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A STUDY OF

GENERALIZED ABSOLUTE NEIGHBORHOOD RETRACTS

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CHAPTER I

INTRODUCTION

The notion of absolute neighborhood retract (ANR) was introduced by Borsuk in 1932. Borsuk considered only compact metric spaces, but the theory has been generalized. In this paper only metric spaces will be used. Since Borsuk's definition, the theory of ANR's has developed to occupy an important position in the area of general topology. Major works in the area include books by Hu [H], and Borsuk [B1].

ANR's can be defined in terms of extensions of maps or retractions.

1.1 Definition. An absolute neighborhood retract (ANR) is a space Y such that for every closed subset A of a space X, if $f : A \to Y$ is a map, then f has an extension over an open subspace U of X which contains A. Y is an absolute retract (AR) if every map $f : A \to Y$ has an extension over all of X.

1.2 Definition. A space Y is an ANR if whenever Y is embedded as a closed subset of a space X, then there is an open subset U of X containing Y and a map $r : U \to Y$ so that $r|Y$ is the identity map. The map r is called a retraction. Y is an AR if there is a retraction $r : X \to Y$.

Examples of ANR's include Euclidean space, all polyhedra, and the Hilbert cube. Any open subspace of an ANR is also an ANR.

Generalized ANR's were introduced by Noguchi in 1953. Also referred to as approximate ANR's, this class of spaces has been studied with various modifications and characterized in different settings. Borsuk called them NE-sets (nearly extendable sets). Mardesic studied approximate polyhedra, that is, any space X so that for $\varepsilon > 0$, there exists a polyhedron P and maps $f : X \to P$ and $g : P \to X$ so that gf is ε-near to the identity on X. Clapp defined an approximate absolute neighborhood retract (AANR) as a space X so that if X is embedded in a metric space Y, $i : X \to Y$, then for $\varepsilon > 0$, there exists a neighborhood U of $i(X)$ and a map $r : U \to i(X)$ so that $d(ri, i) < \varepsilon$. Čerin classified generalized ANR's as P-e-movable compacta. The ANR's, approximate ANR's, etc. used in this paper will be compacta unless otherwise noted.

All of the generalized ANR's mentioned above have been shown to be equivalent and will be referred to in this paper as AANR's. Recently Patten studied a more restrictive generalized ANR which he called a quasi-ANR (q-ANR).

One broad and fairly difficult problem about ANR's that has been studied is the following: Under what conditions is the continuous image of an ANR an ANR? Kozlowski has some results in this area when the map is cell-like. Patten indicates in his work that insight into the problem might be gained by studying the same situation for generalized ANR's. His paper contains a nice result in that direction: The image of a q-ANR under a refinable map is a q-ANR.

The intent of this paper is to add to the known theory of generalized ANR's and to indicate some of the problems encountered in attempts to extend results about ANR's to generalized ANR's

Chapter II gives several characterizations of q-ANR's. Some of these are generalizations of known theorems for the ANR case. Another more interesting characterization relates AANR's and q-ANR's. An AANR X and a q-ANR differ in that the maps connecting X and an ANR do not have to be surjective. Also, X need not be locally connected. The fact that a q-ANR is locally connected seems to give much more structure to the space. Theorem 2.16 shows that a locally connected AANR is also a q-ANR.

ANR's have important homotopy and map extending properties. It is well known that two maps from a metric space into an ANR which are sufficiently close are homotopic. An analogous theorem for AANR's is proved in Chapter III. Another property possessed by any ANR is that any map from a closed subset of a metric space into an ANR can be extended to a neighborhood of the closed subset. This property is also generalized, with appropriate modifications, to AANR's in Chapter III.

While these new theorems show that some of the nice properties of ANR theory can be extended to generalized ANR's, the theorems seem to lack the utility of the ANR case. The reason is that the "extension maps" are not actually extensions, but maps which can be made arbitrarily close to the given maps.

Chapter III also contains the following theorem concerning maps on approximate ANR's.

Theorem 3.8. Let X be an AANR and let $f : X \to Y$ be a cell-like map. Then the following are equivalent:

(i) Y is an AANR.

(ii) f is approximately invertible.

(iii) Y is approximately countable dimensional.

The theorem is, in part, a generalization of a result by Kozlowski which says that if $f : X \to Y$ is a cell-like map, X is an ANR, and Y is a countable dimensional space, then Y is an ANR. The proof is accomplished by using relations to generate continuous functions, a technique developed by Ancel in [A1].

Chapter IV contains mostly examples which illustrate that certain features of ANR theory are not valid when working with approximate ANR's. Most notable among these is an example, due to Borsuk, which shows that a local AANR need not be an AANR.

A hyperspace 2^X, is the set of all compact subspaces of the metric space X. The study of hyperspaces can be traced back to the early 1900's and the works of Hausdorff and Vietoris. In most of this work, 2^X was topologized by the Hausdorff metric. Research on hyperspaces became strongly linked to the study of ANR's when, in 1954, Borsuk defined the homotopy metric on 2^X. This metric has a clear meaning only when the compact ANR subsets of X are considered. For X a finite dimensional space, 2^X, topologized by the homotopy metric, is complete.

Another hyperspace metric, also defined by Borsuk, is the metric of continuity. It has been shown by Čerin that when the hyperspace of AANR subsets of a space X is topologized by the metric of continuity, then 2^X is topologically complete if and only if X is topologically complete.

In Chapter V, the problem of obtaining results for the metric of continuity which are similar to known results for the homotopy metric is explored. This investigation points out some major differences in the two metrics. A new hyperspace metric is defined in Chapter V in an attempt to obtain a metric on 2^X which contains some of the desirable properties of both of Borsuk's metrics. One result concerning the new metric gives a condition that enables one to determine when two ANR's lie in different path components of 2^X.

CHAPTER II

SOME CHARACTERIZATIONS OF q-ANR's

The present chapter is devoted to naming conditions which are necessary and sufficient for a metrizable space to be a q-ANR. In stating definitions and theorems the notion of near maps will be used.

2.1 Definition. Let α be a covering of a space Y. Two maps $f, g : X \to Y$ are said to be α-near if for any $x \in X$, there exists a set $A \in \alpha$ so that $f(x), g(x) \in A$.

2.2 Definition. A space Y is a q-ANR if whenever Y is closed in a space X, and α is an open cover of Y, there exists a neighborhood U of Y in X and a surjective map $h : U \to Y$ such that $h|Y$ is α-near the identity map on Y.

In [H] the notions of ANR and ANE are shown to be equivalent on a wide range of spaces. It is possible to define a quasi-ANE (q-ANE) and establish a similar relationship with q-ANR's.

2.3 Definition. A space Y is a q-ANE if for every open cover α of Y and every surjective map $f : A \to Y$ where A is a closed subset of a metric space X, there exists a neighborhood U of A and a surjective map $h : U \to Y$ such that f and $h|A$ are α-near.

2.4 Theorem. A space Y is a q-ANR if and only if Y is a q-ANE.

Proof: Assume Y is a q-ANE. Let $i : Y \to X$ be an embedding of Y as a closed subset of X. Let α be an open cover of Y. Then $i^{-1}: Y \to Y$ is an onto map from a closed subset of X. Since Y is a q-ANE, there exists a neighborhood U of Y in X and a surjective map $g : U \to Y$ so that i^{-1} and $g|Y$ are $i^{-1}(\alpha)$-near. Define $h : U \to Y$ by $h = ig$. Then h is surjective and $h|Y$ is α-near the identity on Y. Thus, Y is a q-ANR.

Now assume Y is a q-ANR. Embed Y as a closed subset of the convex hull, Z, of Y in $C(Y)$ = {bounded continuous real valued functions on Y}. Let α be an open cover of Y and let $f : A \to Y$ be an onto map where A is a closed subset of a metric space X. Since Y is a q-ANR and is a closed subset of Z, there exists an open neighborhood V of Y in Z and an onto map $g : V \to Y$ such that $g|Y$ is α-near to i, the identity map on Y. Consider f as a map from A into $C(Y)$. Then by Dugundji's extension theorem, [D1], page 188, there exists an extension $P : X \to C(Y)$ of f such that $P(X)$ is contained in the convex hull of $f(A)$. Therefore $P(X)$ is a subset of Z. Let $U = P^{-1}(V)$. Then U is a neighborhood of A in X. Define $h : U \to Y$ by $h(x) = gP(x)$. Clearly h is onto. To show that $h|A$ is α-near f, let $x \in A$. Then since P is an extension of f, $P(x) = f(x)$ in Y. Since $g|Y$ is α-near i, there exists $O \in \alpha$ such that $g(f(x))$ and $f(x)$ are both in O. Thus $g(f(x)) = gP(x) = h(x)$ and $f(x)$ are both in O. Hence $h|A$ and f are α-near. Therefore, Y is a q-ANE and the theorem is proved.

In [P2], page 165, Patten gives a characterization of q-ANR's which is stated here without proof.

2.5 Theorem. A space Y is a q-ANR if and only if whenever Y is a closed subset of a metric space, then for every $\varepsilon > 0$, there exists a neighborhood U of Y and an onto map $r : U \to Y$ such that $d(r(x),x) < \varepsilon$ for all $x \in Y$.

Actually, the definition of q-ANR's used in this paper is just a restatement of 2.5 using near maps.

Before stating the next theorem, the following definitions will be needed.

2.6 Definition. Let α be a covering of a space Y. A homotopy $h_t : X \to Y$ is an α-homotopy if for each $x \in X$, there exists $U \in \alpha$ so that $h_t(x) \in U$ for all t in $I = [0,1]$.

2.7 Definition. Let $\mathcal{U}$ be a cover of a space Y and let f be a mapping of Y into a space X. The map f is a $\mathcal{U}$-map if for every $x \in X$, there exists a neighborhood V of x in X and $U \in \mathcal{U}$ so that $f^{-1}(V)$ is a subset of U.

Using Patten's characterization, the following can be established.

2.8 Theorem. A space Y is a q-ANR if and only if whenever Y is embedded as a closed subset of a metric space, then for each open cover $\mathcal{U}$ of Y there exists a $\mathcal{U}$-homotopy $h_t : Y \to Y$ ($0 \leqq t \leqq 1$) such that h_0 is a $\mathcal{U}$-map and there exists a neighborhood M of Y and an onto map $h : M \to Y$ such that h is an extension of h_1.

Proof: Assume Y is a q-ANR. Let Y be embedded as a closed subset of a metric space. Let $\mathcal{U}$ be an open cover of Y. Let $\mathcal{V}$ be a star refinement of $\mathcal{U}$. Then there is a neighborhood M of Y and an onto map $h : M \to Y$ such that $h|Y$ is $\mathcal{V}$-near the identity on Y. In order to establish that $h|Y$ is a $\mathcal{U}$-map, let $x \in Y$. Then x and h(x) are both in some $V \in \mathcal{V}$. Since $\mathcal{V}$ star refines $\mathcal{U}$, there exists a $U \in \mathcal{U}$ so that the star of V is contained in U. Let $y \in (h|Y)^{-1}(V)$. Then $h(y)$, $y \in V'$ for some $V' \in \mathcal{V}$. Thus since $h(y) \in V$, V' is contained in the star of V. Therefore, $y \in U$. So $(h|Y)^{-1}(V)$ is contained in U and $h|Y$ is a $\mathcal{U}$-map.

Define $h_t : Y \to Y$ by $h_t(y) = h(y)$ for all $y \in Y$ and all $0 \leqq t \leqq 1$. Thus h_0 is a $\mathcal{U}$-map and h_1 extends to an onto map $h : M \to Y$.

Now assume the condition is true. It will be shown that Y is a q-ANR by using Theorem 2.5. Let Y be embedded as a closed subset of a metric space. Let $\varepsilon > 0$. Let $\mathcal{U}$ be an open cover of Y with $\text{mesh}(\mathcal{U}) < \varepsilon/2$. Thus, by assumption, there exists a $\mathcal{U}$-homotopy $h_t : Y \to Y$ such that $d(h_0, 1) < \varepsilon/2$, where 1 is the identity map on Y, and there exists a neighborhood M of Y and an onto extension h of h_1 to M. Let $y \in Y$. Since h_t is a $\mathcal{U}$-homotopy, there exists $U \in \mathcal{U}$ such that $h_t(y) \in U$ for all t. In particular, $h_1(y) \in U$ and $h_0(y) \in U$. Therefore, $d(h_1(y), h_0(y)) < \varepsilon/2$. Now $d(h(y), y) \leqq d(h(y), h_1(y)) + d(h_1(y), h_0(y)) + d(h_0(y), y) < 0 + \varepsilon/2 + \varepsilon/2 = \varepsilon$. Thus there is a neighborhood M of Y and an onto map $h : M \to Y$ such that $d(h(y), y) < \varepsilon$ for all $y \in Y$. Hence, by 2.5, Y is a q-ANR.

2.9 Definition. Let $\mathcal{U}$ be a covering of a space Y. Let $|K|$ be a simplicial polytope. Let $|L|$ be a subpolytope of $|K|$ which contains all vertices of $|K|$. A partial realization of $|K|$ in Y relative to $\mathcal{U}$ defined on $|L|$ is a map $f: |L| \to Y$ such that for every closed simplex S of $|K|$, there exists a $U \in \mathcal{U}$ such that $f(|L| \cap S)$ is a subset of U. If $|L| = |K|$, then f is called a full realization of $|K|$ in Y relative to $\mathcal{U}$.

Notation: Let $\mathcal{U}$ be an open cover of Y. Let $\mathcal{V}$ be an open refinement of $\mathcal{U}$. The statement that every partial realization of a simplicial polytope $|K|$ in Y relative to $\mathcal{V}$ extends to a full realization of $|K|$ in Y relative to $\mathcal{U}$ will be denoted by $L(\mathcal{V},\mathcal{U})$.

It is a theorem of Dugundji [D2] that X is an ANR if and only if for every open cover $\mathcal{U}$ of X there exists an open cover $\mathcal{V}$ of X which refines $\mathcal{U}$ such that $L(\mathcal{V},\mathcal{U})$ is true. This theorem can be generalized to q-ANR's by using, again adapted from [P1], the following characterization of q-ANR's.

2.10 Theorem. A space Y is a q-ANR if and only if for every open cover $\mathcal{U}$ of Y, there exists an ANR P and surjective maps $f : Y \to P$, $g : P \to Y$ so that gf is $\mathcal{U}$-near the identity map on Y.

In order to state efficiently the next theorem, the following notation will be adopted.

Notation: If $\mathcal{U}$ is an open cover of a space Y and $\mathcal{V}$ is an open refinement of $\mathcal{U}$ covering Y, then $qL(\mathcal{V},\mathcal{U})$ is the following statement: If $f : |L| \to Y$ is a partial realization of a simplicial polytope $|K|$ in Y relative to $\mathcal{V}$, then there is a full realization $g : |K| \to Y$ relative to $\mathcal{U}$ such that $g\big|\,|L|$ is $\mathcal{U}$-near f.

2.11 Theorem. Let Y be a Peano continuum. Then Y is a q-ANR if and only if for every open cover $\mathcal{U}$ of Y, there exists an open refinement $\mathcal{V}$ of $\mathcal{U}$ covering Y such that $qL(\mathcal{V},\mathcal{U})$ is true.

Proof: Assume Y is a q-ANR. Let $\mathcal{U}$ be an open cover of Y. There exists an ANR A and maps $h : Y \to A$, $k : A \to Y$ such that kh is $\mathcal{U}$-near the identity on Y. Let $\alpha = k^{-1}(\mathcal{U})$. Since α is an open cover of the ANR A, there exists an open refinement β of α such that $L(\beta,\alpha)$

holds. Let $\mathcal{V} = h^{-1}(\beta) \cap \mathcal{U} = \{M \cap N \mid M \in h^{-1}(\beta),\ N \in \mathcal{U},\ M \cap N \neq \phi\}$. Let $f : |L| \to Y$ be a partial realization in Y of a simplicial polytope $|K|$ relative to $\mathcal{V}$. Then $hf : |L| \to A$ is a partial realization in A of $|K|$ relative to β. Therefore hf extends to a full realization $F : |K| \to A$ relative to α. Define $g : |K| \to Y$ by $g = kF$. To see that g is a full realization relative to $\mathcal{U}$, let σ be a closed simplex of $|K|$. There exists $O \in \alpha$ such that $F(\sigma) \subseteq O$. By the definition of α, $O = k^{-1}(U)$ for some $U \in \mathcal{U}$. Hence, $g(\sigma) = kF(\sigma) \subseteq U$. Note that $g||L| = kF||L| = khf||L|$. Thus, since kh is $\mathcal{U}$-near the identity on Y, $g||L|$ is $\mathcal{U}$-near f. Hence, $qL(\mathcal{V},\mathcal{U})$ holds.

For the converse, assume that for every open cover $\mathcal{U}$ of Y, there exists an open refinement $\mathcal{V}$ of $\mathcal{U}$ covering Y such that $qL(\mathcal{V},\mathcal{U})$ is true. To show that Y is a q-ANR, let $\varepsilon > 0$. Let $\mathcal{U}$ be an open cover of Y with $\text{mesh}(\mathcal{U}) < \varepsilon/3$. Let $\mathcal{V}$ be an open refinement of $\mathcal{U}$ covering Y such that $qL(\mathcal{V},\mathcal{U})$ holds. Let $\mathcal{M}_0$ be a basis of the Peano space Y such that for each $M \in \mathcal{M}_0$, $\bar{M}$ is a Peano space. This choice is possible due to a theorem in [HS], page 219. Let $\mathcal{M} = \{M_1, \ldots, M_n\}$ be a finite subcollection of $\mathcal{M}_0$ so that $\mathcal{M}$ covers Y and $\bar{\mathcal{M}} = \{\bar{M} \mid M \in \mathcal{M}\}$ refines $\mathcal{V}$. By [H-W], page 73, there is an onto $\mathcal{V}$-mapping $f_0 : Y \to P_0$ where P_0 is a subpolytope of the nerve of $\mathcal{V}$. Let $L = \{\text{vertices of } P_0\}$. For each $V \in \mathcal{V}$, let $y_v \in V$. Define $g_0 : |L| \to Y$ by $g_0(V) = y_v$. Then g_0 is a partial realization of P_0 relative to $\mathcal{V}$. Thus there is a full realization $h_0 : P_0 \to Y$ relative to $\mathcal{U}$ so that $h_0||L|$ is $\mathcal{U}$-near g_0.

Since h_0 may not be an onto map, an extension of h_0 must be constructed that is surjective. To do this, relabel the elements of $\mathcal{M}$ so that $M_1, \ldots, M_m$ are the only elements of $\mathcal{M}$ not covered by $h_0(P_0)$.

Since $\mathcal{M}$ refines $\mathcal{V}$, for each $i = 1, \ldots, m$ there is a $V_i \in \mathcal{V}$ so that $\bar{M}_i \subseteq V_i$. To each vertex V_i, $i = 1, \ldots, m$, in the nerve of $\mathcal{V}$ attach an interval as follows:

Let J be m disjoint copies of $[0,1]$. Let 0_i denote 0 in the ith copy of $[0,1] = J_i$. Let $A = \{0_1, \ldots, 0_m\}$. Define $\Psi : A \to P_0$ by $\Psi(0_i) = V_i$. Let P be the adjunction space $J \cup_\Psi P_0$.

Since f_0 is a $\mathcal{V}$-mapping, for each $i = 1, \ldots, m$, there is a neighborhood V_i' in P_0 of V_i such that $f_0^{-1}(V_i') \subseteq V_i$. Clearly, the V_i''s can be chosen to be disjoint. Also, it may be assumed that V_i' is a cone neighborhood of V_i. Each interval in the cone structure of V_i' can be thought of as $[-1,0]$. For $i = 1, \ldots, m$, define an onto map $m_i : V_i' \to V_i' \cup J_i$ as follows:

For $x \in V_i'$, $x \in [-1,0]$. Let $m_i(x) = 1 + 2x$. Then each $[-1,0]$ in V_i' is mapped onto $[-1,1]$ with $m_i(-1) = -1$ and $m_i(0) = 1$.

Define $F : P_0 \to P$ by
$$F(x) = \begin{cases} x & \text{if } x \in P_0 - (\bigcup_{i=1}^m V_i') \\ m_i(x) & \text{if } x \in V_i'. \end{cases}$$

Then F is surjective. Define $f : Y \to P$ by $f = Ff_0$. Thus f is surjective.

By an application of the Hahn-Mazurkiewicz Theorem, for $i = 1, \ldots, m$, there exists a surjective map $k_i : J_i \to \bar{M}_i$. Furthermore, k_i can be defined so that $k_i(V_i) = h_0(V_i)$. Define $h : P \to Y$ by

$$h(x) = \begin{cases} h_0(x) & \text{if } x \in P_0 \\ k_i(x) & \text{if } x \in J_i. \end{cases}$$

Then h is a surjective map. Thus there is an ANR P, and onto maps $f : Y \to P$, $h : P \to Y$.

Finally, it must be shown that hf is ε-near to the identity on Y. Let $x \in Y$. There are two cases to consider.

Case I: $f(x) \in P - (\bigcup_{i=1}^{m} J_i)$.

Now $x \in V$ for some $V \in \mathcal{V}$. Let σ be a closed simplex of P_0 containing $f(x)$ and the vertex V. Since h_0 is a full realization of P_0 relative to $\mathcal{U}$, there exists some $U \in \mathcal{U}$ so that $h_0(\sigma) \subseteq U$. Therefore, $d(h_0(V), h_0(f(x))) < \varepsilon/3$. Recall that $g_0(V) = y_v \in Y$. Therefore, $d(y_v, x) < \varepsilon/3$. Also, since h_0 is $\mathcal{U}$-near g_0, $d(y_v, h_0(V)) < \varepsilon/3$. Thus, from the above inequalities, $d(x, hf(x)) = d(x, h_0(f(x))) \leq d(x, y_v) + d(y_v, h_0(V)) + d(h_0(V), h_0(f(x))) < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$.

Case II: $f(x) \in J_i$ for some $i \in \{1, \ldots, m\}$.

In this case $f_0(x) \in V_i'$. Therefore, since $f_0^{-1}(V_i') \subseteq V_i$, $x \in V_i$. Since $f(x) \in J_i$, $h(f(x)) = k_i(f(x)) \in V_i$. So both x and h(f(x)) are in V_i. Thus, $d(x, h(f(x))) < \varepsilon/3 < \varepsilon$.

In either case x is within an ε distance of hf(x). Hence, Y is a q-ANR and the theorem is proved.

The final characterization of q-ANR's given here will be in terms of AANR's. The proof can be accomplished using the same technique as in Theorem 2.11. Recall the following definition.

2.12 Definition. A space Y is an approximate absolute neighborhood retract (AANR) if for an embedding $i : Y \to X$ where X is a metric space, then for every $\varepsilon > 0$, there exists a neighborhood U of i(Y) and a map $r : U \to i(Y)$ so that $d(ri, i) < \varepsilon$.

In [M], AANR's were characterized as approximate polyhedra.

2.13 <u>Definition</u>. A space Y is an <u>approximate polyhedron</u> if for every $\varepsilon > 0$, there exists a polyhedron P and maps $f : Y \to P$, $g : P \to Y$ so that gf is ε-near the identity on Y.

2.14 <u>Lemma</u>. If Y is a locally connected AANR, then each component of Y is a q-ANR.

Proof: Let Y be a connected, locally connected AANR. Then, under the assumption of compactness, Y is a Peano space. Let $\varepsilon > 0$. Since, by [M], Y is an approximate polyhedron, there exists a polyhedron P_0, and maps $f_0 : Y \to P_0$, and $g_0 : P_0 \to Y$ so that $g_0 f_0$ is $\varepsilon/2$-near the identity on Y.

Let $\beta(P_0)$ be a subdivision of P_0 so that the diameter of each simplex in $\beta(P_0)$ is less than $\varepsilon/2$. Let $\sigma_1, \sigma_2, \ldots, \sigma_k$ be a listing of all the simplices of $\beta(P_0)$ so that $\dim\sigma_1 \geqq \dim\sigma_2 \geqq \cdots \geqq \dim\sigma_k$.

Consider σ_1. If $\sigma_1 \subseteq \text{image}(f_0)$, let $\rho_1 : |\beta(P_0)| \to |\beta(P_0)|$ be the identity map and let $P_1 = P_0$. Otherwise, choose $x \in \text{int}(\sigma_1)$ so that x is not in $\text{image}(f_0)$. Let $r_1 : \sigma_1 - \{x\} \to \text{bdry}(\sigma_1)$ be a radial retraction. Define $\rho_1 : |\beta(P_0)| \to |\beta(P_0)|$ as follows:

$$\rho_1(y) = \begin{cases} y & \text{if } y \text{ is not in } \sigma_1 \\ r_1(y) & \text{if } y \text{ is in } \sigma_1 \end{cases} \qquad \text{for all } y \in \beta(P_0).$$

Let $P_1 = P_0 - \text{int}(\sigma_1)$. Define $f_1 : Y \to P_1$ by $f_1 = \rho_1 f_0$.

Next consider σ_i. If $\sigma_i \subseteq \text{image}(f_0)$, let $\rho_i : |P_{i-1}| \to |P_{i-1}|$ be the identity map and let $P_i = P_{i-1}$. Otherwise, let $x \in \text{int}(\sigma_i)$ so that x is not in $\text{image}(f_0)$. Let $r_i : \sigma_i - \{x\} \to \text{bdry}(\sigma_i)$ be a radial retraction. Define $\rho_i : |P_{i-1}| \to |P_{i-1}|$ as follows:

$$\rho_i(y) = \begin{cases} y & \text{if } y \text{ is not in } \sigma_i \\ r_i(y) & \text{if } y \text{ is in } \sigma_i \end{cases} \qquad \text{for all } y \in P_{i-1}.$$

Let $P_i = P_{i-1} - \text{int}(\sigma_i)$. Define $f_i : Y \to P_i$ by $f_i = \rho_i f_{i-1}$.

Let $f : Y \to P_k$ be defined by $f = f_k$. Let $g : P_k \to Y$ be defined by $g = g_0 | P_k$. Then f is an onto map. It is straightforward to verify that gf is ε-near the identity on Y. Using the same techniques as in Theorem 2.11, enlarge, by forming an adjunction space, P_k to an ANR P, and extend the maps to surjections, $f : Y \to P$, $g : P \to Y$. Thus Y is a q-ANR.

2.15 Corollary. If Y is a locally connected AANR, then Y is a q-ANR.

Proof: A compact, locally connected space has only a finite number of components.

2.16 Theorem. A space Y is a q-ANR if and only if Y is a locally connected AANR.

Proof: As was pointed out in [P1], page 72, every q-ANR is an AANR. The converse is contained in Corollary 2.15.

CHAPTER III

HOMOTOPY PROPERTIES AND EXTENDING MAPS ON GENERALIZED ANR's

Some of the more important elements of the theory of ANR's deal with homotopy properties and extensions of maps. Any two maps into an ANR which are sufficiently close must be homotopic. Also if one of the maps extends to a larger space, then so does the other one. This chapter contains generalizations of the above properties to generalized ANR's. The theorems are, of course, much weaker than their ANR counterparts, but are stated here to give a completeness to the theory of generalized ANR's. Chapter III also includes what will be referred to as the cell-like mapping theorem.

Theorem 3.1 gives what might be thought of as a quasi-homotopy extension property. The proof also shows a good application of Theorem 2.4.

3.1 Theorem. Let Y be a q-ANR. If α is an open cover of Y, $F : X \to Y$, and $h_t : A \to Y$ $(0 \leqq t \leqq 1)$ where A is a closed subset of the metric space X, and h_t is an α-homotopy with $h_0 = F|A$, then there is an α-homotopy $H_t : X \to Y$ $(0 \leqq t \leqq 1)$ such that H_0 is α-near F and $H_t|A$ is α-near h_t for all $t \in [0,1]$.

Proof: Let $P = X \times I$. Let $T = (X \times \{0\}) \cup (A \times [0,1])$. Define

a map $H : T \to Y$ as follows:
$$H(x,t) = \begin{cases} F(x) & \text{if } t = 0 \\ h_t(x) & \text{if } t > 0,\ x \in A. \end{cases}$$
By Theorem 2.4, Y is a q-ANE. Thus, since T is a closed subset of P, there exists an open neighborhood U of T and a map $G : U \to Y$ such that $G|T$ is α-near H. So $G|T$ has an extension, namely G, to an open subset of P containing $A \times [0,1]$. Therefore, by Lemma 2.1 of [H], page 116, G has an extension over all of P. Call the extension G. Define $H_t : X \to Y$ $(0 \leqq t \leqq 1)$ by $H_t(x) = G(x,t)$ for all $x \in X$. Then H_0 is α-near F, and $H_t|A$ is α-near h_t for all $t \in [0,1]$.

Theorem 2.16 says that a locally connected AANR is a q-ANR. Thus, when working in locally connected spaces, a given theorem about q-ANR's can be proved as a corollary to an AANR theorem without the burden of showing that maps are surjective. Therefore, the emphasis here is shifted to AANR's.

3.2 <u>Theorem</u>. If Y is an AANR and α is an open cover of Y, then there exists an open refinement β of α such that if $f, g : X \to Y$ are β-near maps, then there is an α-homotopy $h_t : X \to Y$ $(0 \leqq t \leqq 1)$ with h_0 α-near f, and h_1 α-near g.

Proof: Embed Y as a closed subset of its convex hull Z in $C(Y)$. Let α be an open cover of Y. Let α_0 be a star refinement of α. Since Y is an AANR, there exists a neighborhood U of Y and a map $h : U \to Y$ such that $h|Y$ is α_0-near the identity on Y. Now $h^{-1}(\alpha_0) = \{h^{-1}(V) \mid V \in \alpha_0\}$ is an open cover of U. Since U is an open subset of a locally convex space, an open refinement γ of $h^{-1}(\alpha_0)$ can be constructed so that each set in γ is convex. Let $\beta = \{O \cap Y \mid O \in \gamma\}$. To see that β refines α, let $O \cap Y \in \beta$ where $O \in \gamma$. Since γ refines $h^{-1}(\alpha_0)$, there exists $A \in \alpha_0$ so that $O \subseteq h^{-1}(A)$. Let $x \in O \cap Y$. Then $h(x) \in A$. Since h is α_0-near the identity on Y, there exists $B \in \alpha_0$ so that x and $h(x)$ are in B. Therefore $A \cap B \neq \phi$. Hence $A \cup B \subseteq A^* = \bigcup\{B \in \alpha_0 \mid A \cap B \neq \phi\}$. Thus $O \cap Y \subseteq A^*$. Since α_0 star refines α, $O \cap Y$ is contained in some element of α. Thus β refines α.

Let $f,g : X \to Y$ be β-near maps. For $t \in [0,1]$, define $k_t: X \to Z$ (Z = the convex hull of Y in $C(Y)$) by $k_t(x) = (1 - t)f(x) + t(g(x))$ for all $x \in X$. First it will be shown that k_t $(0 \leq t \leq 1)$ is a γ-homotopy. Let $x \in X$. Since f and g are β-near, there exists $O \cap Y \in \beta$ such that $f(x), g(x) \in O \cap Y$. Therefore, $f(x), g(x) \in O$. Since O is convex, $k_t \in O$ for all t in $[0,1]$. Thus k_t is a γ-homotopy.

For $t \in [0,1]$, define $h_t : X \to Y$ by $h_t(x) = hk_t(x)$ for all $x \in X$. In order to show that h_t is an α-homotopy, let $x \in X$. There exists $O \in \gamma$ such that $k_t(x) \in O$ for all $t \in [0,1]$. Now $O = h^{-1}(U)$ for some $U \in \alpha_0$. Thus, $hk_t(x) = h_t(x) \in U$ for all $t \in [0,1]$. Hence, since α_0 refines α, h_t is an α-homotopy.

To see that f is α-near h_0, note that $f(x) = k_0(x)$ for all $x \in X$. Therefore, since $h|Y$ is α_0-near, hence α-near, the identity on Y, there is a $U \in \alpha$ such that $f(x)$ and $h(f(x)) = h(k_0(x)) = h_0(x)$ are in U. Similarly, g and h_1 are α-near. Thus the proof is complete.

Next is a generalization of Theorem 3.1 of [B1], page 103, on extensions of maps.

3.3 <u>Theorem</u>. If Y is an AANR and $\mathcal{U}$ is an open cover of Y, then there exists an open cover $\mathcal{V}$ of Y such that for every closed subset

A of a metric space X, and for all maps $f, g : A \to Y$ with f $\mathcal{V}$-near g, if there is a map $f' : X \to Y$ so that $f'|A$ is $\mathcal{V}$-near to f, then there is a map $g' : X \to Y$ so that $g'|A$ is $\mathcal{U}$-near to g and g' is $\mathcal{U}$-near f'.

Proof: Let ε be a Lebesgue number for $\mathcal{U}$. Let Y be embedded in an ANR P and let $r : P \to Y$ be a map so that for $y \in Y$, $d(y, r(y)) < \varepsilon/2$. By Theorem 3.1 of [B1], page 103, there exists $\eta > 0$ so that if $f,g: A \to P$ with $d(f, g) < \eta$, if f extends to $f' : X \to P$, then g has an extension $g' : X \to P$ so that $d(f', g') < \varepsilon/2$, where A is a closed subset of a metric space X. Let $\mathcal{V}$ be an open refinement of $\mathcal{U}$ so that $\text{mesh}(\mathcal{V}) < \min\{\varepsilon/2, \eta/2\}$. Let A be a closed subset of a metric space X, and let $f, g : A \to Y$ be maps with f $\mathcal{V}$-near g. Let $f' : X \to Y$ be a map with $f'|A$ $\mathcal{V}$-near f.

Consider f, f', and g as maps from A to P. For $x \in A$, there exists $V_1, V_2 \in \mathcal{V}$ so that $f(x), g(x) \in V_1$, and $f(x), f'(x) \in V_2$. Therefore, since $\text{mesh}(\mathcal{V}) < \eta/2$, $d(f(x), g(x)) < \eta/2$ and $d(f(x), f'(x)) < \eta/2$. Thus, $d(g(x), f'(x)) \leqq d(g(x), f(x)) + d(f(x), f'(x)) < \eta/2 + \eta/2 = \eta$. Therefore, $d(g, f'|A) < \eta$. Thus, since $f'|A$ extends to $f' : X \to P$, there exists $G: X \to P$ which is an extension of g and $d(f', G) < \varepsilon/2$.

Define $g': X \to Y$ by $g' = rG$. To see that $g'|A$ is $\mathcal{U}$-near g, let $x \in A$. Then $d(g'(x), g(x)) = d(rG(x), g(x)) \leqq d(rG(x), G(x)) + d(G(x), g(x)) < \varepsilon/2 + 0 = \varepsilon/2$. Since ε is a Lebesgue number for $\mathcal{U}$, there exists a $U \in \mathcal{U}$ so that $g'(x), g(x) \in U$. Thus $g'|A$ is $\mathcal{U}$-near g.

Finally, it needs to be shown that g' is $\mathcal{U}$-near f'. Let $x \in X$. Then $d(g'(x), f'(x)) \leqq d(g'(x), G(x)) + d(G(x), f'(x)) = d(rG(x), G(x)) + d(G(x), f'(x)) < \varepsilon/2 + \varepsilon/2 = \varepsilon$. Again, this means that there exists a $U \in \mathcal{U}$ that contains both g'(x) and f'(x). Thus, the theorem is proved.

The next theorem will be called the cell-like mapping theorem. This result was motivated by an attempt to generalize some work by Kozlowski. His result [K] says that if $f : X \to Y$ is a cell-like map, X an ANR, and Y a countable dimensional space, then Y is an ANR. By weakening countable dimensional to approximately countable dimensional (see Definition 3.5 below), the expected result that Y is an AANR turns out to be true. Ric Ancel brought to my attention that under the given conditions, Y is an AANR would be equivalent to Y is approximately countable dimensional and f is approximately invertible (see Definition 3.4 below).

Some definitions will be needed before stating the next theorem.

3.4 Definition. Let X be an AANR and let $f : X \to Y$ be an onto map. Then f is approximately invertible if for every open cover $\mathcal{L}$ of Y, there exists a map $g : Y \to X$ such that gf is $f^{-1}(\mathcal{L})$-near the identity on X.

3.5 Definition. A space Y is approximately countable dimensional if for every open cover $\mathcal{L}$ of Y there exists a countable dimensional space Z and maps $\alpha : Y \to Z$, $\beta : Z \to Y$ so that $\beta\alpha$ is $\mathcal{L}$-near the identity on Y.

3.6 Definition. A space X is cell-like if every map of X into an ANR is homotopic to a constant map.

3.7 Definition. A map $f : X \to Y$ is a cell-like mapping if f is proper, onto, and for every $y \in Y$, $f^{-1}(y)$ is cell-like.

3.8 Theorem. (Cell-Like Mapping Theorem)

Let X be an AANR and let $f : X \to Y$ be a cell-like map. Then the following are equivalent:

(i) f is approximately invertible.

(ii) Y is an AANR.

(iii) Y is approximately countable dimensional.

The proof of Theorem 3.8 will require some theorems developed in [A1] and [A2]. Additional terminology will also be needed.

3.9 Definition. A relation R from X to Y is a subset of $X \times Y$ and is denoted $R : X \to Y$.

3.10 Definition. A relation $R : X \to Y$ is continuous if for every closed set K in Y, $R^{-1}(K)$ is closed in X.

3.11 Definition. A relation $R : X \to Y$ is cell-like if it is continuous and R(x) is cell-like for each $x \in R^{-1}(Y)$.

One of the fundamental concepts in [A1] is that of a slice-trivial relation. For the purposes of this paper it is not necessary to state the full definition of slice-triviality. Instead, it is sufficient to know that each slice-trivial relation can be arbitrarily closely approximated by maps. This fact is stated more precisely in the following theorem.

3.12 Theorem. Every slice-trivial relation $R : X \to Y$ has the following property. For every collection $\mathcal{L}$ of open subsets of Y which is refined by $\{R(x) \mid x \in X\}$, there is a map $f : R^{-1}(Y) \to Y$ which is $\mathcal{L}$-near R.

The special case of the main theorem of [A1] needed here is the following:

3.13 Theorem. If $R : X \to Y$ is a cell-like relation where X is countable dimensional and Y is an ANR, then R is slice-trivial.

Proof of Theorem 3.8:

(i) implies (ii).

Assume that f is approximately invertible. To prove that Y is an AANR, let L be an open cover of Y. Let M be a star refinement of L and let $g : Y \to X$ be a map so that gf is $f^{-1}(M)$-near the identity on X. Let P be an ANR with maps $\alpha : X \to P$ and $\beta : P \to X$ so that $\beta\alpha$ is $f^{-1}(M)$-near the identity on X. Define $F : Y \to P$ by $F = \alpha g$. Define $G : P \to Y$ by $G = f\beta$.

To see that GF is L-near the identity on Y, let $y \in Y$. Let $x \in f^{-1}(y)$. Since gf is $f^{-1}(M)$-near the identity on X, there exists $M \in M$ so that x and $gf(x) = g(y)$ are in $f^{-1}(M)$. Thus $f(x) = y \in M$. Since $\beta\alpha$ is $f^{-1}(M)$-near the identity on X, there exists $M' \in M$ so that $g(y)$ and $\beta\alpha(g(y))$ are in $f^{-1}(M')$. Hence $f^{-1}(M) \cap f^{-1}(M') \neq \phi$. Therefore $M \cap M' \neq \phi$, and $f\beta\alpha(g(y)) = GF(y) \in M'$. Since M star refines L, there exists $L \in L$ so that $M \cup M' \subseteq L$. Hence y and GF(y) are in L. Thus, GF is L-near the identity on Y and Y is an AANR.

(ii) implies (iii).

Assume Y is an AANR. Let L be an open cover of Y. There exists an ANR P and maps $\alpha : Y \to P$, $\beta : P \to Y$ so that $\beta\alpha$ is L-near the identity on Y. By Corollary 6.2 of [H], page 139, P is $\beta^{-1}(L)$-dominated by a finite simplicial polytope P'. Thus, by composing maps, one obtains maps between Y and a countable dimensional space, P'. It is routine to verify that these maps will satisfy the necessary conditions.

(iii) implies (i).

Assume that Y is approximately countable dimensional. Let L be an open cover of Y. Choose open covers M and N of Y so that M star refines $\mathcal{L}$ and $\mathcal{N}$ star refines $\mathcal{M}$. Then $f^{-1}(\mathcal{N})$ is an open cover of X. Since X is an AANR, there exists $i : X \to W$, an embedding of X as a closed subset of an ANR W. Also, there exists an open neighborhood O of $i(X)$ in W and a map $r : O \to X$ so that ri is $f^{-1}(\mathcal{N})$-near the identity on X.

Since Y is approximately countable dimensional, there exists a countable dimensional space Z and maps $\alpha : Y \to Z$ and $\beta : Z \to Y$ so that $\beta\alpha$ is $\mathcal{N}$-near the identity on Y.

Now $if^{-1}\beta : Z \to O$ is a continuous relation with cell-like point images. Thus, by Theorem 3.13, $if^{-1}\beta$ is slice-trivial. In order to apply Theorem 3.12, it must be shown that $(fr)^{-1}(\mathcal{M})$ is refined by $\{if^{-1}(y) \mid y \in Y\}$. Let $y \in Y$. Then $y \in N$ for some $N \in \mathcal{N}$. Let $x \in f^{-1}(y)$. There exists $N' \in \mathcal{N}$ so that x and $ri(x)$ are both in $f^{-1}(N')$. Thus, $f(x) \in N \cap N' \subseteq N^* =$ the star of N. Also, $fri(x) \in N' \subseteq N^*$. Therefore, $i(x) \in (fr)^{-1}(N^*)$. Thus $if^{-1}(y) \subseteq (fr)^{-1}(N^*)$, since the choice of N^* does not depend on $x \in f^{-1}(y)$. Therefore, since $\mathcal{N}$ star refines $\mathcal{M}$, $\{if^{-1}(y) \mid y \in Y\}$ refines $(fr)^{-1}(\mathcal{M})$. Thus, $\{if^{-1}\beta(z) \mid z \in X\}$ refines $(fr)^{-1}(\mathcal{M})$. Hence, by Theorem 3.12, there exists a map $\gamma : Z \to O$ that is $(fr)^{-1}(\mathcal{M})$-near $if^{-1}\beta$.

$$\begin{tikzcd}
O \arrow[d, bend right, "r"'] & & Z \arrow[ll, dashed, "\gamma"'] \arrow[d, bend right, "\beta"'] \\
X \arrow[u, bend right, "i"'] \arrow[rr, "f"] & & Y \arrow[u, bend right, "\alpha"']
\end{tikzcd}$$

Define $g : Y \to X$ by $g = r\gamma\alpha$. In order to show that gf is $f^{-1}(\mathcal{L})$-near the identity on X, let $x \in X$. Then $f(x) \in Y$ and $\alpha f(x) \in Z$. Thus, since γ is $(fr)^{-1}(\mathcal{M})$-near $if^{-1}\beta$, there exists $M' \in \mathcal{M}$ so that $\gamma(\alpha f(x))$ and $if^{-1}\beta(\alpha f(x))$ are both contained in $(fr)^{-1}(M')$. Therefore, $f^{-1}(M')$ contains $r\gamma\alpha f(x) = gf(x)$ and $rif^{-1}\beta(\alpha f(x))$. Let $x' \in f^{-1}\beta(\alpha f(x))$. Since ri is $f^{-1}(\mathcal{M})$-near the identity on X, there exists $M \in \mathcal{M}$ so that $ri(x')$ and x' are both in $f^{-1}(M)$. Therefore, $ri(x') \in f^{-1}(M) \cap f^{-1}(M')$. Therefore, $M \cap M' \neq \phi$. Finally, since $\beta\alpha$ is $\mathcal{M}$-near the identity on Y, there exists $M'' \in \mathcal{M}$ so that $\beta\alpha(f(x))$ and $f(x)$ are both in M''. Since $x' \in f^{-1}\beta\alpha f(x)$, $f(x') = \beta\alpha f(x)$. Therefore, $f(x') \in M \cap M''$. Therefore, $M \cap M'' \neq \phi$. Hence, the following have been established: $M' \cup M \cup M'' \subseteq M^* =$ the star of M, $x \in f^{-1}(M'') \subseteq f^{-1}(M^*)$, and $gf(x) \in f^{-1}(M') \subseteq f^{-1}(M^*)$. Since $\mathcal{M}$ star refines $\mathcal{L}$, there exists $L \in \mathcal{L}$ so that $gf(x)$ and x are both in $f^{-1}(L)$. Hence, gf is $f^{-1}(\mathcal{L})$-near the identity on X. Therefore, f is approximately invertible.

CHAPTER IV

LIMITATIONS OF THE THEORY

Many of the nice features of the theory of ANR's are possessed also by AANR's. There are, however, instances where desirable properties of ANR's are not carried over to AANR's. It is instructive to consider some examples which point out the limitations of a generalized ANR theory. This is the topic of Chapter IV.

Some of the useful characterizations of ANR's given in [H] require an application of the following theorem from [Ha]:

Every local ANR is an ANR.

Certainly an analogous theorem for AANR's would be desirable. However, the following is a counterexample.

4.1 Example. In [B2] Borsuk constructs P, a compact connected subset of E^3 which does not have the fixed point property. The details of the construction are too tedious to list here. However, a feeling for P can be gained as follows. Consider a solid cylinder in E^3. Let C_1 and C_2 be circles centered at the center of the cylinder with C_1 on the interior of the top of the cylinder and C_2 on the interior of the bottom of the cylinder. Push out a tube, as shown in Figure 1, which spirals in such a way that C_2 is the limit.

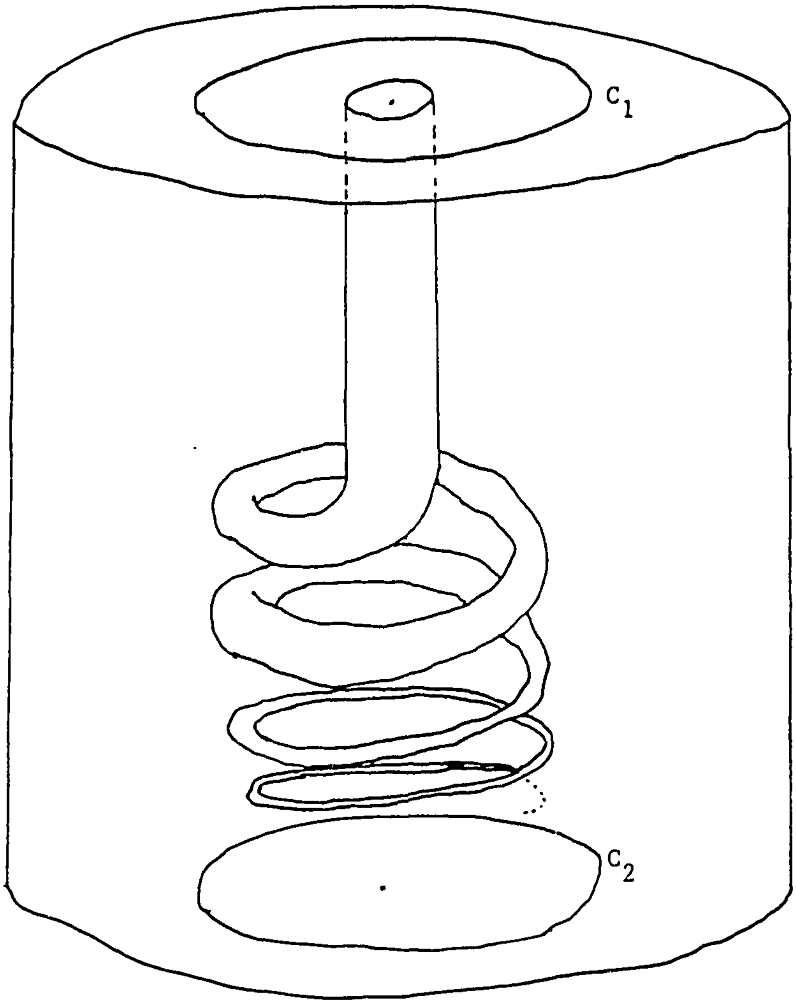


Figure 1.

Similarly, push out a second tube which has C_1 as its limit. The resulting space is illustrated in Figure 2.

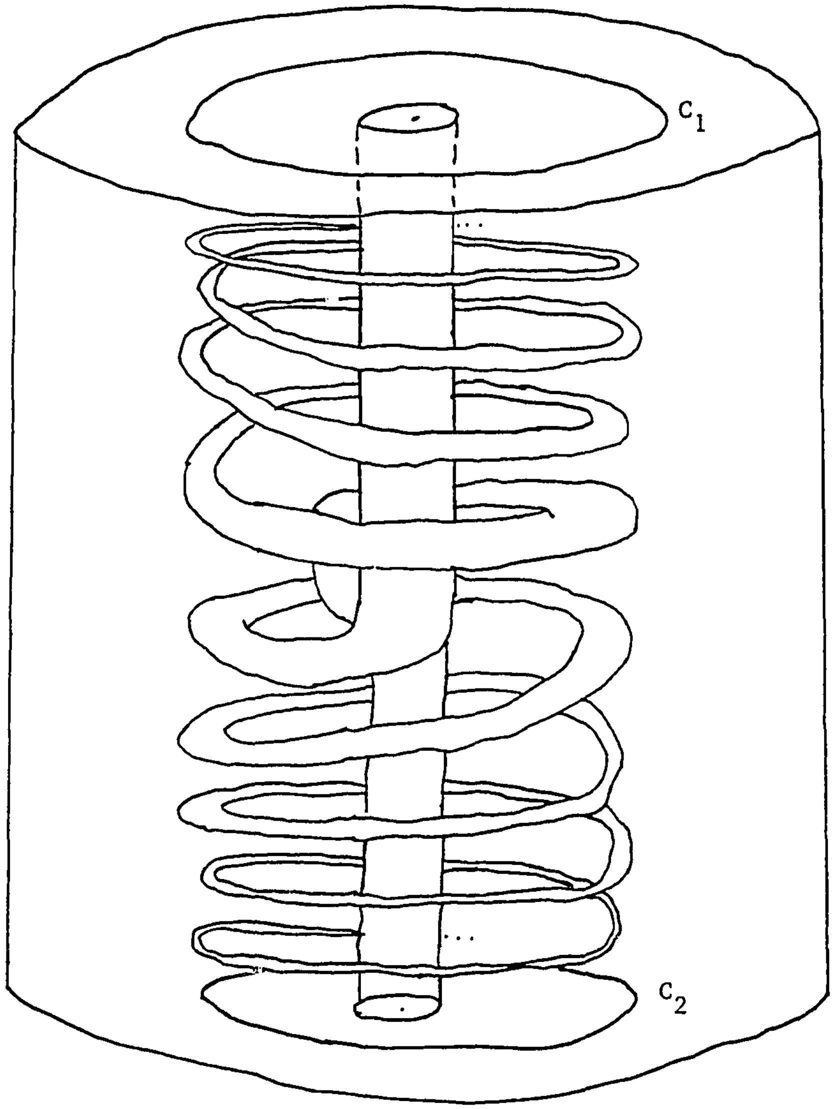


Figure 2.

It is claimed that P is a local-AANR but not an AANR. First note that P is a local AANR, in fact a local ANR, at every point except those on the limiting circles C_1 and C_2. Let x be a point on one of the circles and let N be a neighborhood of x. A cross section of N is shown in Figure 3.

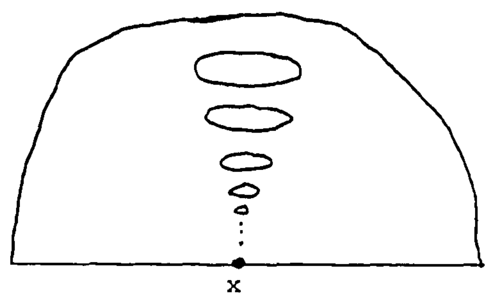


Figure 3.

It will now be established that N is an AANR. Let $\varepsilon > 0$. Let A be the ANR formed by plugging all the holes in N which are within an ε-distance of x. Let $i : N \to A$ be inclusion. Let $r : A \to N$ be the map which retracts the lower portion of A onto the line $y = \varepsilon$ (see Figure 4) then includes the result into N. Then the composition $ri : N \to N$ will move no point more than ε. Thus N is an AANR.

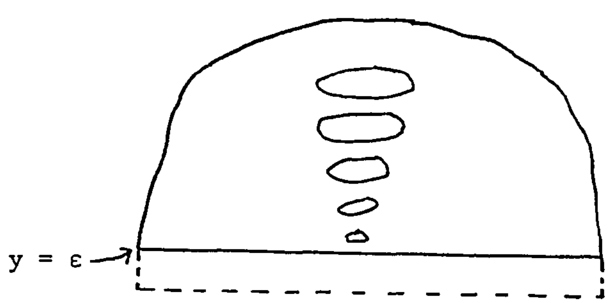


Figure 4.

Next it will be established that P is not an AANR. Borsuk exhibits in [B2] a map $f : P \to P$ that leaves no point fixed. Let $\varepsilon > 0$ be chosen so that for all $x \in P$, $d(x, f(x)) > \varepsilon$. Suppose that P is an AANR. Then there exists a neighborhood O of P and a map $r : O \to P$ so that for all $x \in P$, $d(x, r(x)) < \varepsilon/_2$. Let B be a 3-cell neighborhood of P so that $P \subsetneqq B \subsetneqq O$. Consider $fr : B \to B$. Let $x \in B$. If x is not an element of P, then clearly $d(x, fr(x)) > 0$. If $x \in P$, then $\varepsilon < d(r(x), fr(x)) \leqq d(r(x), x) + d(x, fr(x))$. Therefore, since $d(r(x), x) < \varepsilon/_2$, $\varepsilon < \varepsilon/_2 + d(x, fr(x))$. Thus $d(x, fr(x)) > 0$. Consequently $fr : B \to B$ has no fixed point. This contradicts Brower's fixed point theorem. Thus P is not an AANR.

Hence, P is a local AANR that is not an AANR.

Definition. U is an n-dimensional umbrella if $U = Q \cup L$ where Q is an n-dimensional ball and L is an arc with $Q \cap L$ consisting of exactly one point a which is an endpoint of L and an interior point of Q. The point a is called the center.

Borsuk, in [B1], page 144, proves the following theorem due to Bing and Borsuk. The theorem is used in [B-B] to establish a result about homogeneous spaces.

4.2 Theorem. If X is an n-dimensional ANR, then the set of centers of all n-dimensional umbrellas lying in X is of the first Baire category in X.

The q-ANR version, hence the AANR version, of Theorem 4.2 is not true. Consider the example of Sierpinski's curve below.

4.3 Example. Let $S_0 = I \times I$ where $I = [0,1]$. Divide S_o into nine equal squares. Delete the interior of the following squares: $[0,1/3] \times [1/3,2/3]$, $[1/3,2/3] \times [0,1/3]$, $[1/3,2/3] \times [2/3,1]$, $[2/3,1] \times [1/3,2/3]$. Call the resulting subset of S_o S_1. Let S_2 be the space formed by deleting from each square in S_1 the interiors of four squares analogous to the ones above. S_1 and S_2 are shown in Figure 5.

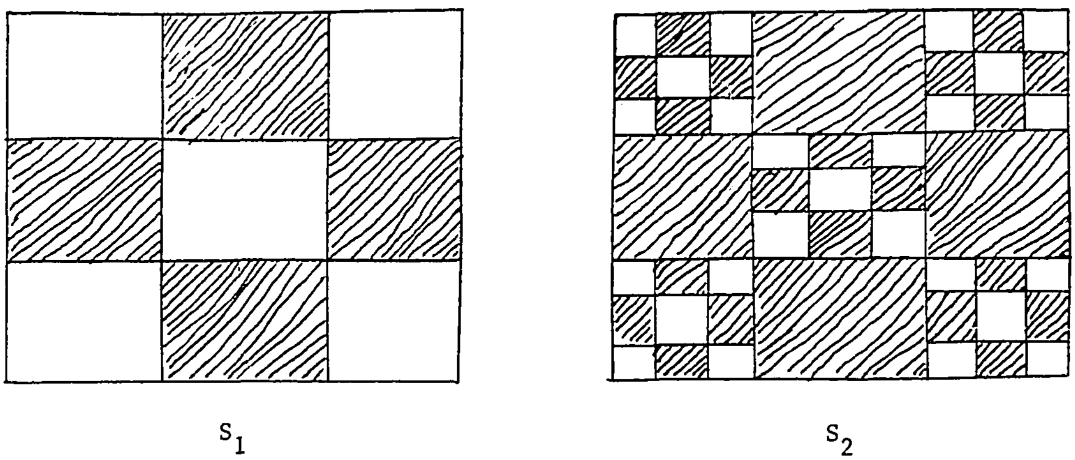


Figure 5.

Continue in this manner. Let $S = \bigcap_{n=1}^{\infty} S_n$. (A slightly different construction of S can be found in [C-V], pages 236-237.)

It is pointed out in [B1], page 144, that S is a one-dimensional space such that each point of S is the center of a one-dimensional umbrella lying in S. However, it will be shown that S is a q-ANR and clearly S is not of the first category.

Let $\varepsilon > 0$. Let n be large enough so that the diameter of each square in S_n is less than $\varepsilon/2$. Now S_{n+1} is an ANR. Let $i : S \to S_{n+1}$

be inclusion. A map r from S_{n+1} into S can be defined by retracting the interior of each square in S_{n+1} to the boundary of the square in S_n that contains it. Then include the result into S. Thus the composition map ri : S $\to$ S will move no point more than ε. Since S is locally connected, by Theorem 2.16, S is a q-ANR.

In order to generalize the umbrellas theorem (Theorem 4.2) to q-ANR's, it appears that local contractibility must be added. However, this makes the space an ANR. The result is then contained in another theorem proved by Bing and Borsuk in [B-B]:

4.4 Theorem. In an n-dimensional locally contractible separable metric space X, (which is of necessity an ANR) the set of all centers of n-dimensional umbrellas contained in X is of the first Baire category.

For subsets X of the plane, it is known ([B3], page 242) that X is an ANR if and only if X is a locally connected compactum so that $E^2 - X$ has only a finite number of components. One might expect to be able to generalize this result to AANR's in the following form:

A subset X of E^2 is an AANR if and only if X is a compactum such that for every $\varepsilon > 0$, $E^2 - X$ has only a finite number of components of diameter greater than ε.

An example that would fit this generalization is the Hawaiian earring, [P1], page 48. However, there are other spaces which show that the statement is false. Consider the following:

4.5 Example. Let X = the boundary of $I^2 \cup \{(1/n, y) \in I^2 \mid n = 1,2,3,\dots\}$. (See Figure 6.) Then X is an AANR.

However, $E^2 - X$ has an infinite number of components, each of which has diameter = 1.

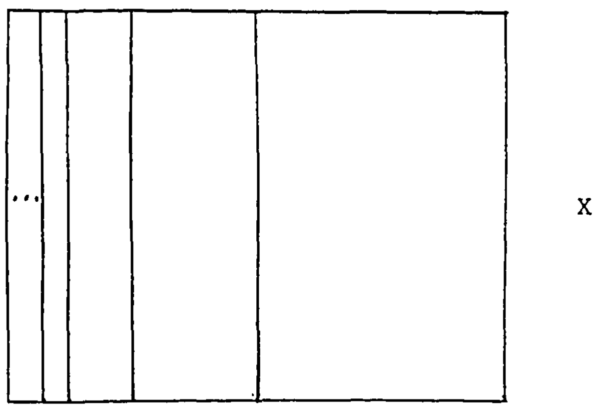


Figure 6.

Remark: Example 4.5 is not a q-ANR since X is not locally connected. The Hawaiian earring is a q-ANR. Thus, the following question remains unanswered:

Is every compact connected subset X of E^2 a q-ANR if and only if for every $\varepsilon > 0$, $E^2 - X$ has only a finite number of components of diameter greater than ε?

In his book Theory of Retracts, Borsuk discusses various "singularities". Roughly speaking, an ANR X has a particular singularity if all polyhedra possess a certain property while X does not. Borsuk gives several examples of ANR's whose topological properties are quite different from the topological properties of polyhedra. Certainly if an ANR X has a particular singularity, then there is an AANR, namely X, with the same singularity. Thus, generally, these singularities are of no interest in this paper. There is, however, one special case which is worthy of investigation.

4.6 Definition. Let X be a compactum. For k = 0, 1,..., the k-th coefficient of Urysohn $d_k(X)$ is the greatest lower bound of the set of all positive numbers ε such that there exists a finite covering α of X by closed sets with diameters less than ε and with the dimension of the nerve of α less than k.

Consider the following condition:

Condition A. For every disjoint pair of subsets A and B of X, with non-empty interiors, there exists $\varepsilon > 0$ such that every closed separator of X between A and B has (n-1)-st coefficient of Urysohn greater than ε where n = dim X.

4.7 Definition. A space which is an ANR and a Cantor manifold but does not satisfy condition A is said to have the singularity of Alexandroff ([B1], page 149).

Lelek has shown in [L] that every 2-dimensional ANR that is a Cantor manifold satisfies condition A. There exist higher dimensional ANR's which do not satisfy condition A (see, for example, [B1], pages 148-149, or [L], pages 244-246).

Thus, the following question is now considered: Does every 2-dimensional q-ANR that is a Cantor manifold satisfy condition A? The following example shows that the answer is no.

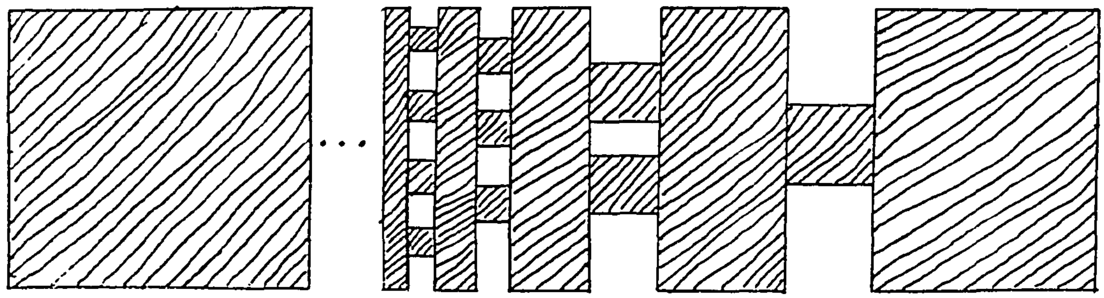
4.8 Example. Define the space X as follows:

Let $M_0 = \{(x,y) \in E^2 \mid -1 \leq x \leq 0,\ -1 \leq y \leq 1\}$.

Let $M_{2k-1} = \{(x,y) \in E^2 \mid \frac{1}{2k} \leq x \leq \frac{1}{2k-1},\ -1 \leq y \leq 1\}$,

and $M_{2k} = \bigcup_{i=1}^{k} \{(x,y) \in E^2 \mid \frac{1}{2k+1} \leq x \leq \frac{1}{2k},\ \frac{2k-4i+1}{2k} \leq y \leq \frac{2k-4i+3}{2k}\}$

for $k = 1,2,3,\ldots$. Let $X = \bigcup_{n=0}^{\infty} M_n$. X is shown in Figure 7.



X

Figure 7.

X is a 2-dimensional locally connected Cantor manifold which does not meet condition A ([L], page 238). X is also a q-ANR.

Remark: Example 4.8 is interesting for another reason also. X is an example of a 2-dimensional q-ANR that is not an ANR.

Next some examples which are similar to Example 4.8 but which are not q-ANR's -- in fact, not even AANR's -- are given.

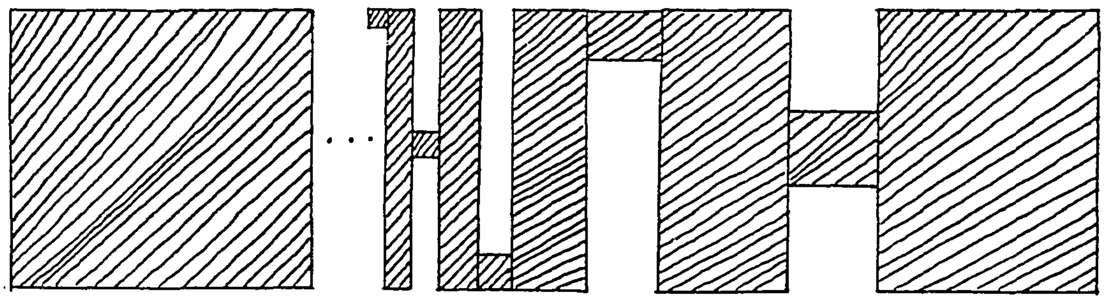
4.9 Example. Let $R = \bigcup_{n=0}^{\infty} R_n$ where R_n for $n = 0,1,2,\ldots$ is is defined as follows:

$R_0 = \{(x,y) \in E^2 \mid -1 \leq x \leq 0,\ -1 \leq y \leq 1\}$,

$$R_{2k-1} = \{(x,y) \in E^2 \mid \frac{1}{2k} \leqq x \leqq \frac{1}{2k-1}, \quad -1 \leqq y \leqq 1\},$$

$$R_{2k} = \begin{cases} \{(x,y) \in E^2 \mid \frac{1}{2k+1} \leqq x \leqq \frac{1}{2k}, \frac{k-1}{k+1} \leqq y \leqq 1\} & \text{if } k = 3n+1 \\ \{(x,y) \in E^2 \mid \frac{1}{2k+1} \leqq x \leqq \frac{1}{2k}, \quad -\frac{1}{k+1} \leqq y \leqq \frac{1}{k+1}\} & \text{if } k = 3n+2 \\ \{(x,y) \in E^2 \mid \frac{1}{2k+1} \leqq x \leqq \frac{1}{2k}, \quad -1 \leqq y \leqq \frac{1-k}{k+1}\} & \text{if } k = 3n. \end{cases}$$

R is shown in Figure 8.



R

Figure 8.

Note that points of R in R_0 cannot be connected by a path to points of R in R_k for $k > 0$.

Clearly R is not a q-ANR since it is not locally connected. Suppose R is an AANR. Then for $\varepsilon = 1/4$, there is a connected ANR neighborhood P of R and a map $r : P \to R$ so that r moves no point of R more than 1/4. Let $x = (-3/4, 0) \varepsilon R_0$ and let $y = (3/4, 0) \varepsilon R_1$. Then there is a path f in P from x to y. However, as has been observed, rf cannot be a path in R. This is a contradiction. Thus R is not an AANR.

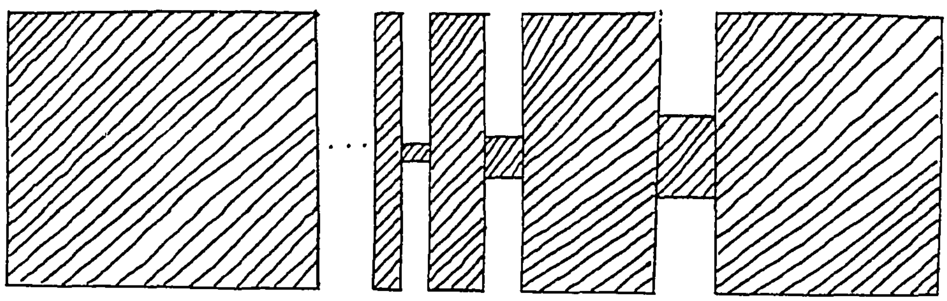
A space similar to R is defined as follows. Let $M = \bigcup_{n=0}^{\infty} M_n$ where M_n is defined by the following:

$M_0 = R_0$,

$M_{2k-1} = R_{2k-1}$,

$M_{2k} = \{(x,y) \in E^2 \mid \frac{1}{2k+1} \leqq x \leqq \frac{1}{2k}, \ \frac{-1}{k+1} \leqq y \leqq \frac{1}{k+1}\}$.

M is shown in Figure 9.



M

Figure 9.

Both M and R are examples of compact connected Cantor manifolds which are not AANR's.

4.10 Definition. A space X is locally connected in the dimension n if for every point $x \in X$ and every neighborhood U of x, there exists a neighborhood $V \subseteq U$ of x such that every map $f : S^n \to V$ extends to a map $g : E^{n+1} \to U$.

4.11 Definition. A space X is locally n-connected (LC^n) if X is locally connected in the dimension q for every $q \leqq n$.

Finite dimensional ANR's can be characterized as LC^n where n is the dimension of the ANR. (see Theorem 7.1 of [H], page 168.) The final topic of this chapter considers the problems encountered when attempting a generalization of the theorem in [H] to AANR's.

4.12 Definition. A space Y is approximately-locally connected in the dimension n if for every $\varepsilon > 0$, if $y \in Y$ and U is a neighborhood of y, there exists a neighborhood $V_\varepsilon \subseteq U$ of y such that if $f : S^n \to V_\varepsilon$, then there is a map $g : E^{n+1} \to U$ such that $g|S^n$ is ε-near f.

4.13 Definition. A space Y is approximately-locally n-connected (A-LC^n) if Y is approximately-locally connected in the dimension q for all $q \leq n$.

4.14 Theorem. Let Y be an AANR. Then Y is A-LC^n for all n.

Proof: Let $\varepsilon > 0$. Let $y \in Y$ and let U be a neighborhood of y. Let $U' = U \cap B_\varepsilon(y)$. ($B_\varepsilon(y) = \{x \in Y \mid d(x, y) < \varepsilon\}$.) Let $\delta = \min\{\varepsilon/2,\ \operatorname{diam}(U')\}$. Since Y is an AANR, an ANR P and maps $\alpha : Y \to P$, $\beta : P \to Y$ can be chosen so that $\beta\alpha$ is δ-near the identity on Y. By the choice of δ, an open set V can be chosen so that $y \in V \subseteq U$, and $\beta\alpha(V) \subseteq U$. Let M be an open set in P containing $\alpha(y)$ so that $\alpha(V) \supseteq M$ and $\beta^{-1}(V) \supseteq M$.

Since P is an ANR, P is LC^m for all m less than a given n. Therefore, there exists an open set M_ε so that $\alpha(y) \in M_\varepsilon \subseteq M$ and if $h : S^m \to M_\varepsilon$, then there is an extension of h from E^{m+1} into M. Let $V_\varepsilon = \alpha^{-1}(M_\varepsilon)$.

Let $f : S^m \to V_\varepsilon$. Then $\alpha f : S^m \to M_\varepsilon$. Thus there is an extension $h : E^{m+1} \to M$. Let $g = \beta h$. Then $g : E^{m+1} \to \beta(M) \subseteq V \subseteq U$, and $g|S^m$ is ε-near f. Hence Y is A-LC^n.

Theorem 4.14 shows that if Y is an n-dimensional AANR, then Y is A-LC^n. However, the converse is not true. This is shown in the following example.

4.15 Example. Let P be the same as in Example 4.1. Then dim P = 3, P is A-LC^3, but P is not an AANR.

Example 4.1 showed that P is not an AANR. In order to see that P is A-LC^3, first note that P is a local ANR everywhere except on the limiting circles. For a point y on a limiting circle, let $\varepsilon > 0$. Let U be a neighborhood of y. Let V_ε be a neighborhood of y so that $V_\varepsilon \subseteq U$ and the diameter of each hole in V_ε is less than ε. Let $f : S^n \to V_\varepsilon$ $(n = 0, 1, 2, 3)$. Let V_ε' be V_ε with the holes filled in. Then V_ε' is an AR. Therefore there exists, by Theorem 11.1 of [H], page 175, an extension $g : E^{n+1} \to V_\varepsilon'$. Now the map g can be adjusted so that the image of g misses the holes in V_ε and no point will have been moved more than ε. Thus g is ε-near f on V_ε. Hence P is A-LC^3.

CHAPTER V

HYPERSPACES OF AANR's

In 1954 Borsuk introduced two metrics, the metric of continuity and the homotopy metric, on 2^X, the hyperspace of compact subsets of a metric space X. The purpose was to define metrics on 2^X so that if a sequence of compacta $\{A_n\}$ converged to a compactum A, then some topological properties of all A_n would be possessed by A.

It will be helpful at this point to recall some definitions from [B4] and to establish the notation that will be used here.

Let X be a metric space with distance function d.

5.1 Definition. $2^X = \{A \subseteq X \mid A \text{ is nonempty and compact}\}$ is topologized by the Hausdorff metric d_s, where $d_s(A,B) =$

$$\inf\left\{\varepsilon > 0 \;\middle|\; \begin{array}{l} A \text{ is contained in an } \varepsilon\text{-neighborhood of } B \text{ and} \\ B \text{ is contained in an } \varepsilon\text{-neighborhood of } A \end{array}\right\}.$$

5.2 Definition. $2^X_c = \{A \subseteq X \mid A \text{ is an AANR}\}$ is topologized by the metric of continuity d_c, where $d_c(A,B)$ is the greatest lower bound of all numbers $t \geq 0$ such that there are maps $f : A \to B$, $g : B \to A$ satisfying the following conditions:

$$d(x, f(x)) \leq t \quad \text{for all } x \in A,$$
$$d(x, g(x)) \leq t \quad \text{for all } x \in B.$$

For the purposes of this paper it will not be necessary to define the homotopy metric d_h. However, it will be important to know that if X is finite dimensional, then d_h induces a topology on $2_h^X = \{A \subseteq X \mid A \text{ is an ANR}\}$ so that $\{A_n\} \xrightarrow{d_h} A$ if and only if:

(i) $\{A_n\} \xrightarrow{d_s} A$ and

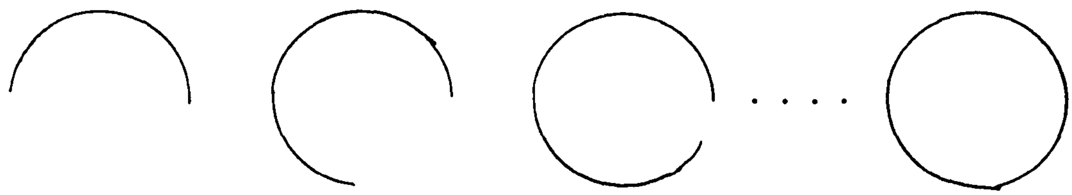
(ii) Given $\varepsilon > 0$, there is a $\delta > 0$ such that for all n, every δ-subset of A_n contracts to a point inside an ε-subset of A_n.

Under the assumption that X is finite dimensional, the homotopy metric makes 2_h^X complete. Example 5.3 below shows that the metric of continuity does not have this desirable property. Consequently, it has been natural for attention to focus on the homotopy metric. However, Čerin has been able to prove in [C] that 2_c^X is topologically complete if and only if X is topologically complete. Thus, our investigation of AANR's now turns to the hyperspace of AANR subsets of a metric space topologized by the metric of continuity, 2_c^X.

Notation: $C(X) \subseteq 2^X$, $C_c(X) \subseteq 2_c^X$, and $C_h(X) \subseteq 2_h^X$ each represent the hyperspace of subcontinua of X with the inherited topology. The unit interval, [0,1], will be denoted by I.

In the following examples, some problems encountered in using the metric of continuity are noted.

5.3 Example. Let $A_n = \{(\cos t, \sin t) \in E^2 \mid 1/n \leq t \leq 2\pi\}$ for n = 1,2,... . Let $A = \{(\cos t, \sin t) \in E^2 \mid 0 \leq t \leq 2\pi\}$. Then the sequence $\{A_n\}$ does not converge to A because $d_c(A_n, A) \geq 2$ for all n. However, $\{A_n\}$ is a Cauchy sequence. Thus 2_c^X is not complete for $X = E^2$.



A_1 A_2 A_3 A

5.4 Example. Let $D_{kj} = \{t \mid 2j/3^k \leq t \leq (2j+1)/3^k\}$ for $k = 1,2,\ldots$, $j = 0,1,\ldots, \frac{1}{2}(3^k-1)$. Let $D_k = \bigcup_{j=0}^{\frac{1}{2}(3^k-1)} D_{kj}$, $H_k = [0,1] - D_k$, and $F_k = D_k \cap \overline{H}_k$. Let $A = I^2 - \bigcup_{k=1}^{\infty} (H_k \times H_k)$. Then A is Sierpinski's curve. Let $A_k = A \cap [(F_k \times I) \cup (I \times F_k)]$ for $k = 1,2,\ldots$. Then $\{A_k\}$ is a sequence of 1-dimensional polyhedra converging to A in 2^X_c, where $X = I^2$.

Sierpinski's curve is not locally contractible. Hence A is not an ANR. Thus, there is a sequence of ANR's converging in 2^X_c to a non ANR.

A_1 A_2

5.5 Example. Let $X = I$. Let $A_n = \{0\} \cup [1/(n+1),\frac{1}{2}]$ and let $A = [0,\frac{1}{2}]$. It will be shown that $\{A_n\} \xrightarrow{d^n_c} A$.

Let $\varepsilon > 0$. Choose N so that $1/(N+1) < \varepsilon$. For $n > N$, define $f : A_n \to A$ by $f(x) = x$, and define $g : A \to A_n$ by mapping $[0,\frac{1}{2}]$ onto

$[1/(n+1),\frac{1}{2}]$. Clearly neither function moves any point more than ε. Therefore, $d_c(A, A_n) < \varepsilon$. Thus $\{A_n\} \xrightarrow{d_c} A$. However, since A has the homotopy type of a point and no A_n has the homotopy type of a point, $\{A_n\}$ cannot converge to A in the homotopy metric.

Next a few basic properties of some simple spaces are observed.

5.6 Proposition. C_c^I is path connected.

Proof: Let A, B ε C_c^I. Being compact connected subsets of I, A and B can be written as closed intervals: $A = [a,b]$, $B = [c,d]$. (The proof is similar if A and B are not homeomorphic. i.e., if A = point and B = interval.) Define a path $f : [0,1] \to C_c^I$ by $f(t) = [(1 - t)a + ct, (1 - t)b + dt]$. Then $f(0) = [a,b]$ and $f(1) = [c,d]$.

It needs to be shown that f is continuous. Let $\{x_i\}$ be a sequence converging to x in I. Let $\varepsilon > 0$. Then there exists an integer N so that for $i \geqq N$, $|x - x_i| < \varepsilon/\max\{|-a + c|, |-b + d|\}$. To establish that for $i \geqq N$, $d_c(f(x_i), f(x)) < \varepsilon$, it is necessary to define a map $\phi : f(x) \to f(x_i)$ so that for $y \,\varepsilon\, f(x)$, $d(\phi(y), y) < \varepsilon$, and define a map $\psi : f(x_i) \to f(x)$ so that for $y \,\varepsilon\, f(x_i)$, $d(\psi(y), y) < \varepsilon$. There is a homeomorphism between $f(x_i)$ and $f(x)$ that sends left endpoints to left endpoints and right endpoints to right endpoints. Let this homeomorphism and its inverse be denoted by ϕ and ψ respectively. Now

$f(x_i) = [(1 - x_i)a + cx_i, (1 - x_i)b + dx_i]$ and

$f(x) = [(1 - x)a + cx, (1 - x)b + dx]$. Note that

$|(1 - x_i)a + cx_i - ((1 - x)a + cx)| = |a - ax_i + cx_i - a + ax - cx| =$

$|x(a - c) - x_i(c - a)| = |(-a + c)(x_i - x)| \leqq |-a + c||x_i - x| <$

$|-a + c|\ \varepsilon/|-a + c| = \varepsilon$. Similarly,

$|(1 - x_i)b + dx_i - ((1 - x)b - dx)| < \varepsilon$. Thus ϕ and ψ move no point more than ε. Therefore, $d_c(f(x_i), f(x)) < \varepsilon$. Hence $\{f(x_i)\}$ converges to $f(x)$ in C_c^I. Thus f is a continuous function. Thus f is a path in C_c^I from A to B. Therefore, C_c^I is path connected.

5.7 Proposition. $A = \{X \in 2_c^I \mid X \text{ is an ANR}\}$ is locally connected.

Proof: Let $X \in A$. X has only a finite number of components. Hence X can be written as follows: $X = [a_1,b_1] \cup \cdots \cup [a_n,b_n]$ where $a_i \leqq b_i < a_{i+1} \leqq b_{i+1}$ for $i = 1,\ldots, n-1$, and $[a_i,b_i]$ is a point if $a_i = b_i$. Let $\varepsilon = \min\{a_{i+1} - b_i\}_{i=1}^{n-1}$. Let $d_c(X,Y) < \varepsilon/2$. Then there maps $f : X \to Y$, $g : Y \to X$ so that $d(x, f(x)) < \varepsilon/2$ for all $x \in X$ and $d(y, g(y)) < \varepsilon/2$ for all $y \in Y$.

Suppose Y has fewer than n components. Then f maps two of the components of X into the same component of Y. Say $f([a_i,b_i] \cup [a_j,b_j])$ is contained in K, where K is one component of Y. Now $g(K) \subseteq [a_k,b_k]$ for some k. But then there must be some $x \in [a_i,b_i] \cup [a_j,b_j]$ so that $d(x, gf(x)) > \varepsilon$. Therefore, since gf can move no point of X more than ε, Y has at least n components. Let these n components be called $[c_i,d_i]$ where $f([a_i,b_i]) \subseteq [c_i,d_i]$ for $i = 1,2,\ldots,n$.

Now Y may have more than n components but for each component K of Y such that $K \neq [c_i,d_i]$ for $i = 1,\ldots,n$, $fg(K) \subseteq [c_i,d_i]$ for some i. Let $K_i = \bigcup\{\text{components } K \text{ of } Y \mid K \neq [c_i,d_i] \text{ and } fg(K) \subseteq [c_i,d_i]\}$. For $i = 1,\ldots,n$, define $F_i : [0,\frac{1}{2}] \to A$ as follows:

$F_i(t) = \{(1 - 2t)x + 2tc_i \mid x \in K_i\} \cup [c_i,d_i]$. Define a path $F : [0,1] \to A$ as follows:

$$F(t) = \begin{cases} \bigcup_{i=1}^{n} F_i(t) & \text{if } t < \frac{1}{2} \\ \bigcup_{i=1}^{n} [2(1-t)c_i + 2a_i(t - \frac{1}{2}),\ 2(1-t)d_i + 2b_i(t - \frac{1}{2})] & \text{if } t \geq \frac{1}{2}. \end{cases}$$

As t varies from 0 to $\frac{1}{2}$, F moves each K_i to c_i while $[c_i, d_i]$ remains fixed. As t varies from $\frac{1}{2}$ to 1, $[c_i, d_i]$ is deformed to $[a_i, b_i]$ for $i = 1,2,\ldots,n$. Clearly, F is continuous. Thus F is a path from Y to X in 2^I_c. Hence the $\varepsilon/2$ ball about X in A is connected. Thus A is locally connected.

It was noted in Example 5.4, that in 2^X_c a sequence of ANR's can converge to a non-ANR. However, if the sequence consists of AANR's, then the following is true.

5.8 <u>Theorem</u>. If $\{A_i\} \xrightarrow{d_c} A$ in 2^X_c and A_i is an AANR for $i = 1,2,\ldots,$ then A is an AANR.

Proof: Let $\varepsilon > 0$. There exists N such that $n \geq N$ implies $d_c(A_n, A) < \varepsilon/3$. Thus for $n \geq N$, there exists maps $f_n : A_n \to A$ and $g_n : A \to A_n$ such that $d(x, f_n(x)) < \varepsilon/3$ for all $x \in A_n$ and $d(x, g_n(x)) < \varepsilon/3$ for all $x \in A$. Since A_n is an AANR, there exists an ANR P and maps $h_n : A_n \to P$ and $k_n : P \to A_n$ so that $k_n h_n$ moves no point of A_n more than $\varepsilon/3$. Let $f : A \to P$ be defined by $f = h_n g_n$ and let $g : P \to A$ be defined by $g = f_n k_n$. Let $x \in A$. Then $d(x, gf(x)) = d(x, f_n k_n h_n g_n(x)) \leq d(x, g_n(x)) + d(g_n(x), k_n h_n(g_n(x))) + d(k_n h_n(g_n(x)), f_n(k_n h_n(g_n(x))) < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$. Thus there are maps between A and the ANR P so that the composition moves no point more than ε. Since ε was arbitrary, A is an AANR.

In view of the many useful results about d_h, a study adapting some of these to d_c would be in order. This investigation focuses on a few of the results in [R].

Note. In [B4], Borsuk's paper defining the homotopy metric and the metric of continuity, it is observed on page 190 that $d_s(X,Y) \leqq d_c(X,Y) \leqq d_h(X,Y)$. Thus if $\{A_i\} \xrightarrow{d_h} A$, then $\{A_i\} \xrightarrow{d_c} A$, and $\{A_i\} \xrightarrow{d_s} A$.

The following lemma is proved in [R].

Lemma 3.6 (page 37). If X is a finite acyclic graph, then $C_h(X) \cong C(X)$.

The lemma is established by showing that a sequence which converges with respect to the Hausdorff metric converges with respect to the homotopy metric. This proof along with the above note yield the following:

5.9 Lemma. If X is a finite acyclic graph, then $C_h(X) \cong C_c(X) \cong C(X)$.

The following theorem can now be proved.

5.10 Theorem. If X is a space such that for every cover α of X, X is α-dominated by a finite acyclic graph, then $C_h(X) \cong C_c(X) \cong C(X)$.

Proof: It suffices to prove that if $\{A_i\} \xrightarrow{d_s} A$, then $\{A_i\} \xrightarrow{d_h} A$. Let $\varepsilon > 0$. Let α be a finite open cover of X with mesh(α) less than ε. X is α-dominated by a finite acyclic graph P. This means that there exists maps $f : X \to P$, $g : P \to X$ so that gf is α-homotopic to the identity on X. Let $h_t : X \to X$ denote the homotopy.

Let $\beta = g^{-1}(\alpha)$. Let $\varepsilon' = \min\{\text{mesh}(\beta), j'\}$ where j' is a Lebesgue number for β. Since $\{A_i\} \xrightarrow{d_s} A$, $\{f(A_i)\} \xrightarrow{d_s} f(A)$. Therefore, by

Lemma 5.9, $\{f(A_i)\} \xrightarrow{d_h} f(A)$. Thus, by the definition of convergence in the homotopy metric, there exists δ' so that for every i, every subset of $f(A_i)$ of diameter less than δ' contracts to a point inside a subset of diameter less than ε'. Let $\gamma = \{f^{-1}(B_{\delta'}(x)) \mid x \in P\}$. ($B_{\delta'}(x)$ = the ball of radius δ' about x.) Then γ is an open cover of X. Let $\delta = \min\{\text{mesh}(\gamma), j, k\}$ where j and k are Lebesgue numbers for γ and α respectively.

Now if Z is any subset of A_i of diameter less than δ, then f(Z) is a subset of $f(A_i)$ of diameter less than δ'. Thus f(Z) contracts to a point inside a subset of $f(A_i)$ of diameter less than ε'. Contractibility is preserved by continuous maps so gf(Z) contracts to a point inside a subset of $gf(A_i)$. By the definition of α, β, and ε', the subset of $gf(A_i)$ in which the contraction of gf(Z) takes place has diameter less than ε. Let $k_t : gf(Z) \to gf(Z)$ denote the homotopy associated with the contraction. Then the homotopy $H_t : Z \to gf(A_i)$ defined by

$$H_t(x) = \begin{cases} h_{2t}(x) & \text{if } t < \tfrac{1}{2} \\ k_{2t-1}(x) & \text{if } t \geqq \tfrac{1}{2} \end{cases}$$

contracts Z to a point inside a subset of A_i of diameter less than 2ε. Since ε was arbitrary, $\{A_i\} \xrightarrow{d_h} A$.

5.11 <u>Corollary</u>. Let X be as in Theorem 5.10. If X is locally connected, then $C_c(X)$ and $C_h(X)$ are locally connected.

Proof: X is locally connected. Thus by 1.92 of [N], page 134, C(X) is locally connected. Hence, by Theorem 5.10, $C_c(X)$ and $C_h(X)$ are locally connected.

The following theorem is proved in [R].

Theorem 3.2 (page 35). If X is a Peano continuum, then every component of 2_h^X is finite dimensional if and only if X is a finite graph. (Recall that 2_h^X = {ANR subsets of X} topologized by the homotopy metric.)

The proof of the theorem makes use of lemma 3.6 (stated above) and the following lemmas:

Lemma 3.7 (page 38). Let X be a graph, Y a subcontinuum of X, and Z_y the component of 2_h^X containing Y. Then Z_y is naturally embedded in C(X).

Lemma 3.1 (page 32). Let L_n = line segment joining (0,0) to $(1/n, 1/n^2)$, and let $T = \bigcup_{n=1}^{\infty} L_n$. Then the subset P = {continua in T containing (0,0)} of 2^T is homeomorphic to the Hilbert cube (Q).

Lemma 3.2 (page 33). The subspace P of 2_h^T is homeomorphic to Q.

Lemma 3.4 (page 35). Let $R = S_1 \cup S_2 =$ $\{(x,y) \mid 0 \leq x \leq 1,\ y = 0\} \cup \{(1/n, y) \mid n = 1,2,\ldots\ ;\ 0 \leq y \leq 1/n\}$, and let S be the collection of all continua in R containing S_1. Then the subset S of 2^R is homeomorphic to Q.

Lemma 3.5 (page 36). The subspace X of 2_h^R is homeomorphic to Q.

Lemmas 3.5 and 3.2 were proved by showing that $\{A_i\} \xrightarrow{d_s} A$ implies $\{A_i\} \xrightarrow{d_h} A$ and thus remain true when the homotopy metric is replaced by the metric of continuity. Lemmas 3.4 and 3.1 are general results which involve neither the homotopy metric nor the metric of continuity.

One might expect to have a theorem similar to Theorem 3.2 for the metric of continuity. However, only the following is true.

5.12 <u>Theorem</u>. If 2_c^X is of finite dimension, then X is a finite graph.

Proof: The proof is accomplished using exactly the same ideas as in [R].

The converse of Theorem 5.12 is not true because Lemma 3.7 does not hold true for the metric of continuity. The following proposition allows an example of a finite graph X such that a component of 2_c^X has infinite dimension.

Note: Each compact subset X of I is an AANR because one can cover X with a finite number of open intervals which are arbitrarily close to the components of X.

5.13 Proposition. Let $X = [0,1]$. Then 2_c^X is connected.

Proof: Let $A \in 2_c^X$. It suffices to prove that there is a path from A to $\{0\}$. Define $f : [0,1] \to 2_c^X$ by $f(t) = \{(1 - t)x \mid x \in A\}$. Then f is a path from A to $\{0\}$. Thus 2_c^X is path connected. Hence 2_c^X is connected.

5.14 Example. It was established in Proposition 5.13 that 2_c^I has only one component. This example shows that the component is infinite dimensional.

Schori and West proved in [S-W] that $2^I \cong Q$. Hence 2^I is infinite dimensional. 2^I can be embedded in 2_c^I as follows: Define $f : 2^I \to 2_c^I$ by $f(A) = A$. Then f is a one-to-one continuous function from a compact metric space. Thus f is a homeomorphism. Therefore, 2_c^I is infinite dimensional.

Example 5.14 shows that d_c is not entirely adequate for obtaining the desired hyperspace results. One problem seems to be that two objects can be close with respect to d_c without having the same number of components. A new hyperspace metric can be defined which eliminates the problem while retaining much of the character of the metric of continuity.

It is well known that d_c can be replaced by an equivalent metric d_c' where $d_c'(A, B) \leq 1$ for all $A, B \in 2^X$. For $A, B \in 2^X$, define $d_{nc}(A, B)$ as follows:

$$d_{nc}(A, B) = \begin{cases} 1 & \text{if the number of components of } A \neq \\ & \text{the number of components of } B \\ d_c' & \text{otherwise.} \end{cases}$$

Certainly closeness with respect to d_{nc} requires the same number of components. It will be shown that this metric does not allow one to have a path between objects with different numbers of components.

Another change that will be helpful is to consider only the ANR subsets of a space X. Thus, $2^X_{nc} = \{A \subseteq X \mid A \text{ is an ANR}\}$.

5.15 __Lemma.__ If $f : I \to 2^X_{nc}$ is a path and $\varepsilon > 0$, then there exists a sequence $0 = a_0 < a_1 < \ldots < a_n = 1$ and ε-maps $\alpha_i\colon f(a_{i-1}) \to f(a_i)$ for $i = 1, \ldots, n$.

Proof: Let $\{x_i\}$ be a sequence in I so that $\{x_i\} \to 0$. Then by continuity, $\{f(x_i)\} \to f(0)$ in 2^X_{nc}. Thus for $\varepsilon > 0$, there exists N so that if $n \geq N$, then $d_{nc}(f(x_n), f(0)) < \varepsilon$. Therefore there exists a map $\alpha_n\colon f(0) \to f(x_n)$ so that $d(y, \alpha_n(y)) < \varepsilon$ for all $y \in f(0)$. Let $a_1 = x_N > 0$, and denote α_N by α_1.

Let $K = \sup M$ where $M = \left\{ k \;\middle|\; \begin{array}{l} \text{there is a finite sequence} \\ 0 = a_0 < a_1 < \ldots < a_m = k, \text{ and} \\ \varepsilon\text{-maps } \alpha_i\colon f(a_{i-1}) \to f(a_i) \end{array} \right\}$.

First observe that $K \varepsilon M$. For there exists a sequence from M, $k_1, k_2, \ldots$, converging to K. Therefore $\{f(k_i)\}$ converges in 2^X_{nc} to $f(K)$. Therefore there exists N so that if $n \geqq N$, there is a map $\alpha : f(k_n) \to f(K)$ so that $d(y, \alpha(y)) < \varepsilon$ for all $y \varepsilon f(k_n)$. Thus there is a finite sequence $0 = a_0 < a_1 < \ldots < k_n \leqq K$. Therefore $K \varepsilon M$.

Now observe that $K = 1$. For if $K < 1$, a sequence $\{x_i\}$ can be chosen so that $\{x_i\} \to K$ with $x_i > K$ for all i. Then, as above, x could be chosen from $\{x_i\}$ satisfying conditions which would contradict $K = \sup M$.

Thus there is a sequence $0 = a_0 < a_1 < \ldots < a_n = 1$ and ε-maps $\alpha_i : f(a_{i-1}) \to f(a_i)$ for $i = 1, \ldots, n$.

The following theorem is due to Ric Ancel.

5.16 Theorem. If J is a simple closed curve in E^3 and U is an open subset of E^3 which intersects J, then there is a compact subset X^* of $U - J$ such that any homotopy which shrinks J to a point must intersect X^*. Furthermore, if X is a one-dimensional compactum in E^3, then X^* can be chosen so that $X^* \cap X = \phi$.

5.17 Theorem. Let X be a one-dimensional space. If $A, B \varepsilon 2^X_{nc}$ such that $H_1(A) \neq H_1(B)$ as subgroups of $H_1(X)$, then A and B lie in different path components of 2^X_{nc}.

Proof: Since A and B are ANR's, their homology is finitely generated. Each of $H_1(A)$ and $H_1(B)$ are generated by the finitely many loops that A and B contain. Since $H_1(A) \neq H_1(B)$, there is a loop J that is contained in exactly one of A or B. Say J is contained in A. Embed X in E^3.

Let U be an open 3-ball so that $U \cap J \neq \phi$ and $U \cap B = \phi$. Then by Theorem 5.16, there exists a compact subset X^* of $U - J$ so that

$X \cap X^* = \phi$ and any homotopy which shrinks J to a point must intersect X^*. Let N be a polyhedral neighborhood of X so that $X^* \cap N = \phi$. Let $\varepsilon > 0$ be chosen so that if f and g are maps into N which are ε-near, then $f \simeq g$.

Suppose there is a path f in 2^X_{nc} from A to B. Then by Lemma 5.15, there exists $0 = a_0 < a_1 < \ldots < a_n = 1$ and ε-maps $\alpha_i : f(a_{i-1}) \to f(a_i)$ for $i = 1,\ldots, n$. Now by the choice of ε, the inclusion map $j_i : f(a_i) \to N$ is homotopic to α_{i+1} for $i = 0,1,\ldots, n-1$. Let H^i denote the homotopies for $i = 0,\ldots, n-1$, where $H^i_t : f(a_i) \to N$, $i/n \leqq t \leqq (i+1)/n$ for $i = 0,\ldots, n-1$, and $H^i_{i/n} = j_i$ for $i = 0,\ldots, n-1$, $H^i_{(i+1)/n} = \alpha_i$ for $i = 1,\ldots, n-1$. Define a homotopy $H : J \to N$ by $H_t(x) = H^i_t\, H^{i-1}_{i/n}\, H^{i-2}_{(i-1)/n} \cdots H^0_{1/n}(x)$ for $t \in [i/n, (i+1)/n]$. This shows that the loop J is homotopic, missing X^*, to a loop in B. In turn, this loop in B is homotopic to a point missing U. Therefore, by Theorem 5.16, X must intersect X^*. But X was chosen so that $X \cap X^* = \phi$, a contradiction.

Hence, there is no path from A to B. Thus A and B are in different path components of 2^X_{nc}.

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