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THE EFFECT OF COGNITIVE STYLE ON THE ACQUISITION OF MATHEMATICAL CONCEPTS PRESENTED THROUGH EMPHASIS ON POSITIVE AND NEGATIVE INSTANCES

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THE EFFECT OF COGNITIVE STYLE ON THE ACQUISITION OF MATHEMATICAL CONCEPTS PRESENTED THROUGH EMPHASIS ON POSITIVE AND NEGATIVE INSTANCES

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THE EFFECT OF COGNITIVE STYLE ON THE ACQUISITION OF MATHEMATICAL CONCEPTS PRESENTED THROUGH EMPHASIS ON POSITIVE AND NEGATIVE INSTANCES

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DEDICATION

This dissertation is dedicated to my beloved wife, Rita, and my three special daughters, Tanya Joy, Julianne Jennifer, and Danielle Jessica.

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CHAPTER I

INTRODUCTION

Rationale

All learning in different subject areas is a highly individual matter and consequently there has been much research in recent years on individualized instruction. Those areas of individualized instruction which have received the greatest emphasis in research are the student's rate of learning, ability, and past achievement level, (De Vault & Kriewall, 1970). However, attempts in mathematics at individualized instruction have met with very limited success, (Schoen, 1976; Miller, 1976). Often the rate at which a student learns, his ability, and consequently his academic achievement level vary from situation to situation and from subject to subject. One important factor in individual difference in performance is the characteristic way individual students process information and react with their surroundings. Research indicates that the cognitive behavior of the individual in a learning situation is greatly influenced by his or her cognitive styles, (Dienes, 1967; Nelson, 1973). Thus, any individualized instruction to be effective should take individual learning styles into consideration.

> ...to teach mathematics effectively demands that one be concerned with the individual, for mathematics learning is a highly individual matter. Mathematics ideals, skills, and structures surely are learned on an individual basis even though we can, as individuals,benefit from assistance given by others or by witnessing their behavior in performing mathematical processes as an example for our own behavior, (De Vault & Kriewall, 1970, p. 409).

Many different styles have been researched and defined. Three major efforts in cognitive style research have received the most attention:

1. Field-independent/Field-dependent (Herman Witkin)
2. Cognitive Controls (George Klein)
3. Conceptual Tempo (Jerome Kagan)

The importance of the research in these areas to both the student and the instructor is exemplified by Witkin (1973). Reporting on the impact of cognitive style research, he stated:

> Cognitive style is a potent variable in students' academic choices and vocational preferences; in students' academic development through their careers; in how students learn and teachers teach; and in how students and teachers interact in the classroom, (1973, p. 1).

The cognitive styles which, by far, have received the most attention are Witkin's field-independent/field-dependent cognitive styles, (Cross, 1976). Since the early fifties, these styles have been studied, refined, and are now characterized as a broad dimension that includes both analysis and structuring in both perceptual and intellectual activities, (Witkin & Goodenough, 1977).

In general, the field-dependent cognitive style has been shown to be detrimental in concept acquisition in analysis and problem solving, especially in the areas of mathematics and science. In contrast the field-independent cognitive style has been shown to be beneficial in these areas. In the social sciences, the exact opposite is true; the field-dependent cognitive style is more beneficial and the field-independent cognitive style is more detrimental. Similarly, the field-dependent individual is more adept at interpersonal relations, while the field-independent individual has a greater degree of personal autonomy, (Witkin & Goodenough, 1977; Witkin, et al, 1977).

Research has also shown that field-dependent learners, in concept learning, are more likely to be affected by just the salient cues of the problem situation, while the field-independent learners are, in general, cognizant of both the salient and non-salient cues, (Witkin & Goodenough, 1977). Thus, it appears that at one end of the continuum, there are field-independent learners who have a learning style that is benign to concept learning in mathematics, and at the other end of the continuum there are field-dependent learners who have a learning style that is malign to mathematical concept learning. Within this continuum, Witkin & Goodenough (1977) have defined what they refer to as the "mobile" cognitive style which is the style of an individual who can function in either the field-independent or field-dependent cognitive styles.

The importance of reinforcement on concept learning has long been understood and implemented by the classroom practitioner. There is now evidence that negative reinforcement has a greater impact (consequentially potential) upon the field-dependent learner than the field-independent learner, while positive reinforcement has little effect upon either learner, (Ferrell, 1971; Fitz, 1970). In addition, McLeod & Adams (1970) have reported that field-dependent students learn better in an expository setting where stress is placed on the structure of the learning situation.

Thus the probability is quite high that in any real-world classroom setting there are individuals whose field-dependent learning style places them at a disadvantage in mathematical concept learning. The challenge is to design a teaching strategy that provides maximal guidance with structure and takes advantage of the impact that negative reinforcement has upon field-dependent learners.

Gagné has stated that "the effect of concept learning is to free the individual from control by specific stimuli", (1970, p. 182). One suggestion for mathematical concept learning is the inclusion of negative instances as an important part of the learning process, (Dienes, 1964; Gelbaum & Olmstead, 1964; Shumway, 1974; Bittinger, 1980). The question now can be posed: Will the inclusion of negative instances as a negative reinforcement in a structured learning situation provide for better learning of mathematical concepts for the field-dependent student, consequently freeing the field-dependent student from the exclusive control of salient cues? This study is designed to attempt to answer this question.

Statement of the Problem

The major purpose of this investigation is to examine the interaction between field-independent/field-dependent cognitive styles and the use of positive/negative instances on the acquisition of mathematical concepts selected from an introductory calculus course, and it is directed specifically to the following questions:

1. Are field-independent subjects superior to field-dependent subjects in the initial achievement and retention of mathematical concepts selected from an introductory calculus course?

2. Does the inclusion of negative instances with positive instances aid in the initial achievement and retention of mathematical concepts selected from an introductory calculus course?

3. Can significant interaction between positive versus positive and negative instances and the field-independent/field-dependent cognitive styles be observed?

Definition of Operational Terms and Variables

Concept Learning - A subject has learned the concept, A, if and only if given any class of objects, X, the subject is able to partition X into two disjoint classes: A and - A where A U - A = X, (Shumway, 1969).

Field-Independent - Subjects who score in the upper one-third of the sample on the Hidden Figures Test (HFT) will be classified field-independent.

Mobile - Subjects who score in the middle one-third of the sample on the Hidden Figures Test (HFT) will be classified mobile.

Field-Dependent - Subjects who score in the lower one-third of the sample on the Hidden Figures Test (HFT) will be classified as field-dependent.

Critical Attribute - An attribute that is a necessary characteristic of the given concept is defined as a critical attribute, (Clark, 1971).

Positive Instance - An example which contains or displays all critical attributes of the concept in their appropriate relationship is defined as a positive instance, (Clark, 1971).

Negative Instance - A counterexample which contains or displays (a) some or none of all the critical attributes of a concept in their appropriate or inappropriate relationship, or (b) all the critical attributes in an inappropriate relation is defined as a negative instance, (Clark, 1971).

Overgeneralization - Overgeneralization occurs when a negative instance is classified as a positive instance.

Undergeneralization - Undergeneralization occurs when a positive instance is classified as a negative instance.

Misconception - Misconception occurs when a positive instance is classified as a negative instance and a negative instance is classified as a positive instance.

Hypotheses

The following hypotheses will be tested:

H1: There is no significant difference in initial achievement between field-independent, mobile, and field-dependent subjects.

H2: There is no significant difference in retention between field-independent, mobile, and field-dependent subjects.

H3: There is no significant difference in initial achievement for subjects receiving an instructional sequence of both positive and negative instances and subjects receiving an instructional sequence of all positive instances.

H4: There is no significant difference in retention for subjects receiving an instructional sequence of both positive and negative instances and subjects receiving an instructional sequence of all positive instances.

H5: There is no significant interaction between the two instructional treatments and field-independent, mobile, and field-dependent cognitive styles as measured by the initial achievement instrument.

H6: There is no significant interaction between the two instructional treatments and field-independent, mobile, and field-dependent cognitive styles as measured by the retention instrument.

Research Design

In order to test the above hypotheses, two sections of Mathematics 1743 will be used. Mathmatics 1743 at The University of Oklahoma is a three-hour credit, first course in calculus designed for the business, life, and social science majors. In each section class size will be approximately forty students.

The text used in Mathematics 1743 is: Calculus: A Modeling Approach, Second Edition, by Marvin L. Bittinger, Indiana University-Purdue University at Indianapolis, Addison-Wesley Publishing Company, Reading, Massachusetts. The first five chapters of this text constitute the subject matter for Mathematics 1743.

The two sections of Mathematics 1743 (Section 900 and Section 901) are intact classes. Section 900 meets from 6:00 to 7:25 PM, Monday and Wednesday, Physical Science Center, room 224, and section 901 meets from 6:00 to 7:25 PM, Tuesday and Thursday, Physical Science Center, room 224. One section will be randomly chosen as the experimental class and the other will be designated as the control class.

During the third week of the semester both sections will be given the Hidden Figures Test (HFT: French, Ekstrom, & Price, 1962) to assess the student's cognitive styles. In the fourth week, students in both sections will be tested on their algebraic skills, and these measures will be used as the covariate in the statistical analysis of covariance.

The unit of learning for experimental consideration will be differentiation using the product and quotient rules and the extended power rule. Some of the common errors resulting from overgeneralization, undergeneralization, or misconception of these rules are:

1. if $p(x) = f(x)g(x)$, then $p'(x) = f'(x)g'(x)$.
2. if $p(x) = f(x)/g(x)$, then $p'(x) = f'(x)/g'(x)$.
3. if $p(x) = (f(x))^n$, then $p'(x) = n(f(x))^{n-1}$.
4. if $p(x) = f(x)/g(x)$, then $p'(x) = \dfrac{f(x)g'(x) - g(x)f'(x)}{(g(x))^2}$.

The experimental variable in this investigation is the use of counterexamples as it relates to the introduction of the concepts, the product, the quotient, and the extended power rules for differentiation. Whenever a concept is introduced to the experimental class, the subjects will be exposed to half examples (positive instances) and half counterexamples (negative instances) both within the classroom setting and through outside assignments. In contrast, when a concept is introduced to the control class, the subjects will be exposed to the same number of instances as the experimental class but all positive instances. The dependent measure will be initial achievement and retention as measured by instruments constructed by the investigator.

A quasi-experimental design for nonequivalent control groups will be used. This design is selected because it is a better approximation of true experimental designs and it allows for more control of internal validity, (Huck, Cromier, & Bounds, p. 303). A 2x3 factorial analysis of covariance will be used to test the stated hypotheses with cells distributed as follows:

EXPERIMENTAL	Field-Independent	Mobile	Field-Dependent
CONTROL	Field-Independent	Mobile	Field-Dependent

Figure 1: Cell Distribution

The experimental design model is:

01	02	X	03	04
01	02		03	04

Figure 2: Nonequivalent Control Group Design

01 - Investigator designed test of algebraic skills used as covariate in analysis of covariance.

02 - Hidden Figures Test (Educational Testing Service, 1962) used to assess cognitive styles.

03 - Investigator designed test of initial achievement of selected mathematical concepts.

04 - Investigator designed test of retention of selected mathematical concepts.

X - Treatment Variable.

CHAPTER II

REVIEW OF THE LITERATURE

Cognitive Styles

The term "cognitive style" has been characterized in the literature as the way a student acquires and processes information. In mathemathics education Romberg and De Vault (1967, p. 98) have stressed that learning style is an important factor that influences cognitive behavior in mathematics. In practical terms, Witkin and Moore (1974) defined cognitive style as:

> ...our typical way of processing information, regardless of whether the information has its primary source in the world outside or within ourselves..., (p. 2).

Witkin, Moore, Goodenough, & Cox (1977), in a review of the literature, described the essential characteristics of cognitive styles. Cognitive styles deal with a self-consistency that individual demonstrate in their cognitive functioning. Cognitive styles are concerned with the process (i.e., the how of it) rather than the context (i.e., the why of it) in concept learning. They are concerned with how we think, perceive, and relate with other individuals and our environment. Cognitive styles are "pervasive" dimensions that transcend individual micro approaches to the study of the human mind and how we learn.

Finally, cognitive styles are relatively stable over a period of time.

Among the different cognitive styles (learning styles), Speer (1979) reported that there are three major areas which have been fairly well defined, and as such are variables which influence concept acquisition in the classroom:

1. field-independence/field-dependence
2. cognitive controls
3. conceptual tempo

Speer listed and described the cognitive styles as follows:

1. Field-independence vs. Field-dependence: Viewing situations as a whole (field-dependent) as opposed to viewing situations as consisting of discrete parts.
2. Cognitive controls:
   a. Leveling vs. Sharpening: Ability to perceive changes over a period of time (sharpener).
   b. Focusing vs. Scanning: Ability to copy or describe a given situation within a minimum amount of time and a minimum number of references (scanning).
   c. Constricted vs. Flexible Control: Susceptibility to distraction or interference and inability to withhold attention from irrelevant information.
   d. Breadth of Categorizing: Ability to ignore subtle differences and use less rigid standards in categorizing (broad inclusiveness).
   e. Tolerance for Incongruous or Unrealistic Experiences: Measured by the willingness to accept situations and experiences that are opposed to what the individual knows to be true.
3. Conceptual Tempo (Reflectiveness vs. Impulsivity): Careful and systematic examination of new information, deliberation in decision making (reflective) as opposed to offering immediate reaction, sometimes thoughtlessly (impulsive), (p. 23).

Correlation Between Cognitive Styles

Edgell (1973) studied the relationship between cognitive styles of third grade children and their cognitive strategies used in the attainment of mathematics concepts. It was reported that field-independent and reflective cognitive styles were positively correlated to each other. Field-independent and reflective subjects tended to exhibit a focus strategy in demonstrating concept attainment.

Ausburn (1976) investigated the relationship of visual perceptual types to cognitive styles. The visual perceptual type is described as an individual who uses his eyes as the main intermediaries for sensory impressions. The haptic perceptual type is a "normally sighted individual who uses his eyes as the primary sensory intermediaries only when compelled to do so, preferring to rely on touch...". It was found that on assessment tasks using visual stimuli, the visual perceptual type tended to display the cognitive style traits of field-independence, reflectivity, and sharpening, while the haptic perceptual type tended to display field-dependence, impulsivity, and leveling.

Field-independence/Field-dependence

The early work with field-independent/field-dependent cognitive styles dealt with an individual's ability to determine the upright in space, (Witkin, 1949, 1950, 1954). Using elaborate mechanical devices such as the Rod and Frame Test, and the Body Adjustment Test, and the Rotating Room Test, individuals were designated as field-independent or field-dependent. Those individuals who used gravitational cues to align with the upright were defined as field-independent and those individuals who used visual cues were defined as field-dependent.

Further studies correlated the "space orientation" of the upright with perceptual performance, (Witkin, 1949, 1950). With this development, in an historical perspective, Witkin & Goodenough (1977) reported that:

> It seemed plausible to hypothesize, therefore, that individual differences in perception of the upright were due to differences in disembedding ability; and it seemed parsimonious to redefine field-independent/ field-dependent dimension as a general ability to overcome embedding contexts in perception, (p. 3).

Still later, it was found that the field-independent/field-dependent cognitive styles were a broader dimension than first anticipated involving both intellectual and perceptual activities. Thus Witkin, et al, (1977) stated that.

> In summary, the field-dependent and field-independent cognitive styles are process rather than content variables; they are pervasive dimensions of the individual's functioning; people tend to be stable over time in their standing on them; they are bi-polar and value neutral, (p. 198).

Witkin (1950) searched for other "situations which would reveal the manner in which people perceive a part within a larger field." Based upon the earlier experiments of Gottschaldt (1926), he designed the Embedded Figures Test where simple figures were embedded in a complex field. Reliability ranged from .74 to .87. Using this test he found a wide range of ability to overcome the embedding. Women, on the average, required more time than men to detect the embedded figures.

Bieri, Bradburn, and Galinsky (1958) examined more closely the sex difference in perceptual behavior. They found significant sex differences in favor of males on Witkin's Embedded Figures Test (EFT). For both men and women there was high correlation between EFT performance and mathematical aptitude. In conclusion, they reported that sex differences in EFT performance are a result of males' superior mathematical aptitude and that males more effectively used that aptitude in combination with conceptual approaches to stimuli resulting in better EFT performance.

Rosenfeld (1958) examined Witkin's perceptual approach to cognitive style and mathematics achievement. It was predicted that proficiency in mathematics would be inversely proportional to field-dependency as measured by the Embedded Figures Test. It was found that the poorer mathematics students were more field-dependent and the better students were more field-independent.

Elkind, Koegler, and Go (1963) hypothesized that field-independent subjects were superior to field-dependent subjects on a test measuring perceptual concept formation. The Embedded Figures Test was used to determine field-independency and field-dependency of the adult subjects. It was found that field-independent subjects scored significantly higher than field-dependent subjects on the concept formation tests. It was concluded that field-independency is an "asset" on tests involving abstractions and relations.

Witkin, Goodenough, and Karp (1967) reported in a longitudinal study that despite shifts toward field-independency with maturation, the field-independency or field-dependency of an individual remains quite stable relative to his peers. It was found that:

> ...despite a marked general increase in differentiation in perceptual functioning with age, each individual tends to maintain his relative position among his peers in the distribution of measures of differentiation from age to age, (p. 291).

Schwartz and Karp (1967) examined Witkin's (1954, 1962, 1967) observation of decreasing field-dependency from childhood to early adulthood. In their longitudinal study investigating field-dependency as it relates from adulthood through old age, it was found that field-dependency increased significantly with age. Males were significantly less field-dependent at the ages of 17 and 30 to 39 years old than at the ages of 50 to 80 years old.

Fitz (1970) studied the effects of praise and censure on serial learning as dependent on locus of control and field-dependency. It was hypothesized that field-dependent persons are more socially oriented and more dependent upon external social reinforcement than field-independent persons; hence field-dependent subjects would show greater differences in achievement as a function of varying external reinforcement. It was found that censure (negative reinforcement) did not affect the field-independent subjects, but censure did have a positive effect on field-dependent subjects. Praise (positive reinforcement) had little effect on either field-independent or field-dependent subjects.

Ferrill (1971) in a similar study to Fitz (1971) found that field-dependent children were more responsive to the effects of reinforcement. Verbal punishment served to significantly increase performance. Immediately following reinforcement field-dependent subjects caught up to and surpassed the response rate of field-independent subjects.

Nelson (1972) studied the effect of field-independent/field-dependent cognitive styles and geometric concepts presented through written lessons which did or did not contain verbal emphasis on the relevant attributes. The HFT (<u>Hidden Figures Test</u>) was used to classify the field-independent and field-dependent 7th grade general mathematics students. While treatment had no effect, the field-independent subjects were shown to be superior in the acquisition of geometric concepts, particularly in questions involving identifying the attributes of a concept.

Bien (1974), in a study involving the relationship of cognitive style and the structure of arithmetic materials to achievement in fourth grade arithmetic, noted that cognitive restructuring techniques using outlining and rephrasing to highlight critical features of the problem increased problem solving success for field-dependent children.

Koback (1975) conducted an aptitude-treatment-interaction study using the field-independent/field-dependent and reflective/impulsive cognitive styles, lessons studied inductively or deductively, and pace (fast or slow). The sample consisted of fifth grade mathematics students. It was found that field-independent and reflective subjects retained more learning when taught deductively than when taught inductively. Field-dependent and impulsive subjects retained more when taught inductively. Field-independent and reflective subjects were superior in measures of cognitive and affective mathematics achievement.

Threadgill (1976) hypothesized that field-independent students would learn better from a discovery approach, while field-dependent students would learn better from a didactic method of instruction in the acquisition of 7th grade mathematics concepts. It was discovered that field-independent subjects using either the discovery approach or the didactic method performed significantly better on concept attainment than field-dependent subjects. There was no significant difference between the two methods of instruction, nor was there a significant interaction between cognitive style and methods of instruction.

Blake (1976), in a laboratory experiment, analyzed mathematical problem solving as it interacts with field-independent/field-dependent cognitive styles. He found that field-independent subjects displayed a wider range of problem solving strategies than field-dependent subjects. When a problem proved difficult, field-independent subjects would trace back over previous attempted solutions, while field-dependent subjects would be more likely to reread the problem and start over.

Witkin, et al, (1977) conducted a longitudinal study examining the role of field-independent/field-dependent cognitive styles in student choice of college majors. It was reported that students whose initial choice of major was not compatible with their cognitive styles tended to select majors that were compatible by their senior year or post graduate study. Also it was found that there was a "tendency" to do better in academic areas congruent with cognitive style; field-independent subjects did better in the physical sciences, and field-dependents did better in the social sciences.

Baldwin (1977) investigated the interaction of field-independent/field-dependent cognitive styles with varying methods of instruction in mathematics using homogeneous groupings, heterogeneous groupings, and individual study. It was found that field-independent subjects achieved equally well in all methods of instruction and that field-dependent subjects did not achieve more in group study as hypothesized.

Fitzgerald (1977) examined, over a two year period, relationships between field-independent/field-dependent cognitive styles, student attitude, and achievement in reading and mathematics at the 4th and 5th grade levels. She found that field-independent subjects outperformed field-dependent subjects in reading but not in mathematics.

Chang (1977) examined the effects of field-independent/field-dependent cognitive style as a function of three instructional methods using: 1. a positive instance, 2. definition and a positive instance, and 3. definition only. The unit of instruction was geometry concepts selected from the elementary grade levels. Field-independent subjects outperformed field-dependent subjects in concept acquisition. The method of instruction using the definition and positive instance was significantly better than the other two methods. There was no significant interaction reported.

Horak (1977) compared the effects of the inductive/deductive methods of instruction and field-independent/field-dependent cognitive styles upon achievement in geometric concepts selected from a pre-service course for elementary education majors. Her major conclusion was that the inductive method of instruction was best for field-dependent subjects for both concept attainment and transfer.

McLeod, et al, (1978) investigated the relationship between field-independent/field-dependent cognitive styles and two methods of instruction differing in the amount of guidance and the level of abstraction. It was reported that field-independent subjects did better with minimal guidance and field-dependent subjects did better with maximal guidance. The investigators surmised that "...this study indicates that measure of cognitive style may be one way to identify the most appropriate levels of guidance for individual students.", (p. 173).

McKay (1978) examined the relationship of locus of control, field-independent/field-dependent cognitive styles, and spatial visualization upon mathematics achievement to compare differences in performance by males and females. She found no significant difference at the fourth grade level, but at the eighth grade level there was a significant difference between males and females on spatial visualization. There was no significant difference between sexes in mathematics achievement when grouped by cognitive styles.

McLeod & Adams (1979) tested the hypothesis that field-independent/field-dependent cognitive styles would interact with treatments of instruction differing in small group or individualized instruction. It was felt that field-dependent subjects would benefit most from small group instruction and field-independent subjects would benefit most from individualized instruction. Three dependent observations were mathematics achievement, retention, and student rating of methods of instruction. While there was a significant interaction between achievement and field-independency, it was surmised that this was not due to the cognitive style of the subjects, but resulted from their general ability since the interaction was not in the direction hypothesized nor supported by the literature.

Adams & McLeod (1979) studied the relationship between field-independent/field-dependent cognitive styles and instructional treatments using high or low levels of guidance. The experimental unit of instruction consisted of mathematical concepts of networks, transversability, and applications. The subjects, elementary education majors, were pretested on cognitive style and mathematics achievement measuring "crystallized ability" (Cattell, 1971). There was reported no significant difference between cognitive styles and the post-tests measuring achievement or retention.

McLeod & Briggs (1980) investigated the interaction between field-independent/field-dependent cognitive styles and general reasoning ability with treatments using an inductive or deductive approach on the mathematical concept of equivalence relations. The subjects were students taking an advanced class in mathematics for elementary education majors. The investigators wanted to determine if the previous reported interactions between cognitive styles and the level of guidance could be extended to the sequence of instruction dimension of discovery learning. They also wanted to substantiate the finding of interactions between general reasoning and the inductive/deductive sequence of instruction. The dependent measures were the scores on an achievement test and a retention test. There were significant interactions between the field-independent cognitive style and the transfer test and between the general reasoning ability and both the achievement test and the transfer test.

McLeod & Adams (1980) reported that:

> ...ATI (aptitude-treatment-interaction) research in mathematics education has found two aptitude variables, general reasoning and field independence, that have produced significant interactions with two dimensions of discovery learning, level of guidance and inductive instruction, (p. 226).

The investigators searched for ATI's between the above aptitudes and treatments that differed in both level of guidance and inductive/deductive methods of instruction. They predicted the field-independent subjects would perform better with the discovery treatment, and subjects who tested high in general reasoning would perform better in the expository treatment. The results showed that there was not a significant interaction between the field-independent cognitive style and the discovery treatment as expected. It was felt that the treatment provided more guidance than intended and this confounded the results. There was a significant interaction between general reasoning and the retention test in the expository treatment as hypothesized.

In general, field-independent subjects use internal referents to organize and restructure the field of learning. Field-dependent subjects tend to use external referents and go along with the prevailing field. Field-independent students are more likely to be introverted, impersonal, and insensitive to social cues. In contrast, field-dependent students are more likely to be extroverted,inter-personal, and attentive to social cues. Field-independent subjects take an active, hypothesis testing, analytic, problem solving approach to learning. Field-dependent subjects are likely to be passive and take a "spectator approach" to learning. In a typical learning situation field-dependent students are affected by cue salience, while field-independent students are more apt to be affected by salient and nonsalient cues. Field-dependent subjects tend to excel in the social sciences, and field-independent subjects tend to excel in the physical sciences.

In particular, field-independent subjects have been shown to be more apt to excel in mathematics concept achievement than field-dependent subjects, (Rosenfeld, 1958;Nelson, 1972;Threadgill, 1976; etc). While cognitive style modification has generally met with little success, and indeed, even if attainable, is of questionable value due to the bi-polar, neutral value of the field-independent/field-dependent cognitive styles, it has been found that treatments featuring an inductive approach or maximal guidance result in better performance for field-dependent subjects in mathematical concept acquisition, (Koback, 1975;Horak, 1977; McLeod, 1978).

Positive & Negative Instances

Shumway (1969) noted that the research of relevant literature in the role of concept acquisition and positive and negative instances revealed basically two distinct groups. In one group we have the research of educational psychologists in the learning of extremely simple concepts in a well-controlled atmosphere. In this area research has revealed inconsistent results. It appears that negative instances do make a measurable contribution to the learning of simple (dimensioned) concepts, but the results are not generalizable to the classroom situation. In the second group, Shumway reported: "We have the mathematics educators researching the effect of specific variations in treatments on the learning of certain mathematical concepts." The results generally showed no significant differences between treatments. It was to bridge this apparent contradiction that Shumway directed his research. He found no studies in 1969 dealing with the effect of counterexamples on the learning of mathematics concepts. In the past decade a number of studies have been conducted in the use of positive and negative instances as they relate to mathematics, but for the most part they have dealt with simple geometric concepts.

Some of the earliest and most often cited studies dealing with positive and negative instances were conducted by Smoke in 1932 and 1933. His first study dealt with serial presentation of six simple geometric concepts using one 16 card sequence of positive instances, and the other, a sequence of both positive and negative instances. The 1933 study was similar to the first study but instead of a serial presentation, he used a simultaneous presentation of the learning materials. His primary concern was the "rapidity" of concept learning. In both experiments there was no statistical significant evidence that one treatment was superior to another. While being antagonistic to the use of negative instances for rapidity of learning, Smoke reported that most of the subjects did prefer positive and negative instances. He stated that:

> There is a tendency for negative instances to discourage "snap judgement." When the subjects were learning from both positive and negative instances they tended to come to an initial wrong conclusion less readily, and to subsequent wrong conclusions less frequently, than when they were learning from positive instances only. Thus, although negative instances may not make for rapidity in learning they tend to make for accuracy, expecially in the case of difficult concepts. It appears that in so far as negative instances assist concept learning they do so largely because of the way in which they prevent the learner from coming to one or more erroneous conclusions while he is still in the midst of the learning process, (p. 587).

Thus Smoke felt that the role of negative instances was to sharpen the concept.

Hovland (1952) cited Smoke's results that negative instances were of little help in concept learning. However, he felt that it was not clear "whether the ineffectiveness of negative instances is primarily attributable to their low value as carriers of information, or whether it is primarily due to the difficulty of assimilating the information which they do convey.", (p. 461). Hovland stressed the value of first analyzing the information transmitted by positive and negative instances, as one instance may transmit more information than another. Secondly, once the information transmitted was equated, he suggested one could experimentally evaluate the merits of positive and negative instances on concept learning. Thus he attempted a mathematical analysis of the number of positive and negative instances required for concept learning under laboratory conditions. In his experiment he used simple geometric shapes allowed to vary on three dimensions (size, shade, and shape). His results showed a wide variation in the positive and negative instances necessary for concept acquisition. One case required only two positive instances and 625 negative instances, while another case required five positive instances and only two negative instances. He concluded that the "relative effectiveness of positive and negative instances cannot be given a categorical answer.", (p. 477). In general, Hovland felt that for conjunctive concepts, positive instances transmit more information than negative instances.

Hovland & Weiss (1953) concluded that positive instances were more beneficial than negative instances in concept learning. The purpose of their investigation was to compare the learning of concepts when the information is transmitted by all positive instances, all negative instances, or mixed positive and negative instances. They attempted to equate the informational content of the different sequences. The results indicated that it was "considerably harder" to define a concept based on a series of negative instances than with a series of positive instances. Their second experiment was similar to the first except they used simultaneous presentation of the instances rather than serial presentation. They found a "consistent superiority" for all positive instances.

In their third experiment, they hypothesized that the greater difficulty using negative instances might be that more negative instances were needed to define a given concept. Hence, they used only concepts which could be defined in terms of the same number of positive or negative instances. The results showed that the all-positive series was superior to the all-negative series. Hovland and Weiss concluded that "the all-negative instances are thus shown to be consistently inferior to all-positive instances,", (p. 182). However, the generalization that concepts cannot be learned from all-negative instances is false, since they observed that over half of their subjects were able to use all negative instances to arrive at the correct conclusion.

Bruner, Goodnow, and Austin (1956) supported the use of positive instances over negative instances. They stated that "...subjects seem not as willing or able to use negative information, instances telling what the concept is not, in the process of obtaining a concept.", (p. 180). They also hypothesized that "most of our environment seems geared to working with positive instances...", (p. 159).

Freiberg & Tulving (1961) using the above hypothesis essentially replicated Hovland & Weiss' (1953) third experiment. They explored the effects of repeated presentations of sequences using positive and negative instances to determine if the observed differences in concept learning would decrease as the subjects became more efficient in using negative instances. They reported two findings:

> 1. A subject's ability to solve concept identification problems (ability to think?) is very greatly affected by practice. This is true for both positive and negative instances, (p. 105).
>
> 2. Although during the early stages of practice the subjects seemed much less capable of assimilating information from negative instances than from positive instances, the difference was very small at the end of 20 trials, (p. 105).

Huttenlocher (1962) studied "the effects of ordering of instances on simple concept formation problems with equivalent amount of information per instance.", (p. 36). She examined four categories of problems using sequences of (+,+), (+,-), (-,+), and (-,-). Huttenlocher found the group receiving two negative instances, (-,-) had significantly poorer performance. It was also reported that the order of the instances was significant. The best performance was by the group which received (-,+), followed by (+,-), and (+,+). She concluded that:

> With one dimensional concepts...all negative instances are harder to use than those containing all-positive instances. The results do not, however, support the findings that (a) the mixed series lies intermediate in difficulty between all-positive and all-negative series or (b) that there are no effects from the ordering of instances in the mixed series, (p. 39).

Fryatt & Tulving (1963) attempted "to clarify the role inter-problem practice (intertask transfer) plays in the utilization of information from positive and negative instances of concepts.", (p. 106). Utilizing four groups of subjects, all subjects solved twenty-four concept identification problems divided into two series of twelve problems. Each conceptual problem used either three positive instances or one positive and two negative instances. Their conclusions supported the work of Freiberg & Tulving (1961) that interproblem practice in the case of mixed instances and less extensively in the case of all positive instances, had a pronounced effect, i.e., concept identification improved greatly with practice problems from the same class.

Conant (1965) reported that the difficulty in learning disjunctive concepts is caused by the necessity of utilizing information from negative instances. He hypothesized that (1) subjects given a limited amount of training can learn to effectively utilize negative instances in disjunctive problems, and (2) their efficiency in using negative instances is a function of the order of presentation of positive and negative instances. Hypothesis (1) was substantiated when it was found that 94 percent of the subjects solved tests containing only negative instances in fewer trials than had been reported in the literature. To test hypothesis (2), Conant utilized six groups characterized by the order of presentation of positive and negative instances as follows:

1. (-,-,-,-,+,+)

2. (-,-,-,+,+,+)

3. (+,+,+,-,-,-)

4. (-,+,-,+,-,+)

5. (+,-,+,-,+,-)

6. (+,+,+,+,+,+)

The results of a permutation test revealed that the rank order of the group did not significantly differ from the predicted order: 1-2-3-4-5-6. It was also reported that the subjects showed a preference for negative instances. Conant remarked that subjects coult utilize negative instances in the acquisition of disjunctive concepts, and that previous reports of difficulty had been overemphasized.

Tavrow (1965) examined the question: Is improvement in performance using negative instances the result of previous practice with negative instances or is it the result of previous familiarization with the universe which contains the concepts? It was felt that if a subject became familiar with the entire universe, the better he would be able to form concepts from the use of negative instances. The results showed that concept acquisition using negative instances was also a function of the amount of previous familiarity with the universe and the amount of practice in the technique specific to the task of concept formation using negative instances.

Haygood & Stevenson (1967) investigated, using geometric designs, the effects of varying the proportion of positive instances from 10% to 80%. It was found that the number of trials to concept acquisition increased as the percent of positive instances decreased. They attributed this to the variation in information rates as the percent of positive instances changed and the difficulty in utilization of information from positive and negative instances.

Bourne & Guy (1968) conducted an experiment to investigate rule learning on the basis of information provided by positive instances only, negative instances only, or a mixture of positive and negative instances. They examined attribute identification (the rule is given and the stimulus attributes are unknown), and rule learning (the attributes are given and the rule is unknown). When the task was to identify the relative attributes, given the rule, the subjects performed best when informational attributes (positive or negative) were representative of a small homogeneous set. In rule learning, the subjects performed best when presented with the greatest variety (positive and negative) of instances.

Taylor (1969) examined the effect of positive and negative instances in an inductive-deductive approach to concept learning in a classroom setting. The subjects were undergraduate elementary school teachers enrolled in a pre-service methods course. The experimental unit of learning dealt with a classification scheme using Bloom's Taxonomy of Educational Objectives. The dependent variable was the proportion of negative instances used in concept acquisition. It was concluded that the subject's performance was not differentially related to the proportion of negative instances, and that a mixture of positive and negative instances can be successfully used in concept learning.

Siegel & Forbes (1969) examined, using geometric shapes, rule structure as it relates to the proportion of positive instances in concept attainment. They hypothesized that:

1. the ability to solve disjunctive concepts increases with age,

2. positive instances are more effective for solving conjunctive concepts and negative instances are more effective for solving disjunctive concepts, and

3. older subjects will show more improvement than younger subjects as the proportion of negative instances increases in disjunctive concept acquisition.

The only significant result was that as the proportion of negative instances increases, disjunctive concept problems were solved more easily.

Shumway (1969) investigated the role of counterexamples in the development of mathematical concepts at the eighth grade level in a classroom setting. He examined the following dependent variables:

1. general mathematics achievement
2. specific mathematics achievement
3. inductive reasoning
4. syllogistic reasoning
5. reading mathematics definitions
6. tendency to overgeneralize

The content areas used included quadrilaterals, exponents, and operations. The experimental group received an equal number of positive and negative instances. The control group received only positive instances. It was found the use of positive and negative instances discourages overgeneralization.

Davidson (1969) felt that previous research in positive and negative instances had been confounded because there is not necessarily more difficulty in using negative instances, but the concept acquisition procedure itself is more complex when negative instances are used. He concluded that "In general, it now appears that positiveness or negativeness per se is of little consequence; the relative usefulness of the two kinds of instances depends primarily on the circumstances ", (p. 372). His search revealed that the supposed detrimental effect of negative instances is either weak or absent.

Anderson (1970) undertook a quarter long investigation of the role of counterexamples in a first year course in calculus. He used sixty students enrolled in two sections. The experimental class was exposed to several examples and counterexamples for each concept. The control class received only positive examples. The numbers of examples used in either class was the same. The investigator was interested in the effects of counterexamples on:

1. general mathematics achievement
2. specific mathematics achievement, and
3. the ability to generalize.

There was no statistical evidence in favor of either treatment. However, both the experimenter and the subjects felt that the use of counterexamples was of "some" benefit. There was no attempt to discuss which specific introductory calculus concepts were or were not affected by the use of counterexamples.

Fraunfelker (1970) examined attribute identification in concept learning as a function of concept, type, mode of presentation, and positive versus negative instances. He used eighty psychology students as subjects in a laboratory experiment involving geometric patterns as stimuli. He found that when both positive and negative instances are present, the attributes of disjunctive concepts are more difficult to identify than those of conjunctive concepts. It was also reported that the initial order of positive followed by negative instances facilitated conjunctive concept identification and inhibited disjunctive concept identification. The exact opposite was true when the initial order was negative instances followed by positive instances.

Jones (1972) investigated the effects of positive and negative instances on the attainment of simple conjunctive mathematical concepts. He hypothesized that subjects who receive a series of positive instances would achieve higher scores on a dependent measure of achievement than those subjects using both positive and negative instances. There was no significant difference between treatments.

Tenneyson, Wooley, & Merrill (1972) reported upon the controversy in concept research concerning the value of negative instances. They examined four instructional strategies for promoting concept acquisition. The independent variables were:

1. probability level of positive and negative instances empirically determined by subjects who correctly classify instances as positive or negative.

2. the matching of a positive instance to a negative instance so that the irrelevant attributes are similar, and

3. divergency of a positive instance with other positive instances so that all irrelevant attributes differ.

Tenneyson, et al, manipulated the above independent variables to examine the effect on overgeneralization, undergeneralization, correct classification, and misconception. It was found that the use of negative instances decreases overgeneralization.

Feldman (1972) and Swanson (1972) both examined the effect of number and type of instances on concept acquisition in the absence of the definition. Both studies revealed that a rational set of both positive and negative instances were superior to a set of all positive instances in subject recognition of new instances and tendency to prevent overgeneralization. They then proposed to replicate the above studies, but this time they included the definition with the treatments. In separate studies, they found that the inclusion of the definition of the concept partially nullified the superiority of the use of negative instances with positive instances. Klausmeier, Ghatala, & Frayer (1974), in summarizing, concluded:

> ...When no definition is presented, a rational set of both positive and negative instances should be presented... When a concept definition is provided, the number and type of instances provided are less important, (p. 202).

Houde (1972) examined the interaction between intelligence and the use of both positive and negative instances on the attainment of the geometric concept of similarity at the sixth grade level. He concluded that:

1. the concept of similarity is attained more efficiently with all positive, or alternating positive and negative instances than with all negative instances,

2. all negative instances tend to confuse subjects with low I.Q., and

3. high I.Q. subjects using positive and negative instances exhibited the best performance on measures of achievement.

Houtz, Moore, and Davis (1973) studied the effect of positive and negative instances on the learning of a non-dimensioned concept selected from 7th and 8th grade mathematics. Positive instances with no common irrelevant attributes and positive and negative instances lacking one common attribute resulted in better performance. It was concluded that the relationship between the structure of positive and negative instances within a learning situation was critical to the acquisition of non-dimensioned concepts.

Hoehn (1973) also investigated the effect of sequences of positive and negative instances upon the attainment of geometric concepts with the subject's level of intelligence. It was replicated that all positive instances or mixed positive and negative instances were significantly more efficient than all negative instances or no instances. Hoehn did not find an interaction between I.Q. and the use of positive and negative instances.

Chernick (1974) varied the proportion of positive and negative instances in an experiment with the learning of three simple concepts from algebra, number theory, and topology. All procedures were carried out in a single class period. Results indicated that there was no significant difference in concept acquisition using all positive instances, half positive and negative instances, and all negative instances.

Shumway (1974) studied the effect of negative instances on the acquisition of commutativity and associativity in the 9th grade. He found that there was a significant difference in achievement favoring the use of both positive and negative instances for associativity over positive instances alone. He also reported a significant effect for negative instances transferring from commutativity to associativity, i.e.,"The acquisition of the concept of associativity was improved by the effect of negative instances for commutativity." He proposed three possible explanations:

1. Negative instances are a necessary and integral part of learning of each concept.

2. Negative instances teach subjects to be skeptical, independent of a given concept.
3. Negative instances simply teach subjects the proportion of the criterion instances that should be classified as negative using guessing, (p. 210).

Curley (1980) studied the effect of varying proportions of positive and negative instances on a student's misinterpretation of the meaning of statistical hypotheses. His work differed from most earlier studies because of his selection of a more complex variable and his use of "interpretation" as the dependent measure rather than concept acquisition. Curley conjectured that increasing the proportion of negative instances up to two-thirds negative would result in increased sensitivity to misinterpretation of statistical hypotheses. The only significant result reported was between the group receiving two-thirds negative instances and the control group who did not receive the treatment. While not statistically significant at the .05 level, it was reported that whichever group received the larger proportion of negative instances performed better than any group receiving fewer negative instances.

For the most part, the literature has not revealed a conclusive finding for the inclusion of negative instances in concept learning. Clark (1971) reviewed over 250 experimental studies in concept attainment and of these, only 50 dealt with positive and negative instances. It was found that 26 of these 50 studies reported a detrimental effect for the use of negative instances, while only 11 of these 50 studies did not report a detrimental effect.

Cognitive Style & Positive and Negative Instances

Nelson & Chavis (1975) investigated the effect of field-independent/field-dependent cognitive style on concept acquisition in non-dimensioned concepts, "real world concepts", from social learning.

They hypothesized that:

1. field-independent subjects are superior to field-dependent subjects in concept acquisition and

2. the inclusion of negative instances will differentially affect the identification of positive instances and the identification of negative instances.

Subjects, tested using the <u>Hidden Figures Test</u>, were classified as field-independent, neutral, or field-dependent. It was found that field-independent subjects are superior to field-dependent subjects if concept acquisition was limited to identification of positive instances. However, there was no significant difference between the field-independent, neutral, and field-dependent subjects on identification of negative instances. There was a significant interaction between the neutral group (subjects who displayed characteristics of both field-independence and field-dependence) and the treatment using both positive and negative instances in the identification of positive instances. It "appears" that the treatment of positive and negative instances hindered identification of positive instances.

Gage (1976) examined the effects of positive and negative instances and field-independent/field-dependent cognitive style on the acquisition of concepts selected from a first year high school level algebra course. The subjects were classified as field-independent or field-dependent by their scores on the <u>Hidden Figures Test</u>. It was found that subjects receiving the treatment of both positive and negative instances outperformed the subjects receiving only positive instances. Field-independent subjects outperformed field-dependent subjects, but it was not statistically significant. There also was no significant interaction between cognitive style and the treatments of positive and negative instances or positive instances only.

CHAPTER III

METHODOLOGY

The Sample

The final sample consisted of sixty-two students enrolled in Mathematics 1743, sections 900 and 901, fall semester, 1980. Mathematics 1743 is an introductory calculus course for business, life, and social sciences majors. Only students completing all phases of the experiment were included in the final sample. The subjects were not told that they were involved in the experiment. Section 900 was randomly chosen as the control class, and section 901 was randomly chosen as the experimental class. The distribution of the final sample for the two sections was as follows:

	Sec 900		Sec 901	
	Male	Female	Male	Female
Freshman	3	4	2	2
Sophmore	6	3	5	8
Junior	3	4	5	1
Senior		4	3	2
Graduate	2	2		3

Table 1: Sample Distribution

Materials and Instructions

The experimental unit of instruction consisted of concepts selected from the introductory calculus course. These concepts dealt with the following rules for differentiation and their application to problem solving:

1. if $p(x) = f(x)g(x)$, then $p'(x) = f(x)g'(x) + g(x)f'(x)$.
2. if $p(x) = f(x)/g(x)$, then $p'(x) = \dfrac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$
3. if $p(x) = (f(x))^n$, then $p'(x) = n(f(x))^{n-1}\ f'(x)$

Many students overgeneralize the rules for differentiation of the sum and difference of two functions and the power function; often they attempt to apply these rules to problem situations requiring the above rules.

The treatments for the experimental and control classes differed in the use of counterexamples (negative instances). Both classes received the same number of instances, but the experimental class received half examples and half counterexamples. The control class received all examples and no counterexamples for each concept introduced.

The treatment for the experimental class extended over a two week period from October 9, 1980, to October 23, 1980. During this time period, the subjects in the experimental class were taught using the lecture-discussion method, supplemented by two in-class handouts containing the lecture notes and all examples and counterexamples used in introducing the concepts. In addition, the experimental subjects received two homework assignments.

On October 9, 1980, the experimental class initially received a review of the definition of the derivative of a function, a review of the fundamental rules for finding the derivative of the sum or difference of two differentiable functions, and a review of the rule for finding the derivative of the power function, i.e., $f(x) = x^n$. Subsequently, the experimental class, using the definitions of the rules and equal numbers of examples and counterexamples, was introduced to the rules for finding the derivative of the product or quotient of two differentiable functions. The examples and counterexamples were selected to emphasize the critical attributes of the concepts.

On October 14, 1980, after a review and a question-answer session dealing with the concepts introduced in the previous class meeting, the experimental subjects spent the remainder of the class time working on the first homework assignment exercise. This homework assignment required the subjects to work with examples and counterexamples of the concepts. At the conclusion of the class period, the subjects were instructed to finish the assignment and to return it the following session.

After reviewing the homework assignment exercises on October 19, 1980, the experimental students were taught the rule for finding the derivative of a function raised to a power, i.e., the extended power rule. Again, the lecture-discussion method of instruction was supplemented by an in-class handout containing the lecture notes and equal numbers of examples and counterexamples used in emphasizing the critical attributes of the concept. At the conclusion of the class session, the subjects were given the second homework assignment exercises, and they were instructed to complete the exercises and return them the following session.

On October 21, 1980, the treatment for the experimental class was concluded after discussing the second homework assignment and reviewing the three concepts introduced in the treatment.

The treatment for the control class extended from October 13, 1980, to October 27, 1980. The method of instruction, using the lecture-discussion approach, supplemented by the in-class handouts and the homework assignments exercises, was identical to the treatment for the experimental class with the following exceptions: each counterexample used in the experimental treatment was replaced, using the same problem, when possible, by an example (positive instance) of the concepts involved in the study. The experimental and control handouts are shown in Appendix A.

The Covariate Measure

Often in experiments in the real-world classroom situation, the investigators do not have the option of randomizing the subjects between the experimental group and the control group. Hence, analysis of covariance provides a statistical analysis which, in effect, adjusts for initial between group differences on selected covariates, (Kerlinger, 1973). The covariate measure chosen for this study was student performance on an achievement test designed to measure algebraic skills necessary for the calculus course.

The same test, as shown in Appendix B, was administered to the experimental class on October 16, 1980, and to the control class on October 17, 1980, respectively. The general form of the Kuder-Richardson formula (Ebel, 1965) was used to determine a reliability coefficient of $r = .9045$.

The Hidden Figures Test

The Hidden Figures Test (HFT), (French, Ekstrom, & Price, 1963) was administered to the control class on September 9, 1980, and to the experimental class on September 10, 1980. The HFT is a group administered test designed to assess the field-independent, the mobile, and field-dependent cognitive styles. Those subjects who scored in the upper third of the distribution of HFT scores were defined as field-independent; the subjects in the middle third of the distribution were defined as mobile, and the subjects in the lower third were defined as field-dependent.

The HFT is comprised of 32 test items in which the subjects are instructed to find a simple closed geometric figure embedded in a complex background. The test is divided into two parts, and the subjects are given ten minutes per part to answer as many items as they can. In scoring, each correct answer is worth one point, and one-fourth the total of the incorrect answers is deducted from the number of correct answers to obtain the final score. The following distributions were obtained:

Cognitive Style	Number	Range
Field-Independent	24	12.00 - 24.50
Mobile	17	6.75 - 11.75
Field-Dependent	21	-3.50 - 6.50

Table 2: HFT Distribution

The general form of the Kuder-Richardson formula (Ebel, 1965) was used to compute a reliability coefficient of r = .8561.

The Initial Achievement Test

The initial achievement test was administered to the experimental class on October 23, 1980, and to the control class on October 27, 1980. The initial achievement test was part of a test designed to measure student achievement in the fundamentals of differentiation. The first part tested differentiation concepts not included in the study; the second part consisted of ten items chosen to assess students' achievement of the concepts selected for the study, (reference Appendix C). The subjects in both classes completed the test in seventy-five minutes or less. In scoring the initial achievement test, each correct item was worth one point, and no credit or partial credit was given to incorrect answers. The general form of the Kuder-Richardson formula (Ebel, 1965) was used to determine a reliability coefficient of r = .8658.

The Retention Test

The retention test was a ten item test administered two weeks after the initial achievement test. The experimental class received the retention test on November 6, 1980, and the control class received the retention test on November 10, 1980. The retention test was unannounced, and it was built into a quiz designed to measure achievement on concepts not included in the treatments, (reference Appendix D). The retention test was scored similarly to the initial achievement test. Credit was given only to correct answers, and the maximum possible score was ten. The general form of the Kuder-Richardson formula (Ebel, 1965) was used to find a reliability coefficient of $r = .8704$.

CHAPTER IV

STATISTICAL ANALYSIS OF THE DATA

Tests of Hypotheses for Initial Achievement

To examine the effect of the treatments on initial achievement, the following hypotheses were investigated:

H1: There is no significant difference in initial achievement between field-independent, mobile, and field-dependent subjects.

H3: There is no significant difference in initial achievement for subjects receiving an instructional sequence of both positive and negative instances and subjects receiving an instructional sequence of all positive instances.

H5: There is no significant interaction between the two instructional treatments and field-independent, mobile, and field-dependent cognitive styles as measured by the initial achievement instrument.

The data from the initial achievement test was subjected to a 2x3 factorial analysis of covariance with unequal cell frequency per cell. The factors were the two treatments (positive and negative vs. positive only) and the three cognitive styles (field-independent, mobile, and field-dependent). The BMDP, biomedical computer program, P2V (BMD, 1979) was used to analyze the data. Since unequal cell frequency was observed, a non-orthogonal analysis of covariance was used. The formulas for this analysis are given in Appendix E.

The observed means and standard deviations for the initial achievement data are given in the following table:

	Field-Independent	Mobile	Field-Dependent
Experimental	$\bar{X}$ = 8.31 SD = 1.49 N = 16	$\bar{X}$ = 7.17 SD = 2.93 N = 6	$\bar{X}$ = 7.22 SD = 2.33 N = 9
Control	$\bar{X}$ = 6.88 SD = 2.70 N = 8	$\bar{X}$ = 7.73 SD = 2.33 N = 11	$\bar{X}$ = 7.42 SD = 2.75 N = 12

Table 3: Summary Table of Initial Achievement Data

Analysis of covariance was used to adjust for between group differences, and the following adjusted means were computed:

	Field-Independent	Mobile	Field-Dependent
Experimental	$\bar{X}$-adj = 8.00	$\bar{X}$-adj = 6.85	$\bar{X}$-adj = 7.36
Control	$\bar{X}$-adj = 7.55	$\bar{X}$-adj = 7.76	$\bar{X}$-adj = 7.40

Table 4: Summary Table of Adjusted Group Means For Initial Achievement Data

The analysis of covariance for the initial achievement test results is shown in the following table:

Source	DF	MS	F	p
Analysis of Covariance				
Treatment	1	.38	.09	.76
Cognitive Style	2	1.26	.31	.74
Interaction	2	2.08	.51	.60
Within (Error)	55	4.07		
Covariate	1	84.00	20.64	.00 *

* - Significant at the .05 level.

Table: 5: Analysis of Covariance for Initial Achievement

The F-ratios obtained above for the treatment, the cognitive style, and their interaction were not statistically significant at the .05 level. Hence the null hypotheses (H1, H3, and H5) for initial achievement were not rejected. The covariate (01), based upon the measurement of pre-calculus algebraic skills, was statistically related to the initial achievement results at the .05 level as expected.

Although the interaction between the cognitive styles and the treatments was not statistically significant at the .05 level, Huck, Cromier, and Bounds (1974) noted that one definition of interaction is the departure from parallelism. Hence the following graph represents the interaction found on the initial achievement test using the adjusted mean scores:

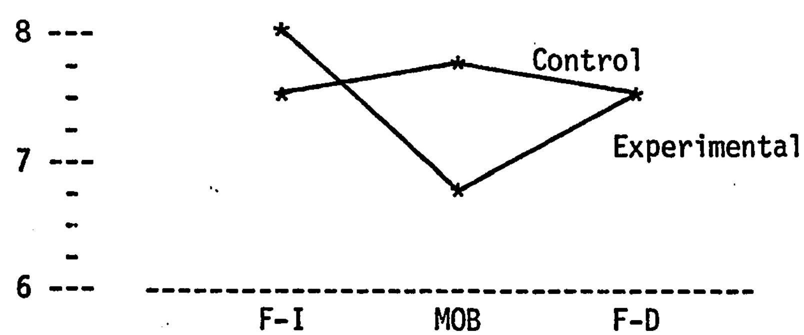

Figure 3: Graph of ITA for Initial Achievement

To examine more closely the effects of the treatments upon the field-independent and field-dependent subjects, a post-hoc analysis of covariance was performed on the initial achievement data by removing the scores of the mobile subjects. The summary table for the adjusted group means was as follows:

	Field-Independent	Field-Dependent
Experimental	$\overline{X}$-adj = 8.00	$\overline{X}$-adj = 7.31
Control	$\overline{X}$-adj = 7.46	$\overline{X}$-adj = 7.37

Table 6: Summary Table of Adjusted Group Means

Post-Hoc Analysis

The analysis of covariance revealed the following results:

Source	DF	MS	F	p
Analysis of Covariance				
Treatment	1	.61	.10	.69
Cognitive Style	1	1.59	.41	.53
Interaction	1	.90	.23	.63
Within (Error)	40	3.88		
Covariate	1	55.39	14.26	.00 *

* - Significant at the .05 level

Table 7: Post-Hoc Analysis of Covariance

For Initial Achievement Data

Since all the F-values for the analysis of covariance were not statistically significant at the .05 level, it was concluded that: removing the middle third of the distribution of scores did not affect the initial decision to accept the null hypotheses for the initial achievement results.

Tests of Hypotheses for Retention

To examine the effect of the treatments on retention, the following hypotheses were investigated:

H2: There is no significant difference in retention between field-independent, mobile, and field-dependent subjects.

H4: There is no significant difference in retention for subjects receiving an instructional sequence of both positive and negative instances and subjects receiving an instructional sequence of all positive instances.

H6: There is no significant interaction between the two instructional treatments and field-independent, mobile, and field-dependent cognitive styles as measured by the retention instrument.

The data from the retention test was subjected to a 2x3 factorial analysis of covariance with unequal cell frequency per cell. The factors were the two treatments (positive and negative versus positive only) and the three cognitive styles (field-independent, mobile, and field-dependent). The BMDP, biomedical computer program, P2V (BMD, 1979) was used to analyze the data. Since unequal cell frequency was observed, a non-orthogonal analysis of covariance was performed.

The observed means and standard deviations for the retention data are given in the following table:

	Field-Independent	Mobile	Field-Dependent
Experimental	$\overline{X}$ = 6.12 SD = 2.47 N = 16	$\overline{X}$ = 6.00 SD = 3.52 N = 6	$\overline{X}$ = 5.89 SD = 3.30 N = 9
Control	$\overline{X}$ = 4.75 SD = 3.01 N = 8	$\overline{X}$ = 6.27 SD = 2.41 N = 11	$\overline{X}$ = 4.50 SD = 3.09 N = 12

Table 8: Summary Table of Retention Data

Analysis of covariance was used to adjust for between group differences, and the following adjusted means were computed:

	Field-Independent	Mobile	Field-Dependent
Experimental	$\overline{X}$-adj = 5.70	$\overline{X}$-adj = 5.57	$\overline{X}$-adj = 6.07
Control	$\overline{X}$-adj = 5.68	$\overline{X}$-adj = 6.32	$\overline{X}$-adj = 4.47

Table 9: Summary Table of Adjusted Group Means

For Retention Data

The analysis of covariance for the retention test data is shown in the following table:

Source	DF	MS	F	p
Analysis of Covariance				
Treatment	1	1.16	.21	.65
Cognitive Style	2	2.12	.38	.69
Interaction	2	6.55	1.16	.32
Within (Error)	55	5.64		
Covariate	1	157.20	27.88	.00 *

*- Significant at the .05 level

Table 10: Analysis of Covariance for Retention Data

The F-ratios obtained above for the treatment, cognitive style, and their interaction were not statistically significant at the .05 level. Hence, the null hypotheses (H2, H4, and H6) for retention were not rejected. The covariate (O1), based upon the measurement of pre-calculus algebraic skills, was statistically related to the retention test results at the .05 level.

The interaction, which is not statistically significant at the .05 level, is graphed below using the adjusted cell means:

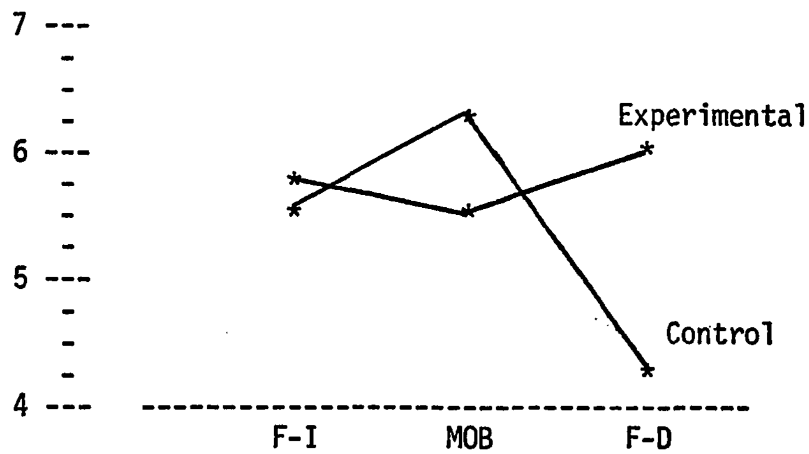

Figure 4: Graph of ITA For Retention

As in the analysis of the inital achievement data, to examine more closely the effects of the treatments upon the field-independent and the field-dependent subjects, a post-hoc analysis of covariance was performed on the retention data by removing the scores of the mobile subjects. The summary table for the adjusted group means was as follows:

	Field-Independent	Field-Dependent
Experimental	$\bar{X}$-adj = 5.68	$\bar{X}$-adj - 6.02
Control	$\bar{X}$-adj = 5.58	$\bar{X}$-adj = 4.43

Table 11: Adjusted Group Means Post-Hoc Analysis for Retention

The analysis of covariance revealed the following results:

Source	DF	MS	F	p
Analysis of Covariance				
Treatment	1	7.30	1.24	.27
Cognitive Style	1	1.73	.30	.59
Interaction	1	5.48	.93	.34
Within (Error)	40	5.87		
Covariate	1	112.29	19.13	.00 *

* - Significant at the .05 level

Table 12: Post-Hoc Analysis of Covariance For Retention Data

All the F-values for the above analysis of covariance were not significant at the .05 level; thus it was concluded that the initial decision to not reject the null hypotheses (H2, H4, and H6) was correct.

Correlation Among Dependent Variables

The BMD, biomedical computer program, P3D (BMD, 1970) was used to compute the following correlation matrix:

	Covariate (01)	HFT	Initial Ach.	Retention
Covariate (01)	1.00	.16	.53 *	.58 *
HFT		1.00	.14	.12
Initial Achievement			1.00	.65 *
Retention				1.00

* - Significant at the .05 level

Table 13: Correlation Matrix for Dependent Variables

The covariate was statistically related at the .05 level to performance on the initial achievement test and to performance on the retention test. The performance on the Hidden Figures Test was not statistically related to performance on the covariate (01) measure, the initial achievement, nor the retention test.

Analysis of Covariance

Analysis of covariance is an extension of analysis of variance; consequently, the assumptions for analysis of covariance are more demanding, (Kirk, 1968; Huck, Cromier, & Bounds, 1974).

Since unequal cell size was observed, a test of the assumption of homogeneity of variance was necessary. The BMDP, biomedical computer program, P9D (BMD, 1979) was used to conduct Bartlett's Test of homogeneity of variance for the initial achievement data and the retention data. The results were as follows:

	Initial Achievement	Retention
Bartlett Test	6.11	2.12
Approximate	1.17	.40
D.F.	5,3146	5,3146
Probability of F	.32	.85

* - Non-significant at the .05 level

Table 14: Test of Homogeneity of Variance

Thus homogeneity of variance was assumed for both the initial achievement data and the retention data.

For analysis of covariance to be valid, the cells must have homogeneity of regression coefficients (i.e., common slope). A test for common slope was conducted using the BMDP, biomedical computer program P1V (BMD, 1979). Since the F-values were non-significant at the .05 level, the use of analysis of covariance for the initial achievement data and the retention data was tenable.

The above computer analysis also plotted the relationshp between the dependent variables and the covariate. Since these scatter diagrams were linear, the basic assumption of linearity was assumed for both the initial achievement data and the retention data.

CHAPTER V

CONCLUSION

Summary

The purpose of this experimental study was to examine the effects of the inclusion of negative instances (counterexamples) on the initial acquisition and retention of mathematical concepts selected from an introductory calculus course as a function of field-independent, mobile, and field-dependent cognitive styles.

The review of the literature has shown continued research in the use of negative instances in the laboratory situation as well as the classroom situation. These investigations have produced differing results; however, it does appear that negative instances can be used to counteract overgeneralization. Since the early fifties, over nine different cognitive styles have been defined and investigated, Messer (1970). These cognitive styles have been characterized as the typical ways an individual acquires and processes information. Much attention has been given to Herman Witkin's field-independent/field-dependent cognitive styles. The field-dependent cognitive style has been shown to be detrimental to concept learning in mathematics and in the physical sciences; in the social sciences, the exact opposite is true, i.e., the field-dependent cognitive style is beneficial.

It was hypothesized that a treatment containing equal numbers of positive and negative instances would prove superior to a treatment containing only positive instances in both initial acquisition and retention. In addition, it was also hypothesized that the treatment containing positive and negative instances would counteract the debilitating affects of the field-dependent cognitive style in mathematical concept learning.

Specifically the following hypotheses were investigated:

H1: There is no significant difference in initial achievement between field-independent, mobile, and field-dependent subjects.

H2: There is no significant difference in retention between field-independent, mobile, and field-dependent subjects.

H3: There is no significant difference in initial achievement for subjects receiving an instructional sequence of both positive and negative instances and subjects receiving an instructional sequence of all positive instances.

H4: There is no significant difference in retention for subjects receiving an instructional sequence of both positive and negative instances and subjects receiving an instructional sequence of all positive instances.

H5: There is no significant interaction between the two instructional treatments and field-independent, mobile, and field-dependent cognitive styles as measured by the initial achievement instrument.

H6: There is no significant interaction between the two instructional treatments and field-independent, mobile, and field-dependent cognitive styles as measured by the retention instrument.

The experimental study was conducted at the University of Oklahoma, Norman, Oklahoma, in the fall semester, 1980. The subjects were college students enrolled in two intact sections of Math 1743. Math 1743 is an introductory calculus course designed for the business, life and social sciences majors. One section was randomly chosen to receive the experimental treatment containing equal numbers of positive and negative instances; the other section was designated the control class and received the same total number of instances but all positive.

The concepts selected for the experiment were the following introductory differentiation properties:

1. if $p(x) = f(x)g(x)$, then $p'(x) = f(x)g'(x) + g(x)f'(x)$.
2. if $p(x) = f(x)/g(x)$, then $p'(x) = \dfrac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$.
3. if $p(x) = (f(x))^n$, then $p'(x) = n(f(x))^{n-1}f'(x)$.

Prior to the treatment, both sections were administered a test of pre-calculus algebraic skills which was used as the covariate in the statistical analysis of the data. Since intact classes were involved, a 2x3 analysis of covariance was used to adjust for between group differences. The Hidden Figures Test (HFT) was used to assess the three levels of cognitive styles (field-independent, mobile, and field-dependent).

Findings

The initial achievement and retention data failed to support statistically any of the six hypotheses, i.e., the null hypotheses of no statistically significant differences between the adjusted cell means were not rejected for differences in treatments nor for cognitive styles. In addition, there was no statistically significant interaction between the treatments and the cognitive styles in initial achievement and retention performances.

Conclusions & Discussion

It would not be prudent to generalize the findings beyond the limited scope of this investigation. It appears that in this setting, an introductory calculus class for non-science majors, dealing with the selected differentiation rules involved in this study, the use of negative instances would not be necessarily beneficial nor detrimental to student learning.

Indeed, with the mixed findings reported in the literature, the impact of negative instances upon the learning of specific mathematical concepts may vary from concept to concept. Thus, the basic question of whether to use or not to use negative instances as a pedagogical strategy cannot be given a categorical answer. Curley (1980) reported that "...it may be that the situation involving positive and negative instances is concept specific." (p. 8).

The fact that there were no statistically significant differences in student performance between the experimental treatment using positive and negative instances and the control treatment using all positive instances may have resulted from one or more of the following factors:

1. The inclusion of the definition with the set of positive

instances in the control treatment may have nullified the suspected superiority of the set of positive and negative instances used in the experimental treatment, Klausmeier,Ghatala, & Frayer (1974).

2. The experiment was conducted over a time frame of three weeks. It might be that a student's ability to use negative instances in new situations depends upon his previous experiences in the use of negative instances. Thus, there exists the possibility that the time frame for this study was too short for some students to become proficient in the use of negative instances.

3. In experimental studies in the "real-world" classroom situation, investigators often do not have the option of randomizing subjects between groups. Thus, when intact classes are used, the experimenter can randomize the treatments within the intact classes, or he can adjust for between group differences statistically. In this experimental study, randomizing treatments within the intact classes using written lessons was ruled out. It was felt that because of the difficult nature of the selected calculus concepts, it would be necessary to use a lecture approach supplemented by written handouts for the experimental and control treatments. Hence, analysis of covariance was used to adjust for between group differences; However, when using analysis of covariance there is a possibility that some confounding variable may be overlooked. This is not the case with "properly" randomized experiments, Kirk (p. 457, 1968).

4. The textbook used in Math 1743 was Marvin L. Bittinger's <u>Calculus: A Modeling Approach</u>, 2nd Edition, 1980. In this edition the author did present one negative instance for each of the three concepts

used in this experimental study. At no time during the treatments were the students referred to the textbook, nor were any assignments made from the textbook during the treatments. However, there exists a small probability that if a student from the control class also used the textbook during the treatment, then this could bias his results on the performance measures.

5. In general, research has shown that field-independent subjects outperform field-dependent subjects on tests of mathematics achievement. Consequently, it was hypothesized that the field-independent subjects would outperform field-dependent subjects and the experimental treatment containing negative instances would have a beneficial interactive effect upon the field-dependent students.

These hypotheses were not supported statistically. However, the treatments might have introduced a confounding variable. The two classes, up to and after the treatments, were conducted by a lecture-discussion method using the textbook as a reference book and a source of problems. During the treatments the lecture-discussion method was supplemented by the in-class handouts containing the lecture notes and the homework assignments. This departure from the normal routine might have provided additional structure for the field-dependent subjects, and consequently their performances on the initial achievement and the retention measures might have been biased.

Recommendations

1. Based upon past research, it is difficult to make generalizations about the use of negative instances in new situations. Much of the research on the use of negative instances has been theoretical in

nature, and even the applied research on the use of negative instances has often dealt with simple geometric concepts. The differentiation concepts used in this present study dealing with negative instances were selected because they represented "real-world" concepts of practical interest to mathematics educators, in general, and calculus instructors, in particular. Thus, it is recommended that additional studies be conducted to examine the effects of negative instances upon other mathematical concepts selected from the "real-world" classroom situations.

2. A replication of this present study should be conducted to investigate students' ability to use negative instances in new situations. Measures of students' general knowledge of negative instances or their ability to use negative instances should be used as covariates in the statistical analysis, in addition to the measure or pre-calculus algebraic skills already used in the present study.

3. A longitudinal study examining the effect of negative instances on student learning over a sequence of secondary or college courses should be conducted.

4. The dependent measures in this study were students' performances on initial achievement and retention instruments. An investigation should be conducted to examine the effects of negative instances on undergeneralization and overgeneralization using the concepts selected for this study.

5. Varying the proportion of negative instances used in the treatment has sometimes produced significant results, (Conant, 1965; Curley, 1980). A replication of this study should be conducted using treatments employing different ratios of positive and negative instances.

6. While there has been reported a correlation between some of the different cognitive styles, it is recommended that additional studies be conducted to examine the effects of negative instances on selected mathematical concepts as a function of other cognitive styles, e.g., the reflective/impulsive or the leveling/sharpening cognitive styles.

This study investigated the interaction of the treatment variable dealing with the use of negative instances and the aptitude variable dealing with the field-independent, mobile, and field-dependent cognitive styles. While no statistically significant interaction was found, each of these variables is important to the mathematics educator, and even more significant to the classroom practitioner.

It is in the classroom where the teacher must decide what strategy to employ, and it is in the classroom where each teacher can and should become an experimenter, trying, synthesizing, and finally accepting or rejecting different strategies. The use of negative instances is just one strategy that might prove beneficial in a given situation. However, with any strategy selected, it must be remembered that each student is an individual perceiving the activities of the classroom environment in a unique way. "The challenge for higher education is to create an array of program alternatives attuned to the diversity of personal styles...", (Messer, p. 326). Ultimately, this challenge is extended to the class room teacher at every level.

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APPENDIX A

EXPERIMENTAL HANDOUTS

CONTROL HANDOUTS

EXPERIMENTAL HOMEWORK EXERCISES

CONTROL HOMEWORK EXERCISES

EXPERIMENTAL HANDOUTS

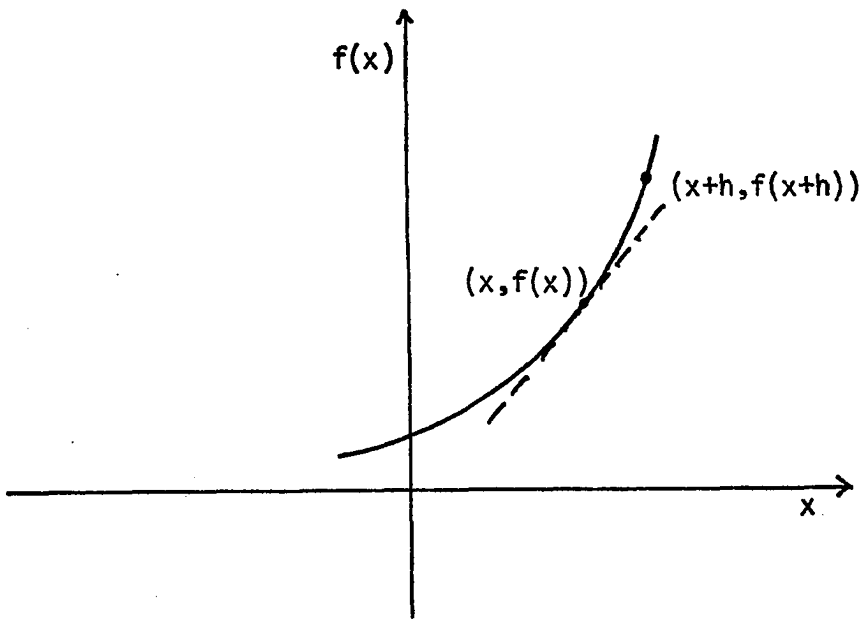
THE PRODUCT AND QUOTIENT RULES

The derivative can be used to better understand many physical phenomena involving changing quantities, such as the speed of a rocket, the increase or decrease in profits, the number of bacteria in a culture, the inflation of currency, the shock wave intensity of an earthquake, etc. We will now examine additional rules for finding the derivative of the product and quotient of two or more functions.

First, let us review some earlier ideas. The derivative of a function $y = f(x)$ at x is:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

This represents the slope of the tangent line to the function f(x) at x as pictured below:



Some of the more common notations used for the derivative of a function $y = f(x)$ are:

$$y' \quad , \quad f'(x) \quad , \quad \frac{dy}{dx} \quad , \quad \text{and } D_x(f) \quad .$$

From the definition of the derivative of a function we have already noted that the following rules for diffentiation can be established:

RULE 1 (POWER RULE)

For any real number n ,

$$\frac{d}{dx} x^n = n \cdot x^{n-1}$$

Example i) $\frac{d}{dx}x^6 = 6x^5$

ii) $\frac{d}{dx} x^{\frac{1}{2}} = \frac{1}{2}x^{-\frac{1}{2}}$

RULE 2 (DERIVATIVE OF A CONSTANT)

For any real number a ,

$$\frac{d}{dx} a = 0 \; .$$

i.e., The derivative of any constant is zero.

Example $\frac{d}{dx} \; 12 = 0$.

RULE 3 (DERIVATIVE OF CONSTANT TIMES A FUNCTION)

For every real number c ,

$$\frac{d}{dx}(c \cdot f(x)) = c \cdot f'(x) .$$

i.e., The derivative of a constant times a function is the constant times the derivative of the function.

Example $\frac{d}{dx}(8x^3) = 8 \cdot \frac{d}{dx}(x^3) = 8 \cdot 3x^2 = 24x^2$

RULE 4 (DERIVATIVE OF THE SUM)

If $f(x)$ and $g(x)$ are differentiable and $p(x) = f(x) + g(x)$,then $p'(x) = f'(x) + g'(x)$.

i.e., The derivative of the sum of two differentiable functions is the sum of the derivatives of the two functions.

Example
$$\frac{d}{dx}(3x^2+7x) = \frac{d}{dx}(3x^2) + \frac{d}{dx}(7x)$$
$$= 3 \cdot \frac{d}{dx}(x^2) + 7 \cdot \frac{d}{dx}(x)$$
$$= 6x + 7 .$$

RULE 5 (DERIVATIVE OF THE DIFFERENCE)

If $f(x)$ and $g(x)$ are differentiable functions and $p(x) = f(x) - g(x)$, then $p'(x) = f'(x) - g'(x)$.

i.e., The derivative of the difference of two differentiable functions is the difference of the derivatives of the two functions.

Example

$$\frac{d}{dx}(7x^4 - 2x^2) = \frac{d}{dx}(7x^4) - \frac{d}{dx}(2x^2)$$

$$= 28x^3 - 4x \,.$$

Using these rules, work the following problems:

Find the derivative $h'(x)$ when,

a) $h(x) = 6x^3-7x+8$

b) $h(x) = -17$

b) $h(x) = -12x^{1/3}$

d) $h(x) = 17x^4+13x^3$

e) $h(x) = (8x^4)(-6x^2)$

f) $h(x) = \dfrac{11x^3}{4x^9}$

g) $h(x) = 6x-x^2$

h) $h(x) = -7\sqrt{x}$

Solutions:

a) $h'(x) = 18x^3-7$

b) $h'(x) = 0$

c) $h'(x) = -4x^{-2/3}$

d) $h'(x) = 68x^3+39x^2$

e) Initially we might be tempted to find the derivative of $(8x^4)(-6x^2)$ as we used rule 4 to find the derivative of the sum of two functions. Thus,

$$h'(x) = \frac{d}{dx}(8x^4)\cdot\frac{d}{dx}(-6x^2)$$

$$= 32x^3\cdot(-12x)$$

$$= -384x^4 \ .$$

However we may also obtain a solution to this problem by first multiplying the two functions and then using the power rule as follows:

$$h(x) = (8x^4)(-6x^2) = -48x^6$$

and

$$h'(x) = \frac{d}{dx}(-48x^6)$$

$$= -288x^5 \ .$$

It is clear that the latter solution is correct. Hence the first solution is not correct, and it can be used as a counterexample to the following false theorem:

$$\frac{d}{dx}\,(f(x)\cdot g(x)) = \frac{d}{dx}\,(f(x))\cdot\frac{d}{dx}\,(g(x)) \quad .$$

Solutions Cont.

f) In the same manner as before, we cannot find the derivative of $h(x) = \frac{11x^3}{4x^9}$ as follows:

$$h'(x) = \frac{\frac{d}{dx}(11x^3)}{\frac{d}{dx}(4x^9)}$$

$$= \frac{33x^2}{36x^8}$$

$$= \frac{11}{12x^6}.$$

This becomes apparent when we examine,

$$h(x) = \frac{11x^3}{4x^9} = \frac{11}{4}x^{-6}$$

Using the power rule as follows:

$$h'(x) = \frac{d}{dx}\left(\frac{11x^{-6}}{4}\right)$$

$$= \frac{-66x^{-7}}{4}$$

$$= \frac{-33}{2x^7}$$

Again the second solution is correct, and the first solution is not correct. Thus by the above counterexample we have shown that the following theorem is false:

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\frac{d}{dx}(f(x))}{\frac{d}{dx}(g(x))}$$

Solutions, Continued:

g) $h'(x) = 6 - 2x$

h) $h'(x) = \frac{d}{dx}(-7x^{\frac{1}{2}})$

$$= \frac{-7}{2} \cdot x^{\frac{1}{2}}$$

$$= \frac{-7}{2\sqrt{x}}$$

Counterexamples play a very important role in the development of mathematics. While examples show how something works, counterexamples can be used to illustrate why something does not work. Often we make statements in the form of theorems or solutions to problems. If we can find just one example where this statement is false, then this type of example is known as a counterexample. For example, consider the following statements and the corresponding counterexamples:

1. $(a+b)^2 = a^2+b^2$

Let $a = 2$ and $b = 3$, then

$$(2+3)^2 = 2^2+3^2$$

$$(5)^2 = 4+9$$

$$25 \neq 13$$

2. $\sqrt{a} + \sqrt{b} = \sqrt{a+b}$

Let $a = 4$ and $b = 16$, then

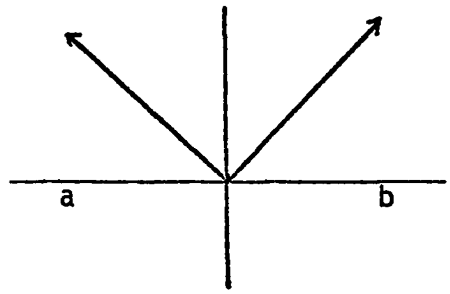
$$\sqrt{4} + \sqrt{16} = \sqrt{4+16}$$

$$2 + 4 = \sqrt{20}$$

$$6 \neq 2 \cdot \sqrt{10}$$

3. If $f(x)$ is continuous on (a,b), then $f'(x)$ exists on (a,b).

Let $f(x) = |x|$, then $f(x)$ is continuous on (a,b), but not differentiable at $x = 0$.



4. $\frac{x + y}{y} = x + 1, y \neq 0$

Let $x = 12$ and $y = 6$, then

$$\frac{12 + 6}{6} = 12 + 1$$

$$\frac{18}{6} = 13$$

$$3 \neq 13$$

5. Theorem: If $f(x)$ and $g(x)$ are differentiable, then the derivative of their product is the product of their derivatives.

Let $f(x) = 4x^2 + 7$ and $g(x) = 5x$, then $\frac{d}{dx}(f \cdot g) = \frac{d}{dx}(4x^2+7) \cdot \frac{d}{dx}(5x) = (8x) \cdot (5) = 40x$. However, $f(x) \cdot g(x) = 20x^3 + 35x$ and $\frac{d}{dx}(f \cdot g) = 60x^2 + 35$.

6. Theorem: If $f(x)$ and $g(x)$ are differentiable, then the derivative of their quotient is the quotient of their derivatives.

Let $f(x) = 6x^3-9x$ and $g(x) = 3x$, then $(f/g) = (6x^3-9x^2)/\frac{d}{dx}(3x) = (18^2-18x)/3 = 6x^2-6x$. However, $f(x)/g(x) = \frac{6x^3-9x^2}{3x} = 2x^2-3x$ and $\frac{d}{dx}(\frac{f}{g}) = 4x-3$.

The rules for finding the derivative of the sum and the derivative of the difference of two functions are relatively simple. However, the rules for finding the derivative of the product or the quotient of two functions are more complex.

These rules are as follows:

RULE 6 (DERIVATIVE OF THE PRODUCT)

If $f(x)$ and $g(x)$ are differentiable and $p(x) = f(x) \cdot g(x)$, then

$$p'(x) = f(x) \cdot g'(x) + g(x) \cdot f'(x) \quad .$$

i.e., The derivative of the product is the first factor times the derivative of the second factor, plus the second factor times the derivative of the first.

RULE 7 (DERIVATIVE OF THE QUOTIENT)

If $f(x)$ and $g(x)$ are differentiable and $p(x) - f(x)/g(x)$, then

$$p'(x) = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{(g(x))^2}$$

i.e., The derivative of a quotient is the denominator times the derivative of the numerator, minus the numerator times the derivative of the denominator, all divided by the square of the denominator

Let's again consider the functions,

$$h(x) = (8x^4)\ (-6x^2) \quad \text{and } h(x) = \frac{11x^3}{4x^9}$$

from page 4. By applying the appropriate rule, we find:

$$h'(x) = \frac{d}{dx}((8x^4)\ (-6x^2)) = 8x^4 \cdot \frac{d}{dx}(-6x^2) + (-6x^2) \cdot \frac{d}{dx}(8x^4)$$

$$= 8x^4(-12x) + (-6x^2)(32x^3)$$

$$= -96x^5 - 192x^5$$

$$= -288x^5$$

NOTE: This agrees with the solution we found when we first multiplied and then used the power rule.

Likewise,

$$h'(x) = \frac{d}{dx}\ \frac{11x^3}{4x^9} = \frac{4x^9 \cdot \frac{d}{dx}(11x^3) - 11x^3 \cdot \frac{d}{dx}(4x^9)}{(4x^9)^2}$$

$$= \frac{4x^9 \cdot 33x^2 - 11x^3 \cdot 36x^8}{16x^{18}}$$

$$= \frac{132x^{11} - 396x^{11}}{16x^{18}}$$

$$= \frac{-264x^{11}}{16x^{18}} = \frac{-33}{2x^7}$$

NOTE: This too agrees with the solution where we first divided and then used the power rule.

Example For $f(x) = (x^2 + 3)(3x^2 + 4x - 9)$, find $f'(x)$.

Solution:

$$f'(x) = (x^2 + 3)\cdot\frac{d}{dx}(3x^2 + 4x - 9)$$

$$+ (3x^2 + 4x - 9) \quad (x^2 + 3)$$

$$= (x^2 + 3)\cdot(6x + 4) + (3x^2 + 4x - 9)\cdot(2x)$$

$$= 6x^3 + 4x^2 + 18x + 12 + 6x^3 + 8x^2 - 18x$$

$$= 12x^3 + 12x^2 + 12.$$

Example For $f(x) = \frac{x}{3x^2 + 9}$, find $f'(x)$.

Solution:

$$f'(x) = \frac{(3x^2 + 9)\cdot\frac{d}{dx}(x) - x\cdot\frac{d}{dx}(3x^2 + 9)}{(3x^2 + 9)^2}$$

$$= \frac{3x^2 + 9)\cdot 1 - x\cdot(6x)}{(3x^2 + 9)^2}$$

$$= \frac{3x^2 + 9 - 6x^2}{(3x^2 + 9)^2}$$

$$= \frac{-3x^2 + 9}{(3x^2 + 9)^2}$$

Example For $h(x) = \frac{3 + x^2}{2x}$, find $h'(x)$.

Solution:

Consider the following solution. It has an error in it, can you detect it?

$$h'(x) = \frac{(3 + x^2)\cdot\frac{d}{dx}(2x) - 2x\cdot\frac{d}{dx}(3 + x^2)}{(2x)^2}$$

$$= \frac{(3 + x^2)\cdot 2 - 2x\cdot 2x}{4x^2}$$

$$= \frac{6 + 2x^2 - 4x^2}{4x^2}$$

$$= \frac{-2x^2 + 6}{4x^2}$$

$$= \frac{-x^2 + 3}{2x^2}$$

Example For $g(x) = (-3x^2+1)(-4x^3+8x)$, find $g'(x)$.

Solution:

Consider the following solution. It has an error in it, can you detect it?

$$g'(x) = \frac{d}{dx}(-3x^2+1) \cdot \frac{d}{dx}(-4x^3+8x)$$

$$= (-6x) \cdot (-12x^2+8)$$

$$= 72x^3 - 48x \quad .$$

For practice work the above problem correctly.

THE EXTENDED POWER RULE

We have seen no matter what the exponent, the power rule allows us to differentiate functions in the form:

$$f(x) = x^n \text{ , where } n \in R$$

and

$$f'(x) = n \cdot x^{n-1} \quad .$$

Using the power rule, find $h'(x)$ when:

a) $h(x) = 4x^{150}$

b) $h(x) = x^{0.83}$

c) $h(x) = 4\sqrt{x}$

d) $h(x) = (x^2+2)^3$

Solution:

a) $h'(x) = 4 \cdot 150x^{149}$

$= 600x^{149}$

b) $h'(x) = .83x^{.83-1}$

$= .83x^{-.17}$

$= \dfrac{.83}{x^{.17}}$

c) $h'(x) = 4(x)^{\frac{1}{2}}$

$= 2x^{-\frac{1}{2}}$

$= \dfrac{2}{\sqrt{x}}$

d) Using the power rule, we find that,

$h'(x) = 3(x^2+2)^{3-1}$
$= 3(x^2+2)^2$

$= 3(x^4+4x^2+4)$

$= 3x^4+12x^2+12$.

However, if we first expand (x^2+2), we obtain

$h(x) = (x^2+2)^3$

$= (x^4+4x^2+4)(x^2+2)$

$= x^6+4x^6+4x^2+2x^4+8x^2+8$

$$= x^6+6x^4+12^2+8$$

and

$$h'(x) = 6x^5+24x^3+24x \ .$$

Thus it became apparent that the power rule does not work for more complex functions.

To differentiate more complex functions such as,

$$h(x) = (x^2+3)^3 \qquad \text{or } g(x) = (2x^2-9)^{4/3} \ ,$$

we use an extension of the power rule as follows:

<u>RULE 8</u> (EXTENDED POWER RULE)

$$\frac{d}{dx}(f(x))^n = n\cdot(f(x))^{n-1}\cdot\frac{d}{dx}(f(x))$$

$$= n\cdot(f(x))^{n-1}\ f'(x) \ .$$

If we apply this rule to problem d on page 1, we find that:

$$\begin{aligned}
\frac{d}{dx}((x^2+2)^3) &= 3(x^2+2)^{3-1}\cdot\frac{d}{dx}(x^2+2) \\
&= 3(x^2+2)^2\cdot 2x \\
&= 3(x^4+4x^2+4)\cdot 2x \\
&= 6x\cdot(x^4+4x^2+4) = 6x^5+24x^3+24x
\end{aligned}$$

NOTE: This is the same result we obtained when we first expanded the function before differentiating it. The extended power rule is a very useful rule for finding the derivative of a function raised to any power. The power rule is, in fact, a special case of the extended power rule. If we allow $f(x) = x$, then

$$\frac{d}{dx}(f(x))^n = \frac{d}{dx}(x)^n = n \cdot x^{n-1} \quad .$$

Example Let $f(x) = (2x-9)^{4/3}$. find $f'(x)$.

Solution:

Consider the following differentiation, what needs to be added or changed to make it correct?

$$f'(x) = \frac{4}{3}(2x-9)^{4/3-1}$$

$$= \frac{4}{3}(2x-9)^{1/3}$$

$$= \sqrt[3]{2x-9)}$$

Example Let $f(x) = (1-x^3)^{2/3}$, find $f'(x)$

Solution:

$$f'(x) = \frac{2}{3}(1-x^3)^{2/3-1} \cdot \frac{d}{dx}(1-x^3)$$

$=$

$=$

Example Let $h(x) = (x-5)^4 \cdot (2x+1)^2$, find $h'(x)$

Solution:

Suppose we differentiate $h(x)$ using the extended power rule as follows:

$$h'(x) = \frac{d}{dx}((x-5)^4) \cdot \frac{d}{dx}((2x+1)^2)$$

$$= 4(x-5)^{4-1} \cdot \frac{d}{dx}(x-5)\ 2(2x+1)^{2-1} \cdot \frac{d}{dx}(2x+1)$$

$$= 4(x-5)^3 \cdot 1 \cdot 2(2x+1) \cdot 2$$

$$= 16(x-5)^3 \cdot (2x+1)$$

However this would be incorrect, why? To correctly differentiate this function we need to use which rules?

Example Let $h(x) = \sqrt[4]{(x+3)/(x-1)}$, find $h'(x)$.

Solution:

$$h'(x) = \frac{d}{dx}(\sqrt{(x+3)/(x-1)}) = \frac{d}{dx}((\frac{x+3}{x-1})^{\frac{1}{4}})$$

$$= \tfrac{1}{4}(\frac{x+3}{x-1})^{\frac{1}{4}-1} \cdot \frac{d}{dx}(\frac{x+3}{x-1})$$

$$= \tfrac{1}{4}(\frac{x+3}{x-1})^{-3/4} \cdot (\frac{(x-1) \cdot \frac{d}{dx}(x+3) - (x+3) \cdot \frac{d}{dx}(x-1)}{(x-1)^2})$$

$$= \tfrac{1}{4}(\frac{x+3}{x-1})^{-3/4} \cdot (\frac{(x-1)(1) - (x+3)(1)}{(x-1)^2})$$

$$h'(x) = \tfrac{1}{4}\left(\frac{x-1}{x+3}\right)^{3/4} \cdot \left(\frac{(x-1) - (x+3)}{(x-1)^2}\right)$$

$$= \tfrac{1}{4}\left(\frac{x-1}{x+3}\right)^{3/4} \cdot \left(\frac{-4}{(x-1)^2}\right)$$

$$= \left(\frac{x-1}{x+3}\right)^{3/4} \cdot \left(\frac{1}{(x-1)^2}\right)$$

CONTROL HANDOUTS

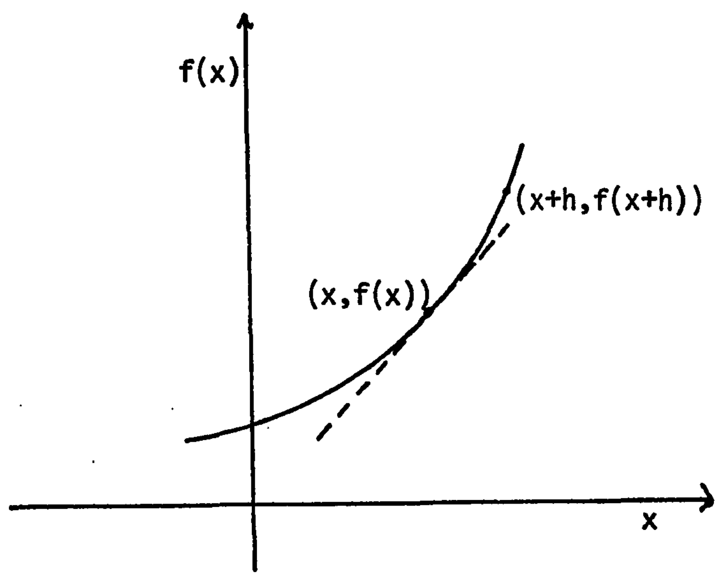
THE PRODUCT AND QUOTIENT RULES

The derivative can be used to better understand many physical phenomena involving changing quantities, such as the speed of a rocket, the increase or decrease in profits, the number of bacteria in a culture, the inflation of currency, the shock wave intensity of an earthquake, etc. We will now examine additional rules for finding the derivative of the product and quotient of two or more functions.

First, let us review some earlier ideas. The derivative of a function $y = f(x)$ at x is:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

This represents the slope of the tangent line to the function f(x) at x as pictured below:



Some of the more common notations used for the derivative of a function $y = f(x)$ are:

$$y' \quad , \quad f'(x) \quad , \frac{dy}{dx} \quad , \text{ and } D_x(f) \quad .$$

From the definition of the derivative of a function we have already noted that the following rules for differentiation can be established:

RULE 1 (POWER RULE)

For any real number n ,

$$\frac{d}{dx} x^n = n \cdot x^{n-1}$$

Example i) $\frac{d}{dx}x^6 = 6x^5$

ii) $\frac{d}{dx} x^{\frac{1}{2}} = \frac{1}{2}x^{-\frac{1}{2}}$

RULE 2 (DERIVATIVE OF A CONSTANT)

For any real number a ,

$$\frac{d}{dx} a = 0 \cdot$$

i.e. The derivative of any constant is zero.

Example $\frac{d}{dx} 12 = 0$.

RULE 3 (DERIVATIVE OF CONSTANT TIMES A FUNCTION)

For every real number c ,

$$\frac{d}{dx}(c \cdot f(x)) = c \cdot f'(x) \quad .$$

i.e., The derivative of a constant times a function is the constant times the derivative of the function.

Example $\frac{d}{dx}(8x^3) = 8 \cdot \frac{d}{dx}(x^3) = 8 \cdot 3x^2 = 24x^2$.

RULE 4 (DERIVATIVE OF THE SUM)

If $f(x)$ and $g(x)$ are differentiable and $p(x) = f(x) + g(x)$, then $p'(x) = f'(x) + g'(x)$.

i.e., The derivative of the sum of two differentiable functions is the sum of the derivatives of the two functions.

Example $\frac{d}{dx}(3x^2+7x) = \frac{d}{dx}(3x^2) + \frac{d}{dx}(7x)$

$$= 3 \cdot \frac{d}{dx}(x^2) + 7 \cdot \frac{d}{dx}(x)$$

$$= 6x + 7 \quad .$$

RULE 5 (DERIVATIVE OF THE DIFFERENCE)

If $f(x)$ and $g(x)$ are differentiable functions and $p(x) = f(x) - g(x)$, then $p'(x) = f'(x) - g'(x)$.

i.e. The derivative of the difference of two differentiable functions is the difference of the derivatives of the two functions.

Example

$$\frac{d}{dx}(7x^4 - 2x^2) = \frac{d}{dx}(7x^4) - \frac{d}{dx}(2x^2)$$

$$= 28x^3 - 4x \ .$$

Using these rules work the following problems:

Find the derivative $h'(x)$ when,

a) $h(x) = 6x^3-7x+8$

b) $h(x) = -17$

c) $h(x) = -12x^{1/3}$

d) $h(x) = 17x^4+13x^3$

e) $h(x) = (8x^4)(-6x^2)$

f) $h(x) = \frac{11x^3}{4x^9}$

g) $h(x) = 6x-x^2$

h) $h(x) = -7\sqrt{x}$

Solutions:

a) $h'(x) = 18x^3 - 7$

b) $h'(x) = 0$

c) $h'(x) = -4x^{-2/3} = \frac{-4}{\sqrt[3]{x^2}}$

d) $h'(x) = 68x^3 + 39x^2$

e) We may obtain a solution to this problem by first multiplying the two functions and using the power rule as follows:

$$h(x) = (8x^4)\ (-6x^2) = -48x^6$$

and

$$h'(x) = \frac{d}{dx}(-48x^6)$$

$$= -288x^5$$

f) In the same manner as above, we find that:

$$h(x) = \frac{11x^3}{4x^9} = \frac{11}{4} \cdot x^{-6}$$

and

$$h'(x) = \frac{d}{dx}\ (\frac{11x^{-6}}{4})$$

$$= \frac{-66x^{-7}}{4}$$

$$= \frac{-33}{2x^7}$$

The rules for finding the derivative of the sum and the derivative of difference of two functions are relatively simple. However, the rules for finding the derivative of the product or the quotient of two functions are more complex.

These rules are as follows:

RULE 6 (DERIVATIVE OF THE PRODUCT)

If $f(x)$ and $g(x)$ are differentiable and

$p(x) = f(x) \cdot g(x)$, then

$$p'(x) = f(x) \cdot g'(x) + g(x) \cdot f'(x) \quad .$$

i.e., The derivative of the product is the first factor times the derivative of the second factor, plus the second factor times the derivative of the first.

RULE 7 (DERIVATIVE OF THE QUOTIENT)

If $f(x)$ and $g(x)$ are differentiable and

$p(x) = f(x)/g(x)$, then

$$p'(x) = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{(g(x))^2}$$

i.e., The derivative of a quotient is the denominator times the derivative of the numerator, minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

Let's again consider the functions,

$$h(x) = (8x^4)(-6x^2) \text{ and } h(x) = \frac{11x^3}{4x^9}$$

from page 4. By applying the appropriate rule, we find:

$$h'(x) = \frac{d}{dx}((8x^4)(-6x^2)) = 8x\cdot\frac{d}{dx}(-6x^2) + (-6x^2)\cdot\frac{d}{dx}(8x^4)$$

$$= 8x^4(-12x) + (-6x^2)(32x^3)$$

$$= -288x^5$$

NOTE: This agrees with the solution we found when we first multiplied and then used the power rule.

Likewise,

$$h'(x) = \frac{d}{dx}\left(\frac{11x^3}{4x^9}\right) = \frac{4x^9\cdot\frac{d}{dx}(11x^3) - 11x^3\cdot\frac{d}{dx}(4x^9)}{(4x^9)^2}$$

$$= \frac{4x^9\cdot 33x^2 - 11x^3\cdot 36x^8}{16x^{18}}$$

$$= \frac{132x^{11} - 396x^{11}}{16x^{18}}$$

$$= \frac{264x^{11}}{16x^{18}} = \frac{-33}{2x^7}$$

NOTE: This too agrees with the solution where we first divided and then used the power rule.

Example For $f(x) = (x^2 + 3)\cdot(3x^2 + 4x -9)$. find $f'(x)$.

Solution:

$$f'(x) = (x^2 + 3)\cdot\frac{d}{dx}(3x^2 + 4x) - 9)$$

$$+ (3x^2 + 4x - 9)\cdot\frac{d}{dx}(x^2 + 3)$$

$$= (x^2 + 3)\cdot(6x + 4) + (3x^2 + 4x - 9)\cdot(2x)$$

$$= 6x^3 + 4x^2 +18x + 12 + 6x^3 + 8x^2 -18x$$

$$= 12x^3 + 12x^2 + 12 .$$

Example For $f(x) = \frac{x}{3x^2 + 9}$, find $f'(x)$.

Solution:

$$f'(x) = \frac{(3x^2 + 9)\cdot\frac{d}{dx}(x) - x\cdot\frac{d}{dx}(3x^2 + 9)}{(3x^2 + 9)^2}$$

$$= \frac{(3x^2 + 9)\cdot 1 - x\cdot(6x)}{(3x^2 + 9)^2}$$

$$= \frac{3x^2 + 9 -6x^2}{(3x^2 + 9)^2}$$

$$= \frac{-3x^2 + 9}{(3x^2 + 9)^2}$$

Example For $h(x) = \dfrac{3 + x^2}{2x}$, find $h'(x)$.

Example For $h(x) = (-3x^2 + 1)\ (-4x^3 + 8x)$, find $g'(x)$

THE EXTENDED POWER RULE

We have seen no matter what the exponent, the power rule allows us to differentiate functions in the form:

$$f(x) = x^n \text{ , where } n \in \mathbf{R} \text{ .}$$

and

$$f'(x) = n \cdot x^{n-1} \text{ .}$$

Using the power rule, find $h'(x)$ when:

a) $h(x) = 4x^{150}$

b) $h(x) = x^{0.83}$

c) $h(x) = 4\sqrt{x}$

d) $h(x) = (x^2+2)^3$

Solution:

a) $h'(x) = 4 \cdot 150x^{149}$

$= 600x^{149}$

b) $h'(x) = .83x^{.83-1}$

$= .83x^{.-17}$

$= \dfrac{.83}{x^{.17}}$

c) $h'(x) = 4 \cdot x^{\frac{1}{2}-1}$

$= 2x^{-\frac{1}{2}}$

$= \dfrac{2}{\sqrt{x}}$

d) If we first expand $(x^2 + 3)$, we obtain

$h(x) = (x^2 + 2)^3$

$= (x^4 + 4x^2 + 4)\ (x^2 + 2)$

$= x^6 + 4x^4 + 2x^4 + 8x^2 + 8$

$= x^6 + 6x^4 + 12x^2 + 8$

and

$h'(x) = 6x^5 + 24x^3 + 24x.$

To differentiate more complex functions such as,

$h(x) = (x^2+2)^3$ Or $g(x) = (2x^2 -9)^{4/3}$.

we use an extension of the power rule as follows:

RULE 8 (EXTENDED POWER RULE)

$$\frac{d}{dx}(f(x))^n = n\cdot(f(x))^{n-1}\cdot\frac{d}{dx}(f(x))$$

$$= n\cdot(f(x))^{n-1}\cdot f'(x).$$

If we apply this rule to problem d on page 1, we find that:

$$\frac{d}{dx}((x^2+2)^3) = 3(x^2+2)^{3-1}\cdot\frac{d}{dx}(x^2+2)$$

$$= 3(x^2+2)^2\cdot 2x$$

$$= 3(x^4+4x^2+4)\cdot 2x$$

$$= 6x\cdot(x^4+4x^2+4)$$

$$= 6x^5+24x^3+24x \quad .$$

NOTE: This is the same result we obtained when we first expanded the function before differentiating it.

The extended power rule is a very useful rule for finding the derivative of a function raised to any power. The power rule is, in fact, a special case of the extended power rule.

If we allow f(x) = x , then

$$\frac{d}{dx}(f(x)) = \frac{d}{dx}(x)^n = n \cdot x^{n-1}$$

Example Let $f(x) = (2x-9)^{4/3}$, find $f'(x)$.

Solution:

$$f'(x) = \frac{d}{dx}((2x-9)^{4/3})$$

$$= \frac{4}{3} \cdot (2x-9)^{4/3-1} \cdot \frac{d}{dx}(2x-9)$$

$$=$$

Example Let $f(x) = (1-x^3)^{2/3}$, find $f'(x)$.

Solution:

Example Let $h(x) = (x-5)^4 \cdot (2x+1)^2$, find $h'(x)$.

Solution:

Example Let $h(x) = \sqrt[4]{(x+3)/(x-1)}$, find $h'(x)$.

Solution:

EXPERIMENTAL HOMEWORK EXERCISES

Mathematics 1743

Name____________________

Homework Exercises # 1

Part I The following functions are differentiated in two different ways. Determine which method is incorrect (if any) and finish the differentiaion for all correct methods:

1. $h(x) = 8x^3(1 - 5x^2)$

   a) $h'(x) = \frac{d}{dx}(8x^3)\cdot\frac{d}{dx}(1 - 5x^2)$

   b) $h'(x) = 8x^3\cdot\frac{d}{dx}(1 - 5x^2) + (1 - 5x^2)\cdot\frac{d}{dx}(8x^3)$

2. $s(x) = \dfrac{3x^2 + 2x}{x^2 + 1}$

   a) $s'(x) = \dfrac{(x^2 + 1)\cdot\frac{d}{dx}(3x^2 +2x) - (3x^2 + 2x)\cdot\frac{d}{dx}(x^2 + 1)}{(x^2 + 1)^2}$

   b) $s'(x) = \dfrac{(3x^2 + 2x)\cdot\frac{d}{dx}(x^2 + 1) - (x^2 + 1)\cdot\frac{d}{dx}(3x^2 + 2x)}{(x^2 + 1)^2}$

3. $f(x) = 5x^2(3x - 8)$

a) $f'(x) = 5x^2 \cdot \frac{d}{dx}(3x - 8) + (3x - 8) \cdot \frac{d}{dx}(5x^2)$

b) $f'(x) = \frac{d}{dx}(15x^3 - 40x^2)$

4. $t(x) = x^9/x^5$

a) $t'(x) = \dfrac{\frac{d}{dx}(x^9)}{\frac{d}{dx}(x^5)}$

b) $t'(x) = \dfrac{x^5 \cdot \frac{d}{dx}(x^9) - x^9 \cdot \frac{d}{dx}(x^5)}{(x^5)^2}$

5. $g(x) = \dfrac{10x}{x-8}$

a) $g'(x) = \dfrac{(x-8)\cdot\frac{d}{dx}(10x) - 10x\cdot\frac{d}{dx}(x-8)}{(x-8)^2}$

b) $g'(x) = \dfrac{(x-8)\cdot\frac{d}{dx}(10x) - 10x\cdot\frac{d}{dx}(x-8)}{(10x)^2}$

6. $f(x) = \dfrac{7x + 3}{x^8}$

a) $f'(x) = \dfrac{x^8\cdot\frac{d}{dx}(7x + 3) - (7x + 3)\cdot\frac{d}{dx}(x^8)}{(x^8)^2}$

b) $f'(x) = (7x + 3)\cdot\frac{d}{dx}(x^{-8}) + x^{-8}\cdot\frac{d}{dx}(7x + 3)$

7. $f(x) = \frac{1}{x - 3}$

a) $f'(x) = \dfrac{(x - 3)\cdot\frac{d}{dx}(1) + 1\cdot\frac{d}{dx}(x - 3)}{(x - 3)^2}$

b) $f'(x) = \dfrac{(x - 3)\cdot\frac{d}{dx}(1) - 1\cdot\frac{d}{dx}(x - 3)}{(x - 3)^2}$

8. $f(x) = \frac{x^4}{\sqrt{x}}$ Hint: rationalize the denominator.

a) $f'(x) = \dfrac{\sqrt{x}\cdot\frac{d}{dx}(x^4) - x^4\cdot\frac{d}{dx}(\sqrt{x}\,)}{(\sqrt{x}^2}$

b) $f'(x) = \sqrt{x}\cdot\frac{d}{dx}(x^3) + x^3\cdot\frac{d}{dx}(\sqrt{x}\,)$

9. $g(x) - 5x(x + 3)\ (x - 3)$

a) $g'(x) = (x+3)(x-3)\cdot\frac{d}{dx}(5x) + 5x(x-3)\cdot\frac{d}{dx}(x+3) + 5x(x+3)\cdot\frac{d}{dx}(x-3)$

b) $g'(x) = \frac{d}{dx}(5x^3 - 45x)$

10. $h(x) = (x^3+8)\cdot\frac{x^2+2}{x+2}$ Hint: $a^3+b^3 = (a+b)(a^2-ab+b^2)$

a) $h'(x) = \frac{x^2+2}{x+2}\cdot\frac{d}{dx}(x^3+8) + (x^3+8)\cdot\frac{d}{dx}(\frac{x^2+2}{x + 2})$

b) $h'(x) = (x^2-2x+4)\cdot\frac{d}{dx}(x^2+2) + (x^2+2)\cdot\frac{d}{dx}(x^2-2x+4)$

Part II Determine which of the following statements are true and which are false. If a statement is true, justify your answer by writing the appropriate rule or giving an example. If the statement is false, give the correct answer or find a counterexample to show why the statement is false.

1. The derivative of the sum is always the sum of the derivatives.
2. If $P(x) = f(x)\cdot g(x)$, then $p'(x) = f(x)\cdot g'(x) - g(x)\cdot f'(x)$.
3. If $r(x) = \frac{s(x)}{t(x)}$, then $r'(x) = \frac{s(x)\cdot t'(x) - t(x) - t(x)\cdot s'(x)}{(s(x))^2}$.
4. The derivative of the quotient is never the quotient of the of the two derivatives.
5. If $p(x) = (6x^3)\cdot(7x^4)$, then $h'(x) = \frac{d}{dx}(42x^7)$
6. The derivative of the product is the product of the derivatives.
7. If $f(x) = \frac{7}{5 - 8x - x^2}$, then

   $f'(x) = \frac{d}{dx}(7) \div \frac{d}{dx}(5) - \frac{d}{dx}(8x) - \frac{d}{dx}(x^2)$
8. If $t(x) = 6x(x-2)(x+5)$, then

   $t'(x) = 6x(x-2)\cdot\frac{d}{dx}(x+5) + 6x(x+5)\cdot\frac{d}{dx}(x-2) + (x+5)(x-2)\cdot\frac{d}{dx}(6x)$.
9. If $f(x) = (x^2+2x+8)^2$, then

   $f'(x) = 2((x^2+2x+8)\cdot\frac{d}{dx}(x^2+2x+8))$.
10. If $h(x) = \frac{1}{x^8}$, then

    $h'(x) = \frac{x^8\cdot\frac{d}{dx}(1) - 1\cdot\frac{d}{dx}(x^8)}{(x^8)^2}$.

Homework Exercises #2

One of the alternate solutions to the following problems may be incorrect. Find the correct solution (s) to each problem and finish finding the derivatives.

1. Let $f(x) = (1-x)^{55}$, find $f'(x)$.

   a) $f'(x) = 55(1-x)^{54}$

   b) $f'(x) = 55(1-x)^{54} \cdot \frac{d}{dx}(1-x)$

2. Let $t(x) = \frac{x^2}{(1+x)^5}$, find $t'(x)$.

   a) $t'(x) = \dfrac{(1+x)^5 \cdot \frac{d}{dx}(x^2) - x^2 \cdot \frac{d}{dx}((1+x)^5)}{((1+x)^5)^2}$

   b) $t'(x) = \frac{d}{dx}(x^2 \cdot (1+x)^{-5})$

3. Let $p(x) = \sqrt{x^4 + 8x^2 + 16}$, find $p'(x)$

   a) $p'(x) = \frac{1}{2}(x^4+8x^2+16)^{\frac{1}{2}-1} \cdot \frac{d}{dx}(x^4+8x^2+16)$

b) $p'(x) = \frac{d}{dx}(x^2+4)$

4. Let $g(x) = x\sqrt{7x+3}$, find $f'(x)$.

a) $g'(x) = (7x+3)^{\frac{1}{2}} \cdot \frac{d}{dx}(x) + x \cdot \frac{d}{dx}((7x+3)^{\frac{1}{2}})$

b) $g'(x) = \frac{d}{dx}(x) \cdot \frac{d}{dx}((7x+3)^{\frac{1}{2}})$

5. Let $f(x) = \frac{1}{(2x+8)^2}$, find $f'(x)$.

a) $f'(x) = \frac{d}{dx}((2x+8)^{-2})$

$= -2(2x+8)^{-2-1} \cdot \frac{d}{dx}(2x+8)$

b) $f'(x) = \frac{d}{dx}\frac{1}{(4x^2+32x+64)}$

$$= \frac{(4x^2+32x+64)\cdot\frac{d}{dx}(1) - 1\cdot\frac{d}{dx}(4x^2+32x+64)}{(4x^2+32x+64)^2}$$

6. Let $h(x) = \sqrt{(4+x)(2-x)}$

a) $h'(x) = \frac{1}{2}((4+x)(2-x))^{\frac{1}{2}-1}\cdot\frac{d}{dx}((4+x)(2-X))$

b) $h'(x) = \frac{1}{2}((4+x)(2-x))^{\frac{1}{2}-1}\cdot\frac{d}{dx}(4+x)\cdot\frac{d}{dx}(2-x)$

7. $p(x) = -7x(2x-3)^3$, find $p'(x)$.

a) $p'(x) = -7x\cdot 3(2x-3)^{3-1} + (2x-3)^3\cdot\frac{d}{dx}(-7x)$

b) $p'(x) = -7x\cdot 3(2x-3)^{3-1}\cdot\frac{d}{dx}(2x-3) + (2x-3)^3\cdot(-7)$

8. Let $u = \frac{(1-2v)^4}{v^4}$, find u'.

a) $u' = \frac{v^4 \cdot \frac{d}{dv}((1-2v)^4) - (1-2v)^4 \cdot \frac{d}{dv}(v^4)}{(v^4)^2}$

b) $u' = (1-2v)^4 \cdot \frac{d}{dv}(v^{-4}) + v^{-4} \cdot \frac{d}{dx}((1-2v)^4)$

9. Let $s(t) = \sqrt{t^2+2t+8}$, find s'(t).

a) $s'(t) = \frac{1}{2}(t^2+2t+8)^{\frac{1}{2}-1} \cdot (t^2+2t+8)$

b) $s'(t) = \frac{1}{2}(t^2+2t+8)^{\frac{1}{2}-1} \cdot \frac{d}{dx}(t \ +2t+8)$

10. Let $f(x) = (x-4)^{1/4} \cdot (x-5)^2$

a) $f'(x) = (x-4)^{1/4} \cdot 2(x-5)^{2-1} \cdot \frac{d}{dx}(x-5) + (x-5)^2 \cdot \frac{d}{dx}((x-4)^{1/4})$

b) $f'(x) = (x-4)^{1/4} \cdot \frac{d}{dx}(x^2-10x+25) + (x^2-10x+25) \cdot \frac{d}{dx}((x-4)^{1/4})$

Part II Use the following functions as an example or counterexample to the following statements:

1. $\frac{d}{dx}(g(x))^n = n\cdot(g(x))^{n-1}$, where $g(x) = (4x^2-12x+9)^{\frac{1}{2}}$.

2. $\frac{d}{dx}(f(x)\cdot g(x)) = \frac{d}{dx}(f(x))\cdot\frac{d}{dx}(g(x))$, where $f(x) = 6x^3$ and $g(x) = x+1$.

3. $\frac{d}{dx}(f(x)/g(x)) = \frac{f(x)\cdot g'(x) - g(x)\cdot f'(x)}{(g(x))^2}$, where $f(x) = x^2+5x+15$ and $g(x) = x+3$.

4. $\frac{d}{dx}(f(x))^n = n\cdot(f(x))^{n-1}\cdot f'(x)$, where $f(x) = 1-3x$ and $n = 2$.

5. $\frac{d}{dx}(f(x)/g(x)) = f'(x)/g'(x)$, where $f(x) = 4x + 1$ and $g(x) = 2x$.

6. $\frac{d}{dx}(f(x)\cdot g(x)) = f(x)\cdot g'(x) + g(x)\cdot f'(x)$, where $f(x) = 2x-1$ and $g(x) = 2x^2+8x-3$.

7. $\frac{d}{dx}(h(x))^n = n\cdot(h(x))^{n-1}\cdot h'(x)$, where $h(x) = (2x+5)$ and $n = 2$.

8. $\frac{d}{dx}(f(x)/g(x)) = \dfrac{g(x)\cdot f'(x) - f(x)\cdot g'(x)}{(g(x))^2}$, where $f(x) = 2x-4$ and $g(x) = 4x^2-16$.

CONTROL HOMEWORK EXERCISES

Mathematics 1743

Name____________________

Homework Exercises #1

Part I Find the derivatives of the following functions:

1. $h(x) = 8x^3(1 - 5x^2)$

2. $s(x) = \dfrac{3x^2 + 2x}{x^2 + 1}$

3. $f(x) = 5x^2(3x - 8)$, find $f'(x)$ in two different ways.

4. $t(x) = x^9/x^5$

5. $g(x) = \dfrac{10x}{x - 8}$

6. $f(x) = \dfrac{7x + 3}{x^8}$, find $f'(x)$ in two different ways.

7. $f(x) = \dfrac{1}{x + 3}$

8. $f(x) = \frac{x^4}{\sqrt{x}}$, find $f'(x)$ in two different ways.

Hint: Rationalize the denominator.

9. $g(x) = 5x(x + 3)(x - 3)$, find $g'(x)$ in two different ways.

10. $h(x) = (x^2-4)\cdot\frac{x^2+2}{x+2}$, find $h'(x)$ in two different ways.

Homework Exercises #1

Part II

1. Give an example of the rule: The derivative of the sum is always the sum of the derivatives.

2. Complete the following: If $p(x) = f(x) \cdot g(x)$, then

$$p'(x) =$$

3. Complete the following and give an example:

If $r(x) = \frac{s(x)}{t(x)}$, then

$$r'(x) =$$

4. If $g(x) = \frac{7x^5}{x + 2}$, then find $g'(x)$.

5. If $p(x) = (6x^3)(7x^4)$, then find $p'(x)$.

6. If $f(x) = (x+5)(x^3-6x^2-9)$, the find $f'(x)$.

7. If $f(x) = \frac{7}{5 - 8x - x^2}$, then find $f'(x)$.

8. If $t(x) = 6x(x-2)(x+5)$, then find $t'(x)$.

9. If $f(x) = (x^2+2x+8)^2$, then find $f'(x)$.

10. If $h(x) = \frac{1}{x^8}$, then find $h'(x)$.

Homework Exercises #2

Part I Find the derivative of the following functions:

1. $f(x) = (1-x)^{55}$

2. $t(x) = \frac{x^2}{(1+x)^5}$, find $t'(x)$ in two different ways.

3. $p(x) = \sqrt{x^4+8x^2+16}$, find $p'(x)$ in two different ways.

   Hint: factor x^4+8x^2+16

4. $g(x) = x \cdot \sqrt{7x+3}$

5. $f(x) = \frac{1}{(2x+8)^2}$, find $f'(x)$ in two different ways.

6. $h(x) = \sqrt{(4+x)/(2-x)}$

7. $p(x) = -7x\cdot(2x-3)^3$

8. $u = \frac{(1-2v)^4}{v^4}$, find u' in two different ways.

9. $s(t) = \sqrt{t^2+2t+8}$

10. $f(x) = (x-4)^{\frac{1}{4}}\cdot(x-5)^2$, find $f'(x)$ in two different ways.

Part II Find the derivative of the following functions:

1. $g(x) = (4x^2-12x+9)^{\frac{1}{2}}$

2. $h(x) = (6x^3)\ (x+1)$

3. $h(x) = \dfrac{x^2+5x+15}{x+3}$

4. $h(x) = (1 - 3x)^2$

5. $h(x) = \dfrac{4x + 1}{2x}$

6. $h(x) = (2x-1)\ (2x^2+8x-3)$

7. $f(x) = (2x+5)^3$

8. $h(x) = \frac{2x-4}{4x^2-16}$

APPENDIX B

THE COVARIATE MEASURE

Mathematics 1743 - Unit 1 Exam

Name ________________________

1. Solve for x.

a) $-6x + 3 = 4x + 5$

b) $7(x + 3) = -2x + 8 + 9x$

c) $x + 13 > 4x - 7$

d) $|6x| + 3 < 12$

e) $3x^2 - 4x = 5$

f) $-6(x + 2) < -12$

2. Define: a) Absolute Value

b) Function

3. Let $f(x) = 3x^2 + x + 8$, Find

a) $f(-3)$

b) $\frac{f(x + h) - f(x)}{h}$

4. Find a quadratic function that fits the data points (1,3), (3,3) and (7,-17).

5. Find a linear function that fits the data points (-3,-7) and (4,-5).

6. Sketch a graph of the following functions:

a) $f(x) = 5x - 7$

b) $g(x) = -2x^2 - 4x + 7$

c) $p(x) = \frac{5}{x^2}$

d) $t(x) = \sqrt{x - 2}$

7. Find the domain and range of the following functions:

a) $f(x) = |x + 1|$

b) $p(x) = \sqrt{1 - x^2}$

c) $g(x) = \sqrt{2x + 10}$

d) $t(x) = x^2 + 5$

8. The weight M of the muscles in a human is directly proportional to body weight, W.

a) It is known that a person who weighs 150 lb. has 60 lb. of muscle. Find an equation of variation expressing M as a function of W.

b) What is the muscle weight of a 180 lb. person?

12. Convert to a fractional exponent: $\frac{1}{\sqrt[3]{x}}$

13. Convert to radical notation: $(y^2+7)^{-5/13}$

14. Evaluate: $32^{-3/5}$

15. Find the point of equilibrium when

$D(x) = (x-3)^2$ and $S(x) = x^2+8x-9$

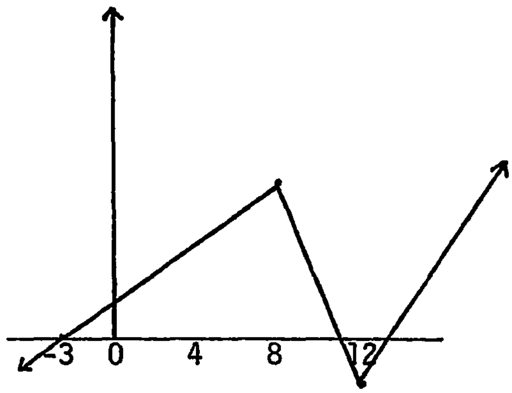
16. Solve for y,

$$(3y^{2/3})^3-18x = -9+27x$$

Does this result represent a function?

9. Find the domain of $g(x) = \frac{5x + 10}{9x^2 - 16}$

10.



For what intervals is the above function increasing and decreasing?

11. Suduki is planning a new line of transistor radios. For the first year the fixed costs are $175,000. Variable costs for producing a radio are estimated at $30. The sales department projects sales of 150,000 radios during the first year at a price of $75.

a) Formulate the cost function $C(x)$ for producing x radios.

b) Formulate the revenue function $R(x)$.

c) Formulate the profit function $P(x)$.

d) Find the break even point.

e) What will be the profit or loss on the sales of only 10,000 radios?

APPENDIX C

THE INITIAL ACHIEVEMENT TEST

Mathematics 1743 Examiniation #2 Name________________

1. Define continuity at a point:

   F(x) is continuous at the point a if and only if

   a) b) c)

2. If $f(x) = x^2-8x+6$, then find the point where the tangent line to f(x) is horizontal.

3. Using the definition of the derivative, find the derivative of $f(x) = 2x^2 + 3$.

4. Find $f'(x)$:

   a) $f(x) = 3x^2-2x+4$ b) $f(x) = 4\sqrt{x}$ c) $f(x) = 7$.

5. Evaluate the following limits:

a) $\lim_{x\to 3} (x^2+4)$

b) $\lim_{x\to 3} 7$

c) $\lim_{x\to 3} \frac{x^2-9}{x-3}$

d) $\lim_{x\to 2} \frac{3x^2-1}{4x^2+2}$

e) $\lim_{x\to \infty} (7 - \frac{9}{x})$

f) $\lim_{x\to \infty} \frac{3x^2-1}{4x^2+2}$

6. Find the derivative for each function:

a) $f(x) = \frac{3x}{x^2+2}$

b) $p(x) = (x^{5/3})(x^2+2)$

c) $t(x) = (x^2+3x-4)^5$

d) $g(x) = \dfrac{x^2+5x+2}{x-1}$

e) $t(x) = m(x) \cdot n(x)$

f) $g(x) = \sqrt{1-3x}$

g) $p(x) = (x^2+1)(2x+3)^3$

h) $f(x) = r(x)/s(x)$

i) $h(x) = (x^2+7x+5)(3x^2-8x+2)$

j) $f(x) = (s(x))^n$

APPENDIX D

THE RETENTION TEST

Mathematics 1743 QUIZ Name________________

1. Let $f(x) = x^3-12x+7$

a) Find the intervals where $f(x)$ is increasing or decreasing.

b) Find the intervals where $f(x)$ is concave upward or concave downward.

c) Find the inflection point.

d) Find the critical values on the closed interval from -3 to 5.

e) Find the maximum and minimum points on the closed interval from -3 to 5.

2. Find $p'(x)$:

a) $p(x) = \frac{1}{\sqrt{3x+5}}$

b) $p(x) = (4x^3-8x+7)\ (6x^4-3x^2+7x-5)$

c) $p(x) = \dfrac{-8x}{x^2+5}$

d) $p(x) = (x^3+5)^7$

e) $p(x) = x^2 \cdot \sqrt{x^2+7}$

f) $p(x) = \dfrac{1 + \sqrt{x}}{x^2 + 3x}$

g) $p(x) = f(1-x^3)^{\frac{2}{3}}$

h) $p(x) = g(x) \cdot f(x)$

i) $p(x) = g(x)/f(x)$

j) $p(x) = (g(x))^m$

APPENDIX E

ANALYSIS OF COVARIANCE

The general algebraic model for analysis of covariance is as follows:

i. $Y_{ij(adj)} = Y_{ij} - \beta(X_{ij} - \overline{X}..) = \mu + \alpha_i + \varepsilon_{ij}$

ii. $\varepsilon_{ij} \sim NID(0,\sigma^2_\varepsilon)$

where,

$Y_{ij(adj)}$ is the adjusted postscore measure for the j-th individual in the i-th treatment group.

Y_{ij} is the postscore for the j-th individual in the i-th treatment group.

β is the within-group linear regression coefficient.

X_{ij} is the covariate measure for the j-th individual in the i-th treatment group.

$\overline{X}..$ is the overall covariate mean.

μ is the overall mean.

α_i is the i-th treatment effect.

ε_{ij} is the error.

$NID(0,\sigma^2_\varepsilon)$ means the errors are normally and independently distributed with mean zero and constant variance, σ^2_ε .

APPENDIX F

THE KUDER-RICHARDSON FORMULA

The general Kuder-Richardson formula (Ebel, 1965) is as follows:

$$r = \frac{K}{K-1}\left(1-\frac{n\Sigma Q^2-\Sigma T^2}{n\Sigma X^2-(\Sigma X)^2}\right)$$

K is the number of items.

n is the number of subjects.

ΣQ^2 is the sum of the squares of the K times n individual questions scores.

ΣT^2 is the sum of the squares of the K question total scores.

ΣX^2 is the sum of the squares of the n student total scores.

ΣX is the sum of the n student total scores.

APPENDIX G

RAW DATA

The raw data for the experimental group is as follows:

Subject	Covariate Test	HFT	Initial Achievement Test	Retention Test
1	128	10.5	9	8
2	139	14	10	9
3	105	-3.5	9	9
4	102	15	10	9
5	106	24.5	9	9
6	125	3.5	8	4
7	117	13.5	8	5
8	103	8.75	10	6
9	110	13.75	9	4
10	93	8.5	3	4
11	140	11.75	9	9
12	135	.5	6	5
13	105	14.75	8	8
14	119	13.75	5	9
15	91	12	10	8
16	37	1	3	0
17	83	3.75	5	5
18	70	15.75	7	2
19	141	3.25	10	10
20	109	11	8	9
21	76	12	8	4
22	144	17	9	5
23	88	18	7	2
24	112	12	10	7
25	108	14	8	4
26	95	9	4	0
27	131	5	9	10
28	141	20	9	6
29	152	21	6	7
30	72	2.25	6	4
31	94	1	9	6

Table 15: Raw Data for the Experimental Group

The raw data for the control group is as follows:

Subject	Covariate Test	HFT	Initial Achievement Test	Retention Test
1	135	2.75	8	5
2	115	9.75	9	7
3	94	9.25	8	7
4	84	12	9	9
5	110	7	7	6
6	111	5.5	8	6
7	124	6	10	9
8	82	2	9	1
9	105	5	8	6
10	68	12	2	1
11	113	6.75	8	2
12	119	8.25	8	5
13	88	-.5	5	3
14	66	2	1	0
15	123	22	10	3
16	129	10	5	4
17	92	8	9	8
18	92	13.75	6	1
19	107	12	10	8
20	95	23	6	6
21	80	10	9	5
22	124	10.75	9	9
23	124	4.75	10	6
24	120	10	9	8
25	77	1.5	8	2
26	77	16.5	6	6
27	106	7.75	10	9
28	138	5.25	10	10
29	89	18	6	4
30	105	1.75	4	4
31	56	8.75	2	1

Table 16: Raw Data for the Control Group