8129420

ROWE, DAVID E.

A STUDY OF BORSUK'S HYPERSPACE 2('X,H)

The University of Oklahoma PH.D. 1981

University
Microfilms
International 300 N. Zeeb Road, Ann Arbor, MI 48106

THE UNIVERSITY OF OKLAHOMA

GRADUATE COLLEGE

A STUDY OF BORSUK'S HYPERSPACE 2^X_h

A DISSERTATION

SUBMITTED TO THE GRADUATE FACULTY

in partial fulfillment of the requirements for the

degree of

DOCTOR OF PHILOSOPHY

BY

DAVID E. ROWE

Norman, Oklahoma

1981

A STUDY OF BORSUK'S HYPERSPACE 2^X_h

APPROVED BY

Leonard R. Rubin

Roger Alperin

Walter S. Kelly

(S. UNGURU)

Darryl McCullough

DISSERTATION COMMITTEE

ACKNOWLEDGMENTS

I would like to take this opportunity to express my gratitude to the many people in the mathematics department who have contributed to the enrichment of my graduate education. My association with the department during the past six years has been a rewarding one both personally and professionally, and I am grateful for having had the opportunity to learn as well as to teach in a stimulating environment that has been conducive to both activities. I would like to thank our chairman, Morris Marx, for the many contributions he has made to the mathematics department that have been beneficial, either directly or indirectly, to the graduate students.

Two members of my original doctoral committee, Jeff Butz and Robert Morris, are no longer with the department, but I would like to thank them for their service. I would also like to thank Roger Alperin and Sabetai Unguru for serving on the present committee. I owe a special thanks to another committee member, Darryl McCullough, who offered essential help with Chapter V of this thesis. Also to be thanked are Ric Ancel and Zvonko Čerin for some useful discussion of ideas that appear here. Finally I wish to give special thanks to my advisor, Leonard Rubin, who oversaw not only the writing of this dissertation, but the major part of my mathematical education. I would also like to thank Carol Wedel for her patience in typing the manuscript.

TABLE OF CONTENTS

CHAPTER I

INTRODUCTION

The study of hyperspaces can be traced back to the early 1900s and the works of Hausdorff and Vietoris. During the 1920s and 1930s many fundamental results were obtained by the Polish school of topologists, e.g., Borsuk, Mazurkiewicz, and Wojdyslawski.[1] Polish mathematicians were also at work during the 1920s on the conjecture that the hyperspace 2^I ($I = [0,1]$) is homeomorphic to the Hilbert cube Q.[2] (2^X denotes the set of all compact subspaces of the (metric) space X, topologized by the Hausdorff metric.) Although they were unsuccessful in proving this, many new results concerning the general structure of hyperspaces suggested that $2^X \simeq Q$ might be true for a fairly wide class of spaces X. Thus in 1938, Wojdyslawski posed the following question, which has since become known as the hyperspace conjecture: If X is a Peano continuum, is 2^X homeomorphic to Q?[3] The converse of this (that if $2^X \simeq Q$, then X is Peanian) had been known as early as 1923.[4]

Closely related to this activity was the interest in studying $C(X) \subseteq 2^X$, the hyperspace of subcontinua of X.

Already in the 1920s Polish topologists had conjectured that $C(B^2)$ ($B^2 = [0,1]^2$) was homeomorphic to Q. In 1923, Vietoris and Wazewski proved that C(X) is Peanian if and only if X is Peanian.[5] Fifteen years later, Wojdyslawski showed that if X is Peanian, then C(X) is contractible and locally contractible. In 1939 he refined this by showing that if X is Peanian, then C(X) is in fact an AR.[6] The dimension of C(X) was studied by Kelley in an important paper of 1942.[7] In this he proves that if X is Peanian, then C(X) is finite dimensional if and only if X is a graph (i.e., a one-dimensional finite simplicial complex), and in this case (Kelley asserted without proof) C(X) is a polyhedron. This latter assertion was eventually proven by R. Duda, who made an intensive investigation of these polyhedra.[8]

The close connection between C(X) and 2^X can be seen from the following result of Wojdyslawski: X is Peanian if and only if C(X) is an AR if and only if 2^X is an AR.[9] There was, however, no further progress on the hyperspace conjecture (X is Peanian if and only if $2^X \simeq Q$) until 1967, when N. Gray attacked the problem by attempting to show that the points of 2^X and C(X) are all unstable. (A point $p \in X$ is unstable if for every neighborhood U of p, there exists a deformation of X which is fixed on X-U, and moves X off of p. This is equivalent to saying that p is a Z-set in X.[10]) It is easy to see that every point of Q is unstable. Gray was able to show that if X is a finite simplicial complex, then C(X) consists entirely of unstable points if and only if X contains no free one-simplex.[11] In 1969, he proved that if X is Peanian, then 2^X consists entirely of unstable points.[12]

By this time a number of powerful tools in infinite-dimensional topology had been developed, which turned out to be of decisive importance in successfully resolving the hyperspace conjecture. In 1970 J. West proved that if $\{X_i\}$ are finite contractible nondegenerate polyhedra, then $\Pi X_i \approx Q$.[13] Thus if G is a connected graph, combining the results of Kelley and Wojdyslawski above, C(G) is a contractible, finite dimensional polyhedron, hence, by West's Theorem, C(G) is a Q-factor (i.e., $C(G) \times Q \approx Q$).

One of the elegant features of hyperspaces is that they behave nicely with respect to inverse limit operations. Thus if $X = \varprojlim\{X_n, f_n\}_{n=1}^{\infty}$ is the limit of an inverse system (assume X and the X_n are continua), then there are naturally induced sequences of maps $f_n^* : 2^{X_{n+1}} \longrightarrow 2^{X_n}$ and $f_n^{\#} : C(X_{n+1}) \longrightarrow C(X_n)$. If $2_\infty^X = \varprojlim\{2^{X_n}, f_n^*\}_{n=1}^{\infty}$ and $C_\infty(X) = \varprojlim\{C(X_n), f_n^{\#}\}_{n=1}^{\infty}$, then the assertion that hyperspaces are well behaved with respect to inverse limits means that $2^X \approx 2_\infty^X$, and $C(X) \approx C_\infty(X)$.[14] An important tool for the resolution of the hyperspace conjecture turned out to be the following theorem of Morton Brown, which provides a key link between inverse limits and a class of mappings that is particularly important in recent work in infinite-dimensional topology:

<u>Theorem</u>: Let (X_i, f_i) be an inverse sequence of compact metric spaces such that each X_i is homeomorphic to Y, and each f_i is a near-homeomorphism (i.e., a uniform limit of homeomorphisms), then $\varprojlim (X_i, f_i)$ is homeomorphic to Y.[15]

The techniques of inverse systems, near homeomorphisms, Z-sets, and other advances in infinite-dimensional topology during the 1960s now led to rapid progress on the hyperspace conjecture. In 1972, West proved that if D is a dendrite, then C(D) is a Q-factor, and that C(D) is homeomorphic to Q if and only if D has a dense set of branch points (i.e., D contains no free arc).[16] Shortly afterward Schori and West sketched a proof that $2^I \approx Q$.[17] This was the decisive breakthrough, and it was soon shown that $2^X \approx Q$ when X is a graph, then a polyhedron, and finally Curtis and Schori proved the result that $2^X \approx Q$ when X is an arbitrary Peano continuum.[18] Concerning C(X), they were able to show that C(X) is a Q-factor if and only if X is Peanian, and that $C(X) \approx Q$ if and only if X contains no free arc.

This brief overview of the activity surrounding work on the hyperspace conjecture is meant to give the reader at least a little feel for the flavor of the general subject of hyperspaces.[19] The result of this activity has not only been the achievement of a singularly beautiful theorem, but a recognition of the applicability of powerful techniques that stress the underlying unity of seemingly diverse aspects of infinite-dimensional topology. As an illustration of this, mention should be made of H. Toruńczyk's recent characterization of Hilbert cube manifolds. Using this profound result, Toruńczyk was able to give a much simplified proof of the hyperspace conjecture that completely avoids the use of inverse limit techniques.[20]

Before turning to subject matter more directly related to the present study, let us briefly consider three results that illustrate interesting applications of hyperspace theory and technique. In 1922 Knaster gave a famous example of an indecomposable continuum.[21] A continuum is indecomposable if it cannot be written as a union of two proper subcontinua. Knaster's continuum was dubbed a <u>pseudo-arc</u> by Moise, who showed that not only is it indecomposable, but every proper subcontinuum contained in it is also indecomposable (i.e., the pseudo-arc is an hereditarily indecomposable continuum).[22] Evidently the pseudo-arc is a highly pathological type of continuum. Thus it was quite a revelation when R. H. Bing proved that if X is E^n or Hilbert space, then the collection of pseudo-arcs constitutes a dense G_δ-subset of $C(X)$![23] This result is typical of a prominent thrust in hyperspace theory, namely to detect the frequency of certain pathologies by application of the Baire Category Theorem to the hyperspace of appropriate subsets.

A second example illustrating the utility of hyperspaces can be found in recent work in dimension theory. In 1978 J. Walsh gave the first example of an infinite-dimensional compactum containing no finite-dimensional subsets (other than zero-dimensional).[24] A key feature in this construction was the use of maps into hyperspaces.[25] Recently, R. Pol used a similar construction in solving Alexandroff's Problem, which had been one of the oldest unsolved problems in dimension theory.[26]

Finally we mention the recent work of West concerning group actions on Q.[27] If $Q = \prod_{i=1}^{\infty} J_i$, $J_i = [-1,1]$, then the reflection map $s_\infty\colon Q \longrightarrow Q$, where $s_\infty(\{t_i\}) = \{-t_i\}$ operates on all the coordinates of Q, is the standard involution of Q with a single fixed point. In 1974 R. Wong proved that an involution T of Q with a single fixed point is conjugate to s_∞ if the fixed point has a basis of T-invariant, contractible neighborhoods.[28] It is still unknown, however, whether there are any involutions of Q with unique fixed point that are not conjugate to s_∞. Although West does not answer this question, he was able to prove that if such an involution does exist, then one will not find it by considering induced involutions on hyperspaces. More precisely, if X is an arbitrary, nondegenerate Peano continuum (hence $2^X \approx Q$) and $T\colon X \longrightarrow X$ an involution, then the fixed point set S of $2^T\colon 2^X \longrightarrow 2^X$ is a Hilbert sub-cube of 2^X. Moreover S is a Z-set (cf. [Ch], p. 2) if and only if the fixed point set of T is nowhere dense in X. In this case $2^X/S \approx Q$ and $\bar{T} : 2^X/S \longrightarrow 2^X/S$ is conjugate to s_∞.[29]

These few remarks must suffice as an introduction to the flavor of the general subject of hyperspaces; we now turn to matters that are of direct relevance to the present study. In 1954 K. Borsuk introduced a new metric in an attempt to refine the usual hyperspace construction.[30] If M is an arbitrary metric space with metric d, then $2^M = \{X \subseteq M \mid X$ nonempty and compact$\}$ is topologized by the Hausdorff metric $d_s : d_s(X,Y) = \inf_{\varepsilon>0}\{X \subseteq N_d(Y,\varepsilon) \text{ and } Y \subseteq N_d(X,\varepsilon)\}$.

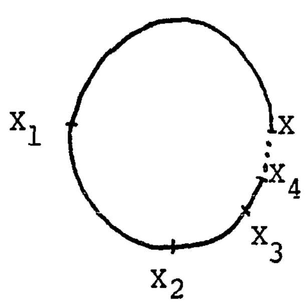
If (M,d) is a complete metric space, then $(2^M, d_s)$ is complete (cf., [N], pp. 35-36). Borsuk developed a new metric, d_h, called the homotopy metric, which, under the assumption that M is finite-dimensional, induces a topology on $2^M_h = \{X \varepsilon 2^M \mid X$ is an ANR[31]$\}$ whereby $X_n \xrightarrow{d_h} X$ if and only if :

i) $X_n \xrightarrow{d_s} X$, and

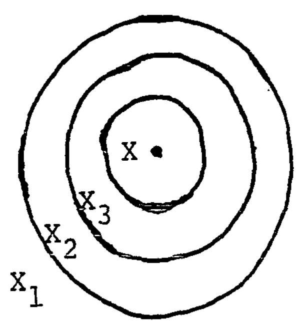
ii) Given $\varepsilon > 0$, there is a $\delta > 0$ such that for all n, every δ-subset of X_n contracts to a point inside an ε-subset of X_n. Condition ii) will be abbreviated $s(X_n, \delta, \varepsilon)$.

The following two examples illustrate the type of convergence in d_s that is ruled out in d_h by condition ii):

<u>Ex. 1</u>:



<u>Ex. 2</u>:



In Example 1, we have arcs X_n which clearly converge to the circle X in the Hausdorff metric d_s. If $\varepsilon > 0$ is chosen smaller than the diameter of the circle, however, then it is easy to see that no matter how small we choose $\delta > 0$, there will be an arc X_n so that two points p_n, q_n in X_n with $d(p_n, q_n) < \delta$ will have to go all the way around X_n (hence outside an ε- neighborhood) to contract to a point. In Example 2, the circles X_n clearly converge to the point X in d_s. But this time the X_n themselves get arbitrarily small, and since none of them is contractible, they cannot possibly converge in d_h.

Borsuk was able to show that if $X_n \xrightarrow{d_h} X$, then all but finitely many of the X_n have the homotopy type of X. This indicates immediately why the two examples above cannot be convergent sequences using the homotopy metric. It also means that $[X] = \{Y \varepsilon 2_h^M \mid Y$ has the homotopy type of $X\}$ is an open subset of 2_h^M, and since $\{[X] \mid X \varepsilon 2_h^M\}$ partition 2_h^M, it follows that $\{[X]\}$ are both closed and open in 2_h^M. In this paper, one question we will consider is how much fragmentation there is within the $[X]$, first for M a one-dimensional Peano continuum, and then for M a closed surface. For these two cases we will characterize the component structure of 2_h^M.

We take this opportunity to summarize some of the main results achieved by Borsuk and subsequent researchers. In Borsuk's original paper, he proves that 2_h^X (X finite dimensional) is a topological invariant (this is also the case with 2^X, cf. [N], pp. 29-30). He furthermore shows that 2_h^X is separable and complete. This last result is the key to Borsuk's original program in constructing 2_h^X, as it raises the possibility of invoking the Baire Theorem[32] in order to answer several natural questions pertaining to ANR's. Borsuk himself asked the following questions:

> Q1: If M is a polyhedron, is the collection B of all subpolyhedra of M dense in 2_h^M? What is the category (in the sense of Baire) of B?
>
> Q2: What is the category (in the sense of Baire) in the space 2_h^M ($M = I^n$) of the collection of all ANR's having the singularity of Brouwer, of Mazurkiewicz, or of Peano?[33]

The motivation behind these questions is an attempt to understand the extent to which ANR's are like nice spaces (i.e., polyhedra, it is known that every compact ANR has the homotopy type of a finite polyhedron[34]). One would likewise be interested to know the extent to which ANR's possess various pathologies (e.g., the singularities referred to in Q2, which cannot occur in polyhedra but can be found in certain ANR's). One way to get a handle on this question is to ask how plentiful (in the sense of Baire) are these nice or unnice spaces when considered as subsets of the hyperspace 2_h^M.

In 1972 Ball and Ford gave the following partial answer to question Q1 ($[B\&F]_1$, p. 40):

> Theorem: If M is a connected polyhedron, then T_M, the set of all subpolyhedra properly contained in M, is a first category subset of 2_h^M if and only if M contains no one-dimensional open subset.

In the case of $M = S^2$, Ball and Ford were also able to show that the subpolyhedra are dense in $2_h^{S^2}$, and that the topological polyhedra constitute a dense G_δ (i.e., second category) subset of $2_h^{S^2}$. These last two results were recently extended to the case where M is a closed two-manifold by L. Boxer.[35]

The most recent work on the homotopy metric has been the extension of Borsuk's original work due to Z. Čerin.[36] Čerin introduced a new notion of strongly e-movable convergence on the hyperspace ANR(X), i.e., ANR subsets of an arbitrary metric space X. He was able to show that this topology can be metrized as a complete, separable metric space, and that, in the case where X is a finite-dimensional compactum, the space ANR(X) with topology induced by strongly e-movable convergence is the same as that induced by the homotopy metric d_h on 2_h^X.

The results obtained in the present study represent a first attempt to obtain topological information about the hyperspace 2_h^X for particular spaces X (or classes of spaces {X}). As such, this work is closer in spirit to early attempts to solve the hyperspace conjecture, or to Duda's study of the topological structure of C(X) when X is a finite graph. In Chapter II we restrict X to being a one-dimensional Peano continuum. For this class of spaces, Theorem 2.1 gives a relatively straightforward criterion for detecting whether or not two ANR's belong to the same component of 2_h^X.

Two important classes for the dimension theory of infinite-dimensional (metric) spaces are the strongly infinite-dimensional (SID) spaces and the countably infinite-dimensional (CID) spaces. The latter class consists of those infinite-dimensional spaces which can be written as a countable union of finite-dimensional subspaces. The class of SID spaces, on the other hand, satisfy the following condition (cf. [R-S-W], pp. 94-95): A space X is SID if there exists a countable collection $\{(A_i,B_i)\}_{i=1}^{\infty}$ of pairs of closed subspaces of X with $A_i \cap B_i = \emptyset$ for all i, and with the property that every collection $\{S_i\}_{i=1}^{\infty}$ of closed subspaces of X such that S_i separates A_i from B_i must have $\bigcap_{i=1}^{\infty} S_i \neq \emptyset$.

It is not difficult to prove that these two classes of infinite-dimensional spaces are mutually exclusive (although it was only recently that R. Pol gave the first example showing that not every infinite-dimensional space belongs to one of these two classes).[37] In Chapter III we prove the following theorems:

<u>Theorem 3.1</u>: For every manifold M with dim M $\geq$ 2, 2^M_h is SID at every non-isolated point.

<u>Theorem 3.2</u>: If X is a Peano continuum, then every component of 2^X_h is finite dimensional if and only if X is a finite graph.

In Chapter IV we give the following extension of Theorem 3.2:

If X is a Peano continuum, then 2^X_h is CID if X is a finite graph, while if X is not a graph, then 2^X_h is SID.

The main emphasis in Chapter IV is to show via examples how 2_h^X can be constructed by means of inductive procedures if X is a simple enough graph.

In Chapter V we prove a theorem analogous to Theorem 2.1 (the main result of Chapter II), but this time for the class of spaces X = {closed 2-manifolds}. Again we obtain a reasonably simple criterion that enables one to determine whether or not two connected ANR's lie in the same component of 2_h^X. We conclude with some remarks related to the following question: If X is a closed 2-manifold, is $AR_h(X) = \{C \in 2_h^X | C \text{ is an AR}\}$ an ℓ_2-manifold? One can attempt to answer this question by appealing to Toruńczyk's characterization of ℓ_2-manifolds.[38] If the answer to the question is affirmative, then the fact that $AR_h(X)$ and X have the same homotopy type (cf. $[Bx]_1$, Cor. 4.7), together with the fact that homotopy equivalent ℓ_2-manifolds are homeomorphic, would imply that $AR_h(X)$ is homeomorphic to $X \times \ell_2$!

NOTES TO CHAPTER I

[1]Cf. K. Borsuk & S. Mazurkiewicz, "Sur l'hyperespace d'un continu," C.R. Soc. Sc. Varsovie, 24(1931), 149-52; S. Mazurkiewicz, "Sur l'hyperespace d'un continu," Fund. Math., 18(1932), 171-77; M. Wojdyslawski, "Retractes absolus et hyperespaces des continus," Fund. Math., 32(1939), 184-192.

[2]This information was conveyed to American topologists by K. Kuratowski. Cf. R. Schori and J. West, "2^I is homeomorphic to the Hilbert cube," BAMS, 78(1972), p. 402.

[3]A Peano continuum is a compact, connected, and locally connected metric space. Cf. M. Wojdyslawski, "Sur la contractilite des hyperespaces des continus localement connexes," Fund. Math., 30(1938), 247-52.

[4]Cf. T. Wazewski, "Sur un continu singulier," Fund. Math., 4(1923), 214-235.

[5]Vietoris ("Continua zweiter Ordnung," Monatshefte für Mathematik und Physik, 33(1923), 49-62) showed that X Peanian implies that 2^X and C(X) are Peanian. The reverse implications were proven by Wazewski, op. cit.

[6]Unless otherwise stated, an AR will mean a compact absolute retract for metric spaces. Thus X is an AR if it is a compact metric space, such that for every imbedding of X as a closed subset of a metric space Y there is a retraction of Y onto the image of X under the imbedding. Cf. Wojdyslawski (1939), op. cit.

[7]J. L. Kelley, "Hyperspaces of a continuum," TAMS, 52(1942), 22-36.

[8]R. Duda, "On the hyperspace of subcontinua of a finite graph, I," Fund. Math., 62(1968), 265-86; "On the hyperspace of subcontinua of a finite graph, II," Fund. Math., 63(1968), 225-255.

[9]M. Wojdyslawski (1939), op. cit.

[10]Cf. T. Chapman, Lectures on Hilbert cube manifolds, p. 2.

[11]N. Gray, "Unstable points in the hyperspace of connected subsets," Pac. J. Math., 23(1967), 515-520.

[12] N. Gray, "On the conjecture $2^X \approx I^\omega$," Fund. Math., 66(1969), 45-52.

[13] J. West, "Infinite Products which are Hilbert cubes," TAMS, 150(1970), 1-25.

[14] Cf. J. Segal, "Hyperspaces of the inverse limit space," PAMS, 10(1959), 706-709.

[15] M. Brown, "Some applications of an approximation theorem for inverse limits," PAMS, 11(1960), 478-83.

[16] J. West, "The subcontinua of a dendron form a Hilbert cube factor," PAMS, 36(1972), 603-608.

[17] R. Schori and J. West, "2^I is homeomorphic to the Hilbert cube," BAMS, 78(1972), 402-406. The complete proof was given in R. Schori and J. West, "The hyperspace of the closed unit interval is a Hilbert cube," TAMS, 213(1975), 217-235.

[18] R. Schori and J. West, "Hyperspaces of graphs are Hilbert cubes," Pac. J. Math., 53(1974), 239-51; D. Curtis and R. Schori, "Hyperspaces of polyhedra are Hilbert cubes," Fund. Math., 99(1978), 189-97; D. Curtis and R. Schori, "2^X and $C(X)$ are homeomorphic to the Hilbert cube," BAMS, 80(1974), 927-931. The complete proof was given in D. Curtis and R. Schori, "Hyperspaces of Peano continua are Hilbert cubes," Fund. Math., 101(1978), 19-38.

[19] For an extensive introduction to the subject, cf. S. Nadler, Hyperspaces of Sets (Marcel Dekker: New York and Basel), 1978.

[20] H. Toruńczyk, "On CE-images of the Hilbert cube and characterization of Q-manifolds," to appear.

[21] B. Knaster, "Un continu dont tout sous-continu est indecomposable," Fund. Math., 3(1922), 247.

[22] E. E. Moise, "An indecomposable plane continuum which is homeomorphic to each of its nondegenerate subcontinua," TAMS, 63(1948), 581-594.

[23] R. H. Bing, "Concerning hereditarily indecomposable continua," Pac. J. Math., 1(1951), 43-51.

[24] J. Walsh, "An infinite dimensional compactum containing no n-dimensional ($n \geq 1$) subsets," Topology, 18(1979), 91-95.

[25]L. Rubin, R. Schori, J. Walsh, "New dimension theory techniques for constructing infinite-dimensional examples," *General Topology and Appl.*, 10(1979), 93-102.

[26]Cf. P. S. Aleksondrov, "The present status of the theory of dimension," *AMS Transl.*, (2)1, (1955), 1-26; R. Pol, "A weakly infinite-dimensional compactum which is not countable-dimensional," preprint.

[27]J. West, "Induced involutions on Hilbert cube hyperspaces," *Proceedings of the 1976 Topology Conference, Auburn University*, 1(1976), 281-293.

[28]R. Wong, "Periodic actions on the Hilbert cube," *Fund. Math.*, 85(1974), 203-210.

[29]T. A. Chapman, "On the structure of Hilbert cube manifolds," *Comp. Math.*, 24(1972), 329-53.

[30]K. Borsuk, "On some metrizations of the hyperspace of compact sets, *Fund. Math.*, 41(1954), 168-202.

[31]By $X \in$ ANR (i.e., absolute neighborhood retract) we mean that X is a compact metric space such that whenever $X \xrightarrow{f} Y$ is embedded as a closed subspace of a metric space Y, there exists a neighborhood $U \subseteq f(X)$ in Y, and a retraction $r: U \longrightarrow f(X)$.

[32]Cf. J. Dugundgi, *Topology*, pp. 249-51.

[33]Cf. K. Borsuk, op. cit., p. 201.

[34]This was first proven in J. West, "Mapping Hilbert cube manifolds to ANR's: A solution to a conjecture of Borsuk," *Ann. of Math.*, 106(1977), 1-18.

[35]L. Boxer, "The space of ANR's of a closed surface," *Pac. J. Math.*, 79(1978), 47-68.

[36]Z. Čerin, "Strongly e-movable convergence and spaces of ANR's," to appear in *Topology*.

[37]R. Pol, *op. cit.*

[38]H. Toruńczyk, "Characterizing Hilbert space topology," Preprint 143, *Inst. of Math.*, Pol. Ac. of Sciences, 1978.

CHAPTER II

COMPONENT STRUCTURE OF 2_h^X FOR X A ONE-DIMENSIONAL PEANO CONTINUUM

We begin with a result which characterizes the component structure of 2_h^X in the case where X is a one-dimensional Peano continuum. The particular homology theory that is used here is of no consequence so long as it satisfies the Eilenberg-Steenrod axioms, since by [Hu] pp. 141-43, any such theory applied to ANR's yields isomorphic homology groups. We will, however, take $R = \mathbb{Z}$ to be our coefficient ring. We now state six preliminary lemmas and the theorem before presenting their proofs:

<u>Lemma 2.1</u>: Let X be a 1-dimensional Peano continuum and $C \subseteq X$ a connected ANR. If $H_1(C) = 0$, then C is contractible via a strongly contracting strong deformation retraction (SCSDR) to a point. (Thus there exists a strong deformation retraction, i.e. a map $H\colon C\times I \longrightarrow C$ with $H_0 = id_C$, $H_1(C) = \{p\}$, and $H_t(p) = p$ taking C to a point p. Saying that H is strongly contracting means that if $0 \leq u \leq v \leq 1$, then $H_u \circ H_v(C) \subset H_v(C) \subset H_u(C)$, cf. $[B\&F]_1$, p. 37).

Lemma 2.2: If C_1, C_2 are connected ANR's contained in X with $H_1(C_1) = H_1(C_2) = 0$, then there exists a path F^*: $I \longrightarrow 2^X_h$ with $F^*(0) = C_1$, $F^*(1) = C_2$.

Lemma 2.3: If C_1, C_2 are ANR's contained in X with $H_1(C_1) = H_1(C_2) = 0$, then C_1 and C_2 are in the same component of 2^X_h if and only if $H_0(C_1) \simeq H_0(C_2)$.

Lemma 2.4: For any connected ANR $C \subseteq X$, there exists a unique ANR $D \subseteq C$ such that:

(i) $[C] = [D]$

(ii) $H_1(C) = H_1(D) \subseteq H_1(X)$ (subgroups induced by inclusions $C \longrightarrow X$, $D \longrightarrow X$)

(iii) D has no points of order one (a point p has order n in a continuum Y if n is the smallest number for which p lies in arbitrarily small neighborhoods whose boundaries in Y consist of at most n points, cf. [Me], p. 97).

Lemma 2.5: The ANR's C and D of Lemma 4 both lie in the same component of 2^X_h .

Lemma 2.6: If C_1, C_2 are ANR's such that $H_1(C_1) \neq H_1(C_2)$ as subgroups of $H_1(X)$, then C_1 and C_2 lie in different components of 2^X_h.

Theorem 2.1: Let X be a 1-dimensional Peano continuum. Then two ANR's C_1, C_2 (with components $\{C_1^j\}$, $\{C_2^j\}$) lie in the same component of 2^X_h if and only if there exists a one-to-one correspondence $C_1^j \longleftrightarrow C_2^j$ between the components of C_1 and C_2 such that:

(i) $[C_1^j] = [C_2^j]$ for all j,

(ii) $H_1(C_1^j) = H_1(C_2^j)$ as subgroups of $H_1(X)$, and

(iii) Let $J_i' = \{j \mid H_1(C_i^j) \neq 0\}$, and let $D_i = \bigcup_{j\varepsilon J'} D_i^j$ where $D_i^j \subseteq C_i^j$ is the unique ANR of Lemma 4. Then $J_1'=J_2'=J'$ and $D_1=D_2=D$ and $H_0(C_1^j) = H_0(C_2^j)$ as subgroups of $H_0(X-D)$, for all $j \not\in J'$.

Pf. Lemma 2.1: Since C is a one-dimensional Peano continuum with $H_1(C) = 0$, C is acyclic. It follows (cf. [Wh], p. 89) that C is a dendrite.

We now make use of the fact that every Peano continuum M has a convex metric d that preserves its topology ([Bi], p. 1109 and [Mo], p. 1119). This means that given any two points p, $q \varepsilon M$, there exists $r \varepsilon M$ such that $d(p,r) = d(r,q) = \frac{1}{2} d(p,q)$. This implies that between any two points of M, there exists an isometrically embedded interval. (If $d(p,q) = s$, then $F: [0,s] \longrightarrow M$ is defined on $\{s \cdot \frac{j}{2^k}\}$, ($j$ odd, $j \leq 2^k$), the dyadic rationals in $[0,s]$, and hence has a unique extension to all of $[0,s]$ which is an embedding).

Let d be such a convex metric for the Peano continuum C. For any given point $p \varepsilon C$, we can assume that for all points

$q \in C$, $d(p,q) \leq 1$. Since C is a dendrite, for each $c \in C$ there exists a unique segment L_c joining p and c ([Me], p. 306). For $0 \leq t \leq 1$ define

$$L_c(t) = \begin{cases} c, \text{ if } t \geq d(p,c) \\ q, \text{ if } t < d(p,c), \text{ where } q \text{ is the} \\ \text{unique point on } L_c \text{ with } d(p,q) = t. \end{cases}$$

Now we can define $F: C \times I \longrightarrow C$ by $F(c,t) = L_c(1-t)$ which we will show is a SCSDR of C to $\{p\}$. First observe that $F_0 = id_C$, $F_1(C) = \{p\}$, and $F_t(p) = p$. F is also clearly strongly contracting. It remains, therefore, to verify that F is continuous. Let $\{(c_i, t_i)\} \longrightarrow (c,t)$ be a convergent sequence in $C \times I$; we wish to show that $\{F(c_i, t_i)\} \longrightarrow F(c,t) = y$. For this purpose, we consider three cases:

1) $t < 1-d(p,c)$
2) $t = 1-d(p,c)$
3) $t > 1-d(p,c)$.

<u>Case 1</u>: If $t < 1-d(p,c)$, then $1-t > d(p,c)$, hence $F(c,t) = c$. Let $1-t-d(p,c) = \eta > 0$. Choose N_1 such that for all $i \geq N_1$, $d(c_i,c) < \eta/2$. Thus $d(p,c_i) < d(p,c) + d(c,c_i) < d(p,c) + \eta/2$.

Now choose $N \geq N_1$ such that for all $i \geq N$, $t_i < 1 - (d(p,c) + \eta/2)$. Then for $i \geq N$, $F(c_i, t_i) = c_i$, since $1 - t_i > d(p,c) + \eta/2 > d(d,c_i)$. Therefore $\{F(c_i, t_i)\} \longrightarrow c = F(c,t)$.

Case 2: Let $\varepsilon > 0$ be given. If $t = 1 - d(p,c)$, then $F(c,t) = c$. Since C is locally connected we can choose a connected neighborhood. U of c with $\text{diam}(U) < \varepsilon$. There exists N_1 such that for all $i \geq N_1$, $c_i \in U$. Now choose $N \geq N_1$ so that $|t_i - t| < \varepsilon$ for all $i \geq N$, and consider a point $x \in L_c$ such that $d(x,c) < \varepsilon$. We claim that if $i \geq N$, then $d(F(x,t_i), F(c,t)) < \varepsilon$. For if $1 - t_i \geq d(p,x)$, then $F(x,t_i) = x$, and hence $d(F(x,t_i), F(c,t)) = d(x,c) < \varepsilon$. On the other hand, if $1 - t_i < d(p,x)$, then $F(x,t_i) = L_x(1-t_i) = q$, where $d(p,q) = 1 - t_i$. But $d(q,c) = |d(p,c) - d(p,q)|$, since $x \in L_c$, while, for $i \geq N$, $|t_i - t| < \varepsilon$. Thus for $i \geq N$, $d(F(x,t_i), F(c,t)) = d(q,c) = |d(p,c) - d(p,q)| = |(1-t) - (1-t_i)| = |t_i - t| < \varepsilon$.

Now consider any $c_i \in U$. Each c_i lies on a unique arc $\ell_i \subseteq U$ which meets L_c in a point x_i. Thus $d(p,c_i) = d(p,x_i) + d(x_i, c_i)$, and since, for i sufficiently large, $d(F(x_i,t_i), F(c,t)) < \varepsilon$, it follows that there exists an integer N, such that $d(F(c_i,t_i), F(c,t)) < \varepsilon$, for all $i \geq N$.

Case 3: If $t > 1 - d(p,c)$, then $F(c,t) = y \neq c$. Let $\varepsilon > 0$ be chosen such that $\varepsilon < d(y,c)$. There exists a connected neighborhood U of c with $\text{diam}(U) < d(y,c) - \varepsilon$. There exists an integer N_1 such that $c_i \in U$ for all $i \geq N_1$, and an integer $N \geq N_1$ such that $|t_i - t| < \varepsilon$, for $i \geq N$. Thus if $x \in L_c$ and $d(x,c) < \varepsilon$, then $d(F(x,t_i), F(c,t)) < \varepsilon$ for all $i \geq N$. Since there exist unique arcs ℓ_i joining c_i and $x_i \in L_c$ such that $\ell_i \subseteq U$, it follows that $d(F(c_i, t_i), F(c,t)) < \varepsilon$ as required.

Thus we have shown that F is continuous, which completes the proof of Lemma 2.1.

<u>Pf. Lemma 2.2</u>: This is a fairly immediate consequence of the following result:

<u>Prop. 2.1 ([B&F]$_1$, Lemma 3.4)</u>: If X is a finite dimensional compactum, and C an ANR contained in X, and if $h: C\times I \longrightarrow C$ is a SCSDR, then $h^*: I \longrightarrow 2^X_h$ given by $h^*(t) = h_t(C)$ is continuous.

Since, by Lemma 1, there exist $F^i: C_i \times I \longrightarrow C_i$, $(i = 1,2)$, SCSDR's that take C_i to a point p_i, the proposition above implies that the F^i induce paths $(F^i)^*: I \longrightarrow 2^X_h$ such that $(F^i)^*(0) = C_i$ and $(F^i)^*(1) = p_i$. Since X is a Peano continuum, it is arcwise connected ([H&Y], p. 118), hence there exists an embedding $f: [0,1] \longrightarrow X$ with $f(0) = p_1$ and $f(1) = p_2$. This f induces a path f^* in 2^X_h by the following result:

<u>Prop. 2.2 ([B&F]$_1$, Lemma 4.2)</u>: Let X be a finite dimensional compactum, $A\varepsilon 2^X_h$, and $f: A \times I \longrightarrow X$ an isotopy. If for each $t\varepsilon I$, $f^*(t) = f_t(A)$, then $f^*: I \longrightarrow 2^X_h$ is continuous.

We can now define $F^*: I \longrightarrow 2^X_h$ as follows:

$$F^*(t) = \begin{cases} (F^1)^*(3t) & 0 \le t \le 1/3 \\ f^*(3t-1) & 1/3 \le t \le 2/3 \\ (F^2)^{*-1}(3t-2) & 2/3 \le t \le 1 \end{cases}$$

Since $F^*(0) = C_1$ and $F^*(1) = C_2$, this proves Lemma 2.2.

Remark: The condition that X be a Peano continuum (i.e., locally connected) cannot be weakened as can readily be seen by considering the topologist's sine curve,

$$X = X_1 \cup X_2 = \{(x, \sin 1/x) \mid 0 < x \leq 1\} \cup \{(x,y) \mid \begin{smallmatrix} x = 0 \\ -1 \leq y \leq 1 \end{smallmatrix}\}.$$

For this X, a connected ANR C is either contained in X_1 or in X_2. Since $H_1(X) = 0$, we merely need to observe that if $C_1 \subseteq X_1$ and $C_2 \subseteq X_2$, then there cannot possibly be a path in 2^X_h joining C_1 and C_2. For suppose $F: I \longrightarrow 2^X_h$ were such a path. Let $t^* = \inf\{t\varepsilon I \mid F(t) \subseteq X_2\}$. Clearly $F(t^*) \subseteq X_2$, and if $F(t^*) \neq X_2$ we can find a compact neighborhood W about $F(t^*)$ in X such that $X_2 \cap W$ is a component of W. The continuity of F implies that there exists a neighborhood V of t^* such that each point in F(V) lies inside of W, and hence inside $X_2 \cap W$. This contradicts the definition of t^*. We can therefore assume that $F(t^*) = C_2 = X_2$, and $F: I \longrightarrow 2^X_h$ is a path joining C_1 and C_2. Since F must be continuous with respect to the Hausdorff metric, it follows that for $\delta > 0$ sufficiently small, any $t \varepsilon [1 - \delta, 1]$ has the property that diam $(\pi_y(F(t)) > 2 - \varepsilon$, where π_y is the projection of X to X_2. Now for each positive integer n, let $X_{n\pi} = \{(x, \sin 1/x) \mid 0 < x \leq \frac{1}{n\pi + \pi/2}\}$. Choose a sequence $\{t_i\} \subseteq [1 - \delta, 1]$ with $\{t_i\} \longrightarrow 1$, and let $F(t_i) = D_i$. Now for each t_i, there exists an n_i such that $D_i \cap X_{n_i\pi} = \emptyset$. On the other hand, since $D_i \xrightarrow{d_s} C_2$, there exists a D_j such that $D_j \subseteq X_{n_i\pi}$. It easily follows, therefore, that there exists t_k with $t_i < t_k < t_j$ such that diam $(F(t_k) \cap X_{n_i\pi}) > \varepsilon$, and diam $(F(t_k) - X_{n_i\pi}) > \varepsilon$. This argument can now be repeated by replacing D_i with D_j, finding an n_j such that $D_j \cap X_{n_j\pi} = \emptyset$, etc., the result being a new t_k. We thus get a sequence $\{D_k\} = \{F(t_k)\}$, where $D_k \xrightarrow{d_s} C_2$. It will further be observed that for each t_k, $\pi_y(D_k) \cap \{-1,1\} \neq \emptyset$. Since each $F(t_k) = D_k$ contains an ε-arc on either side of the endpoint of X_{n_i} (be it a maximum or a minimum), it follows that $D_k \overset{d_h}{\not\longrightarrow} C_2$. For clearly as $n_i \longrightarrow \infty$, the points on opposite sides of the ε-arcs get arbitrarily close to one another. Hence it is impossible to find a uniform δ such that every subset of each D_k with diameter $\leq \delta$ contracts to a point inside a subset of D_k of diameter less than ε. Therefore $F\colon I \longrightarrow 2^X_h$ cannot be continuous.

Pf. Lemma 2.3: Since C_1 and C_2 must have the same homotopy type in order to belong to the same component of 2^X_h (cf. $[B]_2$, p. 200), the condition $H_1(C_1) = H_1(C_2) = 0$ implies $H_0(C_1) \approx H_0(C_2)$ is clearly necessary, as otherwise $[C_1] \neq [C_2]$. Suppose now that C_1 and C_2 each have n-components $\{C_i^j \mid j = 1,2,\ldots,n\}$, $i = 1,2$, and let P_{ji} be any point in C_i^j. Applying Lemma 2.1, there exist SCSDR's $F_i^j\colon C_i^j \times I \longrightarrow C_i^j$ contracting each C_i^j to the point P_{ji}. Thus for $i = 1,2$ we have maps $F_i\colon C_i \times I \longrightarrow C_i$ which are SCSDR's of C_i to $\{P_{ji}\}_{j=1}^n$. By Prop. 2.1, the F_i induce maps $F_i^*\colon I \longrightarrow 2^X_h$ with $F_i^*(0) = C_i$ and $F_i^*(1) = \{P_{ji}\}_{j=1}^n$. It remains, therefore, to show that there exists a $G^*\colon I \longrightarrow 2^X_h$ with $G^*(0) = \{P_{j1}\}$ and

$G^*(1) = \{P_{j2}\}$. Thus, by Prop. 2.2 it suffices to find an isotopy $G: \{P_{j1}\} \times I \longrightarrow X$ with $G_0 = id_{\{P_{j1}\}}$ and $G_1(P_{j1}) = P_{j2}$.

We construct G by induction on n, the number of points $\{P_{j1}\}_{j=1}^{n}$. There are two cases to consider. First suppose X has an endpoint q. Let $\alpha_1: I \longrightarrow X$ be a path joining q and some P_{k1} with the property that $\alpha_1(I) \cap \{P_{j1}\} = P_{k1}$. Next choose a path $\alpha_2: I \longrightarrow X$ joining q and some P_{k2} and such that $\alpha_2(I) \cap \{P_{j2}\} = P_{k2}$. Clearly the path $\alpha_2 \circ \alpha_1^{-1}$ induces an isotopy $G^k: \{P_{k1}\} \times I \longrightarrow X$ with $G^k(P_{k1}, 0) = P_{k1}$, $G^k(P_{k1}, \frac{1}{2}) = q$, and $G^k(P_{k1}, 1) = P_{k2}$. Since q has order 1 we can find a neighborhood U of q in X such that $U \cap \{P_{ji}\} = \emptyset$ for $j \neq k$, and with Bd U = {pt.}. It follows that X - U = X' is a subcontinuum of X, so by our induction hypothesis there is an isotopy for the 2(n-1) points $\{P_{j1}\}_{j \neq k}$, $G^{k'}: \{P_{j1}\} \times I \longrightarrow X'$, with $G^{k'}(P_{j1}, 0) = P_{j1}$ and $G^{k'}(P_{j1}, 1) = P_{j2}$. We now define $G: \{P_{j1}\} \times I \longrightarrow X$ as follows:

$$G(P_{j1},t) = \begin{cases} P_{j1},\ j \neq k \ ; \ G^k(P_{k1},2t),\ j = k \text{ for } 0 \leq t \leq 1/4 \\ G^{k'}(P_{j1},2t-1/2),\ j \neq k \ ; \ q,\ j = k \text{ for } 1/4 \leq t \leq 3/4 \\ P_{j2},\ j \neq k \ ; \ G^k(P_{k1},2t-1),\ j = k \text{ for } 3/4 \leq t \leq 1 \end{cases}$$

This handles case one.

Now suppose X has no points of order one. It follows (cf. [Me], p. 307) that X contains a cycle. Since the points of order 2 are dense in X ([Me], p. 115), we can choose a point q that lies on a cycle of X and has order 2. It follows that X - q is connected, and moreover we can find a neighborhood

U of q in X such that $U \cap \{P_{ij}\}$ consists of at most two points and X - U is a subcontinuum of X (since Bd U = {2 points}). We can therefore use the same construction given above to produce the desired isotopy.

Pf. Lemma 2.4: By Menger's Eindeutigkeitssatz ([Me], p. 336), each point $p \varepsilon C$ that is not an endpoint or a cutpoint lies in exactly one maximal cyclic subcontinuum. Let $C^* = \{p \varepsilon C \mid p$ is neither an endpoint nor a cutpoint$\}$, and let $D' = \bigcup_{p \varepsilon C^*}$ {Max cyclic subcontinua containing p}. If D' is not connected, there exist arcs R_α in C joining certain pairs of components of D'. Each pair of components of D' has at most one R_α joining its elements (otherwise a new cycle in C but not D' would be introduced), and $D = D' \cup \bigcup_\alpha (R_\alpha)$ is connected since C is. Furthermore C - D consists entirely of endpoints and cutpoints.

Now if $C - D \neq \emptyset$, each of its components $\{Q'_\beta\}$ has a single point $p_\beta \varepsilon C$ that lies in the boundary of C - D, i.e., there is a unique point p_β of C that is accessible from a given component Q'_β of C - D (otherwise another cycle of C would be introduced). It follows that $\{Q_\beta\} = \{Q'_\beta \cup p_\beta\}$ is a collection of acyclic Peano continua, i.e., dendrites. Since (cf. [Me], p. 307) a dendrite has at least two endpoints, the condition $C - D \neq \emptyset$ implies that C must have points of order one. Moreover, since every subcontinuum of a dendrite is a dendrite ([Wh], p. 89), it follows that every subcontinuum S of C that properly contains D has endpoints. Since D clearly has no endpoints, it remains to show that every subcontinuum S of C with $D - S \neq \emptyset$ fails to satisfy conditions (i) - (iii).

Let us assume, therefore, that such a subcontinuum S satisfying conditions (i) - (iii) does exist, and that $p \varepsilon D-S$. There are two cases to consider. Suppose first that $p \varepsilon D'$, i.e., that p lies on some cycle of D that is not contained by S. We now show that condition (ii) $H_1(S) = H_1(C) \subseteq H_1(X)$ fails. For purposes of comparing the homology of the ANR's S and C, we are free to use any suitable homology theory (cf. [Hu], pp. 141-43). It will be convenient here to use a homology theory based on taking geometric nerves of open covers (as in [Wi], p. 130). Since dim X = 1, every open cover α of X has a refinement β with $\mathrm{ord}(\beta) = 1$ (cf. [H&W], pp. 54-55). This means that nerve (β) contains no 2-simplexes, and hence $H_1(\text{nerve } \beta) = Z_1(\text{nerve } \beta)$, i.e., the cycles of X are free generators of $H_1(X)$. Since $p \varepsilon C-S$, it follows that some cycle in C containing p is a generator of $H_1(C)$ that is not in $H_1(S)$. Hence $H_1(C) \neq H_1(S)$ as subgroups of $H_1(X)$.

The other possibility, if $D-S \neq \emptyset$, is that $p \varepsilon D-S$ lies on some arc $R_\alpha \subseteq D$. If this is the case, then p is a cut-point of C, and therefore S is contained in one of the components of C - p. Since both of the components of C - p contain cycles (hence $H_1 \neq 0$), it follows that $p \varepsilon R_\alpha$ implies $H_1(S) \neq H_1(C)$, i.e., condition (ii) fails in this case also.

It remains now to show that D itself satisfies conditions (i) and (ii). To see this, note that since each Q_β is a dendrite, by Lemma 2.1 there exists a SCSDR $F^\beta\colon Q_\beta \times I \longrightarrow Q_\beta$ contracting Q_β to $p_\beta \varepsilon D$. Furthermore, since C is an ANR (and hence locally connected) we can perform the contractions F^β simultaneously. That is the map $F\colon C\times I \longrightarrow C$ given by $F(c,t) = \begin{cases} c, & \text{if } c\varepsilon D. \\ F^\beta(c,t), & \text{if } c\varepsilon Q_\beta \end{cases}$ is continuous, and hence a SCSDR of C to D. It follows that (i) $[C] = [D]$, (ii) $H_1(C) = H_1(D)$ as subgroups of $H_1(X)$, and therefore that D is the unique subcontinuum of C satisfying conditions (i) - (iii).

<u>Pf. Lemma 2.5</u>: This follows immediately from the existence of the map F above and Proposition 2.1 (cf. Pf. of Lemma 2.2).

<u>Pf. Lemma 2.6</u>: Since C_1 and C_2 are ANR's their homology is finitely generated ([Hu], pp. 140-41), and, as remarked in the proof of Lemma 2.4, $H_1(C_1)$ and $H_1(C_2)$ are generated by the finitely many cycles of X that C_1 and C_2 each contain. Thus if $H_1(C_1) \neq H_1(C_2)$ as subgroups of $H_1(X)$, then one of the C_i, say C_1, contains a cycle S that is not contained in C_2. Let Y_1, Y_2 be the components of 2^X_h containing C_1, C_2 respectively, and let

$$\mathcal{Z}_1 = \{C\varepsilon 2_h^X \mid S - C \neq \emptyset\}$$
$$\mathcal{Z}_2 = \{C\varepsilon 2_h^X \mid S \subseteq C\}.$$

Thus $\mathcal{Z}_1 \cup \mathcal{Z}_2 = 2_h^X$, while $\mathcal{Z}_1 \cap \mathcal{Z}_2 = \emptyset$. Furthermore $\mathcal{Z}_1$ and $\mathcal{Z}_2$ are both closed and open in 2_h^X. This follows by considering the nature of the metric d_h which is defined as follows: $d_h(C,D) = d_s(C,D) + \sup_t\{|\lambda_C(t) - \lambda_D(t)|\}$. The definition of the λ's is too complicated to elaborate here (cf. $[B]_2$ for details), but it is easy to check that if $C\varepsilon\mathcal{Z}_1$, then $\varepsilon > 0$ can be chosen so that any $D\varepsilon\mathcal{Z}_2$ with $d_s(C,D) \leq \varepsilon$ must have $\sup_t |\lambda_C(t) - \lambda_D(t)| > \varepsilon$, and hence $d_h(C,\mathcal{Z}_2) > \varepsilon$. Similarly if $C\varepsilon\mathcal{Z}_2$, then there exists an $\varepsilon > 0$ such that for any $D\varepsilon\mathcal{Z}_1$ with $d_s(C,D) \leq \varepsilon$, $\sup_t|\lambda_C(t) - \lambda_D(t)| > \varepsilon$, and $d_h(C,\mathcal{Z}_1) \geq \varepsilon$. Finally since $C_1\varepsilon\mathcal{Z}_1$ and $C_2\varepsilon\mathcal{Z}_2$, their respective components must be situated so that $Y_1 \subseteq \mathcal{Z}_1$ and $Y_2 \subseteq \mathcal{Z}_2$. Thus $Y_1 \neq Y_2$ as claimed.

<u>Pf. Thm. 2.1</u>: To see that the above is a necessary condition, suppose C_1 and C_2 lie in the same component of 2_h^X. First observe that card $J_1' =$ card J_2'. For if not, then $H_1(C_1) \neq H_1(C_2)$ as subgroups of $H_1(X)$, and, therefore, by Lemma 2.6, C_1 and C_2 lie in different components of 2_h^X. We

may thus assume that J' is well defined. We now claim that there is a (unique) one-to-one correspondence $C_1^j \longleftrightarrow C_2^j$, $j \varepsilon J'$, satisfying (i) and (ii). For if not, then there is a $k \varepsilon J'$ such that $H_1(C_1^k)$ (resp. $H_1(C_2^k)$) is a subgroup of $H_1(X)$ that lies outside of $H_1(C_2)$ (resp. $H_1(C_1)$). It follows by Lemma 2.6 that C_1 and C_2 lie in different components of 2_h^X. Now for $j \not\in J'$, $H_1(C_i^j) = 0$, and by Lemma 2.1 each of the C_i^j is contractible via a SCSDR. Thus for $j \not\in J'$, $\{C_i^j\}$ automatically satisfy conditions (i) and (ii). Let $C_i = \bigcup_{j \not\in J'} C_i^j$; if condition (iii) fails, then there exists a component R of X-D such that $r_1 = \text{rank } (H_0(R \cap C_1')) \neq \text{rank } (H_0(R \cap C_2')) = r_2$. Let Y_1, Y_2 be the components of 2_h^X containing C_1, C_2 respectively. By Lemma 2.6 and the proof of Lemma 2.4, each point in $Y = Y_1 \cup Y_2$ must contain D. Since D separates R from $X-(D \cup R)$, it follows that every point C in Y_i must have rank $(H_0(R \cap C)) = r_i$. Therefore $Y_1 \cap Y_2 = \emptyset$, and, hence, condition (iii) must hold, and thus we can find a one-to-one correspondence between $C_1^j \longleftrightarrow C_2^j$ satisfying (i)-(iii).

To see that conditions (i) - (iii) are sufficient, note that by Lemma 2.5, C_i^j lies in the same component of 2_h^X as does D_i^j for i = 1, 2 and all $j \varepsilon J'$. Furthermore by conditions (i) and (ii) and the construction of the D_i^j, $[D_1^j] = [C_1^j] = [C_2^j] = [D_2^j]$, and $H_1(D_1^j) = H_1(C_1^j) = H_1(C_2^j) = H_1(D_2^j)$. Thus Lemma 2.4 implies that $D_1^j = D_2^j = D'$ (since D_i^j is unique with respect to the above properties). It now follows from condition (iii) and Lemma 2.3 that each of the C_i^j ($j\varepsilon J'$) lie in the same component of $X - D$, and hence in the same component of 2_h^X. We can therefore join C_1 to C_2 by running a path from C_i^j to D^j for $j\varepsilon J'$. Then run a path between the remaining C_i^j by using Lemma 2.3. This completes the proof.

CHAPTER III

DIMENSION OF 2_h^X

In 1942 Kelley proved that if X is a Peano continuum, then C(X) is finite dimensional if and only if X is a finite graph ([Ke], p. 30). This Theorem is closely related to the results we obtain in this chapter concerning the dimension of 2_h^X. Our starting point is the following result of Ball & Ford ($[B\&F]_1$, p. 48):

If X is an n-manifold, $n \geq 2$, then 2_h^X is infinite dimensional at every non-isolated point. The authors also noted the following fact:

If X is a locally connected continuum that contains a point of order at least n, then $\dim 2_h^X \geq n$. In particular, if $\dim X \geq 2$, then 2_h^X is infinite dimensional.

In this chapter we prove, among other things, the following refinement of the Ball and Ford Theorem:

<u>Theorem 3.1</u>: For every manifold M with $\dim M \geq 2$, the space 2_h^M is strongly infinite dimensional at every non-isolated point.

We will need the following three lemmas.

<u>Lemma 3.1</u>: Let L_n = Line segment joining $(0,0)$ and $(\frac{1}{n},\frac{1}{n^2})$, and let $T = \bigcup_{n=1}^{\infty} L_n$. Then the subset P = {Continua in T containing $(0,0)$} of 2^T is homeomorphic to the Hilbert cube $Q = [0,1]^{\infty}$.

<u>Pf.</u>: For each segment L_n, let $s_n\colon L_n \longrightarrow [0,\frac{1}{n^2}\sqrt{n^2+1}]$ be given by

$$s_n(p) = \begin{cases} 0 & , p = (0,0) \\ \frac{1}{t^2}\sqrt{t^2+1} & , p = (\frac{1}{t},\frac{1}{t^2}). \end{cases}$$

Thus s_n parameterizes L_n by arc length. We can "normalize" this parameterization by defining $f_n\colon L_n \longrightarrow [0,1]$ where $f_n = g_n \circ s_n$; $g_n\colon [0,\frac{1}{n^2}\sqrt{n^2+1}] \longrightarrow [0,1]$ given by $g_n(t) = \dfrac{t}{\frac{1}{n^2}\sqrt{n^2+1}}$. We now define $F_n\colon P \longrightarrow [0,1]$ by $F_n(C) = \max\{f_n(C \cap L_n)\}$. Finally we define $F\colon P \longrightarrow [0,1]^{\infty}$ by $F(C) = \{F_n(C)\}$. Clearly if $C_1, C_2 \varepsilon P$ and $C_1 \neq C_2$, then for some k, $F_k(C_1) \neq F_k(C_2)$, and hence $F(C_1) \neq F(C_2)$. Since F is also clearly onto, it follows that F is a bijection.

Now let $C \varepsilon P$, and Let $N_\varepsilon(C)$ be the ε-ball about C (ε measured in the Hausdorff metric), and choose N such that $\frac{1}{N^2}\sqrt{N^2+1} < \varepsilon$. Since $N_\varepsilon(C) = \{C' \varepsilon P \mid F_n(C)-\varepsilon < F_n(C') < F_n(C)+\varepsilon, \text{ for all } n\}$ and since for $n \geq N$, $F_n(C)-\varepsilon<F_n(C')<F_n(C)+\varepsilon$ holds for all $C' \varepsilon P$, we have

$$N_\varepsilon(C) = \{C' \varepsilon P | F(C') \varepsilon \prod_{n=1}^{N-1} (F_n(C) - \varepsilon, F_n(C) + \varepsilon) \times \prod_{n=N}^{\infty} [0,1]_n\}.$$

Thus $F(N_\varepsilon(C)) = \prod_{n=1}^{N-1} (F_n(C) - \varepsilon, F_n(C) + \varepsilon) \times \prod_{n=N}^{\infty} [0,1]_n$ which is open in Q; hence F is an open map. Essentially the same argument shows that F^{-1} is also open, so $F: P \longrightarrow Q$ is a homeomorphism.

<u>Lemma 3.2</u>: The subspace P of 2^T_h is homeomorphic to Q.

<u>Pf</u>: Given a sequence $p_i \xrightarrow{d_s} p_o$, $(p_i \varepsilon P)$ it suffices to show that $p_i \xrightarrow{d_s} p_o$ (the converse being, of course, automatic). Thus we must show that given any ε, there exists a uniform δ, so that every subset of each p_i with diameter $\leq \delta$ contracts to a point inside a subset of p_i of diameter $\leq \varepsilon$. First note that the ε-ball about (0,0) contains all but finitely many of the branches of T. For the finitely many remaining branches, the angle θ_i between adjacent branches gives a uniform relationship between ε and the distance two points can be apart and still be contractible inside an ε-nbhd, namely $\delta_i(\varepsilon) = (\sin \theta_i) \cdot \varepsilon$. Suppose now that T has n branches of length greater than ε, and let θ be the minimum angle between adjacent branches among these n. Then it is easy to see that $\delta = (\sin \theta) \cdot \varepsilon$ satisfies the condition for convergence of $p_i \xrightarrow{d_h} p_o$.

Lemma 3.3: Let M be a manifold with $\dim M = n \geq 2$. Then for every non-isolated point $C \varepsilon 2_h^M$ and neighborhood $U \subseteq 2_h^M$ there exists an embedding $f: Q \longrightarrow U$ with $f(0,0\ldots) = C$.

Pf.: $M = \bigcup_{j\varepsilon J} M_j$, where $\{M_j\}_{j\varepsilon J_1}$ are the compact components of M, $\{M_j\}_{j\varepsilon J_2}$ the noncompact components, and $J = J_1 \cup J_2$. Note that $C\varepsilon 2_h^M$ is an isolated point if and only if $C = \bigcup_{j\varepsilon K} M_j$, where $K \subseteq J_1$, since no proper subspace of a closed manifold is homotopy equivalent to the manifold itself. Thus if C is a non-isolated point, there exists a $j\varepsilon J$ such that $\emptyset \neq C \cap M_j \neq M_j$, and the manifolds M and $M-C$ satisfy the hypotheses of the following proposition ([H&Y], Thm. 3-18):

> In a locally connected and locally arcwise-connected space M, the set of all points on the boundary of an open set $V(=M-C)$ that are accessible from V is dense in the boundary of V.

Let $p\varepsilon \mathrm{Bd}(C)$ and let B^n be a PL-neighborhood of $p\varepsilon M$. Let $s\varepsilon C$ be any point accessible by a PL-arc L from $B^n \cap (M-C)$. By utilizing the PL-structure we can thicken L up slightly at every point except s to obtain a ball $B^* \supseteq L$ such that $B^* \cap C = s$.

Now let $\varepsilon = d_h(C, 2_h^M - U)$. It follows from the proof of Lemma 3.2 that we can identify the infinite-broom T of Lemma 3.1 with an imbedded copy T' in B^* in such a way that the wedgepoint corresponds to s and $d_h(C, C \cup T') < \varepsilon$. Consider the subspace $D = \{C' \varepsilon 2_h^M | C \subseteq C' \subseteq C \cup T'\}$ of 2_h^M. By the proof of Lemma 3.2 and choice of ε, $D \subseteq U$. Let $D' \subseteq 2_h^M$ be $D' = \{C' - \{C - \{s\}\} | C' \varepsilon D\}$, and let $g: D' \longrightarrow D$ be the map given by $g(C') = C' \cup C$. Clearly g is a homeomorphism. Furthermore, Lemma 3.2 states that there exists a homeomorphism $f'\colon Q \longrightarrow D'$ which can be taken so that $f'(0,0\ldots) = \{s\}$. It follows that $f = g \circ f'\colon Q \longrightarrow D$ is the desired embedding.

Pf. Thm. 3.1: Since Q is strongly infinite dimensional (cf. [R-S-W], p. 95) it follows that every non-isolated point in 2^M_h lies in arbitrarily small neighborhoods that are strongly infinite dimensional, and the Theorem is therefore proved.

As a counterpart to Kelley's Theorem we have the following:

Theorem 3.2: If X is a Peano continuum, then every component of 2^X_h is finite dimensional if and only if X is a finite graph.

We make use of the following four lemmas:

Lemma 3.4: Let $R = S_1 \cup S_2 = \{(x,y) \mid 0 \leq x \leq 1,\ y = 0\} \cup \{(\frac{1}{n}, y) \quad n = 1,2,\ldots;\ 0 \leq y \leq \frac{1}{n}\}$, and let S be the collection of all continua in R containing S_1; then the subset S of 2^R is homeomorphic to the Hilbert cube.

Pf.: Each point in S is uniquely determined by its endpoints (i.e., points of order one). If we express Q as $\prod_{i=n}^{\infty} [1,\frac{1}{n}]$, then there is a one-to-one correspondence between the endpoints of a point in S and the points in Q. Thus if $\{(\frac{1}{n},y_n)\}$ are the endpoints of $C\varepsilon S$, then $F: S \longrightarrow Q$ given by $F(C) = \{y_1, y_2, \ldots\}$ is clearly a bijection. If $p = \{p_n\} \varepsilon Q$, and $\varepsilon > 0$ is given so that $N_{d_s}((F^{-1}(p),\varepsilon)$ is an ε-neighborhood about $F^{-1}(p) \varepsilon S$, then there is an N such that $\frac{1}{N} < \varepsilon$. It follows that $U = \prod_{n=N}^{\infty} [0,1/n] \times \prod_{n=1}^{N-1} (p_n-\varepsilon, p_n+\varepsilon)$ is an open set containing $p\varepsilon Q$ such that $F^{-1}(U) \subseteq N_{d_s}(F^{-1}(p),\varepsilon)$, showing that F^{-1} is continuous. Since Q is compact it follows that F is also continuous, and hence a homeomorphism.

Lemma 3.5: The subspace S of 2^R_h is homeomorphic to Q.

Pf: Given a sequence $C_i \xrightarrow{d_s} C_o$ $(C_i\varepsilon S)$, it suffices to show $C_i \xrightarrow{d_h} C_o$. Let $\varepsilon > 0$ be given, and choose N such that $\frac{1}{N} < \frac{\varepsilon}{2}$. It follows that any subset of C_i that is within $\frac{1}{N}$ distance of (0,0) contracts to a point inside an ε-neighborhood of C_i. The smallest non-convex distance outside this $\frac{1}{N}$ ball is the distance between the Nth and (N-1)st spikes, i.e, $\frac{1}{N-1} - \frac{1}{N} = \frac{1}{N(N-1)} < \frac{1}{N}$. Thus if $\delta < \frac{1}{N(N-1)}$, then every subset of C_i of diameter δ or less contracts to a point inside a subset of C_i of diameter less than ε, for all $i = 0,1,\ldots$. Therefore $C_i \xrightarrow{d_h} C_o$.

<u>Lemma 3.6</u>: Let $AR_h(X)$ denote the component of 2_h^X consisting of AR subsets of X. If X is a finite acyclic graph, then $AR_h(X) \simeq C(X)$.

<u>Pf</u>: Clearly $AR_h(X) = C(X)$ as point sets. Thus it suffices to show that if $C_i \xrightarrow{d_s} C_o$, then $C_i \xrightarrow{d_h} C_o$. Since X is a finite polyhedron, it can be embedded in some Euclidean space E^n so that the edges of X are all straight lines within the affine structure of E^n.

Let $\varepsilon > 0$ be given. Each pair of concurrent edges determines an angle θ_i, whereas for each pair of nonconcurrent edges there is a minimum distance d_j between them.

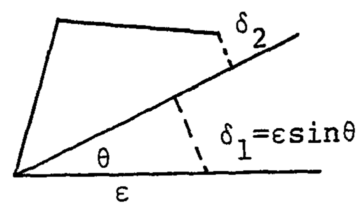


Fig. 3.1

Let $\theta = \min_i \{\theta_i\}$, let $\delta_1 = \varepsilon \sin \theta$, and let $\delta_2 = \min_j \{d_j\}$. It is now easy to see that if $\delta = \min \{\delta_1, \delta_2\}$, then any subset of the C_i (being connected) of diameter δ must contract to a point inside a neighborhood of C_i of diameter ε. Therefore $C_i \xrightarrow{d_s} C_o$ implies $C_i \xrightarrow{d_h} C_o$, and therefore $AR_h(X) \simeq C(X)$.

<u>Lemma 3.7</u>: Let X be a graph, Y a subcontinuum of X, and $\mathcal{Z}_Y$ the component of 2^X_h containing Y. Then $\mathcal{Z}_Y$ is naturally embedded in C(X).

<u>Pf</u>: Let $Y = D \cup E$, where D is a union of cycles and E is the union of a pairwise disjoint collection of arcs. By Theorem 2.1, any other point $Y' \varepsilon \mathcal{Z}_Y$ is of the form $Y' = D \cup E'$, where E' is a pairwise disjoint collection of arcs. Now let $F: \mathcal{Z}_Y \longrightarrow C(X)$ be given by $F(P)=P$; since the topology of $\mathcal{Z}_Y$ is finer than that of C(X), F is obviously continuous. It suffices, therefore, to show that $F^{-1}: F(\mathcal{Z}_Y) \longrightarrow \mathcal{Z}_Y$ is continuous. For this purpose let $C_i \xrightarrow{d_s} C_o$. We wish to show that $C_i \xrightarrow{d_h} C_o$. By the remarks above, for large i each $C_i = D \cup E_i$, where E_i is a pairwise disjoint collection of arcs. Furthermore we can assume that X is embedded in some Euclidean space E^n, so that the edges of the graph X are all straight lines. We consider, as before, pairs of edges of X, but this time we omit those edges lying in D. If the edges are disjoint, we let δ_j be the minimum distance between any two points on the two edges. If the edges intersect, we let $\theta = \min\{\theta_i\}$, where θ_i is the angle between the two edges. Now if $\varepsilon > 0$ is given, we can choose $\delta = \min\{\delta_1, \delta_2\}$ just as in Lemma 3.6, where $\delta_1 = \min\{\delta_j\}$, $\delta_2 = \varepsilon \sin\theta$. It is easy to see that every δ-subset of each C_i contracts to a point inside an ε-subset of C_i, and hence $C_i \xrightarrow{d_h} C_o$.

Pf. Thm. 3.2: To prove the Theorem, we first observe that if X is not a finite graph then either X contains a point p with ord(p) = ∞, or there is an arc in X containing infinitely many branch points. For if X has no points of infinite order, then it must have infinitely many branch points (otherwise X is a graph). Let $A = \{a_i\}_{i=1}^{\infty}$ be an infinite sequence of distinct branch points of X, chosen so as to converge to the limit point $a_o \notin A$. Now suppose that no arc in X contains infinitely many points of A. We now construct a subsequence $\{a_{i_j}\}$ of $\{a_i\}$. For $j \varepsilon \{1, \ldots, n, \ldots\}$ we let α_j be arcs joining a_o and $a_{i_j} \varepsilon A$ with $\text{diam}(\alpha_j)$ less than $\frac{1}{j}$ (cf. [Wh], p. 38), and such that $\alpha_j \cap \alpha_k = \{a_o\}$ for $j \neq k$. If such arcs α_j cannot be found then infinitely many of the $\{a_i\}$ lie on a single arc emanating from a_o. If, on the other hand, the α_j do exist, then $\text{ord}(a_o) = \infty$. Thus X is as claimed.

In case X has a point of infinite order, we can apply the ∞-Beinsatz (cf. [Me], p. 214), which asserts that X contains a copy of the ANR T of Lemma 3.1. It follows from Lemma 3.2 that 2_h^X contains a copy of Q, and hence is (strongly) infinite dimensional. If, on the other hand, X contains an arc with infinitely many branch points, we can inductively apply the n-Bogensatz (cf. [Me], p. 216) to obtain a copy of the ANR R of Lemma 3.4 imbedded in X. Thus, by Lemma 3.5, 2_h^X again contains a copy of Q and is, therefore, (strongly) infinite dimensional.

For the converse, we argue by induction on the number n of components of a point $C\varepsilon 2_h^X$. If n = 1, then by Lemma 3.7, $\mathcal{Z}_C$, the component of 2_h^X containing C can be embedded in C(X). But Kelley's Theorem says that since X is a graph, C(X) is finite dimensional. Therefore $\mathcal{Z}_C$ is finite dimensional, which proves that the subspace $\mathcal{Z}(1)$ of 2_h^X consisting of subcontinua of X is finite dimensional, i.e., n = 1 holds. Now assume that the subspace $\mathcal{Z}(n)$ consisting of points in 2_h^X with n-components is finite dimensional for $n \leq k-1$, and consider the subspace $\mathcal{Z}(k)$. Let $C\varepsilon\mathcal{Z}(k)$ and let $\{C^i\}_{i=1}^k$ be the components of C. Choose an $\varepsilon > 0$ such that $d(\bigcup_{i=1}^{k-1} C_i, C^k) > 2\varepsilon$, and so small that $N_{d_h}(C,\varepsilon)$, the ε-neighborhood about $C\varepsilon 2_h^X$, is contained in $\mathcal{Z}(k)$. We wish to show that $N_{d_h}(C,\varepsilon)$ has dimension less than or equal to $\dim \mathcal{Z}(k-1) + \dim \mathcal{Z}(1)$. To see this, first note that if $D\varepsilon N_{d_h}(C,\varepsilon)$, then $D = \{D^i\}_{i=1}^k$ and $d_s(C^i,D^i) < \varepsilon$ for $i = 1,2,\ldots,k$. It follows that $D^k \cap \bigcup_{i=1}^{k-1} C^i = \emptyset$. Therefore $N_{d_h}(C,\varepsilon)$ can be represented as a product, namely:

$$N_{d_h}(C,\varepsilon) = N_{d_h}(\bigcup_{i=1}^{k-1} C^i,\varepsilon) \times N_{d_h}(C^k,\varepsilon).$$

Since the topology on $N_{d_h}(C,\varepsilon)$ is equivalent to that given by the Hausdorff metric, it is clear that the above product representation is topological and not merely set theoretic.

It follows that $\dim(N_{d_h}(C,\varepsilon)) \leq \dim(N_{d_h}(\bigcup_{i=1}^{k-1} C^i,\varepsilon)) +$ $\dim(N_{d_h}(C^k,\varepsilon)) \leq \dim \mathcal{Z}(k-1) + \dim \mathcal{Z}(1)$. Since both $\mathcal{Z}(k-1)$ and $\mathcal{Z}(1)$ are finite dimensional by hypothesis, we have shown that every point $C\varepsilon\mathcal{Z}(k)$ lies in a neighborhood of dimension less than or equal to $\dim(\mathcal{Z}(k-1)) + \dim(\mathcal{Z}(1))$. Thus $\mathcal{Z}(k)$ is finite dimensional.

Remark: Although each component of 2^X_h is finite dimensional when X is a graph, we will see shortly that the space 2^X_h itself is countably infinite dimensional, that is 2^X_h contains subspaces of arbitrarily large finite dimension.

CHAPTER IV

TOPOLOGICAL STRUCTURE OF 2_h^X FOR X A FINITE GRAPH

In this chapter we consider various simple examples of graphs and their associated hyperspaces 2_h^X. This material is related to previous work by R. Duda (cf. $[D]_1$, $[D]_2$), who studied the hyperspaces $C(X)$ for X a graph. By Theorem 2.1, it is easy to see that the subspace $AR_h(X)$ of 2_h^X consisting of AR-subsets of X is, in fact, a component of 2_h^X. Furthermore, Lemma 3.6 asserts that if X is acyclic, the component $AR_h(X)$ is homeomorphic to $C(X)$. Finally, if $\{C_h^i(X)\}$ are the components of $C_h(X) = \{C\varepsilon 2_h^X | C \text{ is connected}\}$, then Lemma 3.7 states that there is a natural embedding of each $C_h^i(X)$ in $C(X)$. It follows that for acyclic graphs Duda's results pertaining to $C(X)$ carry over verbatim to the component $AR_h(X)$ of 2_h^X, while, for nonacyclic graphs X, the topology of $C_h(X)$ can be determined by identifying the appropriate subspaces of $C(X)$. The topological structure of the remaining components of 2_h^X is very often deducible from the information carried by $C_h(X)$, but, as we shall see, even the most rudimentary examples quickly become quite complicated. We begin with the two simplest cases:

<u>Theorem 4.1</u>: Let $X = I = [0,1]$; then

$$2^I_h \approx I^2 \vee \bigvee_{n=2}^{\infty} (I^{2n-1} \times [0,1)) \text{ , (here V means topological sum).}$$

If $X = S^1$, then $$2^{S^1}_h \approx \{pt.\} \vee \bigvee_{n=1}^{\infty} (S^1 \times I^{2n-2} \times [0,1)) .$$

<u>Pf</u>: Each element T of $AR(I)$ is determined by its midpoint and by its length ($[D]_1$, p. 267). The map $g: AR_h(I) \longrightarrow \Delta \approx I^2$ which sends $T \longrightarrow g(T) =$ (midpoint T, length T) is a homeomorphism. We now show that if $[T] = [n\text{-points}]$, then the component of 2^I_h containing T is homeomorphic to $I^{2n-1} \times [0,1)$. The argument is by induction on n. Thus suppose the component $\{T \varepsilon 2^X_h | T \varepsilon [(n-1)pts.]\}$ is homeomorphic to $I^{2n-3} \times [0,1)$. We prove the result first for $n = 2$. Thus if $T = T_1 \cup T_2$ where T_1 is the component on the left, T_2 the one on the right, then $g(T_2)$ can be any point of the triangle abc except for those that lie on the segment $\overline{ac}$. The left hand endpoint d of T_2 determines a triangle adf, and $g(T_1)$ can be any point in adf that does not lie on segment $\overline{df}$. This situation is completely general

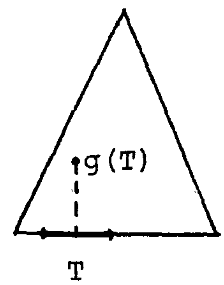


Fig. 4.1

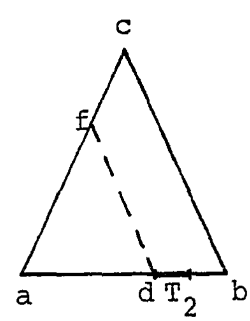


Fig. 4.2

in that each choice of $T_2 \varepsilon \{T | g(T) \not\in \overline{ac}\}$ yields the same topological object for possible choices of T_1, namely $\Delta adf - \overline{df} \simeq I \times [0,1)$. It is clear that a small variation in T_1 leads to only a slight change in the situation regarding choices for T_2, i.e., this procedure is continuous. It follows that $[T] = [T_1 \cup T_2] \simeq (I \times [0,1)) \times (I \times [0,1)) \simeq I^3 \times [0,1)$ as desired. The inductive argument is now clear for $[T] = [\text{n-pts.}]$: choose the right most component T_n; there are topologically $I \times [0,1)$ possibilities. By the inductive assumption, the remaining (n-1) components determine a copy of $I^{2n-3} \times [0,1)$. It follows that the component $[T] = [\text{n-pts.}]$ of 2_h^I is homeomorphic to

$$(I \times [0,1)) \times (I^{2n-3} \times [0,1)) \simeq I^{2n-1} \times [0,1) .$$

Since $\{[\text{n-pts.}]\}_{n \varepsilon N}$ are the components of 2_h^I, which furthermore are clopen sets, it follows that $2_h^I \simeq I^2 \vee \left(\bigvee_{n=2}^{\infty} (I^{2n-1} \times [0,1))\right)$.

The argument for $X = S^1$ is similar. The component $[T] = [S^1]$ is a single point. If $[T] = [\text{1-pt.}]$, then again two parameters, midpoint and length, determine a given T uniquely. Thus, as before, we can define a homeomorphism

$g: AR_h(S^1) \longrightarrow S^1 \times [0,1)$ by $g(T) = (\text{midpoint } T, \frac{\text{length } T}{2\pi})$.

Let $[T] = [\text{n-pts.}]$. If $T = T_1 \cup T_2 \cup \ldots \cup T_n$, then the component T_n can be anything in $AR_h(S^1) \simeq S^1 \times [0,1)$. Once T_n is chosen, the remaining (n-1) components amount to determining $[(\text{n-1) pts.}] \subseteq 2_h^X$ where $X = (0,1)$. By examining the above case where $X = I$, we see that this component is homeomorphic to $[0,1]^{2n-3} \times [0,1)$. It follows that $[T] = [n\text{-points}] \approx S^1 \times [0,1) \times [0,1]^{2n-3} \times [0,1) \approx S^1 \times [0,1]^{2n-2} \times [0,1)$. Therefore $2_h^{S^1} \approx \{pt.\} \vee \bigvee_{n=1}^{\infty} (S^1 \times [0,1]^{2n-2} \times [0,1))$.

Remark: Since every graph (and in fact every nondegenerate Peano continuum) contains an embedded copy of I, it follows that 2_h^X is countably infinite dimensional whenever X is a graph. Thus by Theorem 3.2, if X is a nondegenerate Peano continuum, 2_h^X is countably infinite dimensional if and only if X is a graph, while if X is not a graph, then 2_h^X is strongly infinite dimensional.

Theorem 4.2: Let X be an m-od (i.e., the union of m-arcs $\{B_i\}_{i=1}^{m}$ that meet in a common point p). Then $2_h^X = \bigvee_{n=1}^{\infty} L_n$, where the $L_n = [n\text{-pts.}] \subseteq 2_h^X$ are polyhedra with $\dim L_n = m + 2(n-1)$. Furthermore there is an inductive procedure for constructing polyhedra K_n homeomorphic to the L_n.

Pf: By Theorem 2.1, each $L_n = [n\text{-pts.}]$ is a component of 2_h^X. Furthermore, by Lemma 3.6, $L_1 = AR_h(X) \approx C(X)$. We begin, then, by constructing a polyhedron K_1 with $\dim(K_1) = m$ such that $K_1 \approx C(X)$. To construct K_1, take $X = \bigcup_{i=1}^{m} B_i$ to be a standard m-od where the B_i are all of unit length. The following construction serves as a simple prototype for those that follow. Let $L_o = \{C \varepsilon L_1 | p \varepsilon C\}$. Clearly each $C \varepsilon L_o$ is uniquely determined by the m points $(\partial C \cap B_i)_{i=1}^{m}$, (where ∂ means combinatorial boundary, i.e., endpoints of the B_i are included when $B_i \subseteq C$). Thus we have a map $L_o \longrightarrow [0,1]^m = K_o$ which is clearly a homeomorphism. The remaining continua, those in $L_1 - L_o$ lie entirely on one of the m-branches B_i. We can thus write $L_1 - L_o = \bigcup_{i=1}^{m} L_{1i}$, where $L_{1i} = \{C \varepsilon L_1 | C \subseteq B_i - \{p\}\}$ Since $B_i - \{p\}$ is a half-open interval, each $L_{1i} \simeq [0,1] \times [0,1) = K_{1i}$. It follows that $L_1 = \bigcup_{i=1}^{m} K_{1i} \cup K_o = K_1$ as sets.

We will now determine the topology on K_1 by referring to appropriate subspaces of L_1. First note that no sequence of points in L_o converges to a point in $L_1 - L_o$, and hence K_o must be topologically a closed subspace of K_1. Now let $\{p_i\}$ be a sequence of points in $L_1 - L_o$ such that $p_i \longrightarrow p_o$. If $p_o \varepsilon L_1 - L_o$, then $p_o \varepsilon L_{1j}$ for some $j \varepsilon \{1,\ldots,m\}$. In this case all but finitely many of the p_i lie in L_{1j}, and therefore the topology of $\bigvee_{i=1}^{m} K_{1i} = K_1 - K_o$ must be preserved as an open subspace of K_1. If $p_o \not\varepsilon L_1 - L_o$, however, then p_o must be such that $p_o \subseteq B_j$ for some $j \varepsilon \{1,\ldots,m\}$. Furthermore if $p_o \neq \{p\}$, then all but finitely many of the p_i lie in L_{1j}. Let $\bar{p}_i = (p_i^1, p_i^2)$ in $K_{1j} = [0,1] \times [0,1)$ be the point corresponding to $p_i \varepsilon L_{1j}$. Then $\bar{p}_o = (p_o^1, p_o^2) = (p_o^1, 1)$ in $\bar{K}_{1j} = [0,1]^2$, where $0 \leq p_o^1 \leq 1$. Note that p_o lies entirely on the arc $B_j \subseteq X$, and that the wedge-point $p \varepsilon p_o$. Thus $p_o \varepsilon L_o$ and p_o is determined by one parameter (e.g., its length). It follows that $\bar{p}_o$ can be located in $K_o = [0,1]^m$ via the identification space $\bar{K}_{1j} \underset{\phi_j}{\cup} K_o$, where $\phi_j\colon [0,1] \times \{1\} \longrightarrow [0,1]^m$ is given by $\phi_j(p_o^1,1) = (0,\ldots,\underset{\text{jth entry}}{p_o^1},\ldots,0)$. Finally if $p_o = \{p\} \longleftrightarrow (0,\ldots,0) \varepsilon K_o$, then the p_i can lie anywhere in $L_1 - L_o$ subject to the restriction that $p_i \longrightarrow \{p\}$, i.e., $p_i^1 \longrightarrow 0$ and $p_i^2 \longrightarrow 1$. The topology on K_1 can now be described as follows: Let T be the m-od in $[0,1]^m$ consisting of those points with at least (m-1) coordinates equal to zero. Thus $T = T_1 \cup \ldots \cup T_m$, where $T_i = \{(0,\ldots \underset{\text{ith entry}}{t} \ldots,0)\}$, $0 \leq t \leq 1$.

Then $K_1 \simeq [0,1]^m \underset{\phi}{\cup} (\bigvee_{j=1}^{m} [0,1]_j^2)$, i.e., K_1 is an m-cube with m two-cells attached along an m-od in the boundary. The attaching map is $\phi = \bigvee_{j=1}^{m} \phi_j\colon \bigvee_{j=1}^{m} ([0,1] \times \{1\})_j \longrightarrow [0,1]^m$, where $([0,1] \times \{1\})_j \subseteq \partial(\overline{K_{1j}})$. Thus each 2-cell is attached along a separate branch of the m-od, their only common intersection being at $(0,\ldots,0) \varepsilon K_o$, (cf. Fig. 4.3).

We now indicate how to construct a polyhedron K_2 homeomorphic to $L_2 = [\text{2-pts.}] \subseteq 2_h^X$ from K_1. Let C be an arbitrary point in L_2. If $p \varepsilon C$, then write $C = C_1 \cup C_2$ where the C_i are disjoint subcontinua in X with $p \varepsilon C_2$. If $p \not\varepsilon C$, and

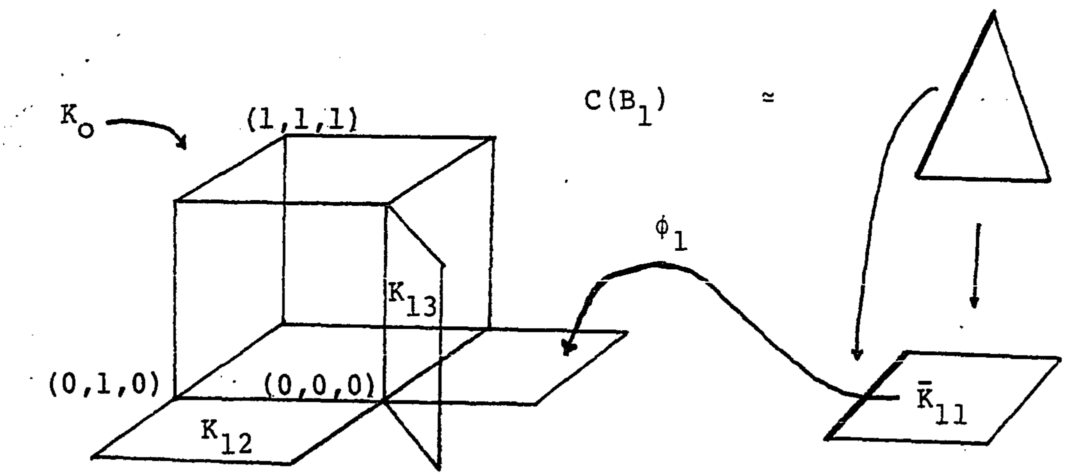


Fig. 4.3

$C \not\subseteq B_i$ for any i, then let $C = C_1 \cup C_2$ where $C_1 \subseteq B_i$, $C_2 \subseteq B_j$ and $i < j$. But if $p \notin C$, and $C \subseteq B_i$, then write $C = C_1 \cup C_2$ where $d(C_2, p) < d(C_1, p)$.

We can now enumerate the points in K_2 as follows. If $C_1 \subseteq B_1$, then $C_1 \varepsilon C([0,1)) \simeq [0,1] \times [0,1)$. For each choice of $C_1 \subseteq B_1$, we have $C_2 \varepsilon C(X-C_1) \simeq C(\{m\text{-od}\} - \{\text{one endpoint}\})$. Clearly this space is homeomorphic to

$$R_{11} = K_1 - \{p = (p^1,\ldots,p^m)\varepsilon[0,1]^m | p^1 = 1\}$$

Thus if $L_{21} = \{C\varepsilon L_2 | C_1 \subseteq B_1\}$, then as a set $L_{21} = K_{21} = [0,1] \times [0,1) \times R_{11}$. Now let $L_{22} = \{C\varepsilon L_2 | C_1 \subseteq B_2\}$ and note that $L_{21} \cap L_{22} = \emptyset$. Again $C_1 \varepsilon C([0,1)) \simeq [0,1] \times [0,1)$. But now $C_2 \subseteq X-C_1$ with the added condition that $C_2 \not\subseteq B_1-\{p\}$. It follows that for each choice of $C_1 \subseteq B_2$, C_2 lies in a space homeomorphic to $R_{12} = K_1 - \{p = (p^1,\ldots,p^m)\varepsilon[0,1]^m | p^2 = 1\} - K_{11}$. Thus we have $K_{22} = [0,1] \times [0,1) \times R_{12}$. Similarly we define

$R_{1i} = K_1 - \{p = (p^1,\ldots,p^m)\varepsilon[0,1]^m | p^i = 1\} - \bigcup_{j=1}^{i-1} K_{1j}$. Then it

is easy to see that the set $L_{2i} = \{C\varepsilon K_2 | C_1 \subseteq B_i\}$ is setwise equivalent to $K_{2i} = [0,1] \times [0,1) \times R_{1i}$, and that $L_2 = \bigcup_{i=1}^{m} L_{2i}$. Now let $K_2 = \bigcup_{i=1}^{m} K_{2i}$. To determine the topology on K_2, first

note that, by construction of the L_{2i}, no sequence of points in L_{2i} can converge to a point in L_{2j} unless i is less than j. It follows that $\bigcup_{i=k}^{m} K_{2i}$ must be closed as subspaces of K_2 for

all k. In particular, the topology of K_{2m} is preserved as a closed subspace of K_2. Let us now consider the topology on $\bar{K}_{21} \subseteq K_2$. First we note that if $\{p_i\}$ is a sequence of points in L_2 such that $p_i \longrightarrow p_o$, where $p_o \varepsilon L_{21}$, then all but finitely many of the p_i lie in L_{21}. Thus the topology of K_{21} is preserved as an open subspace of K_2. If, however, $\{p_i\}$ is a sequence of points with $p_i \varepsilon L_{21}$, and such that $p_i \longrightarrow p_o$, where $p_i \not\varepsilon L_{21}$, then $\bar{p}_o = (q,r,s)\varepsilon[0,1] \times [0,1] \times R_{11}$ must have

$r = 1$ and $s\varepsilon \bigcup_{i=2}^{m} K_{1i} \subseteq R_{11}$. Now if $s = (s_k^1, s_k^2)\varepsilon K_{1k} = [0,1] \times [0,1)$, then $\bar{p}_o$ must be identified with $(s_k^1, s_k^2, (q,0,\ldots,0))\varepsilon[0,1] \times [0,1) \times R_{1k}$. We can therefore form an attaching space

$\bar{K}_{21} \underset{\phi_1}{\cup} \bigcup_{i=2}^{m} K_{2i}$ by defining a map

$\phi_1\colon ([0,1] \times \{1\}_1) \times \bigcup_{i=2}^{m} K_{1i} \longrightarrow \bigcup_{i=2}^{m} K_{2i}$ by

$\phi_1(q,1,s_k^1,s_k^2) = (s_k^1,s_k^2,(q,0,\ldots,0))$. Geometrically ϕ_1 amounts to m-1 attachings of 3-cells, one each for the subspaces K_{2i}, $i = 2,\ldots,m$. From ϕ_1 alone we see that K_2 is connected.

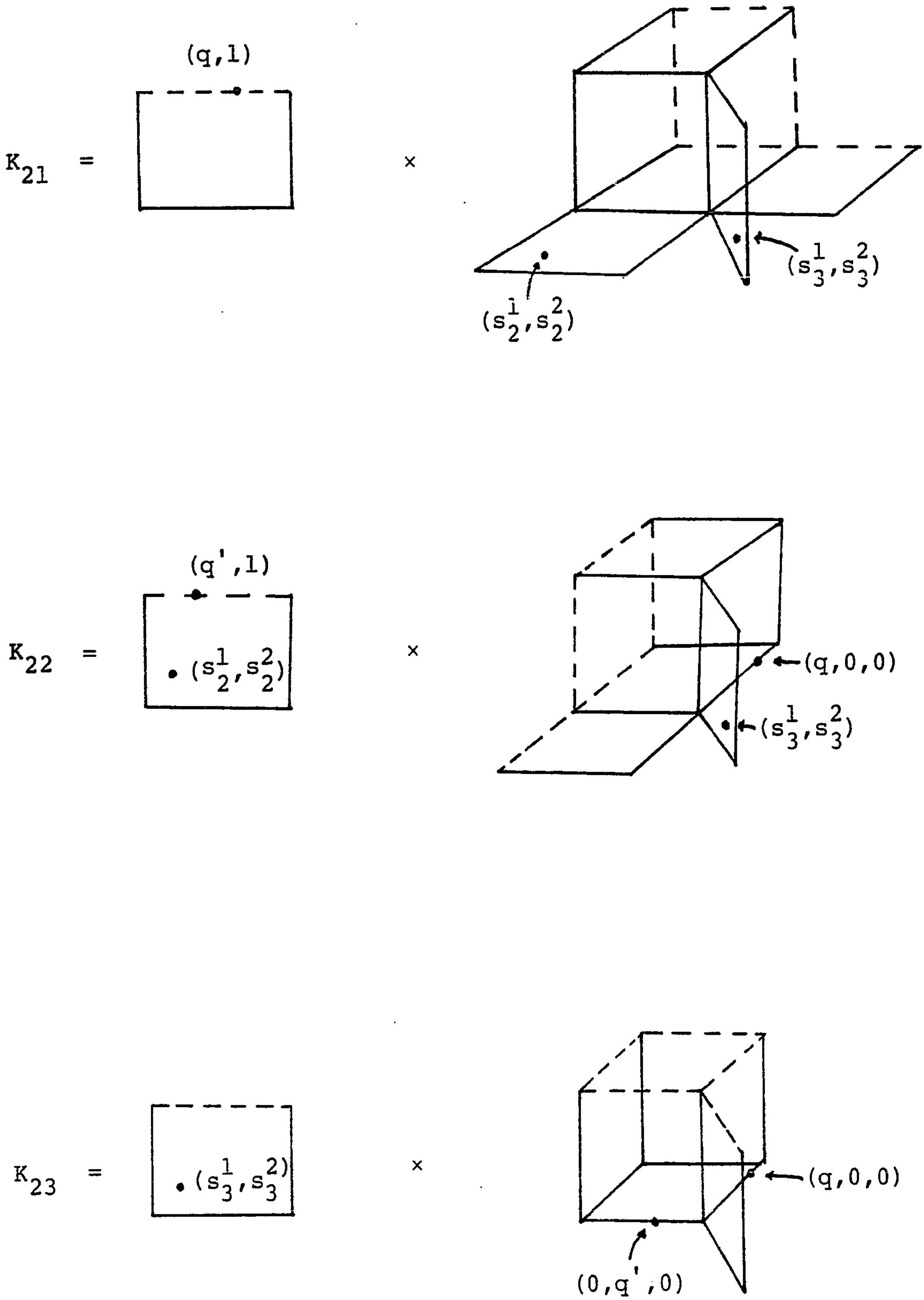
To complete the construction of K_2, we need to define analagous attaching maps ϕ_j, $j = 1,\ldots,m-1$. These are as follows: Let $([0,1] \times \{1\})_j \times \bigcup_{i=j+1}^{m} K_{1i} \subseteq \bar{K}_{2j}$, then

$\phi_j\colon ([0,1] \times \{1\})_j \times \bigcup_{i=j+1}^{m} K_{1i} \longrightarrow \bigcup_{i=j+1}^{m} K_{2i}$ is given by

$\phi_j(q,1,s_k^1,s_k^2) = (s_k^1,s_k^2,(0,\ldots,\underset{\text{jth entry}}{q},\ldots 0))$ where $k \in \{j+1,\ldots,m\}$ is determined by $s = (s_k^1,s_k^2) \in K_{1k}$.

The reader can easily check that the attachings given by the ϕ_j are the only ones possible by referring to the appropriate hyperspace diagram and considering possible sequences of pairs of continua that belong to some L_{2i} but have their limit in some L_{2j}, $i \neq j$. It follows from this that L_2 is homeomorphic to $K_2 = \bigcup_{\substack{i=1\\ \phi}}^{m} K_{2i}$, where $\phi = \bigvee_{j=1}^{m} \phi_j$.

Furthermore, since $\dim K_1 = m$, each R_{1i} is m-dimensional, and therefore each K_{2i} is $(m + 2)$-dimensional. Since the attaching map is defined only along a 3-dimensional subset of $\bigcup_{i=1}^{m} K_{2i}$, it follows that K_2 itself is $(m + 2)$-dimensional.



K_{21} =
(q,1)
×
(s_2^1,s_2^2)
(s_3^1,s_3^2)
K_{22} =
(q',1)
(s_2^1,s_2^2)
×
(q,0,0)
(s_3^1,s_3^2)
K_{23} =
(s_3^1,s_3^2)
×
(q,0,0)
(0,q',0)

Fig. 4.4

Now note that each $K_{2i} = [0,1] \times [0,1) \times R_{1i}$ is a subpolyhedron of K_2. To see that K_2 is a polyhedron, note that $\bar{K}_{2i} \subseteq K_2$ can be obtained from the polyhedron $\bar{K}_{2i}' = \{q\} \times [0,1] \times R_{1i}$ as follows. From $\bar{K}_{2i}'$ remove the cells $\{q\} \times \{1\} \times R_{1i}^*$, where R_{1i}^* is the m-cube together with the ith 2-cell glued to its boundary in R_{1i}, to obtain the polyhedron $\bar{K}_{2i}''$. Then $\bar{K}_{2i} = [0,1] \times \bar{K}_{2i}''$ is a polyhedron. Now note that the ϕ_j simply identify PL-homeomorphic subpolyhedra of the $\bar{K}_{2i}$. Thus K_2 is a polyhedron (cf. [Ro&Sa], p. 26).

To complete the proof we assume that $K_1,\ldots,K_n$ have been constructed following the above scheme and that they satisfy the properties asserted in the theorem. We furthermore assume that for each $j\varepsilon\{1,\ldots,n\}$, $k\varepsilon\{j+1,\ldots,m\}$, $\bigcup_{i=k}^{m} \{K_{ji}\}$ is a subpolyhedron of K_j. We now show that this information is sufficient to construct a polyhedron of dimension $m + 2n$ homeomorphic to $L_{n+1} = [(n+1) - \text{pts.}] \subseteq 2_h^X$. The procedure, in fact, is completely analogous to the previous construction. For if $C\varepsilon L_{n+1}$, then $C = \bigcup_{i=1}^{n+1} C_i$, where the C_i are disjoint continua. Furthermore if j is the smallest number for which there exists a $C_i \subseteq B_j - \{p\}$, then we can let C_1 be the component of C with $C_1 \subseteq B_j - \{p\}$ and $d(C_1, \{p\}) > d(C_i, \{p\})$ for all $C_i \subseteq B_j - \{p\}$. We now use the same scheme as before for identifying the points of L_{n+1}, namely we consider first the case where $C_1 \subseteq B_1$. Here

$C_1 \varepsilon C([0,1)) \simeq [0,1] \times [0,1)$, and for each choice of C_1, the remaining C_i, $i = 2,\ldots,n+1$, lie on an m-od with the endpoint of B_i missing. Clearly this is homeomorphic to an open subspace, call it R_{n1}, of K_n. Thus if $L_{(n+1)1} = \{C \varepsilon L_{n+1} | C_1 \subseteq B_1\}$, then $L_{(n+1)1} \simeq K_{(n+1)1} = [0,1] \times [0,1) \times R_{n1}$. Similarly if $L_{(n+1)i} = \{C \varepsilon L_{n+1} | C_1 \subseteq B_i\}$, then

$L_{(n+1)i} \simeq K_{(n+1)i} = [0,1] \times [0,1) \times R_{ni}$, where $R_{ni} \subseteq K_n$ is the subspace consisting of $\{\bar{C} \varepsilon K_n | \partial B_i \cap C = \emptyset$ and no component of C

lies in $\bigcup_{j=1}^{i-1} B_j - \{p\}\}$, where $\bar{C} \varepsilon K_n$ is the unique point

corresponding to a given $C \varepsilon L_n$. We next construct attaching

maps $\phi_j\colon ([0,1] \times \{1\})_j \times \bigcup_{i=j+1}^{m} K'_{ni} \longrightarrow \bigcup_{i=j+1}^{m} K_{(n+1)i}$, where

$K'_{ni} \subseteq K_{ni}$ is $\{\bar{C} \varepsilon K_{ni} | p \notin C\}$. Let $s = (s_1,\ldots,s_{2n}) \varepsilon K'_{ni}$, then

$\phi_j(q,1,s) = (s_1,s_2,(s_3,\ldots,s_{2n},(0,\ldots,\underset{\text{jth entry}}{q},\ldots,0))) \varepsilon K_{(n+1)i}$.

Just as in the construction of K_2 we now have

$$L_{n+1} \simeq K_{n+1} = \underset{\phi}{\bigcup_{i=1}^{m}} K_{(n+1)i}, \text{ where } \phi = \bigvee_{j=1}^{m} \phi_j.$$

To see that K_{n+1} is a polyhedron, we must first of all show that the R_{ni} are polyhedra. But by construction $R_{ni} = R'_{ni} \cap R''_{ni}$ where $R'_{ni} = \{\bar{C} \varepsilon K_n | \bar{C} \varepsilon \bigcup_{j=i}^{m} K_{nj}\}$ and

$R''_{ni} = \{\bar{C} \varepsilon K_n | \partial B_i \cap C = \emptyset\}$. Since R'_{ni} is a polyhedron by the induction hypothesis, and $R'_{ni} \cap R''_{ni}$ is an open subset of R'_{ni}, it follows that R_{ni} is a polyhedron ([Ro&Sa]), p. 4), and thus so are the $K_{(n+1)i} = [0,1] \times [0,1) \times R_{ni}$. We again note that, as before, $\bar{K}_{(n+1)i} \subseteq K_{n+1}$ can be obtained from the polyhedron $\bar{K}'_{(n+1)i} = \{q\} \times [0,1] \times R_{ni}$ by removing $\{q\} \times \{1\} \times R^*_{ni}$, where $R^*_{ni} = \{\bar{C}\varepsilon R_{ni} | C \not\subseteq \bigcup_{j=i+1}^{m} B_j - \{p\}\}$ to obtain $\bar{K}''_{(n+1)i}$. Note that $\bar{K}''_{(n+1)i}$ is a polyhedron since $R_{ni} - R^*_{ni} = \{\bar{C}\varepsilon R_{ni} | C \subseteq \bigcup_{j=i+1}^{m} B_j - \{p\}\}$ which is clearly a subpolyhedron of R_{ni}. Finally $\bar{K}_{(n+i)i} = [0,1] \times \bar{K}''_{(n+1)i}$, and thus a polyhedron, and since the ϕ_j simply identify homeomorphic subpolyhedra that lie in the boundaries of the $\bar{K}_{(n+1)i}$, it follows that K_{n+1} is a polyhedron ([Ro&Sa], p. 26). Clearly the $\dim(K_{n+1}) = 2 + \dim(K_n) = m + 2n$, which completes the proof.

CHAPTER V

COMPONENT STRUCTURE OF 2_h^M FOR M A CLOSED TWO-MANIFOLD

Borsuk's result that $\{[X]\}_{X\varepsilon 2_h^M}$ partition 2_h^M into clopen (closed and open) sets raises the question as to whether these sets themselves are the components of 2_h^M, or whether there is further fragmentation within each homotopy type. As a first observation we have the following:

<u>Lemma 5.1</u>: The components of 2_h^M (M a closed 2-manifold) are arcwise connected.

<u>Pf</u>: Since 2_h^M is an ANR($\mathfrak{M}$) ($[Bx]_1$, Thm. 4.4), hence locally connected, and a complete metric space ($[B]_2$, Corr. 4, p. 198), the result follows from the fact that a connected, locally connected, complete metric space is arcwise connected. ([H&Y], p. 118).

Thus to determine the component structure of 2_h^M, it suffices to determine necessary and sufficient conditions for the existence of an arc joining two points in 2_h^M. For $M = S^2$, Ball and Ford proved the following result ($[B\&F]_1$, Thm. 4.4):

> Every two homotopically equivalent connected ANRs in S^2 can be joined by an arc in $2_h^{S^2}$.

This result was generalized by Boxer, who was able to drop the condition that the ANR's be connected and enlarge the class of spaces {M} to be {closed 2-manifolds}. He then proves the following ([Bx]$_1$, Thm. 3.1):

> Let $C_i \varepsilon 2_h^M - \{M\}$, $(i = 1,2)$ and let $N_i \varepsilon 2_h^M$ be such that each component of N_i is a bounded surface, $C_i \subseteq \text{int}(N_i)$, and C_i is a strong deformation retract of N_i (the existence of the N_i is proven in [Bx]$_1$, Thm. 2.5). Then there is an arc in 2_h^M from C_1 to C_2 if and only if there is an ambient isotopy of M taking N_1 onto N_2.

The following is a refinement of Boxer's result for the case where the C_i are connected:

<u>Thm. 5.1</u>: Let M be a closed 2-manifold. Then C_1, C_2 (compact, connected ANR's) can be joined by an arc in 2_h^M if and only if $[C_1] = [C_2]$ (i.e., C_1 & C_2 have the same homotopy type) and $\text{im}(\pi_1(C_1))$ and $\text{im}(\pi_1(C_2))$ are conjugate subgroups of $\pi_1(M)$.

<u>Pf</u>: By Boxer's Theorem above, there exist polyhedral bounded surfaces $N_1 \supseteq C_1$, $N_2 \supseteq C_2$ such that C_1 and C_2 are strong deformation retracts of N_1, N_2 respectively. Moreover C_1 and C_2 can be joined by an arc in 2_h^M if and only if there exists an ambient isotopy of M sending N_1 onto N_2. Clearly the existence of such an isotopy implies that $[C_1] = [C_2]$ and

$im(\pi_1(C_1))$ is conjugate to $im(\pi_1(C_2))$ in $\pi_1(M)$. It therefore suffices to show the reverse implication. To do this, we make use of the following result of Jaco and Shalen ([J&S], Lemma 4.2):

> Let M be a surface and $\hat{N}_1$, $\hat{N}_2$ be compact, incompressible submanifolds of M such that no component of either $\hat{N}_1$ or $\hat{N}_2$ has nonnegative Euler characteristic. If $\hat{N}_1$ can be homotoped into $\hat{N}_2$ and $\hat{N}_2$ can be homotoped into $\hat{N}_1$, then there exists an ambient isotopy of M moving $\hat{N}_1$ onto $\hat{N}_2$.

If we examine the proof of this result given in [J&S], it will readily be seen that if $\hat{N}_1$ and $\hat{N}_2$ happen to be connected, then the condition on the Euler characteristic of their components can be dropped. Since C_1 and C_2 are connected, N_1 and N_2 must also be connected, but they need not be incompressible. In order to apply the above lemma, it is necessary to alter N_1 and N_2 slightly in order to obtain incompressible surfaces.

We first observe that ∂N_1 and ∂N_2 consist of pairwise disjoint collections of simple closed curves. For those ∂-components which are null-homotopic in M, we can glue in discs $\{D_1^i\}_{i=1}^p$, $\{D_2^j\}_{j=1}^q$ along these simple closed curves to obtain new surfaces $\hat{N}_1, \hat{N}_2 \subseteq M$ (cf., [Ep], Thm. 1.7).

<u>Lemma 5.2</u>: $\hat{N}_1$, $\hat{N}_2$ are incompressible surfaces in M, i.e., $\pi_1(\hat{N}_i) \longrightarrow \pi_1(M)$ is injective for i = 1,2.

Pf: Let $L: S^1 \longrightarrow \hat{N}_i$ be a loop that contracts to a point in M. Thus there exists a map

$$H: D \longrightarrow M, \quad D = S^1 \times I / S^1 \times \{0\} \quad \text{with } H|\partial D = L.$$

If necessary we can modify H slightly so as to be smooth, keeping $[H|\partial D] = [L] \in \pi_1(\hat{N}_i)$. Furthermore there exists a map $G: D \longrightarrow M$ such that G is smoothly homotopic to H and G intersects $\partial\hat{N}_i$ transversely ($G \pitchfork \partial\hat{N}_i$). Let $F: D \times I \longrightarrow M$ with $F_o = H$, $F_1 = G$ be such a homotopy. By choosing t small enough, F_t will be near $F_o = H$, whereas it follows from Sard's Theorem that $F_t \pitchfork \partial\hat{N}_i$ for almost all $t \in [0,1]$. This means that, without loss of generality, the original map $H: D \longrightarrow M$ can be chosen so that $[H|\partial D] = [L] \in \pi_1(\hat{N}_i)$ and $H \pitchfork \partial\hat{N}_i$. We can furthermore assume that $H(\partial D) \subseteq \text{int}(\hat{N}_i)$.

Since H has been taken transverse to $\partial\hat{N}_i$, $\text{codim}(\partial\hat{N}_i) = \text{codim}(H^{-1}(\partial\hat{N}_i))$, hence $\dim(H^{-1}(\partial\hat{N}_i)) = 1$. Thus $H^{-1}(\partial\hat{N}_i)$ is a submanifold of int D, hence a disjoint collection of simple closed curves. Let C be an innermost simple closed curve amongst those of $H^{-1}(\partial\hat{N}_i)$. The curve C bounds a disc D_C in D, and either $H(D_C) \subseteq \hat{N}_i$ or $H(\text{int } D_C) \cap \hat{N}_i = \emptyset$. In the latter case we wish to modify H so that the new H is homotopic to the original but with $H(D_C) \subseteq \hat{N}_i$.

Now H(C) is a connected subset of $\partial\hat{N}_i$ and, hence, is contained in a ∂-component T (homeomorphic to S^1) of $\partial\hat{N}_i$. We will show that $H|C: C \longrightarrow \complement\hat{N}_i = M\text{-int } \hat{N}_i$ is null-homotopic. Furthermore $\complement\hat{N}_i$ is a compact 2-manifold with boundary, so that the component Q of $\complement\hat{N}_i$ containing $H(D_C)$

is a connected surface with boundary contained in M. Thus $\pi_1(Q)$ is a free group ([Ma], p. 135). Since T is a boundary component of $\hat{N}_i$, T is essential in M, and hence also in Q. It follows that $\pi_1(T) \longrightarrow \pi_1(Q)$ is injective, and, since $[H|C]=0 \in \pi_1(Q)$ it follows from the composition $\pi_1(C) \longrightarrow \pi_1(T) \longrightarrow \pi_1(Q)$ that $[H|C]=0 \in \pi_1(T)$. Thus we can modify H on int D_C so that $H(D_C) \subseteq T \subseteq \hat{N}_i$. Repeating this procedure by choosing each time an innermost curve among the $H^{-1}(\partial\hat{N}_i)$ eventually results in a map $H:D \longrightarrow \hat{N}_i \subseteq M$ such that $[H|\partial D] = [L] \in \pi_1(\hat{N}_i)$. Thus $\pi_1(\hat{N}_i) \longrightarrow \pi_1(M)$ is injective, i.e., $\hat{N}_i$ is an incompressible surface.

<u>Lemma 5.3</u>: The number of discs $\{D_1^i\}_{i=1}^p$ and $\{D_2^j\}_{j=1}^q$ used in constructing $\hat{N}_1$ and $\hat{N}_2$ is equal.

<u>Pf</u>: Since the submanifolds N_1 and N_2 have the same homotopy type, $\pi_1(N_1) = \pi_1(N_2)$. On the other hand, by hypothesis $\mathrm{im}(\pi_1(C_1))$ is conjugate to $\mathrm{im}(\pi_1(C_2))$, whereas $\mathrm{im}(\pi_1(C_i)) = \mathrm{im}(\pi_1(N_i)) = \mathrm{im}(\pi_1(\hat{N}_i))$. It follows that $\pi_1(\hat{N}_1)$ is conjugate to $\pi_1(\hat{N}_2)$ in $\pi_1(M)$, and hence, $\pi_1(\hat{N}_1) \simeq \pi_1(\hat{N}_2)$. But since gluing in a disc adds one to the Euler characteristic of a manifold, the assumption that $p \neq q$ implies $\chi(\hat{N}_1) \neq \chi(\hat{N}_2)$, and since $H_2(\hat{N}_i) = 0$ (consider the Mayer-Vietoris sequence obtained by doubling $\hat{N}_i$) it follows that $H_1(\hat{N}_1) \neq H_1(\hat{N}_2)$ and hence $\pi_1(\hat{N}_1) \neq \pi_1(\hat{N}_2)$, a contradiction.

Lemma 5.4: If $\hat{N}_1$, $\hat{N}_2$ are connected incompressible surfaces in M with $\pi_1(\hat{N}_1)$ conjugate to $\pi_1(\hat{N}_2)$ in $\pi_1(M)$, then there exist homotopies H^1, H^2 such that:

$$H^1: \hat{N}_1 \times I \longrightarrow M \quad \text{with} \quad H^1_o = \mathrm{id}_{\hat{N}_1} \quad \text{and} \quad H^1_1(\hat{N}_1) \subseteq \hat{N}_2$$

$$H^2: \hat{N}_2 \times I \longrightarrow M \qquad\qquad H^2_o = \mathrm{id}_{\hat{N}_2} \qquad\qquad H^2_1(\hat{N}_2) \subseteq \hat{N}_1$$

Pf: By choosing appropriate isotopies of M, we can assume without loss of generality that the base points in $\pi_1(\hat{N}_1,p_1)$, $\pi_1(\hat{N}_2,p_2)$, and $\pi_1(M,p)$ are such that $p_1 = p_2 = p$. We may also assume that $\pi_1(\hat{N}_1) = \pi_1(\hat{N}_2)$ by utilizing a suitable isotopy of M that cancels the conjugacy relation. For if $g\epsilon\pi_1(M)$ is such that $g^{-1}\pi_1(\hat{N}_1)\,g = \pi_1(\hat{N}_2)$, then g can be represented by a (smooth) isotopy of p in M, i.e., a map $g: \{p\} \times I \longrightarrow M$ with $g_o = g_1 = \mathrm{id}_{\{p\}}$. Moreover the map g can be chosen so that $Y = \mathrm{im}(g)$ can be written $Y = \bigcup_{i=1}^{n} Y_i$ where the Y_i are one-manifolds. Let C_i be a compact bicollared neighborhood of Y_i in M_i chosen so that $C_i = C_j$ on $Y_i \cap Y_j \neq \emptyset$; and let $W = \bigcup_{i=1}^{n} C_i$. Now choose an isotopy $G: M \times I \longrightarrow M$ with $G_0 = \mathrm{id}_M$, $G|\{p\} = g$, and $G = \mathrm{id}_M$ off of W (cf. [Mi], pp. 63-64). Now let $x\epsilon\pi_1(\hat{N}_1)$, and note that x can be represented by a loop $x: [0,1] \longrightarrow \hat{N}_1$ with the property that there exist $t_1, t_2 \;\epsilon\; [0,1]$ such that $x([0,t_1])\subseteq W$, $x([t_2,1]) \subseteq W$, and $x((t_1,t_2)) \cap W = \emptyset$. It is easy to see that $x_1 = x|[0,t_1]$ and $x_2 = x|[t_2,1]$ are both homotopic to the constant map c_p: $[0,1] \longrightarrow M$, where $c_p(t) = p$. Moreover by utilizing the bicollarings C_i on the Y_i and the fact that G extends g, we find that $G_1 \circ x_1 \sim g^{-1}$(rel p) and that $G_1 \circ x_2 \sim g$(rel p) (reparameterize $[0,t_1]$ and $[t_2,1]$ and slide $x(t_1)$ and $x(t_2)$ back to the basepoint p). We also note that since $x_3 = x|[t_1,t_2]$ lies outside int(W), $G_1 \circ x_3 = x_3 \sim x$(rel p). It follows that $[G_1 x] = [g^{-1}xg] \in \pi_1(G_1(\hat{N}_1))$, and hence $\pi_1(G_1(\hat{N}_1)) = \pi_1(\hat{N}_2)$. We may therefore assume in what follows that $\pi_1(\hat{N}_1) = \pi_1(\hat{N}_2)$.

Let $\tilde{G}$ be the covering space corresponding to the group $\pi_1(\hat{N}_1,p)$ $(\simeq \pi_1(\hat{N}_2,p))$. There exist liftings $\tilde{i}$, $\tilde{j}$ such that the following diagram commutes:

$$\begin{tikzcd}
& \tilde{G} \arrow[d, "p"] & \\
\hat{N}_1 \arrow[ur, "\tilde{i}"] \arrow[r, "i"] & M & \hat{N}_2 \arrow[ul, "\tilde{j}"'] \arrow[l, "j"']
\end{tikzcd}$$

This induces the following diagram on fundamental groups:

$$\begin{tikzcd}
& \pi_1(\tilde{G}) \arrow[d, "p_\#"] & \\
\pi_1(\hat{N}_1) \arrow[ur, "\tilde{i}_\#"] \arrow[r, "i_\#"'] & \pi_1(M) & \pi_1(\hat{N}_2) \arrow[ul, "\tilde{j}_\#"'] \arrow[l, "j_\#"]
\end{tikzcd}$$

Since $\hat{N}_1$, $\hat{N}_2$ are incompressible, $i_\#$ and $j_\#$ are injective as is $p_\#$, since p is a covering projection. Thus $\tilde{i}_\#$ and $\tilde{j}_\#$ are injective. But they are also surjective, since $p_\#$ is injective and $p_\#(\pi_1(\tilde{G})) = i_\#(\pi_1(\hat{N}_1)) = j_\#(\pi_1(\hat{N}_2))$.

We now use the fact that if $\pi_1(X) \approx \pi_1(Y)$ and Y is a $K(\pi,1)$ (i.e., $\pi_n(Y) = 0$ for $n \neq 1$), then there is a natural one-to-one correspondence between homotopy classes of maps from X to Y and homomorphisms between $\pi_1(X)$ and $\pi_1(Y)$. Thus, since $\hat{N}_2$ is a $K(\pi,1)$*, there is a map $f\colon \hat{N}_1 \longrightarrow \hat{N}_2$ corresponding to (i.e., which induces) the isomorphism $(\tilde{j}_\#)^{-1} \circ \tilde{i}_\#\colon \pi_1(\hat{N}_1) \longrightarrow \pi_1(\hat{N}_2)$. And since $f_\# = (\tilde{j}_\#)^{-1} \circ \tilde{i}_\#$, $\tilde{j}_\# \circ f_\# = \tilde{j}_\# \circ (\tilde{j}_\#)^{-1} \circ \tilde{i}_\# = \tilde{i}_\#$, whereas $\tilde{G}$ is also a $K(\pi,1)$, it follows that $\tilde{j} \circ f \sim \tilde{i}$, hence $p \circ \tilde{j} \circ f \sim p \circ \tilde{i}$, so that $j \circ f \sim i$. Thus $f\colon \hat{N}_1 \longrightarrow \hat{N}_2$ is homotopic to the inclusion $i\colon \hat{N}_1 \longrightarrow M$. It follows that $H^1\colon \hat{N}_1 \times I \longrightarrow M$ with $H^1_o = \mathrm{id}_{\hat{N}_1}$ and $H^1_1(\hat{N}_1) \subseteq \hat{N}_2$ exists, and applying the same argument in reverse shows the existence of H^2 as well.

<u>Pf. Thm. 5.1</u>: Lemma 5.4 together with the Jaco-Shalen Lemma cited earlier now suffice to prove the Theorem. Since C_1 and C_2 are connected so are $\hat{N}_1$ and $\hat{N}_2$, and we can ignore the condition in the Jaco-Shalen result on the Euler characteristics of the components of $\hat{N}_1$ and $\hat{N}_2$. By Lemma 5.4, the incompressible surfaces $\hat{N}_1$ and $\hat{N}_2$ constructed above can be homotoped one into the other so long as $\mathrm{im}(\pi_1(\hat{N}_1))$ $(=\mathrm{im}(\pi_1(C_1)))$

---

*cf. Addendum, p. 63.

and $\mathrm{im}(\pi_1(\hat{N}_2))$ $(=\mathrm{im}(\pi_1(C_2)))$ are conjugate subgroups of $\pi_1(M)$. By the Jaco-Shalen Lemma, it follows that there is an ambient isotopy of M taking $\hat{N}_1$ onto $\hat{N}_2$. Since, by Lemma 5.3, the number of discs $\{D_1^i\}_{i=1}^p$ and $\{D_2^j\}_{j=1}^q$ used in constructing $\hat{N}_1$ and $\hat{N}_2$ was equal, it follows that we may choose an isotopy H of M which sends the collection $\{D_1^i\}$ onto $\{D_2^j\}$ while sending $\hat{N}_1$ onto $\hat{N}_2$. It follows that the isotopy H has the property that $H_1(N_1) = N_2$ as required in Theorem 3.1 of $[Bx]_1$. This completes the proof.

Cor. 5.1: If M is a closed two-manifold and C a connected ANR contained in M, the component structure of $\{[C]\} \subseteq 2_h^M$ is determined by $\pi_1(M)$, i.e., two connected ANR's C_1, C_2 that are homotopically equivalent $([C_1] = [C_2] = [C])$ lie in the same component of 2_h^M if and only if $\mathrm{im}(\pi_1(C_1))$ and $\mathrm{im}(\pi_1(C_2))$ determine the same conjugacy class in $\pi_1(M)$.

Pf: Lemma 5.1 and Theorem 5.1 above.

We close with some observations that concern an interesting open question pertaining to 2_h^M (M a closed two-manifold). First we note that by general results of Borsuk ($[B]_2$, pp. 197-98) the space 2_h^M is separable and complete. Furthermore, by $[B\&F]_1$, p. 17, 2_h^M is not locally compact at every non-isolated point. Finally by $[Bx]_1$, pp. 36-37, 2_h^M is an ANR. Since by Thm. 3.1, 2_h^M is SID, it follows from

Toruńczyk's characterization of ℓ_2-manifolds (cf. [To]$_3$) that 2_h^M is an ℓ_2-manifold at every non-isolated point if and only if 2_h^M satisfies the countable discrete cells property (CDCP). A space X has the CDCP if given any open cover $\mathcal{U}$ of X and mapping $f: \mathbb{Z} \times Q \longrightarrow X$ there is a map $g: \mathbb{Z} \times Q \longrightarrow X$, $\mathcal{U}$-close to f, and such that $\{g(\{n\}\times Q) \mid n\in\mathbb{Z}\}$ is a discrete collection of sets.

In view of the fact that $AR_h(M)$ is a component of 2_h^M and that the natural embedding $M \longrightarrow AR_h(M)$ is a homotopy equivalence, it would be especially interesting to know whether or not $AR_h(M)$ satisfies the CDCP. If it does, then the fact that homotopy equivalent ℓ_2-manifolds are homeomorphic would imply that $AR_h(M) \approx M \times \ell_2$.

---

*Addendum: If $\hat{G}$ were not a $K(\pi,1)$, then $M = S^2$ or RP^2, in which case $\pi_1(M) = 0$ or $Z/2$. Since $\pi_1(\hat{N}_i)$ is a free group and $\pi_1(\hat{N}_i) \longrightarrow \pi_1(M)$ is injective, it follows that $\hat{N}_i$ is a disc. We conclude immediately that there is an isotopy of M taking $\hat{N}_1$ onto $\hat{N}_2$ such that N_1 maps onto N_2.

BIBLIOGRAPHY

[B&F]$_1$ B. J. Ball and Jo Ford, "Spaces of ANR's," Fund. Math. 77(1972), 33-49.

[B&F]$_2$ ________, "Spaces of ANR's, II" Fund. Math. 78(1973), 209-216.

[Bi] R. H. Bing, "Partitioning a Set," BAMS 55(1949), 1101-1110.

[B]$_1$ K. Borsuk, Theory of Retracts, Monografie Matematyczne 44, Warsaw, 1967.

[B]$_2$ ________, "On some metrizations of the hyperspace of compact sets," Fund. Math. 41(1954), 168-202.

[Bx]$_1$ L. Boxer, "The space of ANR's of a closed surface," Pac. J. Math. 79(1978), 47-68.

[Bx]$_2$ ________, "Retraction spaces and the homotopy metric," preprint.

[Bx]$_3$ ________, "Global properties of spaces of AR's," Rocky Mtn. J. Math., to appear.

[Č] Z. Čerin, "Strongly e-movable convergence and spaces of ANR's," Topology, to appear.

[Ch] T. Chapman, Lectures on Hilbert cube manifolds, CBMS Regional Conference Series in Mathematics, #28, Providence, R.I., 1076.

[C&S]$_1$ D. Curtis and R. Schori, "Hyperspaces of polyhedra are Hilbert cubes," Fund. Math. 99(1978), 189-97.

[C&S]$_2$ ________, "2^X and C(X) are homeomorphic to the Hilbert cube," BAMS 80(1974), 927-931.

[C&S]$_3$ ________, "Hyperspaces of Peano continua are Hilbert cubes," Fund. Math. 101(1978), 19-38.

[D]$_1$ R. Duda, "On the hyperspace of subcontinua of a finite graph, I," Fund. Math. 62(1968), 265-286.

[D]$_2$ ________, "On the hyperspace of subcontinua of a finite graph, II," Fund. Math. 63(1968), 225-255.

[Du] J. Dugundji, Topology, Allyn and Bacon, Boston, 1966.

[Ep] D. B. A. Epstein, "Curves on 2-manifolds and isotopies," Acta Math. 115(1966), 83-107.

$[G]_1$ N. Gray, "Unstable points in the hyperspace of connected subsets," Pac. J. Math. 23(1967), 515-520.

$[G]_2$ ________," On the conjecture $2^X \simeq I^\omega$," Fund. Math. 66(1969), 45-52.

[H&Y] J. G. Hocking and G. S. Young, Topology, Addison-Wesley, Reading, Mass., 1961.

[Hu] S. T. Hu, Theory of Retracts, Wayne State Univ. Press, Detroit, 1965.

[H&W] W. Hurewicz and H. Wallman, Dimension Theory, Princeton Univ. Press, Princeton, N.J., 1941.

[J&S] W. Jaco and P. Shalen, "Peripheral structure of 3-manifolds," Invent. Math. 38(1976) no. 1, 55-87.

[Ke] J. L. Kelley, "Hyperspaces of a continuum," TAMS 52(1942), 22-36.

[K] K. Kuratowski, Topologie II, Monogafie Matematyczne, Warsaw, 1950.

[Ma] W. Massey, Algebraic Topology: An Introduction, Harcourt, Brace, and World, New York, 1967.

[Me] K. Menger, Kurventheorie, Chelsea, New York, 1967.

[Mo] E. E. Moise, "Grille decomposition and convexification theorems for locally compact connected continua," BAMS 55(1949), 1111-1121.

[N] S. Nadler, Hyperspaces of Sets, Marcel Dekker, New York and Basel, 1978.

[R&S] J. H. Roberts and N. E. Steenrod, "Monotone transformations of two-dimensional manifolds," Ann. of Math. 39, no. 4 (1938), 851-862.

[Ro&Sa] C. P. Rourke and B. J. Sanderson, Introduction to Piecewise-Linear Topology, Springer, New York/Heidelberg/Berlin, 1972.

[R-S-W] L. Rubin, R. Schori, and J. Walsh, "New dimension theory techniques for constructing infinite-dimensional examples," General Topology and Appl. 10 (1979), 93-102.

$[S\&W]_1$ R. Schori and J. West, "2^I is homeomorphic to the Hilbert cube," BAMS 78(1972), 402-406.

$[S\&W]_2$ ________, "The hyperspace of the closed unit interval is a Hilbert cube," TAMS, 213(1965), 217-235.

$[S\&W]_3$ ________, "Hyperspaces of graphs are Hilbert cubes," Pac. J. Math. 53(1974), 239-51.

$[To]_1$ H. Toruńczyk, "On CE-images of the Hilbert cube and characterization of Q-manifolds," to appear.

$[To]_2$ ________, "Concerning locally homotopy negligible sets and characterization of ℓ_2-manifolds, preprint of the Mathematical Centre at Amsterdam, 1974.

$[To]_3$ ________, "Characterizing Hilbert space topology," Preprint 143, Inst. of Math., Pol. Ac. of Sciences, 1978.

[Wa] N. R. Wagner, "A continuity property with applications to the topology of 2-manifolds," TAMS 200(1974), 369-393.

$[W]_1$ J. West, "Infinite Products which are Hilbert cubes," TAMS 150(1970), 1-25.

$[W]_2$ ________, "The subcontinua of a dendron from a Hilbert cube factor," PAMS 36(1972), 603-608.

$[W]_3$ ________, "Induced involutions on Hilbert cube hyperspaces," Proceedings of the 1976 Topology Conference, Auburn University, 1(1976), 281-293.

[Wh] G. T. Whyburn, Analytic Topology, AMS Colloquium Publications, 28, Providence, R.I.

[Wi] R. L. Wilder, Topology of Manifolds, AMS Colloquium Publications, 32, Providence, R.I.

[Mi] J. Milnor, Lectures on the h-Cobordism Theorem, Princeton Univ. Press, Princeton, N.J., 1965.