geq 0$. Also, since $d \in R_\lambda(b)$ and $b < c < d$, we have $r_\lambda(b, c) \geq 0$. So,

$$0 = r_\lambda(a, c) = r_\lambda(a, b-1) + g_\lambda(b) + r_\lambda(b, d) \geq 0 + 1 + 0 = 1$$

which is a contradiction. $\square$

For $\mu \in \Lambda(n)$, let $s_\mu : \Lambda(n) \times (\mathbb{Z} \backslash \mathbf{c}(\mu)) \to \mathbb{Z}$ by $s_\mu(\lambda, i) = f_\lambda(i) + \sum_{j \in R_\mu(i)} f_\lambda(j)$. This function counts the number of blank spots in $d(\lambda)$ at the labels $\{i\} \cup R_\mu(i)$, for each $i \notin \mathbf{c}(\mu)$. Let $R(\mu) = \{\lambda \in \Lambda(n) : s_\mu(\lambda, i) = 1, \forall i \notin \mathbf{c}(\mu)\}$. Note $R(\mu) \neq \emptyset$ since $\mu \in R(\mu)$.

For example, consider when $\mu =$ . According to our work above, $s_\mu(\lambda, i) = f_\lambda(i) + f_\lambda(-i)$ if $i \in \{-2, -1\}$ and $s_\mu(\lambda, i) = f_\lambda(i)$ otherwise. The table below summarizes the values of $s_\mu(\lambda, i)$ for $\lambda \in \Lambda(4)$ and $i \notin \mathbf{c}(\mu)$.

i	$\emptyset$	$\ydiagram{2}$	$\ydiagram{1,1}$	$\ydiagram{4}$	$\ydiagram{3,1}$	$\ydiagram{2,2}$	$\ydiagram{2,1,1}$	$\ydiagram{1,1,1,1}$
-2	$0+1$	$0+1$	$1+1$	$0+1$	$1+0$	$1+1$	$0+1$	$0+1$
-1	$0+1$	$1+0$	$0+1$	$1+1$	$0+1$	$1+0$	$0+0$	$0+1$
0	1	1	0	1	1	0	1	0
3	1	1	1	0	1	1	1	1
>3	1	1	1	1	1	1	1	1

Table 6

We thus conclude $R(\mu) \supseteq \{\emptyset, \ydiagram{2}, \ydiagram{3,1}, \ydiagram{3,3}\}$. In fact, our results below will reveal this to be an equality of sets. In terms of arc diagrams, we observe that the weight diagrams for $\lambda \in R(\mu)\backslash\{\mu\}$ can be obtained from $\overline{d(\mu)}$ by moving one or more dots in $d(\mu)$ to the left along arcs in $\overline{d(\mu)}$. Indeed, with the arcs in $d(\mu)$ numbered as below,

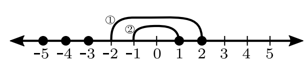


Figure 68

we find three possibilities for λ by collapsing only arc ①, only arc ②, and both arcs ① and ②, respectively:

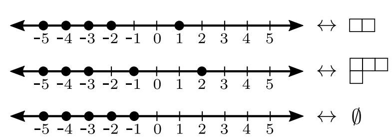


Figure 69

We say an arc in $\overline{d(\mu)}$ has been *collapsed* when the dot connected to it has been moved to the left along its arc. In the event that $i \notin \mathbf{c}(\mu)$ has multiple arcs originating from it, only one arc may be collapsed at a time. Since moving dots to the left in weight diagrams corresponds to removing boxes, it follows that collapsing arcs results in a partition of smaller size. In particular, collapsing an arc reduces the size of a partition by an even number of boxes. We call λ a *k-retract* of μ if

$d(\lambda)$ is obtained from $d(\mu)$ by collapsing one or more arcs in $\overline{d(\mu)}$ such that the total distance traveled by all the dots is $|\mu| - |\lambda| = 2k$.

Lemma 5.12. *$\lambda \in R(\mu)$ if and only if λ is a k-retract of μ.*

Proof. First suppose $\lambda \in R(\mu)$, so that $s_\mu(\lambda, i) = 1$ for all $i \notin \mathbf{c}(\mu)$. If $\lambda = \mu$, then $d(\lambda)$ is obtained from $\overline{d(\mu)}$ by collapsing no arcs, and hence λ is a 0-retract of μ. So, suppose $\lambda \neq \mu$. Then, $\mathbf{c}(\lambda) \neq \mathbf{c}(\mu)$, so let $\mathbf{c}(\lambda)\backslash\mathbf{c}(\mu) = \{a_1, \ldots, a_t\}$; these are the labels for dots in $d(\lambda)$ which resulted from collapsing arcs in $\overline{d(\mu)}$. Let $1 \leq i \leq t$. We have $a_i \notin \mathbf{c}(\mu)$ and hence $s_\mu(\lambda, a_i) = 1$. But, since $a_i \in \mathbf{c}(\lambda)$, we have $f_\lambda(a_i) = 0$, so it follows that there is a unique $b_i \in R_\mu(a_i)$ with $f_\lambda(b_i) = 1$. Hence, $\overline{d(\mu)}$ has an arc connecting a_i and b_i which was collapsed when forming $d(\lambda)$. Since $d(\lambda)$ and $d(\mu)$ have dots at the same labels except for $\{a_1, b_1, \ldots a_t, b_t\}$, we have $|\mu| - |\lambda| = \sum_{i=1}^{t}(b_i - a_i)$, which is an even number by the observation above. Hence, λ is a k-retract of μ, where $k = \frac{1}{2}\left(|\mu| - |\lambda|\right)$.

Now, suppose λ is a k-retract of μ. Then, there are dots at $b_1, \ldots b_t$ in $d(\mu)$ which are moved along arcs in $\overline{d(\mu)}$ to new positions $a_1, \ldots, a_t$, respectively, in $d(\lambda)$. We must show $s_\mu(\lambda, i) = 1$ for all $i \notin \mathbf{c}(\mu)$. First, let $1 \leq i \leq t$. Note $a_i \notin \mathbf{c}(\mu)$. We have $f_\lambda(a_i) = 0$ since $d(\lambda)$ has a dot at a_i. Since $d(\lambda)$ has a dot at $j \in R_\mu(i)\backslash\{b_i\}$ and a blank space at b_i, it follows that $f_\lambda(j) = 0$ for $j \in R_\mu(i)\backslash\{b_i\}$ and $f_\lambda(b_i) = 1$. Hence, $s_\mu(\lambda, a_i) = 1$. Now, take $i \notin \mathbf{c}(\mu)$ and $i \notin \{a_1, \ldots, a_t\}$. Then, we have $i \notin \mathbf{c}(\lambda)$ as well, so $f_\lambda(i) = 1$. Moreover, both $d(\lambda)$ and $d(\mu)$ have dots at $j \in R_\mu(i)$, so $f_\lambda(j) = 0$ for all $j \in R_\mu(i)$. Hence, $s_\mu(\lambda, i) = 1$ for all $i \notin \mathbf{c}(\mu)$, so $\lambda \in R(\mu)$. $\square$

Corollary 5.13. *If $\lambda \in R(\mu)$, then $\lambda \subseteq \mu$. Moreover, if $|\lambda| = |\mu|$, then $\lambda = \mu$.*

Proof. Since collapsing arcs corresponds to removing boxes, it follows that $\lambda \subseteq \mu$ when $\lambda \in R(\mu)$. In particular, if $\lambda \in R(\mu)$ with $|\lambda| = |\mu|$, then $d(\lambda)$ must be obtained from $\overline{d(\mu)}$ by collapsing no arcs, and so $\lambda = \mu$. $\square$

We wish to use arc diagrams to determine precisely when $[\Delta_n(\lambda) : L_n(\mu)] \neq 0$ for $\mu \vdash n$ and $\lambda \in \Lambda(n)$. We begin with an easy lemma.

Lemma 5.14. *Suppose $\mu \vdash n$ has a removable box ε with $c(\varepsilon) = c$. Then,*

$$g_{\mu-\varepsilon}(a) = \begin{cases} g_\mu(a) & \text{if } a \neq c-1, c \\ -1 & \text{if } a = c-1 \\ 1 & \text{if } a = c \end{cases}$$

Hence, if $i < j < c-1$ or $i < c-1 < j$, we have $r_{\mu-\varepsilon}(i,j) = r_\mu(i,j)$. Moreover, if $j > c$, we have $r_{\mu-\varepsilon}(c-2,j) = r_{\mu-\varepsilon}(c,j) = r_\mu(c,j) = r_\mu(c-2,j)$.

Proof. The claimed values for $g_{\mu-\varepsilon}$ follow immediately since $d(\mu)$ and $d(\mu-\varepsilon)$ differ only at $c-1$ and c, with $d(\mu-\varepsilon)$ having a dot at $c-1$ and a blank node at c. If $i < j < c-1$, we have

$$\begin{aligned} r_{\mu-\varepsilon}(i,j) &= \sum_{a=i+1}^{j} g_{\mu-\varepsilon}(a) \\ &= \sum_{a=i+1}^{j} g_\mu(a) \\ &= r_\mu(i,j) \end{aligned}$$

while if $i < c-1 < j$, we have

$$\begin{aligned} r_{\mu-\varepsilon}(i,j) &= \sum_{a=i+1}^{j} g_{\mu-\varepsilon}(a) \\ &= \sum_{a=i+1}^{c-2} g_{\mu-\varepsilon}(a) + g_{\mu-\varepsilon}(c-1) + g_{\mu-\varepsilon}(c) + \sum_{a=c+1}^{j} g_{\mu-\varepsilon}(a) \\ &= \sum_{a=i+1}^{c-2} g_{\mu}(a) + g_{\mu}(c) + g_{\mu}(c-1) + \sum_{a=c+1}^{j} g_{\mu}(a) \\ &= \sum_{a=i+1}^{j} g_{\mu}(a) \\ &= r_{\mu}(i,j) \end{aligned}$$

For the last formula, note that for any $j > c$ we have

$$\begin{aligned} r_{\mu-\varepsilon}(c,j) &= \sum_{a=c+1}^{j} g_{\mu-\varepsilon}(a) \\ &= \sum_{a=c+1}^{j} g_{\mu}(a) \\ &= r_{\mu}(c,j) \end{aligned}$$

while $r_{\mu}(c-2,c) = g_{\mu}(c-1) + g_{\mu}(c) = 1 + (-1) = 0$, so

$$\begin{aligned} r_{\mu}(c-2,j) &= r_{\mu}(c-2,c) + r_{\mu}(c,j) \\ &= r_{\mu}(c,j) \end{aligned}$$

Similarly, we have $r_{\mu-\varepsilon}(c-2,c) = g_{\mu-\varepsilon}(c-1) + g_{\mu-\varepsilon}(c) = (-1)+1 = 0$, so $r_{\mu-\varepsilon}(c,j) = r_{\mu-\varepsilon}(c-2,j)$ as well. □

Suppose $\mu \vdash n$ has a removable box ε with $c(\varepsilon) = c$. Since $d(\mu)$ and $d(\mu-\varepsilon)$

differ only at $c-1$ and c, we have

$$\mathbf{c}(\mu) \cap \mathbf{c}(\mu - \varepsilon) = \mathbf{c}(\mu) \backslash \{c\} = \mathbf{c}(\mu - \varepsilon) \backslash \{c-1\}$$

This suggests that we might consider seven possibilities when comparing c to those $i \notin \mathbf{c}(\mu) \cap \mathbf{c}(\mu - \varepsilon)$. These cases are distinguished in the lemma below. In some of these cases, we must consider further subcases, essentially depending on whether $d(\mu)$ has a dot at $c-2$. These subcases arise in the course of proving later results.

Lemma 5.15. *Suppose $\mu \vdash n$ has a removable box ε with $c(\varepsilon) = c$. Suppose $i \notin \mathbf{c}(\mu)$ or $i \notin \mathbf{c}(\mu - \varepsilon)$.*

1. *If $i > c$, then $R_{\mu-\varepsilon}(i) = R_\mu(i)$.*

2. *Suppose $i = c$.*

   (a) *If $c-2$ is blank in $d(\mu)$, then $R_{\mu-\varepsilon}(c) = R_\mu(c-2) \backslash \{c\}$.*

   (b) *If $d(\mu)$ has a dot at $c-2$ which is not connected to an arc, then $R_{\mu-\varepsilon}(c) \subseteq \{c+2, c+4, \dots\}$.*

   (c) *If $d(\mu)$ has a dot at $c-2$ with $c-2 \in R_\mu(a)$ for some $a < c-2$, then $R_{\mu-\varepsilon}(c) = R_\mu(a) \cap (c, \infty)$.*

3. *If $i = c-1$, then $R_\mu(c-1) = \emptyset$.*

4. *If $i < c-1$ with $R_\mu(i) = \emptyset$, then $i < c-2$ with $R_{\mu-\varepsilon}(i) = \{c-1\}$ or $R_{\mu-\varepsilon}(i) = \emptyset$.*

5. *If $i < \max R_\mu(i) < c-1$, then $\max R_\mu(i) < c-2$ with $R_\mu(i) \subseteq R_{\mu-\varepsilon}(i)$ and $R_{\mu-\varepsilon}(i) \subseteq R_\mu(i) \cup \{c-1\}$.*

6. *Suppose $i < c-1 < \max R_\mu(i)$ with $c \in R_\mu(i)$.*

   (a) *If $c-2$ is blank in $d(\mu)$, then $i = c-2$ and, hence, $R_{\mu-\varepsilon}(c-2) = \emptyset$. Moreover, $R_{\mu-\varepsilon}(c) = R_\mu(c-2) \backslash \{c\}$.*

(b) *If* $d(\mu)$ *has a dot at* $c-2$*, then* $R_{\mu-\varepsilon}(i) = R_\mu(i) \cap (-\infty, c-1)$ *with* $c-2 \in R_{\mu-\varepsilon}(i)$ *and* $R_{\mu-\varepsilon}(c) = R_\mu(i) \cap (c, \infty)$.

7. *If* $i < c-1 < \max R_\mu(i)$ *with* $c \notin R_\mu(i)$*, then* $i < c-2$ *and* $c-2 \notin R_\mu(i)$ *with* $R_\mu(i) \subseteq R_{\mu-\varepsilon}(i)$ *and* $R_{\mu-\varepsilon}(i) \subseteq R_\mu(i) \cup \{c-1\}$.

Proof. If $i > c$, then moving the dot at c in $d(\mu)$ to $c-1$ does not affect i or the arcs connected to i, so we have $R_{\mu-\varepsilon}(i) = R_\mu(i)$. Note that this case does not depend on whether $c-2$ is blank or contains a dot in $d(\mu)$.

Suppose $i = c$.

1. First, suppose that $d(\mu)$ has no dot at $c-2$. Since $c-1$ is blank in $d(\mu)$, this means $\overline{d(\mu)}$ has an arc connecting $c-2$ to c, and possibly other arcs connecting $c-2$ to dots on the right of c. When ε is removed from μ, the arcs originating from $c-2$ are lost since this change places a dot at $c-1$. However, c remains connected in $\overline{d(\mu-\varepsilon)}$ to every other element $j \in R_\mu(c-2)\backslash\{c\}$ since

$$r_{\mu-\varepsilon}(c, j) = r_\mu(c-2, j) = 0$$

and for $c < a < j$, we have

$$r_{\mu-\varepsilon}(c, a) = r_\mu(c-2, a) \geq 0$$

That is, $R_{\mu-\varepsilon}(c) \supseteq R_\mu(c-2)\backslash\{c\}$. Now, if $j \in R_{\mu-\varepsilon}(c)$, we have

- $r_\mu(c-2, j) = r_{\mu-\varepsilon}(c, j) = 0$,
- if $c < a < j$, then $r_\mu(c-2, a) = r_{\mu-\varepsilon}(c, a) \geq 0$,
- $r_\mu(c-2, c) = 0$, and
- $r_\mu(c-2, c-1) = 1$,

which shows $j \in R_\mu(c-2)\backslash\{c\}$. Hence, $R_{\mu-\varepsilon}(c) = R_\mu(c-2)\backslash\{c\}$.

2. If $d(\mu)$ has a dot at $c-2$ with no arc connected to $c-2$, then $R_{\mu-\varepsilon}(c) \subseteq \{c+2, c+4, \ldots\}$ since these are the labels to which the blank node c in $d(\mu-\varepsilon)$ might be connected.

3. Now, suppose $d(\mu)$ has a dot at $c-2$ with $c-2 \in R_\mu(a)$, for some $a < c-2$. Then, we also have $c \in R_\mu(a)$ since $r_\mu(a,c) = r_\mu(a,c-2) + r_\mu(c-2,c) = 0$ and $r_\mu(a,b) \geq 0$ for all $a < b < c$. We claim $R_{\mu-\varepsilon}(c) = R_\mu(a) \cap (c,\infty)$. If $j \in R_\mu(a)$ and $j > c$, then

$$\begin{aligned} r_{\mu-\varepsilon}(c,j) &= r_\mu(c,j) \\ &= r_\mu(a,c) + r_\mu(c,j) \\ &= r_\mu(a,j) \\ &= 0 \end{aligned}$$

and for $c < b < j$ we have

$$\begin{aligned} r_{\mu-\varepsilon}(c,b) &= r_\mu(c,b) \\ &= r_\mu(a,c) + r_\mu(c,b) \\ &= r_\mu(a,b) \\ &\geq 0 \end{aligned}$$

so $j \in R_{\mu-\varepsilon}(c)$. On the other hand, given $j \in R_{\mu-\varepsilon}(c)$, we have

$$\begin{aligned} r_\mu(a,j) &= r_\mu(a,c) + r_\mu(c,j) \\ &= 0 + r_{\mu-\varepsilon}(c,j) \\ &= 0 \end{aligned}$$

and for all $c < b < j$,

$$\begin{aligned} r_\mu(a,b) &= r_\mu(a,c) + r_\mu(c,b) \\ &= 0 + r_{\mu-\varepsilon}(c,b) \\ &\geq 0 \end{aligned}$$

while if $a < b \leq c$, we have $r_\mu(a,b) \geq 0$ since $c \in R_\mu(a)$, so $j \in R_\mu(a)$.

If $i = c-1$, then no arcs can originate from i since there is a dot at c in $d(\mu)$, so $R_\mu(c-1) = \emptyset$. Note that this case does not depend on whether $c-2$ is blank or contains a dot in $d(\mu)$.

Suppose $i < c-1$ with $R_\mu(i) = \emptyset$. If $d(\mu)$ has a dot at $c-2$, then we cannot have $i = c-2$. Also, if $c-2$ is blank in $d(\mu)$, then $c \in R_\mu(c-2)$, so $R_\mu(c-2) \neq \emptyset$. In either case, we have $i < c-2$. If $j \in R_{\mu-\varepsilon}(i)$ with $i < j < c-1$, then $r_\mu(i,j) = r_{\mu-\varepsilon}(i,j) = 0$ and, if $i < a < j$, then $r_\mu(i,a) = r_{\mu-\varepsilon}(i,a) \geq 0$, so $j \in R_\mu(i)$, which is a contradiction. Also, if $j \in R_{\mu-\varepsilon}(i)$ with $i < c-1 < j$, then

- $r_\mu(i,j) = r_{\mu-\varepsilon}(i,j) = 0$
- if $i < a < c-1$, then $r_\mu(i,a) = r_{\mu-\varepsilon}(i,a) \geq 0$
- $r_\mu(i,c-1) = r_\mu(i,c-2) + g_\mu(c-1) = r_{\mu-\varepsilon}(i,c-2) + 1 \geq 0 + 1 = 1$
- if $a > c-1$, then $r_\mu(i,a) = r_{\mu-\varepsilon}(i,a) \geq 0$

so $j \in R_\mu(i)$, which is a contradiction. Hence, we must have either $R_{\mu-\varepsilon}(i) = \{c-1\}$ or $R_{\mu-\varepsilon}(i) = \emptyset$. By considering separate cases on whether or not $c-2$ is blank in $d(\mu)$, we can determine precisely when $R_{\mu-\varepsilon}(i) = \emptyset$. However, we will not need this extra information.

Suppose $i < \max R_\mu(i) < c-1$. If $c-2$ is blank in $d(\mu)$, then we also have $c-2 > \max R_\mu(i)$. On the other hand, if there is a dot at $c-2$ in $d(\mu)$, we note that

$c-2 \notin R_\mu(i)$ since if $c-2 \in R_\mu(i)$, then we would have $c \in R_\mu(i)$, which contradicts $\max R_\mu(i) < c-1$. Hence, $c-2 > \max R_\mu(i)$ in either case. If $j \in R_\mu(i)$, then $r_{\mu-\varepsilon}(i,j) = r_\mu(i,j) = 0$ and for $i < a < j$, we have $r_{\mu-\varepsilon}(i,a) = r_\mu(i,a) \geq 0$, so $j \in R_{\mu-\varepsilon}(i)$. That is, $R_\mu(i) \subseteq R_{\mu-\varepsilon}(i)$. It is possible for $c-1 \in R_{\mu-\varepsilon}(i)$, in the event that $r_\mu(i,c-1) = 2$. However, it is impossible for $j > c-1$ to belong to $R_{\mu-\varepsilon}(i)$, since if this happens,

- $r_\mu(i,j) = r_{\mu-\varepsilon}(i,j) = 0$
- for $i < a < c-1$, we have $r_\mu(i,a) = r_{\mu-\varepsilon}(i,a) \geq 0$
- $r_\mu(i,c-1) = r_\mu(i,c-2) + g_\mu(c-1) = r_{\mu-\varepsilon}(i,c-2) + 1 \geq 0 + 1 = 1$
- for $c \leq a < j$, we have $r_\mu(i,a) = r_{\mu-\varepsilon}(i,a) \geq 0$

so $j \in R_\mu(i)$ and $j > \max R_\mu(i)$, which is a contradiction. On the other hand, if $j \in R_{\mu-\varepsilon}(i)$ and $j < c-1$, then $r_\mu(i,j) = r_{\mu-\varepsilon}(i,j) = 0$ and for $i < a < j$, we have $r_\mu(i,a) = r_{\mu-\varepsilon}(i,a) \geq 0$, so $j \in R_\mu(i)$. Hence, $R_{\mu-\varepsilon}(i) \subseteq R_\mu(i) \cup \{c-1\}$.

Suppose $i < c-1 < \max R_\mu(i)$ with $c \in R_\mu(i)$, so $r_\mu(i,c) = 0$ and $r_\mu(i,a) \geq 0$, for all $i < a < c-2$.

1. Suppose $c-2$ is blank in $d(\mu)$. Then, there is an arc connecting $c-2$ to c in $\overline{d(\mu)}$ since $c-1$ is blank in $d(\mu)$, so we must have $i = c-2$ since two arcs originating at i and $c-2$ cannot intersect at the dot c by Lemma 5.11 unless they are the same arc. Then, since $d(\mu-\varepsilon)$ has a dot at $c-1$, we see that $R_{\mu-\varepsilon}(c-2) = \emptyset$. Now, we claim $R_{\mu-\varepsilon}(c) = R_\mu(c-2) \backslash \{c\}$. If $j \in R_{\mu-\varepsilon}(c)$, then

    - $r_\mu(c-2,j) = r_{\mu-\varepsilon}(c,j) = 0$
    - $r_\mu(c-2,c-1) = g_\mu(c-1) = 1$
    - $r_\mu(c-2,c) = 0$

- if $c < a < j$, then $r_\mu(c-2,a) = r_{\mu-\varepsilon}(c,a) \geq 0$

so $j \in R_\mu(c-2)$ and $j > c$. On the other hand, if $j \in R_\mu(c-2)$ and $j > c$, then

- $r_{\mu-\varepsilon}(c,j) = r_\mu(c-2,j) = 0$
- if $c < a < j$, then $r_{\mu-\varepsilon}(c,a) = r_\mu(c-2,a) \geq 0$

so $j \in R_{\mu-\varepsilon}(c)$, and the claim follows.

2. Suppose $d(\mu)$ has a dot at $c-2$. Then,

$$r_\mu(i,c-2) = r_\mu(i,c) - g_\mu(c-1) - g_\mu(c) = 0 - 1 - (-1) = 0$$

so $c-2 \in R_\mu(i)$ since $c-1$ is blank in $d(\mu)$. Now,

$$\begin{aligned} r_{\mu-\varepsilon}(i,c-1) &= r_{\mu-\varepsilon}(i,c-2) + g_{\mu-\varepsilon}(c-1) \\ &= r_\mu(i,c-2) + (-1) \\ &= 0 - 1 \\ &< 0 \end{aligned}$$

so $c-1 \notin R_{\mu-\varepsilon}(i)$. Moreover, this calculation also shows that if $j > c$, then $j \notin R_{\mu-\varepsilon}(i)$. We claim $R_{\mu-\varepsilon}(i) = R_\mu(i) \cap (-\infty, c-1)$. If $j \in R_\mu(i)$ and $j < c-1$, then $r_{\mu-\varepsilon}(i,j) = r_\mu(i,j) = 0$ and if $i < a < j$, then $r_{\mu-\varepsilon}(i,a) = r_\mu(i,a) \geq 0$ since $i < a < c$ and $c \in R_\mu(i)$. Hence, $j \in R_{\mu-\varepsilon}(i)$. On the other hand, if $j \in R_{\mu-\varepsilon}(i)$, then we have $j < c-1$ by the calculation above, so $r_\mu(i,j) = r_{\mu-\varepsilon}(i,j) = 0$ and if $i < a < j$, then $r_\mu(i,a) = r_{\mu-\varepsilon}(i,a) \geq 0$. So, $j \in R_\mu(i)$, which proves our claim. Finally, we claim $R_{\mu-\varepsilon}(c) = R_\mu(i) \cap (c, \infty)$. If $j \in R_{\mu-\varepsilon}(c)$, then

- $r_\mu(i,j) = r_\mu(i,c) + r_\mu(c,j) = 0 + r_{\mu-\varepsilon}(c,j) = 0$
- if $i < a < c$, then $r_\mu(i,a) \geq 0$ since $c \in R_\mu(i)$
- $r_\mu(i,c) = 0$
- if $c < a < j$, then $r_\mu(i,a) = r_\mu(i,c) + r_\mu(c,a) = 0 + r_{\mu-\varepsilon}(c,a) \geq 0$

so $j \in R_\mu(i)$ and $j > c$. On the other hand, if $j \in R_\mu(i)$ and $j > c$, then

$$r_{\mu-\varepsilon}(c,j) = r_\mu(c,j) = r_\mu(i,c) + r_\mu(c,j) = r_\mu(i,j) = 0$$

and

$$r_{\mu-\varepsilon}(c,a) = r_\mu(c,a) = r_\mu(i,c) + r_\mu(c,a) = r_\mu(i,a) \geq 0$$

for all $c < a < j$, so $j \in R_{\mu-\varepsilon}(c)$.

Finally, suppose $i < c - 1 < \max R_\mu(i)$ with $c \notin R_\mu(i)$. Then, we must have $r_\mu(i,c) > 0$. If $c-2$ is blank in $d(\mu)$, then $c \in R_\mu(c-2)$, so we must have $i < c-2$. If $d(\mu)$ has a dot at $c-2$, then we again have $i < c-2$ and $c-2 \notin R_\mu(i)$ since

$$r_\mu(i,c-2) = r_\mu(i,c) - g_\mu(c-1) - g_\mu(c) > 0 - 1 - (-1) = 0$$

It follows that this case is independent of whether or not $c-2$ is blank in $d(\mu)$. If $j \in R_\mu(i)$, then

- $r_{\mu-\varepsilon}(i,j) = r_\mu(i,j) = 0$ since $j < c-2$ or $j > c$
- if $i < a < c-1$, then $r_{\mu-\varepsilon}(i,a) = r_\mu(i,a) \geq 0$
- $r_{\mu-\varepsilon}(i,c-1) = r_{\mu-\varepsilon}(i,c) - g_{\mu-\varepsilon}(c) = r_\mu(i,c) - 1 \geq 1 - 1 = 0$
- if $c-1 < a < j$, then $r_{\mu-\varepsilon}(i,a) = r_\mu(i,a) \geq 0$

so $j \in R_{\mu-\varepsilon}(i)$. It is possible for $c-1 \in R_{\mu-\varepsilon}(i)$, in the event that $r_\mu(i,c-1) = 2$. If $j \in R_{\mu-\varepsilon}(i)$ and $j < c-1$, then $r_\mu(i,j) = r_{\mu-\varepsilon}(i,j) = 0$ and for $i < a < j$, we have $r_\mu(i,a) = r_{\mu-\varepsilon}(i,a) \geq 0$, so $j \in R_\mu(i)$. On the other hand, if $j \in R_{\mu-\varepsilon}(i)$ and $j > c-1$, then

- $r_\mu(i,j) = r_{\mu-\varepsilon}(i,j) = 0$
- if $i < a < c-1$, then $r_\mu(i,a) = r_{\mu-\varepsilon}(i,a) \geq 0$
- $r_\mu(i,c-1) = r_\mu(i,c) - g_\mu(c) \geq 1-(-1) = 2$
- if $c-1 < a < j$, then $r_\mu(i,a) = r_{\mu-\varepsilon}(i,a) \geq 0$

so $j \in R_\mu(i)$. Hence, $R_{\mu-\varepsilon}(i) \subseteq R_\mu(i) \cup \{c-1\}$. $\square$

Lemma 5.16. *Let $\mu \vdash n$ and $\lambda \in \Lambda(n)$. Let ε be a removable box of μ with $c(\varepsilon) = c$. Suppose λ has a removable box ε' of content c or an addable box ε'' of content $c-1$. Suppose $i \notin \mathbf{c}(\mu)$ or $i \notin \mathbf{c}(\mu-\varepsilon)$.*

1. *If $i > c$, then $s_{\mu-\varepsilon}(\lambda-\varepsilon', i) = s_{\mu-\varepsilon}(\lambda+\varepsilon'', i) = s_\mu(\lambda, i)$.*
2. *Suppose $i = c$.*
   (a) *If $c-2$ is blank in $d(\mu)$, then*

$$s_{\mu-\varepsilon}(\lambda-\varepsilon', c) = s_\mu(\lambda, c-2) - f_\lambda(c-2) + 1$$

   and $s_{\mu-\varepsilon}(\lambda+\varepsilon'', c) = s_\mu(\lambda, c-2)$.

   (b) *If $d(\mu)$ has a dot at $c-2$ which is not connected to an arc, then*

$$1 \leq s_{\mu-\varepsilon}(\lambda-\varepsilon', c) \leq 1 + \sum_{j \in R_{\mu-\varepsilon}(c)} f_\lambda(j)$$

   and

$$f_\lambda(c) \leq s_{\mu-\varepsilon}(\lambda+\varepsilon'', c) \leq f_\lambda(c) + \sum_{j \in R_{\mu-\varepsilon}(c)} f_\lambda(j).$$

(c) *If* $d(\mu)$ *has a dot at* $c-2$ *with* $c-2 \in R_\mu(a)$ *for some* $a < c-2$, *then*

$$s_{\mu-\varepsilon}(\lambda - \varepsilon', c) = 1 + \sum_{j \in R_\mu(a) \cap (c,\infty)} f_\lambda(j)$$

and

$$s_{\mu-\varepsilon}(\lambda + \varepsilon'', c) = f_\lambda(c) + \sum_{j \in R_\mu(a) \cap (c,\infty)} f_\lambda(j).$$

3. *If* $i = c-1$, *then* $s_\mu(\lambda, c-1) = 1$.

4. *If* $i < c-1$ *with* $R_\mu(i) = \emptyset$, *then*

$$s_{\mu-\varepsilon}(\lambda - \varepsilon', i) = s_{\mu-\varepsilon}(\lambda + \varepsilon'', i) = s_\mu(\lambda, i) = f_\lambda(i).$$

5. *If* $i < \max R_\mu(i) < c-1$, *then* $s_{\mu-\varepsilon}(\lambda - \varepsilon', i) = s_{\mu-\varepsilon}(\lambda + \varepsilon'', i) = s_\mu(\lambda, i)$.

6. *Suppose* $i < c-1 < \max R_\mu(i)$ *with* $c \in R_\mu(i)$.

   (a) *If* $c-2$ *is blank in* $d(\mu)$, *then* $i = c-2$ *with* $s_{\mu-\varepsilon}(\lambda - \varepsilon', c-2) = f_\lambda(c-2)$ *and* $s_{\mu-\varepsilon}(\lambda + \varepsilon'', c-2) = 1$. *Moreover, we have*

$$\begin{aligned} s_\mu(\lambda, c-2) &= f_\lambda(c-2) + s_{\mu-\varepsilon}(\lambda - \varepsilon', c) - 1 \\ &= f_\lambda(c-2) + s_{\mu-\varepsilon}(\lambda + \varepsilon'', c). \end{aligned}$$

   (b) *If* $d(\mu)$ *has a dot at* $c-2$, *then*

$$\begin{aligned} s_\mu(\lambda, i) &= s_{\mu-\varepsilon}(\lambda - \varepsilon', i) + s_{\mu-\varepsilon}(\lambda - \varepsilon', c) - 1 \\ &= s_{\mu-\varepsilon}(\lambda + \varepsilon'', i) + s_{\mu-\varepsilon}(\lambda + \varepsilon'', c) - 1. \end{aligned}$$

7. If $i < c-1 < \max R_\mu(i)$ with $c \notin R_\mu(i)$, then

$$s_{\mu-\varepsilon}(\lambda-\varepsilon', i) = s_{\mu-\varepsilon}(\lambda+\varepsilon'', i) = s_\mu(\lambda, i).$$

Proof. In each case, we freely use the results of Lemma 5.15 without reference to relate the sets $R_\mu(i)$ and $R_{\mu-\varepsilon}(i)$. Note that

$$f_{\lambda-\varepsilon'}(a) = \begin{cases} f_\lambda(a) & \text{if } a \neq c-1, c \\ 0 & \text{if } a = c-1 \\ 1 & \text{if } a = c \end{cases}$$

since $d(\lambda)$ and $d(\lambda-\varepsilon')$ differ only at $c-1$ and c, while

$$f_{\lambda+\varepsilon''}(a) = \begin{cases} f_\lambda(a) & \text{if } a \neq c-2, c-1 \\ 1 & \text{if } a = c-2 \\ 0 & \text{if } a = c-1 \end{cases}$$

since $d(\lambda)$ and $d(\lambda+\varepsilon'')$ differ only at $c-2$ and $c-1$. Let $\tau \in \{-\varepsilon', \varepsilon''\}$. We will write $\lambda+\tau$ in calculations which are the same for both $\lambda-\varepsilon'$ and $\lambda+\varepsilon''$.

First, suppose $i > c$. We have $R_{\mu-\varepsilon}(i) = R_\mu(i)$ and $f_{\lambda-\varepsilon'}(x) = f_{\lambda+\varepsilon''}(x) = f_\lambda(x)$ for all $x \in R_\mu(i) \cup \{i\}$ since each such $x > c$, so

$$\begin{aligned} s_{\mu-\varepsilon}(\lambda+\tau, i) &= f_{\lambda+\tau}(i) + \sum_{j \in R_{\mu-\varepsilon}(i)} f_{\lambda+\tau}(j) \\ &= f_\lambda(i) + \sum_{j \in R_\mu(i)} f_\lambda(j) \\ &= s_\mu(\lambda, i) \end{aligned}$$

Next, suppose $i = c$. We have $f_{\lambda-\varepsilon'}(x) = f_{\lambda+\varepsilon''}(x) = f_\lambda(x)$ for all $x \in R_{\mu-\varepsilon}(c)$

since each such $x > c$. So,

$$\begin{aligned} s_{\mu-\varepsilon}(\lambda+\tau, c) &= f_{\lambda+\tau}(c) + \sum_{j \in R_{\mu-\varepsilon}(c)} f_{\lambda+\tau}(j) \\ &= f_{\lambda+\tau}(c) + \sum_{j \in R_{\mu-\varepsilon}(c)} f_{\lambda}(j) \end{aligned}$$

1. If $c-2$ is blank in $d(\mu)$, then $R_{\mu-\varepsilon}(c) = R_\mu(c-2)\backslash\{c\}$. So,

$$\begin{aligned} s_{\mu-\varepsilon}(\lambda+\tau, c) &= f_{\lambda+\tau}(c) + \sum_{j \in R_\mu(c-2)\backslash\{c\}} f_{\lambda}(j) \\ &= f_{\lambda+\tau}(c) + s_\mu(\lambda, c-2) - f_\lambda(c-2) - f_\lambda(c) \end{aligned}$$

2. If $d(\mu)$ has a dot at $c-2$ which is not connected to an arc, then $R_{\mu-\varepsilon}(c) \subseteq \{c+2, c+4, \dots\}$. So,

$$s_{\mu-\varepsilon}(\lambda+\tau, c) \geq f_{\lambda+\tau}(c)$$

which corresponds to $R_{\mu-\varepsilon}(c) = \emptyset$, while

$$s_{\mu-\varepsilon}(\lambda+\tau, c) \leq f_{\lambda+\tau}(c) + \sum_{j \in R_{\mu-\varepsilon}(c)} f_{\lambda}(j)$$

3. If $d(\mu)$ has a dot at $c-2$ with $c-2 \in R_\mu(a)$ for some $a < c-2$, then $R_{\mu-\varepsilon}(c) = R_\mu(a) \cap (c, \infty)$. So,

$$s_{\mu-\varepsilon}(\lambda+\tau, c) = f_{\lambda+\tau}(c) + \sum_{j \in R_\mu(a)\cap(c,\infty)} f_{\lambda}(j)$$

Since $f_{\lambda-\varepsilon'}(c) = 1$ and $f_\lambda(c) = 0$ when λ has a removable box of content c, while $f_{\lambda+\varepsilon''}(c) = f_\lambda(c)$ and $f_\lambda(c-2) = 0$ when λ has an addable box of content $c-1$, this formula further simplifies as claimed.

Now, suppose $i = c-1$. Then, $R_\mu(i) = \emptyset$, so

$$\begin{aligned} s_\mu(\lambda, c-1) &= f_\lambda(c-1) + \sum_{j \in R_\mu(c-1)} f_\lambda(j) \\ &= 1 + 0 \end{aligned}$$

since $c-1$ must be blank in $d(\lambda)$.

Suppose $i < c-1$ with $R_\mu(i) = \emptyset$. Then,

$$\begin{aligned} s_\mu(\lambda, i) &= f_\lambda(i) + \sum_{j \in R_\mu(i)} f_\lambda(j) \\ &= f_\lambda(i) + 0 \end{aligned}$$

Since $i < c-2$ in this case, we have $f_{\lambda-\varepsilon'}(i) = f_{\lambda+\varepsilon''}(i) = f_\lambda(i)$ and $f_{\lambda-\varepsilon}(c-1) = f_{\lambda+\varepsilon''}(c-1) = 0$, so in either case $R_{\mu-\varepsilon}(i) = \{c-1\}$ or $R_{\mu-\varepsilon}(i) = \emptyset$, we have

$$\begin{aligned} s_{\mu-\varepsilon}(\lambda+\tau, i) &= f_{\lambda+\tau}(i) + \sum_{j \in R_{\mu-\varepsilon}(i)} f_{\lambda+\tau}(j) \\ &= f_\lambda(i) + 0 \end{aligned}$$

Next, suppose $i < \max R_\mu(i) < c-1$. We have $\max R_\mu(i) < c-2$, so $f_{\lambda-\varepsilon'}(x) = f_{\lambda+\varepsilon''}(x) = f_\lambda(x)$ for all $x \in R_\mu(i) \cup \{i\}$. Since $R_\mu(i) \subseteq R_{\mu-\varepsilon}(i)$, we have

$$\begin{aligned} s_\mu(\lambda, i) &= f_\lambda(i) + \sum_{j \in R_\mu(i)} f_\lambda(j) \\ &= f_{\lambda+\tau}(i) + \sum_{j \in R_\mu(i)} f_{\lambda+\tau}(j) \\ &\leq f_{\lambda+\tau}(i) + \sum_{j \in R_{\mu-\varepsilon}(i)} f_{\lambda+\tau}(j) \\ &= s_{\mu-\varepsilon}(\lambda+\tau, i) \end{aligned}$$

On the other hand, $R_{\mu-\varepsilon}(i) \subseteq R_\mu(i) \cup \{c-1\}$, so since $f_{\lambda-\varepsilon'}(c-1) = f_{\lambda+\varepsilon''}(c-1) = 0$

we have

$$
\begin{aligned}
s_{\mu-\varepsilon}(\lambda+\tau, i) &= f_{\lambda+\tau}(i) + \sum_{j \in R_{\mu-\varepsilon}(i)} f_{\lambda+\tau}(j) \\
&\leq f_{\lambda+\tau}(i) + \sum_{j \in R_{\mu}(i)} f_{\lambda+\tau}(j) + f_{\lambda+\tau}(c-1) \\
&= f_{\lambda}(i) + \sum_{j \in R_{\mu}(i)} f_{\lambda}(j) + 0 \\
&= s_{\mu}(\lambda, i)
\end{aligned}
$$

Hence, $s_{\mu-\varepsilon}(\lambda+\tau, i) = s_{\mu}(\lambda, i)$ in this case.

Suppose $i < c-1 < \max R_{\mu}(i)$ with $c \in R_{\mu}(i)$.

1. If $c-2$ is blank in $d(\mu)$, then $i = c-2$ and $R_{\mu-\varepsilon}(c-2) = \emptyset$. So,

$$
\begin{aligned}
s_{\mu-\varepsilon}(\lambda+\tau, c-2) &= f_{\lambda+\tau}(c-2) + \sum_{j \in R_{\mu-\varepsilon}(c-2)} f_{\lambda+\tau}(j) \\
&= f_{\lambda+\tau}(c-2) + 0
\end{aligned}
$$

Also, we have $R_{\mu-\varepsilon}(c) = R_{\mu}(c-2) \backslash \{c\}$, so

$$
\begin{aligned}
s_{\mu}(\lambda, c-2) &= f_{\lambda}(c-2) + \sum_{j \in R_{\mu}(c-2)} f_{\lambda}(j) \\
&= f_{\lambda}(c-2) + f_{\lambda}(c) + \sum_{j \in R_{\mu-\varepsilon}(c)} f_{\lambda}(j) \\
&= f_{\lambda}(c-2) + f_{\lambda}(c) + \sum_{j \in R_{\mu-\varepsilon}(c)} f_{\lambda+\tau}(j) \\
&= f_{\lambda}(c-2) + f_{\lambda}(c) - f_{\lambda+\tau}(c) + s_{\mu-\varepsilon}(\lambda+\tau, c)
\end{aligned}
$$

These simplify further to the claimed formulas since $f_{\lambda-\varepsilon'}(c-2) = f_{\lambda}(c-2)$, $f_{\lambda}(c) = 1$, and $f_{\lambda-\varepsilon}(c) = 1$ when λ has a removable box of content c, while $f_{\lambda+\varepsilon''}(c-2) = 1$ and $f_{\lambda+\varepsilon''}(c) = f_{\lambda}(c)$ when λ has an addable box of content $c-1$.

2. If $d(\mu)$ has a dot at $c-2$, then $R_{\mu-\varepsilon}(i) = R_\mu(i) \cap (-\infty, c-1)$ and $c-2 \in R_{\mu-\varepsilon}(i)$. We have $f_{\lambda-\varepsilon'}(x) = f_{\lambda+\varepsilon''}(x) = f_\lambda(x)$ for all $x \in \{i\} \cup R_\mu(i)$ with $x < c-2$ or $x > c$, so

$$\begin{aligned}
s_{\mu-\varepsilon}(\lambda+\tau, i) &= f_{\lambda+\tau}(i) + \sum_{j \in R_{\mu-\varepsilon}(i)} f_{\lambda+\tau}(j) \\
&= f_{\lambda+\tau}(i) + \sum_{j \in R_\mu(i) \cap (-\infty, c-1)} f_{\lambda+\tau}(j) \\
&= f_\lambda(i) + f_{\lambda+\tau}(c-2) - f_\lambda(c-2) + \sum_{j \in R_\mu(i) \cap (-\infty, c-1)} f_\lambda(j)
\end{aligned}$$

Now, $R_{\mu-\varepsilon}(c) = R_\mu(i) \cap (c, \infty)$, so

$$\begin{aligned}
s_{\mu-\varepsilon}(\lambda+\tau, c) &= f_{\lambda+\tau}(c) + \sum_{j \in R_{\mu-\varepsilon}(c)} f_{\lambda+\tau}(j) \\
&= f_{\lambda+\tau}(c) + \sum_{j \in R_\mu(i) \cap (c, \infty)} f_{\lambda+\tau}(j) \\
&= f_{\lambda+\tau}(c) + \sum_{j \in R_\mu(i) \cap (c, \infty)} f_\lambda(j)
\end{aligned}$$

Since $R_\mu(i) = (R_\mu(i) \cap (-\infty, c-1)) \cup (R_\mu(i) \cap (c, \infty)) \cup \{c\}$, we have

$$\begin{aligned}
s_{\mu-\varepsilon}(\lambda+\tau, i) + s_{\mu-\varepsilon}(\lambda+\tau, c) &= f_{\lambda+\tau}(c) + f_{\lambda+\tau}(c-2) - f_\lambda(c-2) \\
&\quad + f_\lambda(i) + \sum_{j \in R_\mu(i)} f_\lambda(j) - f_\lambda(c) \\
&= f_{\lambda+\tau}(c-2) - f_\lambda(c-2) + f_{\lambda+\tau}(c) \\
&\quad - f_\lambda(c) + s_\mu(\lambda, i)
\end{aligned}$$

Note that this simplifies further to the claimed formulas since $f_\lambda(c) = 0$, $f_{\lambda-\varepsilon'}(c) = 1$, and $f_{\lambda+\tau}(c-2) = f_\lambda(c-2)$ when λ has a removable box of content c, while $f_\lambda(c-2) = 0$, $f_{\lambda+\varepsilon''}(c-2) = 1$, and $f_{\lambda+\varepsilon''}(c) = f_\lambda(c)$ when λ has an addable box of content $c-1$.

Finally, suppose $i < c-1 < \max R_\mu(i)$ with $c \notin R_\mu(i)$. We have $i < c-2$ and $c-2, c-1 \notin R_\mu(i)$ in this case, so $f_{\lambda-\varepsilon'}(x) = f_{\lambda+\varepsilon''}(x) = f_\lambda(x)$ for all $x \in R_\mu(i) \cup \{i\}$. Since $R_\mu(i) \subseteq R_{\mu-\varepsilon}(i)$, we have

$$\begin{aligned} s_\mu(\lambda, i) &= f_\lambda(i) + \sum_{j \in R_\mu(i)} f_\lambda(j) \\ &= f_{\lambda+\tau}(i) + \sum_{j \in R_\mu(i)} f_{\lambda+\tau}(j) \\ &\leq f_{\lambda+\tau}(i) + \sum_{j \in R_{\mu-\varepsilon}(i)} f_{\lambda+\tau}(j) \\ &= s_{\mu-\varepsilon}(\lambda+\tau, i) \end{aligned}$$

Also, since $R_{\mu-\varepsilon}(i) \subseteq R_\mu(i) \cup \{c-1\}$ and $f_{\lambda-\varepsilon'}(c-1) = f_{\lambda+\varepsilon''}(c-1) = 0$, we have

$$\begin{aligned} s_{\mu-\varepsilon}(\lambda+\tau, i) &= f_{\lambda+\tau}(i) + \sum_{j \in R_{\mu-\varepsilon}(i)} f_{\lambda+\tau}(j) \\ &\leq f_{\lambda+\tau}(i) + \sum_{j \in R_\mu(i)} f_{\lambda+\tau}(j) + f_{\lambda+\tau}(c-1) \\ &= f_\lambda(i) + \sum_{j \in R_\mu(i)} f_\lambda(j) + 0 \\ &= s_\mu(\lambda, i) \end{aligned}$$

Hence, $s_{\mu-\varepsilon}(\lambda+\tau, i) = s_\mu(\lambda, i)$ in this case. $\square$

Proposition 5.17. *Let $\mu \vdash n$ and $\lambda \in \Lambda(n)$. Let ε be a removable box of μ with $c(\varepsilon) = c$. Suppose λ has a removable box ε' of content c with $\lambda - \varepsilon' \in R(\mu-\varepsilon)$ or an addable box ε'' of content $c-1$ with $\lambda + \varepsilon'' \in R(\mu - \varepsilon)$. Then, $\lambda \in R(\mu)$.*

Proof. If $\lambda - \varepsilon' \in R(\mu-\varepsilon)$ or $\lambda + \varepsilon'' \in R(\mu-\varepsilon)$, then we have $s_{\mu-\varepsilon}(\lambda-\varepsilon', j) = 1$ or $s_{\mu-\varepsilon}(\lambda+\varepsilon'', j) = 1$, respectively, for all $j \notin \mathbf{c}(\mu-\varepsilon)$. We must show that $s_\mu(\lambda, i) = 1$ for all $i \notin \mathbf{c}(\mu)$. Note that $\mathbf{c}(\mu) = (\mathbf{c}(\mu-\varepsilon) \backslash \{c-1\}) \cup \{c\}$, so the possible values of $i \notin \mathbf{c}(\mu)$ are those $i \notin \mathbf{c}(\mu-\varepsilon)$ with $i \neq c$ or $i = c-1$. We therefore consider six possibilities for $i \notin \mathbf{c}(\mu)$.

By Lemma 5.16, we have

$$s_\mu(\lambda, i) = s_{\mu-\varepsilon}(\lambda - \varepsilon', i) = s_{\mu-\varepsilon}(\lambda + \varepsilon'', i)$$

in each of the following cases:

- $i > c$,
- $i < c - 1$ with $R_\mu(i) = \emptyset$,
- $i < \max R_\mu(i) < c - 1$, or
- $i < c - 1 < \max R_\mu(i)$ with $c \notin R_\mu(i)$.

So, if $\lambda - \varepsilon' \in R(\mu - \varepsilon)$ or $\lambda + \varepsilon'' \in R(\mu - \varepsilon)$, then we have $s_\mu(\lambda, i) = 1$ for all i in these cases. The lemma also shows $s_\mu(\lambda, c-1) = 1$ when $i = c-1$. In the remaining case, we freely use the formulas from Lemmas 5.15 and 5.16 without individual citations.

Suppose $i < c - 1 < \max R_\mu(i)$ with $c \in R_\mu(i)$. Note that since c is blank in $d(\mu - \varepsilon)$, we have $c \notin \mathbf{c}(\mu - \varepsilon)$, so $s_{\mu-\varepsilon}(\lambda - \varepsilon', c) = 1$. First, further suppose $c - 2$ is blank in $d(\mu)$, so $i = c - 2 \notin \mathbf{c}(\mu)$ with $s_{\mu-\varepsilon}(\lambda - \varepsilon', c - 2) = f_\lambda(c - 2)$ and $s_{\mu-\varepsilon}(\lambda + \varepsilon'', c - 2) = 1$.

1. If $\lambda - \varepsilon' \in R(\mu - \varepsilon)$, then we have $1 = s_{\mu-\varepsilon}(\lambda - \varepsilon', c - 2) = f_\lambda(c - 2)$, so $c - 2$ is blank in $d(\lambda)$ and $d(\lambda - \varepsilon')$. Hence,

$$\begin{aligned} s_\mu(\lambda, c-2) &= f_\lambda(c-2) + s_{\mu-\varepsilon}(\lambda - \varepsilon', c) - 1 \\ &= 1 + 1 - 1 \\ &= 1 \end{aligned}$$

2. If $\lambda + \varepsilon'' \in R(\mu - \varepsilon)$, then $f_\lambda(c-2) = 0$ since $d(\lambda)$ has a dot at $c-2$. So,

$$\begin{aligned} s_\mu(\lambda, c-2) &= f_\lambda(c-2) + s_{\mu-\varepsilon}(\lambda + \varepsilon'', c) \\ &= 0 + 1 \\ &= 1 \end{aligned}$$

Finally, suppose further that $d(\mu)$ has a dot at $c-2$. If $\lambda - \varepsilon' \in R(\mu - \varepsilon)$ or $\lambda + \varepsilon'' \in R(\mu - \varepsilon)$, then we have $s_{\mu-\varepsilon}(\lambda + \tau, i) = 1$, where $\tau \in \{-\varepsilon', \varepsilon''\}$, so

$$\begin{aligned} s_\mu(\lambda, i) &= s_{\mu-\varepsilon}(\lambda + \tau, i) + s_{\mu-\varepsilon}(\lambda + \tau, c) - 1 \\ &= 1 + 1 - 1 \\ &= 1 \end{aligned}$$

In all cases, we have $s_\mu(\lambda, i) = 1$, hence $\lambda \in R(\mu)$. $\square$

Corollary 5.18. *If $\mu \vdash n$ and $\lambda \in \Lambda(n)$ have a common weight of length n, then $\lambda \in R(\mu)$. In particular, if $t \in UD_n(\lambda)$ and $u \in UD_n(\mu)$ with $wt(t) = wt(u)$, then $t_i \in R(u_i)$ for all $i = 1, \ldots, n$.*

Proof. By induction on n. When $n = 1$, this is trivially true since we must have $\lambda = \mu = \square$ and there is only one up-down tableau of length 1. Suppose $n > 1$. Let $t \in UD_n(\lambda)$ and $u \in UD_n(\mu)$ with $\mathrm{wt}(t) = \mathrm{wt}(u)$. By induction, we have $t_{n-1} \in R(u_{n-1})$ since $u_{n-1} \vdash n-1$ and $\mathrm{wt}(t_1, \ldots, t_{n-1}) = \mathrm{wt}(u_1, \ldots, u_{n-1})$, so, by the previous result, this forces $\lambda \in R(\mu)$. For the last statement, note that for each $i = 1, \ldots, n$, we have $\mathrm{wt}\left((t_1, \ldots, t_i)\right) = \mathrm{wt}\left((u_1, \ldots, u_i)\right)$ with $u_i \vdash i$, so $t_i \in R(u_i)$ by the statement we just proved. $\square$

Theorem 5.19. *Suppose $\mu \vdash n$ and $\lambda \in \Lambda(n)$. If $[\Delta_n(\lambda) : L_n(\mu)] \neq 0$, then $\lambda \in R(\mu)$.*

Proof. By Proposition 5.9, we know λ and μ have a common weight of length n, so $\lambda \in R(\mu)$ by the previous result. $\square$

We wish to strengthen this result in two ways: first, by proving the reverse implication, and then by showing that the nonzero decomposition numbers must be equal to 1. We have partial results in this direction, which appear below. Following those, we outline a strategy for proving this stronger result.

Lemma 5.20. *Suppose $\mu \vdash n$ and $\lambda \in R(\mu)$. For each removable box ε of μ, there is a removable box ε' or an addable box ε'' of λ with $c(\varepsilon) = c(\varepsilon') = c(\varepsilon'') + 1$.*

Proof. Suppose λ has neither an addable nor a removable box of the correct content. If ε is a removable box of μ with content c, then $c-1$ must be blank in $d(\mu)$, so $c-1 \notin \mathbf{c}(\mu)$. Since $d(\mu)$ has a dot at c, we have $R_\mu(c-1) = \emptyset$ and so $s_\mu(\lambda, c-1) = f_\lambda(c-1)$. Now, $\lambda \in R(\mu)$ so this forces $f_\lambda(c-1) = 1$, and hence $c-1$ must also be blank in $d(\lambda)$. Since λ does not have an addable or removable box of the correct weight, this forces $c-2$ and c to also be blank in $d(\lambda)$. So, $f_\lambda(c-2) = f_\lambda(c) = 1$. In particular, since c is blank in $d(\lambda)$ but not in $d(\mu)$, there must exist $i_0 \notin \mathbf{c}(\mu)$ with $c \in R_\mu(i_0)$ such that the arc connecting i_0 and c was collapsed when forming $d(\lambda)$ from $d(\mu)$. Then, since $s_\mu(\lambda, i_0) = 1$, we must have $f_\lambda(i_0) = 0$, $f_\lambda(c) = 1$, and $f_\lambda(j) = 0$ for all $j \in R_\mu(i_0) \backslash \{c\}$. We consider two subcases. First, suppose $f_\mu(c-2) = 0$ so that μ has a dot at $c-2$. Then,

$$r_\mu(i_0, c-2) = r_\mu(i_0, c) - g_\mu(c-1) - g_\mu(c) = 0 - 1 - (-1) = 0$$

and if $i_0 < a < c-2$ then $r_\mu(i_0, a) \geq 0$ since $i_0 < a < c$ and $c \in R_\mu(i_0)$, so we have $c-2 \in R_\mu(i_0)$. But, then $f_\lambda(c-2) = 0$, which is a contradiction. Now, suppose $f_\mu(c-2) = 1$ so that μ does not have a dot at $c-2$. Then, $r_\mu(c-2, c) = g_\mu(c-1) + g_\mu(c) = 1 - 1 = 0$ and $r_\mu(c-2, c-1) = g_\mu(c-1) = 1 > 0$, so $c \in R_\mu(c-2)$. Since arcs can only intersect at left endpoints, this forces $i_0 = c-2$. Since the arc connecting i_0 to c was collapsed when forming $d(\lambda)$ from $d(\mu)$, this means $d(\lambda)$ must have a dot at $i_0 = c - 2$, hence $f_\lambda(c-2) = 0$, which is a contradiction. $\square$

Proposition 5.21. *Let $\mu \vdash n$ and $\lambda \in R(\mu)$. Suppose ε is a removable box of μ with content c. By the lemma above, we know λ has a removable box ε' with content c or an addable box ε'' with content $c - 1$.*

1. *If λ has a removable box of content c but no addable box of content $c-1$, then $\lambda - \varepsilon' \in R(\mu - \varepsilon)$.*

2. *If λ has an addable box of content $c - 1$ but no removable box of content c, then $\lambda + \varepsilon'' \in R(\mu - \varepsilon)$.*

3. *Suppose λ has both a removable box of content c and an addable box of content $c - 1$.*

   (a) *If $\overline{d(\mu)}$ has dots at both $c - 2$ and c with*

      i. *no arcs connected to either dot, or*

      ii. *arcs connecting both dots to $a < c - 2$ with either no arc connected to a collapsed or an arc connecting a to $b < c - 2$ collapsed when forming $d(\lambda)$,*

      then $\lambda - \varepsilon' \in R(\mu - \varepsilon)$ and $\lambda + \varepsilon'' \notin R(\mu - \varepsilon)$.

   (b) *If $\overline{d(\mu)}$ has either*

      i. *dots at both $c - 2$ and c connected by arcs to $a < c - 2$ with an arc connecting a to $b > c$ collapsed when forming $d(\lambda)$, or*

      ii. *$c-2$ blank and more than one arc originating from $c-2$ with an arc connecting $c - 2$ to $b > c$ collapsed when forming $d(\lambda)$,*

      then $\lambda + \varepsilon'' \in R(\mu - \varepsilon)$ and $\lambda - \varepsilon' \notin R(\mu - \varepsilon)$.

Hence, it is always the case that λ has either a removable box ε' of content c with $\lambda - \varepsilon' \in R(\mu - \varepsilon)$ or an addable box ε'' of content $c-1$ with $\lambda + \varepsilon'' \in R(\mu - \varepsilon)$, but not simultaneously both.

Proof. Suppose $\lambda \in R(\mu)$, so that $s_\mu(\lambda, j) = 1$ for all $j \notin \mathbf{c}(\mu)$. We must show that $s_{\mu-\varepsilon}(\lambda \pm \tau, i) = 1$ for all $i \notin \mathbf{c}(\mu - \varepsilon)$. Note that $\mathbf{c}(\mu - \varepsilon) = (\mathbf{c}(\mu)\backslash\{c\}) \cup \{c-1\}$, so if $i \notin \mathbf{c}(\mu - \varepsilon)$, then $i \notin \mathbf{c}(\mu)$ with $i \neq c-1$ or $i = c$. We must therefore consider six cases on $i \notin \mathbf{c}(\mu - \varepsilon)$.

By Lemma 5.16, we have

$$1 = s_\mu(\lambda, i) = s_{\mu-\varepsilon}(\lambda - \varepsilon', i) = s_{\mu-\varepsilon}(\lambda + \varepsilon'', i)$$

in each of the following cases:

- $i > c$,
- $i < c - 1$ with $R_\mu(i) = \emptyset$,
- $i < \max R_\mu(i) < c - 1$, or
- $i < c - 1 < \max R_\mu(i)$ with $c \notin R_\mu(i)$.

This leaves two cases, $i = c$ and $i < c - 1 < \max R_\mu(i)$ with $c \in R_\mu(i)$, which we now check. In each case, we freely use the results of Lemma 5.16 without reference.

We proceed by three cases on λ, and consider the two cases above in each main case.

Case 1. λ has a removable box of content c but no addable box of content $c-1$.

In this case, we have $f_\lambda(c-2) = 1$, since $c-2$ is blank in $d(\lambda)$, while $f_\lambda(c) = 0$. Since $\lambda \in R(\mu)$, it must be the case that either

- $c-2$ is blank in $d(\mu)$ and no arc originating from $c-2$ is collapsed

- $d(\mu)$ has a dot at $c-2$ with both $c-2, c \in R_\mu(a)$, for some $a < c-2$, and the arc connecting a to $c-2$ in $\overline{d(\mu)}$ is collapsed when forming $d(\lambda)$.

Suppose $i = c$. First, further suppose that $c-2$ is blank in $d(\mu)$. Since $c-2 \notin \mathbf{c}(\mu)$, we have $s_\mu(\lambda, c-2) = 1$, so

$$\begin{aligned} s_{\mu-\varepsilon}(\lambda - \varepsilon', c) &= s_\mu(\lambda, c-2) - f_\lambda(c-2) + 1 \\ &= 1 - 1 + 1 \\ &= 1 \end{aligned}$$

Now, suppose $d(\mu)$ has a dot at $c-2$. Then, we have $c-2 \in R_\mu(a)$ with the arc from a to $c-2$ in $\overline{d(\mu)}$ collapsed when forming $d(\lambda)$, so $f_\lambda(x) = 0$ for all $x \in R_\mu(a) \backslash \{c-2\}$. Hence,

$$\begin{aligned} s_{\mu-\varepsilon}(\lambda - \varepsilon', c) &= 1 + \sum_{j \in R_\mu(a) \cap (c, \infty)} f_\lambda(j) \\ &= 1 + 0 \end{aligned}$$

Next, suppose $i < c-1 < \max R_\mu(i)$ with $c \in R_\mu(i)$. If $c-2$ is blank in $d(\mu)$, then $i = c-2$ and

$$s_{\mu-\varepsilon}(\lambda - \varepsilon', c-2) = f_\lambda(c-2) = 1$$

If $d(\mu)$ has a dot at $c-2$, then

$$\begin{aligned} s_{\mu-\varepsilon}(\lambda - \varepsilon', i) &= s_\mu(\lambda, i) - s_{\mu-\varepsilon}(\lambda - \varepsilon', c) + 1 \\ &= 1 - 1 + 1 \\ &= 1 \end{aligned}$$

since we showed above that $s_{\mu-\varepsilon}(\lambda - \varepsilon', c) = 1$.

Hence, $s_{\mu-\varepsilon}(\lambda - \varepsilon', i) = 1$ for all $i \notin \mathbf{c}(\mu - \varepsilon)$, so $\lambda - \varepsilon' \in R(\mu - \varepsilon)$ in this case.

Case 2. λ has an addable box of content $c-1$, but no removable box of content c.

In this case, we have $f_\lambda(c)=1$, since c is blank in $d(\lambda)$, while $f_\lambda(c-2)=0$. Since c is blank in $d(\lambda)$ but not in $d(\mu)$, it must be the case that the dot at c in $d(\mu)$ is collapsed along an arc in $\overline{d(\mu)}$ when forming $d(\lambda)$. This can happen in one of two ways:

- $c-2$ is blank in $d(\mu)$ and connected to c by an arc in $\overline{d(\mu)}$, which is collapsed when forming $d(\lambda)$, or
- $d(\mu)$ has a dot at $c-2$ with both $c-2, c \in R_\mu(a)$, for some $a<c-2$, and the arc connecting a to c in $\overline{d(\mu)}$ collapsed when forming $d(\lambda)$.

Suppose $i=c$. If $c-2$ is blank in $d(\mu)$, then $c-2 \notin \mathbf{c}(\mu)$, so

$$s_{\mu-\varepsilon}(\lambda+\varepsilon'', c) = s_\mu(\lambda, c-2) = 1$$

Now, if $d(\mu)$ has a dot at $c-2$, then $f_\lambda(x)=0$ for all $x \in R_\mu(a)\backslash\{c\}$. So,

$$\begin{aligned} s_{\mu-\varepsilon}(\lambda+\varepsilon'', c) &= f_\lambda(c) + \sum_{j \in R_\mu(a)\cap(c,\infty)} f_\lambda(j) \\ &= 1+0 \end{aligned}$$

Next, suppose $i < c-1 < \max R_\mu(i)$ with $c \in R_\mu(i)$. If $c-2$ is blank in $d(\mu)$, then $i=c-2$ and $s_{\mu-\varepsilon}(\lambda+\varepsilon'', c-2)=1$. If $d(\mu)$ has a dot at $c-2$, then

$$\begin{aligned} s_{\mu-\varepsilon}(\lambda+\varepsilon'', i) &= s_\mu(\lambda, i) - s_{\mu-\varepsilon}(\lambda+\varepsilon'', c) + 1 \\ &= 1-1+1 \\ &= 0 \end{aligned}$$

since we showed above that $s_{\mu-\varepsilon}(\lambda+\varepsilon'', c)=1$.

Hence, $s_{\mu-\varepsilon}(\lambda+\varepsilon'', i) = 1$ for all $i \notin \mathbf{c}(\mu-\varepsilon)$, so $\lambda+\varepsilon'' \in R(\mu-\varepsilon)$ in this case.

Case 3. λ has both a removable box of content c and an addable box of content $c-1$.

In this case, we have $f_\lambda(c-2) = f_\lambda(c) = 0$. Since $\lambda \in R(\mu)$, it must be the case that either

- $c-2$ is blank in $d(\mu)$ and connected to c and $b > c$ by arcs in $\overline{d(\mu)}$, with the arc connected to b collapsed,
- $d(\mu)$ has a dot at $c-2$ with no arc connected to $c-2$, or
- $d(\mu)$ has a dot at $c-2$ with $c-2, c \in R_\mu(a)$, for some $a < c-2$, and neither arc connecting a to $c-2$ nor c in $\overline{d(\mu)}$ is collapsed. This can happen in three ways:
  - an arc connecting a to $b > c$ in $\overline{d(\mu)}$ is collapsed,
  - an arc connecting a to $b < c-2$ in $\overline{d(\mu)}$ is collapsed, or
  - no arc originating from a in $\overline{d(\mu)}$ is collapsed.

For this final case, we proceed by subcases depending on the configuration of dots in $d(\mu)$.

Case 3a. $c-2$ is blank in $d(\mu)$.

Since $c-2 \notin \mathbf{c}(\mu)$, we have $s_\mu(\lambda, c-2) = 1$. Then

$$s_{\mu-\varepsilon}(\lambda+\varepsilon'', c) = s_\mu(\lambda, c-2) = 1$$

while

$$
\begin{aligned}
s_{\mu-\varepsilon}(\lambda-\varepsilon',c) &= s_\mu(\lambda,c-2)-f_\lambda(c-2)+1\\
&= 1-0+1\\
&= 2
\end{aligned}
$$

If $i<c-1<\max R_\mu(i)$ with $c\in R_\mu(i)$, then $i=c-2$ and $s_{\mu-\varepsilon}(\lambda+\varepsilon'',c-2)=1$. These calculations show that $\lambda+\varepsilon''\in R(\mu-\varepsilon)$, while $\lambda-\varepsilon'\notin R(\mu-\varepsilon)$.

Case 3b. $c-2,c\in R_\mu(a)$ with an arc connecting a to $b>c$ collapsed.

Then, $f_\lambda(x)=0$ for all $x\in R_\mu(a)\backslash\{b\}$, so

$$
\begin{aligned}
s_{\mu-\varepsilon}(\lambda-\varepsilon',c) &= 1+\sum_{j\in R_\mu(a)\cap(c,\infty)} f_\lambda(j)\\
&= 1+1\\
&= 2
\end{aligned}
$$

and

$$
\begin{aligned}
s_{\mu-\varepsilon}(\lambda+\varepsilon'',c) &= f_\lambda(c)+\sum_{j\in R_\mu(a)\cap(c,\infty)} f_\lambda(j)\\
&= 0+1
\end{aligned}
$$

since $b>c$. If $i<c-1<\max R_\mu(i)$ with $c\in R_\mu(i)$, then

$$
\begin{aligned}
s_{\mu-\varepsilon}(\lambda+\varepsilon'',i) &= s_\mu(\lambda,i)-s_{\mu-\varepsilon}(\lambda+\varepsilon'',c)+1\\
&= 1-1+1\\
&= 1
\end{aligned}
$$

since we showed above that $s_{\mu-\varepsilon}(\lambda+\varepsilon'',c)=1$. These calculations show that

$\lambda + \varepsilon'' \in R(\mu - \varepsilon)$, while $\lambda - \varepsilon' \notin R(\mu - \varepsilon)$.

Case 3c. $c-2, c \in R_\mu(a)$ with an arc connecting a to $b < c-2$ collapsed.

Then, $f_\lambda(x) = 0$ for all $x \in R_\mu(a) \backslash \{b\}$, so

$$\begin{aligned} s_{\mu-\varepsilon}(\lambda - \varepsilon', c) &= 1 + \sum_{j \in R_\mu(a) \cap (c,\infty)} f_\lambda(j) \\ &= 1 + 0 \end{aligned}$$

and

$$\begin{aligned} s_{\mu-\varepsilon}(\lambda + \varepsilon'', c) &= f_\lambda(c) + \sum_{j \in R_\mu(a) \cap (c,\infty)} f_\lambda(j) \\ &= 0 + 0 \end{aligned}$$

since $b < c-2$. If $i < c-1 < \max R_\mu(i)$ with $c \in R_\mu(i)$, then

$$\begin{aligned} s_{\mu-\varepsilon}(\lambda - \varepsilon', i) &= s_\mu(\lambda, i) - s_{\mu-\varepsilon}(\lambda - \varepsilon', c) + 1 \\ &= 1 - 1 + 1 \\ &= 1 \end{aligned}$$

since we showed above that $s_{\mu-\varepsilon}(\lambda + \varepsilon'', c) = 1$. These calculations show that $\lambda - \varepsilon' \in R(\mu - \varepsilon)$, while $\lambda + \varepsilon'' \notin R(\mu - \varepsilon)$.

Case 3d. $c-2, c \in R_\mu(a)$ with no arc connected to a collapsed.

Then, $f_\lambda(x) = 0$ for all $x \in R_\mu(a)$, so

$$\begin{aligned} s_{\mu-\varepsilon}(\lambda - \varepsilon', c) &= 1 + \sum_{j \in R_\mu(a) \cap (c,\infty)} f_\lambda(j) \\ &= 1 + 0 \end{aligned}$$

and

$$\begin{aligned} s_{\mu-\varepsilon}(\lambda+\varepsilon'',c) &= f_\lambda(c) + \sum_{j\in R_\mu(a)\cap(c,\infty)} f_\lambda(j) \\ &= 0+0 \end{aligned}$$

If $i < c-1 < \max R_\mu(i)$ with $c \in R_\mu(i)$, then

$$\begin{aligned} s_{\mu-\varepsilon}(\lambda-\varepsilon',i) &= s_\mu(\lambda,i) - s_{\mu-\varepsilon}(\lambda-\varepsilon',c)+1 \\ &= 1-1+1 \\ &= 1 \end{aligned}$$

since we showed above that $s_{\mu-\varepsilon}(\lambda+\varepsilon'',c) = 1$. These calculations show that $\lambda-\varepsilon' \in R(\mu-\varepsilon)$, while $\lambda+\varepsilon'' \notin R(\mu-\varepsilon)$.

Case 3e. $d(\mu)$ has a dot at $c-2$ which is not connected to an arc.

Since $r_\mu(c-2,c) = g_\mu(c-1)+g_\mu(c) = 1+(-1) = 0$ and $r_\mu(c-2,c-1) = g_\mu(c-1) > 0$, it follows that c is also not connected to an arc, since if it were, these calculations would show that $c-2$ has to be connected to the same blank node as c. Suppose $x \in R_{\mu-\varepsilon}(c)$ belongs to $R_\mu(a)$. If $a > c$, then since a will remain blank in $d(\mu-\varepsilon)$, we have two arcs, originating at c and a, which both terminate at x. This situation is impossible by Lemma 5.11. Since $d(\mu)$ has dots at $c-2$ and c, this means we must have $a < c-2$. Now, $r_\mu(c-2,c) = 0$ and $r_\mu(c,x) = r_{\mu-\varepsilon}(c,x) = 0$, so

$$r_\mu(a,c-2) = r_\mu(a,c-2) + r_\mu(c-2,c) + r_\mu(c,x) = r_\mu(a,x) = 0$$

and, $r_\mu(a,j) \geq 0$ for all $a < j < c-2$ since $c-2 < x$, so $c-2 \in R_\mu(a)$, which is impossible. Hence, $x \in R_{\mu-\varepsilon}(c)$ cannot be connected to an arc in $\overline{d(\mu)}$, so $f_\lambda(x) = 0$

for all $x \in R_{\mu-\varepsilon}(c)$. Then, we have

$$1 \leq s_{\mu-\varepsilon}(\lambda - \varepsilon', c) \leq 1 + \sum_{j \in R_{\mu-\varepsilon}(c)} f_\lambda(j) = 1 + 0$$

and

$$0 = f_\lambda(c) \leq s_{\mu-\varepsilon}(\lambda + \varepsilon'', c) \leq f_\lambda(c) + \sum_{j \in R_{\mu-\varepsilon}(c)} f_\lambda(j) = 0 + 0$$

so $s_{\mu-\varepsilon}(\lambda - \varepsilon', c) = 1$ while $s_{\mu-\varepsilon}(\lambda + \varepsilon'', c) = 0$. That is, we have $\lambda - \varepsilon' \in R(\mu - \varepsilon)$, while $\lambda + \varepsilon'' \notin R(\mu - \varepsilon)$.

For the last statement about uniqueness of the situation $\lambda - \varepsilon' \in R(\mu - \varepsilon)$ or $\lambda + \varepsilon'' \in R(\mu - \varepsilon)$, we simply observe that the above bulleted list exhausts all possibilities for the configuration of dots and arcs at $c - 2$ in $\overline{d(\mu)}$ which yield $\lambda \in R(\mu)$. $\square$

Corollary 5.22. *Let $\mu \vdash n$. If $\lambda \in R(\mu)$, then for all $u \in UD_n(\mu)$, there exists a unique $t \in UD_n(\lambda)$ such that $wt(t) = wt(u)$. Hence, λ and μ have exactly one common weight of length n.*

Proof. By induction on n. When $n = 1$, this is trivially true since we must have $\lambda = \mu = \square$ and there is only one up-down tableau of length 1. Suppose $n > 1$. Given $u \in UD_n(\mu)$, we have that $u_{n-1} = \mu - \varepsilon$, for some removable box ε of μ with content $c = c(\varepsilon)$. By the proposition above, there is a unique choice of $\tau \in \{-\varepsilon', \varepsilon''\}$,where $c(\varepsilon') = c$ and $c(\varepsilon'') = c-1$, for which $t_{n-1} := \lambda + \tau \in R(u_{n-1})$. By induction, there is a unique up-down tableau $(t_1, \ldots, t_{n-1})$ which has the same weight as $(u_1, \ldots, u_{n-1})$. By taking $t_n := \lambda$, we obtain a unique up-down tableau t of length n for λ with the same weight as u. $\square$

Theorem 5.23. *If $\mu \vdash n$ and $\lambda \in R(\mu)$, then $[\Delta_n(\lambda) : L_n(\mu)] \in \{0, 1\}$.*

Proof. Let $u \in UD_n(\mu)$. If $[\Delta_n(\lambda) : L_n(\mu)] > 1$, then we can find a chain of submodules, $M_1 < M_2 < M_3 < M_4$, of $\Delta_n(\lambda)$ such that $M_2/M_1 \cong M_4/M_3 \cong L_n(\mu)$.

By Lemma 5.7 and Proposition 5.9, M_2, and hence M_3, contains a weight vector of weight $\mathrm{wt}(u)$. Also, by Lemma 5.7, we can find a weight vector of weight $\mathrm{wt}(u)$ in M_4 which is not contained in M_3. This means $\dim\left(\Delta_n(\lambda)\right)_{\mathrm{wt}(u)} \geq 2$, so, by Lemma 5.8, we can find two up-down tableau $t, t' \in UD_n(\lambda)$ of weight $\mathrm{wt}(\mu)$, which contradicts the previous corollary. □

5.5 A Second Look at the Decomposition Matrix for $B_n(0,-1)$

Recall from Theorem 5.2 that the decomposition matrix, D_n, for $B_n(0,-1)$ has the following block form:

$$\begin{pmatrix} I & 0 \\ * & D_{n-2} \end{pmatrix}$$

Understanding this matrix thus reduces, by induction, to understanding the starred block. The decomposition numbers which appear in this block are precisely those $[\Delta_n(\lambda) : L_n(\mu)]$ with $\mu \vdash n$ and $|\lambda| < n$. In the previous section, we studied this block using the combinatorics of arc diagrams and we found, in Theorems 5.19 and 5.23, that

Theorem. *If $\mu \vdash n$ and $|\lambda| < n$ with $[\Delta_n(\lambda) : L_n(\mu)] \neq 0$, then $\lambda \in R(\mu)$ and $[\Delta_n(\lambda) : L_n(\mu)] = 1$.*

In order to fully understand the starred block, we need to reverse this implication, as in the conjecture below.

Conjecture 5.24. *Let $\mu \vdash n$ and $\lambda \in \Lambda(n)$. If $\lambda \in R(\mu)$, then $[\Delta_n(\lambda) : L_n(\mu)] = 1$.*

In light of Theorem 5.23, we need only show $[\Delta_n(\lambda) : L_n(\mu)] \neq 0$. This conjecture has been proven independently by Coulembier and Ehrig in [13] using combinatorics of skew Young diagrams, however we seek a proof using only the combinatorics of weights and arc diagrams. This claim would follow from the following stronger conjecture.

Conjecture 5.25. *If $\mu \vdash n$ and $\lambda \in R(\mu)$, then $Hom_{B_n}(L_n(\mu), \Delta_n(\lambda)) \neq 0$.*

Proving this result will proceed by induction on n and will require adjointness of $i - \mathrm{ind}_{n-1}^{n}$ and $i - \mathrm{res}_{n-1}^{n}$ as well as a careful study of the effect of $i - \mathrm{ind}_{n-1}^{n}$ on the simple B_n-modules. With this result, the above claim about decomposition numbers follows immediately.

Chapter 6

Future Work

The present work inspires many additional questions about the marked Brauer algebra and its representation theory. As we saw in Section 2.4, the center of $B_n(0,-1)$ is yet unknown in general, although it is presumed to be trivial for all $n \neq 3$. It will also be interesting to extend the results of this work to fields other than the complex numbers. Coulembier [12] provides several results over fields whose characteristic is larger than n, however fields with characteristic between 2 and n remain unexplored.

Proving Conjecture 5.25 is of immediate interest since this claim, along with Theorem 5.2, provides a complete description of the decomposition matrix for $B_n(0,-1)$ in terms of arc diagrams. Since this matrix can be used to obtain the Cartan decomposition matrix for B_n (see [4], Section 1.9), we should be able to provide an alternate proof using arc diagrams of the block result obtained by Coulembier in [12] and the Cartan multiplicities computed by Coulembier and Ehrig in [13].

The decomposition data reported in Section 5.1 provides evidence that the composition factors $L_n(\gamma)$ of $\Delta_n(\lambda)$ are precisely those whose label γ can be obtained from λ by adding t-many "dominoes" of the form $\square\square$, for $t \in \left\{0, 1, \ldots, \frac{1}{2}(n - |\lambda|)\right\}$, in all ways which result in a valid partition. If true, this gives an easy characterization of the partitions $\gamma \vdash n, n-2, \ldots, |\lambda|$ which label composition factors of $\Delta_n(\lambda)$.

While these decomposition numbers can be obtained from the decomposition matrix, we suggest that an alternate definition of arc diagrams (see [1]) in which the arcs originate (on the left) from dots and terminate (on the right) at blank spaces, along with a suitable collapsing procedure may provide a more elementary way to obtain these multiplicities. This would give a more direct way to obtain the composition factors, and hence the composition series, of any standard module.

Somewhat related to this problem is that of describing the spaces of B_n-module homomorphisms between two standard modules. The standard based structure provides the partial result

$$\dim \operatorname{Hom}_{B_n}(\Delta_n(\lambda), \Delta_n(\mu)) = \delta_{\lambda\mu}$$

when $\lambda \geq \mu$ in the partial order on Λ (see Theorem 3.2, Property 1). However, we obtain little information when $\lambda < \mu$. Recent conversations with Gordon Brown suggest that we may be able to provide descriptions of these spaces using webs, which are another type of diagrammatic object. See [36] for some background information on webs.

It is also of interest to obtain more information about the simple B_n-modules. In particular, we would like to know the dimensions of the simples and, if possible, provide a concrete way to construct these modules. Partial results are available for the ordinary Brauer algebra in [15] and [40], and these techniques may carry over to our setting. Empirical evidence along with Corollary 5.22 suggest that we may be able to compute the dimensions of simple modules in terms of the up-down tableaux. Recent work by Benkart and Moon [3] may provide further insight into this approach to the dimension problem.

Finally, we wish to further explore the connection between $B_n(0, -1)$ and the Lie superalgebra $\mathfrak{p}(r)$ (see [29] and [33]). For instance, it will be interesting to understand the kernel and image of the algebra homomorphism

$$B_n(0,-1) \to \mathrm{End}_{\mathfrak{p}(r)}\left(W^{\otimes n}\right)$$

where W is a superspace as in the Introduction. Recent work [1] has drawn attention to $\mathfrak{p}(r)$, whose representation theory is not well understood. We hope that the results contained herein along with those of Coulembier [12] and Coulembier-Ehrig [13] will become useful tools in the study of the representation theory of $\mathfrak{p}(r)$ via the Schur-Weyl duality linking these two algebras.

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