STATISTICS OF THE NUMBER OF REAL ZEROS OF RANDOM ORTHOGONAL POLYNOMIALS

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Abstract: We study the expected number of real zeros for random linear combinations of orthogonal polynomials

$$P_n(x) = \sum_{j=0}^{n} c_j p_j(x),$$

where $\{p_j(x)\}_{j=0}^{\infty}$ is a set of orthonormal polynomials with respect to some measure, supported on the real line, and $\{c_j\}_{j=0}^{\infty}$ is a set of i.i.d. (independently identically distributed) random variables. It is well known that Kac polynomials

$$P_n(x) = \sum_{j=0}^{n} c_j x^j,$$

where $\{c_j\}_{j=0}^{\infty}$ is a set of i.i.d. Gaussian coefficients, have only $(2/\pi + o(1)) \log n$ expected real zeros in terms of the degree n. If the basis $\{p_j(x)\}_{j=0}^{\infty}$ is given by the orthonormal polynomials associated with a compactly supported Borel measure μ on the real line or associated with a Freud weight defined on the whole real line, then random linear combinations have $n/\sqrt{3}+o(n)$ expected real zeros. We also prove that the same asymptotic relation holds for all random orthogonal polynomials on the real line associated with a large class of exponential weights. It reveals the universality of the expected number of real zeros for random orthogonal polynomials. On the other hand, we give local results on the expected number of real zeros in all considered cases and show that the normalized counting measures of (properly scaled) real zeros of these random polynomials converge weakly.

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LIST OF SYMBOLS

$\mathbb{N}$	Set of natural numbers: $1, 2, 3, \ldots$
$\mathbb{R}$	Set of real numbers
$\mathbb{C}$	Set of complex numbers
$\mathbb{R}^n$	n-dimensional real vector space
$z_n \cong w_n$	Two sequences z_n and w_n of complex numbers have the property that $w_n \neq 0$ and $z_n/w_n \to 1$ as $n \to \infty$
$C, C_1, C_2, \ldots$	Positive constants independent of n, x, t and polynomials P of degree at most n unless otherwise stated
$f(x) \sim g(x)$	There are positive constants C_1, C_2 such that for the relevant range of x, $C_1 \leq f(x)/g(x) \leq C_2$. Similar notation is used for sequences and sequences of functions
$z_n = \mathrm{O}\big(a_n\big)$	If $a_n > 0$ and z_n/a_n is bounded as $n \to \infty$
$z_n = o(a_n)$	If $a_n > 0$ and z_n/a_n tends to 0 as $n \to \infty$
$a_n \ll b_n$	$a_n = \mathrm{O}\big(b_n\big)$
$\Im(z)$	The imaginary part of a complex number z
$\Re(z)$	The real part of a complex number z

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CHAPTER 1

Development of Random Polynomials

1.1 Zeros of random polynomials

Let $\{c_j\}_{j=0}^{\infty}$ be a sequence of independent and identically distributed (i.i.d.) random variables. The expected number of real zeros $\mathbb{E}[N_n(\mathbb{R})]$ for random algebraic polynomials

$$P_n(z) = \sum_{j=0}^{n} c_j z^j \tag{1.1.1}$$

was studied since the 1930's. In 1932, Bloch and Pólya [10] established the upper bound

$$\mathbb{E}[N_n(\mathbb{R})] = O(\sqrt{n}) \quad \text{as } n \to \infty,$$

for polynomials with coefficients selected from the set $\{-1, 0, 1\}$ with equal probabilities. Their method can be extended to other discrete distributions such as Bernoulli (uniform on the set $\{-1, 1\}$); see [22]. This upper bound is not sharp, and it was a surprise that random algebraic polynomials actually have a remarkably small number of real zeroes. In a series of fundamental papers [55, 56, 57, 58], published between 1938 and 1948, Littlewood and Offord proved the strong bounds

$$\frac{\log n}{\log\log\log n} \ll N_n(\mathbb{R}) \ll \log^2 n$$

with probability $1 - o(1)$ as $n \to \infty$, for coefficients from many basic distributions (such as Bernoulli, real Gaussian, and uniform on $[-1, 1]$).

During that time, in 1943, Kac [44] established the important asymptotic result

$$\mathbb{E}[N_n(\mathbb{R})] = (2/\pi)\log n + o(\log n), \tag{1.1.2}$$

for random polynomials with standard real Gaussian coefficients. In fact, Kac [44]-[45] found the exact formula for $\mathbb{E}[N_n(\mathbb{R})]$ in this case:

$$\mathbb{E}[N_n(\mathbb{R})] = \frac{4}{\pi}\int_0^1 \frac{\sqrt{A(x)C(x)-B^2(x)}}{A(x)}\,dx,$$

where

$$A(x)=\sum_{j=0}^{n} x^{2j},\quad B(x)=\sum_{j=1}^{n} jx^{2j-1}\quad\text{and}\quad C(x)=\sum_{j=1}^{n} j^2x^{2j-2}.$$

It took much effort to extend (1.1.2) to other distributions. In a subsequent paper [46], Kac managed to extend (1.1.2) to the uniform distribution on $[-1,1]$. Stevens [81] extended it further to cover a large class of continuous and smooth distributions with certain regularity properties (see [81, page 457] for details).

Jamrom [41, 42] improved the Kac asymptotic result and showed that

$$\lim_{n\to\infty}\{\mathbb{E}[N_n(\mathbb{R})]-(2/\pi)\log(n+1)\} = A_0, \tag{1.1.3}$$

where the constant A_0 is given by

$$A_0=\frac{2}{\pi}\left(\log 2+\int_0^1\sqrt{1-t^2\operatorname{csch}^2 t}\,\frac{dt}{t}-\int_1^\infty\left(1-\sqrt{1-t^2\operatorname{csch}^2 t}\right)\frac{dt}{t}\right).$$

Wang [93] also derived (1.1.3); his value of A_0 is

$$\frac{8}{\pi}\int_0^1(1-y^2)^{-1}\left(1+\sqrt{\frac{1-y^2-2y\log y}{1-y^2+2y\log y}}\right)^{-1}dy.$$

(The substitution $y=e^{-t}$ and some simple manipulations show that the two values of A_0 are the same.) Wilkins [94] gave an asymptotic series expansion for $\mathbb{E}[N_n(\mathbb{R})]$ in the case of standard real Gaussian coefficients of the form

$$\mathbb{E}[N_n(\mathbb{R})] = \frac{2}{\pi}\log(n+1)+\sum_{r=0}^{\infty}A_r(n+1)^{-r},$$

where $A_1=A_3=A_5=0$; A_0 (as defined above), A_2, A_4 are explicitly defined constants by integrals whose numerical values are approximately 0.625735815, −0.24261274, and -0.08794067, respectively. More refined forms of the Kac asymptotic were developed by Hammersley [33], Edelman and Kostlan [20], and others.

In 1956, Erdős and Offord [21] found a new approach to handle discrete distributions. Considering Bernoulli coefficients, they proved that with probability $1 - o((\log \log n)^{-1/2})$,

$$N_n(\mathbb{R}) = (2/\pi) \log n + o((\log n)^{2/3} \log \log n).$$

In late 1960s and early 1970s, Ibragimov and Maslova [35, 36, 37] successfully refined Erdős-Offord method. They extended the result to all distributions in the domain of attraction of the normal law: if each $\mathbb{E}[c_j] = 0$, then the same asymptotic holds:

$$\mathbb{E}[N_n(\mathbb{R})] = (2/\pi) \log n + o(\log n),$$

though, if $\mathbb{E}[c_j] \neq 0$, one expects half as many zeros as in the previous case. In 2015, H. Nguyen, O. Nguyen and Vu [68] improved Ibragimov and Maslova's result by showing that the error term in the case of mean zero coefficients equals $\mathrm{O}(1)$. Many additional references and further directions of work on the expected number of real zeros may be found in the books of Bharucha-Reid and Sambandham [7], and of Farahmand [24].

In all these instances cited above, the main focus is real roots of a random algebraic polynomial. It is natural to generalize the study to consider complex roots of a random algebraic polynomial. Shparo and Shur [79], Arnold [2], and many other authors showed that most of zeros of random algebraic polynomials are accumulating near the unit circumference, being equidistributed in the angular sense, under mild conditions on the probability distribution of the coefficients. Introducing modern terminology, we define the zero counting measure

$$\tau_n = \frac{1}{n} \sum_{k=1}^{n} \delta_{Z_k},$$

where $\{Z_k\}_{k=1}^n$ are the zeros of a polynomial of degree n, and δ_{Z_k} is the unit point mass at Z_k. The fact of equidistribution for the zeros of random polynomials is expressed

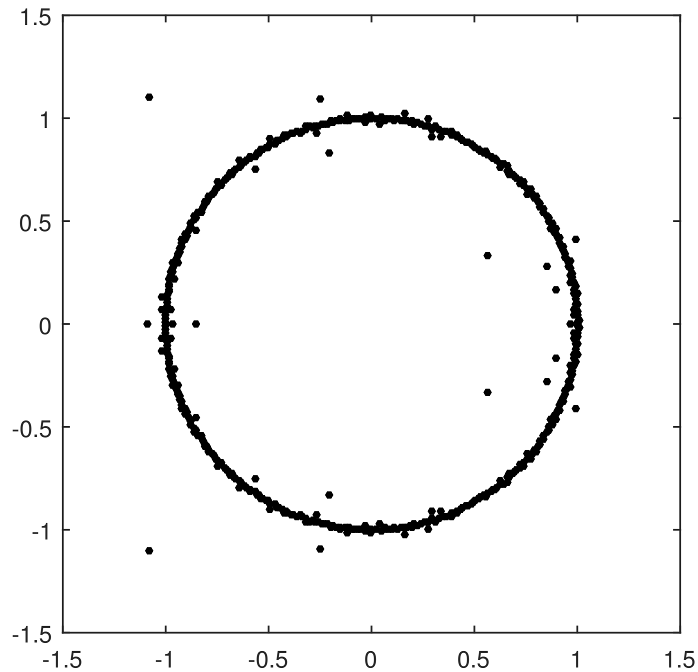

Figure 1.1: *Zeros of a random Kac polynomial of degree 500.*

via the weak convergence of τ_n to the normalized arclength measure $\mu_{\mathbb{T}}$ on the unit circumference, where

$$d\mu_{\mathbb{T}}\left(e^{it}\right) := dt/(2\pi).$$

Namely, we have that $\tau_n \to \mu_{\mathbb{T}}$ weakly with probability 1 (abbreviated as a.s. or almost surely). More recent work on the global limiting distribution of zeros of random polynomials (1.1.1) include papers of Hughes and Nikeghbali [34], Ibragimov and Zeitouni [38], Ibragimov and Zaporozhets [39], Kabluchko and Zaporozhets [47, 48], etc. In particular, Ibragimov and Zaporozhets [39] proved that if the coefficients are independent and identically distributed, then the condition

$$\mathbb{E}[\log^+|c_0|] < \infty$$

is necessary and sufficient for $\tau_n \to \mu_{\mathbb{T}}$ weakly almost surely. The results of Shepp and Vanderbei [82] provide asymptotics for the expected number of complex zeros, when random polynomials have real standard Gaussian coefficients. Ibragimov and Zeitouni [38] obtained generalizations of those results for random coefficients from the domain of attraction of a stable law. A Java program that computes and plots the complex roots of random polynomials may be found on the web page of Vanderbei [92].

The majority of available results on the equidistribution of zeros of random algebraic polynomials require the coefficients $\{c_j\}_{j=0}^n$ to be independent and identically distributed (i.i.d.) random variables. This assumption is certainly natural from probabilistic point of view. However, it is not necessary as Pritsker pointed out in his paper [69]. He proved results on the zero distribution of random algebraic polynomials whose coefficients need not have identical distributions and may be even dependent. Pritsker and Sola [71] showed that the expected discrepancy of roots of a random algebraic polynomial of degree n, with not necessarily independent coefficients, decays like $\sqrt{\log n/n}$. This result was further generalized by Pritsker and Yeager [72].

Another interesting line of investigation involves the local scaling limit results on the complex zeros of random polynomials which we do not discuss in details but refer to Bleher and Di [6], Tao and Vu [91], and Sinclair and Yattselev [83].

The expected number of zeros $\mathbb{E}[N_n([0,2\pi])]$ on $[0,2\pi]$ for a random trigonometric polynomial

$$T_n(x) = \sum_{j=1}^{n} c_j \cos(jx), \quad x \in [0,2\pi], \tag{1.1.4}$$

with real standard Gaussian coefficients was considered by Dunnage [19] in 1966. He proved that

$$\mathbb{E}[N_n([0,2\pi])] \cong (2/\sqrt{3})n \quad \text{as } n \to \infty.$$

In 1967, Das [17] considered a more general ensemble with real standard Gaussian coefficients:

$$\sum_{j=0}^{n} b_j c_j \cos(jx),$$

where $\{b_j\}_{j=0}^{\infty}$ is a sequence of positive constants. In the particular case of

$$\sum_{j=0}^{n} c_j \cos(jx),$$

he has shown that for large n,

$$\mathbb{E}[N_n([0,2\pi])] = 2n/\sqrt{3} + \mathrm{O}\big(\sqrt{n}\big).$$

Then in 1991, Wilkins [95] improved the Das result by showing that the error term $\mathrm{O}\big(\sqrt{n}\big)$ in the Das result is actually $\mathrm{O}\big(1\big)$. More precisely, Wilkins proved that for large n,

$$\mathbb{E}[N_n([0,2\pi])] = 3^{-1/2}(2n+1)\sum_{r=0}^{3}(2n+1)^{-r}D_r + \mathrm{O}\big((2n+l)^{-3}\big),$$

where $D_0 = 1$ and D_1, D_2, and D_3 are explicitly defined constants whose numerical values are approximately 0.232423, -0.25973, and 0.2172, respectively.

The expected number of zeros $\mathbb{E}[N_n([0,2\pi])]$ on $[0,2\pi]$ for a random polynomial

$$X_n(x) = \frac{1}{\sqrt{n}}\sum_{j=1}^{n}\left(c_j \sin(jx) + d_j \cos(jx)\right), \quad x \in [0,2\pi], \tag{1.1.5}$$

where both $\{c_j\}_{j=0}^{\infty}$ and $\{d_j\}_{j=0}^{\infty}$ are sequences of independent and identically distributed real standard Gaussian coefficients, was first considered by Qualls [74] in 1970. He found that

$$\mathbb{E}[N_n([0,2\pi])] = 2\sqrt{\frac{(n+1)(2n+1)}{6}} \cong (2/\sqrt{3})n \quad \text{as } n \to \infty.$$

He also proved that

$$|N_n([0,2\pi]) - \mathbb{E}[N_n([0,2\pi])]| \leq Cn^{3/4}$$

for some $C > 0$ with probability $1 - o(1)$.

1.2 Random orthogonal polynomials

Let μ denote a positive Borel measure on the real line, with infinitely many points in its support, and with finite power moments of all orders. For $n \geq 0$, let

$$p_n(x) = \gamma_n x^n + \ldots$$

denote the nth orthonormal polynomial for μ, with $\gamma_n > 0$, so that

$$\int p_n p_m d\mu = \delta_{mn}.$$

We consider

$$d\mu = w\, dx,$$

where $w = W^2$ is the weight function. Using the orthonormal polynomials $\{p_j\}_{j=0}^{\infty}$ as the basis, we study the ensemble of random polynomials of the form

$$P_n(x) = \sum_{j=0}^{n} c_j p_j(x), \quad n \in \mathbb{N}, \tag{1.2.1}$$

where the coefficients $c_0, c_1, \ldots, c_n$ are i.i.d. random variables. Such a family is often called random orthogonal polynomials. Interesting computations and pictures of zeros of random orthogonal polynomials may be found on the CHEBFUN web page of Trefethen [90].

The case where μ has compact support was first studied. Das [16] considered random Legendre polynomials (that is, $d\mu(x) = dx$ on $[-1, 1]$), and found that $\mathbb{E}[N_n(-1, 1)]$ is asymptotically equal to $n/\sqrt{3}$. Wilkins [96] improved the error term

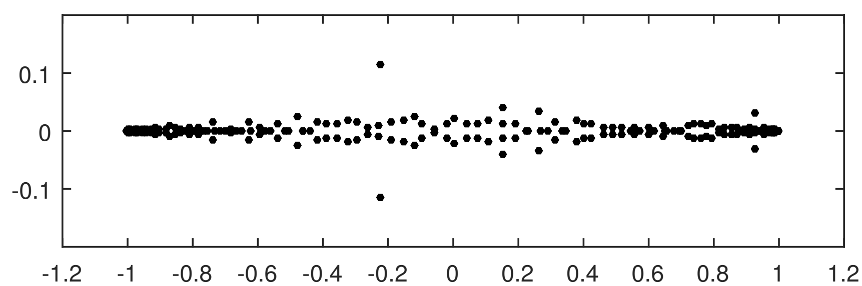

Figure 1.2: *Zeros of a random Legendre polynomial of degree 200.*

in this asymptotic relation by showing that $\mathbb{E}[N_n(-1, 1)] = n/\sqrt{3} + o(n^{\varepsilon})$ for any $\varepsilon > 0$. Farahmand [24, 25, 27] considered various generalizations of these results for the level crossings of random Legendre polynomials with coefficients that may have different distributions. Das and Bhatt [18] proved that for random Jacobi polynomials (that is, $d\mu(x) = (1-x)^\alpha(1+x)^\beta\,dx$ on $[-1,1]$, where $\alpha, \beta > -1$), $\mathbb{E}[N_n(-1,1)]$ is asymptotically equal to $n/\sqrt{3}$ as well. Finally, Lubinsky, Pritsker, and Xie [59]

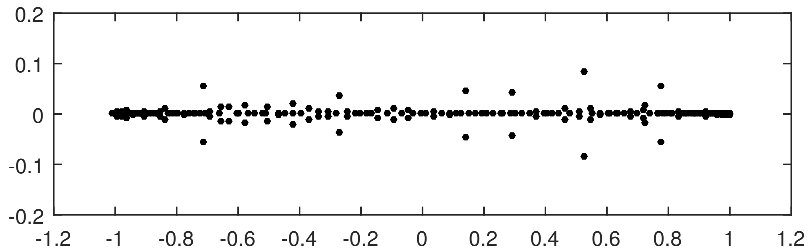

Figure 1.3: *Zeros of a random Chebyshev polynomial of degree 200.*

studied the case of any compactly supported weights on the real line (in particular, including Jacobi weights) and proved that these random linear combinations have $n/\sqrt{3}+o(n)$ expected real zeros under mild conditions. We also gave local results on the expected number of real zeros. Detailed exposition of these results is contained in Chapter 2 (for example, see Theorem 2.1.1 and Theorem 2.1.2).

Weights with unbounded support have been studied as well. Das and Bhatt [18] provided estimates for the expected number of real zeros of random Hermite polynomials (that is, $d\mu(x) = e^{-x^2}\,dx$ on $\mathbb{R}$), and concluded that $\mathbb{E}[N_n(-1,1)]$ is asymptotically equal to $n/\sqrt{3}$. They also provided estimates for random Laguerre polynomials (that is, $d\mu(x) = x^\alpha e^{-x}\,dx$ on $[0,\infty)$, where $\alpha > -1$), but those arguments contain significant gaps.

As a special case of weights with unbounded support, Pritsker and Xie [73] have studied the case of Freud weights (in particular, including Hermite weights) $W(x) = e^{-c|x|^\lambda}$, $x \in \mathbb{R}$, where $c > 0$ and $\lambda > 1$ are constants. We showed that these random linear combinations have $n/\sqrt{3} + o(n)$ expected real zeros, and gave local results on the expected number of real zeros. We also proved that the counting measures of properly scaled zeros of random Freud polynomials converge weakly to the

Ullman distribution. Details of these results are contained in Chapter 3 (for example, Theorem 3.1.1, Theorem 3.1.2 and Theorem 3.1.3). Later on, Lubinsky, Pritsker, and Xie [60] considered the general exponential weights (that is, weights in the form of $W = e^{-Q}$). We proved that the same asymptotic relation still holds for a wide class of exponential weights, and gave local results on the expected number of real zeros. We also showed that the counting measures of properly scaled zeros of these random polynomials converge weakly to either the Ullman distribution or the arcsine distribution. For detailed statements of these results, see Theorem 4.1.1, Theorem 4.1.2 and Theorem 4.1.3 in Chapter 4. Note that the work of Lubinsky, Pritsker, and Xie revealed the universal phenomenon on the expected number of real zeros for random orthogonal polynomials - there is on average $n/\sqrt{3}$ number of real zeros for such polynomials.

Another interesting direction is related to the study of the limiting distribution of the zeros of random polynomials spanned by various general bases, e.g., by orthogonal polynomials over a contour or over an area. These questions were considered by Shiffman and Zelditch [85, 86, 87], Bloom [11] and [12], Bloom and Shiffman [14], Bloom and Levenberg [13], Bayraktar [4] and others. Many of the mentioned papers used potential theoretic approach to study the limiting zero distribution, which is well developed for deterministic polynomials, see Blatt, Saff and Simkani [9], and Andrievskii and Blatt [1]. Pritsker [69] considered random polynomials spanned by general bases which include random orthogonal polynomials and random Faber polynomials on various sets in the plane. He showed almost sure convergence of the zero counting measures to the corresponding equilibrium measures for associated sets in the plane, and quantified this convergence. Those results were further generalized in his work [70]. We point out that in his results, random coefficients may be dependent and need not have identical distributions.

1.3 The variance of the number of real zeros of random polynomials

The variance $\mathrm{Var}[N_n(\mathbb{R})]$ of the number of real zeros of a random algebraic polynomial (1.1.1) was first studied by Maslova [63] in 1974, who treated any distribution of the coefficients c_j as long as they belong to the domain of attraction of the normal law. She established the only heretofore known asymptotic

$$\mathrm{Var}[N_n(\mathbb{R})] \cong \frac{4}{\pi}(1 - \frac{2}{\pi}) \ln n \quad \text{as } n \to \infty.$$

In her next paper [64], she went further to establish an even more striking result, proving the normal limiting distribution for $N_n(\mathbb{R})$ as $n \to \infty$.

In 1996, the variance $\mathrm{Var}[N_n([0, 2\pi])$ of the number of zeros of a random trigonometric polynomial (1.1.5) with real standard Gaussian coefficients on $[0, 2\pi]$ was shown by Bogomolny, Bohigas, and Leboeuf to satisfy

$$\mathrm{Var}[N_n([0, 2\pi])] \cong c_1 n \quad \text{as } n \to \infty,$$

where c_1 is a positive constant approximated by $c_1 \approx 0.55826$. Later in 1997, the variance $\mathrm{Var}[N_n([0, 2\pi])]$ of the number of zeros of a random trigonometric polynomial (1.1.4) with real standard Gaussian coefficients on $[0, 2\pi]$ was shown by Farahmand to be $\mathrm{O}\big(n^{3/2}\big)$ in [26]. In 2010, Granville and Wigman [32] found the asymptotic variance $\mathrm{Var}[N_n([0, 2\pi])$ for a random trigonometric polynomial (1.1.5) in the case of real standard Gaussian coefficients, as well as the central limit theorem for $N_n([0, 2\pi])$:

$$\mathrm{Var}[N_n([0, 2\pi])] \cong c_2 n \quad \text{as } n \to \infty, \text{ where } c_2 \approx 0.55826,$$
$$\frac{N_n([0, 2\pi]) - \mathbb{E}[N_n([0, 2\pi])]}{\sqrt{c_2 n}} \xrightarrow{d} \mathcal{N}(0, 1).$$

The latest result on the variance of the number of zeros for a random trigonometric polynomial (1.1.4) was found by Su and Shao [78] in 2012. They proved in the case of real standard Gaussian coefficients that

$$\lim_{n\to\infty} \mathrm{Var}[N_n([0, 2\pi])]/n = c_0 \quad \text{as } n \to \infty,$$

where the complicated constant $c_0 \approx 3.148$ is explicitly defined.

1.4 Plan of this dissertation

The main focus of this dissertation will be on examining the expected number of real zeros of random orthogonal polynomials associated with different weights. For the orthonormal polynomials $\{p_j(x)\}_{j=0}^{\infty}$ associated with the measure μ, define the reproducing kernel and the differentiated kernels by

$$K_n(x,y) = \sum_{j=0}^{n-1} p_j(x)p_j(y), \quad K_n^{(k,l)}(x,y) = \sum_{j=0}^{n-1} p_j^{(k)}(x)p_j^{(l)}(y), \quad k,l \in \mathbb{N}\cup\{0\}. \tag{1.4.1}$$

Our strategy is to apply the formula in Proposition 2.1.1 from Chapter 2:

$$\mathbb{E}[N_n(a,b)] = \frac{1}{\pi}\int_a^b \frac{\sqrt{K_{n+1}(x,x)K_{n+1}^{(1,1)}(x,x) - K_{n+1}^{(0,1)}(x,x)^2}}{K_{n+1}(x,x)}\,dx,$$

where (a,b) is an interval on the real line. We will use universality limits for the reproducing kernels of orthogonal polynomials (see Levin and Lubinsky [52, 53, 54], Lubinsky [61]-[62], and Totik [88]-[89]), and asymptotic results on zeros of random polynomials (cf. Pritsker [69]) to give asymptotics for the expected number of real zeros for a wide class of random orthogonal polynomials. We will begin by considering weights with compact support in Chapter 2. We will introduce some potential theory background there. In Chapter 3, we will discuss Freud weights which have unbounded support. In Chapter 4, we consider general weights with unbounded support which include Freud weights as special cases. In chapter 5, we study the variance of the number of real zeros of random orthogonal polynomials.

CHAPTER 2

Random Orthogonal Polynomials for Weights with Compact Support

This chapter is based on the joint work with D. S. Lubinsky and I. E. Pritsker [59]. We investigate the expected number of real zeros for the following random orthogonal polynomial:

$$P_n(x) = \sum_{j=0}^{n} c_j p_j(x), \tag{2.0.1}$$

where the coefficients $c_0, c_1, \cdots, c_n$ are i.i.d. Gaussian random variables with distribution $\mathcal{N}(0, \sigma^2), \sigma > 0$, and $\{p_j\}_{j=0}^{\infty}$ are orthonormal polynomials with respect to a compactly supported measure μ:

$$\int p_n p_m \, d\mu = \delta_{mn}.$$

2.1 The expected number of real zeros

Before we state our main results on the expected number of real zeros of random orthogonal polynomials, we state a result on the number of real zeros for the random linear combinations of rather general functions. It originated in the papers of Kac [44]-[46], who used the monomial basis, and was extended to trigonometric polynomials and other bases, see Adler and Taylor [3], Cramér and Leadbetter [15, p. 284], Farahmand [24], Ibragimov and Zaporozhets [40, Theorem 2], and Das [16]-[17]. For any set $E \subset \mathbb{C}$, we use the notation $N_n(E)$ for the number of zeros of random functions (2.1.1) (or random orthogonal polynomials of degree at most n) located in E. The expected number of zeros in E is denoted by $\mathbb{E}[N_n(E)]$, with $\mathbb{E}[N_n(a, b)]$ being the expected number of zeros in $(a, b) \subset \mathbb{R}$.

Proposition 2.1.1 *Let $[a,b] \subset \mathbb{R}$, and consider real valued functions $g_j(x) \in C^1([a,b]),\ j = 0,\dots,n$, with $g_0(x)$ being a nonzero constant. Define the random function*

$$G_n(x) = \sum_{j=0}^{n} c_j g_j(x), \tag{2.1.1}$$

where the coefficients c_j are i.i.d. random variables with Gaussian distribution $\mathcal{N}(0,\sigma^2), \sigma > 0$. If there is $M \in \mathbb{N}$ such that $G_n'(x)$ has at most M zeros in (a,b) for all choices of coefficients, then the expected number of real zeros of $G_n(x)$ in the interval (a,b) is given by

$$\mathbb{E}[N_n(a,b)] = \frac{1}{\pi}\int_a^b \frac{\sqrt{A(x)C(x) - B^2(x)}}{A(x)}\,dx \tag{2.1.2}$$

where

$$A(x) = \sum_{j=0}^{n} g_j^2(x), \quad B(x) = \sum_{j=1}^{n} g_j(x)g_j'(x) \quad \text{and} \quad C(x) = \sum_{j=1}^{n} [g_j'(x)]^2. \tag{2.1.3}$$

Clearly, the original formula of Kac follows from this proposition for $g_j(x) = x^j,\ j = 0,1,\dots,n$. We note that multiple zeros are counted only once by the standard convention in all our results on real zeros. However, the probability of having a multiple zero for a polynomial with Gaussian coefficients is equal to 0, so that we have the same result on the expected number of zeros regardless whether they are counted with or without multiplicities.

Theorem 2.1.1 *Let $K \subset \mathbb{R}$ be a finite union of closed and bounded intervals, and let μ be a positive Borel measure supported on K such that $d\mu(x) = w(x)dx$ and $w > 0$ a.e. on K. If for every $\varepsilon > 0$ there is a closed set $S \subset K$ of Lebesgue measure $|S| < \varepsilon$, and a constant $C > 1$ such that $C^{-1} < w < C$ a.e. on $K \setminus S$, then the expected number of real zeros of random orthogonal polynomials* (1.2.1) *satisfies*

$$\lim_{n\to\infty} \frac{1}{n}\mathbb{E}[N_n(\mathbb{R})] = \frac{1}{\sqrt{3}}. \tag{2.1.4}$$

A simple example of the orthogonality measure μ satisfying the above conditions is given by the density w that is continuous on K except for finitely many points, and has finitely many zeros on K. More specifically, one may consider the generalized Jacobi weight of the form $w(x) = v(x)\prod_{j=1}^{J}|x-x_j|^{\alpha_j}$, where $v(x)>0,\ x\in K$, and $\alpha_j>-1,\ j=1,\dots,J$.

Theorem 2.1.1 is a consequence of more precise and general local results given below. In order to state them, we need the notion of the equilibrium measure ν_K of a compact set $K\subset\mathbb{C}$. This is the unique probability measure supported on K that minimizes the energy

$$I[\nu] = -\iint \log|z-t|\,d\nu(t)d\nu(z)$$

amongst all probability measures ν with support on K. The logarithmic capacity of K is

$$\mathrm{cap}(K) = \exp\left(-I[\nu_K]\right).$$

When we say that a compact set K is regular, this means regularity in the sense of Dirichlet problem (or potential theory). See Ransford [75] for further orientation.

We also need the notion of a measure μ regular in the sense of Stahl, Totik, and Ullman [80]. If $K=\operatorname{supp}\mu$ and

$$\lim_{n\to\infty}\gamma_n^{1/n} = \frac{1}{\mathrm{cap}(K)},$$

where γ_n is the leading coefficient of p_n, then we say that μ is STU-regular. A sufficient condition for this is that K consists of finitely many intervals and $\mu' = w > 0$ a.e. in those intervals.

Theorem 2.1.2 *Let μ be an STU regular measure with compact support $K\subset\mathbb{R}$, which is regular in the sense of potential theory. Let O be an open set in which μ is absolutely continuous, and such that for some $C>1$*

$$C^{-1}\le\mu'\le C \text{ a.e. in } O. \tag{2.1.5}$$

Then given any compact subinterval $[a,b]$ of O, we have

$$\lim_{n\to\infty}\frac{1}{n}\mathbb{E}\left[N_n\left([a,b]\right)\right] = \frac{1}{\sqrt{3}}\nu_K([a,b]), \tag{2.1.6}$$

where ν_K is the equilibrium measure of K.

This is a special case of the following result, where μ does not need to be STU regular. The asymptotic lower bound requires very little of μ.

Theorem 2.1.3 *Let μ be a measure on the real line with compact support K.*
(a) Assume that $\mu' > 0$ a.e. in the interval $[a,b]$. Then

$$\liminf_{n\to\infty} \frac{1}{n}\mathbb{E}\left[N_n\left([a,b]\right)\right] \geq \frac{1}{\sqrt{3}}\,\nu_K([a,b]). \tag{2.1.7}$$

(b) Suppose in addition that (2.1.5) *holds, and that $[a,b] \subset O$. Then*

$$\limsup_{n\to\infty} \frac{1}{n}\mathbb{E}\left[N_n\left([a,b]\right)\right] \leq \frac{1}{\sqrt{3}}\inf_L\ \nu_L([a,b]), \tag{2.1.8}$$

where the inf *is taken over all regular compact sets $L \subset K$ such that $L \supset [a,b]$, and the restriction $\mu|_L$ of μ to L is STU regular.*

It is plausible that the right hand sides of (2.1.7) and (2.1.8) are equal under mild assumptions such as the one of part (a). An interesting open problem is to find rates of convergence in the limit relations (2.1.4) and (2.1.6).

2.2 Proofs

Proof of Proposition 2.1.1. Various generalizations of Kac's integral formula for the expected number of real zeros were obtained by many authors; see e.g. Cramér and Leadbetter [15, p. 284], Ibragimov and Zaporozhets [40, Theorem 2]. We sketch the proof of Proposition 2.1.1 using ideas of Kac [44, pp. 318–320] and Das [17]. The joint probability density of $\mathbf{c} = (c_0, c_1, \cdots, c_n)$ is

$$dP(\mathbf{c}) = (2\pi)^{-(n+1)/2}\sigma^{-(n+1)}e^{-\frac{\|\mathbf{c}\|^2}{2\sigma^2}}\,dc_0dc_1\cdots dc_n,$$

where $\|\mathbf{c}\|^2 = c_0^2 + c_1^2 + \cdots + c_n^2$. Since $G_n(x)$ has at most $M+1$ zeros in (a,b) for all $\mathbf{c}$ by Rolle's theorem, $N_n(a,b)$ is integrable over $\mathbb{R}^{n+1}$ with respect to $dP(\mathbf{c})$. Define

$$N_n^*(a,b) = N_n(a,b) - (\kappa(a) + \kappa(b))/2,$$

where

$$\kappa(x) = \begin{cases} 1 & \text{if } G_n(x) = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Since $G_n(a)$ and $G_n(b)$ are continuous random variables, we have

$$\mathbb{E}[N_n(a,b)] = \int_{\mathbb{R}^{n+1}} N_n^*(a,b)\, dP(\mathbf{c}).$$

We state the following result from Kac [45, Theorem 1]. This lemma has been generalized by many authors; see e.g. Farahmand [24, p. 11], Leadbetter [50], Cramér and Leadbetter [15].

Lemma 2.2.1 *If $f(x)$ is continuous for $\alpha \le x \le \beta$ and continuously differentiable for $\alpha < x < \beta$, and $f'(x)$ vanishes only at a finite number of points in $\alpha < x < \beta$, then the number of zeros of $f(x)$ in $\alpha < x < \beta$ (multiple zeros are counted once and if either α or β is a zero, it is counted as $1/2$) is equal to*

$$\text{P.V. } \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{\alpha}^{\beta} \cos(yf(x))\, |f'(x)| \; dx\, dy.$$

Note that P.V. throughout this proof is understood in the Cauchy principal value sense. In our notation, this gives

$$N_n^*(a,b) = \text{P.V. } \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{a}^{b} \cos(yG_n(x))\, |G_n'(x)| \; dx\, dy.$$

Thus

$$\begin{aligned} \mathbb{E}[N_n(a,b)] &= (2\pi)^{-\frac{n+1}{2}} \sigma^{-(n+1)} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} N_n^*(a,b) e^{-\frac{\|\mathbf{c}\|^2}{2\sigma^2}}\, dc_0 dc_1 \cdots dc_n \\ &= \frac{\sigma^{-(n+1)}}{2\pi} \int_a^b \int_{-\infty}^{\infty} R_n(x,y)\, dy\, dx, \qquad (2.2.1) \end{aligned}$$

where

$$R_n(x,y) = (2\pi)^{-\frac{n+1}{2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\frac{\|\mathbf{c}\|^2}{2\sigma^2}} \cos(yG_n(x))\, |G_n'(x)| \; dc_0 dc_1 \cdots dc_n. \quad (2.2.2)$$

The interchange of the integration order is justified by the fact that the integrand is dominated by

$$e^{-\frac{\|\mathbf{c}\|^2}{2\sigma^2}} \sum_{j=0}^{n} |c_j| \left| g_j'(x) \right|,$$

which is exponentially small outside bounded sets in $\mathbb{R}^{n+1}$. We use the known relation

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1 - \cos(uv)}{u^2}\, du = |v| \tag{2.2.3}$$

to write (2.2.2) as

$$R_n(x,y) = \text{P.V. } \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{du}{u^2} \times$$
$$(2\pi)^{-\frac{n+1}{2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\frac{\|\mathbf{c}\|^2}{2\sigma^2}} \left(\cos(yG_n(x)) - \cos(yG_n(x))\cos(uG_n'(x))\right)\, dc_0 dc_1 \cdots dc_n, \tag{2.2.4}$$

where the interchange of orders of the integration can be justified as above, and (2.2.4) is interpreted as

$$\lim_{N\to\infty} \lim_{\epsilon\to 0} \frac{1}{\pi} \left(\int_{-N}^{-\epsilon} + \int_{\epsilon}^{N} \right) (\cdots) \frac{du}{u^2}. \tag{2.2.5}$$

Noting that

$$\cos(yG_n(x))\cos(uG_n'(x)) = \frac{1}{2} \Re \left(e^{iyG_n(x) + iuG_n'(x)} + e^{iyG_n(x) - iuG_n'(x)} \right),$$

we obtain with help of [30, 3.323(2) on p. 337] that

$$(2\pi)^{-\frac{n+1}{2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\frac{\|\mathbf{c}\|^2}{2\sigma^2}} \cos(yG_n(x))\cos(uG_n'(x))\, dc_0 \cdots dc_n$$
$$= \frac{(2\pi)^{-\frac{n+1}{2}}}{2} \times$$
$$\Re \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\frac{\|\mathbf{c}\|^2}{2\sigma^2}} \left(e^{i\sum_{j=0}^{n} [yc_j g_j(x) + uc_j g_j'(x)]} + e^{i\sum_{j=0}^{n} [yc_j g_j(x) - uc_j g_j'(x)]} \right) dc_0 \cdots dc_n$$
$$= \frac{(2\pi)^{-\frac{n+1}{2}}}{2} \Re \left(\prod_{j=0}^{n} \int_{-\infty}^{\infty} e^{-\frac{c_j^2}{2\sigma^2} + i[yg_j(x) + ug_j'(x)]c_j}\, dc_j + \prod_{j=0}^{n} \int_{-\infty}^{\infty} e^{-\frac{c_j^2}{2\sigma^2} + i[yg_j(x) - ug_j'(x)]c_j}\, dc_j \right)$$
$$= \frac{(2\pi)^{-\frac{n+1}{2}}}{2} \Re \left(\prod_{j=0}^{n} (2\pi)^{\frac{1}{2}} \sigma e^{-\frac{1}{2}[yg_j(x) + ug_j'(x)]^2 \sigma^2} + \prod_{j=0}^{n} (2\pi)^{\frac{1}{2}} \sigma e^{-\frac{1}{2}[yg_j(x) - ug_j'(x)]^2 \sigma^2} \right)$$
$$= \frac{\sigma^{n+1}}{2} e^{-\frac{\sigma^2}{2} \sum_{j=0}^{n} [yg_j(x) + ug_j'(x)]^2} + \frac{\sigma^{n+1}}{2} e^{-\frac{\sigma^2}{2} \sum_{j=0}^{n} [yg_j(x) - ug_j'(x)]^2}.$$

For $u = 0$, we have

$$(2\pi)^{-\frac{n+1}{2}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\frac{\|\mathbf{c}\|^2}{2\sigma^2}} \cos(yG_n(x))\, dc_0 \cdots dc_n = \sigma^{n+1} e^{-\frac{\sigma^2}{2} \sum_{j=0}^{n} [yg_j(x)]^2}.$$

Using abbreviations $A = A(x)$, $B = B(x)$ and $C = C(x)$, we rewrite

$$\begin{aligned} R_n(x,y) &= \frac{\sigma^{n+1}}{\pi} \int_{-\infty}^{\infty} \frac{e^{-\frac{\sigma^2}{2}Ay^2}}{u^2}\, du \\ &\quad - \frac{\sigma^{n+1}}{2\pi} \left(\int_{-\infty}^{\infty} \frac{e^{-\frac{\sigma^2}{2}(Ay^2+Cu^2+2yuB)}}{u^2}\, du + \int_{-\infty}^{\infty} \frac{e^{-\frac{\sigma^2}{2}(Ay^2+Cu^2-2yuB)}}{u^2}\, du \right) \\ &= \frac{\sigma^{n+1}}{\pi} e^{-\frac{\sigma^2}{2}Ay^2} \int_{-\infty}^{\infty} \frac{1 - e^{-\frac{\sigma^2}{2}Cu^2 + yuB\sigma^2}}{u^2}\, du, \end{aligned}$$

where the integral exists as a principal value, in the sense indicated in (2.2.5). If $C(x) = 0$ for some x then $B(x) = 0$ and $R(x, y) = 0$ for the same x and all y. Thus we set $By\sigma^2 = t$ and $\sigma^2 C = h > 0$, so that

$$R_n(x,y) = \frac{\sigma^{n+1}}{\pi} e^{-\frac{\sigma^2}{2}Ay^2} \int_{-\infty}^{\infty} \frac{1 - e^{-\frac{1}{2}hu^2 + tu}}{u^2}\, du.$$

The Taylor expansion

$$e^{tu} = 1 + tu + \sum_{m=2}^{\infty} \frac{t^m u^m}{m!},$$

together with known identities

$$\int_{-\infty}^{\infty} \frac{1 - e^{-\frac{1}{2}hu^2}}{u^2}\, du = \sqrt{2\pi h} \quad \text{and} \quad \text{P.V.} \int_{-\infty}^{\infty} \frac{tu}{u^2} e^{-\frac{1}{2}hu^2}\, du = 0,$$

gives that

$$R_n(x,y) = \frac{\sigma^{n+1}}{\pi} e^{-\frac{\sigma^2}{2}Ay^2} \left(\sqrt{2\pi h} - \int_{-\infty}^{\infty} \left(\sum_{m=2}^{\infty} \frac{t^m u^m}{m!} \right) \frac{e^{-\frac{1}{2}hu^2}}{u^2}\, du \right).$$

Assuming that $h > 0$, we further obtain that

$$\begin{aligned} &\int_{-\infty}^{\infty} \left(\sum_{m=2}^{\infty} \frac{t^m u^m}{m!} \right) \frac{e^{-\frac{1}{2}hu^2}}{u^2}\, du \\ &= \sum_{m=1}^{\infty} \frac{t^{2m}}{(2m)!} \int_{-\infty}^{\infty} u^{2(m-1)} e^{-\frac{1}{2}hu^2}\, du \\ &= \sum_{m=1}^{\infty} \frac{t^{2m}}{(2m)!} \frac{(2(m-1))!}{2^{m-1}(m-1)!} \sqrt{2\pi} h^{-(m-1)-\frac{1}{2}} \quad \text{(by [30, 3.461(2) on p. 364])} \\ &= \sum_{m=1}^{\infty} \frac{\sqrt{2\pi h}}{m!(2m-1)} \left(\frac{t^2}{2h} \right)^m. \end{aligned}$$

Hence

$$R_n(x,y) = \frac{\sigma^{n+1}}{\pi} e^{-\frac{\sigma^2}{2}Ay^2} \left(\sqrt{2\pi h} - \sum_{m=1}^{\infty} \frac{\sqrt{2\pi h}}{m!(2m-1)} \left(\frac{t^2}{2h}\right)^m \right)$$
$$= \sqrt{\frac{2C}{\pi}}\ \sigma^{n+2} e^{-\frac{\sigma^2}{2}Ay^2} \left(1 - \sum_{m=1}^{\infty} \frac{1}{m!(2m-1)} \left(\frac{B^2\sigma^2}{2C}\right)^m y^{2m} \right).$$

Applying [30, 3.461(2) on p. 364] again, we obtain that

$$\int_{-\infty}^{\infty} R_n(x,y)\,dy$$
$$= \sqrt{\frac{2C}{\pi}}\ \sigma^{n+2} \int_{-\infty}^{\infty} e^{-\frac{\sigma^2}{2}Ay^2} \left(1 - \sum_{m=1}^{\infty} \frac{1}{m!(2m-1)} \left(\frac{B^2\sigma^2}{2C}\right)^m y^{2m} \right) dy$$
$$= \sqrt{\frac{2C}{\pi}}\ \sigma^{n+2} \left(\sqrt{\frac{2\pi}{A\sigma^2}} - \sum_{m=1}^{\infty} \frac{(\frac{B^2\sigma^2}{2C})^m}{m!(2m-1)} \frac{(2m)!}{m!2^m} \sqrt{\frac{2\pi}{A\sigma^2}} \frac{1}{(A\sigma^2)^m} \right)$$
$$= 2\sqrt{\frac{C}{A}}\ \sigma^{n+1} \left(-\sum_{m=0}^{\infty} \left(\frac{B^2}{AC}\right)^m \frac{(2m)!}{(m!)^2(2m-1)4^m} \right)$$
$$= 2\sqrt{\frac{C}{A}}\sqrt{1-\frac{B^2}{AC}}\ \sigma^{n+1}.$$

Then (2.2.1) gives us the desired formula

$$\mathbb{E}[N_n(a,b)] = \frac{1}{\pi}\int_a^b \frac{\sqrt{AC-B^2}}{A}\,dx,$$

where $AC - B^2 \geq 0$ by (2.1.3) and the Cauchy-Schwarz inequality. □

In addition to the reproducing kernels

$$K_n(x,y) = \sum_{j=0}^{n-1} p_j(x)p_j(y), \quad K_n^{(k,l)}(x,y) = \sum_{j=0}^{n-1} p_j^{(k)}(x)p_j^{(l)}(y), \quad k,l \in \mathbb{N}\cup\{0\},$$

we also use their weighted versions in the proofs below:

$$\tilde{K}_n^{(k,\ell)}(x,y) = \mu'(x)^{1/2}\,\mu'(y)^{1/2} \sum_{j=0}^{n-1} p_j^{(k)}(x)\,p_j^{(\ell)}(y).$$

Lemma 2.2.2 *Let μ be a measure with compact support and with infinitely many points in its support. Let O be an open set in which μ is absolutely continuous, and such that for some $C > 1$ (2.1.5) holds. Then given any compact subinterval $[a,b]$ of O, we have*

$$\frac{1}{n}\mathbb{E}\left[N_n([a,b])\right] = \frac{1+o(1)}{\sqrt{3}} \int_a^b \frac{1}{n} K_{n+1}(x,x)\,d\mu(x). \tag{2.2.6}$$

Proof. First note that the hypothesis that $\mu' \geq C^{-1}$ in O gives [29, Theorem 3.3, p. 104]

$$C_1 = \sup_{n \geq 1} \sup_{x \in [a,b]} \frac{1}{n} K_{n+1}(x,x) < \infty.$$

Next, we use Corollary 1.4 in [62, p. 224]. It gives for all $j, k \geq 0$,

$$\lim_{n \to \infty} \int_a^b \left| \frac{\tilde{K}_{n+1}^{(j,k)}(x,x)}{\tilde{K}_{n+1}(x,x)^{j+k+1}} - \pi^{j+k} \tau_{j,k} \right| dx = 0. \tag{2.2.7}$$

Here

$$\tau_{j,k} = \begin{cases} 0, & j+k \text{ odd}, \\ (-1)^{(j-k)/2} \frac{1}{j+k+1}, & j+k \text{ even}. \end{cases}$$

Applying (2.1.2) in a modified form, we obtain that

$$\frac{1}{n} \mathbb{E} \left[N_n \left([a,b] \right) \right] = \frac{1}{\pi} \int_a^b \sqrt{ \frac{\tilde{K}_{n+1}^{(1,1)}(x,x)}{\tilde{K}_{n+1}(x,x)^3} - \left(\frac{\tilde{K}_{n+1}^{(0,1)}(x,x)}{\tilde{K}_{n+1}(x,x)^2} \right)^2 } \frac{1}{n} \tilde{K}_{n+1}(x,x) \, dx. \tag{2.2.8}$$

Since $\frac{1}{n} \tilde{K}_{n+1}(x,x)$ is bounded uniformly in n and in $x \in [a,b]$, we can use (2.2.7) above to obtain

$$\begin{aligned} \frac{1}{n} E \left[N_n([a,b]) \right] &= \frac{1}{\pi} \int_a^b \left(\sqrt{\pi^2 \tau_{1,1} - (\pi \tau_{0,1})^2} + o(1) \right) \frac{1}{n} \tilde{K}_{n+1}(x,x) \, dx \\ &= \frac{1 + o(1)}{\sqrt{3}} \int_a^b \frac{1}{n} \tilde{K}_{n+1}(x,x) \, dx. \end{aligned}$$

□

Proof of Theorem 2.1.2. Note that since $\mu' > 0$ a.e. in $[a,b]$, this interval is contained in $\operatorname{supp} \nu_K$. In [88, p. 287, Theorem 1], under weaker conditions, Totik proved that for a.e. $x \in [a,b]$,

$$\lim_{n \to \infty} \frac{1}{n} K_{n+1}(x,x) = \frac{d\nu_K}{d\mu}(x).$$

Since

$$\lim_{n \to \infty} \frac{1}{n} \tilde{K}_{n+1}(x,x) = \frac{d\nu_K}{d\mu}(x) \, \mu'(x) = \nu_K'(x),$$

the uniform boundedness of $\left\{ \frac{1}{n} \tilde{K}_{n+1}(x,x) \right\}_{n=1}^{\infty}$ and Lemma 2.2.2 then give the result.

□

Proof of Theorem 2.1.3. We start with part (a). Given $r > 0$, and $j, k \geq 0$, with $\tau_{j,k}$ as above, it follows from [62, p. 250, Proof of Corollary 1.4] that

$$\left| \frac{\tilde{K}_{n+1}^{(j,k)}(x,x)}{\tilde{K}_{n+1}(x,x)^{j+k+1}} - \pi^{j+k}\tau_{j,k} \right|$$
$$\leq \frac{j!k!}{r^{j+k}} \sup_{|u|,|v|\leq r} \left| \frac{K_{n+1}\left(x+\frac{u}{\tilde{K}_{n+1}(x,x)}, x+\frac{v}{\tilde{K}_{n+1}(x,x)}\right)}{K_{n+1}(x,x)} - \frac{\sin(\pi(u-v))}{\pi(u-v)} \right|.$$

Next, using that $\mu' > 0$ a.e. in $[a,b]$, we have from [62, p. 223, Theorem 1.1] that

$$\operatorname{meas}\left\{ x \in [a,b] : \sup_{|u|,|v|\leq r} \left| \frac{K_{n+1}\left(x+\frac{u}{\tilde{K}_{n+1}(x,x)}, x+\frac{v}{\tilde{K}_{n+1}(x,x)}\right)}{K_{n+1}(x,x)} - \frac{\sin(\pi(u-v))}{\pi(u-v)} \right| \geq \varepsilon \right\}$$
$$\to 0$$

as $n \to \infty$, for any given $\varepsilon, r > 0$. Thus also

$$\operatorname{meas}\left\{ x \in [a,b] : \left| \frac{\tilde{K}_{n+1}^{(j,k)}(x,x)}{\tilde{K}_{n+1}(x,x)^{j+k+1}} - \pi^{j+k}\tau_{j,k} \right| \geq \varepsilon \right\} \to 0$$

as $n \to \infty$. Now let $\varepsilon > 0$, and for $n \geq 1$, let

$$\mathcal{E}_n = \left\{ x \in [a,b] : \sqrt{\frac{\tilde{K}_{n+1}^{(1,1)}(x,x)}{\tilde{K}_{n+1}(x,x)^3} - \left(\frac{\tilde{K}_{n+1}^{(0,1)}(x,x)}{\tilde{K}_{n+1}(x,x)^2} \right)^2} \leq \sqrt{\pi^2/3} - \varepsilon \right\}.$$

Then it follows that

$$\operatorname{meas}(\mathcal{E}_n) \to 0 \text{ as } n \to \infty.$$

Using [89, p. 118, Thm. 2.1], we have for a.e. $x \in [a,b]$ that

$$\liminf_{n\to\infty} \frac{1}{n}\tilde{K}_{n+1}(x,x) \geq \nu_K'(x).$$

It then follows, that given $\varepsilon > 0$,

$$\mathcal{F}_n = \left\{ x \in [a,b] : \frac{1}{n}\tilde{K}_{n+1}(x,x) \leq \nu_K'(x) - \varepsilon \right\}$$

has

$$\operatorname{meas}(\mathcal{F}_n) \to 0 \text{ as } n \to \infty. \tag{2.2.9}$$

Indeed, if we set

$$f_n(x) = \min\left\{\frac{1}{n}\tilde{K}_{n+1}(x,x) - \nu_K'(x), 0\right\},$$

then by Totik's result,

$$\lim_{n\to\infty} f_n(x) = 0 \text{ a.e. in } [a,b],$$

while f_n is bounded below by $-\nu_K'$, so Lebesgue's Dominated Convergence Theorem gives

$$0 = \lim_{n\to\infty} \int_a^b f_n \le \liminf_{n\to\infty} (-\varepsilon)\mathrm{meas}\left(\mathcal{F}_n\right).$$

Thus (2.2.9) holds. Then by (2.1.2), (2.2.8) and the definitions of $\mathcal{E}_n$ and $\mathcal{F}_n$, we have

$$\begin{aligned}
\frac{1}{n}\mathbb{E}\left[N_n\left([a,b]\right)\right] &= \frac{1}{\pi}\int_a^b \sqrt{\frac{\tilde{K}_{n+1}^{(1,1)}(x,x)}{\tilde{K}_{n+1}(x,x)^3} - \left(\frac{\tilde{K}_{n+1}^{(0,1)}(x,x)}{\tilde{K}_{n+1}(x,x)^2}\right)^2}\,\frac{1}{n}\tilde{K}_{n+1}(x,x)\,dx \\
&\ge \frac{1}{\pi}\int_{[a,b]\setminus(\mathcal{E}_n\cup\mathcal{F}_n)} \left(\sqrt{\pi^2/3} - \varepsilon\right)\left(\nu_K'(x) - \varepsilon\right) dx \\
&\to \frac{1}{\pi}\int_a^b \left(\sqrt{\pi^2/3} - \varepsilon\right)\left(\nu_K'(x) - \varepsilon\right) dx \quad \text{as } n\to\infty.
\end{aligned}$$

Now we can let $\varepsilon \to 0$.

We pass to the proof of part (b). Let $L \subset K$ be a regular compact set such that the restriction $\mu|_L$ of μ to L is STU regular, and L contains $[a,b]$ in its interior. By monotonicity of the reproducing kernel (Christoffel function), if $K_n\left(\mu|_L,\cdot,\cdot\right)$ denotes the reproducing kernel of the measure $\mu|_L$, then for a.e. $x\in[a,b]\subset L$, Totik's result [88, p. 287, Theorem 1] gives

$$\begin{aligned}
&\limsup_{n\to\infty}\frac{1}{n}K_{n+1}(x,x)\mu'(x) \\
\le\ &\limsup_{n\to\infty}\frac{1}{n}K_{n+1}\left(\mu|_L,x,x\right)\mu'(x) = \nu_L'(x).
\end{aligned}$$

Moreover, $\left\{\frac{1}{n}K_{n+1}\left(\mu|_L,x,x\right)\mu'(x)\right\}_{n=1}^{\infty}$ is uniformly bounded in $[a,b]$. Then Lemma 2.2.2 implies that

$$\limsup_{n\to\infty}\frac{1}{n}\mathbb{E}\left[N_n\left([a,b]\right)\right] \le \frac{1}{\sqrt{3}}\int_a^b \nu_L'(x)\,dx.$$

Finally, taking the inf over all L gives the result. $\square$

Lemma 2.2.3 *Let μ be an STU regular measure on the real line with compact support K, and let ν_K be the equilibrium measure of K. Suppose that the coefficients of random orthogonal polynomials* (2.0.1) *are complex i.i.d. random variables such that* $\mathbb{E}[|\log|c_0||] < \infty$. *If $E \subset \mathbb{C}$ is any compact set satisfying $\nu_K(\partial E) = 0$, then*

$$\lim_{n\to\infty} \frac{1}{n}\mathbb{E}\left[N_n(E)\right] = \nu_K(E). \tag{2.2.10}$$

Proof. Consider the normalized counting measure $\tau_n = \frac{1}{n}\sum_{k=1}^{n} \delta_{z_k}$ for a polynomial (2.0.1), where $\{z_k\}_{k=1}^{n}$ are the zeros of that polynomial, and δ_z denotes the unit point mass at z. Theorem 2.2 of [69] implies that measures τ_n converge weakly to ν_K with probability one. Since $\nu_K(\partial E) = 0$, we obtain that $\tau_n|_E$ converges weakly to $\nu_K|_E$ with probability one by Theorem 0.5′ of [49] and Theorem 2.1 of [8]. In particular, we have that the random variables $\tau_n(E) \to \nu_K(E)$ a.s. Hence this convergence holds in L^p sense by the Dominated Convergence Theorem, as $\tau_n(E)$ are uniformly bounded by 1, see Chapter 5 of [31]. It follows that

$$\lim_{n\to\infty} \mathbb{E}[|\tau_n(E) - \nu_K(E)|] = 0$$

for any compact set E such that $\nu_K(\partial E) = 0$, and

$$|\mathbb{E}[\tau_n(E) - \nu_K(E)]| \le \mathbb{E}[|\tau_n(E) - \nu_K(E)|] \to 0 \quad \text{as } n \to \infty.$$

But $\mathbb{E}[\tau_n(E)] = \mathbb{E}[N_n(E)]/n$ and $\mathbb{E}[\nu_K(E)] = \nu_K(E)$, which immediately gives (2.2.10).

□

Proof of Theorem 2.1.1. Given any $\varepsilon > 0$, we find a closed set S satisfying the assumptions, and obtain from Theorem 2.1.2 that

$$\lim_{n\to\infty} \frac{1}{n}\mathbb{E}\left[N_n\left([a,b]\right)\right] = \frac{1}{\sqrt{3}}\nu_K([a,b])$$

for any interval $[a,b] \subset K^\circ \setminus S$, where K° is the interior of K. Note that both $\mathbb{E}\left[N_n\left(H\right)\right]$ and $\nu_K(H)$ are additive functions of the set H. Moreover, they both vanish when H is a single point by (2.2.10), because ν_K is absolutely continuous with respect to Lebesgue measure on K, see [80, Lemma 4.4.1, p. 117]. Hence (2.2.10) gives that

$$\lim_{n\to\infty} \frac{1}{n}\mathbb{E}\left[N_n\left(\mathbb{R}\setminus S\right)\right] = \frac{1}{\sqrt{3}}\nu_K(\mathbb{R}\setminus S).$$

We can find finitely many open intervals $I_k \subset \mathbb{R},\ k = 1, \ldots, m$, covering S, with total length $\sum_{k=1}^{m} |I_k| < 2\varepsilon$. Let $R_k = \{x + iy : x \in I_k,\ |y| < 1\},\ k = 1, \ldots, m$, so that for $R = \cup_{k=1}^{m} R_k$ we have $S \subset R$ and $\nu_K(\partial R) = 0$. Applying Lemma 2.2.3 again, we obtain that

$$\limsup_{n\to\infty} \frac{1}{n}\mathbb{E}\left[N_n\left(S\right)\right] \le \limsup_{n\to\infty} \frac{1}{n}\mathbb{E}\left[N_n\left(\overline{R}\right)\right] = \nu_K(\overline{R}\cap\mathbb{R}) = \nu_K\left(\cup_{k=1}^{m}\overline{I_k}\right),$$

Absolute continuity of ν_K with respect to dx implies that the last term in the above estimate tends to 0 as $\varepsilon \to 0$. Thus (2.1.4) follows. □

CHAPTER 3

Random Freud Orthogonal Polynomials

This section is based on the joint work with I. E. Pritsker [73]. In Chapter 2, we considered weights with compact support on the real line. It is natural to study weights with unbounded support as well. In this chapter, we consider the Freud weights

$$W(x) = e^{-c|x|^{\lambda}},\ x \in \mathbb{R},$$

where $c > 0$ and $\lambda > 1$ are constants. For $n \geq 0$, let

$$p_n(x) = p_n\left(W^2, x\right) = \gamma_n x^n + ...$$

denote the nth orthonormal polynomial with $\gamma_n > 0$, so that

$$\int p_n p_m W^2 = \delta_{mn}.$$

Using the orthonormal polynomials $\{p_j\}_{j=0}^{\infty}$ as the basis, we consider the ensemble of random polynomials of the form

$$P_n(x) = \sum_{j=0}^{n} c_j p_j(x), \quad n \in \mathbb{N}, \tag{3.0.1}$$

where the coefficients $c_0, c_1, \ldots, c_n$ are i.i.d. random variables. We call such a family random Freud orthogonal polynomials. Note that when $\lambda = 2$, we have random Hermite orthogonal polynomials.

3.1 The expected number of real zeros

Theorem 3.1.1 *The expected number of real zeros of random Freud orthogonal polynomials* (3.0.1) *with mean-zero Gaussian random variables satisfies*

$$\lim_{n\to\infty} \frac{1}{n}\mathbb{E}[N_n(\mathbb{R})] = \frac{1}{\sqrt{3}}. \tag{3.1.1}$$

We note that asymptotic relation (3.1.1) is new even in the classical case of Hermite weight $W(x) = e^{-\frac{1}{2}x^2}, \quad x \in \mathbb{R}$. Theorem 3.1.1 is a combination of two results on zeros of random orthogonal polynomials given below. Define the constant

$$\gamma_\lambda = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{\lambda}{2})}{2\Gamma(\frac{\lambda+1}{2})}, \tag{3.1.2}$$

and the contracted version of P_n:

$$P_n^*(s) := P_n(a_n s), \quad n \in \mathbb{N}, \tag{3.1.3}$$

where $a_n = \gamma_\lambda^{\frac{1}{\lambda}} c^{-\frac{1}{\lambda}} n^{\frac{1}{\lambda}}$ is a positive number.

For any set $E \subset \mathbb{C}$, we use the notation $N_n^*(E)$ for the number of (complex) zeros of the random function $P_n^*(s)$ located in E. The expected number of (complex) zeros of $P_n^*(s)$ in E is denoted by $\mathbb{E}[N_n^*(E)]$, with $\mathbb{E}[N_n^*([a,b])]$ being the expected number of (real) zeros of $P_n^*(s)$ in $[a,b] \subset \mathbb{R}$.

Theorem 3.1.2 *If* $[a,b] \subset (-1,1)$ *is any closed interval, then for* $P_n^*(s)$ *with mean-zero Gaussian random variables, we have*

$$\lim_{n\to\infty} \frac{1}{n}\mathbb{E}\left[N_n^*([a,b])\right] = \frac{1}{\sqrt{3}}\mu_w([a,b]), \tag{3.1.4}$$

where the measure μ_w *is given by*

$$d\mu_w(s) = \left(\frac{\lambda}{\pi}\int_{|s|}^1 \frac{y^{\lambda-1}}{\sqrt{y^2-s^2}}\,dy\right) ds, \;\; s \in [-1,1].$$

Note that μ_w is the weighted equilibrium measure for the weight $w(x) = e^{-\gamma_\lambda |x|^\lambda}$ on $\mathbb{R}$, see [80] and the next section for details. This measure is often called the Ullman distribution.

Define the normalized zero counting measure

$$\tau_n = \frac{1}{n} \sum_{k=1}^{n} \delta_{z_k}$$

for the scaled polynomial $P_n^*(s)$ defined by (3.1.3), where $\{z_k\}_{k=1}^n$ are its zeros, and δ_z denotes the unit point mass at z. We can determine the weak limit of τ_n for random polynomials with quite general random coefficients $\{c_k\}_{k=0}^{\infty}$.

Theorem 3.1.3 *If the coefficients $\{c_k\}_{k=0}^{\infty}$ of random Freud orthogonal polynomials* (3.0.1) *are complex i.i.d. random variables such that $\mathbb{E}[|\log|c_0||] < \infty$, then the normalized zero counting measures τ_n for the scaled polynomials $P_n^*(s)$ converge weakly to μ_w with probability one.*

Closely related results on the asymptotic zeros distribution of random orthogonal polynomials with varying weights were proved by Bloom [11] and Bloom and Levenberg [13], but they are not directly applicable to our case because of different normalization. Theorem 3.1.3 allows us to find asymptotics for the expected number of zeros in various sets. In particular, we need the following corollary for the proof of Theorem 3.1.1.

Corollary 3.1.1 *Suppose that the coefficients $\{c_k\}_{k=0}^{\infty}$ of random Freud orthogonal polynomials* (3.0.1) *are complex i.i.d. random variables such that $\mathbb{E}[|\log|c_0||] < \infty$. If $E \subset \mathbb{C}$ is any compact set satisfying $\mu_w(\partial E) = 0$, then*

$$\lim_{n\to\infty} \frac{1}{n}\mathbb{E}\left[N_n^*(E)\right] = \mu_w(E). \tag{3.1.5}$$

It is of interest to develop similar results with relaxed conditions on random coefficients c_k, e.g., by considering probability distributions from the domain of attraction of normal law as in [35, 36].

3.2 Potential theory with external fields generated by Freud weights

We consider the general Freud weights

$$W(x) = e^{-c|x|^\lambda},\ x \in \mathbb{R},$$

where $c > 0$ and $\lambda > 1$ are constants. Set $Q(x) := -\log W(x) = c\,|x|^\lambda, \quad x \in \mathbb{R}$. The weighted equilibrium measure μ_W associated with the weight W is the unique probability measure with compact support $S_W =$ supp $\mu_W \subset \mathbb{R}$ that minimizes the energy functional

$$I[\nu] = -\iint \log|z-t|\, d\nu(t) d\nu(z) + 2\int Q\, d\nu$$

amongst all probability measures ν with support on $\mathbb{R}$. It satisfies

$$\int \log \frac{1}{|z-t|}\, d\mu_W(t) + Q(z) = C, \quad z \in S_W,$$

and

$$\int \log \frac{1}{|z-t|}\, d\mu_W(t) + Q(z) \geq C, \quad z \in \mathbb{R},$$

where C is a constant. In the case of the standard Freud weight $w(x) = e^{-\gamma_\lambda |x|^\lambda}$, where γ_λ is defined by (3.1.2), the weighted equilibrium measure μ_w is given by

$$d\mu_w(s) = \left(\frac{\lambda}{\pi} \int_{|s|}^1 \frac{y^{\lambda-1}}{\sqrt{y^2 - s^2}}\, dy \right) ds,\ s \in [-1,1],$$

by Theorem 5.1 of [80, p. 240].

The Mhaskar-Rakhmanov-Saff number

$$a_n > 0$$

is defined for $n \geq 1$ by the relation

$$\frac{2}{\pi} \int_0^1 \frac{a_n t Q'(a_n t)}{\sqrt{1-t^2}}\, dt = n.$$

Existence and uniqueness of these numbers is established in the monographs [51], [65], [80], but goes back to earlier work of Mhaskar, Saff, and Rakhmanov. Let P be any polynomial of degree at most n, then one illustration of the Mhaskar-Rakhmanov-Saff number's role is the Mhaskar-Saff identity:

$$||PW||_{\mathbb{R}} = ||PW||_{[-a_n,a_n]},$$

where $\|\cdot\|$ is the supremum norm. For a Freud weight $W(x) = e^{-c|x|^\lambda}$, where $c > 0$ and $\lambda > 1$ are constants, it is known that the Mhaskar-Rakhmanov-Saff number is given by

$$a_n = \gamma_\lambda^{\frac{1}{\lambda}} c^{-\frac{1}{\lambda}} n^{\frac{1}{\lambda}}.$$

See [80, p. 308] for further details. We define the Mhaskar-Rakhmanov-Saff interval $\Delta_n := [-a_n, a_n]$. The linear transformation

$$L_n(x) = \frac{x}{a_n},\ x \in \mathbb{R},$$

maps Δ_n onto $[-1,1]$. Its inverse is

$$L_n^{[-1]}(s) = a_n s,\ s \in \mathbb{R}.$$

For $\varepsilon \in (0,1)$, we let

$$J_n(\varepsilon) = L_n^{[-1]}[-1+\varepsilon, 1-\varepsilon] = (1-\varepsilon)[-a_n, a_n].$$

Then the equilibrium density is defined as

$$\sigma_n(x) = \frac{\sqrt{(a_n+x)(a_n-x)}}{\pi^2} \int_{-a_n}^{a_n} \frac{Q'(s)-Q'(x)}{s-x} \frac{ds}{\sqrt{(a_n+s)(a_n-s)}},\ x \in \Delta_n.$$

The equilibrium density satisfies [51, p. 41]:

$$\int_{-a_n}^{a_n} \log \frac{1}{|x-s|} \sigma_n(s)\, ds + Q(x) = C,\ x \in \Delta_n,$$

and

$$\int_{-a_n}^{a_n} \log \frac{1}{|x-s|} \sigma_n(s)\, ds + Q(x) \geq C,\ x \in \mathbb{R}.$$

Note that the measure $\sigma_n(x)dx$ has total mass n:

$$\int_{-a_n}^{a_n} \sigma_n(x)\, dx = n.$$

We also define the normalized version of σ_n as follows:

$$\sigma_n^*(s) := \frac{a_n}{n}\sigma_n(a_n s), \quad s \in [-1,1].$$

Note that

$$\int_{-1}^{1} \sigma_n^*(s)\, ds = 1.$$

For details on σ_n, one should consult the book [51] by Levin and Lubinsky.

3.3 Proofs

Lemma 3.3.1 *For a Freud weight $W(x) = e^{-c|x|^\lambda}$, where $c > 0$ and $\lambda > 1$ are constants, the normalized equilibrium density satisfies*

$$\sigma_n^*(s)\, ds = d\mu_w(s) \text{ for all } n \in \mathbb{N}, \quad s \in [-1,1].$$

That is,

$$\sigma_n^*(s) = \frac{\lambda}{\pi}\int_{|s|}^{1} \frac{y^{\lambda-1}}{\sqrt{y^2 - s^2}}\, dy \text{ for all } n \in \mathbb{N}, \quad s \in [-1,1].$$

Proof. Recall that σ_n satisfies

$$\int_{-a_n}^{a_n} \log\frac{1}{|x-y|}\sigma_n(y)\, dy + c\,|x|^\lambda = C_1, \quad x \in [-a_n, a_n],$$

and

$$\int_{-a_n}^{a_n} \log\frac{1}{|x-y|}\sigma_n(y)\, dy + c\,|x|^\lambda \geq C_1, \quad x \in \mathbb{R},$$

where $a_n = \gamma_\lambda^{\frac{1}{\lambda}} c^{-\frac{1}{\lambda}} n^{\frac{1}{\lambda}}$. The changes of variables $x = a_n s$ and $y = a_n t$ reduce the above relations to

$$\int_{-1}^{1} \log\frac{1}{|s-t|}\sigma_n^*(t)\, dt + \gamma_\lambda\,|s|^\lambda = C_2,\ s \in [-1,1],$$

and

$$\int_{-1}^{1} \log\frac{1}{|s-t|}\sigma_n^*(t)\, dt + \gamma_\lambda\,|s|^\lambda \geq C_2,\ s \in \mathbb{R}.$$

Invoking Theorem 3.1 of [80, p. 43], we deduce that

$$\sigma_n^*(s)\,ds = d\mu_w(s) \text{ for all } n \in \mathbb{N},\ s \in [-1,1].$$

On the other hand, recalling that

$$d\mu_w(s) = \left(\frac{\lambda}{\pi}\int_{|s|}^{1} \frac{y^{\lambda-1}}{\sqrt{y^2-s^2}}\,dy\right) ds,\ s \in [-1,1],$$

we obtain that

$$\sigma_n^*(s) = \frac{\lambda}{\pi}\int_{|s|}^{1} \frac{y^{\lambda-1}}{\sqrt{y^2-s^2}}\,dy \text{ for all } n \in \mathbb{N},\ s \in [-1,1].$$

□

Proof of Theorem 3.1.2. In this case, $W(x) = e^{-c|x|^\lambda}, x \in \mathbb{R}$, where $c > 0$ and $\lambda > 1$ are constants. The strategy is to apply Theorem 1.6 of [52]. It states that for all $r, s \geq 0$ and any $\varepsilon \in (0,1)$, we have

$$\frac{W^2(x)K_n^{(r,s)}(x,x)}{(\sigma_n(x))^{r+s+1}} = \sum_{j=0}^{r}\begin{pmatrix} r \\ j \end{pmatrix}\sum_{k=0}^{s}\begin{pmatrix} s \\ k \end{pmatrix}\tau_{j,k}\pi^{j+k}\left(\frac{Q'(x)}{\sigma_n(x)}\right)^{r+s-j-k} + o(1) \quad \text{as } n \to \infty, \tag{3.3.1}$$

uniformly for $x \in J_n(\varepsilon)$, where

$$\tau_{j,k} = \begin{cases} 0, & j+k \text{ odd}, \\ (-1)^{(j-k)/2}\frac{1}{j+k+1}, & j+k \text{ even}. \end{cases}$$

That is, uniformly in $x \in J_{n+1}(\varepsilon)$,

$$\frac{W^2(x)K_{n+1}^{(0,0)}(x,x)}{\sigma_{n+1}(x)} = 1 + o(1) \quad \text{as } n \to \infty,$$

$$\frac{W^2(x)K_{n+1}^{(0,1)}(x,x)}{(\sigma_{n+1}(x))^2} = \frac{Q'(x)}{\sigma_{n+1}(x)} + o(1) \quad \text{as } n \to \infty,$$

and

$$\frac{W^2(x)K_{n+1}^{(1,1)}(x,x)}{(\sigma_{n+1}(x))^3} = \left(\frac{Q'(x)}{\sigma_{n+1}(x)}\right)^2 + \frac{\pi^2}{3} + o(1) \quad \text{as } n \to \infty.$$

Notice that applying Proposition 2.1.1 with $g_j = p_j$ gives us that

$$A(x) = K_{n+1}(x,x), \quad B(x) = K_{n+1}^{(0,1)}(x,x) \quad \text{and} \quad C(x) = K_{n+1}^{(1,1)}(x,x).$$

Applying (2.1.2), we obtain that

$$\frac{1}{n}\mathbb{E}\left[N_n\left([l,q]\right)\right] = \frac{1}{\pi}\int_l^q \sqrt{\frac{K_{n+1}^{(1,1)}(x,x)}{K_{n+1}^{(0,0)}(x,x)} - \left(\frac{K_{n+1}^{(0,1)}(x,x)}{K_{n+1}^{(0,0)}(x,x)}\right)^2}\frac{1}{n}dx \tag{3.3.2}$$

for any closed interval $[l,q] \subset J_{n+1}(\varepsilon)$ (l, q may depend on n). Now, uniformly for $x \in J_{n+1}(\varepsilon)$,

$$\frac{K_{n+1}^{(1,1)}(x,x)}{K_{n+1}^{(0,0)}(x,x)} = \sigma_{n+1}^2(x)\left(\left(\frac{Q'(x)}{\sigma_{n+1}(x)}\right)^2 + \frac{\pi^2}{3} + o(1)\right)(1+o(1))^{-1}$$

and

$$\begin{aligned}\left(\frac{K_{n+1}^{(0,1)}(x,x)}{K_{n+1}^{(0,0)}(x,x)}\right)^2 &= \sigma_{n+1}^2(x)\left(\frac{Q'(x)}{\sigma_{n+1}(x)} + o(1)\right)^2(1+o(1))^{-2} \\ &= \sigma_{n+1}^2(x)\left(\left(\frac{Q'(x)}{\sigma_{n+1}(x)}\right)^2 + \frac{Q'(x)}{\sigma_{n+1}(x)}o(1) + o(1)\right)(1+o(1))^{-1}.\end{aligned}$$

We thus obtain that

$$\begin{aligned}&\frac{1}{n}\mathbb{E}\left[N_n([l,q])\right] \\ &= \frac{1}{\pi}\int_l^q \frac{1}{n}\sqrt{\sigma_{n+1}^2(x)\frac{\left(\frac{Q'(x)}{\sigma_{n+1}(x)}\right)^2 + \frac{\pi^2}{3} + o(1)}{1+o(1)} - \sigma_{n+1}^2(x)\frac{\left(\frac{Q'(x)}{\sigma_{n+1}(x)}\right)^2 + \frac{Q'(x)}{\sigma_{n+1}(x)}o(1) + o(1)}{1+o(1)}}\,dx \\ &= \frac{1}{\pi}\int_l^q \frac{\sigma_{n+1}(x)}{n}\sqrt{\frac{\pi^2}{3} + \left(\frac{Q'(x)}{\sigma_{n+1}(x)}\right)^2 o(1) + \frac{Q'(x)}{\sigma_{n+1}(x)}o(1) + o(1)}\,dx \quad \text{as } n \to \infty.\end{aligned}$$

Note that the number $N_n(E)$ of real zeros of $P_n(x)$ in E equals the number $N_n^*(E^*)$ of real zeros of $P_n^*(s)$ in $E^* := E/a_{n+1}$, since L_{n+1} is a bijection. Since $[a,b] \subset (-1,1)$ is a closed interval, we have that $[a_{n+1}a, a_{n+1}b] \subset J_{n+1}(\varepsilon) = (1-\varepsilon)[-a_{n+1}, a_{n+1}]$ provided $\max\{|a|,|b|\} \le 1-\varepsilon$ for some constant $\varepsilon \in (0,1)$. Hence

$$\begin{aligned}\frac{1}{n}\mathbb{E}\left[N_n^*([a,b])\right] &= \frac{1}{n}\mathbb{E}\left[N_n([a_{n+1}a, a_{n+1}b])\right] \\ &= \frac{1}{\pi}\int_{a_{n+1}a}^{a_{n+1}b}\frac{\sigma_{n+1}(x)}{n}\sqrt{\frac{\pi^2}{3} + \left(\frac{Q'(x)}{\sigma_{n+1}(x)}\right)^2 o(1) + \frac{Q'(x)}{\sigma_{n+1}(x)}o(1) + o(1)}\,dx \\ &= \frac{n+1}{\pi n}\int_a^b \sigma_{n+1}^*(s)\sqrt{\frac{\pi^2}{3} + \left(\left(\frac{Q'(a_{n+1}s)}{\sigma_{n+1}(a_{n+1}s)}\right)^2 + \frac{Q'(a_{n+1}s)}{\sigma_{n+1}(a_{n+1}s)} + 1\right)o(1)}\,ds.\end{aligned} \tag{3.3.3}$$

We show that $|Q'(x)/\sigma_{n+1}(x)| \le C$ on $x \in [a_{n+1}a, a_{n+1}b]$ for some constant $C > 0$ independent of n. Recall that $[a_{n+1}a, a_{n+1}b] \subset J_{n+1}(\varepsilon) = (1-\varepsilon)[-a_{n+1}, a_{n+1}]$, for some $\varepsilon \in (0,1)$. It is clear that $Q'(x)/\sigma_{n+1}(x)$ is an odd function of $x \in [a_{n+1}, 0) \cup (0, a_{n+1}]$, so that we only need to consider the interval $(0, a_{n+1}(1-\varepsilon)]$. First note that for $x \in (0, a_{n+1})$,

$$\frac{Q'(x)}{\sigma_{n+1}(x)} = c^{1-1/\lambda}\gamma_\lambda^{1/\lambda}\pi \frac{x^{\lambda-1}}{(n+1)^{1-1/\lambda}} \left(\int_{x/a_{n+1}}^{1} \frac{y^{\lambda-1}}{\sqrt{y^2 - x^2/a_{n+1}^2}}\, dy \right)^{-1}.$$

Since $\lambda > 1$, we can estimate the integral in the above formula for $x \in (0, a_{n+1}(1-\varepsilon)]$ as follows:

$$\int_{x/a_{n+1}}^{1} \frac{y^{\lambda-1}}{\sqrt{y^2 - x^2/a_{n+1}^2}}\, dy \ge \int_{x/a_{n+1}}^{1} \frac{y^{\lambda-1}}{\sqrt{y^2}}\, dy = \frac{1-(x/a_{n+1})^{\lambda-1}}{\lambda-1}.$$

Using this estimate, we see that for $x \in (0, a_{n+1}(1-\varepsilon)]$,

$$\begin{aligned}
\left| \frac{Q'(x)}{\sigma_{n+1}(x)} \right| &\le c^{1-1/\lambda}\gamma_\lambda^{1/\lambda}\pi \frac{x^{\lambda-1}}{(n+1)^{1-1/\lambda}} \frac{\lambda-1}{1-(x/a_{n+1})^{\lambda-1}} \\
&\le c^{1-1/\lambda}\gamma_\lambda^{1/\lambda}\pi \frac{(a_{n+1}(1-\varepsilon))^{\lambda-1}}{(n+1)^{1-1/\lambda}} \frac{\lambda-1}{1-(1-\varepsilon)^{\lambda-1}} \\
&= \frac{\gamma_\lambda \pi(\lambda-1)(1-\varepsilon)^{\lambda-1}}{1-(1-\varepsilon)^{\lambda-1}} =: C.
\end{aligned}$$

Applying (3.3.3) and Lemma 3.3.1, we obtain that

$$\begin{aligned}
\frac{1}{n}\mathbb{E}\left[N_n^*([a,b])\right] &= \frac{1}{\pi}\left(1+\frac{1}{n}\right) \int_a^b \sigma_{n+1}^*(s) \sqrt{\frac{\pi^2}{3} + o(1)}\, ds \\
&= \frac{1+o(1)}{\sqrt{3}} \int_a^b \sigma_{n+1}^*(s)\, ds \\
&= \frac{1+o(1)}{\sqrt{3}} \int_a^b d\mu_w(s).
\end{aligned}$$

To complete the proof, we pass to the limit as $n \to \infty$:

$$\lim_{n\to\infty} \frac{1}{n}\mathbb{E}\left[N_n^*\left([a,b]\right)\right] = \frac{1}{\sqrt{3}} \int_a^b d\mu_w(s) = \frac{1}{\sqrt{3}}\mu_w\left([a,b]\right).$$

□

Proof of Theorem 3.1.3. Following [80], we call a sequence of monic polynomials $\{Q_n\}_{n=1}^{\infty}$, with $\deg(Q_n) = n$, asymptotically extremal with respect to the weight w if it satisfies

$$\lim_{n\to\infty} \|w^n Q_n\|_{\mathbb{R}}^{1/n} = e^{-F_w},$$

where $\|\cdot\|_{\mathbb{R}}$ is the supremum norm on $\mathbb{R}$ and $F_w = \log 2 + 1/\lambda$ is the modified Robin constant corresponding to w, see [80, p. 240]. Theorem 4.2 of [80, p. 170] states that any sequence of such asymptotically extremal monic polynomials have their zeros distributed according to the measure μ_w. Namely, the normalized zero counting measures of Q_n converge weakly to μ_w. We show that the monic polynomials

$$Q_n^*(x) := P_n^*(x)/(c_n \gamma_n a_n^n), \quad n \in \mathbb{N},$$

are asymptotically extremal in this sense with probability one, so that the result of Theorem 3.1.3 follows.

Using orthogonality, we obtain for polynomials defined in (3.0.1) that

$$\int_{-\infty}^{\infty} |P_n(x)|^2 W^2(x)\, dx = \sum_{k=0}^{n} |c_k|^2.$$

Hence

$$\max_{0\le k\le n} |c_k| \le \left(\int_{-\infty}^{\infty} |P_n(x)|^2 W^2(x)\, dx\right)^{1/2} \le (n+1) \max_{0\le k\le n} |c_k|.$$

Lemma 4.2 of [69] (see (4.6) there) implies that

$$\lim_{n\to\infty} \left(\int_{-\infty}^{\infty} |P_n(x)|^2 W^2(x)\, dx\right)^{1/(2n)} = \lim_{n\to\infty} \left(\max_{0\le k\le n} |c_k|\right)^{1/n} = 1$$

with probability one. Applying the Nikolskii-type inequalities of Theorem 6.1 and Theorem 6.4 from [67], we obtain that the same holds for the supremum norm:

$$\lim_{n\to\infty} \|w^n P_n^*\|_{\mathbb{R}}^{1/n} = \lim_{n\to\infty} \|P_n W\|_{\mathbb{R}}^{1/n} = 1$$

with probability one. Recall that the leading coefficients of orthonormal polynomials p_n satisfy

$$\lim_{n\to\infty} \gamma_n^{1/n}\, n^{1/\lambda} = 2c^{1/\lambda} \gamma_\lambda^{-1/\lambda} e^{1/\lambda}$$

by Theorem 1.2 of [80, p. 362]. We also use below that $\lim_{n\to\infty} |c_n|^{1/n} = 1$ with probability one by Lemma 4.2 of [69]. It follows that

$$\lim_{n\to\infty} \|w^n Q_n^*\|_{\mathbb{R}}^{1/n} = \lim_{n\to\infty} \|w^n P_n^*\|_{\mathbb{R}}^{1/n} \lim_{n\to\infty} |c_n\gamma_n|^{-1/n} |a_n|^{-1} = \lim_{n\to\infty} \left(\gamma_n^{1/n} n^{1/\lambda} c^{-1/\lambda} \gamma_\lambda^{1/\lambda}\right)^{-1}$$
$$= \left(2c^{1/\lambda}\gamma_\lambda^{-1/\lambda} e^{1/\lambda} c^{-1/\lambda}\gamma_\lambda^{1/\lambda}\right)^{-1} = e^{-(\log 2 + 1/\lambda)} = e^{-F_w}.$$

□

Proof of Corollary 3.1.1. Consider the normalized zero counting measure $\tau_n = \frac{1}{n}\sum_{k=1}^n \delta_{z_k}$ for the scaled polynomial $P_n^*(s)$ defined by (3.1.2), where $\{z_k\}_{k=1}^n$ are the zeros of that polynomial, and δ_z denotes the unit point mass at z. Theorem 3.1.3 implies that measures τ_n converge weakly to μ_w with probability one. Since $\mu_w(\partial E) = 0$, we obtain that $\tau_n|_E$ converges weakly to $\mu_w|_E$ with probability one by Theorem 0.5′ of [49] and Theorem 2.1 of [8]. In particular, we have that the random variables $\tau_n(E)$ converge to $\mu_w(E)$ with probability one. Hence this convergence holds in L^p sense by the Dominated Convergence Theorem, as $\tau_n(E)$ are uniformly bounded by 1, see Chapter 5 of [31]. It follows that

$$\lim_{n\to\infty} \mathbb{E}[|\tau_n(E) - \mu_w(E)|] = 0$$

for any compact set E such that $\mu_w(\partial E) = 0$, and

$$|\mathbb{E}[\tau_n(E) - \mu_w(E)]| \le \mathbb{E}[|\tau_n(E) - \mu_w(E)|] \to 0 \quad \text{as } n \to \infty.$$

But $\mathbb{E}[\tau_n(E)] = \mathbb{E}[N_n^*(E)]/n$ and $\mathbb{E}[\mu_w(E)] = \mu_w(E)$, which immediately gives (3.1.5).

□

Proof of Theorem 3.1.1. Theorem 3.1.2 gives that

$$\lim_{n\to\infty} \frac{1}{n}\mathbb{E}\left[N_n^*\left([a,b]\right)\right] = \frac{1}{\sqrt{3}}\mu_w([a,b])$$

for any interval $[a,b] \subset (-1,1)$. Note that both $\mathbb{E}\left[N_n^*\left(H\right)\right]$ and $\mu_w(H)$ are additive functions of the set H. Moreover, they both vanish when H is a single point by (3.1.5) and the absolute continuity of μ_w with respect to Lebesgue measure on $S_w = [-1, 1]$. Hence (3.1.5) gives that

$$\lim_{n\to\infty} \frac{1}{n} \mathbb{E}\left[N_n^* \left(\mathbb{R} \setminus (-1, 1)\right)\right] = \mu_w(\mathbb{R} \setminus (-1, 1)) = 0.$$

It now follows that

$$\lim_{n\to\infty} \frac{1}{n} \mathbb{E}[N_n^*(\mathbb{R})] = \frac{1}{\sqrt{3}} \mu_w((-1, 1)) = \frac{1}{\sqrt{3}}.$$

To complete the proof, observe that $N_n^*(\mathbb{R}) = N_n(\mathbb{R})$, so that $\mathbb{E}[N_n^*(\mathbb{R})] = \mathbb{E}[N_n(\mathbb{R})]$, since L_{n+1} is a bijection for each fixed n. Therefore (3.1.1) is proved. $\square$

CHAPTER 4

Random Orthogonal Polynomials for Exponential Weights

The chapter is based on the joint work with D. S. Lubinsky and I. E. Pritsker [60]. We studied Freud weights whose support is $\mathbb{R}$ in Chapter 3. In this chapter, we will study general weights with unbounded support. Let $W = e^{-Q}$, where $Q : \mathbb{R} \longrightarrow [0, \infty)$ is continuous, and assume that all moments

$$\int_{\mathbb{R}} x^j W^2(x)\, dx, \;\; j = 0, 1, 2, \ldots,$$

are finite. For $n \geq 0$, let

$$p_n(x) = p_n\left(W^2, x\right) = \gamma_n x^n + \ldots$$

denote the nth orthonormal polynomial with $\gamma_n > 0$, so that

$$\int p_n p_m W^2 = \delta_{mn}.$$

Using the orthonormal polynomials $\{p_j\}_{j=0}^{\infty}$ as the basis, we consider the ensemble of random polynomials of the form

$$P_n(x) = \sum_{j=0}^{n} c_j p_j(x), \quad n \in \mathbb{N}, \tag{4.0.1}$$

where the coefficients $c_0, c_1, \ldots, c_n$ are i.i.d. random variables.

We will use the weight class $\mathcal{F}(C^2)$ from [51].

Definition 4.0.1 *Let $W = e^{-Q}$, where $Q : \mathbb{R} \to [0, \infty)$ satisfies the following conditions:*

(a) Q' is continuous in $\mathbb{R}$ and $Q(0) = 0$.

(b) Q' *is non-decreasing in* $\mathbb{R}$*, and* Q'' *exists in* $\mathbb{R} \setminus \{0\}$.

(c)

$$\lim_{|t|\to\infty} Q(t) = \infty.$$

(d) The function

$$T(t) = \frac{tQ'(t)}{Q(t)},\ t \neq 0,$$

is quasi-increasing in $(0, \infty)$*, in the sense that for some* $C_1 > 0$,

$$0 < x < y \Rightarrow T(x) \leq C_1 T(y).$$

We assume an analogous restriction for $y < x < 0$*. In addition, we assume that for some* $\Lambda > 1$,

$$T(t) \geq \Lambda \text{ in } \mathbb{R} \setminus \{0\}.$$

(e) There exists $C_2 > 0$ *such that*

$$\frac{Q''(x)}{|Q'(x)|} \leq C_2 \frac{|Q'(x)|}{Q(x)},\ x \in \mathbb{R} \setminus \{0\}.$$

Then we write $W \in \mathcal{F}(C^2)$.

Note that Freud weights we considered in Chapter 3 are in this weight class.

4.1 The expected number of real zeros

Theorem 4.1.1 *Let* $W = e^{-Q} \in \mathcal{F}(C^2)$*, where* Q *is even. If the function* T *in the definition of* $\mathcal{F}(C^2)$ *satisfies*

$$\lim_{x\to\infty} T(x) = \alpha \in (1, \infty], \tag{4.1.1}$$

then the expected number of real zeros of random orthogonal polynomials (4.0.1) *with mean-zero Gaussian coefficients satisfies*

$$\lim_{n\to\infty} \frac{1}{n}\mathbb{E}[N_n(\mathbb{R})] = \frac{1}{\sqrt{3}}. \tag{4.1.2}$$

Theorem 4.1.1 is a combination of Theorem 4.1.2 and Corollary 4.1.1 given below. Define the Ullman distribution μ_α for $0 < \alpha < \infty$ by

$$\mu_\alpha'(x) = \frac{\alpha}{\pi} \int_{|x|}^{1} \frac{t^{\alpha-1}}{\sqrt{t^2 - x^2}}\, dt, \quad x \in [-1, 1],$$

and for $\alpha = \infty$, the arcsine distribution μ_∞ by

$$\mu_\infty'(x) = \frac{1}{\pi\sqrt{1 - x^2}}, \quad x \in [-1, 1],$$

see [80] and [51]. We use the contracted version of P_n:

$$P_n^*(s) := P_n(a_n s), \quad n \in \mathbb{N}, \tag{4.1.3}$$

where a_n is the Mhaskar-Rakhmanov-Saff number associated with the weight W, see [51], [65], [80] and the next section below.

For any set $E \subset \mathbb{C}$, $N_n^*(E)$ denotes the number of (complex) zeros of a random polynomial $P_n^*(s)$ located in E. The expected number of (complex) zeros of $P_n^*(s)$ in E is denoted by $\mathbb{E}[N_n^*(E)]$ with $\mathbb{E}[N_n^*([a, b])]$ being the expected number of (real) zeros of $P_n^*(s)$ in $[a, b] \subset \mathbb{R}$. We now state the local result on the asymptotic of $\mathbb{E}[N_n^*([a, b])]$ for intervals $[a, b] \subset (-1, 1)$.

Theorem 4.1.2 *Let $W = e^{-Q} \in \mathcal{F}(C^2)$, where Q is even. Assume that the function T in the definition of $\mathcal{F}(C^2)$ satisfies* (4.1.1). *If $[a, b] \subset (-1, 1)$ is any closed interval, then*

$$\lim_{n\to\infty} \frac{1}{n} \mathbb{E}\left[N_n^*([a, b])\right] = \frac{1}{\sqrt{3}} \mu_\alpha([a, b]). \tag{4.1.4}$$

We will establish a generalization of Theorem 4.1.2 for non-even weights in the next section. Define the normalized zero counting measure

$$\tau_n = \frac{1}{n} \sum_{k=1}^{n} \delta_{z_k}$$

for the scaled polynomial $P_n^*(s)$ defined by (4.1.3), where $\{z_k\}_{k=1}^n$ are its zeros, and δ_z denotes the unit point mass at z. We determine the weak limit of τ_n for random polynomials with quite general random coefficients $\{c_j\}_{j=0}^\infty$.

Theorem 4.1.3 *Let the coefficients* $\{c_j\}_{j=0}^{\infty}$ *of random orthogonal polynomials* (4.0.1) *be complex i.i.d. random variables such that* $\mathbb{E}[|\log|c_0||] < \infty$. *If* $W = e^{-Q} \in \mathcal{F}(C^2)$, *where* Q *is even, and if the function* T *in the definition of* $\mathcal{F}(C^2)$ *satisfies* (4.1.1), *then the normalized zero counting measures* τ_n *for the scaled polynomials* $P_n^*(s)$ *converge weakly to* μ_α *with probability one.*

Theorem 4.1.3 permits us to find asymptotics for the expected number of zeros in various sets. In particular, we need the following corollary for the proof of Theorem 4.1.1.

Corollary 4.1.1 *Suppose that the assumptions of Theorem 4.1.3 hold. If* $E \subset \mathbb{C}$ *is any compact set satisfying* $\mu_\alpha(\partial E) = 0$, *then*

$$\lim_{n\to\infty} \frac{1}{n}\mathbb{E}\left[N_n^*(E)\right] = \mu_\alpha(E). \tag{4.1.5}$$

It is of interest to relax conditions on random coefficients c_j, e.g., by considering probability distributions from the domain of attraction of the normal law as in [35, 36].

4.2 Potential theory with external fields

Let W be a continuous nonnegative weight function on $\mathbb{R}$ such that W is not identically zero and $\lim_{|x|\to\infty} |x|\, W(x) = 0$. Set $Q(x) := -\log W(x)$. The weighted equilibrium measure μ_W associated with the weight W is the unique probability measure with compact support $S_W =$ supp $\mu_W \subset \mathbb{R}$ that minimizes the energy functional

$$I[\nu] = \iint \log \frac{1}{|z-t|}\, d\nu(t)d\nu(z) + 2\int Q\, d\nu$$

amongst all probability measures ν with support on $\mathbb{R}$. It satisfies

$$\int \log \frac{1}{|z-t|}\, d\mu_W(t) + Q(z) = C, \quad z \in S_W,$$

and

$$\int \log \frac{1}{|z-t|}\, d\mu_W(t) + Q(z) \geq C, \quad z \in \mathbb{R},$$

where C is a constant.

For a weight function $W(x) = e^{-Q(x)}$, where Q is often assumed convex on $\mathbb{R}$, the Mhaskar-Rakhmanov-Saff numbers

$$a_{-n} < 0 < a_n$$

are defined for $n \geq 1$ by the relations

$$n = \frac{1}{\pi} \int_{a_{-n}}^{a_n} \frac{xQ'(x)}{\sqrt{(x - a_{-n})(a_n - x)}} \, dx$$

and

$$0 = \frac{1}{\pi} \int_{a_{-n}}^{a_n} \frac{Q'(x)}{\sqrt{(x - a_{-n})(a_n - x)}} \, dx.$$

We also let

$$\delta_n = \frac{1}{2}(a_n + |a_{-n}|) \text{ and } \beta_n = \frac{1}{2}(a_n + a_{-n}).$$

For even Q, $a_{-n} = -a_n$, and we may define a_n by

$$\frac{2}{\pi} \int_0^1 \frac{a_n t Q'(a_n t)}{\sqrt{1 - t^2}} \, dt = n. \tag{4.2.1}$$

Existence and uniqueness of these numbers are established in the monographs [51], [65], [80], but go back to earlier work of Mhaskar, Saff, and Rakhmanov. Let P be any polynomial of degree at most n, then one illustration of the Mhaskar-Rakhmanov-Saff numbers' role is the Mhaskar-Saff identity:

$$\|PW\|_{L^\infty(\mathbb{R})} = \|PW\|_{L^\infty([a_{-n}, a_n])}.$$

We define the Mhaskar-Rakhmanov-Saff interval Δ_n as $\Delta_n := [a_{-n}, a_n]$. The linear transformation

$$L_n(x) = \frac{x - \beta_n}{\delta_n}, \ x \in \mathbb{R},$$

maps Δ_n onto $[-1, 1]$. Its inverse is

$$L_n^{[-1]}(s) = \beta_n + \delta_n s, \ s \in \mathbb{R}.$$

For $\varepsilon \in (0,1)$, we let

$$J_n(\varepsilon) = L_n^{[-1]}[-1+\varepsilon, 1-\varepsilon] = [a_{-n} + \varepsilon\delta_n, a_n - \varepsilon\delta_n].$$

The equilibrium density is defined as

$$\sigma_n(x) = \frac{\sqrt{(x-a_{-n})(a_n-x)}}{\pi^2} \int_{a_{-n}}^{a_n} \frac{Q'(s)-Q'(x)}{s-x} \frac{ds}{\sqrt{(s-a_{-n})(a_n-s)}}, \; x \in \Delta_n.$$

It satisfies the following equilibrium equations [51, p. 41]:

$$\int_{a_{-n}}^{a_n} \log \frac{1}{|x-s|} \sigma_n(s)\, ds + Q(x) = C, \; x \in \Delta_n,$$

and

$$\int_{a_{-n}}^{a_n} \log \frac{1}{|x-s|} \sigma_n(s)\, ds + Q(x) \geq C, \; x \in \mathbb{R}.$$

Note that the measure $\sigma_n(x)\, dx$ has total mass n on Δ_n:

$$\int_{a_{-n}}^{a_n} \sigma_n(x)\, dx = n.$$

We also define the normalized version of σ_n as follows:

$$\sigma_n^*(s) := \frac{\delta_n}{n} \sigma_n(L_n^{[-1]}(s)), \quad s \in [-1,1].$$

Note that $\sigma_n^*(s)\, ds$ is a unit measure supported on $[-1,1]$:

$$\int_{-1}^{1} \sigma_n^*(s)\, ds = 1.$$

For details on σ_n and σ_n^*, one should consult the book [51].

In particular, the Ullman distribution μ'_α is the normalized equilibrium density for the standard Freud weight $w(x) = e^{-\gamma_\alpha |x|^\alpha}$ on $\mathbb{R}$, see Theorem 5.1 of [80, p. 240], where

$$\gamma_\alpha = \frac{\Gamma(\frac{\alpha}{2})\Gamma(\frac{1}{2})}{2\Gamma(\frac{\alpha}{2}+\frac{1}{2})},$$

An alternative formula for the Ullman distribution follows from that for σ_n above, namely,

$$\mu'_\alpha(x) = \frac{2\sqrt{1-x^2}}{\pi^2 B_\alpha} \int_0^1 \frac{t^\alpha - x^\alpha}{t^2-x^2} \frac{dt}{\sqrt{1-t^2}}, \quad x \in [-1,1], \tag{4.2.2}$$

where

$$B_\alpha = \frac{2}{\pi} \int_0^1 \frac{t^\alpha}{\sqrt{1-t^2}} \, dt.$$

For $n \geq 1$, we also define the square root factor

$$\rho_n(x) = \sqrt{(x - a_{-n})(a_n - x)}, \quad x \in \Delta_n. \tag{4.2.3}$$

For more detailed knowledge of potential theory with external fields, see [51] and [80].

4.3 Proofs

We start with a general result, our only one that allows non-even weights. In this more general setting, P_n^* is given by

$$P_n^*(s) = P_n\left(L_n^{[-1]}(s)\right),$$

rather than by (4.1.3).

Theorem 4.3.1 *If $W = e^{-Q} \in \mathcal{F}(C^2)$ and $[a, b] \subset (-1, 1)$ is any given closed interval, then as $n \to \infty$,*

$$\frac{1}{n} \mathbb{E}\left[N_n^*\left([a, b]\right)\right] = \frac{1 + o(1)}{\sqrt{3}} \int_a^b \sigma_{n+1}^*(y) \, dy.$$

Proof. The strategy is to apply Theorem 1.6 of [52]. It states that for all $r, s \geq 0$, and any $\varepsilon \in (0, 1)$, we have uniformly for $x \in J_n(\varepsilon)$ as $n \to \infty$,

$$\frac{W^2(x) K_n^{(r,s)}(x, x)}{(\sigma_n(x))^{r+s+1}} = \sum_{j=0}^{r} \begin{pmatrix} r \\ j \end{pmatrix} \sum_{k=0}^{s} \begin{pmatrix} s \\ k \end{pmatrix} \tau_{j,k} \pi^{j+k} \left(\frac{Q'(x)}{\sigma_n(x)}\right)^{r+s-j-k} + o(1),$$

where

$$\tau_{j,k} = \begin{cases} 0, & j + k \text{ odd}, \\ (-1)^{(j-k)/2} \frac{1}{j+k+1}, & j + k \text{ even}. \end{cases}$$

In particular, uniformly in $x \in J_{n+1}(\varepsilon)$,

$$\frac{W^2(x) K_{n+1}^{(0,0)}(x, x)}{\sigma_{n+1}(x)} = 1 + o(1),$$

$$\frac{W^2(x)K_{n+1}^{(0,1)}(x,x)}{(\sigma_{n+1}(x))^2} = \frac{Q'(x)}{\sigma_{n+1}(x)} + o(1),$$

and

$$\frac{W^2(x)K_{n+1}^{(1,1)}(x,x)}{(\sigma_{n+1}(x))^3} = \left(\frac{Q'(x)}{\sigma_{n+1}(x)}\right)^2 + \frac{\pi^2}{3} + o(1).$$

Next, from Proposition 2.1.1, for any closed interval $[l,q] \subset J_{n+1}(\varepsilon)$ (where l, q may depend on n),

$$\frac{1}{n}\mathbb{E}\left[N_n\left([l,q]\right)\right] = \frac{1}{n\pi}\int_l^q \sqrt{\frac{K_{n+1}^{(1,1)}(x,x)}{K_{n+1}^{(0,0)}(x,x)} - \left(\frac{K_{n+1}^{(0,1)}(x,x)}{K_{n+1}^{(0,0)}(x,x)}\right)^2}\,dx.$$

Substituting the above asymptotics, and cancelling, yields

$$\begin{aligned}&\frac{1}{n}\mathbb{E}\left[N_n([l,q])\right]\\ &= \frac{1}{n\pi}\int_l^q \sigma_{n+1}(x)\sqrt{\frac{\pi^2}{3} + \left(\frac{Q'(x)}{\sigma_{n+1}(x)}\right)^2 o(1) + \frac{Q'(x)}{\sigma_{n+1}(x)}o(1) + o(1)}\,dx \quad \text{as } n \to \infty.\end{aligned}$$

We note that [52, p. 87, Lemma 5.1(a),(d)] uniformly for $x \in J_{n+1}(\varepsilon)$,

$$\sigma_{n+1}(x) \geq C_1 \frac{n+1}{\delta_{n+1}}$$

and

$$|Q'(x)| \leq C_2 \frac{n+1}{\rho_{n+1}(x)},$$

so that

$$\left|\frac{Q'(x)}{\sigma_{n+1}(x)}\right| \leq C_3 \frac{\delta_{n+1}}{\rho_{n+1}(x)} \leq \frac{C_3}{\sqrt{\varepsilon(2-\varepsilon)}}, \quad x \in J_{n+1}(\varepsilon).$$

Thus, uniformly for all intervals $[l,q] \subset J_{n+1}(\varepsilon)$, as $n \to \infty$,

$$\frac{1}{n}\mathbb{E}\left[N_n([l,q])\right] = \frac{1}{n\pi}\int_l^q \sigma_{n+1}(x)\sqrt{\frac{\pi^2}{3} + o(1)}\,dx = \frac{1+o(1)}{n\sqrt{3}}\int_l^q \sigma_{n+1}(x)\,dx.$$

Note that the number $N_n(E)$ of real zeros of $P_n(x)$ in E equals the number $N_n^*(E^*)$ of real zeros of $P_n^*(s)$ in $E^* := L_n(E) = \{L_n(x) : x \in E\}$, since L_n is a bijection. We recall that a_n is increasing to $+\infty$ and a_{-n} is decreasing to $-\infty$ as $n \to \infty$. It is also known that

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = 1, \quad \lim_{n\to\infty}\frac{a_{-(n+1)}}{a_{-n}} = 1 \quad \text{and} \quad \lim_{n\to\infty}\frac{\delta_{n+1}}{\delta_n} = 1,$$

see Lemma 3.11(a) of [51, p. 81]. Hence we have

$$L_{n+1}\left(L_n^{[-1]}(s)\right) = L_{n+1}(\beta_n + \delta_n s) = \frac{\delta_n}{\delta_{n+1}} s + \frac{\beta_n - \beta_{n+1}}{\delta_{n+1}} \to s \quad \text{as } n \to \infty,$$

uniformly for s in compact subsets of $\mathbb{R}$. If $[a,b] \subset (-1,1)$, then for large $n \in \mathbb{N}$,

$$L_n^{[-1]}([a,b]) = [a_{-n} + \delta_n(1+a), a_n - \delta_n(1-b)] \subset J_{n+1}(\varepsilon),$$

provided $0 < \varepsilon < \min\{1+a, 1-b\}$. It follows that

$$\begin{aligned}
\frac{1}{n}\mathbb{E}\left[N_n^*([a,b])\right] &= \frac{1}{n}\mathbb{E}\left[N_n\left(L_n^{[-1]}([a,b])\right)\right] \\
&= \frac{1+o(1)}{(n+1)\sqrt{3}} \int_{L_n^{[-1]}(a)}^{L_n^{[-1]}(b)} \sigma_{n+1}(x)\,dx \\
&= \frac{1+o(1)}{\sqrt{3}} \int_{L_{n+1}\left(L_n^{[-1]}(a)\right)}^{L_{n+1}\left(L_n^{[-1]}(b)\right)} \sigma_{n+1}^*(s)\,ds \quad (\text{where } s = L_{n+1}(x)) \\
&= \frac{1+o(1)}{\sqrt{3}} \int_a^b \sigma_{n+1}^*(s)\,ds \quad \text{as } n \to \infty,
\end{aligned}$$

where we used on the last step that

$$\sigma_{n+1}^*(s) \le \frac{C}{\sqrt{1-s^2}}, \quad s \in (-1,1),$$

by Theorem 1.11(V) of [51, p. 18]. □

Lemma 4.3.1 *Let $W = e^{-Q} \in \mathcal{F}(C^2)$, where Q is even. Let $\alpha \in (1,\infty]$. If the function T in the definition of $\mathcal{F}(C^2)$ satisfies*

$$\lim_{x\to\infty} T(x) = \alpha,$$

then

$$\lim_{n\to\infty} \sigma_n^*(x) = \mu_\alpha'(x), \quad x \in (-1,1) \setminus \{0\}.$$

Remark 4.3.1 *An equivalent form of*

$$\lim_{x\to\infty} T(x) = \alpha \in (1,\infty)$$

is that uniformly for t in compact subsets of $(0,1]$,

$$\lim_{x\to\infty}\frac{Q'(xt)}{Q'(x)}=t^{\alpha-1}. \tag{4.3.1}$$

Indeed, if this last condition holds, then as $x\to\infty$,

$$\begin{aligned}T(x)^{-1}&=\frac{Q(x)}{xQ'(x)}=\frac{1}{xQ'(x)}\int_0^x Q'(u)\,du\\&=\int_0^1\frac{Q'(xt)}{Q'(x)}\,dt\to\int_0^1 t^{\alpha-1}\,dt=\frac{1}{\alpha}.\end{aligned}$$

Here we also used $0\le Q'(xt)/Q'(x)\le 1$ and dominated convergence. In the other direction, as $x\to\infty$,

$$\begin{aligned}\frac{Q'(xt)}{Q'(x)}&=\frac{T(xt)}{T(x)}\frac{Q(xt)}{tQ(x)}=\frac{T(xt)}{tT(x)}\exp\left(-\int_{xt}^x\frac{Q'(u)}{Q(u)}\,du\right)\\&=\frac{T(xt)}{tT(x)}\exp\left(-\int_{xt}^x\frac{T(u)}{u}\,du\right)\\&=\frac{T(xt)}{tT(x)}\exp\left(-\int_{xt}^x\frac{\alpha+o(1)}{u}\,du\right)\\&=\frac{1+o(1)}{t}\exp\left(-(\alpha+o(1))\log\frac{1}{t}\right)=t^{\alpha-1}(1+o(1)).\end{aligned}$$

Given any $\varepsilon\in(0,1)$, this holds uniformly for $t\in[\varepsilon,1]$.

Proof of Lemma 4.3.1. We prove the case $1<\alpha<\infty$ first:

From (4.2.1), as $n\to\infty$,

$$\begin{aligned}\frac{n}{a_nQ'(a_n)}&=\frac{2}{\pi}\int_0^1\frac{tQ'(a_nt)}{Q'(a_n)\sqrt{1-t^2}}\,dt\\&\to\frac{2}{\pi}\int_0^1\frac{t^\alpha}{\sqrt{1-t^2}}\,dt=B_\alpha.\end{aligned} \tag{4.3.2}$$

Indeed, the integrand converges pointwise, and because Q is convex, so

$$Q'(a_nt)/Q'(a_n)\le 1,$$

and we can apply Lebesgue's Dominated Convergence Theorem. In particular, for $n\ge 1$, and some $C_1>1$ independent of n,

$$C_1^{-1}n\le a_nQ'(a_n)\le C_1n. \tag{4.3.3}$$

Next, we know that for $x \in (0,1)$,

$$\sigma_n^*(x) = \frac{2\sqrt{1-x^2}}{\pi^2} \int_0^1 \frac{a_n t Q'(a_n t) - a_n x Q'(a_n x)}{n(t^2 - x^2)} \frac{dt}{\sqrt{1-t^2}}.$$

For $t \in (0,1) \setminus \{x\}$, we obtain from (4.3.1) and (4.3.2) that

$$\begin{aligned} &\lim_{n\to\infty} \frac{a_n t Q'(a_n t) - a_n x Q'(a_n x)}{n(t^2 - x^2)} \\ &= B_\alpha^{-1} \lim_{n\to\infty} \frac{a_n t Q'(a_n t) - a_n x Q'(a_n x)}{a_n Q'(a_n)(t^2 - x^2)} \\ &= B_\alpha^{-1} \frac{t^\alpha - x^\alpha}{t^2 - x^2}. \end{aligned}$$

We need a bound on the integrand so as to apply dominated convergence. First, $T(u)$ is bounded above. Next, for some ξ between t and x,

$$\begin{aligned} &\left| \frac{a_n t Q'(a_n t) - a_n x Q'(a_n x)}{n(t^2 - x^2)} \right| \\ &= \left| \frac{\frac{d}{du}(a_n u Q'(a_n u))|_{u=\xi}}{n(t+x)} \right| \\ &\leq \frac{a_n Q'(a_n \xi) + a_n^2 \xi Q''(a_n \xi)}{n(t+x)}. \end{aligned}$$

Here (4.3.3) gives (since Q' is increasing)

$$\frac{a_n Q'(a_n \xi)}{n(t+x)} \leq \frac{a_n Q'(a_n)}{n(t+x)} \leq \frac{C_1}{x}.$$

By definition of $\mathcal{F}(C^2)$ and boundedness of T, we have

$$0 \leq \frac{Q''(y)}{Q'(y)} \leq \frac{C_2 T(y)}{y} \leq \frac{C_3}{y}, \quad y > 0,$$

so that

$$\frac{a_n^2 \xi Q''(a_n \xi)}{n(t+x)} \leq C_3 \frac{a_n^2 \xi Q'(a_n \xi)}{a_n \xi n(t+x)} \leq C_3 \frac{a_n Q'(a_n \xi)}{n(t+x)} \leq \frac{C_4}{x}.$$

Thus, for all $t \in (0,1)$,

$$\left| \frac{a_n t Q'(a_n t) - a_n x Q'(a_n x)}{n(t^2 - x^2)} \right| \leq \frac{C_5}{x},$$

and we can apply dominated convergence to deduce that

$$\lim_{n\to\infty} \sigma_n^*(x) = \frac{2\sqrt{1-x^2}}{\pi^2 B_\alpha} \int_0^1 \frac{t^\alpha - x^\alpha}{t^2 - x^2} \frac{dt}{\sqrt{1-t^2}} = \mu_\alpha^{'}(x).$$

Next, we deal with the case $\alpha = \infty$:

Let $0 < r < s < 1$. We consider $x \in (0, r]$ and split

$$\begin{aligned} \sigma_n^*(x) &= \frac{2\sqrt{1-x^2}}{\pi^2} \left(\int_0^s + \int_s^1 \right) \frac{a_n t Q'(a_n t) - a_n x Q'(a_n x)}{n(t^2 - x^2)} \frac{dt}{\sqrt{1-t^2}} \\ &=: I_1 + I_2. \end{aligned} \tag{4.3.4}$$

We shall show that the main contribution to σ_n^* comes from I_2. Since the integrand in the integral defining σ_n^* is nonnegative, we have for $x \in (0, r]$ that

$$\begin{aligned} I_2 &= \frac{2\sqrt{1-x^2}}{\pi^2} \int_s^1 \frac{a_n t Q'(a_n t) - a_n x Q'(a_n x)}{n(t^2 - x^2)} \frac{dt}{\sqrt{1-t^2}} \\ &\le \frac{2\sqrt{1-x^2}}{\pi^2} \int_s^1 \frac{a_n t Q'(a_n t)}{n(t^2 - x^2)} \frac{dt}{\sqrt{1-t^2}} \\ &\le \frac{2\sqrt{1-x^2}}{\pi^2 (s^2 - x^2) n} \int_s^1 a_n t Q'(a_n t) \frac{dt}{\sqrt{1-t^2}} \\ &\le \frac{\sqrt{1-x^2}}{\pi (s^2 - x^2) n} \frac{2}{\pi} \int_0^1 a_n t Q'(a_n t) \frac{dt}{\sqrt{1-t^2}} \\ &= \frac{\sqrt{1-x^2}}{\pi (s^2 - x^2)}. \end{aligned} \tag{4.3.5}$$

Next, note that by the lower bound in (3.5) of [51, p. 64], for $t \in [0, r]$,

$$\begin{aligned} 0 &\le \frac{a_n t Q'(a_n t)}{a_n s Q'(a_n s)} \le \frac{a_n r Q'(a_n r)}{a_n s Q'(a_n s)} \le \frac{T(a_n r)}{T(a_n s)} \left(\frac{r}{s} \right)^{\max\{\Lambda, C_6 T(a_n r)\}} \\ &\le C_7 \left(\frac{r}{s} \right)^{C_8 T(a_n r)}, \end{aligned}$$

since T is quasi-increasing. Our hypothesis

$$\lim_{x\to\infty} T(x) = \infty$$

gives

$$\lim_{n\to\infty} \max_{t\in[0,r]} \frac{a_n t Q'(a_n t)}{a_n s Q'(a_n s)} = 0. \tag{4.3.6}$$

It also then follows easily from (4.2.1) that for each fixed $\tau \in (0,1)$,

$$\lim_{n\to\infty} \frac{a_n \tau Q'(a_n\tau)}{n} = 0. \tag{4.3.7}$$

Now uniformly for $x \in [0,r]$,

$$\begin{aligned} I_2 &\geq \frac{2\sqrt{1-x^2}}{\pi^2(1-x^2)} \int_s^1 \frac{a_n t Q'(a_n t) - a_n x Q'(a_n x)}{n} \frac{dt}{\sqrt{1-t^2}} \\ &\geq \frac{1}{\pi\sqrt{1-x^2}} \frac{2}{\pi n} \int_s^1 a_n t Q'(a_n t)(1+o(1)) \frac{dt}{\sqrt{1-t^2}} \\ &= \frac{1+o(1)}{\pi\sqrt{1-x^2}} \frac{2}{\pi n} \int_0^1 a_n t Q'(a_n t) \frac{dt}{\sqrt{1-t^2}} \\ &= \frac{1+o(1)}{\pi\sqrt{1-x^2}} \quad \text{as } n \to \infty, \end{aligned} \tag{4.3.8}$$

by (4.2.1) and using (4.3.6). Now we deal with I_1 - it clearly suffices to show only an upper bound. Let $s < \rho < 1$. By definition of the class $\mathcal{F}(C^2)$ and (4.3.7), we have that

$$\begin{aligned} I_1 &= \frac{2\sqrt{1-x^2}}{\pi^2} \int_0^s \frac{a_n t Q'(a_n t) - a_n x Q'(a_n x)}{n(t^2-x^2)} \frac{dt}{\sqrt{1-t^2}} \\ &\leq \frac{2\sqrt{1-x^2}}{\pi^2 n x} \max_{u\in[0,s]} \left| \frac{d}{du}(a_n u Q'(a_n u)) \right| \int_0^s \frac{dt}{\sqrt{1-t^2}} \\ &\leq \frac{C_9}{nx}[a_n Q'(a_n s) + \max_{u\in[0,s]} a_n^2 u Q''(a_n u)] \\ &\leq o(1) + \frac{C_9}{nx} \max_{u\in[0,s]} a_n Q'(a_n u) T(a_n u) \quad \text{as } n \to \infty. \end{aligned}$$

Using the fact that T is quasi-increasing and the lower bound in (3.5) of [51, p. 64], we continue this as

$$\begin{aligned} I_1 &\leq o(1) + \frac{C_9}{nx} a_n Q'(a_n s) T(a_n s) \\ &\leq o(1) + \frac{C_9}{nx} a_n Q'(a_n \rho) \frac{T(a_n s)}{T(a_n \rho)} \left(\frac{s}{\rho}\right)^{\max\{\Lambda, C_6 T(a_n s)\}-1} T(a_n s) \\ &\leq o(1) + \frac{C_9}{nx} a_n Q'(a_n \rho) \sup_{y\in[0,\infty)} \left(\frac{s}{\rho}\right)^{\max\{\Lambda, C_6 y\}-1} y = o(1) \text{ as } n \to \infty, \end{aligned}$$

by (4.3.7) and as $s/\rho < 1$. Together with the fact that $I_1 \geq 0$, and using (4.3.4), (4.3.5), (4.3.8), we have shown that for $x \in (0,r]$,

$$\frac{1}{\pi\sqrt{1-x^2}} \leq \liminf_{n\to\infty} \sigma_n^*(x) \leq \limsup_{n\to\infty} \sigma_n^*(x) \leq \frac{\sqrt{1-x^2}}{\pi(s^2-x^2)}.$$

As s is independent of r, we can let $s \to 1-$ to deduce that for $x \in (0, r]$,

$$\lim_{n\to\infty} \sigma_n^*(x) = \frac{1}{\pi\sqrt{1-x^2}} = \mu_\infty^{'}(x).$$

□

Proof of Theorem 4.1.2. We know from Theorem 4.3.1 that

$$\frac{1}{n}\mathbb{E}\left[N_n^*\left([a,b]\right)\right] = \frac{1+o(1)}{\sqrt{3}} \int_a^b \sigma_{n+1}^*(y)\, dy.$$

Lemma 4.3.1 gives for $1 < \alpha \leq \infty$ that

$$\lim_{n\to\infty} \sigma_{n+1}^*(y) = \mu_\alpha^{'}(y), \quad y \in (-1,1) \setminus \{0\}.$$

Next, by Theorem 1.11(V) of [51, p. 18],

$$\sigma_{n+1}^*(s) \leq \frac{C}{\sqrt{1-s^2}}, \quad s \in (-1,1).$$

Lebesgue's Dominated Convergence Theorem now implies that

$$\lim_{n\to\infty} \frac{1}{n}\mathbb{E}\left[N_n^*\left([a,b]\right)\right] = \frac{1}{\sqrt{3}} \int_a^b \lim_{n\to\infty} \sigma_{n+1}^*(y)\, dy = \frac{1}{\sqrt{3}}\mu_\alpha([a,b]).$$

□

Lemma 4.3.2 *If $W = e^{-Q} \in \mathcal{F}(C^2)$ then*

$$\lim_{n\to\infty} a_n^{1/n} = 1.$$

Proof. Lemma 3.5(c) of [51, p. 72] implies that there is a constant $C > 0$ such that

$$1 \leq \frac{a_n}{a_1} \leq Cn^{1/\Lambda} \text{ for all } n \geq 1,$$

which immediately gives the needed result. □

Lemma 4.3.3 *Let $W = e^{-Q} \in \mathcal{F}(C^2)$, where Q is even. If the coefficients $\{c_j\}_{j=0}^\infty$ of random orthogonal polynomials* (4.0.1) *are complex i.i.d. random variables such that $\mathbb{E}[|\log|c_0||] < \infty$, then*

$$\lim_{n\to\infty} \|P_n W\|_{L^\infty(\mathbb{R})}^{1/n} = 1 \textit{ with probability one.}$$

Proof. Using orthogonality, we obtain for polynomials defined in (4.0.1) that

$$\int_{-\infty}^{\infty} |P_n(x)|^2 W^2(x)\, dx = \sum_{j=0}^{n} |c_j|^2.$$

Hence

$$\max_{0\le j\le n} |c_j| \le \left(\int_{-\infty}^{\infty} |P_n(x)|^2 W^2(x)\, dx\right)^{1/2} \le (n+1) \max_{0\le j\le n} |c_j|.$$

Lemma 4.2 of [69] (see (4.6) there) implies that

$$\lim_{n\to\infty} \left(\int_{-\infty}^{\infty} |P_n(x)|^2 W^2(x)\, dx\right)^{1/(2n)} = \lim_{n\to\infty} \left(\max_{0\le j\le n} |c_j|\right)^{1/n} = 1$$

with probability one. That is,

$$\lim_{n\to\infty} \|P_n W\|_{L^2(\mathbb{R})}^{1/n} = 1 \text{ with probability one.} \tag{4.3.9}$$

We use the Nikolskii inequalities of Theorem 10.3 of [51, p. 295] stated as

$$\|P_n W\|_{L^\infty(\mathbb{R})} \le C_1 \left(\frac{n}{a_n}\right)^{1/2} (T(a_n))^{1/4} \|P_n W\|_{L^2(\mathbb{R})}$$

and

$$\|P_n W\|_{L^2(\mathbb{R})} \le C_2 a_n^{1/2} \|P_n W\|_{L^\infty(\mathbb{R})}.$$

Since $T(a_n) = O(n^2)$ by Lemma 3.7 of [51, p. 76], we obtain that

$$\frac{1}{C_2}\frac{1}{\sqrt{a_n}} \|P_n W\|_{L^2(\mathbb{R})} \le \|P_n W\|_{L^\infty(\mathbb{R})} \le C_3 n \|P_n W\|_{L^2(\mathbb{R})},$$

and the result follows by applying Lemma 4.3.2 and (4.3.9). □

Lemma 4.3.4 *Let $W = e^{-Q} \in \mathcal{F}(C^2)$, where Q is even. If the function T in the definition of $\mathcal{F}(C^2)$ satisfies*

$$\lim_{x\to\infty} T(x) = \infty,$$

then

$$\lim_{n\to\infty} \gamma_n^{1/n} a_n = 2,$$

where γ_n is the leading coefficient of the orthonormal polynomial $p_n(x)$ associated with the weight W^2.

Proof. Theorem 1.22 of [51, p. 25] gives

$$\gamma_n = \frac{1}{\sqrt{2\pi}} \left(\frac{a_n}{2}\right)^{-n-\frac{1}{2}} e^{\frac{1}{\pi}\int_{-a_n}^{a_n} \frac{Q(s)}{\sqrt{a_n^2-s^2}}\,ds} (1+o(1)) \text{ as } n \to \infty,$$

so that

$$\gamma_n^{1/n} a_n = 2a_n^{-\frac{1}{2n}} e^{\frac{1}{n\pi}\int_{-a_n}^{a_n} \frac{Q(s)}{\sqrt{a_n^2-s^2}}\,ds} (1+o(1)) \text{ as } n \to \infty. \tag{4.3.10}$$

Since Q is increasing on $(0,\infty)$, we have that

$$0 \le \lim_{n\to\infty} \frac{1}{n\pi} \int_{-a_n}^{a_n} \frac{Q(s)}{\sqrt{a_n^2-s^2}}\,ds \le \frac{Q(a_n)}{n\pi} \int_{-a_n}^{a_n} \frac{ds}{\sqrt{a_n^2-s^2}} = \frac{Q(a_n)}{n} \le \frac{C}{\sqrt{T(a_n)}} \to 0 \tag{4.3.11}$$

as $n \to \infty$, by Lemma 3.4 of [51, p. 69]. Thus

$$\lim_{n\to\infty} \frac{1}{n\pi} \int_{-a_n}^{a_n} \frac{Q(s)}{\sqrt{a_n^2-s^2}}\,ds = 0,$$

and (4.3.10) together with Lemma 4.3.2 imply the result. □

Lemma 4.3.5 *Let $W = e^{-Q} \in \mathcal{F}(C^2)$, where Q is even. If the function T in the definition of $\mathcal{F}(C^2)$ satisfies*

$$\lim_{x\to\infty} T(x) = \alpha \in (1,\infty),$$

then

$$\lim_{n\to\infty} \gamma_n^{1/n} a_n = 2e^{1/\alpha},$$

where γ_n is the leading coefficient of the orthonormal polynomial $p_n(x)$ associated with the weight W^2.

Proof. Considering Lemma 4.3.2 and (4.3.10), we only need to show

$$\lim_{n\to\infty} \frac{1}{n\pi} \int_{-a_n}^{a_n} \frac{Q(s)}{\sqrt{a_n^2-s^2}}\,ds = \lim_{n\to\infty} \frac{1}{n} \int_{-1}^{1} \frac{Q(a_n t)}{\pi\sqrt{1-t^2}}\,dt = 1/\alpha.$$

In terms of the function T, we can recast this as

$$\lim_{n\to\infty} \frac{1}{n} \int_{-1}^{1} \frac{1}{T(a_n t)} \frac{a_n t Q'(a_n t)}{\pi\sqrt{1-t^2}}\,dt = 1/\alpha.$$

Using our assumption that

$$\lim_{t\to\infty} T(t) = \alpha \in (1, \infty),$$

we have uniformly for $|t| \geq a_n^{-1/2}$, that $T(a_n t) = \alpha(1 + o(1))$, so as the integrand is non-negative,

$$\frac{1}{n}\int_{a_n^{-1/2}\leq|t|\leq 1} \frac{1}{T(a_n t)} \frac{a_n t Q'(a_n t)}{\pi\sqrt{1-t^2}}\, dt = \frac{1+o(1)}{\alpha}\frac{1}{n}\int_{a_n^{-1/2}\leq|t|\leq 1} \frac{a_n t Q'(a_n t)}{\pi\sqrt{1-t^2}}\, dt. \quad (4.3.12)$$

The integral over the remaining range is small: for $j = 0, 1$, using (4.3.2) and $\lim_{n\to\infty} a_n = \infty$,

$$\begin{aligned} 0 &\leq \frac{1}{n}\int_{|t|\leq a_n^{-1/2}} \frac{1}{T(a_n t)^j} \frac{a_n t Q'(a_n t)}{\pi\sqrt{1-t^2}}\, dt \\ &\leq \frac{1}{n}\frac{a_n^{1/2} Q'(a_n^{1/2})}{\Lambda^j \pi\sqrt{1-a_n^{-1}}} 2a_n^{-1/2} \leq C\frac{Q'(a_n^{1/2})}{n} \leq C\frac{Q'(a_n)}{n} = o(1). \end{aligned}$$

Thus (4.3.12) and (4.2.1) yield

$$\frac{1}{n}\int_{-1}^{1} \frac{1}{T(a_n t)} \frac{a_n t Q'(a_n t)}{\pi\sqrt{1-t^2}}\, dt = \frac{1+o(1)}{\alpha}\frac{1}{n}\int_{-1}^{1} \frac{a_n t Q'(a_n t)}{\pi\sqrt{1-t^2}}\, dt = \frac{1+o(1)}{\alpha}.$$

□

Proof of Theorem 4.1.3. We first deal with the case

$$\lim_{x\to\infty} T(x) = \infty,$$

and show that the normalized zero counting measures τ_n for the scaled polynomials $P_n^*(s)$ converge weakly to the arcsine distribution μ_∞ with probability one. Theorem 2.1 of [9, p. 310] states that if $\{M_n\}_{n=1}^\infty$ is any sequence of monic polynomials of degree $\deg(M_n) = n$ satisfying

$$\limsup_{n\to\infty} \|M_n\|_{L^\infty([-1,1])}^{1/n} \leq \frac{1}{2}, \quad (4.3.13)$$

then the normalized zero counting measures τ_n for the polynomials M_n converge weakly to μ_∞. Note that $1/2$ in the above equation is the logarithmic capacity of $[-1, 1]$, see Corollary 5.2.4 of [75, p. 134]. We show that the monic polynomials

$$M_n(x) := P_n^*(x)/(c_n \gamma_n a_n^n), \quad n \in \mathbb{N},$$

satisfy (4.3.13) with probability one, so that the result of Theorem 4.1.3 follows for $\alpha = \infty$. We know from Lemma 4.3.3 that

$$\limsup_{n\to\infty} \|P_n W\|_{L^\infty(\mathbb{R})}^{1/n} \le 1 \text{ with probability one.}$$

Using the contracted weight

$$w_n(s) := \sqrt[n]{W(a_n s)} = e^{-\frac{Q(a_n s)}{n}}, \quad s \in \mathbb{R},$$

and the properties of a_n [51, p. 4], we obtain that

$$\|P_n^* w_n^n\|_{L^\infty([-1,1])} = \|P_n W\|_{L^\infty([-a_n,a_n])} = \|P_n W\|_{L^\infty(\mathbb{R})}.$$

It follows that

$$\limsup_{n\to\infty} \|P_n^* w_n^n\|_{L^\infty([-1,1])}^{1/n} \le 1 \text{ with probability one.}$$

Since $\lim_{n\to\infty} Q(a_n)/n = 0$ (recall (4.3.11)), we have that

$$\limsup_{n\to\infty} \|P_n^*\|_{L^\infty([-1,1])}^{1/n} \le \limsup_{n\to\infty} \|P_n^* w_n^n\|_{L^\infty([-1,1])}^{1/n} e^{Q(a_n)/n} \le 1$$

with probability one. We use below that $\lim_{n\to\infty} \gamma_n^{1/n} a_n = 2$ by Lemma 4.3.4, and that $\lim_{n\to\infty} |c_n|^{1/n} = 1$ with probability one by Lemma 4.2 of [69]. This implies that

$$\begin{aligned}\limsup_{n\to\infty} \|M_n\|_{L^\infty([-1,1])}^{1/n} &= \limsup_{n\to\infty} \left\| \frac{P_n^*}{c_n \gamma_n a_n^n} \right\|_{L^\infty([-1,1])}^{1/n} \\ &= \limsup_{n\to\infty} \|P_n^*\|_{L^\infty([-1,1])}^{1/n} \frac{1}{|c_n|^{1/n}} \frac{1}{\gamma_n^{1/n} a_n} \le \frac{1}{2}\end{aligned}$$

with probability one.

Next, we prove the case

$$\lim_{x\to\infty} T(x) = \alpha \in (1, \infty).$$

Recall that the standard Freud weight with index α is given by

$$w(s) = e^{-\gamma_\alpha |s|^\alpha}, \quad s \in \mathbb{R},$$

where

$$\gamma_\alpha = \frac{\Gamma(\frac{\alpha}{2})\Gamma(\frac{1}{2})}{2\Gamma(\frac{\alpha}{2}+\frac{1}{2})} = \int_0^1 \frac{t^{\alpha-1}}{\sqrt{1-t^2}}\,dt$$

see [80, p. 239]. Since $\gamma_{\alpha+1} = B_\alpha \pi/2$, we apply $\Gamma(1/2) = \sqrt{\pi}$ and $\Gamma(t+1) = t\Gamma(t)$ to obtain that

$$\gamma_\alpha B_\alpha = \gamma_\alpha \frac{2\gamma_{\alpha+1}}{\pi} = \frac{2}{\pi}\frac{\Gamma(\frac{\alpha}{2})\Gamma(\frac{1}{2})}{2\Gamma(\frac{\alpha}{2}+\frac{1}{2})}\frac{\Gamma(\frac{\alpha+1}{2})\Gamma(\frac{1}{2})}{2\Gamma(\frac{\alpha+1}{2}+\frac{1}{2})} = \frac{1}{\alpha}.$$

Note that by [80, p. 240], $F_w = \log 2 + 1/\alpha$ is the modified Robin constant and $\mu_w = \mu_\alpha$ is the equilibrium measure corresponding to w. Following [80], we call a sequence of monic polynomials $\{M_n\}_{n=1}^\infty$, with $\deg(M_n) = n$, asymptotically extremal with respect to the weight w if it satisfies

$$\lim_{n\to\infty} \|w^n M_n\|_{L^\infty(\mathbb{R})}^{1/n} = e^{-F_w} = e^{-1/\alpha}/2. \tag{4.3.14}$$

Theorem 4.2 of [80, p. 170] states that asymptotically extremal monic polynomials have their zeros distributed according to the measure μ_w. Namely, the normalized zero counting measures of M_n converge weakly to $\mu_w = \mu_\alpha$. On the other hand, by Corollary 2.6 of [80, p. 157] and Theorem 5.1 of [80, p. 240],

$$\|w^n M_n\|_{L^\infty(\mathbb{R})} = \|w^n M_n\|_{L^\infty([-1,1])}.$$

Together with Theorem 3.6 of [80, p. 46], (4.3.14) is equivalent to

$$\limsup_{n\to\infty} \|w^n M_n\|_{L^\infty([-1,1])}^{1/n} \le e^{-F_w} = e^{-1/\alpha}/2.$$

We show that the monic polynomials

$$M_n(x) := P_n^*(x)/(c_n \gamma_n a_n^n), \quad n \in \mathbb{N},$$

are asymptotically extremal in this sense with probability one, so that the result of Theorem 4.1.3 follows. Note that

$$\lim_{n\to\infty} \|P_n W\|_{L^\infty(\mathbb{R})}^{1/n} = 1 \text{ with probability one}$$

by Lemma 4.3.3, and that

$$\|P_n^* w_n^n\|_{L^\infty([-1,1])} = \|P_n W\|_{L^\infty([-a_n,a_n])} = \|P_n W\|_{L^\infty(\mathbb{R})}$$

by [51, p. 4]. Hence

$$\limsup_{n\to\infty} \|P_n^* w_n^n\|_{L^\infty([-1,1])}^{1/n} \leq 1 \text{ with probability one.}$$

By Lemma 4.3.5, and since $\lim_{n\to\infty} |c_n|^{1/n} = 1$ with probability one by Lemma 4.2 of [69], it follows that

$$\begin{aligned}
\limsup_{n\to\infty} \|M_n w^n\|_{L^\infty([-1,1])}^{1/n} &= \limsup_{n\to\infty} \|P_n^* w^n\|_{L^\infty([-1,1])}^{1/n} \frac{1}{c_n^{1/n} \gamma_n^{1/n} a_n} \\
&= \frac{1}{2e^{1/\alpha}} \limsup_{n\to\infty} \|P_n^* w^n\|_{L^\infty([-1,1])}^{1/n} \\
&= e^{-F_w} \limsup_{n\to\infty} \|P_n^* w^n\|_{L^\infty([-1,1])}^{1/n} .
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\limsup_{n\to\infty} \|P_n^* w^n\|_{L^\infty([-1,1])}^{1/n} &\leq \limsup_{n\to\infty} \|P_n^* w_n^n\|_{L^\infty([-1,1])}^{1/n} \|w/w_n\|_{L^\infty([-1,1])} \\
&\leq \limsup_{n\to\infty} \|w/w_n\|_{L^\infty([-1,1])} .
\end{aligned}$$

Since w_n and w are both even, it remains to show that

$$\limsup_{n\to\infty} \|w/w_n\|_{L^\infty([0,1])} \leq 1.$$

Let $\varepsilon \in (0,1)$. For $x \in [\varepsilon, 1]$, (4.3.2) and then (4.3.1) give that

$$\begin{aligned}
\frac{Q(a_n s)}{n} &= \frac{1+o(1)}{B_\alpha} \int_0^s \frac{a_n Q'(a_n x)}{a_n Q'(a_n)}\, dx = \frac{1+o(1)}{B_\alpha} \int_0^s x^{\alpha-1}(1+o(1))\, dx \\
&= \frac{s^\alpha}{\alpha B_\alpha}(1+o(1)) = \gamma_\alpha s^\alpha (1+o(1)) \quad \text{as } n\to\infty.
\end{aligned}$$

This holds uniformly for $s \in [\varepsilon, 1]$ as (4.3.1) does. Hence

$$\|w/w_n\|_{L^\infty([\varepsilon,1])} = \sup_{s\in[\varepsilon,1]} \exp\left(\frac{Q(a_n s)}{n} - \gamma_\alpha s^\alpha\right) \to 1 \quad \text{as } n\to\infty.$$

Since Q is increasing, we also have that

$$\|w/w_n\|_{L^\infty([0,\varepsilon])} \leq \exp\left(\frac{Q(a_n \varepsilon)}{n}\right) \to \exp(\gamma_\alpha \varepsilon^\alpha).$$

We finish the proof by letting $\varepsilon \to 0$. □

Proof of Corollary 4.1.1. Consider the normalized zero counting measure

$$\tau_n = \frac{1}{n} \sum_{k=1}^{n} \delta_{z_k}$$

for the scaled polynomial $P_n^*(s)$ of (4.1.3), where $\{z_k\}_{k=1}^n$ are the zeros of that polynomial, and δ_z denotes the unit point mass at z. Theorem 4.1.3 implies that the measures τ_n converge weakly to μ_α with probability one. Since $\mu_\alpha(\partial E) = 0$, we obtain that $\tau_n|_E$ converges weakly to $\mu_\alpha|_E$ with probability one by Theorem 0.5′ of [49] and Theorem 2.1 of [8]. In particular, we have that the random variables $\tau_n(E)$ converge to $\mu_\alpha(E)$ with probability one. Hence this convergence holds in L^p sense by the Dominated Convergence Theorem, as $\tau_n(E)$ are uniformly bounded by 1, see Chapter 5 of [31]. It follows that

$$\lim_{n\to\infty} \mathbb{E}[|\tau_n(E) - \mu_\alpha(E)|] = 0$$

for any compact set E such that $\mu_\alpha(\partial E) = 0$, and

$$|\mathbb{E}[\tau_n(E) - \mu_\alpha(E)]| \le \mathbb{E}[|\tau_n(E) - \mu_\alpha(E)|] \to 0 \quad \text{as } n \to \infty.$$

But $\mathbb{E}[\tau_n(E)] = \mathbb{E}[N_n^*(E)]/n$ and $\mathbb{E}[\mu_\alpha(E)] = \mu_\alpha(E)$, which immediately gives (4.1.5).
□

Proof of Theorem 4.1.1. Theorem 4.1.2 gives that

$$\lim_{n\to\infty} \frac{1}{n} \mathbb{E}\left[N_n^*\left([a,b]\right)\right] = \frac{1}{\sqrt{3}} \mu_\alpha([a,b])$$

for any interval $[a,b] \subset (-1,1)$. Note that both $\mathbb{E}\left[N_n^*\left(H\right)\right]$ and $\mu_\alpha(H)$ are additive functions of the set H. Moreover, they both vanish when H is a single point by (4.1.5) and the absolute continuity of μ_α with respect to Lebesgue measure on $[-1,1]$. Hence (4.1.5) gives that

$$\lim_{n\to\infty} \frac{1}{n} \mathbb{E}\left[N_n^*\left(\mathbb{R} \setminus (-1,1)\right)\right] = \mu_\alpha(\mathbb{R} \setminus (-1,1)) = 0.$$

It now follows that

$$\lim_{n\to\infty} \frac{1}{n}\mathbb{E}[N_n^*(\mathbb{R})] = \frac{1}{\sqrt{3}}\mu_\alpha((-1,1)) = \frac{1}{\sqrt{3}}.$$

To complete the proof, observe that $N_n^*(\mathbb{R}) = N_n(\mathbb{R})$, so that $\mathbb{E}[N_n^*(\mathbb{R})] = \mathbb{E}[N_n(\mathbb{R})]$, since $L_n(x) = x/a_n$ is a bijection for each fixed n. Therefore (4.1.2) is proved. □

CHAPTER 5

The Asymptotic Variance of the Number of Real Zeros

We study the asymptotic variance of the number of real zeros for the following ensemble of random orthogonal polynomial for a fixed $n \in \mathbb{N}$:

$$P_n(x) = \sum_{j=0}^{n} c_j p_j(x), \tag{5.0.1}$$

where the coefficients $c_0, c_1, \cdots, c_n$ are i.i.d. Gaussian random variables with distribution $\mathcal{N}(0,1)$, and $\{p_j\}_{j=0}^{\infty}$ are orthonormal polynomials with respect to a finite positive Borel measure μ supported on $[-1,1]$, with all finite power moments:

$$\int_{-1}^{1} p_n p_m \, d\mu = \delta_{mn}.$$

We write $p_j(x)$ as

$$p_j(x) = k_j x^j + \cdots, \quad k_j > 0.$$

For this ensemble (5.0.1),

$$\mathbb{E}[P_n(x)] = 0, \quad \mathrm{Var}[P_n(x)] = \sum_{j=0}^{n} p_j^2(x) = K_{n+1}(x,x) > 0, \quad x \in \mathbb{R}.$$

That is, for fixed $x \in \mathbb{R}$ [43, chapter 16],

$$P_n(x) \text{ has Gaussian distribution } \mathcal{N}(0, K_{n+1}(x,x)).$$

Let $N_n(a,b)$ denote the number of real zeros of $P_n(x)$ in the interval $(a,b) \subset \mathbb{R}$ with $N_n(\mathbb{R})$ being the total number of zeros of $P_n(x)$ on $\mathbb{R}$. It is known that for the expected number of real zeros, one has

$$\lim_{n\to\infty} \frac{\mathbb{E}[N_n(\mathbb{R})]}{n} = \frac{1}{\sqrt{3}},$$

see Theorem 2.1.1. Our goal is to study the variance $\mathrm{Var}[N_n(\mathbb{R})]$ of the number of real zeros for this ensemble and determine its behavior as $n \to \infty$.

The ensemble $P_n(x)$ is a centered non-stationary Gaussian process with covariance function

$$r_{P_n}(x,y) = \mathbb{E}[P_n(x)P_n(y)] = \mathbb{E}\Bigg[\sum_{j,k=0}^{n} c_j c_k g_j(x) g_k(y)\Bigg] = K_{n+1}(x,y),$$

where we used the fact that

$$\mathbb{E}[c_j c_k] = \begin{cases} 1 \quad \text{if } j = k, \\ 0 \text{ otherwise}, \end{cases} \tag{5.0.2}$$

and the standard definition of (weighted) reproducing kernels of orthonormal polynomials:

$$K_{n+1}(x,y) := \sum_{j=0}^{n} p_j(x)p_j(y), \quad \tilde{K}_{n+1}(x,y) := \sqrt{\mu'(x)\mu'(y)} \sum_{j=0}^{n} p_j(x)p_j(y).$$

We also use the so-called differentiated kernels:

$$K_{n+1}^{(k,l)}(x,y) := \sum_{j=0}^{n} p_j^{(k)}(x) p_j^{(l)}(y), \tag{5.0.3}$$

$$\tilde{K}_{n+1}^{(k,l)}(x,y) := \sqrt{\mu'(x)\mu'(y)} \sum_{j=0}^{n} p_j^{(k)}(x) p_j^{(l)}(y), \quad k,l = 0,1,2,\ldots.$$

From the Cauchy-Schwarz inequality, it follows

$$\left| K_{n+1}^{(k,l)}(x,y) \right| \le \sqrt{K_{n+1}^{(k,k)}(x,x) K_{n+1}^{(l,l)}(y,y)},$$

where equality holds if and only if $x = y$.

5.1 Variance formulas

We begin by stating the following result from Farahmand [24, p. 22] (below x, y, y_1, y_2 denote four distinct parameters).

Lemma 5.1.1 *If $(a,b) \subset \mathbb{R}$, then*

$$\mathbb{E}[N_n(a,b)\,(N_n(a,b)-1)] = \lim_{\varepsilon\to 0}\iint_{D(\varepsilon)}\int_{\mathbb{R}}\int_{\mathbb{R}} |y_1 y_2|\, p_{x,y}(0,0,y_1,y_2)\,dy_1\,dy_2\,dx\,dy,$$

where $p_{x,y}(z_1,z_2,y_1,y_2)$ is the four-dimensional density function of the random vector $(P_n(x), P_n(y), P_n'(x), P_n'(y))^T$, and

$$D(\varepsilon) = \{(x,y)\in\mathbb{R}^2 |\ a < x, y < b,\ |x-y| > \varepsilon\}, \quad \varepsilon > 0.$$

Here and below T denotes the transpose of a matrix. We give a more explicit representation of the above expectation below.

Proposition 5.1.1 *Let (a,b) and $D(\varepsilon)$ be the same as in Lemma 5.1.1. Then*

$$\begin{aligned}&\mathbb{E}[N_n(a,b)(N_n(a,b)-1)]\\ &= \frac{1}{\pi^2}\lim_{\varepsilon\to 0}\iint_{D(\varepsilon)}\left(\sqrt{\Omega_{11}\Omega_{22}-\Omega_{12}^2} + \Omega_{12}\arcsin\left(\frac{\Omega_{12}}{\sqrt{\Omega_{11}\Omega_{22}}}\right)\right)\frac{dx\,dy}{\sqrt{\Delta}},\end{aligned} \tag{5.1.1}$$

where

$$\Delta(x,y) := K_{n+1}(x,x)K_{n+1}(y,y) - K_{n+1}^2(x,y)$$

and

$$\Omega = \begin{bmatrix}\Omega_{11} & \Omega_{12}\\ \Omega_{12} & \Omega_{22}\end{bmatrix}$$

is the covariance matrix of the random vector $(P_n'(x), P_n'(y))^T$, conditional upon $P_n(x) = P_n(y) = 0$, with

$$\begin{aligned}\Omega_{11}(x,y) :=& K_{n+1}^{(1,1)}(x,x) - \frac{1}{\Delta}\Big(K_{n+1}(y,y)(K_{n+1}^{(0,1)}(x,x))^2\\ &- 2K_{n+1}(x,y)K_{n+1}^{(0,1)}(x,x)K_{n+1}^{(0,1)}(y,x) + K_{n+1}(x,x)(K_{n+1}^{(0,1)}(y,x))^2\Big),\end{aligned}$$

$$\begin{aligned}\Omega_{22}(x,y) :=& K_{n+1}^{(1,1)}(y,y) - \frac{1}{\Delta}\Big(K_{n+1}(y,y)(K_{n+1}^{(0,1)}(x,y))^2\\ &- 2K_{n+1}(x,y)K_{n+1}^{(0,1)}(x,y)K_{n+1}^{(0,1)}(y,y) + K_{n+1}(x,x)(K_{n+1}^{(0,1)}(y,y))^2\Big),\end{aligned}$$

$$\Omega_{12}(x,y) := K_{n+1}^{(1,1)}(x,y) - \frac{1}{\Delta}\Big(K_{n+1}(y,y)K_{n+1}^{(0,1)}(x,x)K_{n+1}^{(0,1)}(x,y) - K_{n+1}(x,y)K_{n+1}^{(0,1)}(x,y)K_{n+1}^{(0,1)}(y,x)$$
$$- K_{n+1}(x,y)K_{n+1}^{(0,1)}(x,x)K_{n+1}^{(0,1)}(y,y) + K_{n+1}(x,x)K_{n+1}^{(0,1)}(y,x)K_{n+1}^{(0,1)}(y,y)\Big).$$

Proposition 5.1.1 indicates that we need to study the asymptotics of the kernels in order to know the asymptotic of $\mathrm{Var}[N_n(\mathbb{R})]$.

5.2 The asymptotic variance of the number of real zeros for random orthogonal polynomials

Using Proposition 5.1.1, we can prove the following result.

Theorem 5.2.1 *Let μ be a positive Borel measure supported on $[-1,1]$ such that $d\mu(x) = w(x)dx$ and $w > 0$ a.e. on $[-1,1]$. Let $w(\cos\theta)\left|\sin\theta\right|,\ \theta\in[-\pi,\pi]$ satisfy the Lipschitz-Dini condition*

$$\left|w(\cos(\theta+\delta))\left|\sin(\theta+\delta)\right| - w(\cos\theta)\left|\sin\theta\right|\right| < L\left|\log\delta\right|^{-1-\lambda},$$

where $L > 0$ and $\lambda > 0$ are fixed numbers. Assume that $O \subset [-1,1]$ is an open set and that there exists a constant $C > 1$ such that a.e. in O,

$$C^{-1} \le w \le C.$$

Then for any $[a,b] \subset O$,

$$\lim_{n\to\infty} \frac{\mathbb{E}[N_n^2([a,b])]}{n^2} = \frac{1}{3}\nu_{[-1,1]}([a,b])^2,$$

where

$$d\nu_{[-1,1]}(x) = \frac{dx}{\pi\sqrt{1-x^2}}, \quad x\in[-1,1],$$

is the equilibrium measure of $(-1,1)$. Moreover, as $n\to\infty$,

$$\mathrm{Var}[N_n([a,b])] = o(n^2).$$

Applying Theorem 5.2.1, we have the following result.

Theorem 5.2.2 *Let μ be a positive Borel measure supported on $[-1,1]$ such that $d\mu(x) = w(x)dx$ and $w > 0$ a.e. on $[-1,1]$. Let $w(\cos\theta)\left|\sin\theta\right|,\ \theta \in [-\pi,\pi]$ satisfy the Lipschitz-Dini condition*

$$\left|w(\cos(\theta+\delta))\left|\sin(\theta+\delta)\right| - w(\cos\theta)\left|\sin\theta\right|\right| < L\left|\log\delta\right|^{-1-\lambda},$$

where $L > 0$ and $\lambda > 0$ are fixed numbers. If for any closed interval $[a,b] \subset (-1,1)$ there is a constant $C > 1$ such that $C^{-1} < w < C$ a.e. on $[a,b]$, then

$$\lim_{n\to\infty} \frac{\mathbb{E}[N_n^2([-1,1])]}{n^2} = \frac{1}{3}.$$

Moreover, as $n \to \infty$,

$$\mathrm{Var}[N_n([-1,1])] = o(n^2).$$

Theorem 5.2.3 *Under the assumptions of Theorem 5.2.2,*

$$\lim_{n\to\infty} \frac{\mathbb{E}[N_n^2(\mathbb{R})]}{n^2} = \frac{1}{3},$$

and, as $n \to \infty$,

$$\mathrm{Var}[N_n(\mathbb{R})] = o(n^2).$$

I believe that the orthogonality measure μ, produced by the density w that is continuous on $[-1,1]$ except for finitely many points, and has finitely many zeros on $[-1,1]$, will also give the above results . More specifically, one may consider the generalized Jacobi weight of the form $d\mu(x) = v(x)\prod_{j=1}^{J}|x-x_j|^{\alpha_j}\,dx$, where $v(x) > 0,\ x \in [-1,1]$, and $\alpha_j > -1,\ j = 1,\ldots,J$.

5.3 Proof of Proposition 5.1.1

Since we do not assume to have a stationary process in our setting, we cannot apply Lemma 2.2 and Corollary 2.5 from [32] directly. Instead, we adapt ideas of [32] in the proof. For fixed $x, y \in \mathbb{R}$, define the random vector

$$V = V(x,y) := (P_n(x), P_n(y), P_n'(x), P_n'(y))^T .$$

It is easy to see that

$$P_n'(x) = \sum_{j=0}^{n} c_j p_j'(x).$$

It is clear that

$$\mathbb{E}[P_n'(x)] = \mathbb{E}[P_n'(y)] = 0,$$

$$\mathrm{Var}(P_n'(x)) = K_{n+1}^{(1,1)}(x,x) > 0, \quad \mathrm{Var}(P_n'(y)) = K_{n+1}^{(1,1)}(y,y) > 0.$$

Summarizing,

$$P_n(x) \text{ has Gaussian distribution } \mathcal{N}(0, K_{n+1}(x,x));$$

$$P_n(y) \text{ has Gaussian distribution } \mathcal{N}(0, K_{n+1}(y,y));$$

$$P_n'(x) \text{ has Gaussian distribution } \mathcal{N}(0, K_{n+1}^{(1,1)}(x,x));$$

$$P_n'(y) \text{ has Gaussian distribution } \mathcal{N}(0, K_{n+1}^{(1,1)}(y,y)).$$

The covariance matrix Σ of V is defined by

$$\Sigma = \Sigma(x,y)$$

$$:= \begin{bmatrix} \mathrm{Var}[P_n(x)] & \mathrm{Cov}[P_n(x), P_n(y)] & \mathrm{Cov}[P_n(x), P_n'(x)] & \mathrm{Cov}[P_n(x), P_n'(y)] \\ \mathrm{Cov}[P_n(y), P_n(x)] & \mathrm{Var}[P_n(y)] & \mathrm{Cov}[P_n(y), P_n'(x)] & \mathrm{Cov}[P_n(y), P_n'(y)] \\ \mathrm{Cov}[P_n'(x), P_n(x)] & \mathrm{Cov}[P_n'(x), P_n(y)] & \mathrm{Var}[P_n'(x)] & \mathrm{Cov}[P_n'(x), P_n'(y)] \\ \mathrm{Cov}[P_n'(y), P_n(x)] & \mathrm{Cov}[P_n'(y), P_n(y)] & \mathrm{Cov}[P_n'(y), P_n'(x)] & \mathrm{Var}[P_n'(y)] \end{bmatrix}.$$

When $x = y$, the first row of Σ is the same as the second row, and hence $\det \Sigma = 0$. Before we start the proof of Proposition 5.1.1, we state a lemma which implies that when $x \neq y$ and $n \geq 3$, $\det \Sigma > 0$.

Lemma 5.3.1 *Let $\{p_j(x)\}_{j=0}^n$ be a polynomial basis for the vector space of all polynomials with real coefficients of degree at most n. Define*

$$P_n(x) = \sum_{j=0}^{n} c_j p_j(x),$$

where c_j's are i.i.d. real random variables with zero mean and unit variance. Let Σ be the covariance matrix of the random vector

$$V = V(x,y) := \left(P_n(x), P_n(y), P_n'(x), P_n'(y)\right)^T.$$

If $x \neq y$ and $n \geq 3$, then Σ is positive definite (hence $\det \Sigma > 0$).

Proof. By definition of positive definite matrix, we only need to show that $\vec{a}^T \Sigma \vec{a} > 0$, for all $\vec{a} \in \mathbb{R}^4 \setminus \{\vec{0}\}$. Note that any covariance matrix is positive semi-definite [43, Theorem 12.4]: $\vec{a}^T \Sigma \vec{a} \geq 0$, for all $\vec{a} \in \mathbb{R}^4 \setminus \{\vec{0}\}$. This means that we only need to demonstrate that $\vec{a}^T \Sigma \vec{a} = 0$ implies $\vec{a} = \vec{0}$. Indeed, observe that $\vec{a}^T \Sigma \vec{a} = \mathrm{Var}(\vec{a}^T V)$, and write the column vector

$$\vec{a} = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \end{bmatrix}^T.$$

Then we see that

$$\begin{aligned}
\vec{a}^T V &= \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \end{bmatrix} \begin{bmatrix} P_n(x) \\ P_n(y) \\ P_n'(x) \\ P_n'(y) \end{bmatrix} \\
&= a_1 P_n(x) + a_2 P_n(y) + a_3 P_n'(x) + a_4 P_n'(y) \\
&= [a_1 p_0(x) + a_2 p_0(y) + a_3 p_0'(x) + a_4 p_0'(y)] c_0 \\
&+ [a_1 p_1(x) + a_2 p_1(y) + a_3 p_1'(x) + a_4 p_1'(y)] c_1 \\
&+ [a_1 p_2(x) + a_2 p_2(y) + a_3 p_2'(x) + a_4 p_2'(y)] c_2 \\
&+ \cdots + [a_1 p_n(x) + a_2 p_n(y) + a_3 p_n'(x) + a_4 p_n'(y)] c_n.
\end{aligned}$$

Thus

$$\begin{aligned}\mathrm{Var}(\vec{a}^T V) &= [a_1 p_0(x) + a_2 p_0(y) + a_3 p_0'(x) + a_4 p_0'(y)]^2 \\ &+ [a_1 p_1(x) + a_2 p_1(y) + a_3 p_1'(x) + a_4 p_1'(y)]^2 \\ &+ [a_1 p_2(x) + a_2 p_2(y) + a_3 p_2'(x) + a_4 p_2'(y)]^2 \\ &+ \cdots + [a_1 p_n(x) + a_2 p_n(y) + a_3 p_n'(x) + a_4 p_n'(y)]^2.\end{aligned}$$

Now it is clear that $\vec{a}^T \Sigma \vec{a} = 0$ if and only if

$$\begin{cases} a_1 p_0(x) + a_2 p_0(y) + a_3 p_0'(x) + a_4 p_0'(y) = 0 \\ a_1 p_1(x) + a_2 p_1(y) + a_3 p_1'(x) + a_4 p_1'(y) = 0 \\ a_1 p_2(x) + a_2 p_2(y) + a_3 p_2'(x) + a_4 p_2'(y) = 0 \\ \vdots \\ a_1 p_n(x) + a_2 p_n(y) + a_3 p_n'(x) + a_4 p_n'(y) = 0 \end{cases} \tag{5.3.1}$$

We need to show this system of equations has only trivial solution $\vec{a} = \vec{0}$. Indeed, if we write

$$Q_n(t) = \sum_{j=0}^{n} b_j p_j(t),$$

where $\{b_j\}_{j=0}^n \subset \mathbb{R}$ is any sequence, then the system of equations (5.3.1) implies that

$$a_1 Q_n(x) + a_2 Q_n(y) + a_3 Q_n'(x) + a_4 Q_n'(y) = 0. \tag{5.3.2}$$

Since $\{p_j(x)\}_{j=0}^n$ is a polynomial basis for the vector space of all polynomials with real coefficients of degree at most n and $\{b_j\}_{j=0}^n \subset \mathbb{R}$ is any sequence, $Q_n(t)$ can be any polynomial in t with real coefficients of degree at most n. In particular, since $n \geq 3$ and $x \neq y$, because of (5.3.2),

$$\begin{aligned} Q_n(t) = (t-x)(t-y)^2 &\Rightarrow a_3 = 0; \\ Q_n(t) = (t-x)^2(t-y) &\Rightarrow a_4 = 0; \\ Q_n(t) = t - y &\Rightarrow a_1 = 0; \\ Q_n(t) = t - x &\Rightarrow a_2 = 0. \end{aligned}$$

That is, the system of equations (5.3.1) has only trivial solution $\vec{a} = \vec{0}$, as required.

$\square$

Now it is clear that Lemma 5.3.1 implies that for $n \geq 3$, $\det \Sigma = 0$ if and only if $x = y$.

Proof of Proposition 5.1.1. By Lemma 5.3.1 and [43, Corollary 16.2], V has a multivariate normal distribution with mean zero and the covariance matrix Σ, given by

$$\begin{bmatrix} \mathrm{Var}[P_n(x)] & \mathrm{Cov}[P_n(x), P_n(y)] & \mathrm{Cov}[P_n(x), P_n'(x)] & \mathrm{Cov}[P_n(x), P_n'(y)] \\ \mathrm{Cov}[P_n(y), P_n(x)] & \mathrm{Var}[P_n(y)] & \mathrm{Cov}[P_n(y), P_n'(x)] & \mathrm{Cov}[P_n(y), P_n'(y)] \\ \mathrm{Cov}[P_n'(x), P_n(x)] & \mathrm{Cov}[P_n'(x), P_n(y)] & \mathrm{Var}[P_n'(x)] & \mathrm{Cov}[P_n'(x), P_n'(y)] \\ \mathrm{Cov}[P_n'(y), P_n(x)] & \mathrm{Cov}[P_n'(y), P_n(y)] & \mathrm{Cov}[P_n'(y), P_n'(x)] & \mathrm{Var}[P_n'(y)] \end{bmatrix}.$$

We will express all entries of the matrix Σ through the reproducing kernels. From (5.0.2) and (5.0.3), it immediately follows that

$$\Sigma = \begin{bmatrix} K_{n+1}(x,x) & K_{n+1}(x,y) & K_{n+1}^{(0,1)}(x,x) & K_{n+1}^{(0,1)}(x,y) \\ K_{n+1}(x,y) & K_{n+1}(y,y) & K_{n+1}^{(0,1)}(y,x) & K_{n+1}^{(0,1)}(y,y) \\ K_{n+1}^{(0,1)}(x,x) & K_{n+1}^{(0,1)}(y,x) & K_{n+1}^{(1,1)}(x,x) & K_{n+1}^{(1,1)}(x,y) \\ K_{n+1}^{(0,1)}(x,y) & K_{n+1}^{(0,1)}(y,y) & K_{n+1}^{(1,1)}(x,y) & K_{n+1}^{(1,1)}(y,y) \end{bmatrix} =: \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}, \qquad (5.3.3)$$

where A, B and C are the corresponding 2×2 matrices. Note that $\det A = \Delta = 0$ if and only if $x = y$ by the equality case in the Cauchy-Schwarz inequality. Thus we define $\Omega = C - B^T A^{-1} B$ for $(x, y) \in D(\varepsilon)$, and write

$$\Sigma = \begin{bmatrix} A & \mathbf{0} \\ B^T & \mathbf{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{I} & A^{-1}B \\ \mathbf{0} & \Omega \end{bmatrix}.$$

The latter implies that

$$\det \Sigma = \det A \, \det \Omega = \Delta \, \det \Omega.$$

Since Σ is invertible in $D(\varepsilon)$ by Lemma 5.3.1, so is Ω and thus $\det \Omega > 0$ in $D(\varepsilon)$. It also follows from (5.3.3) by direct algebraic manipulations that the elements of the matrix

$$\Omega = C - B^T A^{-1} B = \begin{bmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{12} & \Omega_{22} \end{bmatrix}$$

are defined as stated in Proposition 5.1.1.

As the random vector $V = V(x, y)$ has the multivariate normal distribution $\mathcal{N}(\mathbf{0}, \Sigma)$ with a non-singular covariance matrix Σ, we find the density of its distribution [43, p. 130] to be

$$\begin{aligned} p_{x,y}(0,0,y_1,y_2) &= \frac{\exp\left(-\frac{1}{2}(0,0,y_1,y_2)\,\Sigma^{-1}(0,0,y_1,y_2)^T\right)}{(2\pi)^2(\det\Sigma)^{1/2}} \\ &= \frac{\exp\left(-\frac{1}{2}(y_1,y_2)\,\Omega^{-1}(y_1,y_2)^T\right)}{(2\pi)^2(\det\Sigma)^{1/2}}. \end{aligned}$$

Using matrix algebra, we further obtain that

$$\Sigma^{-1} = \begin{bmatrix} [A - BC^{-1}B^T]^{-1} & -A^{-1}B[C - B^T A^{-1} B]^{-1} \\ -C^{-1}B^T[A - BC^{-1}B^T]^{-1} & [C - B^T A^{-1} B]^{-1} \end{bmatrix}.$$

Lemma 5.1.1 now gives that

$$\begin{aligned} &\mathbb{E}[N_n(a,b)\{N_n(a,b)-1\}] \\ &= \lim_{\varepsilon\to 0} \iint_{D(\varepsilon)} \int_{\mathbb{R}} \int_{\mathbb{R}} |y_1 y_2| \, \frac{\exp\left(-\frac{1}{2}(y_1,y_2)\Omega^{-1}(y_1,y_2)^T\right)}{(2\pi)^2(\det\Sigma)^{1/2}} \, dy_1\, dy_2\, dx\, dy, \\ &= \lim_{\varepsilon\to 0} \iint_{D(\varepsilon)} \int_{\mathbb{R}} \int_{\mathbb{R}} |y_1 y_2| \, \frac{\exp\left(-\frac{1}{2}(y_1,y_2)\Omega^{-1}(y_1,y_2)^T\right)}{(2\pi)^2(\Delta\ \det\Omega)^{1/2}} \, dy_1\, dy_2\, dx\, dy, \\ &= \frac{1}{4\pi^2} \lim_{\varepsilon\to 0} \iint_{D(\varepsilon)} \frac{I(x_1,x_2)}{\sqrt{\Delta\ \det\Omega}} \, dx_1\, dx_2, \end{aligned}$$

where the inner integral is

$$I(x,y) = \int_{\mathbb{R}} \int_{\mathbb{R}} |y_1 y_2| \exp\left(-\frac{1}{2}(y_1,y_2)\Omega^{-1}(y_1,y_2)^T\right) dy_1\, dy_2.$$

Note that in $D(\varepsilon)$,

$$\det\Omega = \Omega_{11}\Omega_{22} - \Omega_{12}^2 > 0$$

and

$$\Omega^{-1} = \frac{1}{\det\Omega} \begin{bmatrix} \Omega_{22} & -\Omega_{12} \\ -\Omega_{12} & \Omega_{11} \end{bmatrix}.$$

We compute that

$$(y_1, y_2)\,\Omega^{-1}\,(y_1, y_2)^T = \frac{\Omega_{22}}{\det\Omega}\,y_1^2 + 2\,\frac{-\Omega_{12}}{\det\Omega}\,y_1 y_2 + \frac{\Omega_{11}}{\det\Omega}\,y_2^2.$$

Applying the result of [6, (3.9)], we evaluate the inner integral as

$$I(x, y) = \frac{4(\det\Omega)^2}{\Omega_{11}\Omega_{22}(1-\delta^2)}\left(1 + \frac{\delta}{\sqrt{1-\delta^2}}\arcsin\delta\right),$$

with

$$\delta = -\frac{\Omega_{12}}{\sqrt{\Omega_{11}\Omega_{22}}}.$$

Finally, putting everything together, we obtain

$$\begin{aligned}
&\mathbb{E}[N_n(a,b)\{N_n(a,b)-1\}] \\
&= \frac{1}{4\pi^2}\lim_{\varepsilon\to 0}\iint_{D(\varepsilon)} \frac{4(\det\Omega)^2}{\Omega_{11}\Omega_{22}(1-\delta^2)}\left(1 + \frac{\delta}{\sqrt{1-\delta^2}}\arcsin\delta\right)\frac{dx\,dy}{\sqrt{\Delta\,\det\Omega}} \\
&= \frac{1}{\pi^2}\lim_{\varepsilon\to 0}\iint_{D(\varepsilon)} \sqrt{\Omega_{11}\Omega_{22}-\Omega_{12}^2}\left(1 - \frac{\Omega_{12}}{\sqrt{\Omega_{11}\Omega_{22}-\Omega_{12}^2}}\arcsin\left(-\frac{\Omega_{12}}{\sqrt{\Omega_{11}\Omega_{22}}}\right)\right)\frac{dx_1\,dx_2}{\sqrt{\Delta}} \\
&= \frac{1}{\pi^2}\lim_{\varepsilon\to 0}\iint_{D(\varepsilon)} \left(\sqrt{\Omega_{11}\Omega_{22}-\Omega_{12}^2} + \Omega_{12}\arcsin\frac{\Omega_{12}}{\sqrt{\Omega_{11}\Omega_{22}}}\right)\frac{dx\,dy}{\sqrt{\Delta}}.
\end{aligned}$$

□

5.4 Proofs of the main theorems

Throughout this section, $[a, b] \subset (-1, 1)$ is a fixed interval independent of n.

5.4.1 Preliminary results on kernels and equilibrium measures

We first recall some useful facts about weak* convergence from Totik's paper [88, Theorem 1 and Corollary 1]:

Lemma 5.4.1 *Assume that $w > 0$ a.e. in $(-1, 1)$. Then for a.e. $x \in [-1, 1]$,*

$$\lim_{n\to\infty}\frac{n}{K_{n+1}(x,x)} = \frac{d\mu(x)}{d\nu_{[-1,1]}};$$

more precisely, for every bounded and measurable function g on $[-1,1]$,

$$\lim_{n\to\infty}\int g(x)\frac{K_{n+1}(x,x)}{n}\,d\mu(x)=\int g(x)\,d\nu_{[-1,1]}(x),$$
$$\lim_{n\to\infty}\int g(x)\frac{\tilde{K}_{n+1}(x,x)}{n}\,dx=\int g(x)\,d\nu_{[-1,1]}(x).$$

Corollary 1.4 in [62, p. 224] gives the following fact: for all $j,k\geq 0$,

$$\lim_{n\to\infty}\int_a^b\left|\frac{\tilde{K}_{n+1}^{(j,k)}(x,x)}{\tilde{K}_{n+1}(x,x)^{j+k+1}}-\pi^{j+k}\tau_{j,k}\right|dx=0,$$

where

$$\tau_{j,k}=\begin{cases}0 & j+k \text{ odd},\\ \frac{(-1)^{\frac{j-k}{2}}}{j+k+1} & j+k \text{ even}.\end{cases}$$

This implies the following lemma.

Lemma 5.4.2 *For a.e. $x\in[a,b]$,*

$$\lim_{n\to\infty}\frac{\tilde{K}_{n+1}^{(0,1)}(x,x)}{\tilde{K}_{n+1}(x,x)^2}=0,\quad \lim_{n\to\infty}\frac{\tilde{K}_{n+1}^{(1,0)}(x,x)}{\tilde{K}_{n+1}(x,x)^2}=0,\quad \lim_{n\to\infty}\frac{\tilde{K}_{n+1}^{(1,1)}(x,x)}{\tilde{K}_{n+1}(x,x)^3}=\frac{\pi^2}{3}.$$

We recall an important fact proved by Rakhmanov [76], and later reproved by Mate, Nevai, and Totik in [66] (see also [51, page 3, (1.10)]).

Lemma 5.4.3 *If $w>0$ a.e. in $(-1,1)$, then*

$$\lim_{n\to\infty}\frac{k_n}{k_{n+1}}=\frac{1}{2}, \tag{5.4.1}$$

where k_j is the positive leading coefficient of $p_j(x)$.

Finally we give a lemma about the various asymptotics of kernels $K_{n+1}^{(k,l)}(x,y)$ with $x\neq y$.

Lemma 5.4.4 *Under the assumptions of Theorem 5.2.1, for a.e. $x\in[a,b]$, a.e. $y\in[a,b]$ such that $x\neq y$,*

$$\lim_{n\to\infty}\frac{K_{n+1}(x,y)}{n}=\lim_{n\to\infty}\frac{K_{n+1}^{(0,1)}(x,y)}{n^2}=\lim_{n\to\infty}\frac{K_{n+1}^{(1,0)}(x,y)}{n^2}=\lim_{n\to\infty}\frac{K_{n+1}^{(1,1)}(x,y)}{n^3}=0.$$

Note that the exceptional set in the above Lemma is a dimension-two null set.

Proof of Lemma 5.4.4. Having the Szegő's condition on the real line

$$\int_{-1}^{1} \frac{\log w(x)}{\sqrt{1-x^2}}\, dx > -\infty,$$

one can form the Szegő function

$$D(z) = \exp\left(-\frac{1}{4\pi} \int_{-\pi}^{\pi} \log \sigma'(\theta) \frac{z+e^{i\theta}}{z-e^{i\theta}}\, d\theta \right), \quad |z| < 1.$$

Here and below σ' is defined by

$$\sigma'(\theta) := w(\cos\theta) \left|\sin\theta\right|,\ \theta \in [-\pi, \pi].$$

The standard Szegő asymptotic for p_n has the form

$$p_n(x) = \sqrt{2/\pi}\, \Re\left(\frac{z^n}{D(z^{-1})} \right) + o(1), \qquad \text{as } n \to \infty, \tag{5.4.2}$$

where $x = \cos\theta$, $z = e^{i\theta}$. The Szegő condition guarantees that (5.4.2) holds in L^2 sense, but not necessarily pointwise. For pointwise or uniform asymptotics, one typically needs some smoothness on $\sigma'(\theta)$, such as a Lipschitz condition. The following theorem is a modified version of a theorem of Szegő [84, pp. 297–299].

Theorem 5.4.1 *Let w be positive a.e. in $(-1,1)$, and satisfy the Lipschitz-Dini condition*

$$\left|w(\cos(\theta+\delta)) \left|\sin(\theta+\delta)\right| - w(\cos\theta) \left|\sin\theta\right|\right| < L \left|\log\delta\right|^{-1-\lambda},$$

where $L > 0$ and $\lambda > 0$ are fixed numbers. Then, uniformly for $x = \cos\theta \in [-1,1]$, or $\theta \in [0,\pi]$,

$$(1-x^2)^{1/4}\sqrt{w(x)}p_n(x) = \sqrt{2/\pi}\cos(n\theta + \gamma(\theta)) + O((\log n)^{-\lambda}),$$

where

$$\gamma(\theta) = \frac{1}{4\pi} \int_{-\pi}^{\pi} [\log\sigma'(t) - \log\sigma'(\theta)] \cot\frac{\theta - t}{2}\, dt,$$

and the constant factor in the O-term depends only on L, λ, the minimum and maximum of $\sigma'(\theta)$.

This theorem says that under the assumptions of Theorem 5.4.1, uniformly for $x = \cos\theta \in (-1,1)$ with $w(x) \neq 0$, one has

$$p_n(x) = \frac{1}{(1-x^2)^{1/4}} \frac{1}{\sqrt{w(x)}} \left(\sqrt{2/\pi} \cos(n\theta + \gamma(\theta)) + o(1) \right). \tag{5.4.3}$$

The next theorem is based on a recent result from Levin and Lubinsky [54, Theorem 2 and Corollary 3].

Theorem 5.4.2 *Under the assumptions of Theorem 5.4.1, if* $[a,b] \subset (-1,1)$, *and* $w > 0$ *a.e. in* $(-1,1)$, *then uniformly for* $x = \cos\theta \in [a,b]$,

$$\frac{\sqrt{1-x^2}}{n} p_n'(x) = \Im\left(z^n f(z)\right) + o(1),$$

where

$$f(z) := \frac{\sqrt{2/\pi}}{D(z^{-1})}.$$

Note that f is a bounded function on $\{z = e^{i\theta} : \theta \in [0,\pi], \cos\theta \in [a,b]\}$. In other words, this theorem asserts that uniformly for $x = \cos\theta \in [a,b]$,

$$p_n'(x) = \frac{n}{\sqrt{1-x^2}} \left(\cos(n\theta)\, \Im\left(f(e^{i\theta})\right) + \sin(n\theta)\, \Re\left(f(e^{i\theta})\right) + o(1) \right). \tag{5.4.4}$$

Note that here $\Im\left(f(e^{i\theta})\right)$ and $\Re\left(f(e^{i\theta})\right)$ are both bounded. Now we finish the proof of Lemma 5.4.4. Applying the Christoffel-Darboux formula [84, p. 43], we derive the following relations:

$$\begin{aligned}
K_{n+1}(x,y) &= \frac{k_n}{k_{n+1}} \frac{p_{n+1}(y)p_n(x) - p_{n+1}(x)p_n(y)}{y-x}, \qquad (5.4.5)\\
K_{n+1}^{(0,1)}(x,y) &= \frac{k_n}{k_{n+1}} \frac{p_{n+1}'(y)p_n(x) - p_{n+1}(x)p_n'(y)}{y-x} - \frac{K_{n+1}(x,y)}{y-x},\\
K_{n+1}^{(0,1)}(y,x) &= \frac{k_n}{k_{n+1}} \frac{p_{n+1}'(x)p_n(y) - p_{n+1}(y)p_n'(x)}{x-y} - \frac{K_{n+1}(x,y)}{x-y},\\
K_{n+1}^{(1,1)}(x,y) &= \frac{k_n}{k_{n+1}} \frac{p_{n+1}'(y)p_n'(x) - p_{n+1}'(x)p_n'(y)}{y-x} + \frac{K_{n+1}^{(0,1)}(x,y) - K_{n+1}^{(0,1)}(y,x)}{y-x}.
\end{aligned}$$

Making use of (5.4.3), we have for $x = \cos\theta_1 \neq y = \cos\theta_2$,

$$p_{n+1}(y)p_n(x) - p_{n+1}(x)p_n(y) = (1-x^2)^{-1/4}(1-y^2)^{-1/4}(w(x)w(y))^{-1/2}\times$$
$$\Bigg(\frac{2}{\pi}\cos((n+1)\theta_2 + \gamma(\theta_2))\cos(n\theta_1 + \gamma(\theta_1))$$
$$-\frac{2}{\pi}\cos((n+1)\theta_1 + \gamma(\theta_1))\cos(n\theta_2 + \gamma(\theta_2)) + o(1)\Bigg).$$

Therefore, (5.4.1) and (5.4.5) give us that for a.e. $x \in [a,b]$, a.e. $y \in [a,b]$ such that $x \neq y$,

$$\lim_{n\to\infty} \frac{K_{n+1}(x,y)}{n} = 0.$$

In the same fashion, we can justify the other limits in Lemma 5.4.4 by exploiting (5.4.4), (5.4.1), and the other three Christoffel-Darboux formulas from the above. □

5.4.2 Behavior of $\Delta, \Omega_{11}, \Omega_{12}$ and Ω_{22}

Proposition 5.4.1 *Under the assumptions of Theorem 5.2.1, for a.e. $x \in [a,b]$, a.e. $y \in [a,b]$ such that $x \neq y$,*

$$\lim_{n\to\infty} \frac{\Delta}{n^2} = \frac{d\nu_{[-1,1]}}{d\mu}(x)\frac{d\nu_{[-1,1]}}{d\mu}(y).$$

Proof. From Proposition 5.1.1, it follows that

$$\frac{\Delta}{n^2} = \frac{K_{n+1}(x,x)}{n}\frac{K_{n+1}(y,y)}{n}\left(1 - \left(\frac{K_{n+1}(x,y)}{n}\right)^2\left(\frac{K_{n+1}(x,x)}{n}\frac{K_{n+1}(y,y)}{n}\right)^{-1}\right).$$

Applying Lemma 5.4.1, 5.4.2, and 5.4.4 gives us the desired limit. □

Proposition 5.4.2 *Under the assumptions of Theorem 5.2.1, for a.e. $x \in [a,b]$, a.e. $y \in [a,b]$ such that $x \neq y$,*

$$\lim_{n\to\infty} \frac{\Omega_{11}}{n^3} = \frac{\pi^2}{3}\frac{(\nu'_{[-1,1]}(x))^3}{w(x)}, \quad \lim_{n\to\infty} \frac{\Omega_{22}}{n^3} = \frac{\pi^2}{3}\frac{(\nu'_{[-1,1]}(y))^3}{w(y)}.$$

Proof. By Proposition 5.1.1,

$$\begin{aligned}
\frac{\Omega_{11}}{n^3} &= \frac{1}{w(x)}\frac{\tilde{K}^{(1,1)}_{n+1}(x,x)}{\tilde{K}_{n+1}(x,x)^3}\left(\frac{\tilde{K}_{n+1}(x,x)}{n}\right)^3 \\
&- \left(\frac{\Delta}{n^2}\right)^{-1}\left\{\frac{1}{(w(x))^2}\frac{K_{n+1}(y,y)}{n}\left(\frac{\tilde{K}^{(0,1)}_{n+1}(x,x)}{\tilde{K}_{n+1}(x,x)^2}\right)^2\left(\frac{\tilde{K}_{n+1}(x,x)}{n}\right)^4\right. \\
&- \frac{2}{w(x)}\frac{K_{n+1}(x,y)}{n}\frac{\tilde{K}^{(0,1)}_{n+1}(x,x)}{\tilde{K}_{n+1}(x,x)^2}\frac{K^{(0,1)}_{n+1}(y,x)}{n^2}\left(\frac{\tilde{K}_{n+1}(x,x)}{n}\right)^2 \\
&\left. + \frac{K_{n+1}(x,x)}{n}\left(\frac{K^{(0,1)}_{n+1}(y,x)}{n^2}\right)^2\right\}.
\end{aligned}$$

Applying Lemma 5.4.1, 5.4.2, and 5.4.4, we derive the first limit. A similar approach can be used to justify the second limit. □

Proposition 5.4.3 *Under the assumptions of Theorem 5.2.1, for a.e.* $x \in [a,b]$, *a.e.* $y \in [a,b]$ *such that* $x \neq y$,

$$\lim_{n\to\infty}\frac{\Omega_{12}}{n^3} = 0.$$

Proof. Invoking Proposition 5.1.1, we have

$$\begin{aligned}
\frac{\Omega_{12}}{n^3} &= \frac{K^{(1,1)}_{n+1}(x,y)}{n^3} - \left(\frac{\Delta}{n^2}\right)^{-1}\left\{\frac{1}{w(x)}\frac{K_{n+1}(y,y)}{n}\frac{\tilde{K}^{(0,1)}_{n+1}(x,x)}{\tilde{K}_{n+1}(x,x)^2}\frac{\tilde{K}_{n+1}(x,x)^2}{n^2}\frac{K^{(0,1)}_{n+1}(x,y)}{n^2}\right. \\
&- \frac{K_{n+1}(x,y)}{n}\frac{K^{(0,1)}_{n+1}(x,y)}{n^2}\frac{K^{(0,1)}_{n+1}(y,x)}{n^2} \\
&- \frac{1}{w(x)w(y)}\frac{K_{n+1}(x,y)}{n}\frac{\tilde{K}^{(0,1)}_{n+1}(x,x)}{\tilde{K}_{n+1}(x,x)^2}\frac{\tilde{K}^{(0,1)}_{n+1}(y,y)}{\tilde{K}_{n+1}(y,y)^2}\frac{\tilde{K}_{n+1}(x,x)^2}{n^2}\frac{\tilde{K}_{n+1}(y,y)^2}{n^2} \\
&\left. + \frac{1}{w(y)}\frac{K_{n+1}(x,x)}{n}\frac{\tilde{K}^{(0,1)}_{n+1}(y,y)}{\tilde{K}_{n+1}(y,y)^2}\frac{\tilde{K}_{n+1}(y,y)^2}{n^2}\frac{K^{(0,1)}_{n+1}(y,x)}{n^2}\right\}.
\end{aligned}$$

To complete the proof, we apply Lemma 5.4.1, 5.4.2, and 5.4.4. □

5.4.3 Proofs of the main theorems

Proof of Theorem 5.2.1. By Proposition 5.1.1,

$$\frac{\mathbb{E}[N_n(a,b)(N_n(a,b)-1)]}{n^2} = \frac{1}{\pi^2}\times$$

$$\lim_{\varepsilon\to 0}\iint_{D(\varepsilon)}\left(\sqrt{\frac{(\Omega_{11}/n^3)(\Omega_{22}/n^3)-(\Omega_{12}/n^3)^2}{\Delta/n^2}} + \frac{\Omega_{12}/n^3}{\sqrt{\Delta/n^2}}\arcsin\left(\frac{\Omega_{12}}{\sqrt{\Omega_{11}\Omega_{22}}}\right)\right)dx\,dy.$$

Invoking the Dominated Convergence Theorem, we see that we only need to compute the limit of the integrand when $n \to \infty$. Note that by Propostion 5.4.1,

$$\lim_{n\to\infty} \frac{\Delta}{n^2} = \frac{d\nu_{[-1,1]}}{d\mu}(x)\frac{d\nu_{[-1,1]}}{d\mu}(y)$$

is not identically zero. Note also that the arcsine function is uniformly bounded. Applying Proposition 5.4.1, 5.4.2, and 5.4.3 gives us that

$$\lim_{n\to\infty} \frac{\mathbb{E}[N_n([a,b])(N_n([a,b])-1)]}{n^2} = \frac{1}{\pi^2}\lim_{\varepsilon\to 0}\iint_{D(\varepsilon)} \frac{\pi^2}{3}\nu'_{[-1,1]}(x)\nu'_{[-1,1]}(y)\,dx\,dy.$$

Note that

$$\lim_{\varepsilon\to 0} D(\varepsilon) = \{(x,y)\in\mathbb{R}^2 | x,y\in[a,b], \quad x\neq y\}.$$

The set

$$\{(x,y)\in\mathbb{R}^2 | x,y\in[a,b], \quad x=y\}$$

has (dimension two) Lebesgue measure zero. Thus,

$$\lim_{n\to\infty} \frac{\mathbb{E}[N_n([a,b])(N_n([a,b])-1)]}{n^2} = \frac{1}{3}\iint_{[a,b]\times[a,b]} \nu'_{[-1,1]}(x)\nu'_{[-1,1]}(y)\,dx\,dy = \frac{1}{3}\nu_{[-1,1]}([a,b])^2.$$

Now we recall that Theorem 2.1.2 implies that

$$\lim_{n\to\infty} \frac{\mathbb{E}[N_n([a,b])]}{n} = \frac{1}{\sqrt{3}}\nu_{[-1,1]}([a,b]).$$

Therefore, we obtain

$$\lim_{n\to\infty} \frac{\mathbb{E}[N_n^2([a,b])]}{n^2} = \frac{1}{3}\nu_{[-1,1]}([a,b])^2.$$

Furthermore, since

$$\mathrm{Var}[N_n([a,b])] = \mathbb{E}[N_n^2([a,b])] - \left(\mathbb{E}[N_n([a,b])]\right)^2,$$

we have as $n \to \infty$,

$$\mathrm{Var}[N_n([a,b])] = o(n^2).$$

□

In order to prove Theorem 5.2.2 and Theorem 5.2.3, we give a lemma first.

Lemma 5.4.5 *Under the assumptions of Theorem 5.2.1, if $E \subset \mathbb{C}$ is any set satisfying $\nu_{[-1,1]}(\partial E) = 0$, then*

$$\lim_{n\to\infty} \frac{\mathbb{E}[N_n(E)]}{n} = \nu_{[-1,1]}(E), \quad \lim_{n\to\infty} \frac{\mathbb{E}[N_n^2(E)]}{n^2} = \nu_{[-1,1]}^2(E). \tag{5.4.6}$$

Proof. Consider the normalized counting measure $\tau_n = \frac{1}{n}\sum_{k=1}^n \delta_{z_k}$ for a polynomial (5.0.1), where $\{z_k\}_{k=1}^n$ are the zeros of that polynomial, and δ_z denotes the unit point mass at z. Theorem 2.2 of [69] implies that measures τ_n converge weakly to $\nu_{[-1,1]}$ with probability one. Since $\nu_{[-1,1]}(\partial E) = 0$, we obtain that $\tau_n|_E$ converges weakly to $\nu_{[-1,1]}|_E$ with probability one by Theorem 0.5′ of [49] and Theorem 2.1 of [8]. In particular, we have that the random variables $\tau_n(E) \to \nu_{[-1,1]}(E)$ a.s. Then the random variables $\tau_n^2(E) \to \nu_{[-1,1]}^2(E)$ a.s. Hence in both cases convergence holds in L^p sense by the Dominated Convergence Theorem, as $\tau_n^2(E)$ and $\tau_n(E)$ are both uniformly bounded by 1, see Chapter 5 of [31]. It follows that

$$\lim_{n\to\infty} \mathbb{E}[|\tau_n(E) - \nu_{[-1,1]}(E)|] = 0, \quad \lim_{n\to\infty} \mathbb{E}[|\tau_n^2(E) - \nu_{[-1,1]}^2(E)|] = 0$$

for any set E such that $\nu_{[-1,1]}(\partial E) = 0$, and

$$\left|\mathbb{E}[\tau_n(E) - \nu_{[-1,1]}(E)]\right| \le \mathbb{E}[|\tau_n(E) - \nu_{[-1,1]}(E)|] \to 0 \quad \text{as } n \to \infty;$$

$$\left|\mathbb{E}[\tau_n^2(E) - \nu_{[-1,1]}^2(E)]\right| \le \mathbb{E}[|\tau_n^2(E) - \nu_{[-1,1]}^2(E)|] \to 0 \quad \text{as } n \to \infty.$$

But $\mathbb{E}[\tau_n^2(E)] = \mathbb{E}[N_n^2(E)]/n^2$, $\mathbb{E}[\tau_n(E)] = \mathbb{E}[N_n(E)]/n$, $\mathbb{E}[\nu_{[-1,1]}(E)] = \nu_{[-1,1]}(E)$, and $\mathbb{E}[\nu_{[-1,1]}^2(E)] = \nu_{[-1,1]}^2(E)$, which immediately gives the desired results. □

Proof of Theorem 5.2.2. Under the assumptions, we obtain from Theorem 5.2.1 and Theorem 2.1.2 that

$$\lim_{n\to\infty} \frac{1}{n}\mathbb{E}\left[N_n([a,b])\right] = \frac{1}{\sqrt{3}}\nu_{[-1,1]}([a,b]),$$

$$\lim_{n\to\infty} \frac{1}{n^2}\mathbb{E}\left[N_n^2([a,b])\right] = \frac{1}{3}\nu_{[-1,1]}^2([a,b])$$

for any closed interval $[a,b] \subset (-1,1)$. Now we take a sequence of nested closed intervals

$$[-1+1/m, 1-1/m] \quad , m = 2,3,4,\cdots,$$

converging to $(-1,1)$. Note that $\mathbb{E}\left[N_n\left(H\right)\right]$, $\mathbb{E}\left[N_n^2\left(H\right)\right]$, and $\nu_{[-1,1]}(H)$ vanish when H is a single point by (2.2.10) and Lemma 5.4.5, because $\nu_{[-1,1]}$ is absolutely continuous with respect to Lebesgue measure on $[-1,1]$, see [80, Lemma 4.4.1, p. 117]. Hence by monotonicity and [77, Theorem 7.11] using Lemma 5.4.5, we obtain

$$\begin{aligned}\lim_{n\to\infty} \frac{\mathbb{E}[N_n^2([-1,1])]}{n^2} &= \lim_{n\to\infty} \frac{\mathbb{E}[N_n^2((-1,1))]}{n^2} \\ &= \lim_{n\to\infty}\lim_{m\to\infty} \frac{\mathbb{E}[N_n^2((-1+1/m,1-1/m))]}{n^2} \\ &= \lim_{m\to\infty} \frac{1}{3}\nu_{[-1,1]}^2([-1+1/m,1-1/m]) = \frac{1}{3}\nu_{[-1,1]}^2((-1,1)) = \frac{1}{3}.\end{aligned}$$

$$\begin{aligned}\lim_{n\to\infty} \frac{\mathbb{E}[N_n([-1,1])]}{n} &= \lim_{n\to\infty} \frac{\mathbb{E}[N_n((-1,1))]}{n} \\ &= \lim_{n\to\infty}\lim_{m\to\infty} \frac{\mathbb{E}[N_n((-1+1/m,1-1/m))]}{n} \\ &= \lim_{m\to\infty} \frac{1}{\sqrt{3}}\nu_{[-1,1]}([-1+1/m,1-1/m]) \\ &= \frac{1}{\sqrt{3}}\nu_{[-1,1]}((-1,1)) = \frac{1}{\sqrt{3}}.\end{aligned}$$

Thus we have

$$\lim_{n\to\infty} \frac{\mathbb{E}[N_n^2([-1,1])]}{n^2} = \frac{1}{3}, \quad \lim_{n\to\infty} \frac{\mathbb{E}[N_n([-1,1])]}{n} = \frac{1}{\sqrt{3}}. \tag{5.4.7}$$

Moreover,

$$\mathrm{Var}[N_n([-1,1])] = o(n^2)$$

is proved by applying (5.4.7) to the relation

$$\frac{\mathrm{Var}[N_n([-1,1])]}{n^2} = \frac{\mathbb{E}[N_n^2([-1,1])]}{n^2} - \left(\frac{\mathbb{E}[N_n([-1,1])]}{n}\right)^2.$$

□

Proof of Theorem 5.2.3. We already know from Theorem 5.2.2 that

$$\lim_{n\to\infty} \frac{\mathbb{E}[N_n^2([-1,1])]}{n^2} = \frac{1}{3}.$$

Applying Lemma 5.4.5, since $\nu_{[-1,1]}(\partial(\mathbb{R}\setminus[-1,1])) = \nu_{[-1,1]}(\mathbb{R}\setminus[-1,1]) = 0$ and the support of $\nu_{[-1,1]}$ is $[-1,1]$,

$$\lim_{n\to\infty} \frac{\mathbb{E}[N_n^2(\mathbb{R}\setminus[-1,1])]}{n^2} = \nu_{[-1,1]}^2(\mathbb{R}\setminus[-1,1]) = 0.$$

On the other hand, we have that

$$\begin{aligned}\frac{\mathbb{E}[N_n^2(\mathbb{R})]}{n^2} &= \frac{\mathbb{E}[N_n^2((\mathbb{R}\setminus[-1,1])\cup[-1,1])]}{n^2} \\ &= \frac{\mathbb{E}[N_n^2(\mathbb{R}\setminus[-1,1])]}{n^2} + \frac{\mathbb{E}[N_n^2([-1,1])]}{n^2} + 2\frac{\mathbb{E}[N_n([-1,1])N_n(\mathbb{R}\setminus[-1,1])]}{n^2},\end{aligned}$$

and for all n,

$$0 \le \frac{\mathbb{E}[N_n([-1,1])N_n(\mathbb{R}\setminus[-1,1])]}{n^2} \le \frac{\mathbb{E}[N_n(\mathbb{R}\setminus[-1,1])]}{n}.$$

Using Lemma 5.4.5 and noting that the support of $\nu_{[-1,1]}$ is $[-1,1]$, we get

$$\frac{\mathbb{E}[N_n(\mathbb{R}\setminus[-1,1])]}{n} = \nu_{[-1,1]}(\mathbb{R}\setminus[-1,1]) = 0.$$

Thus we have the desired result

$$\lim_{n\to\infty} \frac{\mathbb{E}[N_n^2(\mathbb{R})]}{n^2} = \frac{1}{3}.$$

Note that

$$\frac{\mathrm{Var}[N_n(\mathbb{R})]}{n^2} = \frac{\mathbb{E}[N_n^2(\mathbb{R})]}{n^2} - \left(\frac{\mathbb{E}[N_n(\mathbb{R})]}{n}\right)^2.$$

By Theorem 2.1.1,

$$\lim_{n\to\infty} \frac{\mathbb{E}[N_n(\mathbb{R})]}{n} = \frac{1}{\sqrt{3}}.$$

Thus

$$\lim_{n\to\infty} \frac{\mathrm{Var}[N_n(\mathbb{R})]}{n^2} = 0.$$

□

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