ON UNIPOTENT ORBITAL INTEGRALS
FOR p-ADIC GROUPS

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Abstract: In this thesis, we compute unipotent orbital integrals of spherical functions for $\mathrm{SL}_2(F)$ where F is a p-adic field.

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CHAPTER 1

INTRODUCTION

Let F be a p-adic field with residue field of order $q = p^f$, $\mathbf{G}$ a connected reductive linear algebraic group defined over F, $G = \mathbf{G}(F)$ the group of F-points of $\mathbf{G}$, and $\mathfrak{g} = \mathrm{Lie}(G)$. Let $C_c(G)$ denote the space of locally constant, complex valued functions on G with compact support. For $x \in G$, let $C_G(x)$ denote the centralizer of x in G and let $\mathcal{O}_x \cong G/C_G(x)$ be the G-orbit of x. Since G is unimodular locally compact and $C_G(x)$ is unimodular, the quotient space $G/C_G(x)$ has a G-invariant measure dg^*, which is unique up to a scalar [Kna05, Theorem 6.18]. For $f \in C_c(G)$, one may consider the integral

$$\int_{G/C_G(x)} f(gxg^{-1})dg^*. \tag{1.1}$$

This is called an orbital integral. It is known that the integral (1.1) converges [Rao72] for all $x \in G$ and $f \in C_c(G)$.

Orbital integrals are fundamental objects in harmonic analysis on G. They define distributions on the group which are invariant under conjugation. They also appear in the "geometric side" of the Arthur-Selberg's trace formula; whereas the "spectral side" contains traces of representations.

Among the general orbital integrals, the unipotent orbital integrals play an important role. In [Sha72], Shalika showed that the orbital integrals of small regular elements can be expressed as combinations of unipotent orbital integrals, with coefficients called Shalika germs.

Orbital integrals in some cases have been calculated explicitly. For example, in [Rep84a] and [Rep84b], Repka calculated the elliptic orbital integrals and unipotent orbital integrals for some characteristic function of certain compact subgroup of $\mathrm{GL}_n(F)$.

The purpose of this thesis is to calculate unipotent orbital integrals of all spherical functions for p-adic SL_2. The main tool available for such calculation is a theorem of Rao in [Rao72]. Similar calculations have been achieved for p-adic 4×4 symplectic groups Sp(4) and GSp(4) in [Ass93].

The main result of this thesis is the following theorem.

Theorem 1.1 *Let F be a p-adic field with ring of integers $\mathcal{O}_F$, a uniformalizer π, and the order of the residue field $q = p^f$. We assume that $p \neq 2$. Let $G = SL_2(F)$, $K = SL_2(\mathcal{O}_F)$. For $m \in \mathbb{N}$, let f_m be the characteristic function of $K \begin{pmatrix} \pi^m & 0 \\ 0 & \pi^{-m} \end{pmatrix} K$. Let $d\mu$ be a G-invariant measure on the G-orbit $\mathcal{O}_u$ where $u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then there exists some constant $c \neq 0$ such that*

$$\int_{\mathcal{O}_u} f_m d\mu = \begin{cases} c \cdot \frac{1}{2}\frac{q}{q+1}, & \text{if } m = 0, \\ 0, & \text{if } m > 0 \text{ is odd}, \\ c \cdot \frac{1}{2} q^m (1 - q^{-1}), & \text{if } m > 0 \text{ is even.} \end{cases}$$

The organization of this thesis is as follows. In Section 2, we begin by introducing Rao's data associated to a unipotent orbit in a reductive group defined over F and Rao's theorem for unipotent orbital integrals. This will be our main tool to calculate orbital integrals. We then explain our choices of the test functions f so that we reduce a triple integral to a double integral.

In Section 3, we apply the results in Section 2 to calculate orbital integrals of SL_2. In 3.1 we calculate Rao's data for nontrivial unipotent orbit in SL_2. In 3.2 we change the orbital integral into the volume of a specific set and use the action of SL_2 on F^2 to determine such a set. In 3.3 we provide another proof for our calculation, by determining the set directly.

One hopes to generalize the results from SL_2 to the exceptional group of type G_2. I may pursue this in the future.

1.1 Notation

Throughout this thesis, F will denote a p-adic field, $\mathcal{O}_F$ the ring of integers, and $\mathcal{P}_F$ the maximal ideal of $\mathcal{O}_F$. The order of the residue field $\mathcal{O}_F/\mathcal{P}_F$ is denoted by $q = p^f$. We assume that $p \neq 2$. Let π be a uniformizer. We denote by val the valuation function on F. We normalize the absolute value $|\cdot|$ on F such that $|\pi| = q^{-1}$.

CHAPTER 2

RAO'S THEOREM AND UNIPOTENT ORBITAL INTEGRALS

Our main tool for calculating unipotent orbital integrals will be a theorem of Rao in [Rao72].

Let $\mathbf{G}$ be a connected reductive linear algebraic group defined over a p-adic field F, and write the F-rational points of $\mathbf{G}$ as $G = \mathbf{G}(F)$. The Lie algebra of G is denoted as $\mathfrak{g} = \mathrm{Lie}(G)$.

We denote by $\mathbf{B}$ a Borel subgroup of $\mathbf{G}$ containing a maximal torus $\mathbf{T}$. Recall that the characters and co-characters of $\mathbf{T}$ are defined as

$$X^* = X^*(\mathbf{T}) = \mathrm{Hom}(\mathbf{T}, \mathbf{G}_m),$$

$$X_* = X_*(\mathbf{T}) = \mathrm{Hom}(\mathbf{G}_m, \mathbf{T}).$$

We denote the set of roots for $\mathbf{G}$ by Φ, the set of positive roots by Φ^+, and the set of simple roots by Δ. Then, the positive Weyl chamber P^+ in X_* is defined to be

$$P^+ = \{\lambda \in X_* = \mathrm{Hom}(\mathbf{G}_m, \mathbf{T}) : \langle \alpha, \lambda \rangle \geq 0 \text{ for all } \alpha \in \Delta\}.$$

Let $u \in G$ be a unipotent element and let $X_0 \in \mathfrak{g}$ such that $\exp(X_0) = u$. By the Jacobson-Morozov Theorem, there is a $\mathfrak{sl}_2$-triple (H_0, X_0, Y_0) containing X_0. For each $i \in \mathbb{Z}$, let $\mathfrak{g}_i := \{X \in \mathfrak{g} : \mathrm{ad}_{H_0}(X) = iX\}$. Then we have a $\mathbb{Z}$-grading $\mathfrak{g} = \oplus_{i\in\mathbb{Z}}\mathfrak{g}_i$. Let $\mathfrak{p} := \sum_{i\geq 0} \mathfrak{g}_i$ and $\mathfrak{n}_j := \sum_{i>j} \mathfrak{g}_i$ for $j \in \mathbb{Z}$. Then $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$ with $\mathfrak{n}_0$ as its nilradical. Let $\mathbf{P}$ be the corresponding parabolic subgroup of $\mathbf{G}$ and $\mathbf{N}$ its unipotent radical. Let $P = G \cap \mathbf{P}$, $N = G \cap \mathbf{N}$. Then P and N are closed subgroups of G with Lie algebras $\mathfrak{p}$ and $\mathfrak{n}_0$ respectively.

Let M be the centralizer of H_0 in G, i.e., $M = \{m \in G : \mathrm{Ad}_m(H_0) = H_0\}$, then $P = MN$. Moreover, there is a maximal compact subgroup K of G so that we have the decomposition $G = KP$.

Let $V(X_0) := \{\mathrm{Ad}_m(X_0) : m \in M\}$. Then $V(X_0) \subset \mathfrak{g}_2$. To see this, let $m \in M$, then

$$\begin{aligned}\mathrm{ad}_{H_0}(\mathrm{Ad}_m(X_0)) &= [H_0, mX_0m^{-1}] = H_0mX_0m^{-1} - mX_0m^{-1}H_0\\ &= mH_0X_0m^{-1} - mX_0H_0m^{-1} = 2mX_0m^{-1} = 2\mathrm{Ad}_m(X_0).\end{aligned}$$

Moreover, $V(X_0)$ is open in the Hausdorff topology of $\mathfrak{g}_2$.

By a lemma of Rao [Rao72], the G-orbit of X_0 is $\mathrm{Ad}_K(V(X_0) + \mathfrak{n}_2)$, and is locally compact.

Now we define the Rao function φ on $\mathfrak{g}_2$. Let $Z_1, \cdots, Z_r$ and $Z_1', \cdots, Z_r'$ be bases for $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ respectively such that $B(Z_i, Z_j') = \delta_{ij}$ for a symmetric, non-degenerate, G-invariant, bilinear form B on $\mathfrak{g}$ which coincides with the Killing form on the derived algebra of $\mathfrak{g}$. For $X \in \mathfrak{g}_2$, let $[X, Z_j'] = \sum_{i=1}^r c_{ji}(X)Z_i$ and define $\varphi(X) := |\det(c_{ji}(X))|$.

The G-orbit $\mathcal{O}_u$ is locally compact, and homeomorphic to G/G_u. The isotropy subgroup G_u is unimodular, and so G/G_u has a Haar measure. Let $d\mu$ be a G-invariant measure on the G-orbit $\mathcal{O}_u$ of u.

We normalize measures as follows. We normalize the Haar measure on G such that K has measure one. We assume that $\mathfrak{n}_2(\mathcal{O}_F)$ and $\mathfrak{g}_2(\mathcal{O}_F)$ both have measure one. We also normalize the measure on F by assuming that $\mathcal{O}_F$ has measure one.

Theorem 2.1 ([Rao72]) *There exists a constant $c \neq 0$ such that for all $f \in C_c(G)$ the following holds:*

$$\int_{\mathcal{O}_u} f d\mu = c \int_{V(X_0)+\mathfrak{n}_2} \varphi(X)\overline{f}(X+Z)dXdZ$$

where dX is a Haar measure on $\mathfrak{g}_2$, dZ is a Haar measure on $\mathfrak{n}_2$, and

$$\overline{f}(X) := \int_K f \circ \exp(Ad_k(X))dk, \quad X \in \mathfrak{g}$$

where dk is a Haar measure on K.

We define the orbital integral

$$\Phi(f, \mathcal{O}_u) := \int_{V(X_0)+\mathfrak{n}_2} \varphi(X)\overline{f}(X+Z)dXdZ.$$

In this thesis we only consider the functions f which are K-spherical. In other words, f is K-bi-invariant. The spherical Hecke algebra $\mathcal{H}(G, K)$ is by definition the space of all locally constant, compactly supported functions $f : G \to \mathbb{C}$ which are K-bi-invariant: $f(k_1 x k_2) = f(x)$ for all $k_1, k_2 \in K$. The multiplication in $\mathcal{H}(G, K)$ is the convolution

$$(f_1 * f_2)(g) = \int_G f_1(gh^{-1})f_2(h)dh,$$

where dh is the unique Haar measure on G giving K volume 1. Then $\mathcal{H}(G, K)$ becomes a commutative ring. The group G is the disjoint union of the double cosets $K\lambda(\pi)K$, where λ runs through the co-characters in the positive Weyl chamber P^+ [Gno98, Proposition 2.6]. The spherical Hecke algebra is spanned by the characteristic functions of $K\lambda(\pi)K$, $\lambda \in P^+$. Thus, it suffices to consider each $\Phi(f, \mathcal{O}_u)$ where f is the characteristic function of $K\lambda(\pi)K$, as λ runs through P^+.

If $f \in \mathcal{H}(G, K)$, then for $X \in \mathfrak{g}$,

$$\begin{aligned}\overline{f}(X) &= \int_K f \circ \exp(\mathrm{Ad}_k(X))dk = \int_K f(k\exp(X)k^{-1})dk \\ &= \int_K f(\exp(X))dk \qquad = f \circ \exp(X) \int_K dk \\ &= f \circ \exp(X).\end{aligned}$$

Then the orbital integral reduces to

$$\Phi(f, \mathcal{O}_u) = \int_{V(X_0)+\mathfrak{n}_2} \varphi(X) f \circ \exp(X+Z)dXdZ. \tag{2.1}$$

We will use (2.1) to calculate unipotent orbital integrals of spherical functions.

CHAPTER 3

UNIPOTENT ORBITAL INTEGRALS OF SPHERICAL FUNCTIONS FOR SL_2

In this section, we calculate the unipotent orbital integrals of all spherical functions for SL_2. Our first goal is to express the orbital integral explicitly.

3.1 Rao's Data for SL_2

Let $G = \mathrm{SL}_2(F)$, $\mathfrak{g} = \mathrm{Lie}(G)$, $K = \mathrm{SL}_2(\mathcal{O}_F)$. There is a one-to-one correspondence between the set of unipotent orbits in G and the set of partitions of 2 [CM93, 3.1]; hence there are two unipotent orbits in G: the trivial orbit $\mathcal{O}_I = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ and the orbit $\mathcal{O}_u$ through $u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. We are interested in the non-trivial orbit. Let

$$H_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, X_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, Y_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then $\exp(X_0) = u$, and (H_0, X_0, Y_0) is a $\mathfrak{sl}_2$-triple containing X_0. We have a grading $\mathfrak{g} = \oplus_{i\in\mathbb{Z}}\mathfrak{g}_i$ where $\mathfrak{g}_i = \{X \in \mathfrak{g} : \mathrm{ad}_{H_0}(X) = iX\}$. Specifically,

$$\mathfrak{g}_i = \begin{cases} \left\{\begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix} : a \in F\right\} = \{aY_0 : a \in F\}, i = -2, \\ \left\{\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} : a \in F\right\} = \{aH_0 : a \in F\}, i = 0, \\ \left\{\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} : a \in F\right\} = \{aX_0 : a \in F\}, i = 2, \\ \{0\}, \quad \text{otherwise}. \end{cases}$$

Moreover,

$$M = \left\{\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a \in F^\times\right\},$$

$$V(X_0) = \left\{\begin{pmatrix} 0 & a^2 \\ 0 & 0 \end{pmatrix} : a \in F^\times\right\}.$$

Since $\mathfrak{g}_1 = \mathfrak{g}_{-1} = \{0\}$, the Rao function $\varphi \equiv 1$ on $\mathfrak{g}_2$. Also note that $\mathfrak{n}_2 = \{0\}$. Thus, for any K-spherical function f, we have

$$\begin{aligned} \Phi(f, \mathcal{O}_u) &= \int_{V(X_0)+\mathfrak{n}_2} \varphi(X) f \circ \exp(X+Z) dX dZ \\ &= \int_{V(X_0)+\mathfrak{n}_2} f \circ \exp(X) dX dZ \\ &= \int_{V(X_0)} f \circ \exp(X) dX \int_{\mathfrak{n}_2} dZ \\ &= \int_{V(X_0)} f \circ \exp(X) dX \\ &= \mathrm{vol}(V(X_0) \cap \mathrm{supp}(f \circ \exp)). \end{aligned}$$

3.2 Orbital Integrals of Spherical Functions

Recall that the spherical Hecke algebra $\mathcal{H}(G, K)$ is spanned by the set of functions

$$\{f_m : m \in \mathbb{N}\}$$

where f_m is the characteristic function of $K \begin{pmatrix} \pi^m & 0 \\ 0 & \pi^{-m} \end{pmatrix} K$. Thus, it suffices to consider $f = f_m$ for each $m \in \mathbb{N}$. To simplify our notation, denote

$$S(m) := V(X_0) \cap \operatorname{supp}(f_m \circ \exp).$$

Then,

$$\begin{aligned} S(m) &= \left\{ \begin{pmatrix} 0 & x^2 \\ 0 & 0 \end{pmatrix} : \exp\left(\begin{pmatrix} 0 & x^2 \\ 0 & 0 \end{pmatrix} \right) \in K \begin{pmatrix} \pi^m & 0 \\ 0 & \pi^{-m} \end{pmatrix} K, x \in F^\times \right\} \\ &= \left\{ \begin{pmatrix} 0 & x^2 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} 1 & x^2 \\ 0 & 1 \end{pmatrix} \in K \begin{pmatrix} \pi^m & 0 \\ 0 & \pi^{-m} \end{pmatrix} K, x \in F^\times \right\}. \end{aligned}$$

If $m = 0$, then

$$\begin{aligned} S(0) &= \left\{ \begin{pmatrix} 0 & x^2 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} 1 & x^2 \\ 0 & 1 \end{pmatrix} \in K, x \in F^\times \right\} \\ &= \left\{ \begin{pmatrix} 0 & x^2 \\ 0 & 0 \end{pmatrix} : x^2 \in \mathcal{O}_F \backslash \{0\} \right\} \\ &= \left\{ \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} : x \in \coprod_{n=0}^{\infty} \pi^{2n} (\mathcal{O}_F^\times)^2 \right\}. \end{aligned}$$

Hence,

$$\begin{aligned} \Phi(f_0, \mathcal{O}_u) = \operatorname{vol}(S(0)) &= \frac{1}{2} \operatorname{vol}(\coprod_{n=0}^{\infty} \pi^{2n} \mathcal{O}_F^\times) \\ &= \frac{1}{2} \sum_{n=0}^{\infty} q^{-2n} (1 - q^{-1}) = \frac{q}{2(q+1)}. \end{aligned} \tag{3.1}$$

Now we determine the set $S(m)$ for $m > 0$.

Proposition 3.2.1 *Let $m > 0$ be an integer. Then*

$$S(m) = \begin{cases} \emptyset, & \text{if } m > 0 \text{ is odd,} \\ \left\{ \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} : x \in \pi^{-m}(\mathcal{O}_F^\times)^2 \right\}, & \text{if } m > 0 \text{ is even.} \end{cases}$$

Proof. G acts on F^2 via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (v_1, v_2) = (av_1 + bv_2, cv_1 + dv_2)$$

for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$, $(v_1, v_2) \in F^2$. We define a norm $||\cdot||$ on F^2 by

$$||(v_1, v_2)|| := \max\{|v_1|, |v_2|\}.$$

Note that for any $k = \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix} \in K$, $v = (v_1, v_2) \in F^2$, we have

$$\begin{aligned} ||k \cdot v|| &= ||(k_1v_1 + k_2v_2, k_3v_1 + k_4v_2|| \\ &= \max\{|k_1v_1 + k_2v_2|, |k_3v_1 + k_4v_2|\} \\ &\le ||v|| \end{aligned} \tag{3.2}$$

since $|k_1v_1 + k_2v_2| \le \max\{|k_1v_1|, |k_2v_2|\} \le \max\{|v_1|, |v_2|\} = ||v||$ and similarly $|k_3v_1 + k_4v_2| \le ||v||$. Then for any $k \in K$, $w \in F^2$,

$$||k^{-1} \cdot w|| \le ||w||.$$

Replacing w by $k \cdot v$ we get

$$||v|| \le ||k \cdot v||. \tag{3.3}$$

Combining (3.2) and (3.3) we conclude that

$$||k \cdot v|| = ||v|| \tag{3.4}$$

for any $k \in K$, $v \in F^2$. Thus, $||\cdot||$ is invariant under the action of K.

Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$, and consider the function

$$\phi(g) := \max_{v \in \mathcal{O}_F^2}\{||g \cdot v||\}.$$

If g lies in the double K-coset $K \begin{pmatrix} \pi^m & 0 \\ 0 & \pi^{-m} \end{pmatrix} K$, then

$$\begin{aligned}
\phi(g) &= \max_{v \in \mathcal{O}_F^2}\left\{ || \begin{pmatrix} \pi^m & 0 \\ 0 & \pi^{-m} \end{pmatrix} \cdot v|| \right\} \\
&= \max_{(v_1,v_2)\in \mathcal{O}_F^2} \left\{||(\pi^m v_1, \pi^{-m} v_2)||\right\} \\
&= \max_{(v_1,v_2)\in \mathcal{O}_F^2} \left\{|\pi^m v_1|, |\pi^{-m} v_2|\right\} \\
&= q^m.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\phi\left(\begin{pmatrix} 1 & x^2 \\ 0 & 1 \end{pmatrix}\right) &= \max_{(v_1,v_2)\in \mathcal{O}_F^2} \left\{||(v_1 + x^2 v_2, v_2)||\right\} \\
&= \max_{(v_1,v_2)\in \mathcal{O}_F^2} \left\{|v_1 + x^2 v_2|, |v_2|\right\} \\
&= \begin{cases} 1 & \text{if } |x| \leq 1, \\ |x|^2 & \text{if } |x| > 1. \end{cases}
\end{aligned}$$

Therefore, $\begin{pmatrix} 1 & x^2 \\ 0 & 1 \end{pmatrix} \in K \begin{pmatrix} \pi^m & 0 \\ 0 & \pi^{-m} \end{pmatrix} K$ if and only if $|x|^2 = q^m$. Therefore,

$$S(m) = \begin{cases} \emptyset, & \text{if } m \text{ is odd,} \\ \left\{ \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} : x \in \pi^{-m}(\mathcal{O}_F^\times)^2 \right\}, & \text{if } m \text{ is even.} \end{cases}$$

■

Theorem 3.1 *We have*

$$\Phi(f_m, \mathcal{O}_u) = \begin{cases} \frac{1}{2}\frac{q}{q+1}, & \text{if } m = 0, \\ 0, & \text{if } m > 0 \text{ is odd}, \\ \frac{1}{2}q^m(1 - q^{-1}), & \text{if } m > 0 \text{ is even}. \end{cases}$$

Proof. Under our normalization, the set $\{x \in F^\times : x \in \pi^{-m}(\mathcal{O}_F^\times)^2\} = \pi^{-m}(\mathcal{O}_F^\times)^2$ has measure $\frac{1}{2}q^m(1 - q^{-1})$. The result follows from (3.1) and Proposition 3.2.1. ■

Since $\{f_m, m \in \mathbb{N}\}$ spans all the K-spherical functions, we have determined the orbital integrals of all spherical functions for SL_2.

3.3 Another Determination of the Set $S(m)$

We could actually determine the set $S(m)$ for $m > 0$ directly. Note

$$S(m) = \left\{ \begin{pmatrix} 0 & x^2 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} 1 & x^2 \\ 0 & 1 \end{pmatrix} \in K \begin{pmatrix} \pi^m & 0 \\ 0 & \pi^{-m} \end{pmatrix} K, x \in F^\times \right\}$$

$$= \left\{ \begin{pmatrix} 0 & x^2 \\ 0 & 0 \end{pmatrix} : \begin{cases} \begin{pmatrix} 1 & x^2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \pi^m & 0 \\ 0 & \pi^{-m} \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} \\ ad - bc = 1 \\ eh - fg = 1 \\ a, \cdots, h \in \mathcal{O}_F, x \in F^\times \end{cases} \right\}$$

$$= \left\{ \begin{pmatrix} 0 & x^2 \\ 0 & 0 \end{pmatrix} : \begin{cases} \begin{pmatrix} 1 & x^2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} ae\pi^m + bg\pi^{-m} & af\pi^m + bh\pi^{-m} \\ ce\pi^m + dg\pi^{-m} & cf\pi^m + dh\pi^{-m} \end{pmatrix} \\ ad - bc = 1 \\ eh - fg = 1 \\ a, \cdots, h \in \mathcal{O}_F, x \in F^\times \end{cases} \right\}$$

$$= \left\{ \begin{pmatrix} 0 & x^2 \\ 0 & 0 \end{pmatrix} : \begin{cases} x^2 = af\pi^m + bh\pi^{-m} \\ ce\pi^m + dg\pi^{-m} = 0 \\ ae\pi^m + bg\pi^{-m} = 1 \\ cf\pi^m + dh\pi^{-m} = 1 \\ ad - bc = 1 \\ eh - fg = 1 \\ a, \cdots, h \in \mathcal{O}_F, x \in F^\times \end{cases} \right\}$$

Thus, we need to determine the set of $x \in F^\times$ satisfying

$$x^2 = af\pi^m + bh\pi^{-m}, \tag{3.5}$$

$$ce\pi^m + dg\pi^{-m} = 0, \tag{3.6}$$

$$ae\pi^m + bg\pi^{-m} = 1, \tag{3.7}$$

$$cf\pi^m + dh\pi^{-m} = 1, \tag{3.8}$$

$$ad - bc = 1, \tag{3.9}$$

$$eh - fg = 1, \tag{3.10}$$

where

$$m > 0, a, \cdots, h \in \mathcal{O}_F. \tag{3.11}$$

Lemma 3.1 *Suppose* $x \in F^\times$ *with* $\begin{pmatrix} 1 & x^2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \pi^m & 0 \\ 0 & \pi^{-m} \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix}$, *where* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} e & f \\ g & h \end{pmatrix} \in K$. *Then* $abcdefgh = 0$.

Proof. Suppose not. Then we can write uniquely $a = \pi^{\mathrm{val}(a)} u_a, b = \pi^{\mathrm{val}(b)} u_b, \cdots, h = \pi^{\mathrm{val}(h)} u_h$, where $0 \le \mathrm{val}(a), \cdots, \mathrm{val}(h) < \infty$ and $\mathrm{val}(u_a) = \cdots = \mathrm{val}(u_h) = 0$. By (3.6), we must have

$$\mathrm{val}(c) + \mathrm{val}(e) + m = \mathrm{val}(d) + \mathrm{val}(g) - m, \tag{3.12}$$

and

$$u_c u_e + u_d u_g = 0.$$

From (3.7) we also know that

$$0 = \mathrm{val}(1) \ge \min\{\mathrm{val}(a) + \mathrm{val}(e) + m, \mathrm{val}(b) + \mathrm{val}(g) - m\}.$$

If $\mathrm{val}(a) + \mathrm{val}(e) + m \le \mathrm{val}(b) + \mathrm{val}(g) - m$, then

$$0 \ge \mathrm{val}(a) + \mathrm{val}(e) + m \ge m > 0,$$

which is a contradiction. Hence, we must have $\mathrm{val}(a)+\mathrm{val}(e)+m > \mathrm{val}(b)+\mathrm{val}(g)-m$, and

$$\mathrm{val}(b) + \mathrm{val}(g) - m = 0. \tag{3.13}$$

Similarly, from (3.8), we have

$$\mathrm{val}(d) + \mathrm{val}(h) - m = 0. \tag{3.14}$$

Also, by (3.9),

$$0 = \mathrm{val}(1) \geq \min\{\mathrm{val}(a) + \mathrm{val}(d), \mathrm{val}(b) + \mathrm{val}(c)\}.$$

Hence, either $\mathrm{val}(a) = \mathrm{val}(d) = 0$ or $\mathrm{val}(b) = \mathrm{val}(c) = 0$. A similar argument for (3.10) implies that either $\mathrm{val}(e) = \mathrm{val}(h) = 0$ or $\mathrm{val}(f) = \mathrm{val}(g) = 0$.

Now if $\mathrm{val}(a) = \mathrm{val}(d) = 0$, then we get $\mathrm{val}(h) = m$ by (3.14), and $\mathrm{val}(f) = \mathrm{val}(g) = 0$ since $\mathrm{val}(h) \neq 0$. Then by (3.12),

$$\mathrm{val}(c) + \mathrm{val}(e) + m = -m < 0,$$

which is a contradiction again. Thus we have

$$\mathrm{val}(b) = \mathrm{val}(c) = 0. \tag{3.15}$$

Then we have $\mathrm{val}(g) = m$ by (3.13), $\mathrm{val}(e) = \mathrm{val}(h) = 0$ since $\mathrm{val}(g) \neq 0$, and $\mathrm{val}(d) = m$ by (3.12). Now $\mathrm{val}(bh\pi^{-m}) = -m < 0 < \mathrm{val}(af\pi^m)$ and we get $2\mathrm{val}(x) = -m$ from (3.5). Since $\mathrm{val}(x) \in \mathbb{Z}$, we must have $S(m) = \emptyset$ unless m is even.

Now let $m > 0$ be an even integer. Rewrite (3.7) and (3.9) as

$$\begin{cases} \pi^{\mathrm{val}(a)+m} u_a u_e + u_b u_g = 1, \\ \pi^{\mathrm{val}(a)+m} u_a u_d - u_b u_c = 1. \end{cases}$$

So

$$\pi^{\mathrm{val}(a)+m} u_a (u_e - u_d) + u_b (u_c + u_g) = 0.$$

Taking valuations, one gets

$$\mathrm{val}(a) + m = \mathrm{val}(u_c + u_g) - \mathrm{val}(u_e - u_d).$$

Similarly, combining (3.8) and (3.10) we get

$$\mathrm{val}(f) + m = \mathrm{val}(u_d - u_e) - \mathrm{val}(u_c + u_g).$$

Hence,

$$\mathrm{val}(a) + \mathrm{val}(f) + 2m = 0.$$

On the other hand, one has

$$\mathrm{val}(a) + \mathrm{val}(f) + 2m \geq 2m > 0,$$

a contradiction. ■

Lemma 3.2 *Suppose* $x \in F^\times$ *with* $\begin{pmatrix} 1 & x^2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \pi^m & 0 \\ 0 & \pi^{-m} \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix}$, *where* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} e & f \\ g & h \end{pmatrix} \in K$. *Then* $bcdehg \neq 0$.

Proof. We prove by contradiction.

Suppose $bg = 0$. Then $ae\pi^m = 1$ by (3.7). Considering valuations on both sides we see that it contradicts the fact that $m > 0$.

Suppose $dh = 0$. Then $cf\pi^m = 1$ by (3.8). This is a also contradiction by the same argument.

Suppose $c = 0$. Then by (3.6), (3.8),

$$\begin{cases} dg\pi^{-m} = 0, \\ dh\pi^{-m} = 1. \end{cases}$$

Hence $g = 0$. Contradiction.

Suppose $e = 0$. Then by (3.6), (3.7),

$$\begin{cases} dg\pi^{-m} = 0, \\ bg\pi^{-m} = 1. \end{cases}$$

Hence $d = 0$. Contradiction. ∎

By Lemma 3.2, we can write $b = \pi^{\mathrm{val}(b)}u_b$, $c = \pi^{\mathrm{val}(c)}u_c$, $d = \pi^{\mathrm{val}(d)}u_d$, $e = \pi^{\mathrm{val}(e)}u_e$, $h = \pi^{\mathrm{val}(h)}u_h$, $g = \pi^{\mathrm{val}(g)}u_g$.

Next, we consider a and f. There are only three cases.

Case I: $a = f = 0$;

Case II: $a = 0, f \neq 0$;

Case III: $f = 0, a \neq 0$.

The following three lemmas are devoted to each of the above cases.

Lemma 3.3 *Let $m > 0$ be an integer. Then*

$$\left\{ x \in F^\times : \begin{pmatrix} 1 & x^2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \pi^m & 0 \\ 0 & \pi^{-m} \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix}, a = f = 0, \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} e & f \\ g & h \end{pmatrix} \in K \right\}$$

$$= \begin{cases} \emptyset, & m \text{ odd}, \\ \pi^{-m/2}\mathcal{O}_F^\times, & m \text{ even}. \end{cases}$$

Proof. If $a = f = 0$, we can rewrite equations (3.5)-(3.10) as

$$x^2 = bh\pi^{-m} = \pi^{\mathrm{val}(b)+\mathrm{val}(h)-m}u_b u_h, \tag{3.16}$$

$$\pi^{\mathrm{val}(c)+\mathrm{val}(e)+m}u_c u_e + \pi^{\mathrm{val}(d)+\mathrm{val}(g)-m}u_d u_g = 0, \tag{3.17}$$

$$\pi^{\mathrm{val}(b)+\mathrm{val}(g)-m}u_b u_g = 1, \tag{3.18}$$

$$\pi^{\mathrm{val}(d)+\mathrm{val}(h)-m}u_d u_h = 1, \tag{3.19}$$

$$-\pi^{\mathrm{val}(b)+\mathrm{val}(c)}u_b u_c = 1, \tag{3.20}$$

$$\pi^{\mathrm{val}(e)+\mathrm{val}(h)}u_e u_h = 1, \tag{3.21}$$

Then we have

$$\mathrm{val}(b) = \mathrm{val}(c) = \mathrm{val}(e) = \mathrm{val}(h) = 0, \mathrm{val}(d) = \mathrm{val}(g) = m,$$

$$u_c = -u_b^{-1}, u_g = u_b^{-1}, u_d = u_e = u_h^{-1}.$$

By (3.16), $\mathrm{val}(x) = \frac{\mathrm{val}(b)+\mathrm{val}(h)-m}{2} = -\frac{m}{2}$. If m is odd, then the equations (3.5)-(3.10) have no solution since $\mathrm{val}(x) \in \mathbb{Z}$. If m is even, then $x \in \pi^{-m/2}\mathcal{O}_F^\times$. On the other hand, since there are no restrictions on u_b and u_h as elements of $\mathcal{O}_F^\times$, x can be any element of $\pi^{-m/2}\mathcal{O}_F^\times$ by taking $u_h = u_b$. ■

Lemma 3.4 *Let $m > 0$ be an integer. Then*

$$\left\{ x \in F^\times : \begin{pmatrix} 1 & x^2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \pi^m & 0 \\ 0 & \pi^{-m} \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix}, a = 0, f \neq 0, \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} e & f \\ g & h \end{pmatrix} \in K \right\}$$

$$= \begin{cases} \emptyset, & m \text{ odd}, \\ \pi^{-m/2}\mathcal{O}_F^\times, & m \text{ even}. \end{cases}$$

Proof. Suppose $a = 0, f \neq 0$. We rewrite equations (3.5)-(3.10) as

$$x^2 = bh\pi^{-m} = \pi^{\mathrm{val}(b)+\mathrm{val}(h)-m}u_b u_h, \tag{3.22}$$

$$\pi^{\mathrm{val}(c)+\mathrm{val}(e)+m}u_c u_e + \pi^{\mathrm{val}(d)+\mathrm{val}(g)-m}u_d u_g = 0, \tag{3.23}$$

$$\pi^{\mathrm{val}(b)+\mathrm{val}(g)-m}u_b u_g = 1, \tag{3.24}$$

$$\pi^{\mathrm{val}(c)+\mathrm{val}(f)+m}u_c u_f + \pi^{\mathrm{val}(d)+\mathrm{val}(h)-m}u_d u_h = 1, \tag{3.25}$$

$$-\pi^{\mathrm{val}(b)+\mathrm{val}(c)}u_b u_c = 1, \tag{3.26}$$

$$\pi^{\mathrm{val}(e)+\mathrm{val}(h)}u_e u_h - \pi^{\mathrm{val}(f)+\mathrm{val}(g)}u_f u_g = 1, \tag{3.27}$$

Then we have

$$\mathrm{val}(b) = \mathrm{val}(c) = \mathrm{val}(e) = \mathrm{val}(h) = 0, \mathrm{val}(d) = \mathrm{val}(g) = m,$$

$$u_c = -u_b^{-1}, u_g = u_b^{-1}, u_d = u_e.$$

By (3.22), $\mathrm{val}(x) = \frac{\mathrm{val}(b)+\mathrm{val}(h)-m}{2} = -\frac{m}{2}$. If m is odd, then no such x exists. If m is even, then $x \in \pi^{-m/2}\mathcal{O}_F^\times$. It's easy to check that there is no restrictions on u_b and u_h. Hence x can be any element of $\pi^{-m/2}\mathcal{O}_F^\times$. ∎

Lemma 3.5 *Let $m > 0$ be an integer. Then*

$$\left\{ x \in F^\times : \begin{pmatrix} 1 & x^2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \pi^m & 0 \\ 0 & \pi^{-m} \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix}, a \neq 0, f = 0, \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} e & f \\ g & h \end{pmatrix} \in K \right\}$$
$$= \begin{cases} \emptyset, & m \text{ odd,} \\ \pi^{-m/2}\mathcal{O}_F^\times, & m \text{ even.} \end{cases}$$

Proof. Suppose $a \neq 0, f = 0$. We rewrite equations (3.5)-(3.10) as

$$x^2 = bh\pi^{-m} = \pi^{\mathrm{val}(b)+\mathrm{val}(h)-m}u_b u_h, \tag{3.28}$$

$$\pi^{\mathrm{val}(c)+\mathrm{val}(e)+m}u_c u_e + \pi^{\mathrm{val}(d)+\mathrm{val}(g)-m}u_d u_g = 0, \tag{3.29}$$

$$\pi^{\mathrm{val}(a)+\mathrm{val}(e)+m}u_a u_e + \pi^{\mathrm{val}(b)+\mathrm{val}(g)-m}u_b u_g = 1, \tag{3.30}$$

$$\pi^{\mathrm{val}(d)+\mathrm{val}(h)-m}u_d u_h = 1, \tag{3.31}$$

$$\pi^{\mathrm{val}(a)+\mathrm{val}(d)}u_a u_d - \pi^{\mathrm{val}(b)+\mathrm{val}(c)}u_b u_c = 1, \tag{3.32}$$

$$\pi^{\mathrm{val}(e)+\mathrm{val}(h)}u_e u_h = 1, \tag{3.33}$$

Then we have

$$\mathrm{val}(b) = \mathrm{val}(c) = \mathrm{val}(e) = \mathrm{val}(h) = 0, \mathrm{val}(d) = \mathrm{val}(g) = m,$$

$$u_d = u_e = u_h^{-1}, u_c = -u_g.$$

The rest of the proof is essentially the same as in Lemma 3.3 and Lemma 3.4. By (3.28), $\text{val}(x) = \frac{\text{val}(b)+\text{val}(h)-m}{2} = -\frac{m}{2}$. If m is odd, then no such x exists. If m is even, then $x \in \pi^{-m/2}\mathcal{O}_F^\times$. It's easy to check that there is no restrictions on u_b and u_h. Hence x can be any element of $\pi^{-m/2}\mathcal{O}_F^\times$. ■

We conclude that

$$S(m) = \begin{cases} \emptyset, & \text{if } m > 0 \text{ is odd,} \\ \left\{ \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} : x \in \pi^{-m}(\mathcal{O}_F^\times)^2 \right\}, & \text{if } m > 0 \text{ is even.} \end{cases}$$

We see that this result is consistent with Proposition 3.2.1.

Remark 3.1 *Although we could attack this problem directly for SL_2, the computation would be prohibitive for other reductive groups such as G_2. Thus, we need a similar method as in the proof of Proposition 3.2.1 to avoid computations.*

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