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THE STRUCTURE OF INJECTIVE HULLS OF LIE MODULES

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THE STRUCTURE OF INJECTIVE HULLS

OF

LIE MODULES

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INTRODUCTION

Let A be a finite dimensional abelian Lie algebra over the algebraically closed field k of characteristic zero. It is a classical result that the injective hull $E_A(k)$ of the 1-dimensional trivial A-module k is isomorphic to the k-algebra $k[X_1,\ldots,X_n]$ of polynomials in n indeterminates where n is the k-dimension of A. The representation of A in $E_A(k)$ is defined by sending the basis element e_i of A to the partial derivative with respect to the indeterminate X_i $(1 \leq i \leq n)$ and extending linearly to all of A. We generalize this result to finite dimensional nilpotent Lie algebras N. We show that the injective hull $E_N(k)$ of the 1-dimensional trivial N-module k is isomorphic to $k[X_1,\ldots,X_n]$ where n is the k-dimension of N. The representation of N in $E_N(k)$ is defined by sending the basis elements of N to first-order partial differential operators with polynomial coefficients whose degrees are bounded above by the integer $d-1$ where d is the index of nilpotency of N. We then extend by linearity to all of N.

We say that a finite dimensional Lie algebra L is nilpotent-by-finite if L is the semi-direct product

$N \rtimes H$ of a nilpotent Lie algebra N and an arbitrary Lie algebra H (both N and H are necessarily finite dimensional). We prove that the injective hull $E_L(k)$ of the 1-dimensional trivial L-module k is isomorphic to the tensor product over k of the injective hulls $E_N(k)$ and $E_H(k)$, where the latter modules are equipped with suitable L-module structures. In particular, if L is nilpotent-by-abelian then we can construct the representations of N and H in $E_N(k)$ and $E_H(k)$ respectively by the result mentioned above. We show that in this case $E_L(k)$ is also a polynomial algebra over k in the number of indeterminates equal to the dimension of L. Furthermore, L is represented in $E_L(k)$ by derivations. We shall give several examples of these representations for various solvable and nilpotent Lie algebras.

A left module V for a k-algebra A is said to be locally finite dimensional (locally finite) if the k-dimension of Av is finite for each element v in V. We prove that $E_N(V)$ is a locally finite U(N)-module where U(N) is the universal enveloping algebra of a finite dimensional nilpotent Lie algebra N, and V is a locally finite N-module. It follows immediately by the preceding paragraph that the injective hull $E_L(k)$ will be locally finite if L is nilpotent-by-abelian since the tensor product of two locally finite modules is again locally finite. Furthermore, we obtain this result for any finite dimensional solvable Lie algebra L. Finally, we use the local finiteness of $E_L(k)$ to prove that $E_L(V)$ is locally finite for any locally finite Lie module V over a finite dimensional solvable Lie algebra L. This result has been obtained independently by Stephen Donkin [Do, p. 3, 1.1.1]. Donkin adapts an argument given by K. A. Brown which shows that the injective hull of a locally finite kG-module is locally finite when G is a polycyclic-by-finite group. In fact, it is claimed in [Do, p. 37] that if V is a locally finite Lie module for a finite dimensional Lie algebra L over a field of characteristic zero, then $E_L(V)$ is locally finite if and only if L is solvable.

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0. PRELIMINARIES

All Lie algebras considered in this paper are finite dimensional over an algebraically closed field k of characteristic zero. All unadorned tensor products $\otimes$ will be taken over k .

(0.1) Given any Lie algebra L there is a pair (U,i) consisting of an associative k-algebra with identity and a Lie algebra homomorphism $i : L \to U$ (U has a Lie algebra structure given by $[u_1,u_2] = u_1u_2 - u_2u_1$, $u_1,u_2 \in U$). This pair satisfies the following universal property: given any Lie algebra homomorphism $f : L \to A$ where A is an associative k-algebra with identity, there is a unique algebra homomorphism $F : U \to A$ which maps 1 onto 1 and makes the following diagram commute.

$$\begin{tikzcd}
U \arrow[dr, "F"] & \\
L \arrow[u, "i"] \arrow[r, "f"] & A
\end{tikzcd}$$

It follows from the universal property that U is unique up to isomorphism, and U is generated by the image $i(L)$. U is called the universal enveloping algebra of L and will be denoted by $U(L)$ [J, p. 152].

(0.2) We will give a brief outline of the construction of $U(L)$ for any Lie algebra L. Let $T(L)$ denote the tensor algebra of the k-vector space L. Recall that $T(L) = k1 \oplus T_1(L) \oplus T_2(L) \oplus \ldots \oplus T_j(L) \oplus \ldots$ where $T_1(L) = L$ and $T_j(L) = L \otimes L \otimes \ldots \otimes L$, j times. We have the usual vector space operations in $T(L)$, and $T(L)$ has a multiplication indicated by $\otimes$ and defined by

$$(x_1 \otimes \ldots \otimes x_p) \otimes (y_1 \otimes \ldots \otimes y_q)$$
$$= x_1 \otimes \ldots \otimes x_p \otimes y_1 \otimes \ldots \otimes y_q .$$

We then extend this operation linearly to all of $T(L)$.

Let I be the two-sided ideal in $T(L)$ generated by all the elements of the form $x \otimes y - y \otimes x - [x,y]$, $x,y \in L$. Set $U(L) = T(L)/I$, and let i be the restriction to $L_1 = L$ of the canonical projection of $T(L)$ onto $U(L)$. We have

$$i(x) \otimes i(y) - i(y) \otimes i(x) - i([x,y])$$
$$= (x \otimes y - y \otimes x - [x,y]) + I$$
$$= I = 0 \quad (\text{in } U(L)).$$

Hence, $i([x,y]) = i(x) \otimes i(y) - i(y) \otimes i(x) = [i(x),i(y)]$ in $U(L)$; thus $i : L \to U(L)$ is a Lie algebra homomorphism. The pair $(U(L),i)$ is a universal enveloping algebra for L [J, p. 156, Theorem 2].

(0.3) Let J be a linearly ordered set and suppose that $\{e_j \mid j \in J\}$ is a basis for L. Since $U(L)$ is generated by $i(L)$, $U(L)$ is spanned by 1 and the products $i(e_{j_1})\dots i(e_{j_r})$, $j_1, \dots, j_r \in J$. In fact, $U(L)$ is spanned by 1 and the products $i(e_{j_1})\dots i(e_{j_r})$, where $j_1 \leqslant \dots \leqslant j_r$, since we may rearrange the order of the factors by using the formula

$$i(e_q)\,i(e_p) = i(e_p)\,i(e_q) + i([e_q,e_p]) .$$

The Poincaré-Birkhoff-Witt Theorem states that $U(L)$ has as basis over k 1 and the products $i(e_{j_1})\dots i(e_{j_r})$, $j_1 \leqslant \dots \leqslant j_r$ [J, p. 159, Theorem 3].

A consequence of this theorem is the fact that $i : L \to U(L)$ is injective [J, p. 160, Corollary 1]. Hence we can and will identify L with $i(L)$ in $U(L)$.

(0.4) A Lie module V over a Lie algebra L is a k-vector space together with a Lie algebra homomorphism $\rho : L \to gl(V)$, where $gl(V)$ is the endomorphism ring $End_k(V)$ equipped with a Lie algebra structure given by $[A,B] = AB - BA$, where AB is the usual multiplication of A and B in $End_k(V)$. If $x \in L$ and $v \in V$ we shall often write $x.v$ or xv for $\rho(x)(v)$. The homomorphism ρ is called a representation of L in V.

(0.5) Given a representation $\rho : L \to gl(V)$, there is a unique algebra homomorphism $\sigma : U(L) \to End_k(V)$ such that $\sigma|L = \rho$. Hence V has a unique $U(L)$-module structure determined by ρ . Conversely, if V is a U(L)-module, then there is a representation (i.e. an algebra homomorphism) $\sigma : U(L) \to \mathrm{End}_k(V)$, and σ restricted to L defines a representation of L in V . Hence V is also an L-module. Thus the notions of an L-module and a U(L)-module essentially coincide. In this paper, we will write L-module for U(L)-module, $\mathrm{Hom}_L(\ , \ )$ for $\mathrm{Hom}_{U(L)}(\ , \ )$, and module will always mean left-module.

Definition 0.6 [HS, p. 36]. Let R be any ring with 1 and let M and N be R-modules. An injective R-module homomorphism $\mu : M \to N$ is called essential if for any submodule H of N , $H \neq 0$ implies that $H \cap \mu(M) \neq 0$. If M is regarded as a submodule of N , then N is called an essential extension of M .

(0.7) It is easy to see that N is an essential extension of M if and only if for any $n \in N$, $n \neq 0$, there exists some $r \in R$ such that $rn \in M$, $rn \neq 0$ [HS, p. 37, 9.1].

(0.8) Given any R-module M there is an injective R-module E containing M such that every injective R-module F containing M contains an isomorphic copy of E . This property defines E up to isomorphism and E is called the injective hull (injective envelope) of M [HS, p. 36].

(0.9) If M is a submodule of the injective module F , then there is a maximal essential extension E of M

contained in F . Any such maximal essential extension E is an injective R-module isomorphic to the injective hull of M [HS, p. 37, 9.2].

(0.10) If V is an L-module then the injective hull of V will be denoted by $E_L(V)$.

We shall need the following facts:

(0.11) Let R be a left Noetherian ring with 1 . Then

(i) A direct limit of injective R-modules is injective.

(ii) A direct sum of injective R-modules is injective [Ma, p. 512, 1.2].

Definition 0.12. Let L be a Lie algebra over k and let I be an ideal (two-sided ideal) in U(L) . We say that I is a cofinite ideal if the k-dimension of U(L)/I is finite.

Proposition 0.13. Let V be a finitely generated L-module such that IM = 0 for the cofinite ideal I in U(L) . Then V is finite dimensional over k .

Proof:

Since V is annihilated by I , V may be considered as an R = U(L)/I-module. Let $v_1,\dots,v_n$ be a set of generators for V over U(L) . Then $v_1,\dots,v_n$ also form a set of generators for V over R . We have a surjective homomorphism:

$$\pi : Rv_1 \oplus Rv_2 \oplus \ldots \oplus Rv_n \to V$$

$$(r_1v_1, \ldots, r_nv_n) \longrightarrow \sum_{i=1}^{n} r_iv_i \; .$$

The cyclic modules Rv_i are clearly finite dimensional since the algebra R is finite dimensional and Rv_i is the image of the homomorphism

$$\pi_i : R \to Rv_i \quad (1 \leqslant i \leqslant n)$$
$$1 \to v_i \; .$$

Thus $Rv_1 \oplus \ldots \oplus Rv_n$ is finite dimensional and we conclude that V is also finite dimensional.

THE STRUCTURE OF INJECTIVE HULLS
OF LIE MODULES

CHAPTER I

INJECTIVE HULLS OVER NILPOTENT LIE ALGEBRAS

The main result of this chapter is a description of the representation of a nilpotent Lie algebra N in the injective hull of the 1-dimensional trivial N-module k. The representation space is the k-algebra of polynomials $k[X_1, \ldots, X_n]$, n = k-dimension of N, and the action of N on this space is given by k-derivations. We begin by exhibiting the module structure for the injective hull of the 1-dimensional trivial A-module in the case where A is a 1-dimensional Lie algebra. Indeed, this example provided the motivation to generalize to the nilpotent case.

Proposition 1.1. Let A be a 1-dimensional Lie algebra over k with basis e, and let $k[X]$ be the k-algebra of polynomials in the indeterminate X. Then $k[X]$ with the A-module structure given by $(\alpha e).p(X) = \alpha\frac{d}{dX}\, p(X)$, $\alpha \in k$, $p(X) \in k[X]$, is isomorphic to the injective hull, $E_A(k)$, of the 1-dimensional trivial A-module k .

Proof:

Let $\rho : A \to gl(k[X])$ be the corresponding representation of A given by $\rho(e) = \frac{d}{dX}$ and extending linearly to all of A . Then ρ extends uniquely to a representation of the universal enveloping algebra $\sigma : U(A) \to \mathrm{End}(k[X])$. Since $U(A)$ may be identified with the k-algebra of polynomials in the basis element e , σ is uniquely defined by $\sigma(f(e)) = f(\frac{d}{dX})$, i.e. if $f(e) = a_n e^n + \ldots + a_0 \in U(A)$, then $f(\frac{d}{dX}) = a_n \frac{d^n}{dX^n} + \ldots + a_0 1$, where 1 is the identity map on $k[X]$. We note that $k[X]$, with the U(A)-module structure given by σ , is an essential extension of k . Namely, if $p(X)$ is any nonzero polynomial in $k[X]$ of degree n , then $\sigma(e^n)(p(X)) = a_n n! \neq 0$, where a_n is the leading coefficient of $p(X)$.

To complete the proof we must show that $k[X]$ is an injective U(A)-module. Since $U(A)$ is a principal ideal domain, it suffices to show that $k[X]$ is a divisible U(A)-module [HS, p. 31, 7.1]. Thus let $p(X) \in k[X]$ and $f(e) \in U(A)$ be given. We will show that there exists some $q(X)$ in $k[X]$ satisfying $\sigma(f(e))(q(X)) = p(X)$. It suffices to show that the differential operator $F = f(\frac{d}{dX})$ is a surjective endomorphism of $k[X]$. Since F is clearly k-linear, we show by induction that X^n lies in the image of F for all integers $n \geq 0$. Suppose

$F = a_m \frac{d^m}{dX^m} + \ldots + a_0 1$ and let j be the least integer such that $a_j \neq 0$, $0 \leqslant j \leqslant m$. Then $F\left((a_j j!)^{-1} X^j\right) = 1$. Assume now that all polynomials of degree less than n lie in the image of F . Then $F\left[n!(a_j(n+j)!)^{-1} X^{n+j}\right] = X^n + r(X)$, where $r(X)$ is in $k[X]$ and has degree strictly less than n . By induction, there exists some $s(X)$ in $k[X]$ such that $F(s(X)) = r(X)$. Thus $F\left[n!(a_j(n+j)!)^{-1} X^{n+j} - s(X)\right] = X^n$. Hence F is surjective. Thus $k[X]$ is a divisible, and hence injective $U(A)$-module. Since $k[X]$ is also an essential extension of k , $k[X]$ is isomorphic to the injective hull of k .

Remark 1.2. Note that the representation of A in (1.1) is faithful. We will see later that the representation of N in $E_N(k)$ is faithful when the Lie algebra N is nilpotent.

The following lemmas will be used to prove a structure theorem for the injective hull $E_N(k)$ where N is a nilpotent Lie algebra. In particular, Lemma 1.4 is of independent interest and will be extended to solvable Lie algebras.

Definition 1.3. Let R be a ring and I a two-sided ideal in R . I has the weak Artin-Rees property (weak AR property) if for any finitely generated left R-module V and submodule W of V there exists some positive integer

n such that $I^nV \cap W \leq IW$ [P, pp. 485-486].

Lemma 1.4. Let N be a nilpotent Lie algebra over the field k and let V be a locally finite N-module. Then $E_N(V)$ is locally finite.

Proof:

We first assume that the k-dimension of V is finite. Let I be the kernel of the structure map $U(N) \to End(V)$, where $U(N)$ is the universal enveloping algebra of N. Then I is a two-sided cofinite ideal in $U(N)$ with the weak AR property [Mc, p. 497, 4.2]. Let S be any finite subset of $E_N(V)$ and form the finitely generated submodule M which is generated by V and the subset S. Since I has the weak AR property, there is an integer n such that $I^nM \cap V \leq IV = 0$. Thus M is finite dimensional because I^n is also a cofinite ideal in $U(N)$ [D, p. 82, 2.5.1], and hence every finite subset of $E_N(V)$ is contained in a finite dimensional submodule of $E_N(V)$. Thus $E_N(V)$ is locally finite.

Now suppose V is locally finite. Any finitely generated submodule V_i of V is finite dimensional. Since V is a direct limit of its finitely generated submodules, we have that $V = \sum V_i$ where each V_i is finite dimensional. Now the inclusions $V_i \to V_j$ for $i \leqslant j$ induce injective maps $E_N(V_i) \to E_N(V_j)$. To see this, consider the diagram below, where f_{ij} is the composite $V_i \to V_j \to E_N(V_j)$ and g is inclusion.

$$\begin{tikzcd} V_i \arrow[r, "g"] \arrow[d, "f_{ij}"'] & E_N(V_i) \arrow[dl, "F_{ij}"] \\ E_N(V_j) & \end{tikzcd}$$

The injectivity of $E_N(V_j)$ implies the existence of the map F_{ij} and furthermore, $\ker(F_{ij}) \cap g(V_i) = 0$ because f_{ij} is injective. But $E_N(V_i)$ is an essential extension of V_i and hence $\ker(F_{ij}) = 0$. Thus the $E_N(V_i)$ together with the induced injective maps F_{ij} define a directed system and $\text{dir lim } E_N(V_i) = \sum E_N(V_i)$. Clearly $\sum E_N(V_i)$ is an essential extension of V. We claim that $\sum E_N(V_i)$ is an injective N-module. But this follows from the fact that $U(N)$ is Noetherian and (0.11). Thus $\sum E_N(V_i)$ is an injective N-module which is an essential extension of V and hence is isomorphic to $E_N(k)$. It is now clear that $E_N(V)$ is locally finite since each $E_N(V_i)$ is locally finite by the previous paragraph.

Lemma 1.5. Let N be a nilpotent Lie algebra over the field k, and suppose that V is a finite dimensional essential extension of k. Then

(i) The representation $\rho : N \to gl(V)$ defined by $\rho(x)(v) = xv$, where x is in N and v is in V, is a nil representation, i.e. $\rho(x)$ is a nilpotent endomorphism of V for every x in N.

(ii) There is a positive integer d such that $I^d V = 0$ where I denotes the augmentation ideal of $U(N)$.

Proof:

(i) $V = V_0 \oplus V_1$ where V_0 and V_1 are submodules of V such that for every x in N, $\rho(x) \mid V_0$ is nilpotent and $\rho(x) \mid V_1$ is an automorphism [J, p. 39, 4]. Suppose $0 \neq v$ is an element in V_1. Since V is essential over k, there exists some u in $U(N)$ such that uv is a nonzero element in k. Now uv lies in V_1, but for all x in N we have $0 = x(uv) = \rho(x)(uv)$, contradicting the definition of V_1. Thus $V_1 = 0$.

(ii) Let $0 = V_0 \leq V_1 \leq \cdots \leq V_d = V$ be a composition series for V. Since the factor modules V_i/V_{i-1} are finite dimensional simple modules, they are 1-dimensional [D, p. 13, 1.3.13]. Suppose that for x in N, $\rho(x)(v_i) = \alpha v_i$ (modulo V_{i-1}), where $v_i \in V_i$ is a coset representative for a basis of V_i/V_{i-1}, $1 \leqslant i \leqslant d$, and $\alpha \in k$. Since ρ is nil by (i), $\alpha^r v_i = \rho(x)^r(v_i) = 0$ (modulo V_{i-1}); hence $\alpha = 0$. Thus for any x in N, $xV_i = \rho(x)(V_i) \leq V_{i-1}$, $1 \leqslant i \leqslant d$, and hence $I^d V \leq V = 0$.

Lemma 1.6. Let I be the augmentation ideal in the universal enveloping algebra $U(N)$ of the nilpotent Lie algebra N over the field k. Then $\bigcup_{n\geqslant 0} (U(N)/I^n)^*$ may be identified with the submodule of $U(N)^*$ consisting of those linear functionals which vanish on some power of the ideal I.

Proof:

The surjections $U(N)/I^m \twoheadrightarrow U(N)/I^n$, $m \geq n$, give rise to the injective maps $f_m^n : (U(N)/I^n)^* \to (U(N)/I^m)^*$, and the modules $(U(N)/I^n)^*$ together with the homomorphisms f_m^n form a direct system of N-modules. Since the f_m^n are injective, the direct limit
$\operatorname{dir\,lim} (U(N)/I^n)^* = \bigcup_{n\geq 0} (U(N)/I^n)^*$, and hence the latter set is an N-module. If f is an element of $U(N)^*$ such that $f(I^n) = 0$, then f induces a linear functional $\bar{f}$ in $(U(N)/I^n)^*$. Conversely, if $\bar{g}$ is in $(U(N)/I^n)^*$ we may define a linear functional g in $U(N)^*$ by $g(I^n) = 0$ and $g(u) = \bar{g}(u + I^n)$ for u in $U(N)$. This correspondence is clearly an N-module isomorphism and allows us to identify $(U(N)/I^n)^*$ with those functionals in $U(N)^*$ which vanish on I^n .

<u>Theorem 1.7</u>. Let N be a nilpotent Lie algebra over the field k . Then $E_N(k) = \operatorname{dir\,lim} (U(N)/I^n)^* = \bigcup_{n\geq 0} (U(N)/I^n)^*$.

Proof:

First of all, we show that $(U(N)/I^n)^*$ is an essential extension of k (which we identify with k^* under the usual isomorphism) for every positive integer n . We will make the identifications guaranteed by (1.6). Let f be a nonzero linear functional in $U(N)^*$ which vanishes on I^n . Choose an integer m , $1 \leq m \leq n$, such that $f(I^m) = 0$ and $f(I^{m-1}) \neq 0$. Hence there is some u in I^{m-1} such that $f(u) \neq 0$. Now $U(N)^*$ is an N-module by the action $(x.f)(w) = -f(xw)$, where x is in N , f is in $U(N)^*$, w is in $U(N)$, and the product xw is taken in $U(N)$. This action extends uniquely to an action of $U(N)$ on $U(N)^*$: $(w.f)(v) = f(w^T v)$, where w and v are in $U(N)$ and $w \to w^T$ is the unique anti-automorphism of $U(N)$ such that $x^T = -x$ for all elements x in N [D, p. 73, 2.2.18]. Thus the functional $u.f$ is nonzero and $(u.f)(v) = 0$ for all v in I since $u^T v$ lies in I^m . Hence $u.f$ is an element in $(U(N)/I)^* = k^* = k$.

Next we observe that $U(N)^*$ is an injective N-module. If M is any N-module, then

$$\begin{aligned} \mathrm{Hom}_N(M,U(N)^*) &= \mathrm{Hom}_N\left(M,\mathrm{Hom}_k(U(N),k)\right) \\ &= \mathrm{Hom}_k\left(U(N) \otimes_N M,k\right) \\ &= \mathrm{Hom}_k(M,k) \end{aligned}$$

where the second line follows from adjoint associativity [HS, p. 111, Exercise 7.3], and the third line follows from the natural isomorphism $U(N) \otimes_N M = M$. Since $\mathrm{Hom}_k(\ ,k)$ is an exact functor on N-modules so is $\mathrm{Hom}_N(\ ,U(N)^*)$ and hence $U(N)^*$ is injective.

Thus we have the following inclusions:

$$k = k^* \leq \bigcup_{n\geqslant 0} (U(N)/I^n)^* \leq U(N)^* .$$

Since $U(N)^*$ is injective and contains an isomorphic copy of k , $U(N)^*$ contains a submodule isomorphic to $E_N(k)$ [HS, p. 38, 9.3]. Hence $\bigcup_{n\geq 0} (U(N)/I^n)^* \subseteq E_N(k)$, if we identify $E_N(k)$ with its image in $U(N)^*$ under the isomorphism, because by our first argument, $\bigcup_{n\geq 0} (U(N)/I^n)^*$ is an essential extension of k and $E_N(k)$ is the maximal essential extension of k contained inside of $U(N)^*$. By (1.4), $E_N(k)$ is locally finite, and hence by (1.5), if f is an element of $E_N(k)$ there is a positive integer n such that $I^n.f = 0$. But this implies that f vanishes on I^n , and hence $f \in \bigcup_{n\geq 0} (U(N)/I^n)^*$. Thus the result follows.

In order to make explicit calculations of the injective hull $E_N(k)$ when N is a nilpotent Lie algebra, we will need the following results on the dual of the universal enveloping algebra of an arbitrary Lie algebra. The relevant facts will be summarized here for a finite dimensional Lie algebra. However, these facts may be extended to the case where the Lie algebra is of arbitrary dimension over a field k of characteristic zero [B1, pp. 118-119].

Let L be a finite dimensional Lie algebra over the field k of characteristic zero, and suppose that $e_1, \ldots ,e_n$ is a basis for L over k . The diagonal map $L \to L \times L$ given by $x \to (x,x)$ induces a Lie algebra homomorphism $L \to U(L) \otimes U(L)$ given by $x \to x \otimes 1 + 1 \otimes x$, and hence induces an algebra homomorphism (in fact, a unique algebra homomorphism) $U(L) \to U(L) \otimes U(L)$ called the coproduct of $U(L)$. The coproduct c together with the augmentation map $\varepsilon : U(L) \to k$ give $U(L)$ the structure of a cocommutative bialgebra [B, p. 115]. Thus the dual $U(L)^*$ is an algebra with the multiplication given as follows: if f and g are elements in $U(L)^*$ and u is in $U(L)$, then $fg(u) = f \otimes g(c(u))$. In order to elucidate the structure of this algebra, we choose the following basis for $U(L)$: for $\nu = (\nu_1, \ldots, \nu_n)$ any n-tuple of nonnegative integers define $e_\nu = e_1^{\nu_1} e_2^{\nu_2} \ldots e_n^{\nu_n}/\nu_1! \ldots \nu_n!$. Then $\{e_\nu \mid \nu \in \mathbb{N}\}$ is a basis over k for $U(L)$ by the Poincaré-Birkhoff-Witt Theorem. The following lemma shows that the value of the coproduct c on e_ν has a particularly simple expression.

Lemma 1.8. [D, p. 90, 2.7.2] $c(e_\nu) = \sum_{\mu+\lambda=\nu} e_\mu \otimes e_\lambda$.

Proof:

We first observe that

$$\begin{aligned}
c(e_i^{\nu_i}/\nu_i!) &= \frac{1}{\nu_i!} c(e_i)^{\nu_i} \\
&= \frac{1}{\nu_i!} (e_i \otimes 1 + 1 \otimes e_i)^{\nu_i} \\
&= \frac{1}{\nu_i!} \sum_{\mu_i+\lambda_i=\nu_i} \binom{\nu_i}{\mu_i} e_i^{\mu_i} \otimes e_i^{\lambda_i} \\
&= \sum_{\mu_i+\lambda_i=\nu_i} e_i^{\mu_i} \otimes e_i^{\lambda_i}/\mu_i!\lambda_i! \\
&= \sum_{\mu_i+\lambda_i=\nu_i} e_i^{\mu_i}/\mu_i! \otimes e_i^{\lambda_i}/\lambda_i! \ .
\end{aligned}$$

Hence the result is clear in this case. Finally,

$$
\begin{aligned}
c(e_\nu) &= c(e_1^{\nu_1} e_2^{\nu_2} \ldots e_n^{\nu_n}/\nu_1!\nu_2! \ldots \nu_n!) \\
&= \prod_{i=1}^{n} \sum_{\mu_i+\lambda_i=\nu_i} e_i^{\mu_i}/\mu_i! \otimes e_i^{\lambda_i}/\lambda_i! \\
&= \sum_{\mu_i+\lambda_i=\nu_i} e_1^{\mu_1} e_2^{\mu_2} \ldots e_n^{\mu_n}/\mu_1! \ldots \mu_n! \\
&\qquad \otimes e_1^{\lambda_1} e_2^{\lambda_2} \ldots e_n^{\lambda_n}/\lambda_1! \ldots \lambda_n! \\
&= \sum_{\mu+\lambda=\nu} e_\mu \otimes e_\lambda .
\end{aligned}
$$

Theorem 1.9. [D, p. 90, 2.7.5] Let $(e_1, \ldots, e_n)$ be a basis for a Lie algebra L over a field k of characteristic zero, and let $k[[X_1, \ldots, X_n]]$ denote the algebra of formal power series over k in the n indeterminates $X_1, \ldots, X_n$. If $f \in U(L)^*$, denote the formal power series $\sum_{\nu \in \mathbb{N}^n} f(e_\nu) X^\nu$ by s_f where $X^\nu = X_1^{\nu_1} X_2^{\nu_2} \ldots X_n^{\nu_n}$. Then $f \to s_f$ is an isomorphism of the algebra $U(L)^*$ onto the algebra $k[[X_1, \ldots, X_n]]$.

Proof:

Since the e_ν for all $\nu \in \mathbb{N}^n$ form a basis of $U(L)$, it is clear that s is a k-linear isomorphism of $U(L)^*$ onto $k[[X_1, \ldots, X_n]]$. To see that s is an algebra homomorphism let f and g be elements in $U(L)^*$. Then,

$$
\begin{aligned}
s_{fg} &= \sum_{\nu\in\mathbb{N}^n} fg(e_\nu)X^\nu \\
&= \sum_{\nu\in\mathbb{N}^n} f\otimes g\big(c(e_\nu)\big)X^\nu \\
&= \sum_{\nu\in\mathbb{N}^n}\ \sum_{\mu+\lambda=\nu} f(e_\mu)g(e_\lambda)X^\nu \\
&= \sum_{\mu\in\mathbb{N}^n} f(e_\mu)X^\mu \sum_{\lambda\in\mathbb{N}^n} g(e_\lambda)X^\lambda \\
&= s_f s_g\ .
\end{aligned}
$$

Lemma 1.10. [D, p. 91, 2.7.7] The Lie algebra L in Theorem 1.9 acts via the left corregular representation as a Lie algebra of derivations on the algebra $U(L)^*$.

Proof:

Let f and g be elements in $U(L)^*$ and let x be in L . Then

$$
\begin{aligned}
\big(x.(fg)\big)(e_\nu) &= -(fg)(xe_\nu) \\
&= -(f\otimes g)\big(c(xe_\nu)\big) \\
&= -(f\otimes g)\big((x\otimes 1 + 1\otimes x)\sum_{\mu+\lambda=\nu} e_\mu\otimes e_\lambda \\
&= -\sum_{\mu+\lambda=\nu} f(xe_\mu)g(e_\lambda) - \sum_{\mu+\lambda=\nu} f(e_\mu)g(xe_\lambda) \\
&= \big((x.f)g + f(x.g)\big)(e_\nu)\ .
\end{aligned}
$$

Thus we have a representation $L \to \mathrm{Der}\,\big(U(L)^*\big)$ by (1.10). Furthermore, the image of $\bigcup_{n\geq 0} (U(L)/I^n)^*$ under the isomorphism s in (1.9) is contained in $k[X_1,\ldots,X_n]$. For if f vanishes on I^m for some integer m , then $f(e_\nu) = 0$ for all ν with $|\nu| = \sum_{i=1}^{n} \nu_i \geq m$, and clearly from this one sees that s_f is a polynomial. If the Lie algebra L is nilpotent, then we will show below that the converse is also true, namely that s maps $\bigcup_{n\geq 0} (U(L)/I^n)^*$ onto $k[X_1,\ldots,X_n]$. Using this isomorphism s, we can then identify $E_L(k)$ with $k[X_1,\ldots,X_n]$.

We will first develop some preliminary concepts and notation due to M. Vergne [V, pp. 87-88]. Let N be a nilpotent Lie algebra with corresponding lower central series $N = N_1 \supseteq N_2 \supseteq \ldots \supseteq N_{d+1} = 0$. Recall that $N_i = [N,N_{i-1}]$ for all $i > 1$. Using the Jacobi identity, one can easily show that the N_i are ideals in N, and, by induction on i, one can also show that $[N_i,N_j] \subseteq N_{i+j}$ for all pairs i and j. Hence the lower central series forms a finite filtration on N in the sense that $N_i = N$ for $i \leqslant m$ and $N_i = 0$ for $i \geqslant M$; in fact, in this case take $m = 1$ and $M = d+1$. Thus the graded vector space $gr(N) = \sum N_i/N_{i+1}$ has the structure of a Lie algebra: $[x + N_{i+1}, y + N_{j+1}] = [x,y] + N_{i+j+1}$. Let $x_1,\ldots,x_n$ be a basis of $gr(N)$ consisting of homogeneous elements and define α_i to be the degree of homogeneity of x_i, i.e. $x_i \in N_{\alpha_i}/N_{\alpha_{i+1}}$. Choose a representative e_i in N for each x_i and call α_i the weight of e_i, $wt(e_i) = \alpha_i$. Then $e_1,\ldots,e_n$ form a basis of N. For the sequence $M = \{i_1,\ldots,i_r\}$ of integers from the set $\{1,2,\ldots,n\}$ we define the length of M, $\ell(M) = r$, and define the weight of M, $wt(M) = \alpha_{i_1} + \alpha_{i_2} + \ldots + \alpha_{i_r}$. We denote by M^*

the sequence consisting of the same indices as M but rearranged in ascending order. Finally, set

$e_M = e_{i_1} e_{i_2} \cdots e_{i_r}$.

Lemma 1.11. [V, p. 88, Lemma 3] Let M be a sequence of indices from the set $\{1,2,\ldots,n\}$. Then $e_M = e_{M^*} + \sum C_{A^*} e_{A^*}$ with $\ell(A^*) < \ell(M)$ and $wt(A^*) \geqslant wt(M)$.

Proof:

We will use induction on $\ell(M)$. The case $\ell(M) = 1$ is trivial. Suppose $M = \{i_1, i_2, \ldots, i_r\}$ and let i_s be the smallest index appearing in M . Then

$$e_M = e_{i_s} e_{M'} + \sum_{t=1}^{s-1} e_{i_1} e_{i_2} \cdots [e_{i_{s-t}}, e_{i_s}] \cdots \hat{e}_{i_s} \cdots e_{i_r} ,$$

where $M' = \{i_1, i_2, \ldots, i_{s-1}, i_{s+1}, \ldots, i_r\}$, and $\hat{}$ means delete that term. Now each of the terms in the sum on the right is of length one less than $\ell(M)$, and since $[e_{i_{s-t}}, e_{i_s}]$ is a sum of basis elements of weight greater than or equal to $\alpha_{i_{s-t}} + \alpha_{i_s}$, it follows that each of these terms has weight greater than or equal to $wt(M)$. By induction, each of these terms can be expressed as a sum of ordered monomials of length less than $\ell(M)$ and of weight greater than or equal to $wt(M)$. Since $\ell(M') < \ell(M)$, again it follows by induction that $e_{M'} = e_{(M')^*} + \sum C_{B^*} e_{B^*}$ where $\ell(B^*) < \ell(M')$ and $wt(B^*) \geqslant wt(M')$. Hence, $e_{i_s} e_{M'} = e_{M^*} + \sum C_{B^*} e_{i_s} e_{B^*}$. If we let B' denote the sequence whose first term is i_s and the following terms are just the terms in the sequence B^* arranged in the same order, then $e_{i_s}e_{B^*} = e_{B'}$, and we have $\ell(B') = \ell(B^*) + 1 < \ell(M') + 1 = \ell(M)$. Also, $wt(B') = wt(B^*) + wt(e_{i_s}) \geq wt(M') + wt(e_{i_s}) = wt(M)$. Thus, by induction, each term $e_{B'}$ can be written as a sum of ordered monomials of length less than $\ell(M)$ and of weight greater than or equal to $wt(M)$. Thus the result follows.

Corollary 1.12. Let N be a nilpotent Lie algebra. Then $\bigcup_{n\geq 0} (U(N)/I^n)^*$ is isomorphic to the subalgebra $k[X_1,\ldots,X_n]$ of $k[[X_1,\ldots,X_n]]$ under the isomorphism s in (1.9).

Proof:

It suffices to show that s maps $\bigcup_{n\geq 0} (U(N)/I^n)^*$ onto $k[X_1,\ldots,X_n]$. Let $f = s^{-1}(p) \in U(N)^*$ where p is in $k[X_1,\ldots,X_n]$. Then there exists some positive integer m such that $f(e_\nu) = 0$ for all ordered monomials e_ν with $\ell(\nu) \geq m$. Suppose that the index of nilpotency of N is equal to d, i.e. $N_{d+1} = 0$. Then $1 \leq wt(e_i) \leq d$ for $i = 1,2,\ldots,n$. Furthermore, if $wt(M) \geq dm$, then $e_M = e_{M^*} + \sum C_{A^*}e_{A^*}$ where $\ell(A^*) < \ell(M)$ and $wt(A^*) \geq wt(M)$ by (1.11). Since $d\ell(A^*) \geq wt(A^*) \geq wt(M) \geq dm$, we have that $\ell(A^*) \geq m$, and hence $f(e_M) = 0$. Since I^{dm} consists of linear combinations of monomials e_M with $wt(M) \geq \ell(M) \geq dm$, it follows that $f(I^{dm}) = 0$, whence $f \in \bigcup_{n\geq 0} (U(N)/I^n)^*$.

We can now identify $k[X_1,\ldots,X_n]$ with $\bigcup_{n\geq 0} (U(N)/I^n)^* = E_N(k)$ under the isomorphism s and define an N-module action in the following way: if e_i is a member of a basis of N and $p \in k[X_1,\ldots,X_n]$, let $f = s^{-1}(p)$ and define $e_i.p = s(e_i.f) = \sum_\nu (e_i.f)(e_\nu)X^\nu$. We then extend this action linearly to all of N. Since N acts via derivations on $k[X_1,\ldots,X_n]$ by (1.10), it suffices to determine the polynomials $e_i.X_j$ $(1 \leq i,j \leq n)$. These polynomials correspond under the isomorphism s to the linear functionals $e_i.f_j$ where f_j is the linear functional in $U(N)^*$ which takes on the value 1 at e_j and 0 otherwise. Thus, $e_i.X_j = \sum_\nu (e_i.f_j)(e_\nu)X^\nu$ and we need only to compute the values $(e_i.f_j)(e_\nu)$. By the proof of (1.12), we have $f_j(e_\nu) = 0$ whenever $wt(\nu) > d$. If $\ell(\nu) > d$, then clearly $wt(\nu) > d$, so we need only to compute the value of $e_i.f_j$ on standard monomials e_ν with $\ell(e_\nu) \leq d-1$.

We now give a series of examples of the computation of the representation of N in $E_N(k)$ for various nilpotent Lie algebras N.

(1.13). Let N be an abelian Lie algebra of dimension n over the field k. Let $e_1,\ldots,e_n$ be a basis of N. Then with f_j in $U(N)^*$ defined as above, we have $(e_i.f_j)(1) = -f_j(e_i) = -\delta_{ij}$, where δ_{ij} is defined to be 1 if $i = j$ and 0 otherwise. Note that the index of nilpotency here is 1 so we only had to compute the values $(e_i.f_j)(e_\nu)$ for $\ell(\nu) \leq 0$, i.e. for $e_\nu = 1$. Thus we have a representation $\rho : N \to \mathrm{Der}(k[X_1,\ldots,X_n])$ with $\rho(e_i) = -\frac{\partial}{\partial X_i}$ $(1 \leq i \leq n)$ and extending linearly to all of N. This generalizes (1.1).

(1.14). Let N be a 4-dimensional nilpotent Lie algebra over k with basis e_1,e_2,e_3,e_4 and nonzero brackets given by $[e_2,e_3] = e_1 = [e_2,e_4]$, $[e_3,e_4] = e_1 + e_2$, and e_1 is a central element. Note that the index of nilpotency of N is 3. Thus we need only to compute the values $(e_i.f_j)(e_\nu)$ for $|\nu| \leq 2$. Since e_1 is a central element, $e_1.f_j = 0$ for $j \neq 1$ and $(e_1.f_1)(1) = -f_1(e_1) = -1$. It is clear that $(e_1.f_1)(e_\nu) = 0$ for $|\nu| \geq 1$. Thus $e_1.X_1 = -1$ and $e_1.X_j = 0$ for $j = 2,3,4$. If we let $\rho : N \to \mathrm{Der}(k[X_1,\ldots,X_4])$ be the corresponding representation of N, it is now clear that $\rho(e_1) = -\frac{\partial}{\partial X_1}$. In a similar manner, one can show that $\rho(e_2) = -\frac{\partial}{\partial X_2}$. Next consider $e_3.f_j$. For $j = 2,4$, we have $e_3.f_j = 0$. Also, as above, we have $(e_3.f_3)(1) = -1$, and $(e_3.f_3)(e_\nu) = 0$ for $|\nu| \geq 1$. Finally, $(e_3.f_1)(e_2) = -f_1(e_3e_2) = -f_1(e_2e_3 + [e_3,e_2]) = -f_1(e_2e_3 - e_1) = 1$, and $(e_3.f_1)(e_\nu) = 0$ for all $e_\nu \neq e_2$. Thus, $\rho(e_3) = X_2\frac{\partial}{\partial X_1} - \frac{\partial}{\partial X_3}$.

To complete this example, we look at $e_4.f_j$. One can easily check that the only nonzero values are given by

$$(e_4.f_1)(e_2) = -f_1(e_4e_2) = -f_1(e_2e_4+[e_4,e_2])$$
$$= -f_1(-e_1) = 1 ,$$
$$(e_4.f_1)(e_3) = -f_1(e_4e_3) = -f_1(e_3e_4+[e_4,e_3])$$
$$= -f_1(-e_1-e_2) = 1 .$$

$$\begin{aligned}(e_4.f_1)(e_3^2/2!) &= -f_1(e_4e_3^2/2) \\ &= -\tfrac{1}{2}f_1(e_3e_4e_3 + [e_4,e_3]e_3) \\ &= -\tfrac{1}{2}f_1\big(e_3(e_3e_4+[e_4,e_3]) - e_1e_3 - e_2e_3\big) \\ &= -\tfrac{1}{2}f_1(e_3^2e_4 - e_3e_1 - e_3e_2 - e_1e_3 - e_2e_3) \\ &= -\tfrac{1}{2}f_1(e_3^2e_4 - 2e_1e_3 - 2e_2e_3 - [e_3,e_2]) \\ &= -\tfrac{1}{2}f_1(e_3^2e_4 - 2e_1e_3 - 2e_2e_3 + e_1) \\ &= -\tfrac{1}{2} ,\end{aligned}$$

$$(e_4.f_2)(e_3) = -f_2(e_4e_3) = -f_2(e_3e_4+[e_4,e_3])$$
$$= -f_2(-e_1-e_2) = 1 ,$$

$$(e_4.f_4)(1) = -1 .$$

Thus, $\rho(e_4) = (X_2 + X_3 - \frac{1}{2}X_3^2)\frac{\partial}{\partial X_1} + X_3\frac{\partial}{\partial X_2} - \frac{\partial}{\partial X_4}$.

We summarize:

$$\begin{aligned}\rho : N &\to \mathrm{Der}\big(k[X_1,\ldots,X_4]\big) \\ e_1 &\to -\frac{\partial}{\partial X_1} \\ e_2 &\to -\frac{\partial}{\partial X_2} \\ e_3 &\to X_2\frac{\partial}{\partial X_1} - \frac{\partial}{\partial X_3} \\ e_4 &\to (X_2 + X_3 - \tfrac{1}{2}X_3^2)\frac{\partial}{\partial X_1} + X_3\frac{\partial}{\partial X_2} - \frac{\partial}{\partial X_3} .\end{aligned}$$

We then extend linearly to all of N .

(1.15). Let $\eta(3,k)$ denote the nilpotent Lie algebra of strictly upper triangular 3×3 matrices over k. $\eta(3,k)$ has a basis consisting of the matrices e_{ij} $(1 \leqslant i < j \leqslant 3)$ where e_{ij} is the 3×3 matrix having a 1 in the (i,j) position and zeros elsewhere. The only nonzero commutator is $[e_{12},e_{23}] = e_{13}$. The index of nilpotency is 2 so one only needs to determine the values $(e_{ij}.f_{k\ell})(e_\nu)$ for $|\nu| \leqslant 1$. If we choose e_{13},e_{12},e_{23} as the ordered basis, then the representation has the following form:

$$
\begin{aligned}
\rho : \eta(3,k) &\to \mathrm{Der}\big(k[X_1,\dots,X_4]\big)\\
e_{13} &\to -\frac{\partial}{\partial X_1}\\
e_{12} &\to -\frac{\partial}{\partial X_2}\\
e_{23} &\to X_2\frac{\partial}{\partial X_1} - \frac{\partial}{\partial X_3}\quad .
\end{aligned}
$$

Again we extend linearly to all of $\eta(3,k)$.

(1.16). Let $\eta(4,k)$ have as ordered basis $e_{14},e_{24},e_{13},e_{34},e_{23},e_{12}$. The index of nilpotency is 3 and we have

$$
\begin{aligned}
\rho : \eta(4,k) &\to \mathrm{Der}\big(k[X_1,\dots,X_6]\big)\\
e_{14} &\to -\frac{\partial}{\partial X_1}\\
e_{24} &\to -\frac{\partial}{\partial X_2}\\
e_{13} &\to -\frac{\partial}{\partial X_3}\\
e_{34} &\to X_3\frac{\partial}{\partial X_1} - \frac{\partial}{\partial X_4}
\end{aligned}
$$

$$e_{23} \to -X_4\frac{\partial}{\partial X_2} - \frac{\partial}{\partial X_5}$$

$$e_{12} \to (-X_2 + X_4X_5)\frac{\partial}{\partial X_1} - X_5\frac{\partial}{\partial X_3} - \frac{\partial}{\partial X_6} \ .$$

Again, we extend linearly to all of $\eta(4,k)$.

In all of the examples above, the degree of the polynomial coefficients of the various partial derivatives are bounded above by the integer $d-1$ where d is the index of nilpotency. Also notice that each of these representations of the nilpotent Lie algebra N are faithful.

CHAPTER II

A STRUCTURE THEOREM FOR $E_L(k)$ WHEN L IS NILPOTENT-BY-FINITE

In this chapter we consider Lie algebras L which are the semi-direct product $N \rtimes H$ of nilpotent Lie algebras N by arbitrary (finite dimensional) Lie algebras H. We are assuming that N is an H-module such that H acts as a Lie algebra of derivations on N. We identify N with a nilpotent ideal of L and H with a subalgebra of L [J, p. 18]. The goal of this chapter is to describe the structure of the injective hull of the 1-dimensional trivial L-module $E_L(k)$ in terms of the injective hulls $E_N(k)$ and $E_H(k)$.

Proposition 2.1. Let L be an arbitrary finite dimensional Lie algebra over k with universal enveloping algebra $U(L)$. If K is a subalgebra of L and $U(K)$ is the universal enveloping algebra of K, then any injective L-module is also an injective K-module.

Proof:

$U(L)$ is a free right $U(K)$-module [D, p. 71, 2.2.7], so the functor $M \to U(L) \underset{U(K)}{\otimes} M$ is exact on left K-modules. If E is any injective L-module, then $P \to \mathrm{Hom}_L(P,E)$ is an exact functor on left L-modules P. Hence the composite functor $M \to \mathrm{Hom}_L\left(U(L) \underset{U(K)}{\otimes} M, E\right) = \mathrm{Hom}_K(M,E)$ [HS, p. 111, exercise 7.3] is exact on left K-modules. Thus E is an injective K-module.

Remark 2.2. (2.1) implies that $E_L(k)$ is an injective K-module and hence contains a submodule isomorphic to $E_K(k)$ [HS, p. 38, 9.3].

Now assume that $L = N \rtimes H$ where N is a nilpotent ideal in L and H is a subalgebra of L. In order to prove our structure theorem for $E_L(k)$, we will need to show that the ideal in $U(L)$ generated by N has the weak AR property. The following results will allow us to establish this fact.

Lemma 2.3. Let R be a k-algebra containing 1 which is generated by a finite dimensional Lie subalgebra H and a left Noetherian subalgebra S containing the same 1 as R and satisfying $[H,S] \subseteq S$. Then R is left Noetherian.

Proof:

We first observe that any element of R can be written as a finite sum of terms $s_1 h_{i_1} s_2 \ldots s_d h_{i_d} s_{d+1}$, where either $s_i \in S$ or is omitted, and $\{h_{i_1}, \ldots, h_{i_d}\}$ is any subset of a basis $h_1, \ldots, h_n$ of H. We define the number of inversions of an s_i in this term to be the number of h's preceding s_i, and we define the number of inversions in a term of the above type to be the sum of the inversions of the s_i's appearing in the term [compare Mc, pp. 488-489].

We show that any term $s_1 h_{i_1} \ldots s_d h_{i_d} s_{d+1}$ can be written as a sum of terms of the form $s h_{j_1} h_{j_2} \ldots h_{j_e}$, where $s \in S$ and $\{h_{j_1}, \ldots, h_{j_e}\}$ is a subset of the basis for H. We induct on the number of inversions appearing in a term and on the number d of h's appearing in a term. For $d = 1$, the term is either $s_1 h_{i_1}$, in which case we are done, or $s_1 h_{i_1} s_2$. By hypothesis, $[h_{i_1}, s_2] = s' \in S$; hence $s_1 h_{i_1} s_2 = s_1 s_2 h_{i_1} + s_1 s'$, and the terms on the right are in the desired form. Suppose now that the result holds for all terms with the same number of h's but fewer inversions, and for all terms with fewer than d h's. Again, by hypothesis, $[h_{i_d}, s_{d+1}] = s' \in S$; hence,

$$s_1 h_{i_1} s_2 \ldots s_d h_{i_d} s_{d+1} = s_1 h_{i_1} s_2 \ldots h_{i_{d-1}} s_d s_{d+1} h_{i_d} + s_1 h_{i_1} s_2 \ldots h_{i_{d-1}} s_d s' .$$

The first term on the right has fewer inversions than the left-hand side, and the second term on the right has less than d h's. By induction, these terms can be rewritten as a finite sum of terms in the desired form. Hence the term on the left can also be expressed in this form.

Next, given any term $sh_{j} \ldots h_{j_e}$, by using the switching process employed in the Poincaré-Birkhoff-Witt Theorem [J, pp. 157-159], we may express this term as a sum of ordered monomials in the $h_1,\ldots,h_n$ with coefficients on the left in S . Note, however, that this expression is not necessarily unique.

We define a filtration on R in the following way: $R_p = 0$ for any negative integer p ; $R_0 = S$; and R_p = sums of ordered monomials in $h_1,\ldots,h_n$ with coefficients on the left in S whose total degree in the h's is less than or equal to the positive integer p . By the above paragraph, $R = \bigcup_p R_p$, and clearly $\bigcap_p R_p = 0$. Since $R_{-1} = 0$, the induced topology on R is discrete (hence complete) and Hausdorff. If $sh_1^{i_1}\ldots h_n^{i_n} \in R_p$ and $s'h_1^{j_1}\ldots h_n^{j_n} \in R_q$, then $sh_1^{i_1}\ldots h_n^{i_n} s'h_1^{j_1}\ldots h_n^{j_n} = ss'h_1^{i_1+j_1}\ldots h_n^{i_n+j_n}$ modulo lower degree terms, since switching s' with the h's does not introduce any new h's , and rearranging the h's does not increase the total degree [compare J, p. 157, 1]. Thus $R_pR_q \subseteq R_{p+q}$ and R is a filtered ring.

We can now make use of the associated graded algebra $gr(R) = \sum R_p/R_{p-1}$. Given any two basis elements of H , say h_i and h_j , we have

$$(h_i+R_0)(h_j+R_0) - (h_j+R_0)(h_i+R_0) = [h_i,h_j] + R_1 = R_1 .$$

Also, given $s \in S = R_0/R_{-1}$, and an h_i , we have

$(h_i+R_0)(s+R_{-1}) - (s+R_{-1})(h_i+R_0) = [h_i,s] + R_0 = R_0$, because $[h_i,s] \in S = R_0$. Thus gr(R) is isomorphic to the algebra over S generated by the central elements $h_i + R_0$ $(1 \leq i \leq n)$. If $y_1,\ldots,y_n$ are indeterminates which commute with S and with each other, then gr(R) is a homomorphic image of the polynomial algebra over S , $S[y_1,\ldots,y_n]$. Thus gr(R) is left Noetherian because $S[y_1,\ldots,y_n]$ is left Noetherian by the Hilbert Basis Theorem [L, pp. 70-71]. This implies that R is left Noetherian [B2, p. 42, Corollary 2].

Remark 2.4. If, in Lemma 2.3, S is right Noetherian, then the same proof shows that R is right Noetherian.

We shall need some additional notions related to the weak AR property for ideals in a ring R . Following the terminology given in Passman [P, p. 488], we say that an ideal I in a ring R with 1 is polycentral of height t if there exists a finite series of ideals in R

$$I = I_0 \supseteq I_1 \supseteq \ldots \supseteq I_t = 0$$

such that for each j , $0 \leq j \leq t-1$, I_j/I_{j+1} is a centrally generated ideal of R/I_{j+1} . Given such an ideal I , we define $\tilde{R}(I) = \tilde{R}(I:I_0,I_1,\ldots,I_t)$ to be the subring of the polynomial ring $R[Z]$ generated by R and $I_jZ, I_jZ^2,\ldots,I_jZ^{2^j}$ for $j = 0,1,\ldots,t$. The following two lemmas show the relation between polycentral ideals in a left (resp. right) Noetherian ring and the weak AR property.

Lemma 2.5. [P, p. 489, 2.6] Let R be a left (resp. right) Noetherian ring, and let I be a polycentral ideal of height t with corresponding central series $I = I_0 \supseteq I_1 \supseteq \dots \supseteq I_t = 0$. Then $\tilde{R}(I : I_0, I_1, \dots, I_t)$ is left (resp. right) Noetherian.

Lemma 2.6. [P, p. 491, 2.7] Let R be a ring as in (2.5), and let $I = I_0 \supseteq I_1 \supseteq \dots \supseteq I_t = 0$ be a chain of ideals in R. If $\tilde{R}(I) = \tilde{R}(I : I_0, I_1, \dots, I_t)$ is left (resp. right) Noetherian, then I has the weak AR property.

The next proposition is essentially a theorem of Roseblade [P, p. 492, 2.9]. Roseblade's theorem deals with a ring R with 1 which is generated by a right Noetherian subring S with the same 1 as R and by a polycyclic-by-finite group of units G such that the action of G on S by conjugation stabilizes S, i.e. $S^G = S$ where $S^G = \left\{x^{-1}sx \mid x \in G, \ s \in S\right\}$. Here we are replacing R by a k-algebra with 1, S by a subalgebra with the same 1 as R, and G by a finite dimensional Lie subalgebra H such that the commutator in R of an element in H with an element in S lies in S. This result will be useful in proving the structure theorem for $E_L(k)$ where L is the semi-direct product $N \rtimes H$ as described above.

<u>Proposition 2.7</u>. Let R be a k-algebra containing 1 which is generated as a ring by a finite dimensional Lie sub-algebra H and a left Noetherian subalgebra S containing the same 1 as R and satisfying $[H,S] \subseteq S$. Suppose that I is a polycentral ideal in S such that $[H,I] \subseteq I$. If V is a finitely generated left R-module and if U is a sub-module, then there exists an integer d such that $I^d V \cap U \subseteq IU$. Furthermore, $RI = IR$ has the weak AR property.

Proof:

We will follow the proof of Roseblade's theorem as given in [P, p. 492, 2.9] and make the necessary adjustments wherever needed.

Let $I = Y_0 \supseteq Y_1 \supseteq \dots \supseteq Y_t = 0$ be the given central series for I such that Y_j is an ideal of S that is centrally generated modulo Y_{j+1}. Let $h_1, \dots, h_n$ be a basis for H and define

$$I_j = \sum_{h \in H} [h, Y_j] + Y_j = \sum_{i=1}^{n} [h_i, Y_j] + Y_j \,,$$
$$0 \leqslant j \leqslant t \,.$$

It is clear that I_j is an additive subgroup of S since Y_j is an ideal in S and the Lie bracket is bilinear. To show that I_j is an ideal in S, it suffices to show that for $s, s' \in S$, we have $s[h_i, y_j]s' \in I_j$ for any $y_j \in Y_j$ and $1 \leqslant i \leqslant n$, $0 \leqslant j \leqslant t$. But,

$$\begin{aligned} s[h_i,y_j]s' &= sh_iy_js' - sy_jh_is' \\ &= h_isy_js' + s_i'y_js' - sy_js'h_i - sy_js_i'' \\ &= [h_i,sy_js'] + (s_i'y_js' - sy_js_i'') \in I_j , \end{aligned}$$

where $s_i' = [s,h_j] \in S$ and $s_i'' = [h_i,s'] \in S$. Next, we claim that I_j is centrally generated modulo I_{j+1} , where $0 \leqslant j \leqslant t-1$. If $y_{j1} + Y_{j+1}, \ldots ,y_{jn_j} + Y_{j+1}$ are central generators for Y_j/Y_{j+1} in S/Y_{j+1} , it is clear that $y_{j1} + I_{j+1}, \ldots ,y_{jn_j} + I_{j+1}$ and $[h_i,y_{jk}] + I_{j+1}$ generate I_j/I_{j+1} for $1 \leqslant k \leqslant n_j$ and $1 \leqslant i \leqslant n$. It remains to show that these elements are central in S/I_{j+1} . This is clear for the $y_{jk} + I_{j+1}$. On the other hand, if $s + I_{j+1} \in S/I_{j+1}$, we have

$$\begin{aligned} &[[h_i,y_{jk}] + I_{j+1} , s + I_{j+1}] \\ &\quad = [[h_i,y_{jk}],s] + I_{j+1} \\ &\quad = [[h_i,s],y_{jk}] + [h_i,[y_{jk},s]] + I_{j+1} \quad \text{by the Jacobi identity} \\ &\quad = I_{j+1} , \end{aligned}$$

since $[h_i,s] \in S$ and $[y_{jk},s] \in Y_{j+1}$. Thus, $I = I_0 \supseteq I_1 \supseteq \ldots \supseteq I_t = 0$ is a series of ideals in S such that I_j/I_{j+1} is centrally generated in S/I_{j+1} , and, by construction, $[H,I_j] \subseteq I_j$ for $0 \leqslant j \leqslant t$.

Since $[H,S] \subseteq S$ and $[H,I_j] \subseteq I_j$ for $0 \leqslant j \leqslant t$, by using an argument similar to that given in the first part of the proof of (2.3), one can show that $RI_j = I_jR$ (recall that R is generated by S and H). Thus, $X = IR$ and $X_j = I_jR$ are ideals in R. The following computations all occur in the polynomial algebra $R[Z]$.

$$\begin{aligned}\tilde{R}(X) &= \tilde{R}(X : X_0,\ldots,X_t)\\ &= \langle R, X_jZ^q \mid 0 \leqslant j \leqslant t\ , 1 \leqslant q \leqslant 2^j \rangle\\ &= \langle R, I_jZ^q \mid 0 \leqslant j \leqslant t\ , 1 \leqslant q \leqslant 2^j \rangle\\ &= \langle S, I_jZ^q, H \mid 0 \leqslant j \leqslant t\ , 1 \leqslant q \leqslant 2^j \rangle\\ &= \langle \tilde{S}(I), H \rangle .\end{aligned}$$

Since I is polycentral in S, $\tilde{S}(I)$ is left Noetherian by (2.5), and furthermore, it is easily seen that $[H,\tilde{S}(I)] \subseteq \tilde{S}(I)$. Thus (2.3) implies that $\tilde{R}(X) = \langle \tilde{S}(I), H \rangle$ is also left Noetherian, and we conclude by (2.6) that X has the weak AR property. Finally, if V is any left R-module, then, because $X = RI = IR$, we have $X^nV = I^nV$, and the result follows.

<u>Lemma 2.8</u>. Let $L = N \rtimes H$ where N is a nilpotent Lie algebra and H is an arbitrary (finite dimensional) Lie algebra. Then $E_N(k) \otimes E_H(k)$ is an L-module and is an essential extension of the 1-dimensional trivial L-module k.

Proof:

Let $E = E_N(k) \otimes E_H(k)$, $X = E_N(k)$, and $Y = E_H(k)$. By (1.7), $X = \bigcup_{n\geqslant 0} (U(N)/I^n)^*$ where I is the augmentation ideal of $U(N)$. When we make the usual identifications of H with a subalgebra of L and N with an ideal of L, H acts on N as a Lie algebra of derivations of $U(N)$ [D, p. 79, 2.4.9]. Let D_h denote the derivation of $U(N)$ which extends $ad_L(h)$, where $h \in H$. Then $D_h(u) = hu - uh$ for any $u \in U(N)$ [D, p. 79, 2.4.9]. We show that $D_h(I^n) \subseteq I^n$ for every positive integer n. For $n = 1$, it suffices to show that $D_h(n_1 n_2 \ldots n_q) \in I$ where $n_i \in N$, $1 \leq i \leq q$, since I consists of sums of products of elements in N_q. But,

$$D_h(n_1 n_2 \ldots n_q) = \sum_{i=1}^{q} n_1 \ldots n_{i-1}[h,n_i]n_{i+1} \ldots n_q \in I$$

since $[h,n_i] \in N$ for all i. In general, $D_h(I^n) = D_h(I)I^{n-1} + ID_h(I^{n-1}) \subseteq I^n$ by induction. Thus $U(N)/I^n$ and also $(U(N)/I^n)^*$ are L-modules. This implies that X is an L-module. Furthermore, I is an L-module via projection of L onto H, and since the kernel of this projection is N, it is clear that Y is a trivial N-module. Thus the tensor product E is an L-module.

We now show that E is an essential extension of k. Since Y is a trivial N-module, $E \cong X^\alpha$ as an N-module, where α is the k-dimension of Y. By (1.4) and (1.5), every element of X is annihilated by a power of I. Hence the same is true for elements of E. Given an element $x \in E$, $x \neq 0$, we can find a least integer n with the property that $I^n x = 0$. Then there exists some $v \in I^{n-1}$ with $vx \neq 0$. Since $I(vx) = 0$ implies that $z(vx) = 0$ for every $z \in N$, vx is an element of the

N-invariants of E , E^N . Since E^N is canonically an $L/N = H$-module which is clearly essential over k as an H-module, we have $E^N \leq Y$. But since $k \leq X$, $E = X \otimes Y$ contains a copy of Y , namely $1 \otimes Y$, and this implies that $Y = Y^N \leq E^N \leq Y$, and hence $Y = E^N$. Thus we can find $w \in U(H)$ (considered as a subalgebra of $U(L)$) such that $w(vx) = (wv)x \neq 0$ and $(wv)x \in k$. Since $wv \in U(L)$, the result follows.

We are now prepared to prove the structure theorem for $E_L(k)$.

Theorem 2.9. Let $L = N \rtimes H$ where N is a nilpotent Lie algebra and H is an arbitrary Lie algebra (N and H are finite dimensional). Then $E_N(k) \otimes E_H(k)$ and $E_L(k)$ are isomorphic as L-modules.

Proof:

We will use the notation set up in the proof of (2.8). By (2.8), $E \leq E_L(k)$. Since $E \simeq X^\alpha$ as an N-module where α is the k-dimension of Y and $U(N)$ is left Noetherian, it follows that E is an injective N-module because $X = E_N(k)$ is an injective N-module and the direct sum of injective modules is injective in a left Noetherian ring (0.11). Hence the injective hull $E_L(k)$, when considered as an N-module, is isomorphic to the direct sum $X^\beta \oplus E'$, where β is some cardinal number greater than or equal to the k-dimension of Y , and E' is a direct sum of indecomposable, injective N-submodules of $E_L(k)$ which are not isomorphic to $E_N(k) = X$ [Ma, p. 516, 2.5].

We claim that $E' = 0$. Suppose that $x \in E_\gamma$ where E_γ is one of the indecomposable, injective summands in E'. Form the finitely generated L-submodule $V = k + U(L)x \leq E_L(k)$. Since N is a nilpotent Lie algebra, the augmentation ideal I of $U(N)$ is a polycentral ideal [Mc, p. 498, 4.2]. Also, $[H,N] \leq N$ implies that $[H,I] \leq I$, since I is generated by products of elements in N. It is clear that $U(L) = \langle U(N),H \rangle$, so by (2.7) with $S = U(N)$, we have $I^nV \cap k \leq Ik = 0$ for some positive integer n. Now if J denotes the ideal in $U(L)$ generated by I, then $J = U(L)I = IU(L)$ [D, p. 72, 2.2.14], and $I^nV = J^nV$ implies that I^nV is an L-submodule of $E_L(k)$. Thus $I^nV = 0$. We can therefore choose d such that $I^dx \neq 0$ but $I^{d+1}x = 0$, and hence there exists some $u \neq 0$ in I^d such that $ux \neq 0$. Since $I(ux) = 0$, the k-span of ux is isomorphic to the 1-dimensional trivial N-module k. Let T denote the k-span of ux. Since E_γ is an injective hull of every one of its nonzero submodules [Ma, p. 514, 2.2], $E_\gamma = E_N(T) = E_N(k)$, contradicting the definition of E'. Thus $T = 0$, and hence $x = 0$. Since x was chosen arbitrarily from E_γ and E_γ was any one of the summands of E', it follows that $E_\gamma = 0$; hence, $E' = 0$. Therefore, as an N-module, $E_L(k) = X^\beta$.

Next, by the proof of (2.8), we have $Y = E^N$. It is clear that $E_L(k)^N$ is an essential extension of k when considered as an $L/N = H$-module; hence $E_L(k)^N \subseteq E_H(k) = Y$. But $Y = E^N \subseteq E_L(k)^N$ and hence $Y = E_L(k)^N$. By the previous paragraph, $E_L(k) = X^\beta$ as an N-module, where $\beta \geqslant \alpha$ = k-dimension of Y . But $X^\beta = E_N(k)^\beta = E_N(k^\beta)$ [Ma, p. 514, 2.1]; hence $Y = E_L(k)^N = E_N(k^\beta)^N = k^\beta$ as N-modules, and therefore the k-dimension of Y is equal to β . Thus $E_L(k) = E_N(Y) = E_N(E^N)$ as N-modules. Since E is an injective N-module containing E^N , we have $E_N(E^N) \subseteq E$, and this implies that $E = E_L(k)$.

CHAPTER III

INJECTIVE HULLS OVER SOLVABLE LIE ALGEBRAS

In Chapter I we showed that the injective hull of a locally finite module over a nilpotent Lie algebra is locally finite. By using the structure theorem (2.9), we will be able to extend this result to locally finite modules over solvable Lie algebras. We will also give some examples of $E_L(k)$ in the case where L is a solvable Lie algebra of the form $N \rtimes A$. In this case N denotes a nilpotent ideal and A an abelian subalgebra of L. In particular, the solvable Lie algebra $\tau(n,k)$ of upper triangular $n \times n$ matrices with entries in the field k is of this form: take N to be $\eta(n,k)$ = strictly upper triangular $n \times n$ matrices, and $A = \delta(n,k)$ = diagonal $n \times n$ matrices.

Lemma 3.1. Let L be a solvable Lie algebra over the algebraically closed field k of characteristic zero. Then $E_L(k)$ is locally finite.

Proof:

Since we are assuming that L is finite dimensional over k, Ado's theorem implies that there is a finite dimensional representation $\rho : L \to g\ell(n,k)$, where n = dimension of the representation [J, p. 202]. The image $\rho(L)$ lies in the solvable subalgebra $\tau(n,k)$ of $g\ell(n,k)$. We identify L with $\rho(L)$ and consider ρ as an inclusion. Then by (2.1), $E_{\tau(n,k)}(k)$ is an injective L-module, and hence there is a submodule of $E_{\tau(n,k)}(k)$ isomorphic to $E_L(k)$ [HS, p. 38, 9.3]. Therefore, if $E_{\tau(n,k)}(k)$ is locally finite then so is $E_L(k)$. But since $\tau(n,k) = \eta(n,k) \rtimes \delta(n,k)$, by (2.9), $E_{\tau(n,k)}(k) = E_{\eta(n,k)}(k) \otimes E_{\delta(n,k)}(k)$ and both $E_{\eta(n,k)}(k)$ and $E_{\delta(n,k)}(k)$ are locally finite by (1.4). Therefore $E_{\tau(n,k)}(k)$ is locally finite because it is a tensor product of two locally finite modules.

Theorem 3.2. Let S be an irreducible module over a solvable Lie algebra L . Then $E_L(S) = S \otimes E_L(k)$ and hence $E_L(S)$ is locally finite.

Proof:

By Lie's theorem [J, p. 50], S is 1-dimensional. We first show that $S \otimes E_L(k)$ is an injective L-module. Let W be any L-module. Then

$$\begin{aligned}\mathrm{Hom}_L(W, S\otimes E_L(k)) &= \mathrm{Hom}_L(W,(S^*)^*\otimes E_L(k))\\ &= \mathrm{Hom}_L(W,\mathrm{Hom}_k(S^*,E_L(k)))\\ &= \mathrm{Hom}_L(S^*\otimes W, E_L(k)) .\end{aligned}$$

Since S^* is a free k-module and $E_L(k)$ is an injective L-module, the composite functor $W \to \mathrm{Hom}_L(S^*\otimes W, E_L(k))$ is exact. Thus, $\mathrm{Hom}_L(\ ,S\otimes E_L(k))$ is an exact functor and $S\otimes E_L(k)$ is an injective L-module [HS, p. 105, 5.6].

Now $S\otimes E_L(k)$ contains an isomorphic copy of S since $k \leq E_L(k)$ and hence $E_L(S) \leq S\otimes E_L(k)$ because $S\otimes E_L(k)$ is injective [HS, p. 38, 9.3]. Since $E_L(S)$ is also injective, we have $S\otimes E_L(k) = E_L(S)\oplus E'$ for some L-module E'. We must show that $E' = 0$. But, $S^*\otimes(S\otimes E_L(k)) = (S^*\otimes E_L(S)) \oplus (S^*\otimes E')$; since $\dim_k S = 1$, we have $S^*\otimes S = k$, and hence, $S^*\otimes(S\otimes E_L(k)) = (S^*\otimes S)\otimes E_L(k) = k\otimes E_L(k) = E_L(k)$. We conclude that $E_L(k) = (S^*\otimes E_L(S)) \oplus (S^*\otimes E')$. Now $E_L(k) = E_L(U(L)/I)$ where I is the augmentation ideal of $U(L)$; therefore, $E_L(k)$ is an indecomposable, injective L-module since I is an irreducible ideal in $U(L)$ [Ma, p. 515, 2.4]. Thus $E_L(k)$ cannot have any proper nonzero direct summands, and we conclude that $S^*\otimes E' = 0$; hence $E' = 0$. Thus $S\otimes E_L(k) = E_L(S)$. Finally, it is clear that $E_L(S)$ is locally finite because both $E_L(k)$ and S are locally finite.

<u>Corollary 3.3</u>. Let L be a solvable Lie algebra over the algebraically closed field k of characteristic zero. Let V be a locally finite L-module. Then $E_L(V)$ is also locally finite.

Proof:

By using an argument similar to that given in the proof of (1.4), it suffices to consider the case where V

is finite dimensional. Since the field k is algebraically closed and characteristic zero, V contains irreducible submodules by Lie's theorem. If $\{S_\lambda\}_{\lambda\in\Lambda}$ is the collection of all irreducible submodules of V, then the socle of V, $\mathrm{Soc}(V)$, is equal to the direct sum of a certain subcollection of the S's, say $\mathrm{Soc}(V) = \bigoplus_{\alpha\in A} S_\alpha$ where $A \subseteq \Lambda$ [L, p. 60]. Thus $E_L(\mathrm{Soc}(V)) = \bigoplus_\alpha E_L(S_\alpha)$ [Ma, p. 514, 2.1]. By (3.2) each $E_L(S_\alpha)$ is locally finite and hence $E_L(\mathrm{Soc}(V))$ is also locally finite since a direct sum of locally finite modules is locally finite.

To complete the proof, we show that $E_L(V) = E_L(\mathrm{Soc}(V))$. If U is any nonzero submodule of V, then U has a composition series $U = U_0 \supseteq U_1 \supseteq \dots \supseteq U_d \supseteq U_{d+1} = 0$ and U_d is irreducible by Lie's theorem. Thus $U \cap \mathrm{Soc}(V) \neq 0$, and hence V is an essential extension of $\mathrm{Soc}(V)$. Therefore, $V \subseteq E_L(\mathrm{Soc}(V))$, and since $E_L(\mathrm{Soc}(V))$ is an injective L-module containing V, $E_L(V) \subseteq E_L(\mathrm{Soc}(V))$. But clearly, $E_L(\mathrm{Soc}(V))$ is contained in $E_L(V)$ since $\mathrm{Soc}(V) \subseteq V$. Thus $E_L(V) = E_L(\mathrm{Soc}(V))$ and the result follows.

<u>Remark 3.4</u>. The local finiteness of $E_L(V)$ when V is locally finite has also been obtained independently by Stephen Donkin. See [Do].

We will now give some examples of $E_L(k)$ where L is a solvable Lie algebra of the form $N \rtimes A$. These calculations will make use of (1.7) and (2.9).

(3.5). Let L be the 2-dimensional non-abelian Lie algebra over the field k with basis e_1, e_2 and bracket $[e_1, e_2] = e_1$. Then $L = N \rtimes A$, where $N = [L,L] = ke_1$, and $A = ke_2$. By (1.13), $E_N(k) = k[X_1]$ with the basis element e_1 sent to $-\frac{\partial}{\partial X_1}$, and $E_A(k) = k[X_2]$ with $e_2 \to -\frac{\partial}{\partial X_2}$. By (2.9), $E_L(k) = E_N(k) \otimes E_A(k) = k[X_1, X_2]$. We only need to determine the action of e_2 on $E_N(k)$, and it suffices to calculate $e_2.X_1$ since A acts on N via derivations. Recall that X_1 corresponds to the linear functional f_1 in $U(N)^*$ satisfying $f_1(e_1) = 1$ and $f_1(e_\nu) = 0$ for all $e_\nu \neq e_1$. Now $e_2.f_1(e_1) = -f_1([e_2, e_1]) = -f_1(-e_1) = 1$. Furthermore, $e_2.f_1(e_1^{\nu_1}/\nu_1!) = 0$ for all $\nu_1 \neq 1$. Thus we have the representation

$$\begin{aligned} L &\to \mathrm{Der}(k[X_1, X_2]) \\ e_1 &\to -\frac{\partial}{\partial X_1} \\ e_2 &\to X_1\frac{\partial}{\partial X_1} - \frac{\partial}{\partial X_2} \end{aligned}$$

obtained by extending linearly to all of L.

(3.6). Let L be a 3-dimensional solvable Lie algebra over the field k with basis e_1, e_2, e_3. The brackets are given by $[e_1, e_3] = -e_1, [e_2, e_3] = -e_1 + e_2$, and all other brackets are zero. We have $[L,L] = ke_1 + ke_2$ is an abelian ideal in L and L is the semi-direct product

$[L,L] \rtimes ke_3$. Since $N = [L,L]$ is abelian, by (1.13) we know that $E_N(k) = k[X_1,X_2]$ with $e_1 \to -\frac{\partial}{\partial X_1}$ and $e_2 \to -\frac{\partial}{\partial X_2}$. Also by (1.13), $E_A(k) = k[X_3]$ where $A = ke_3$ and $e_3 \to -\frac{\partial}{\partial X_3}$. Thus we only have to determine the action of e_3 on X_1 and X_2. Let f_1 (resp. f_2) be the linear functional in $U(N)^*$ which takes the value 1 at e_1 (resp. e_2) and zero otherwise. Note that f_1 (resp. f_2) corresponds to X_1 (resp. X_2) under the isomorphism given in (1.12). If $f \in U(N)^*$, $u \in U(N)$, then $e_3.f(u) = -f([e_3,u]) = -f(e_3u - ue_3)$. We have

$$e_3.f_3(e_2) = -f_2([e_3,e_2]) = -f_2(e_1-e_2) = 1 ,$$

$$\begin{aligned}
e_3.f_2(e_1^{\nu_1}e_2^{\nu_2}/\nu_1!\nu_2!) &= \frac{-1}{\nu_1!\nu_2!}f_2(\nu_1[e_3,e_1]e_1^{\nu_1-1}e^{\nu_2} \\
&\quad + \nu_2 e^{\nu_1}[e_3,e_2]e_2^{\nu_2-1}) \\
&= \frac{-1}{\nu_1!\nu_2!}f_2(\nu_1 e_1^{\nu_1}e_2^{\nu_2} \\
&\quad + \nu_2 e_1^{\nu_1}(e \; - e_2)e_2^{\nu_2-1}) \\
&= \frac{-1}{\nu_1!\nu_2!}f_2((\nu_1-\nu_2)e_1^{\nu_1}e_2^{\nu_2} \\
&\quad + \nu_2 e_1^{\nu_1+1}e_2^{\nu_2-1}) \\
&= 0 \quad \text{if} \quad (\nu_1,\nu_2) \neq (0,1) .
\end{aligned}$$

Thus $e_3.X_2 = X_2$. Similarly, one can show that the only nonzero values for $e_3.f_1$ are given by

$$e_3.f_1(e_2) = -f_1([e_3,e_2]) = -f_1(e_1 - e_2) = -1 ,$$

and

$$e_3.f_1(e_1) = -f_1([e_3,e_1]) = -f_1(e_1) = -1 .$$

Thus we have the representation

$$\begin{aligned} L &\to \mathrm{Der}(k[X_1,X_2,X_3]) \\ e_1 &\to -\frac{\partial}{\partial X_1} \\ e_2 &\to -\frac{\partial}{\partial X_2} \\ e_3 &\to -(X_1 + X_2)\frac{\partial}{\partial X_1} - \frac{\partial}{\partial X_2} . \end{aligned}$$

The next examples are calculated in the same manner as the previous ones. However, since these calculations become exceedingly tedious, we will omit the details. The general form of the representation of $\tau(n,k)$ in $E_{\tau(n,k)}(k)$ is unknown to the author at this time, although theoretically these calculations can be accomplished using the method outlined in the previous examples.

(3.7). $L = \tau(2,k)$. L has the basis e_{12},e_{11},e_{22} with brackets $[e_{12},e_{22}] = e_{12}$, $[e_{12},e_{11}] = -e_{12}$, and all other brackets are zero. The representation is given by

$$\begin{aligned} L &\to \mathrm{Der}(k[X_1,X_2,X_3]) \\ e_{12} &\to -\frac{\partial}{\partial X_1} \\ e_{11} &\to -X_1\frac{\partial}{\partial X_1} - \frac{\partial}{\partial X_2} \\ e_{22} &\to X_1\frac{\partial}{\partial X_1} - \frac{\partial}{\partial X_3} \end{aligned}$$

and extending linearly to all of L .

(3.8). $L = \tau(3,k)$. L has the basis $e_{13}, e_{12}, e_{23}, e_{11}, e_{22}, e_{33}$. The bracket relations are given by $[e_{ij}, e_{k\ell}] = \delta_{jk}e_{i\ell} - \delta_{i\ell}e_{kj}$ for $1 \leq i,k < j,\ell \leq 3$. We have the representation

$$
\begin{aligned}
L &\to \mathrm{Der}(k[X_1, X_2, \ldots, X_6]) \\
e_{13} &\to -\frac{\partial}{\partial X_1} \\
e_{12} &\to -\frac{\partial}{\partial X_2} \\
e_{23} &\to X_2\frac{\partial}{\partial X_1} - \frac{\partial}{\partial X_3} \\
e_{11} &\to -X_1\frac{\partial}{\partial X_1} - X_2\frac{\partial}{\partial X_2} - \frac{\partial}{\partial X_4} \\
e_{22} &\to X_2\frac{\partial}{\partial X_1} - X_3\frac{\partial}{\partial X_3} - \frac{\partial}{\partial X_5} \\
e_{33} &\to X_1\frac{\partial}{\partial X_1} + X_3\frac{\partial}{\partial X_3} - \frac{\partial}{\partial X_6} .
\end{aligned}
$$

We then extend by linearity to all of L .

(3.9). $L = \tau(4,k)$. L has the basis $e_{14}, e_{24}, e_{13}, e_{34}, e_{23}, e_{12}, e_{11}, e_{22}, e_{33}, e_{44}$. The bracket relations are given by $[e_{ij}, e_{k\ell}] = \delta_{jk}e_{i\ell} - \delta_{i\ell}e_{kj}$ for $1 \leq i,k < j,\ell \leq 4$. The injective hull $E_{\tau(4,k)}(k)$ has for its representation space the algebra $k[X_1, X_2, \ldots, X_{10}]$ and the representation is defined by extending linearly to all of L the function whose values on the basis are listed below:

$$
\begin{aligned}
L &\to \mathrm{Der}(k[X_1, X_2, \ldots, X_{10}]) \\
e_{14} &\to -\frac{\partial}{\partial X_1} \\
e_{24} &\to -\frac{\partial}{\partial X_2}
\end{aligned}
$$

$$e_{13} \to -\frac{\partial}{\partial X_3}$$

$$e_{34} \to X_3\frac{\partial}{\partial X_1} - \frac{\partial}{\partial X_4}$$

$$e_{23} \to -X_4\frac{\partial}{\partial X_2} - \frac{\partial}{\partial X_5}$$

$$e_{12} \to (-X_2 + X_4X_5)\frac{\partial}{\partial X_1} - X_5\frac{\partial}{\partial X_3} - \frac{\partial}{\partial X_6}$$

$$e_{11} \to -X_1\frac{\partial}{\partial X_1} - X_3\frac{\partial}{\partial X_3} - X_6\frac{\partial}{\partial X_6} - \frac{\partial}{\partial X_7}$$

$$e_{22} \to -X_2\frac{\partial}{\partial X_2} - X_5\frac{\partial}{\partial X_5} + X_6\frac{\partial}{\partial X_6} - \frac{\partial}{\partial X_8}$$

$$e_{33} \to X_3\frac{\partial}{\partial X_3} - X_4\frac{\partial}{\partial X_4} + X_5\frac{\partial}{\partial X_5} - \frac{\partial}{\partial X_9}$$

$$e_{44} \to X_1\frac{\partial}{\partial X_1} + X_2\frac{\partial}{\partial X_2} + X_4\frac{\partial}{\partial X_4} - \frac{\partial}{\partial X_{10}} .$$

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