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THE VIBRATIONAL RESPONSE OF THE RECTANGULAR
PARALLELEPIPED

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THE VIBRATIONAL RESPONSE OF THE RECTANGULAR PARALLELEPIPED

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ABSTRACT

This work presents exact normal mode solutions for the forced vibrational response of the rectangular parallelepiped with three sets of boundary conditions: (1) completely rigid-lubricated boundaries; (2) two stress-free and four rigid-lubricated boundaries; and (3) two elastically restrained and four rigid-lubricated boundaries. Both analytical and numerical verifications of these solutions are provided. Applications are discussed in the fields of acoustic emission non-destructive testing and the calibration of piezoelectric transducers.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

CHAPTER I

INTRODUCTION

1.1 Background

The forced vibrational response of the rectangular parallelepiped is of particular interest in the study of wave propagation in solids and especially in the characterization of acoustic emission sources. Acoustic emission are the stress waves generated by the rapid release or redistribution of stored energy that accompany many deformation and fracture processes. The two major sources of acoustic emission are plastic deformation and crack growth. There has been considerable interest in studying the mechanisms associated with these sources in order to predict, and eventually perhaps control, flaw growth in structural materials.

Understanding the relationship between source and receiver in acoustic emission experiments has been the motivation for several recent papers [1-3] which have addressed the dynamic response of plates. However, many acoustic emission applications involve specimens of finite dimensions which are not accurately modeled by a plate. Some experimental work has been done on the source-receiver problem in finite bodies, but very little analytical work due mainly to the complexity of the mathematics describing the specimen response. In fact, there are no forced vibration solutions for parallelepipeds in the literature and only a few free vibration solutions [4-9,12-17].

It is the purpose of this work to help bridge the gap between the experimental and the analytical by providing normal mode solutions for the forced vibrational response of the rectangular parallelepiped with boundary conditions sufficiently realistic in a physical sense to allow inferences to be made concerning the source event. Obviously, the more realistic the boundary conditions, the more accurate the inferences. The two sets of boundary conditions considered here are (1) all six faces rigid-lubricated and (2) four rigid-lubricated and two stress-free faces. These represent approximations to the completely stress-free case, which has not, as yet, been solved by the classical normal mode technique. A third set of boundary conditions consisting of four rigid-lubricated and two elastically restrained faces is considered in Chapter IV. The solution to this problem is of interest because, by adjusting the value of the elastic modulus, the solutions for the previous two sets of boundary conditions can be recovered.

1.2 System Response

Much of what is known about the nature of acoustic emission sources has been learned through the use of piezoelectric transducers coupled to rectangular parallelepiped or plate type specimens. Unfortunately, by the time an acoustic emission signal is displayed on an output device, the waveform has undergone some very complex transformations. An example of these complexities is demonstrated in the simple crack growth monitoring system of Figure 1.1. Here, the specimen is under some type of loading which causes a material flaw to grow,

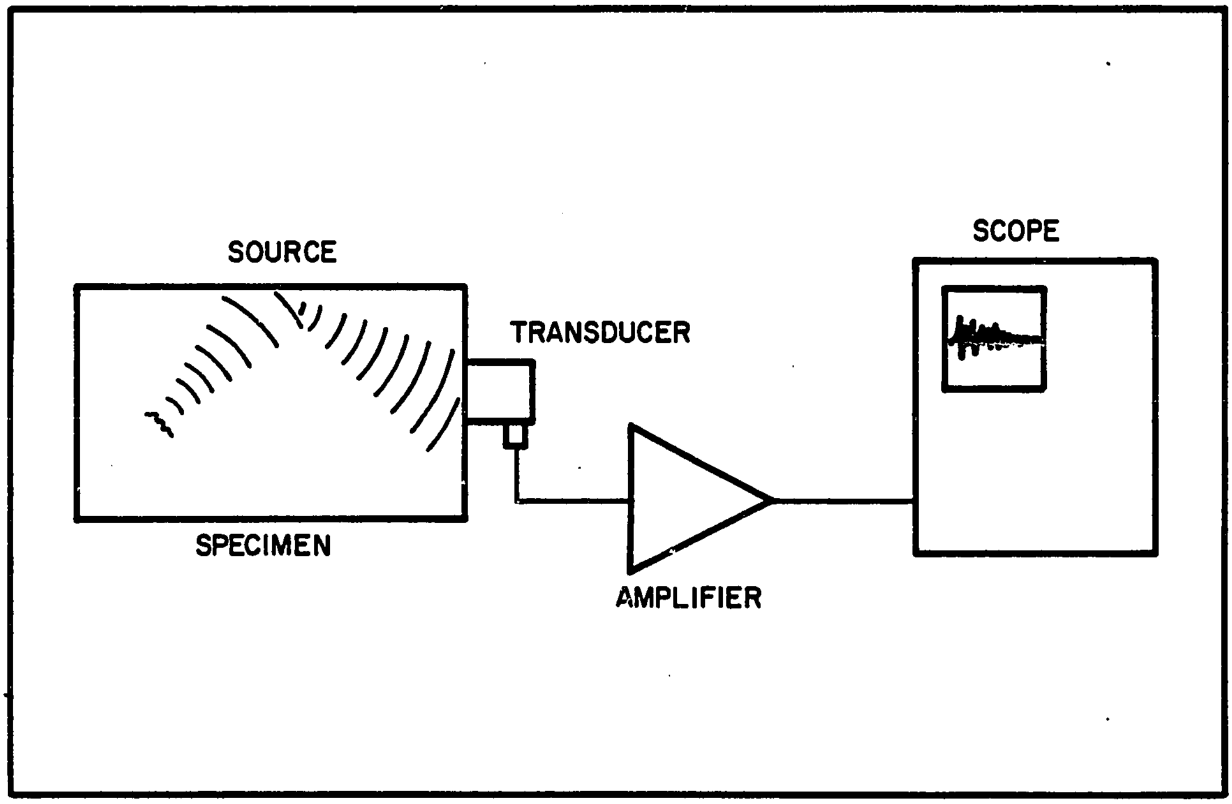

FIGURE 1.1
CRACK GROWTH MONITORING SYSTEM

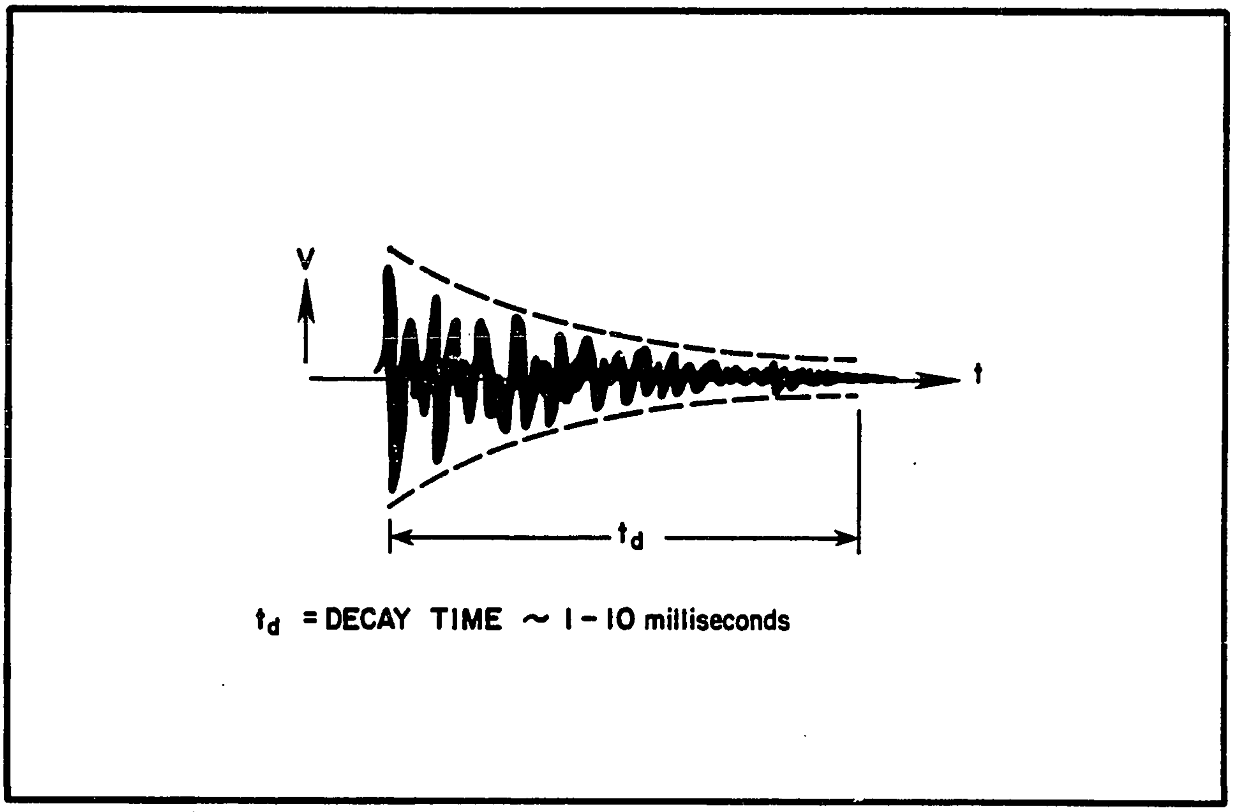

FIGURE 1.2

TYPICAL VOLTAGE VS TIME OSCILLOSCOPE DISPLAY
PRODUCED BY AN ACOUSTIC EMISSION BURST

thereby releasing energy in the form of acoustic emission. These waves reflect off the specimen boundaries and are sensed by the piezoelectric transducer. The piezoelectric crystal generates an electrical signal in proportion to the strength of the received stress wave. This signal is then amplified and displayed on an oscilloscope. A typical voltage versus time output trace for an acoustic emission burst is shown in Figure 1.2. Its attenuation is due primarily to damping at the specimen-transducer interface [7] and depends, to a much lesser extent, on the material properties.

Using a systems analysis approach, Spanner [8] postulated a linear response for the crack detection system as a whole; Houghton, Townsend and Packman [9] confirmed this experimentally. If it is assumed that the amplifier and the oscilloscope introduce no appreciable distortion to the transducer output, there are still three sources of distortion: the specimen, the specimen-transducer interface, and the transducer itself. The measured voltage response of the crack growth monitoring system as a function of frequency is then expressed as

$$H_{MEASURED}(\omega) = H_{TRANSDUCER/INTERFACE}(\omega)\ H_{SPECIMEN}(\omega)\ H_{SOURCE}(\omega), \quad (1.2.1)$$

where

$$H_{TRANSDUCER/INTERFACE}(\omega) = H_{TRANSDUCER}(\omega)\ H_{INTERFACE}(\omega)$$

is the combined transfer function for the transducer and the specimen-transducer interface.

To begin with, the only known quantity in equation (1.2.1) is $H_{MEASURED}(\omega)$. This is the frequency spectrum of the time-domain oscilloscope output (Fig. 1.2) and can be measured experimentally. Given a known source and point of application, $H_{MEASURED}(\omega)$ may be calculated. For example, an impulse function has a uniform frequency spectrum from DC to 20 MHz; the frequency response of a step function decays exponentially. Then assuming that the transfer function for the specimen (specimen response) $H_{SPECIMEN}(\omega)$ can be determined, the transducer/interface response, $H_{TRANSDUCER/INTERFACE}(\omega)$ may be calculated according to equation (1.2.1), i.e., the transducer/interface can be calibrated. Once the transducer/interface is calibrated, the transfer function for any unknown source (of known location) can be obtained according to the expression

$$H_{SOURCE}(\omega) = \frac{H_{MEASURED}(\omega)}{H_{TRANSDUCER/INTERFACE}(\omega)\ H_{SPECIMEN}(\omega)}\ , \qquad (1.2.2)$$

which is simply a rearrangement of equation (1.2.1). $H_{SOURCE}(\omega)$ can then be deconvoluted to obtain the time-domain source waveform. Inferences can then be made concerning the nature of the acoustic emission source and the mechanisms involved in its production.

Perhaps the system model which is the most physically realistic is a simply supported specimen with a uniform loading at the specimen-transducer interface and otherwise stress-free boundaries. One approximation to this system would be a specimen with completely stress-free boundaries. This approximate system is defined and discussed in the ensuing sections of this chapter along with two further simplifications of lesser mathematical difficulty.

1.3 Specimen Response

The specimen is assumed to be a homogeneous, isotropic, perfectly elastic solid. Its wave propagation is, therefore, governed by the linear three-dimensional theory of elastodynamics [10,11]. The coordinate system, dimensions, and stress convention are given in Figure 1.3, and the governing equation of motion is Navier's equation, which may be expressed in terms of wave speeds as

$$c_t^2\nabla^2\bar{u} + (c_\ell^2 - c_t^2)\nabla\nabla\cdot\bar{u} + \bar{f} = \frac{\partial^2\bar{u}}{\partial t^2} \tag{1.3.1}$$

with

$$\bar{u} = u_1\hat{e}_1 + u_2\hat{e}_2 + u_3\hat{e}_3 = \text{displacement}$$

$$\bar{f} = f_1\hat{e}_1 + f_2\hat{e}_2 + f_3\hat{e}_3 = \text{body force per unit mass}$$

$$u_i = u_i(x_1,x_2,x_3,t)$$

$$f_i = f_i(x_1,x_2,x_3,t) \qquad i=1,2,3 \;;$$

$$\nabla = \frac{\partial}{\partial x_1}\,\hat{e}_1 + \frac{\partial}{\partial x_2}\,\hat{e}_2 + \frac{\partial}{\partial x_3}\,\hat{e} \;;$$

and

$$c_\ell = \left[\frac{\lambda + 2\mu}{\rho}\right]^{\frac{1}{2}} = \text{longitudinal wave speed}$$

$$c_t = \left[\frac{\mu}{\rho}\right]^{\frac{1}{2}} = \text{transverse wave speed} \;.$$

Here, ρ is the density and λ and μ are the Lamé elastic constants. The body force term $\bar{f}$ is used to represent acoustic emission bursts. No surface forces are considered here since acoustic emission is primarily a body force phenomenon.

The boundary conditions for the completely stress-free rectangular parallelepiped are as follows:

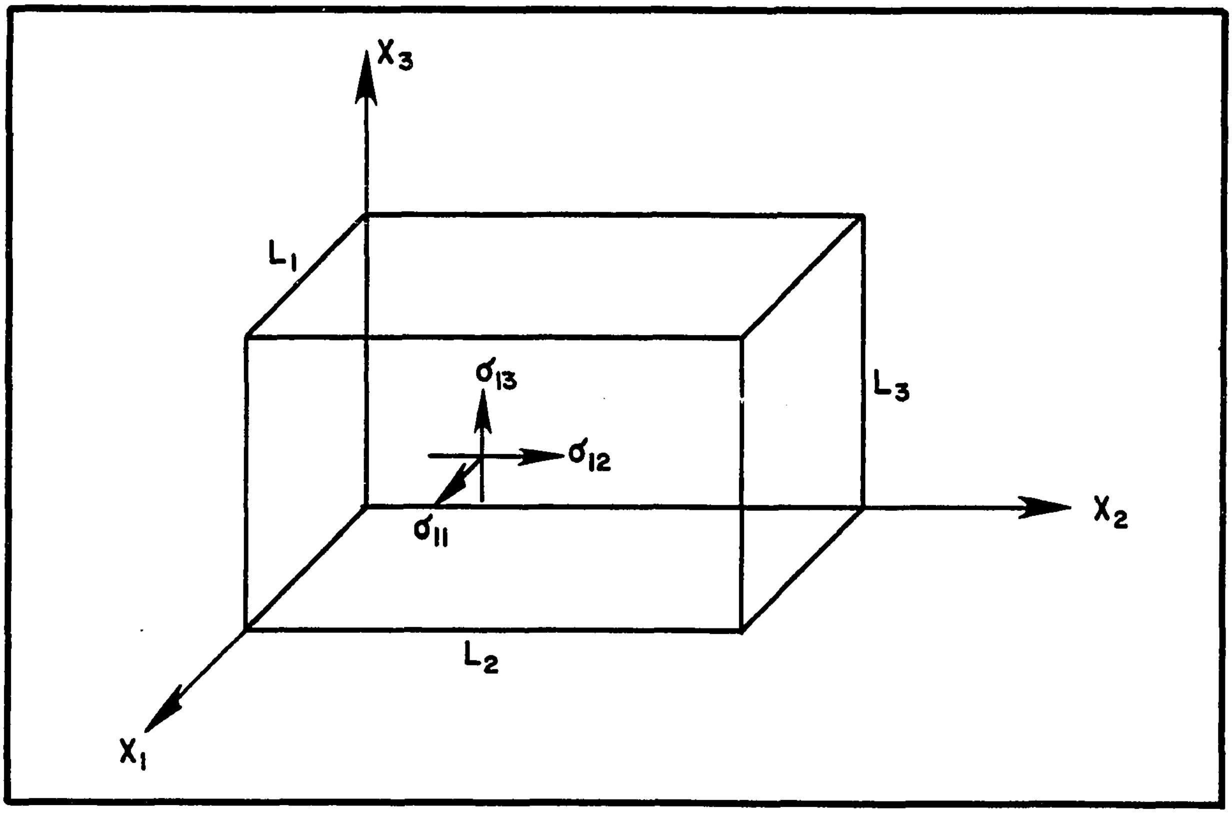

FIGURE 1.3
COORDINATE SYSTEM, DIMENSIONS AND STRESS CONVENTION

$$
\begin{aligned}
x_1 &= 0, L_1 \qquad & \sigma_{11} &= \sigma_{12} = \sigma_{13} = 0 \\
x_2 &= 0, L_2 \qquad & \sigma_{22} &= \sigma_{21} = \sigma_{23} = 0 \\
x_3 &= 0, L_3 \qquad & \sigma_{33} &= \sigma_{31} = \sigma_{32} = 0 \; .
\end{aligned}
$$

Writing the stresses in terms of displacements gives

$$
\begin{aligned}
\frac{\sigma_{11}}{\lambda} &= \gamma \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \\
\frac{\sigma_{22}}{\lambda} &= \frac{\partial u_1}{\partial x_1} + \gamma \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \\
\frac{\sigma_{33}}{\lambda} &= \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \gamma \frac{\partial u_3}{\partial x_3}
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{12} &= \sigma_{21} = \mu\left[\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right] \\
\sigma_{13} &= \sigma_{31} = \mu\left[\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right] \\
\sigma_{23} &= \sigma_{32} = \mu\left[\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right] ,
\end{aligned}
$$

where $\gamma = 1 + \frac{2\mu}{\lambda}$. Therefore, the stress-free boundary conditions in terms of displacements become

$$
\begin{aligned}
x_1 &= 0, L_1 \quad \gamma \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} = \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} = 0 \\
x_2 &= 0, L_2 \quad \frac{\partial u_1}{\partial x_1} + \gamma \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2} = \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} = 0 \qquad (1.3.2) \\
x_3 &= 0, L_3 \quad \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \gamma \frac{\partial u_3}{\partial x_1} = \frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3} = \frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3} = 0
\end{aligned}
$$

Within the parallelepiped there are two types of waves propagating, dilatational and equivoluminal, both of which are three-dimensional in nature. Any three-dimensional wave front, no matter what its shape, can be represented by an infinite set of contiguous points, each point being the limiting case of a planar wave front. Accordingly, dilatational waves can be expressed in terms of an infinite sum of plane longitudinal waves propagating in every direction and equivoluminal waves in terms of a similar set of plane transverse waves; hence, the notation for the wave speeds, c_ℓ and c_t. These two wave types are depicted in Figure 1.4.

When either a longitudinal or a shear wave reflects off a stress-free surface, depending upon the angle of incidence, any one of three things can happen. The incident wave can either reflect unchanged; a portion of it can be mode converted into the other wave type, in which case two waves are reflected; or the incident wave can be entirely converted into a third wave type, the inhomogeneous wave. The most common type of inhomogeneous wave is the Rayleigh surface wave. These mode conversions at stress-free boundaries are only part of what make wave propagation problems in solids so difficult. The other part is the occurrence of multiple reflections between the boundaries in finite specimens.

The wave propagation problem can be simplified by assuming rigid-lubricated boundaries. This is because reflections from rigid-lubricated surfaces are specular, i.e., no mode conversions occur, only phase changes. Therefore, there are no inhomogeneous waves (imaginary wave numbers), and the only difficulty is multiple reflections. Physically, these boundary conditions suggest a problem in which a body is vibrating inside a container with infinitely rigid, frictionless walls. Although this is not representative of the typical acoustic emission experiment, the solution does provide a first step in solving

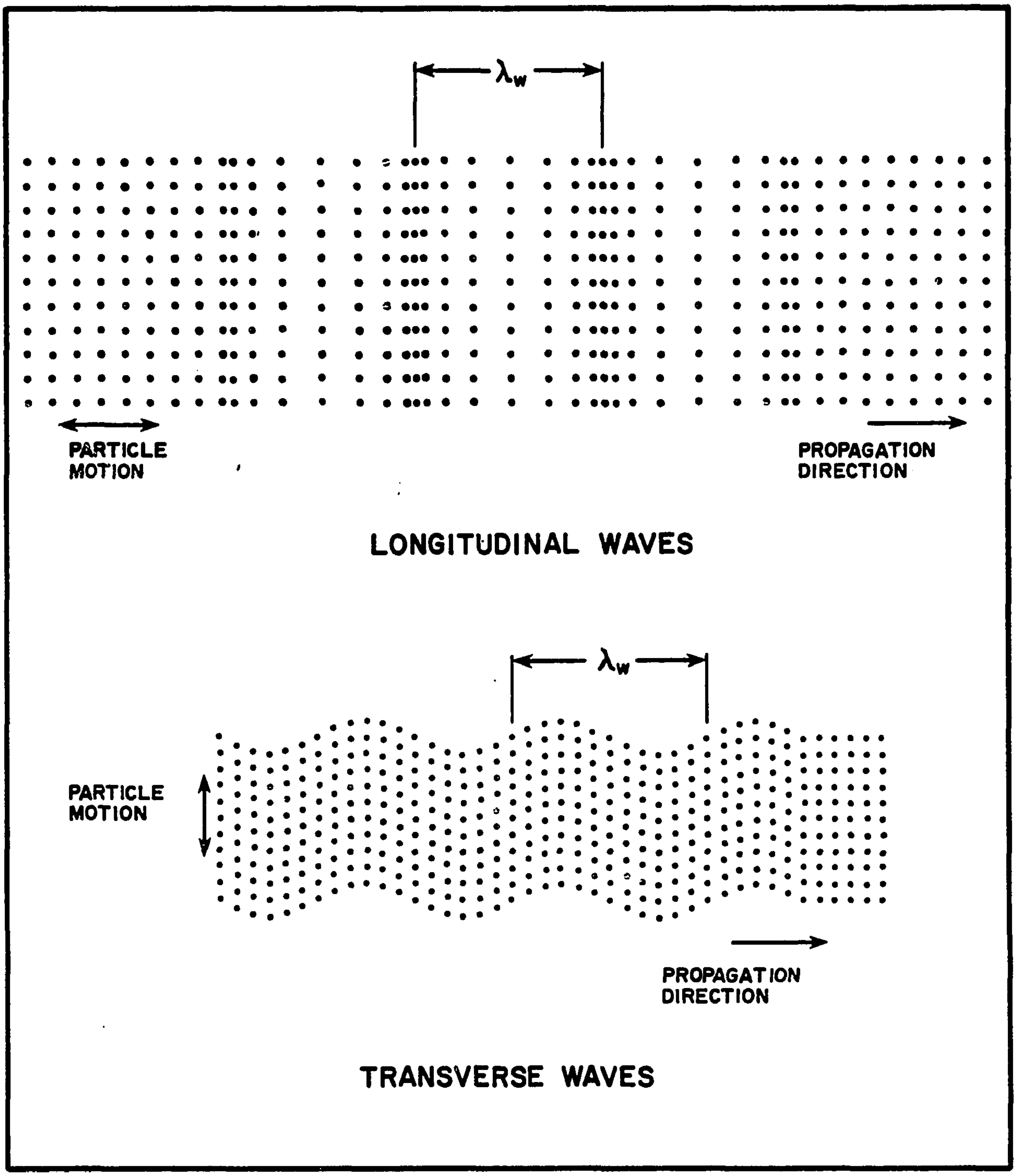

FIGURE 1.4

WAVES PROPAGATING WITHIN AN ELASTIC SOLID for the more difficult stress-free cases.

Next in complexity is the solution for the problem of a rectangular parallelepiped with four rigid-lubricated and two stress-free boundaries. This problem is considerably more involved than the previous one due to the mode conversions on the two stress-free faces; on the other hand, it is also more realistic. Here, there are longitudinal-shear and shear-longitudinal conversions corresponding to the real wave numbers and shear-inhomogeneous conversions associated with the imaginary wave numbers, and as before, there are multiple reflections.

The problem of the parallelepiped with completely stress-free boundaries is the most complex of the three presented and also the most realistic. It allows for mode conversions at all the boundaries as well as multiple reflections. The next section discusses normal mode solutions to these three problems.

1.4 Normal Mode Solutions

The normal mode technique is appropriate for solving vibration or wave propagation problems in finite bodies. This is true because finite bodies only vibrate at discrete frequencies as opposed to infinite bodies which respond to the whole frequency spectrum. The displacement pattern associated with each of the natural frequencies is called a normal mode, and all the normal modes combine to give the total vibration or displacement pattern of the body. Separation of variables is the approach typically used to determine the natural frequencies and normal modes of a given system.

In order to solve for the forced vibrational response of any system using the normal mode approach, it is necessary to first solve for the free vibrational response. The free vibration problem for a rectangular parallelepiped with rigid-lubricated boundaries was first solved by Ortway [4] in 1913 and repeated by Nadeau [5] in 1964. In an effort to preserve continuity, Nadeau's solution is recast in Chapter II. The forced vibration problem is solved by first uncoupling the equations of motion using a vector displacement function, then utilizing a normal mode approach to obtain the desired displacements.

The free vibration solution for the case of four rigid-lubricated and two stress-free boundaries is the work of Kaliski as presented by Malecki [12]. Kaliski's original work [13] is in Polish; Malecki's text provides an English translation. Several significant errors were discovered in this presentation. As such, the free vibration problem is reworked in its entirety in Chapter III. This includes many of the details omitted by Malecki. The forced vibration problem is then solved by the normal mode technique.

Using a straightforward normal mode approach to solve the problem of the rectangular parallelepiped with completely stress-free boundaries, one obtains trivial solutions only. This is because separation of variables assumes factored solutions of the form

$$u_i(x_1,x_2,x_3,t) = X_{1i}(x_1)X_{2i}(x_2)X_{3i}(x_3)T(t), \quad i=1,2,3 \qquad (1.4.1)$$

and no member of this set can satisfy the completely stress-free boundary conditions. In fact, the use of such solutions leads to the situation where there are more equations than unknowns.

The additional unknowns can be generated systematically using the method of associated periodicity developed by Fromme and Leissa [14,15]. They applied this technique to obtain a periodic extension of Navier's equation (1.2.1) and the stress-free boundary conditions and then employed Fourier analysis to reduce the partial differential equations to a set of algebraic equations. These equations were then solved to obtain the complete eigenspectrum for the free vibration problem. One significant drawback to this technique is the need to solve an infinite matrix in order to determine the natural frequencies.

Budanov and Orlov [16] obtained a portion of the eigenspectrum by assuming a particular form for $\nabla\cdot\bar{u}$ and solving for the symmetric modes. The antisymmetric modes were not considered, nor were any other forms for $\nabla\cdot\bar{u}$; moreover, several simplifying approximations were made in their numerical computations. In spite of all this, their computed natural frequencies for the rigid body modes did compare favorably with experimental beam data. There was no indication as to how well this analysis worked on rectangular parallelepipeds having dimensions of similar magnitude.

The two solutions discussed above are the only known exact analytical solutions for the free vibrational response of the rectangular parallelepiped with stress-free boundaries. Both of them are algebraically very complex, which may explain why neither work has been referenced in any recent publications. Because of this complexity, and the need for a forced vibration solution to model acoustic emission activity, the author made several attempts to solve this problem using other approaches. Unfortunately, none of them were successful. As a consequence, the solution developed herein for the forced vibration of the rectangular parallelepiped with four rigid-lubricated and two stress-free boundaries probably represents the best available analytical tool to model acoustic emission activity in parallelepipeds with stress-free boundaries. The completely rigid-lubricated problem mainly provides a first step in obtaining the more difficult rigid-lubricated/stress-free solution.

CHAPTER II

RIGID-LUBRICATED BOUNDARIES

2.1 Free Vibration Solution

The equations of motion for the free vibration solution are Navier's equations (1.3.1) with the body force terms set equal to zero:

$$c_t^2\nabla^2\bar{u} + (c_\ell^2 - c_t^2)\nabla\nabla\cdot\bar{u} = \frac{\partial^2\bar{u}}{\partial t^2} . \qquad (2.1.1)$$

The rigid-lubricated boundary conditions are given as

$$\begin{array}{lll} x_1 = 0, L_1 & u_1 = 0 & \sigma_{12} = \sigma_{13} = 0 \\ x_2 = 0, L_2 & u_2 = 0 & \sigma_{21} = \sigma_{23} = 0 \\ x_3 = 0, L_3 & u_3 = 0 & \sigma_{31} = \sigma_{32} = 0 \quad , \end{array}$$

which, in terms of displacements, become

$$\begin{array}{lll} x_1 = 0, L_1 & u_1 = 0 & \dfrac{\partial u_2}{\partial x_1} = \dfrac{\partial u_3}{\partial x_1} = 0 \\ x_2 = 0, L_2 & u_2 = 0 & \dfrac{\partial u_1}{\partial x_2} = \dfrac{\partial u_3}{\partial x_2} = 0 \\ x_3 = 0, L_3 & u_3 = 0 & \dfrac{\partial u_1}{\partial x_3} = \dfrac{\partial u_2}{\partial x_3} = 0 \end{array} \qquad (2.1.2)$$

The problem may be solved by assuming simple harmonic motion of the parallelepiped and normal mode displacement components of the form [4-6]

$$u_{1N} = A_{1N} \sin k_1 x_1 \cos k_2 x_2 \cos k_3 x_3 \sin \omega_N t$$

$$u_{2N} = A_{2N} \cos k_1x_1 \sin k_2x_2 \cos k_3x_3 \sin\omega_N t \qquad (2.1.3)$$

$$u_{3N} = A_{3N} \cos k_1x_1 \cos k_2x_2 \sin k_3x_3 \sin\omega_N t ,$$

where the ω_N are the natural frequencies or eigenvalues of the system. In order to satisfy the boundary conditions, the wave numbers must be $k_1 = n_1\Pi/L_1$, $k_2 = n_2\Pi/L_2$, and $k_3 = n_3\Pi/L_3$ with n_1, n_2, n_3 being the integers from zero to infinity. The direction of propagation of each component wave is determined by the set of integer indices $N(n_1,n_2,n_3)$. Substituting the above assumed normal modes into the equations of motion (2.1.1), one obtains for each set N,

$$\begin{bmatrix} \beta_N + k_1^2 & k_1k_2 & k_1k_3 \\ k_1k_2 & \beta_N + k_2^2 & k_2k_3 \\ k_1k_3 & k_2k_3 & \beta_N + k_3^2 \end{bmatrix} \begin{Bmatrix} A_{1N} \\ A_{2N} \\ A_{3N} \end{Bmatrix} = 0 \qquad (2.1.4)$$

with $\beta_N = (c_t^2\alpha_N^2 - \omega_N^2)/(c_\ell^2 - c_t^2)$ and $\alpha_N^2 = k_1^2 + k_2^2 + k_3^2$. This set of equations has a nontrivial solution if and only if the determinent of the 3x3 matrix is equal to zero, i.e.,

$$(\beta_N + \alpha_N^2)\,\beta_N^2 = 0 \ . \qquad (2.1.5)$$

Equation (2.1.5) is the characteristic equation. Corresponding to its roots, $\beta_{1N} = -\alpha_N^2$ and $\beta_{2N}=\beta_{3N}=0$, are the natural frequencies of the system

$$\omega_{1N} = c_\ell \alpha_N \qquad (2.1.6)$$

$$\omega_{2N} = \omega_{3N} = c_t \alpha_N \ . \qquad (2.1.7)$$

It can be shown [11,14] that ω_{1N} is associated with dilatational waves and $\omega_{2N}=\omega_{3N}$ with the two orthogonal polarizations of equivoluminal waves. Thus, each displacement component (u_1, u_2, u_3) is made up of three contributions, one due to dilatational waves and the other two due to equivoluminal waves. However, as was mentioned previously, any three-dimensional wave front can be expressed in terms of infinite sums of plane longitudinal and transverse wave components so that $\omega_{1N}=\omega_{\ell N}$ and $\omega_{2N}=\omega_{3N}=\omega_{tN}$, and therefore

$$
\begin{aligned}
u_1 &= \sum_N \sin k_1x_1 \cos k_2x_2 \cos k_3x_3 \left[(A_{1N})_\ell \sin\omega_{\ell N}t + (A_{1N})_t \sin\omega_{tN}t\right] \\
u_2 &= \sum_N \cos k_1x_1 \sin k_2x_2 \cos k_3x_3 \left[(A_{2N})_\ell \sin\omega_{\ell N}t + (A_{2N})_t \sin\omega_{tN}t\right] \\
u_3 &= \sum_N \cos k_1x_1 \cos k_2x_2 \sin k_3x_3 \left[(A_{3N})_\ell \sin\omega_{\ell N}t + (A_{3N})_t \sin\omega_{tN}t\right] ,
\end{aligned}
\tag{2.1.8}
$$

with the notation $\sum\limits_N = \sum\limits_{n_1=0}^{\infty} \sum\limits_{n_2=0}^{\infty} \sum\limits_{n_3=0}^{\infty}$.

The longitudinal wave amplitude relations are determined by substituting $\omega_N=\omega_{\ell N}$ back into equation (2.1.4). This gives the result

$$
\begin{aligned}
(A_{1N})_\ell &= (A_{1N})_\ell \\
(A_{2N})_\ell &= \frac{k_2}{k_1} (A_{1N})_\ell \\
(A_{3N})_\ell &= \frac{k_3}{k_1} (A_{1N})_\ell .
\end{aligned}
\tag{2.1.9}
$$

A similar procedure determines the amplitude relations for the transverse waves:

$$
\begin{aligned}
(A_{1N})_t &= (A_{1N})_t \\
(A_{2N})_t &= (A_{2N})_t \\
(A_{3N})_t &= -\frac{k_1}{k_3} (A_{1N})_t - \frac{k_2}{k_3} (A_{2N})_t .
\end{aligned}
\tag{2.1.10}
$$

Thus, displacements (2.1.8) become

$$u_1 = \sum_N \sin k_1x_1 \cos k_2x_2 \cos k_3x_3 \{(A_{1N})_\ell \sin\omega_{\ell N}t + (A_{1N})_t \sin\omega_{tN}t\}$$

$$u_2 = \sum_N \cos k_1x_1 \sin k_2x_2 \cos k_3x_3 \{\frac{k_2}{k_1}(A_{1N})_\ell \sin\omega_{\ell N}t + (A_{2N})_t \sin\omega_{tN}t\}$$

$$u_3 = \sum_N \cos k_1x_1 \cos k_2x_2 \sin k_3x_3 \{\frac{k_3}{k_1}(A_{1N})_\ell \sin\omega_{\ell N}t - [\frac{k_1}{k_3}(A_{1N})_t + \frac{k_2}{k_3}(A_{2N})_t]\sin\omega_{tN}t\} \ . \tag{2.1.11}$$

These equations represent the free vibration displacements of any point within or on the surface of the rectangular parallelepiped as a function of time. The displacements $(A_{1N})_\ell$, $(A_{1N})_t$, and $(A_{2N})_t$ must be determined from the initial conditions of the problem.

2.2 Forced Vibration Solution

The equation of motion governing the forced vibration problem is (1.3.1). On writing this equation in component form,

$$c_t^2\nabla^2u_1 + (c_\ell^2 - c_t^2)\frac{\partial}{\partial x_1}(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}) + f_1 = \frac{\partial^2 u_1}{\partial t^2}$$

$$c_t^2\nabla^2u_2 + (c_\ell^2 - c_t^2)\frac{\partial}{\partial x_2}(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}) + f_2 = \frac{\partial^2 u_2}{\partial t^2} \tag{2.2.1}$$

$$c_t^2\nabla^2u_3 + (c_\ell^2 - c_t^2)\frac{\partial}{\partial x_3}(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}) + f_3 = \frac{\partial^2 u_3}{\partial t^2} \ ,$$

it can be seen that the three equations are elastically coupled (u_1,u_2, and u_3 appear in each equation). This prevents a straightforward solution, in that the equations must first be uncoupled, an algebraically cumbersome project, even with the use of Laplace transforms. These difficulties can be minimized by expressing the displacement vector in terms of a vector displacement function $\bar{\Psi}$ [14,17,18]:

$$\bar{u} = \rho[c_\ell^2\nabla^2\bar{\Psi} - (c_\ell^2 - c_t^2)\nabla\nabla\cdot\bar{\Psi} - \frac{\partial^2\bar{\Psi}}{\partial t^2}] , \tag{2.2.2}$$

with $\bar{\Psi} = \Psi_1\hat{e}_1 + \Psi_2\hat{e}_2 + \Psi_3\hat{e}_3$

$$\Psi_i = \Psi_i(x_1,x_2,x_3,t) \qquad i=1,2,3$$

Substituting equation (2.2.2) into equation (2.2.1), one obtains the result

$$(c_\ell^2\nabla^2 - \frac{\partial^2}{\partial t^2})(c_t^2\nabla^2 - \frac{\partial^2}{\partial t^2})\bar{\Psi} = -\frac{\bar{f}}{\rho} , \tag{2.2.3}$$

which is the uncoupled equation of motion in terms of the displacement functions. Solving for the Ψ_i(i=1,2,3) from equation (2.2.3) then allows the determination of the displacements from equation (2.2.2).

The solution begins by assuming displacement functions having the same spatial form as the previously assumed normal modes (2.1.3) but now being a general function of time (instead of being restricted to simple harmonic motion):

$$\begin{aligned} \Psi_1 &= \sum_N \sin k_1x_1\cos k_2x_2\cos k_3x_3 T_{1N}(t) \\ \Psi_2 &= \sum_N \cos k_1x_1\sin k_2x_2\cos k_3x_3 T_{2N}(t) \\ \Psi_3 &= \sum_N \cos k_1x_1\cos k_2x_2\sin k_3x_3 T_{3N}(t) . \end{aligned} \tag{2.2.4}$$

These expressions satisfy the boundary conditions, equations (2.1.2). Substituting equation (2.2.4a) into the appropriate uncoupled equation of motion (2.2.3a) and performing the necessary algebraic manipulations, one gets the following results:

$$c_\ell^2c_t^2\sum_N \alpha_N^4\sin k_1x_1\cos k_2x_2\cos k_3x_3 T_{1N} + (c_\ell^2 + c_t^2)\sum_N \alpha_N^2\sin k_1x_1\cos k_2x_2\cos k_3x_3\ddot{T}_{1N}$$
$$+\sum_N \sin k_1x_1\cos k_2x_2\cos k_3x_3 \ddot{\ddot{T}}_{1N} = -\frac{f_1}{\rho} , \tag{2.2.5}$$

where $\ddot{T}_{1N}$ represents the second derivative of $T_{1N}(t)$ with respect to time, etc. The next step is to multiply both sides of equation (2.2.5) by $\sin\ell_1 x_1 \cos\ell_2 x_2 \cos\ell_3 x_3$ and integrate over the spatial domain. However, due to the orthogonality of the normal modes, the following is true:

$$\int_0^{L_1}\int_0^{L_2}\int_0^{L_3} \sin k_1x_1 \sin\ell_1x_1 \cos k_2x_2 \cos\ell_2x_2 \cos k_3x_3 \cos\ell_3x_3\, dx_1 dx_2 dx_3$$

$$i=1,2,3, \qquad = \begin{cases} 0 & \text{when } \ell_i \neq k_i \\ \dfrac{\eta_1 V}{8} & \text{when } \ell_i = k_i \end{cases}$$

with $\eta_1=(1+\delta_{k_2 0})(1+\delta_{k_3 0})$ and $V=L_1L_2L_3$ is the volume of the parallelepiped. Therefore, performing the integrations on equation (2.2.5) gives

$$\ddot{\ddot{T}}_{1N} + (c_\ell^2 + c_t^2)\alpha_N^2 \ddot{T}_{1N} + c_\ell^2 c_t^2 \alpha_N^4 T_{1N}$$
$$= -\frac{8}{\rho\eta_1 V}\int_0^{L_1}\int_0^{L_2}\int_0^{L_3} f_1 \sin k_1x_1 \cos k_2x_2 \cos k_3x_3\, dx_1 dx_2 dx_3. \tag{2.2.6}$$

This expression may be solved by using Laplace transforms and assuming that the motion starts from rest ($T_{1N}(0)=\dot{T}_{1N}(0)=\ddot{T}_{1N}(0)=\dddot{T}_{1N}(0)=0$). Thus,

$$\bar{T}_{1N}(s) = \bar{F}_{1N}(s)\bar{G}_{1N}(s), \tag{2.2.7}$$

where $\bar{F}_{1N}(s)$ is the transform of the forcing function on the right-hand side of equation (2.2.6) and

$$\bar{G}_{1N}(s) = \frac{1}{(s^2+c_\ell^2\alpha_N^2)(s^2+c_t^2\alpha_N^2)}.$$

Using the convolution property and the fact that $\omega_{\ell N}=c_\ell\alpha_N$ and $\omega_{tN}=c_t\alpha_N$, one can write the inverse transform of equation (2.2.7) as

$$T_{1N}(t) = \int_0^t F_{1N}(\tau)G_{1N}(t-\tau)d\tau, \tag{2.2.8}$$

with

$$F_{1N}(\tau) = -\frac{8}{\rho\eta_1 V}\int_0^{L_1}\int_0^{L_2}\int_0^{L_3} f_1(x_1,x_2,x_3,\tau)\sin k_1x_1\cos_2x_2\cos k_3x_3dx_1dx_2dx_3 \tag{2.2.9}$$

$$G_{1M}(t-\tau) = \frac{1}{\omega_{tN}^2-\omega_{\ell N}^2}\left[\frac{\sin\omega_{\ell N}(t-\tau)}{\omega_{\ell N}} - \frac{\sin\omega_{tN}(t-\tau)}{\omega_{tN}}\right]. \tag{2.2.10}$$

The other two uncoupled equations of motion, (2.2.3b) and (2.2.3c), may be treated in a similar fashion with the results

$$T_{2N}(t) = \int_0^t F_{2N}(\tau)G_{2N}(t-\tau)d\tau \tag{2.2.11}$$

$$T_{3N}(t) = \int_0^t F_{3N}(\tau)G_{3N}(t-\tau)d\tau \tag{2.2.12}$$

and

$$F_{2N}(\tau) = -\frac{8}{\rho\eta_2 V}\int_0^{L_1}\int_0^{L_2}\int_0^{L_3} f_2(x_1,x_2,x_3,\tau)\cos k_1x_1\sin k_2x_2\cos k_3x_3dx_1dx_2dx_3 \tag{2.2.13}$$

$$F_{3N}(\tau) = -\frac{8}{\rho\eta_3 V}\int_0^{L_1}\int_0^{L_2}\int_0^{L_3} f_3(x_1,x_2,x_3,\tau)\cos k_1x_1\cos k_2x_2\sin k_3x_3dx_1dx_2dx_3 \tag{2.2.14}$$

$$G_{2N}(t-\tau) = G_{3N}(t-\tau) = \frac{1}{\omega_{tN}{}^2-\omega_{\ell N}{}^2}\left[\frac{\sin\omega_{\ell N}(t-\tau)}{\omega_{\ell N}} - \frac{\sin\omega_{tN}(t-\tau)}{\omega_{tN}}\right], \tag{2.2.15}$$

where

$$\eta_2 = (1+\delta_{k_1 0})(1+\delta_{k_3 0})$$

$$\eta_3 = (1+\delta_{k_1 0})(1+\delta_{k_2 0}) .$$

Finally, equations (2.2.8) through (2.2.15) are substituted into the displacement functions (2.2.4). These in turn are substituted into equation (2.2.2) to arrive at the forced vibration dispalcement $\bar{u}(x_1,x_2,x_3,t)$ for any generalized body force (per unit mass) $\bar{f}(x_1,x_2,x_3,t)$.

2.3 Response to an Impulse

According to Stephens and Pollock [19], acoutic emission source waves are pulselike functions of stress (force) which are produced by the step displacements associated with material yielding. This model is physically consistent with both plastic deformation and crack propagation, the two major sources of acoustic emission. Assuming a very short duration source event within the body, the Dirac delta function provides an extremely simple mathematical approximation of the resulting impulsive body force. In general, this body force will be three-dimensional; however, here for simplicity it is assumed to be one-dimensional in the x_3 direction and of amplitude F_0. This may be expressed mathematically as

$$\bar{f} = f_3\hat{e}_3$$

$$f_1 = f_2 = 0 \qquad f_3 = F_o\delta(x_1-\xi_1)\delta(x_2-\xi_2)\delta(x_3-\xi_3)\delta(t). \tag{2.3.1}$$

Note that this is an impulsive load applied at the point (ξ_1,ξ_2,ξ_3) and at time $t=0$ (Fig. 2.1).

Substitution of the above impulse into the results of the previous sections gives, from equation (2.2.14),

$$F_3(\tau) = -\frac{8F_o}{\rho n_3 V}\cos k_1\xi_1\cos k_2\xi_2\sin k_3\xi_3 . \tag{2.3.2}$$

This result is then combined with equation (2.2.15) and substituted into equation (2.2.12) to produce the time varying portion of the assumed displacement function:

$$T_3(t) = \frac{8F_o\cos k_1\xi_1\cos k_2\xi_2\sin k_3\xi_3}{\rho n_3 V(\omega_{\ell N}{}^2-\omega_{tN}{}^2)}\left[\frac{\sin\omega_{\ell N}t}{\omega_{\ell N}}-\frac{\sin\omega_{tN}t}{\omega_{tN}}\right] . \tag{2.3.3}$$

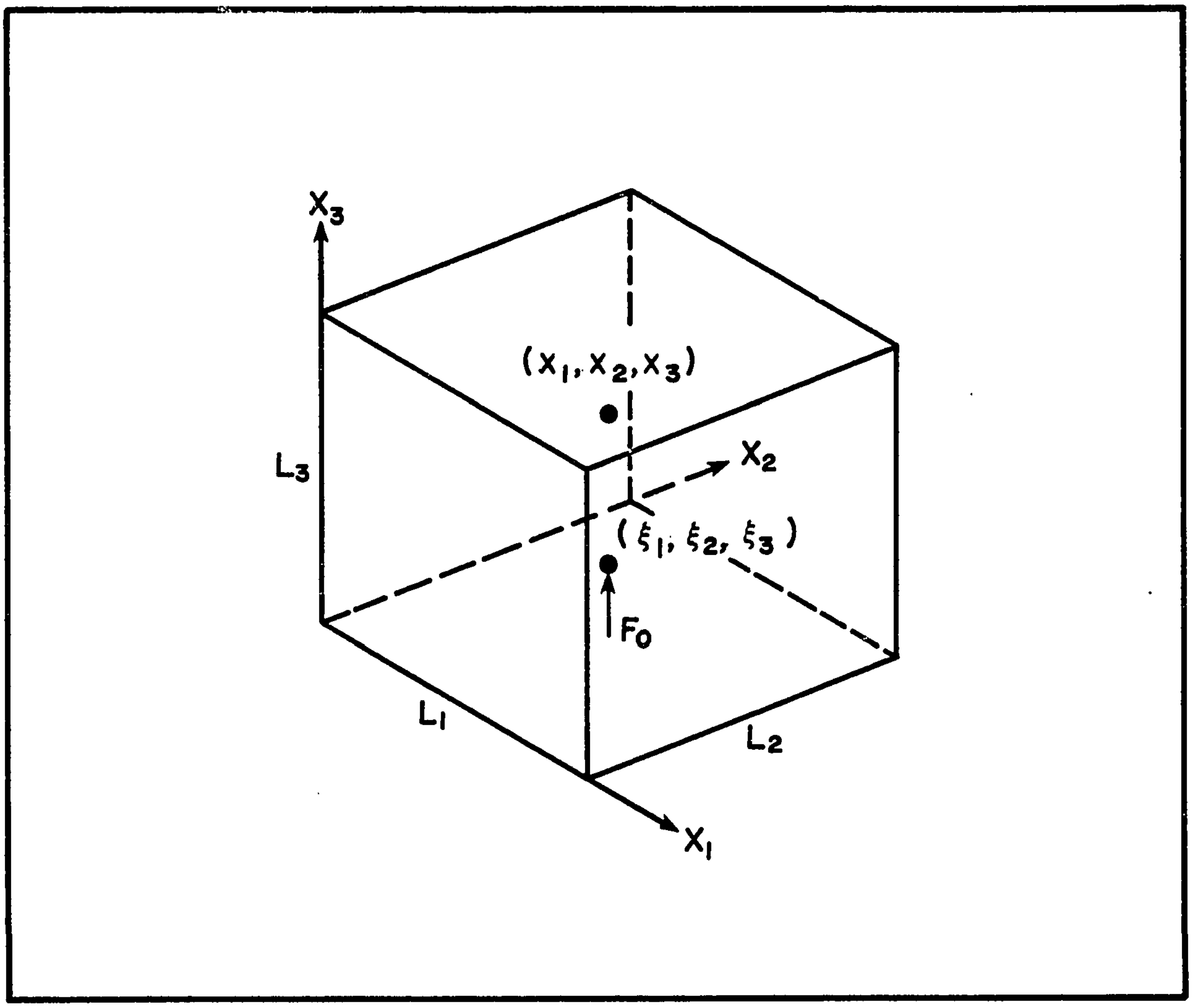

FIGURE 2.1

IMPULSIVE BODY FORCE APPLIED AT THE POINT (ξ_1, ξ_2, ξ_3) AND SENSED AT (x_1, x_2, x_3)

Then from equation (2.2.4) the displacement function becomes

$$\Psi_3 = \sum_N \frac{8F_o \cos k_1\xi_1 \cos k_2\xi_2 \sin k_3\xi_3}{\rho\eta_3 V(\omega_{\ell N}{}^2 - \omega_{tN}{}^2)} \cos k_1x_1 \cos k_2x_2 \sin k_3x_3 \left[\frac{\sin\omega_{\ell N}t}{\omega_{\ell N}} - \frac{\sin\omega_{tN}t}{\omega_{tN}}\right]. \tag{2.3.4}$$

Finally, substituting the above into equation (2.2.2) gives the three forced vibration displacement components shown below:

$$\begin{aligned}
u_1 &= \sum_N F_o \Phi_{3N} \sin k_1x_1 \cos k_2x_2 \cos k_3x_3 \left[\frac{k_1k_3}{\alpha_N{}^2}(\Omega_\ell - \Omega_t)\right] \\
u_2 &= \sum_N F_o \Phi_{3N} \cos k_1x_1 \sin k_2x_2 \cos k_3x_3 \left[\frac{k_2k_3}{\alpha_N{}^2}(\Omega_\ell - \Omega_t)\right] \qquad (2.3.5) \\
u_3 &= \sum_N F_o \Phi_{3N} \cos k_1x_1 \cos k_2x_2 \sin k_3x_3 \left[\Omega_t + \frac{k_3{}^2}{\alpha_N{}^2}(\Omega_\ell - \Omega_t)\right] \quad ;
\end{aligned}$$

here

$$\Omega_\ell = \frac{\sin\omega_{\ell N}t}{\omega_{\ell N}}$$

$$\Omega_t = \frac{\sin\omega_{tN}t}{\omega_{tN}}$$

and

$$\Phi_{3N} = \frac{8}{\eta_3 V} \cos k_1\xi_1 \cos k_2\xi_2 \sin k_3\xi_3 \ .$$

Not surprisingly, these results are the same as those obtained by Hill and Egle [20] using a Green's function approach to the problem.

2.4 Symmetric Boundary Conditions

Algebraically, it is often advantageous to work a problem over a symmetric interval. As such, the free and forced vibration solutions for the rectangular parallelepiped with completely rigid-lubricated faces are presented here for the symmetric boundary conditions

$$x_1 = -\frac{L_1}{2}, \frac{L_1}{2} \qquad u_1 = 0 \qquad \frac{\partial u_2}{\partial x_1} = \frac{\partial u_3}{\partial x_1} = 0$$

$$x_2 = -\frac{L_2}{2}, \frac{L_2}{2} \qquad u_2 = 0 \qquad \frac{\partial u_1}{\partial x_2} = \frac{\partial u_3}{\partial x_2} = 0 \tag{2.4.1}$$

$$x_3 = -\frac{L_3}{2}, \frac{L_3}{2} \qquad u_3 = 0 \qquad \frac{\partial u_1}{\partial x_3} = \frac{\partial u_2}{\partial x_3} = 0$$

For the free vibration solution, the equation of motion is again equation (2.1.1). Assuming normal modes of the form

$$u_{1N} = A_{1N} \sin k_1(x_1 + \frac{L_1}{2}) \cos k_2(x_2 + \frac{L_2}{2}) \cos k_3(x_3 + \frac{L_3}{2}) \sin \omega_N t$$

$$u_{2N} = A_{2N} \cos k_1(x_1 + \frac{L_1}{2}) \sin k_2(x_2 + \frac{L_2}{2}) \cos k_3(x_3 + \frac{L_3}{2}) \sin \omega_N t \tag{2.4.2}$$

$$u_{3N} = A_{3N} \cos k_1(x_1 + \frac{L_1}{2}) \cos k_2(x_2 + \frac{L_2}{2}) \sin k_3(x_3 + \frac{L_3}{2}) \sin \omega_N t$$

and proceeding as in Section 2.1, identical results are obtained for the wave numbers, characteristic equation, natural frequencies and the amplitude relations. Therefore, the free vibration displacements for the symmetric boundary conditions may be written as

$$u_1 = \sum_N \sin k_1(x_1 + \frac{L_1}{2}) \cos k_2(x_2 + \frac{L_2}{2}) \cos k_3(x_3 + \frac{L_3}{2}) \{(A_{1N})_\ell \sin \omega_{\ell N} t + (A_{1N})_t \sin \omega_{tN} t\}$$

$$u_2 = \sum_N \cos k_1(x_1 + \frac{L_1}{2}) \sin k_2(x_2 + \frac{L_2}{2}) \cos k_3(x_3 + \frac{L_3}{2}) \{\frac{k_2}{k_1}(A_{1N})_\ell \sin \omega_{\ell N} t + (A_{2N})_t \sin \omega_{tN} t\} \tag{2.4.3}$$

$$u_3 = \sum_N \cos k_1(x_1 + \frac{L_1}{2}) \cos k_2(x_2 + \frac{L_2}{2}) \sin k_3(x_3 + \frac{L_3}{2}) \{\frac{k_3}{k_1}(A_{1N})_\ell \sin \omega_{\ell N} t - [\frac{k_1}{k_3}(A_{1N})_t + \frac{k_2}{k_3}(A_{2N})_t] \sin \omega_{tN} t\} \ .$$

The forced vibration solution may be handled similarly. Assuming that the displacement functions can be expressed as

$$
\begin{aligned}
\Psi_1 &= \sum_N \sin k_1\left(x_1+\frac{L_1}{2}\right)\cos k_2\left(x_2+\frac{L_2}{2}\right)\cos k_3\left(x_3+\frac{L_3}{2}\right)T_{1N}(t) \\
\Psi_2 &= \sum_N \cos k_1\left(x_1+\frac{L_1}{2}\right)\sin k_2\left(x_2+\frac{L_2}{2}\right)\cos k_3\left(x_3+\frac{L_3}{2}\right)T_{2N}(t) \qquad (2.4.4) \\
\Psi_3 &= \sum_N \cos k_1\left(x_1+\frac{L_1}{2}\right)\cos k_2\left(x_2+\frac{L_2}{2}\right)\sin k_3\left(x_3+\frac{L_3}{2}\right)T_{3N}(t)
\end{aligned}
$$

and following the same procedure as in Section 2.2, one obtains similar results, the only difference being in the arguments of the spatial sin and cos terms. Instead of k_1x_1, k_2x_2, and k_3x_3, these arguments should be $k_1(x_1+\frac{L_1}{2})$, $k_2(x_2+\frac{L_2}{2})$ and $k_3(x_3+\frac{L_3}{2})$. In all other respects the functions are identical.

CHAPTER III

STRESS-FREE/RIGID-LUBRICATED BOUNDARIES

3.1 Free Vibration Solution

The applicable equation of motion for the free vibration solution is (2.1.1), which is repeated here for convenience:

$$c_t^2\nabla^2\bar{u} + (c_\ell^2 - c_t^2)\nabla\nabla\cdot\bar{u} = \frac{\partial^2\bar{u}}{\partial t^2} \,. \tag{3.1.1}$$

The boundary conditions consist of two stress-free faces and four rigid-lubricated faces and can be written as

$$\begin{aligned}
&x_1 = 0,L_1 && u_1 = 0 && \frac{\partial u_2}{\partial x_1} = \frac{\partial u_3}{\partial x_1} = 0 \\
&x_2 = 0,L_2 && u_2 = 0 && \frac{\partial u_1}{\partial x_2} = \frac{\partial u_3}{\partial x_2} = 0 \\
&x_3 = 0,L_3 && \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \gamma\,\frac{\partial u_3}{\partial x_3} = 0 && \frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3} = \frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3} = 0
\end{aligned} \tag{3.1.2}$$

A pictorial presentation of this system is shown in Figure 3.1. The specimen has a finite elastic modulus and is enclosed on four sides by an infinitely rigid medium such that normal displacements at these four surfaces are zero. However, due to lubrication between the contacting surfaces, transverse motion is uninhibited. The two x_3 faces (cross-hatched) are stress-free and, as a result, incident waves will mode convert on reflection. The two x_1 faces and the two x_2 faces, being rigid-lubricated, will reflect with no mode conversion. Thus, as

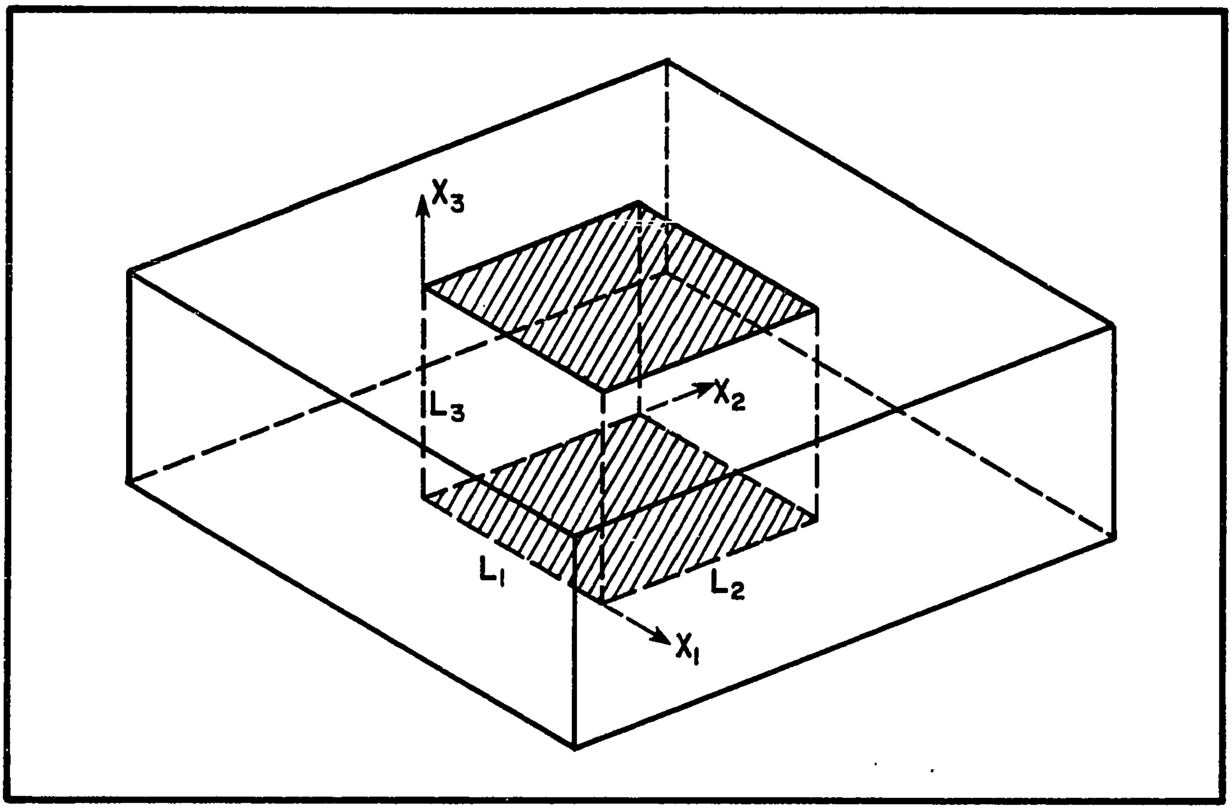

FIGURE 3.1
STRESS-FREE/RIGID-LUBRICATED BOUNDARIES

the boundary conditions become more complex, so also does the wave propagation. This increase in complexity holds true for the normal modes and the characteristic equation as well. Where in Chapter II it was possible to determine by inspection the exact form of the normal modes, this no longer holds true; rather, considerable calculation is required.

These calculations begin with the Helmholtz resolution [11, 21], which says that any vector field may be resolved into the gradient of a scalar and the curl of a zero-divergence vector. The vector field of interest here is displacement; hence,

$$\bar{u} = \nabla S + \nabla \times \bar{V} \tag{3.1.3}$$

$$\nabla \cdot \bar{V} = 0 \ , \tag{3.1.4}$$

where

$S = S(x_1,x_2,x_3,t)$ = scalar potential

$\bar{V} = V_1\hat{e}_1 + V_2\hat{e}_2 + V_3\hat{e}_3$ = vector potential

and $V_i = V_i(x_1,x_2,x_3,t)$, $i=1,2,3$. Substitution of equation (3.1.3) into the equation of motion (3.1.1) leads to the separated wave equations (c.f. Appendix A):

$$\nabla^2 S = \frac{1}{c_\ell^{\,2}} \frac{\partial^2 S}{\partial t^2} \tag{3.1.5}$$

$$\nabla^2 \bar{V} = \frac{1}{c_t^{\,2}} \frac{\partial^2 \bar{V}}{\partial t^2} \ . \tag{3.1.6}$$

From the above equations it can be seen that the scalar potential is associated with dilatational (or infinite sums of component plane longitudinal) waves and the vector potential with equivoluminal (transverse) waves. As such, equation (3.1.3) may then be rewritten as

$$\bar{u} = \bar{u}^D + \bar{u}^E \tag{3.1.7}$$

with

$$\bar{u}^D = \nabla S = \frac{\partial S}{\partial x_1}\,\hat{e}_1 + \frac{\partial S}{\partial x_2}\,\hat{e}_2 + \frac{\partial S}{\partial x_3}\,\hat{e}_3 = u_1^D\hat{e} + u_2^D\hat{e} + u_3^D\hat{e} \tag{3.1.8}$$

$$\bar{u}^E = \nabla \times \bar{V} = \left(\frac{\partial V_3}{\partial x_2} - \frac{\partial V_2}{\partial x_3}\right)\hat{e}_1 + \left(\frac{\partial V_1}{\partial x_3} - \frac{\partial V_3}{\partial x_1}\right)\hat{e}_2 + \left(\frac{\partial V_2}{\partial x_1} - \frac{\partial V_1}{\partial x_2}\right)\hat{e}_3 = u_1^E\hat{e}_1 + u_2^E\hat{e}_2 + u_3^E\hat{e}_3. \tag{3.1.9}$$

The superscripts D and E denote the dilatational and equivoluminal components respectively. Thus, the Helmholtz resolution mathematically uncouples the wave motion such that the dilatational and equivoluminal components can be dealt with separately. The price for this convenience is one additional equation, the zero divergence gauge condition (3.1.4). In other words, there are now four equations to solve, instead of three, for the three displacement components (u_1,u_2,u_3).

The general solutions of the separated wave equations, (3.1.5) and (3.1.6), as developed in Appendix A, may be particularized to fit the boundary conditions (3.1.2) and the Helmholtz resolution (3.1.3). Hence, the scalar and vector potentials must be of the form

$$S_N = -\cos k_1x_1 \cos k_2x_2 (A_{1N}\cos k_\ell x_3 + A_{2N}\sin k_\ell x_3)\sin\omega_N t \tag{3.1.10}$$

and

$$\begin{aligned} V_{1N} &= \cos k_1x_1 \sin k_2x_2 (B_{1N}\cos k_t x_3 + B_{2N}\sin k_t x_3)\sin\omega_N t \\ V_{2N} &= \sin k_1x_1 \cos k_2x_2 (C_{1N}\cos k_t x_3 + C_{2N}\sin k_t x_3)\sin\omega_N t \\ V_{3N} &= \sin k_1x_1 \sin k_2x_2 (D_{1N}\cos k_t x_3 + D_{2N}\sin k_t x_3)\sin\omega_N t\ . \end{aligned} \tag{3.1.11}$$

These are corrections to those presented by Kaliski [12]. The associated wave numbers are given as

$$k_\ell = \left[\frac{\omega_N^2}{c_\ell^2} - (k_1^2 + k_2^2)\right]^{1/2} \tag{3.1.12}$$

$$k_t = \left[\frac{\omega_N^2}{c_t^2} - (k_1^2 + k_2^2)\right]^{1/2} , \tag{3.1.13}$$

where $k_1 = \frac{n_1 \Pi}{L_1}$ and $k_2 = \frac{n_2 \Pi}{L_2}$ as before ($n_i = 0,1,2,\ldots;\ i=1,2$). On substituting the assumed potentials, equations (3.1.10) and (3.1.11), into equations (3.1.8) and (3.1.9), one finds the normal modes of the dilatational and equivoluminal displacement components to be

$$\begin{aligned}
u_{1N}^D &= \sin k_1x_1 \cos k_2x_2[k_1(A_{1N}\cos k_\ell x_3 + A_{2N}\sin k_\ell x_3)]\sin\omega_N t = \Phi_{1N}^D \sin\omega_N t \\
u_{2N}^D &= \cos k_1x_1 \sin k_2x_2[k_2(A_{1N}\cos k_\ell x_3 + A_{2N}\sin k_\ell x_3)]\sin\omega_N t = \Phi_{2N}^D \sin\omega_N t \\
u_{3N}^D &= \cos k_1x_1 \cos k_2x_2[k_\ell(A_{1N}\sin k_\ell x_3 - A_{2N}\cos k_\ell x_3)]\sin\omega_N t = \Phi_{3N}^D \sin\omega_N t
\end{aligned} \tag{3.1.14}$$

$$\begin{aligned}
u_{1N}^E &= \sin k_1x_1 \cos k_2x_2[(k_2D_{1N} - k_tC_{2N})\cos k_tx_3 + (k_2D_{2N} + k_tC_{1N})\sin k_tx_3]\sin\omega_N t \\
u_{2N}^E &= \cos k_1x_1 \sin k_2x_2[(k_tB_{2N} - k_1D_{1N})\cos k_tx_3 - (k_1D_{2N} + k_tB_{1N})\sin k_tx_3]\sin\omega_N t \\
u_{3N}^E &= \cos k_1x_1 \cos k_2x_2[(k_1C_{1N} - k_2B_{1N})\cos k_tx_3 + (k_1C_{2N} - k_2B_{2N})\sin k_tx_3]\sin\omega_N t .
\end{aligned} \tag{3.1.15}$$

Application of the zero divergence gauge condition (3.1.4) to the vector potential $\bar{V}$ leads to the result that

$$D_{1N} = -\frac{1}{k_t}(k_1B_{2N} + k_2C_{2N})$$

$$D_{2N} = \frac{1}{k_t}(k_1B_{1N} + k_2C_{1N}) ,$$

which allows a simplification of equation (3.1.15) to

$$u_{1N}^{E} = \sin k_1x_1 \cos k_2x_2\{-\frac{1}{k_t}[k_1k_2B_{2N} + (k_2^2+k_t^2)C_{2N}]\cos k_tx_3 + \frac{1}{k_t}[k_1k_2B_{1N} + (k_2^2+k_t^2)C_{1N}]\sin k_tx_3\}\sin\omega_N t$$

$$u_{2N}^{E} = \cos k_1x_1 \sin k_2x_2\{\frac{1}{k_t}[(k_1^2+k_t^2)B_{2N} + k_1k_2C_{2N}]\cos k_tx_3 - \frac{1}{k_t}[(k_1^2+k_t^2)B_{1N} + k_1k_2C_{1N}]\sin k_tx_3\}\sin\omega_N t \tag{3.1.16}$$

$$u_{3N}^{E} = \cos k_1x_1 \cos k_2x_2\{(k_1C_{1N} - k_2B_{1N})\cos k_tx_3 + (k_1C_{2N} - k_2B_{2N})\sin k_tx_3\}\sin\omega_N t.$$

The equivoluminal displacement components may be put into the same form as Kaliski's [12] by letting

$$A_{3N} = -\frac{1}{k_t}[k_1k_2B_{2N} + (k_2^2+k_t^2)C_{2N}]$$

$$A_{4N} = \frac{1}{k_t}[k_1k_2B_{1N} + (k_2^2+k_t^2)C_{1N}]$$

$$A_{5N} = \frac{1}{k_t}[(k_1^2+k_t^2)B_{2N} + k_1k_2C_{2N}]$$

$$A_{6N} = -\frac{1}{k_t}[(k_1^2+k_t^2)B_{1N} + k_1k_2C_{1N}],$$

and substituting these amplitudes into equation (3.1.16). This yields the results

$$u_{1N}^{E} = \sin k_1x_1 \cos k_2x_2(A_{3N}\cos k_tx_3 + A_{4N}\sin k_tx_3)\sin\omega_N t = \Phi_{1N}^{E}\sin\omega_N t$$

$$u_{2N}^{E} = \cos k_1x_1 \sin k_2x_2(A_{5N}\cos k_tx_3 + A_{6N}\sin k_tx_3)\sin\omega_N t = \Phi_{2N}^{E}\sin\omega_N t$$

$$u_{3N}^{E} = \cos k_1 x_1 \cos k_2 x_2 \{\frac{1}{k_t}[(k_1 A_{4N} + k_2 A_{6N})\cos k_t x_3$$

$$- (k_1 A_{3N} + k_2 A_{5N})\sin k_t x_3]\}\sin\omega_N t = \Phi_{2N}^{E}\sin\omega_N t. \qquad (3.1.17)$$

The dilatational and equivoluminal displacements, equations (3.1.14) and (3.1.17) may then be combined according to equation (3.1.7) to generate the normal mode displacement components obtained by Kaliski [12]:

$$u_{1N} = (\phi_{1N}^{D} + \phi_{1N}^{E})\sin\omega_N t = \phi_{1N}\sin\omega_N t$$

$$u_{2N} = (\phi_{2N}^{D} + \phi_{2N}^{E})\sin\omega_N t = \phi_{2N}\sin\omega_N t \qquad (3.1.18)$$

$$u_{3N} = (\phi_{3N}^{D} + \phi_{3N}^{E})\sin\omega_N t = \phi_{3N}\sin\omega_N t.$$

The next step is to determine the natural frequencies of the system. This is accomplished by substituting the above normal modes into the boundary conditions (3.1.2). Twelve of the eighteen boundary conditions are satisfied exactly, leaving six equations in the six unknowns $A_{iN}(i=1,2,\ldots,6)$:

$$(k_1^2 + k_2^2 + \gamma k_\ell^2)A_{1N} - (\gamma-1)k_1 A_{3N} - (\gamma-1)k_2 A_{5N} = 0$$

$$(k_1^2 + k_2^2 + \gamma k_\ell^2)\cos k_\ell L_3 A_{1N} + (k_1^2 + k_2^2 + \gamma k_\ell^2)\sin k_\ell L_3 A_{2N} - (\gamma-1)k_1 \cos k_t L_3 A_{3N}$$

$$- (\gamma-1)k_1 \sin k_t L_3 A_{4N} - (\gamma-1)k_2 \cos k_t L_3 A_{5N} - (\gamma-1)k_2 \sin k_t L_3 A_{6N} = 0$$

$$2k_1 k_\ell k_t A_{2N} - (k_1^2 - k_t^2)A_{4N} - k_1 k_2 A_{6N} = 0 \qquad (3.1.19)$$

$$- 2k_1k_\ell k_t \sin k_\ell L_3 A_{1N} + 2k_1k_\ell k_t \cos k_\ell L_3 A_{2N} + (k_1^2 - k_t^2)\sin k_t L_3 A_{3N}$$
$$- (k_1^2 - k_t^2)\cos k_t L_3 A_{4N} + k_1k_2 \sin k_t L_3 A_{5N} - k_1k_2 \cos k_t L_3 A_{6N} = 0$$

$$2k_2k_\ell k_t A_{2N} - k_1k_2A_{4N} - (k_2^2 - k_t^2)A_{6N} = 0$$

$$- 2k_2k_\ell k_t \sin k_\ell L_3 A_{1N} + 2k_2k_\ell k_t \cos k_\ell L_3 A_{2N} + k_1k_2 \sin k_t L_3 A_{3N}$$
$$- k_1k_2 \cos k_t L_3 A_{4N} + (k_2^2 - k_t^2)\sin k_t L_3 A_{5N} - (k_2^2 - k_t^2)\cos k_t L_3 A_{6N} = 0 .$$

The amplitude relations and frequency equations are determined from these six expressions. There are several appropriate combinations depending upon the values of $\sin k_t L_3$ and the wave numbers k_1 and k_2. These are summarized in Table 3.1.

The first combination includes amplitude relations (3.1.20) and the frequency equation (3.1.21); this applies when $\sin k_t L_3 = 0$ and $k_1 > 0$, $k_2 > 0$. It represents horizontally polarized (displacements in $x_1 - x_2$ plane only) shear waves and is sometimes referred to as an SH wave solution. From equations (3.1.14), (3.1.17), and (3.1.18), it can be seen that this solution contributes nothing to the u_3 displacement component and allows for no mode conversions at the boundaries.

The amplitude relations and frequency equation associated with $\sin k_t L_3 \neq 0$ and $k_1 = k_2 = 0$ are (3.1.22) and (3.1.23), respectively. The latter is derived from the fact that the only meaningful solution to equations (3.1.19) comes when $\sin k_\ell L_3 = 0$ and $A_{2N} \neq 0$. These are longitudinal waves propagating in the x_3 direction, and because they are normally incident on the stress-free surfaces, there are no mode conversions. They simply reflect back and forth between the two faces.

TABLE 3.1. Appropriate Modal Coefficients and Frequency Equations, Stress-Free/Rigid-Lubricated Boundaries

Modal Coefficients	$\sin k_t L_3 = 0$	$\sin k_t L_3 \neq 0$		
	$k_1 > 0,\ k_2 > 0$	$k_1 = k_2 = 0$	$k_1 > 0,\ k_2 = 0$	$k_1 \geq 0,\ k_2 = 0$
A_{1N}	0	0	$-\frac{P}{R}\left(\frac{k_1^2 - k_t^2}{2k_1 k_\ell k_t}\right) A_{3N}$	$-\frac{P}{R}\left(\frac{k_1^2 + k_2^2 - k_t^2}{2k_2 k_\ell k_t}\right) A_{5N}$
A_{2N}	0	*	$\frac{k_1^2 - k_t^2}{2k_1 k_\ell k_t} A_{4N}$	$\frac{k_1^2 + k_2^2 - k_t^2}{2k_2 k_\ell k_t} A_{6N}$
A_{3N}	$-\frac{k_2}{k_1} A_{5N}$	0	$-\frac{R(\cos k_\ell L_3 - \cos k_t L_3)}{P\sin k_\ell L_3 + R\sin k_t L_3} A_{4N}$	$\frac{k_1}{k_2} A_{5N}$
A_{4N}	0	0	*	$\frac{k_1}{k_2} A_{6N}$
A_{5N}	*	0	0	$-\frac{R(\cos k_\ell L_3 - \cos k_t L_3)}{P\sin k_\ell L_3 + R\sin k_t L_3} A_{6N}$
A_{6N}	0	0	0	*
	(3.1.20)	(3.1.22)	(3.1.24)	(3.1.25)
Frequency Equations	$\omega_N = c_t \alpha$ (3.1.21)	$\omega_N = c_\ell \alpha$ (3.1.23)	$(P^2+R^2)\sin k_\ell L_3 \sin k_t L_3 + 2PR(1 - \cos k_\ell L_3 \cos k_t L_3) = 0$ (3.1.26)	
	$\alpha^2 = k_1^2 + k_2^2 + k_3^2$		$P = 4(k_1^2 + k_2^2) k_\ell k_t$	$R = (k_1^2 + k_2^2 - k_t^2)^2$

* For the free vibration problem, these values are determined from the initial conditions; in the forced vibration problem, from the forcing function.

The last two sets of amplitude relations, (3.1.24) and (3.1.25), share the frequency equation (3.1.26). This equation is obtained by setting the determinant of equations (3.1.19) equal to zero and dividing the result by $\sin k_t L_3$, since $\sin k_t L_3 \neq 0$. Whereas for the completely rigid-lubricated problem the natural frequencies of each of the plane wave components $N(n_1,n_2,n_3)$ could be determined explicitly from the frequency equation (2.1.5), here they must be solved for implicitly because (3.1.26) is a transcendental equation, which allows for mode conversions at the two stress-free surfaces. These mode conversions are responsible for the increased complexity in the amplitude relations. Equations (3.1.24), (3.1.25), and (3.1.26) thus describe the motion of the mode converting longitudinal and vertically polarized shear waves. This solution is also referred to as the SV/P wave solution.

Notice that the case $\sin k_t L_3 \neq 0$ and $k_1 > 0$, $k_2 = 0$ corresponds to modes in which the shear waves propagate in $x_1 - x_3$ planes only. Consequently, there are no equivoluminal displacements in the x_2 direction, i.e., $u_{2N}^E = 0$ (ref. equation (3.1.17b)). When $\sin k_t L_3 \neq 0$ and $k_1 = 0$, $k_2 > 0$, the inverse condition exists: shear waves propagate in $x_2 - x_3$ planes only, and as a result, $u_{1N}^E = 0$.

For the free vibration problem, the amplitudes designated by the asterisks in Table 3.1 are determined from the initial conditions; in the forced vibration problem of the ensuing section, they are determined from the forcing function. The expressions for P and R and the amplitude relations A_{1N}, A_{2N}, and A_{5N} from (3.1.25) are all corrections to Kaliski's free vibration solution [12], as are the assumed scalar and vector potentials, equations (3.1.10) and (3.1.11).

Each of the normal modes defined by equations (3.1.18) represents a plane wave component traveling in a direction determined by the set $N(n_1,n_2,n_3)$, where n_1 and n_2 specify the wave numbers $k_1 = n_1\pi/L_1$ and $k_2 = n_2\pi/L_2$ and n_3 refers to the infinite set of natural frequencies. The three-dimensional free vibration displacement components are then made up of the infinity of plane wave components N traveling in all directions:

$$
\begin{aligned}
u_1(x_1,x_2,x_3) &= \sum_N u_{1N}(x_1,x_2,x_3) \\
u_2(x_1,x_2,x_3) &= \sum_N u_{2N}(x_1,x_2,x_3) \\
u_3(x_1,x_2,x_3) &= \sum_N u_{3N}(x_1,x_2,x_3) \quad ,
\end{aligned}
\tag{3.1.27}
$$

with $\sum\limits_N = \sum\limits_{n_1=0}^{\infty} \sum\limits_{n_2=0}^{\infty} \sum\limits_{n_3=0}^{\infty}$, as before.

3.2 Forced Vibration Solution

The equation of interest for the forced vibration solution is

$$c_t^2\nabla^2\bar{u} + (c_\ell^2 - c_t^2)\nabla\nabla\cdot\bar{u} + \bar{f} = \frac{\partial^2\bar{u}}{\partial t^2} \quad , \tag{3.2.1}$$

and the displacement vector is assumed to be of the form [22]

$$\bar{u} = \sum_N \bar{\phi}_N(x_1,x_2,x_3)T_N(t) \quad , \tag{3.2.2}$$

where

$$\bar{\phi}_N = \phi_{1N}\hat{e}_1 + \phi_{2N}\hat{e}_2 + \phi_{3N}\hat{e}_3 \quad .$$

and the $\phi_{iN}(i=1,2,3)$ are the modal functions defined by equations (3.1.14), (3.1.17), and (3.1.18). Note that these functions represent the spatial portion of the normal modes, and as such, they satisfy the rigid-lubricated/stress-free boundary conditions, equations (3.1.2).

Substitution of the assumed displacement (3.2.2) into the governing equation of motion (3.2.1) produces the result

$$\sum_N [c_t^2\nabla^2\bar{\phi}_N + (c_\ell^2 - c_t^2)\nabla\nabla\cdot\bar{\phi}_N]T_N + \bar{f} = \sum_N \bar{\phi}_N\ddot{T}_N \quad . \tag{3.2.3}$$

The bracketed term on the left hand side of this expression may be simplified by substituting the free vibration displacements of equations (3.1.23),

$$\bar{u} = \sum_N \bar{\phi}_N \sin\omega_N t \quad , \tag{3.2.4}$$

into the free vibration equation of motion (3.1.1); thus

$$c_t^2\nabla^2\bar{\phi}_N + (c_\ell^2 - c_t^2)\nabla\nabla\cdot\bar{\phi}_N = -\omega_N\bar{\phi}_N \quad . \tag{3.2.5}$$

Equation (3.2.5) is then substituted into equation (3.2.3) and the results rearranged:

$$\sum_N \bar{\phi}_N(\ddot{T}_N + \omega_N^2 T_N) = \bar{f}. \tag{3.2.6}$$

Taking the scalar product of both sides of this equation with $\bar{\phi}_M$, where $M(m_1,m_2)$ denotes another modal function, and integrating over the spatial domain, one gets

$$\sum_N (\ddot{T}_N + \omega_N^2 T_N) \int_V \bar{\phi}_N \cdot \bar{\phi}_M dV = \int_V \bar{f} \cdot \bar{\phi}_M dV \ . \tag{3.2.7}$$

It can be shown that the governing equations are self-adjoint; consequently, the $\bar{\phi}_N$ must be orthogonal [23]. This means that

$$\int_V \bar{\phi}_N \cdot \bar{\phi}_M dV = 0 \qquad N(n_1,n_2) \neq M(m_1,m_2), \tag{3.2.8}$$

and therefore,

$$\ddot{T}_N + \omega_N^2 T_N = W_N \tag{3.2.9}$$

with

$$W_N(t) = \frac{1}{E_N} \int_0^{L_1}\int_0^{L_2}\int_0^{L_3} \bar{f}(x_1,x_2,x_3,t) \cdot \bar{\phi}_N(x_1,x_2,x_3) dx_1 dx_2 dx_3 \tag{3.2.10}$$

$$E_N = \int_0^{L_1}\int_0^{L_2}\int_0^{L_3} \bar{\phi}_N \cdot \bar{\phi}_N dx_1 dx_2 dx_2 . \tag{3.2.11}$$

The quantity E_N represents the generalized mass per unit density.

The generalized time varying function $T_N(t)$ is found by assuming that the motion starts from rest ($T_N(0)=\dot{T}_N(0)=0$) and taking Laplace transforms:

$$\bar{T}_N(s) = \bar{V}_N(s)\bar{W}_N(s) \ . \tag{3.2.12}$$

Here

$$\bar{V}_N(s) = \frac{1}{s^2 + \omega_N^2} \tag{3.2.13}$$

and $\bar{W}_N(s)$ is the transform of equation (3.2.10). The inverse transforms of equations (3.2.12) and (3.2.13), when combined, yield the desired result

$$T_N(t) = \frac{1}{\omega_N} \int_0^t W_N(\tau)\sin\omega_N(t-\tau)d\tau \ . \tag{3.2.14}$$

The forced vibration displacements are obtained by substituting the above back into equation (3.2.2). Hence, in component form they become

$$\begin{aligned} u_1(x_1,x_2,x_3,t) &= \sum_N \phi_{1N}(x_1,x_2,x_3)T_N(t) \\ u_2(x_1,x_2,x_3,t) &= \sum_N \phi_{2N}(x_1,x_2,x_3)T_N(t) \\ u_3(x_1,x_2,x_3,t) &= \sum_N \phi_{3N}(x_1,x_2,x_3)T_N(t) \ . \end{aligned} \tag{3.2.15}$$

The time varying function T_N may be determined for any generalized body force (per unit mass) according to equations (3.2.10), (3.2.11), and (3.2.14).

3.3 Response to an Impulse

The impulsive body force assumed here is the same as that employed in Chapter II and is written as

$$\bar{f}(x_1,x_2,x_3,t) = F_o\delta(x_1 - \xi_1)\delta(x_2 - \xi_2)\delta(x_3 - \xi_3)\delta(t)\hat{e}_3 \ . \tag{3.3.1}$$

On substituting this expression into equation (3.2.10), one obtains

$$W_N(t) = \frac{F_o}{E_N}\phi_{3N}(\xi_1,\xi_2,\xi_3)\delta(t) \ , \tag{3.3.2}$$

In order to get the time varying function T_N, the above is substituted into equation (3.2.14). Integration then yields

$$T_N(t) = \frac{F_o}{E_N\omega_N}\,\phi_{3N}(\xi_1,\xi_2,\xi_3)\sin\omega_N t \ . \tag{3.3.3}$$

This is then combined with equation (3.2.15) to obtain the displacement components produced by a one-dimensional impulse of magnitude F_o applied in the x_3 direction at the point (ξ_1,ξ_2,ξ_3):

$$\begin{aligned}
u_1(x_1,x_2,x_3,t) &= \sum_N \frac{F_o}{E_N\omega_N}\,\phi_{3N}(\xi_1,\xi_2,\xi_3)\phi_{1N}(x_1,x_2,x_3)\sin\omega_N t \\
u_2(x_1,x_2,x_3,t) &= \sum_N \frac{F_o}{E_N\omega_N}\,\phi_{3N}(\xi_1,\xi_2,\xi_3)\phi_{2N}(x_1,x_2,x_3)\sin\omega_N t \\
u_3(x_1,x_2,x_3,t) &= \sum_N \frac{F_o}{E_N\omega_N}\,\phi_{3N}(\xi_1,\xi_2,\xi_3)\phi_{3N}(x_1,x_2,x_3)\sin\omega_N t .
\end{aligned} \tag{3.3.4}$$

In performing the calculations, the ω_N are determined from the characteristic equation (3.1.20) and the quantity E_N is evaluated in Appendix B.

3.4 Symmetric Boundary Conditions

The symmetric boundary conditions for the parallelepiped with two stress-free and four rigid-lubricated boundaries are

$$\begin{array}{lll}
x_1 = -\dfrac{L_1}{2}, \dfrac{L_1}{2} & u_1 = 0 & \dfrac{\partial u_2}{\partial x_1} = \dfrac{\partial u_3}{\partial x_1} = 0 \\
x_2 = -\dfrac{L_2}{2}, \dfrac{L_2}{2} & u_2 = 0 & \dfrac{\partial u_1}{\partial x_2} = \dfrac{\partial u_3}{\partial x_2} = 0 \\
x_3 = -\dfrac{L_3}{2}, \dfrac{L_3}{2} & \dfrac{\partial u_1}{\partial x_1}+\dfrac{\partial u_2}{\partial x_2}+\gamma\dfrac{\partial u_3}{\partial x_3} = 0 & \dfrac{\partial u_3}{\partial x_1}+\dfrac{\partial u_1}{\partial x_3} = \dfrac{\partial u_3}{\partial x_2}+\dfrac{\partial u_2}{\partial x_3} = 0 \ .
\end{array} \tag{3.4.1}$$

Once again, the equation of motion for the free vibration solution is equation (3.1.1). Here the scalar and vector potentials are assumed to be

$$S_N = -\cos k_1(x_1+\frac{L_1}{2})\cos k_2(x+\frac{L_2}{2})[A_{1N}\cos k_\ell(x_3+\frac{L_3}{2})+A_{2N}\sin k_\ell(x_3+\frac{L_3}{2})]\sin\omega_N t \tag{3.4.2}$$

and

$$\begin{aligned}
V_{1N} &= \cos k_1(x_1+\frac{L_1}{2})\sin k_2(x_2+\frac{L_2}{2})[B_{1N}\cos k_t(x_3+\frac{L_3}{2})+B_{2N}\sin k_t(x_3+\frac{L_3}{2})]\sin\omega_N t \\
V_{2N} &= \sin k_1(x_1+\frac{L_1}{2})\cos k_2(x_2+\frac{L_2}{2})[C_{1N}\cos k_t(x_3+\frac{L_3}{2})+C_{2N}\sin k_t(x_3+\frac{L_3}{2})]\sin\omega_N t \\
V_{3N} &= \sin k_1(x_1+\frac{L_1}{2})\sin k_2(x_2+\frac{L_2}{2})[D_{1N}\cos k_t(x_3+\frac{L_3}{2})+D_{2N}\sin k_t(x_3+\frac{L_3}{2})]\sin\omega_N t
\end{aligned} \tag{3.4.3}$$

Following the same procedure as in Section 3.1 results in the normal mode displacement components

$$\begin{aligned}
u_{1N} &= \sin k_1(x_1+\frac{L_1}{2})\cos k_2(x_2+\frac{L_2}{2})\{k_1[A_{1N}\cos k_\ell(x_3+\frac{L_3}{2})+A_{2N}\sin k_\ell(x_3+\frac{L_3}{2})] \\
&\qquad + A_{3N}\cos k_t(x_3+\frac{L_3}{2})+A_{4N}\sin k_t(x_3+\frac{L_3}{2})\}\sin\omega_N t \\
u_{2N} &= \cos k_1(x_1+\frac{L_1}{2})\sin k_2(x_2+\frac{L_2}{2})\{k_2[A_{1N}\cos k_\ell(x_3+\frac{L_3}{2})+A_{2N}\sin k_\ell(x_3+\frac{L_3}{2})] \\
&\qquad + A_{5N}\cos k_t(x_3+\frac{L_3}{2})+A_{6N}\sin k_t(x_3+\frac{L_3}{2})\}\sin\omega_N t \\
u_{3N} &= \cos k_1(x_1+\frac{L_1}{2})\cos k_2(x_2+\frac{L_2}{2})\{k_\ell[A_{1N}\sin k_\ell(x_3+\frac{L_3}{2})-A_{2N}\cos k_\ell(x_3+\frac{L_3}{2})] \\
&\qquad + \frac{1}{k_t}[(k_1A_{4N}+k_2A_{6N})\cos k_t(x_3+\frac{L_3}{2})-(k_1A_{3N}+k_2A_{5N})\sin k_t(x_3+\frac{L_3}{2})]\}\sin\omega_N t\ ,
\end{aligned} \tag{3.4.4}$$

which may then be substituted into the boundary conditions (3.4.1) to obtain the same characteristic equation and amplitude relations as before.

The results for the forced vibration case are developed in like fashion and yield similar results, again the only difference is in the arguments of the spatial sin and cos terms. Thus, $k_1(x_1+\frac{L_1}{2})$, $k_2(x_2+\frac{L_2}{2})$, $k_\ell(x_3+\frac{L_3}{2})$, and $k_t(x_3+\frac{L_3}{2})$ should be substituted for k_1x_1, k_2x_2, $k_\ell x_3$, and k_tx_3, respectively; otherwise, the results are the same.

CHAPTER IV

ELASTICALLY RESTRAINED/RIGID-LUBRICATED BOUNDARIES

4.1 Free and Forced Vibration Solutions

For the free vibration problem, the two stress-free boundaries of Chapter III are replaced by two elastically restrained boundaries, and the four rigid-lubricated boundaries remain unchanged:

$$\begin{array}{lll} x_1 = 0, L_1 & u_1 = 0 & \dfrac{\partial u_2}{\partial x_1} = \dfrac{\partial u_3}{\partial x_1} = 0 \\ x_2 = 0, L_2 & u_2 = 0 & \dfrac{\partial u_1}{\partial x_2} = \dfrac{\partial u_3}{\partial x_2} = 0 \\ x_3 = 0, L_3 & \dfrac{\partial u_1}{\partial x_1} + \dfrac{\partial u_2}{\partial x_2} + \gamma\dfrac{\partial u_3}{\partial x_3} = \pm\dfrac{e_3 u_3}{\lambda} & \dfrac{\partial u_3}{\partial x_1} + \dfrac{\partial u_1}{\partial x_3} = \dfrac{\partial u_3}{\partial x_2} + \dfrac{\partial u_2}{\partial x_3} = 0. \end{array} \tag{4.1.1}$$

Here e_3 is the elastic modulus of the upper and lower restraints and $\gamma = 1 + \frac{2\mu}{\lambda}$. As before, the free vibration equation of motion is

$$c_t^2 \nabla^2 \bar{u} + (c_\ell^2 - c_t^2)\nabla\nabla\cdot\bar{u} = \frac{\partial^2 \bar{u}}{\partial t^2}\ . \tag{4.12}$$

The development of the normal mode displacement components is the same as that in Chapter III, and as such, only the results are presented here:

$$\begin{aligned} u_{1N} &= \sin k_1x_1 \cos k_2x_2[k_1(A_{1N}\cos k_\ell x_3 + A_{2N}\sin k_\ell x_3) + A_{3N}\cos k_t x_3 + A_{4N}\sin k_t x_3]\sin\omega_N t \\ u_{2N} &= \cos k_1x_1 \sin k_2x_2[k_2(A_{1N}\cos k_\ell x_3 + A_{2N}\sin k_\ell x_3) + A_{5N}\cos k_t x_3 + A_{6N}\sin k_t x_3]\sin\omega_N t \\ u_{3N} &= \cos k_1x_1 \cos k_2x_2\{k_\ell(A_{1N}\sin k_\ell x_3 - A_{2N}\cos k_\ell x_3) \\ &\quad + \frac{1}{k_t}[(k_1A_{4N} + k_2A_{6N})\cos k_t x_3 - (k_1A_{3N} + k_2A_{5N})\sin k_t x_3]\ \sin\omega_N t \end{aligned} \tag{4.1.3}$$

These components are then substituted into boundary conditions (4.1.1) with the result that the twelve boundary conditions on the x_1 and x_2 faces are satisfied identically and the remaining six boundary conditions on the x_3 faces are satisfied when

$$\lambda(k_1^2 + k_2^2 + \gamma k_\ell^2)k_t A_{1N} + e_3 k_\ell k_t A_{2N} - 2\mu k_1 k_t A_{3N} - e_3 k_1 A_{4N} - 2\mu k_2 k_t A_{5N} - e_3 k_2 A_{6N} = 0$$

$$\begin{aligned}
&[\lambda(k_1^2 + k_2^2 + \gamma k_\ell^2)k_t \cos k_\ell L_3 + e_3 k_\ell k_t \sin k_\ell L_3]A_{1N} \\
&\qquad + [\lambda(k_1^2 + k_2^2 + \gamma k_\ell^2)k_t \sin k_\ell L_3 - e_3 k_\ell k_t \cos k_\ell L_3]A_{2N} \\
&- [2\mu k_1 k_t \cos k_t L_3 + e_3 k_1 \sin k_t L_3]A_{3N} - [2\mu k_1 k_t \sin k_t L_3 - e_3 k_1 \cos k_t L_3]A_{4N} \\
&- [2\mu k_2 k_t \cos k_t L_3 + e_3 k_2 \sin k_t L_3]A_{5N} - [2\mu k_2 k_t \sin k_t L_3 - e_3 k_2 \cos k_t L_3]A_{6N} = 0
\end{aligned}$$

$$2k_1 k_\ell k_t A_{2N} - (k_1^2 - k_t^2)A_{4N} - k_1 k_2 A_{6N} = 0 \tag{4.1.4}$$

$$\begin{aligned}
&- 2k_1 k_\ell k_t \sin k_\ell L_3 A_{1N} + 2k_1 k_\ell k_t \cos k_\ell L_3 A_{2N} + (k_1^2 - k_t^2)\sin k_t L_3 A_{3N} \\
&\qquad - (k_1^2 - k_t^2)\cos k_t L_3 A_{4N} + k_1 k_2 \sin k_t L_3 A_{5N} - k_1 k_2 \cos k_t L_3 A_{6N} = 0
\end{aligned}$$

$$2k_2 k_\ell k_t A_{2N} - k_1 k_2 A_{4N} - (k_2^2 - k_t^2)A_{6N} = 0$$

$$\begin{aligned}
&- 2k_2 k_\ell k_t \sin k_\ell L_3 A_{1N} + 2k_2 k_\ell k_t \cos k_\ell L_3 A_{2N} + k_1 k_2 \sin k_t L_3 A_{3N} - k_1 k_2 \cos k_1 L_3 A_{4N} \\
&\qquad + (k_2^2 - k_t^2)\sin k_t L_3 A_{5N} - (k_2^2 - k_t^2)\cos k_t L_3 A_{6N} = 0.
\end{aligned}$$

These expressions are valid for finite values of e_3.

As in Chapter 3, the appropriate amplitude relations and frequency equations are determined from the above equations and depend upon the values of $\sin k_t L_3$ and the wave numbers k_1 and k_2. Table 4.1 lists these combinations. Equations (4.1.5) and (4.1.6) correspond to the SH wave motion and are identical to equations (3.1.20) and (3.1.21) from Table 3.1. The longitudinal wave motion described by equations (4.1.7)

TABLE 4.1. Appropriate Modal Coefficients and Frequency Equations, Elastically Restrained/Rigid-Lubricated Boundaries ($0<e_3<\infty$)

Modal Coefficients	$\sin k_t L_3=0$	$\sin k_t L_3\neq 0$		
	$k_1>0,\ k_2>0$	$k_1=k_2=0$	$k_1>0,\ k_2=0$	$k_1\geq 0,\ k_2=0$
A_{1N}	0	$-\frac{e_3}{\delta}A_{2N}$	$-\frac{P}{R}\left(\frac{k_1^2-k_t^2}{2k_1k_\ell k_t}\right)\left[A_{3N}-\frac{e_3(k_1^2+k_t^2)k_\ell}{R}A_{4N}\right]$	$\left(\frac{k_1^2+k_2^2-k_t^2}{2k_2k_\ell k_t}\right)\left[\frac{\sin k_tL_3}{\sin k_\ell L_3}A_{5N}+\frac{(\cos k_\ell L_3-\cos k_tL_3)}{\sin k_\ell L_3}A_{6N}\right]$
A_{2N}	0	*	$\frac{k_1^2-k_t^2}{2k_1k\ k_t}A_{4N}$	$\frac{k_1^2+k_2^2-k_t^2}{2k_2k_\ell k_t}A_{6N}$
A_{3N}	$-\frac{k_2}{k_1}A_{5N}$	0	$\left[\frac{e_3(k_1^2+k_2^2)k_\ell\sin k_\ell L_3-R(\cos k_\ell L_3-\cos k_tL_3)}{P\sin k_\ell L_3+R\sin k_tL_3}\right]A_{4N}$	$\frac{k_1}{k_2}A_{5N}$
A_{4N}	0	0	*	$\frac{k_1}{k_2}A_{6N}$
A_{5N}	*	0	0	$\frac{e_3(k_1^2+k_2^2+k_t^2)k_\ell\sin k_\ell L_3-R(\cos k_\ell L_3-\cos k_tL_3)}{P\sin k_\ell L_3+R\sin k_tL_3}$
A_{6N}	0	0	0	*
	(4.1.5)	(4.1.7)	(4.1.9)	(4.1.10)
Frequency Equations	$\omega_N=c_t\alpha$ (4.1.6) $\alpha^2=k_1^2+k_2^2+k_3^2$	$e_3^2+2e_3\delta\cot k_\ell L_3-\delta^2=0$ (4.1.8) $\delta=\lambda\gamma k_\ell$	$\{e_3^2(k_1^2+k_2^2+k_t^2)^2k_\ell^2\sin k_\ell L_3\sin k_tL_3-2e_3(k_1^2+k_2^2+k_t^2)k_\ell(P\sin k_\ell L_3\cos k_tL_3+R\cos k_\ell L_3\sin k_tL_3)$ $-[(P^2+R^2)\sin k_\ell L_3\sin k_tL_3+2PR(1-\cos k_\ell L_3\cos k_tL_3)]\}\sin k_tL_3=0$ (4.1.11) $P=4(k_1^2+k_2^2)k_\ell k_t$ $\quad R=(k_1^2+k_2^2-k_t^2)^2$	

* For the free vibration problem, these values are determined from the initial conditions; in the forced vibration problem, from the forcing function.

and (4.1.8) is similar to that described by equations (3.1.22) and (3.1.23), except more complicated, in that the natural frequencies are now a function of both the elastic modulus e_3 and the wave number k_ℓ, instead of just the latter. Finally, the SV/P motion is given by equations (4.1.9), (4.1.10), and (4.1.11). Here again, the amplitudes denoted by the asterisks are determined from the intiál conditions for the free vibration problem or from the forcing function for the forced vibration problem.

A superposition of the displacement due to the wave components traveling in all directions yields the three-dimensional displacement components

$$u_1 = \sum_N u_{1N}$$
$$u_2 = \sum_N u_{2N} \tag{4.1.12}$$
$$u_3 = \sum_N u_{2N}$$

with the u_{iN} (i=1,2,3) as given by equations (4.1.3). This completes the free vibration solution for the rectangular parallelepiped with elastically restrained/rigid-lubricated boundaries. The forced vibration solution proceeds exactly as in Section 3.2, and the results are the same.

4.2 Reduction to the Previous Cases

The free and forced vibration solution for the parallelepiped with elastically restrained/rigid-lubricated boundaries is particularly interesting in that by allowing $e_3 \to \infty$, the normal restraints on the x_3 faces become rigid. Thus, all of the boundaries become rigid-lubricated as in Chapter II, and the same results should be obtained. Conversely, letting $e_3 \to \infty$, the normal stresses acting on the x_3 faces approach zero, and the stress-free solution of Chapter III should be recovered. From the above, it can be seen that the elastically restrained/rigid-lubricated solution can serve as a check on the previous solutions.

We begin by dividing the normal stress boundary conditions on the x_3 faces (4.1.1c) by e_3, and then we let $e_3 \to \infty$. The result is $u_3=0$, which means that the shear stress conditions can be written as $\partial u_1/\partial x_3=\partial u_2/\partial x_3=0$, and the completely rigid-lubricated boundary conditions (2.1.2) have been recovered. The characteristic equations, on the other hand, must be divided by e_3^2 before allowing $e_3 \to \infty$. For the transcendental equation (4.1.11), this gives

$$(k_1^2+k_2^2+k_t^2)^2\ k_\ell^2 \sin k_\ell L_3 \sin k_t L_3 \sin k_t L_3 = 0$$

but since $(k_1^2+k_2^2+k_\ell^2)^2 k_\ell^2 \neq 0$, the frequency equation becomes

$$\sin k_\ell L_3 \sin k_t L_3 \sin k_t L_3 = 0. \tag{4.2.1}$$

The implication here is that either

$$k_\ell = \frac{n\Pi}{L_3},$$

$$(k_t)_1 = \frac{n\Pi}{L_3}$$

or

$$(k_t)_2 = \frac{n\Pi}{L_3} \qquad n=0,1,2,\ldots;$$

but, $k_3 = \dfrac{n_3\Pi}{L_3}$ $(n_3=0,1,2,\ldots)$, meaning that

$$k_\ell=(k_t)_1=(k_t)_2=k_3. \tag{4.2.2}$$

However, by definition,

$$k_\ell^2 = \frac{\omega_N^2}{c_\ell^2} - (k_1^2 + k_2^2) \tag{4.2.3}$$

$$k_t^2 = \frac{\omega_N^2}{c_t^2} - (k_1^2 + k_2^2) \tag{4.2.4}$$

and

$$\alpha_N^2 = k_1^2 + k_2^2 + k_3^2 \ . \tag{4.2.5}$$

Combing equations (4.2.2) through (4.2.5), one obtains the frequency equations

$$\omega_{1N} = c_\ell \alpha_N \tag{4.2.6}$$

$$\omega_{2N} = \omega_{3N} = c_t \alpha_N \ , \tag{4.2.7}$$

which are identical to (2.1.6) and (2.1.7), as expected.

The appropriate amplitude relations are recovered by performing a division and limiting operating similar to the above on equations (4.1.4), keeping in mind the relationships given in equation (4.2.2). This procedure leads to the result that

$$A_{2N} = A_{4N} = A_{6N} = 0 \ , \tag{4.2.8}$$

which means that the displacement components of equations (4.1.3) reduce to

$$\begin{aligned} u_{1N} &= \sin k_1x_1 \cos k_2x_2 \cos k_3x_3[k_1A_{1N}\sin\omega_{\ell N}t + A_{3N}\sin\omega_{tN}t] \\ u_{2N} &= \cos k_1x_1 \sin k_2x_2 \cos k_3x_3[k_2A_{1N}\sin\omega_{\ell N}t + A_{5N}\sin\omega_{tN}t] \\ u_{3N} &= \cos k_1x_1 \cos k_2x_2 \sin k_3x_3[k_3A_{1N}\sin\omega_{\ell N}t - (\frac{k_1}{k_3}A_{3N} + \frac{k_2}{k_3}A_{5N})\sin\omega_{tN}t] . \end{aligned} \tag{4.2.9}$$

If we let $k_1A_{1N} = (A_{1N})_\ell$, $A_{3N} = (A_{1N})_t$, and $A_{5N} = (A_{2N})_t$ and add up all the normal modes, the displacements obtained are the same as those obtained in Chapter II, equations (2.1.11):

$$
\begin{aligned}
u_1 &= \sum_N \sin k_1x_1 \cos k_2x_2 \cos k_3x_3 \{(A_{1N})_\ell \sin\omega_{\ell N}t + (A_{1N})_t \sin\omega_{tN}t\} \\
u_2 &= \sum_N \cos k_1x_1 \sin k_2x_2 \cos k_3x_3 \{\frac{k_2}{k_1}(A_{1N})_\ell \sin\omega_{\ell N}t + (A_{2N})_t \sin\omega_{tN}t\} \qquad (4.2.10) \\
u_3 &= \sum_N \cos k_1x_1 \cos k_2x_2 \sin k_3x_3 \{\frac{k_3}{k_1}(A_{1N})_\ell \sin\omega_{\ell N}t - [\frac{k_1}{k_3}(A_{1N})_t + \frac{k_2}{k_3}(A_{2N})_t] \sin\omega_{tN}t\}.
\end{aligned}
$$

Therefore, allowing the elastic modulus of the boundary restraint on the two x_3 faces to become infinitely large eliminates any displacements normal to these surfaces, and the boundary conditions become completely rigid-lubricated.

Substitution of $e_3 \to 0$ into equations (4.1.1), (4.1.4), and the equations of Table 4.1 yields equations (3.1.2), (3.1.19) and the equations of Table 3.1. Hence, when the elastic modulus of the restraint on the x_3 faces approaches zero, the boundaries become stress-free, and the complete stress-free/rigid-lubricated solution of Chapter III is recovered.

CHAPTER V

RESULTS AND CONCLUSIONS

5.1 Numerical Results

Numerical results were computed for the response of a parallelepiped with stress-free/rigid-lubricated boundaries to an impulsive body force (c.f. Section 3.3 and Appendices B and C). Since the infinite series solutions of equations (3.3.4) had to be truncated and a displacement vs. time curve was the desired result, it was decided that the truncation should be performed so as to include all the resonant frequencies from DC to some cutoff frequency f_{co}. The specimen was a .0254 x .0254 x .0254 m (1 x 1 x 1 in.) aluminum (ρ=2700 kg/m^3) block. The two cutoff frequencies chosen were 1.25 MHz and 2.0 MHz.

It was found that the first eleven wave numbers had to be considered in order to include all the normal modes with natural frequencies up to 1.25 MHz. Thus, the normal mode indices $N(n_1,n_2,n_3)$ varied from $N(0,0,n_3)$ to $N(10,10,n_3)$. Associated with these indices were 480 frequencies (modes) which produced significant displacements in the x_3 direction. The 2.0 MHz cutoff frequency required the inclusion of the first seventeen wave numbers in the x_1 and x_2 directions. This resulted in 1574 contributing modes.

Figure 5.1 shows the u_3 displacement vs. time history at the position .0127, .0127, .0147 m due to a 1 N-sec Dirac delta function

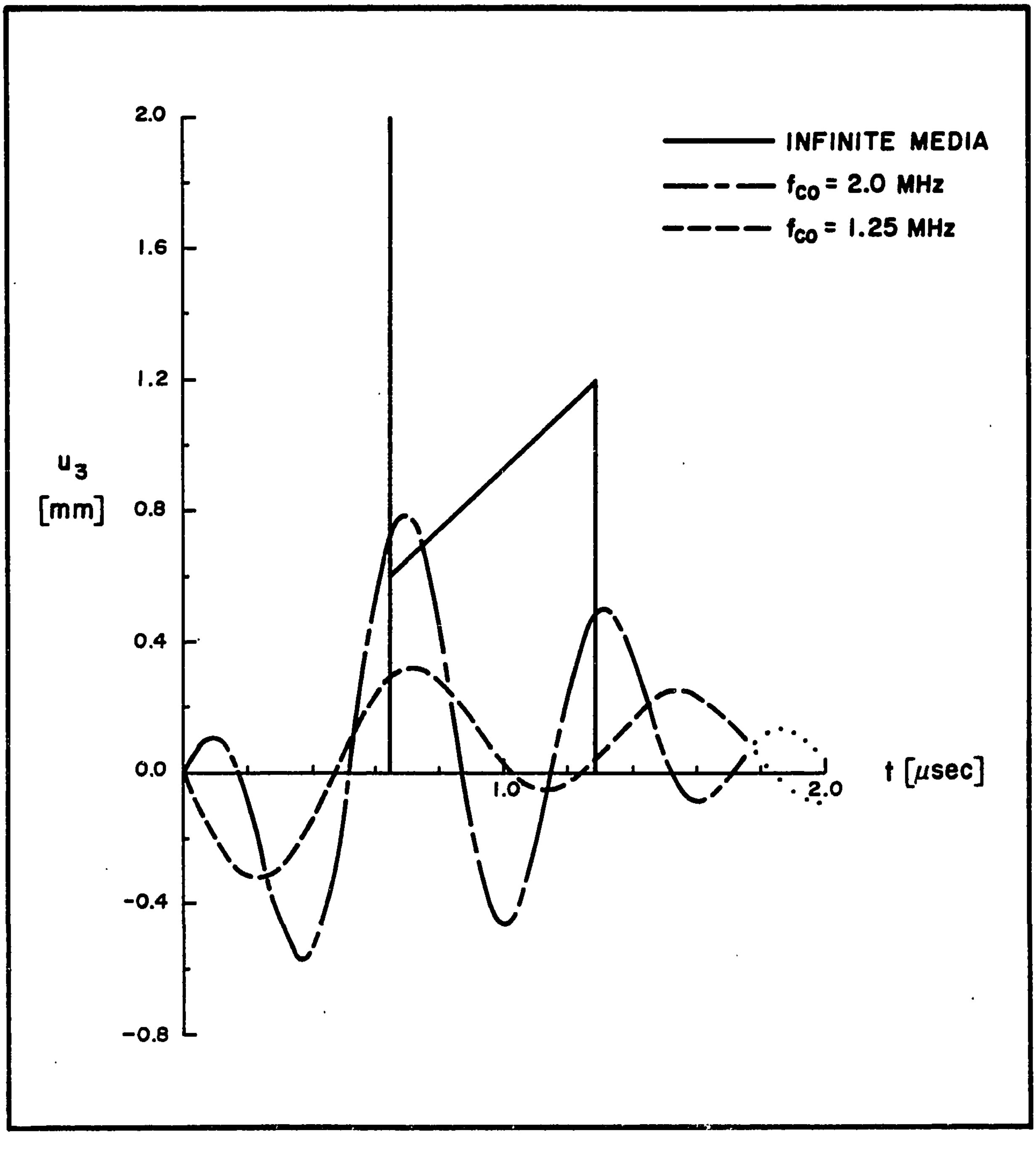

FIGURE 5.1

RESPONSE OF A RECTANGULAR PARALLELEPIPED WITH TWO STRESS-FREE AND FOUR RIGID-LUBRICATED FACES TO AN IMPULSIVE POINT LOAD — TRUNCATED NORMAL MODE SOLUTIONS — COMPARED TO THE INFINITE MEDIA RESPONSE impulse acting in the x_3 direction at .0127, .0127, .0107 m. Since the first 1.74 μsec of time history is reflection free, this part of the solution may be compared to the exact solution for an infinite body subjected to the same loading [24,25]. This solution consists of an infinite spike corresponding to the arrival of the longitudinal wave followed by a ramp which drops to zero when the shear wave arrives. The other two curves are truncated normal mode solutions to the parallelepiped problem. The dotted portions of these curves begin at 1.74 μsec, the time at which the first reflection occurs and indicate that these values cannot be compared with the infinite media solution. The f_{co} = 2.0 MHz solution is obviously a better approximation to the infinite media response than is the f_{co} = 1.25 MHz solution, but neither of these truncated solutions does a very good job. More frequencies (normal modes) are needed.

To illustrate this point, consider the frequency spectrum (Fourier transform) of a Dirac delta function impulse. It has a constant amplitude for all frequencies from DC to infinity. This means that all frequency components contribute equally to the computed results. Thus, any truncated representation will obviously contain distortions; the fewer the frequencies, the greater the distortion.

Figure 5.2 shows the inverted Fourier transform of the infinite space solution. It was calculated using a Fast Fourier Transform (FFT) routine which was truncated at 10 MHz. The difference between this solution and the infinite media response is due primarily to the Gibbs phenomenon [26] which manifests itself as a ripple in the output. This phenomenon arises whenever a spectral representation is truncated

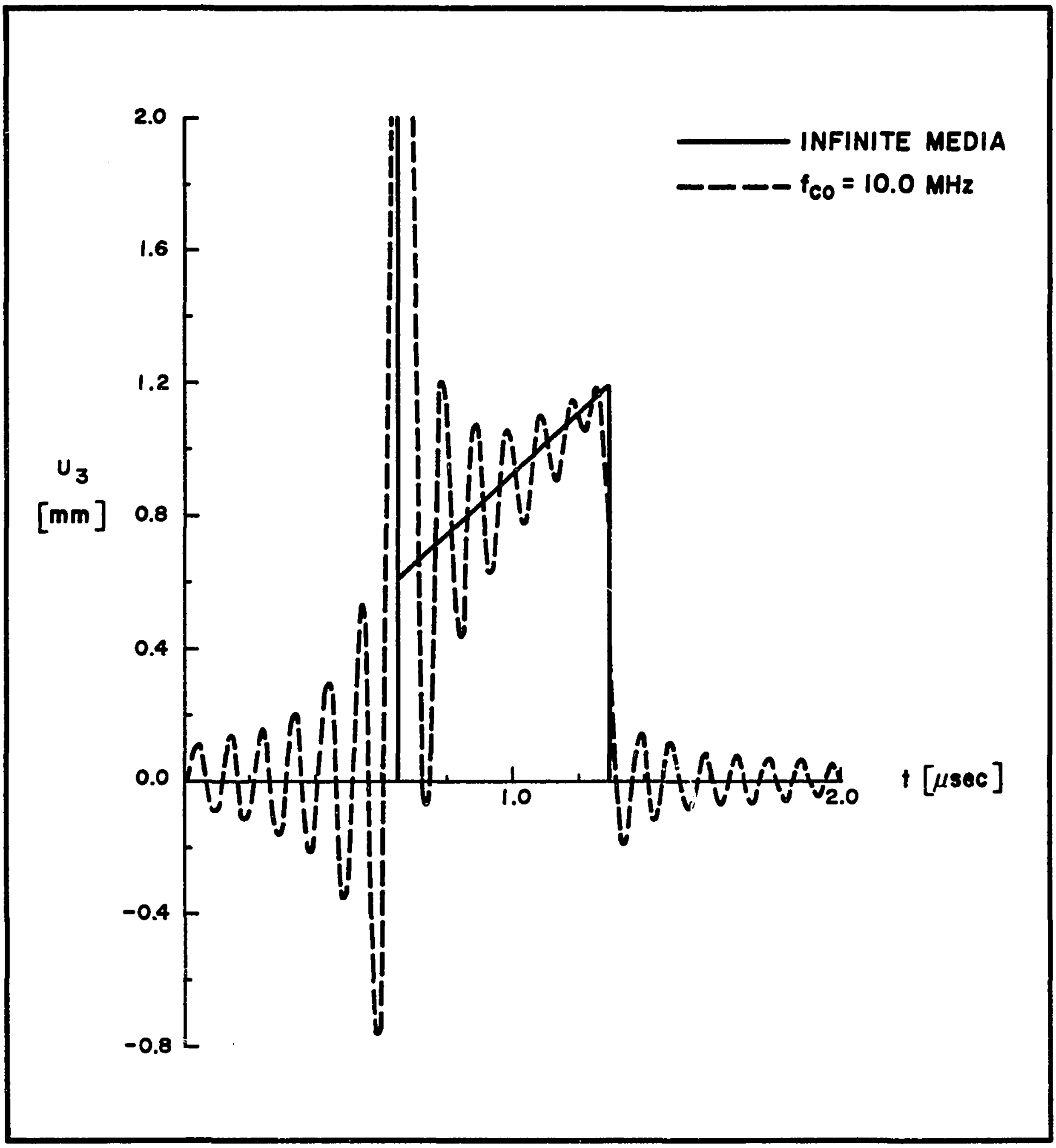

FIGURE 5.2

TRUNCATED FFT REPRESENTATION OF THE INFINITE MEDIA RESPONSE TO AN IMPULSIVE POINT LOAD abruptly. In this case, the frequencies above 10 MHz are eliminated from consideration. It can be seen from both Figures 5.1 and 5.2 that the ripple frequency is roughly equal to the cutoff frequency f_{co}.

In numerical computations such as this, where infinite series are involved, the accuracy of the final output is obviously limited by the available CPU time. This is compounded by the fact that the transcendental frequency equations (those involving mode conversions) must be solved iteratively. For the 1.25 MHz cutoff frequency, the CPU time required was 9.5 minutes; the 2.0 MHz solution required 32.25 minutes. The majority of this time was taken in solving for the natural frequencies. The larger the specimen, the more resonant frequencies there are to find. In spite of these limitations, the results seem to be headed in the right direction.

A similar analysis was performed by Hill and Egle [20] for the rectangular parallelepiped with completely rigid-lubricated boundaries. The nearly 2600 contributing modes (out of the 2×10^5 considered) were solved for explicitly from the two rather simple frequency equations, $\omega_N = c_\ell \alpha$ and $\omega_N = c_t \alpha$, (2.1.6) and (2.1.7). Because they could be solved for explicitly, rather than implicitly, far more natural frequencies were considered. The additional frequencies improved the accuracy to the point where the normal mode solution and the truncated FFT solution were virtually identical.

5.2 Conclusions and Future Directions

Presented here are exact normal mode solutions for the forced vibrational response of the rectangular parallelepiped with three sets of boundary conditions: (1) completely rigid-lubricated boundaries;

(2) two stress-free and four rigid-lubricated boundaries; and (3) two elastically restrained and four rigid-lubricated boundaries. For cases (1) and (2), the response is calculated for a Dirac delta function impulsive body force. An analytical verification for both solutions is obtained from the elastically restrained solution.

By allowing the elastic modulus of the restraint e_3 to approach infinity, the completely rigid-lubricated results are obtained. When the elastic modulus is allowed to approach zero, the stress-free/rigid-lubricated solution is recovered. The fact that these reductions can be made indicates that these solutions are probably correct, and although not as conclusive in the stress-free/rigid-lubricated case, the numerical results tend to reinforce this conclusion.

The forced vibration solution for the rectangular parallelepiped with completely rigid-lubricated boundaries might be used to model the vibration of a lubricated rubber block completely enclosed in a rigid metal container and stimulated by some internal source. In this case, the normal displacements at the surface are negligible; as a result, there are no mode conversions and hence no inhomogeneous (surface) waves. Though not a physically commonplace problem, the chief value of this solution is the insight it provides into solving the more difficult stress-free and elastically restrained cases.

The stress-free/rigid-lubricated solution, on the other hand, if programmed on a state-of-the-art scientific computer, could prove to be very useful in studying fracture mechanics, and in particular, acoustic emission source events. The elastically restrained/rigid-lubricated solution might be even better in this regard. Both solutions allow for mode conversions on two of the six faces, and both therefore take into account all three major wave types — shear, longitudinal and surface waves — which provides a better model for the typical acoustic emission application.

All three solutions were developed on the premise that acoustic emission is primarily a body force phenomenon. Although this is true, acoustic emission is more conveniently simulated on the surface of a specimen. If the equation of motion was modified to include surface forces [22] and the length dimensions L_1 and L_2 were allowed to become very large in comparison to L_3, the above solutions could be compared to the work on simulated acoustic emission in plates by Pardee and Graham [2] and Hsu, Simmons and Hardy [2]. Such a formulation would also lend itself better to experimental verification since surface forces of known location are easily generated, whereas body forces are neither easy to generate nor to locate. Including the surface forces would also allow the weight of the transducer to be modeled.

The ultimate goal is to completely bridge the gap between the experimental and analytical, such that flaw growth in structural materials can be predicted and, perhaps to some extent, controlled. The forced vibration solution for the rectangular parallelepiped with completely stress-free boundaries would be a significant step in this direction. Unfortunately, none of the three solutions could be extended by superposition to attain the completely stress-free solution. In each instance, the twelve shear stress boundary conditions were satisfied but not the six normal stress conditions. However, the three solutions presented here do represent a meaningful contribution to the field.

With the inclusion of surface forces, these solutions could very well provide a means of extending Hatano's Rayleigh wave calibration of piezoelectric transducers [26] to include all three wave types. They might also serve to verify the diffuse field calibration technique [27]. In conclusion, if these three solutions do nothing else, they will have at least broadened the author's horizons.

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APPENDICES

APPENDIX A

Separated Wave Equations

The governing equation for wave propagation in solids is Navier's equation, which may be expressed in terms of the longitudinal and transverse wave speeds as

$$c_t^2\nabla^2\bar{u} + (c_\ell^2 - c_t^2)\nabla\nabla\cdot\bar{u} + \bar{f} = \frac{\partial^2\bar{u}}{\partial t^2} \,. \tag{A1}$$

Substituting the Helmholtz resolutions of displacement

$$\bar{u} = \nabla S + \nabla \times \bar{V} \tag{A2}$$

$$\nabla\cdot\bar{V} = 0 \tag{A3}$$

and body force

$$\bar{f} = \nabla f + \nabla \times \bar{F} \tag{A4}$$

$$\nabla\cdot\bar{F} = 0 \tag{A5}$$

into the equations of motion (A1) gives

$$c_t^2\nabla^2(\nabla S + \nabla \times \bar{V}) + (c_\ell^2 - c_t^2)\nabla\nabla\cdot(\nabla S + \nabla \times \bar{V}) + (\nabla f + \nabla \times \bar{F}) = \frac{\partial^2}{\partial t^2}(\nabla S + \nabla \times \bar{V}) \tag{A6}$$

But since

$$\nabla^2(\nabla S) = \nabla(\nabla^2 S)$$

$$\nabla\cdot\nabla S = \nabla^2 S$$

and

$$\nabla^2(\nabla \times \bar{V}) = \nabla \times (\nabla^2\bar{V})$$

$$\nabla\cdot\nabla \times \bar{V} = 0 \,,$$

equation (A6) may be rewritten as

$$\nabla(c_\ell^2\nabla^2 S + f - \frac{\partial^2 S}{\partial t^2}) + \nabla \times (c_t^2\nabla^2\bar{V} + \bar{F} - \frac{\partial^2 \bar{V}}{\partial t^2}) = 0 \tag{A7}$$

This equation is satisfied if each of the terms in parenthesis vanishes. Hence, the three original equations of motion (A1), each of which included both longitudinal and transverse waves, are separated into the four independent equations

$$c_\ell^2\nabla^2 S + f = \frac{\partial^2 S}{\partial t^2} \tag{A8}$$

$$c_t^2\nabla^2\bar{V} + \bar{F} = \frac{\partial^2 \bar{V}}{\partial t^2} . \tag{A9}$$

Equation (A8) defines the longitudinal wave motion and equation (A9) the transverse wave motion. These are the separated wave equations. Conditions (A3) and (A5) allow a unique determination of the three components of $\bar{u}$ from the four components of S and $\bar{V}$ and the four components of f and $\bar{F}$.

Free Vibration Case

For the free vibration case, the body force terms vanish and the separated wave equations may be rearranged as

$$\nabla^2 S = \frac{1}{c_\ell^2}\frac{\partial^2 S}{\partial t^2} \tag{A10}$$

$$\nabla^2 \bar{V} = \frac{1}{c_t^2}\frac{\partial^2 \bar{V}}{\partial t^2} \tag{A11}$$

Both wave equations may be solved by separation of variables. The longitudinal wave equation (A10) may be solved by assuming

$$S(x_1,x_2,x_3,t) = W(x_1,x_2,x_3)T(t); \tag{A12}$$

substitution of this expression into equation (A10) yields

$$\frac{\nabla^2 W}{W} = \frac{T''}{c_\ell^2 T} = -\alpha_\ell^2 \tag{A13}$$

from which

$$\nabla^2 W + \alpha_\ell^2 W = 0 \tag{A14}$$

and

$$T'' + c_\ell^2 \alpha_\ell^2 T = 0. \tag{A15}$$

Equation (A14) is known as the Helmholtz equation. Its solution is obtained by substituting into it

$$W = X_1(x_1) X_2(x_2) X_3(x_3) \tag{A16}$$

with the result

$$\frac{X_1''}{X_1} + \frac{X_2''}{X_2} + \frac{X_3''}{X_3} = -\alpha_\ell^2. \tag{A17}$$

Letting

$$\frac{X_1''}{X_1} = -k_1^2 \tag{A18}$$

$$\frac{X_2''}{X_2} = -k_2^2 \tag{A19}$$

gives the third equation

$$\frac{X_3''}{X_3} = -[\alpha_\ell^2 - (k_1^2 + k_2^2)] = -k_\ell^2. \tag{A20}$$

The frequencies may be defined as $\omega_\ell = c_\ell \alpha_\ell$ and $\omega_t = c_t \alpha_t$; hence, the solutions to equations (A15), (A18), (A19), and (A20) are

$$T(t) = A_1 \cos\omega_\ell t + A_2 \sin\omega_\ell t \tag{A21}$$

and

$$X_1(x_1) = B_1 \cos k_1 x_1 + B_2 \sin k_1 x_1 \tag{A22}$$

$$X_2(x_2) = B_3 \cos k_2 x_2 + B_4 \sin k_2 x_2 \tag{A23}$$

$$X_3(x_3) = B_5 \cos k_3 x_3 + B_6 \sin k_3 x_3\ , \tag{A24}$$

which, along with the initial condition $T(0)=0$, may be combined according to equations (A12) and (A16) to give

$$S = (C_1 \cos k_1 x_1 + C_2 \sin k_2 x_2)(C_3 \cos k_2 x_2 + C_4 \sin k_2 x_2)(C_5 \cos k_\ell x_3 + C_6 \sin k_\ell x_3) \sin \omega_\ell t, \tag{A25}$$

the general solution for the free vibration scalar potential. The vector potential components are determined analogously:

$$V_1 = (D_1 \cos k_1 x_1 + D_2 \sin k_1 x_1)(D_3 \cos k_2 x_2 + D_4 \sin k_2 x_2)(D_5 \cos k_t x_3 + D_6 \sin k_t x_3) \sin \omega_t t$$
$$V_2 = (E_1 \cos k_1 x_1 + E_2 \sin k_1 x_1)(E_3 \cos k_2 x_2 + E_4 \sin k_2 x_2)(E_5 \cos k_t x_3 + E_6 \sin k_t x_3) \sin \omega_t t$$
$$V_3 = (F_1 \cos k_1 x_1 + F_2 \sin k_1 x_1)(F_3 \cos k_2 x_2 + F_4 \sin k_2 x_2)(F_5 \cos k_t x_3 + F_6 \sin k_t x_3) \sin \omega_t t. \tag{A26}$$

In view of the frequency definitions above, the longitudinal and transverse wave numbers may be written as

$$k_\ell^2 = \frac{\omega_\ell^2}{c_\ell^2} - (k_1^2 + k_2^2) \tag{A27}$$

$$k_t^2 = \frac{\omega_t^2}{c_t^2} - (k_1^2 + k_2^2) \ . \tag{A28}$$

APPENDIX B

Calculating the Generalized Mass Term

The generalized mass term, E_N, given by equation (3.2.11), is expanded here for computational use:

$$E_N = \int_0^{L_1}\int_0^{L_2}\int_0^{L_3} \bar{\phi}_N \cdot \bar{\phi}_N dx_1 dx_2 dx_3; \tag{B1}$$

but $\bar{\phi}_N \cdot \bar{\phi}_N = \phi_{1N}^2 + \phi_{2N}^2 + \phi_{3N}^2$. Consequently, equation (B1) may be rewritten as

$$E_N = E_{1N} + E_{2N} + E_{3N} , \tag{B2}$$

where

$$E_{iN} = \int_0^{L_1}\int_0^{L_2}\int_0^{L_3} \phi_{iN}^2 dx_1 dx_2 dx_3 , \qquad i=1,2,3. \tag{B3}$$

The ϕ_{iN} (i=1,2,3) are the modal functions defined by equations (3.1.14), (3.1.17), and (3.1.18).

Substituting the modal functions into equations (B3) and performing the indicated integrations results in the following expressions:

$$\begin{aligned} E_{1N} = \frac{\eta_1 L_1 L_2}{4} \{ & k_1^2 (A_{1N}^2 \Delta_1 + 2A_{1N}A_{2N}\Delta_2 + A_{2N}^2 \Delta_3) \\ & + 2k_1 [A_{1N}(A_{3N}\Delta_4 + A_{4N}\Delta_5) + A_{2N}(A_{3N}\Delta_6 + A_{4N}\Delta_7)] \\ & + (A_{3N}^2 \Delta_8 + 2A_{3N}A_{4N}\Delta_9 + A_{4N}^2 \Delta_{10}) \} \end{aligned} \tag{B4}$$

$$E_{2N} = \frac{\eta_2 L_1 L_2}{4} \{k_2^2(A_{1N}^2\Delta_1 + 2A_{1N}A_{2N}\Delta_2 + A_{2N}^2\Delta_3)$$
$$+ 2k_2[A_{1N}(A_{5N}\Delta_4 + A_{6N}\Delta_5) + A_{2N}(A_{5N}\Delta_6 + A_{6N}\Delta_7)]$$
$$+ (A_{5N}^2\Delta_8 + 2A_{5N}A_{6N}\Delta_9 + A_{6N}^2\Delta_{10})\} \qquad (B5)$$

$$E_{3N} = \frac{\eta_3 L_1 L_2}{4} \{k_\ell^2(A_{1N}^2\Delta_3 + 2A_{1N}A_{2N}\Delta_2 + A_{2N}^2\Delta_1)$$
$$+ \frac{k_\ell}{k_t}[A_{1N}(k_1A_{4N} + k_2A_{6N})\Delta_6 - A_{1N}(k_1A_{3N} + k_2A_{5N})\Delta_7$$
$$- A_{2N}(k_1A_{4N} + k_2A_{6N})\Delta_4 + A_{2N}(k_1A_{3N} + k_2A_{5N})\Delta_5]$$
$$+ \frac{1}{k_t^2}[(k_1A_{4N} + k_2A_{6N})^2\Delta_8 - 2(k_1A_{4N} + k_2A_{6N})(k_1A_{3N} + k_2A_{5N})\Delta_9$$
$$+ (k_1A_{3N} + k_2A_{5N})^2\Delta_{10}]\}. \qquad (B6)$$

Here,

$$\eta_1 = (1 - \delta_{k_1 0})(1 + \delta_{k_2 0}) \qquad (B7)$$

$$\eta_2 = (1 + \delta_{k_1 0})(1 - \delta_{k_2 0}) \qquad (B8)$$

$$\eta_3 = (1 + \delta_{k_1 0})(1 + \delta_{k_2 0}) \quad , \qquad (B9)$$

and

$$\Delta_1 = \frac{L_3}{2} + \frac{\sin 2k_\ell L_3}{4k_\ell} \qquad (B10)$$

$$\Delta_2 = \frac{\sin^2 k_\ell L_3}{2k_\ell} \qquad (B11)$$

$$\Delta_3 = \frac{L_3}{2} - \frac{\sin 2k_\ell L_3}{4k_\ell} \qquad (B12)$$

$$\Delta_4 = \frac{\sin(k_\ell - k_t)L_3}{2(k_\ell - k_t)} \qquad (B13)$$

$$\Delta_5 = \frac{1-\cos(k_\ell + k_t)L_3}{2(k_\ell + k_t)} - \frac{1-\cos(k_\ell - k_t)L_3}{2(k_\ell - k_t)} \tag{B14}$$

$$\Delta_6 = \frac{1-\cos(k_\ell - k_t)L_3}{2(k_\ell - k_t)} + \frac{1-\cos(k_\ell + k_t)L_3}{2(k_\ell + k_t)} \tag{B15}$$

$$\Delta_7 = \frac{\sin(k_\ell - k_t)L_3}{2(k_\ell - k_t)} - \frac{\sin(k_\ell + k_t)L_3}{2(k_\ell + k_t)} \tag{B16}$$

$$\Delta_8 = \frac{L_3}{2} + \frac{\sin k_t L_3}{4k_t} \tag{B17}$$

$$\Delta_9 = \frac{\sin^2 k_t L_3}{2k_t} \tag{B18}$$

$$\Delta_{10} = \frac{L_3}{2} - \frac{\sin 2k_t L_3}{4k_t} \quad . \tag{B19}$$

Finally, all of the above may be combined according to equation (B2) to obtain E_N.

APPENDIX C

A COMPUTER PROGRAM FOR CALCULATING THE X_3-AXIS DISPLACEMENT RESPONSE DUE TO AN IMPULSIVE BODY FORCE

```
          RRRRRRRRRRR   EEEEEEEEEEEE   SSSSSSSSSS   PPPPPPPPPPP   OOOOOOOOOOO  NN        NN   SSSSSSSSSS   EEEEEEEEEEEE
         RRRRRRRRRRRR  EEEEEEEEEEEE  SSSSSSSSSSSS  PPPPPPPPPPPP  OOOOOOOOOOOO  NNN       NN  SSSSSSSSSSSS  EEEEEEEEEEEE
        RR        RR  EE            SS        SS  PP        PP  OO        OO  NNNN      NN  SS        SS  EE
       RR        RR  EE            SS            PP        PP  OO        OO  NN NN     NN  SS            EE
      RR        RR  EE            SSS           PP        PP  OO        OO  NN  NN    NN  SSS           EE
     RRRRRRRRRRRR  EEEEEEEE       SSSSSSSSS    PPPPPPPPPPPP  OO        OO  NN   NN   NN   SSSSSSSSS    EEEEEEEE
    RRRRRRRRRRR   EEEEEEEE        SSSSSSSSS   PPPPPPPPPPP   OO        OO  NN    NN  NN    SSSSSSSSS   EEEEEEEE
   RR    RR      EE                    SSS  PP            OO        OO  NN     NN NN          SSS  EE
  RR     RR     EE                     SS  PP            OO        OO  NN      NNNN           SS  EE
 RR      RR    EE            SS        SS  PP            OO        OO  NN       NNN  SS       SS  EE
RR       RR   EEEEEEEEEEEE  SSSSSSSSSSSS  PP            OOOOOOOOOOOO  NN        NN  SSSSSSSSSSSS  EEEEEEEEEEEE
RR       RR  EEEEEEEEEEEE   SSSSSSSSSS   PP            OOOOOOOOOOOO  NN         N   SSSSSSSSSS   EEEEEEEEEEEE

   JJJJJJJJJJ      11       2222222222   777777777777                          00000000      00000000
   JJJJJJJJJJ     111      222222222222  77777777777                          0000000000    0000000000
           JJ    1111      22        22  77        77                         00     0000  00     0000
           JJ      11                22           77                          00    00 00  00    00 00
           JJ      11                22          77                           00   00  00  00   00  00
           JJ      11               22          77                            00  00   00  00  00   00
           JJ      11              22           77                            00 00    00  00 00    00
           JJ      11             22            77                            0000     00  0000     00
   JJ      JJ      11            22             77                            000      00  000      00
   JJ      JJ      11           22              77                            0000     00  0000     00
   JJJJJJJJJ    1111111111   222222222222       77                            0000000000    0000000000
    JJJJJJ      1111111111   222222222222       77                             00000000      00000000

PDE=RM00.DEST=RM00  START PRT  20.38.41  PRINTER1  80.230  HILL HLD AT MERRICK        (17 AUG 80)  UNIVERSITY OF OKLAHOMA
PDE=RM00.DEST=RM00  START PRT  20.38.41  PRINTER1  80.230  HILL HLD AT MERRICK        (17 AUG 80)  UNIVERSITY OF OKLAHOMA
PDE=RM00.DEST=RM00  START PRT  20.38.41  PRINTER1  80.230  HILL HLD AT MERRICK        (17 AUG 80)  UNIVERSITY OF OKLAHOMA
PDE=RM00.DEST=RM00  START PRT  20.38.41  PRINTER1  80.230  HILL HLD AT MERRICK        (17 AUG 80)  UNIVERSITY OF OKLAHOMA
PDE=RM00.DEST=RM00  START PRT  20.38.41  PRINTER1  80.230  HILL HLD AT MERRICK        (17 AUG 80)  UNIVERSITY OF OKLAHOMA
PDE=RM00.DEST=RM00  START PRT  20.38.41  PRINTER1  80.230  HILL HLD AT MERRICK        (17 AUG 80)  UNIVERSITY OF OKLAHOMA

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```
              COMMON XL1,XL2,XL3,XC1,XC2,XC3,CL,WND,IZERO,CT
              COMMON N1,N2,XK1,XK2,X1,X2,X3,ASYMP
              DIMENSION U3(200),U3S(200),FN(15000),D3N(15000)
              COMPLEX A,COEFF,DIFL,DIFT,XKL,XKT,RS,PST,RST,T1,T2,ZER
        C
        C     LONGITUDINAL AND TRANSVERSE WAVE SPEEDS
        C
              CL=6150.
              CT=3190.
        C
        C     SPECIMEN DIMENSIONS
        C
              XL1=.0254
              XL2=.0254
              XL3=.0254
        C
        C     POINT OF APPLICATION OF THE IMPULSIVE LOAD
        C
              XC1=XL1/2.
              XC2=XL2/2.
              XC3=.0107
        C
        C     POINT AT WHICH DISPLACEMENTS ARE SENSED
        C
              X1=XC1
              X2=XC2
              X3=.0147
        C
        C     LOWER AND UPPER FREQUENCY BOUNDS
        C
              PI=3.1415926536
              WNL=(1.0E4)*2.*PI
              WNU=(1.25E6)*2.*PI
        C
        C     P AND SV WAVE CHARACTERISTIC FREQUENCIES AND U3 MODAL DISPLACEMENT
        C     COEFFICIENTS
        C
              N=1
              ZERO=0.
              IZERO=0
              DELF=250.*PI
              DMIN=1.0E-3
              A=CMPLX(0.,1.)
              ASYMP=18.
              DO 16 J=1,13,2
              N2=J-1
              XN2=N2
              XK2=XN2*PI/XL2
              DO 15 I=1,13,2
              K=1
              L=1
              WN=WNL
              N1=I-1
              XN1=N1
              XK1=XN1*PI/XL1
              WND=XK1*XK1+XK2*XK2
              IF(WND.GT.ZERO) GO TO 2
        39    XN3=K
```

```
            WFN=CL*XN3*PI/XL3
            IF(WFN.GT.WNU) GO TO 15
            GO TO 14
      2     WFL=WN*WN/(CL*CL)
            WFT=WN*WN/(CT*CT)
            DIFL=WFL-WND
            CIFT=WFT-WND
            XKL=CSQRT(DIFL)
            XKT=CSQRT(DIFT)
            RS=WND-DIFT
            PST=-4.*WND*XKL*XKT
            RST=-RS*RS
            T1=XKL*XL3
            T2=XKT*XL3
            RT1=CABS(T1)
            RT2=CABS(T2)
            COEFF=.5*((PST/RST)+(RST/PST))
            IF(WND.GT.WFL.AND.WND.LT.WFT.AND.RT1.GT.ASYMP) GO TO 40
            IF(WND.GT.WFT.AND.RT2.GT.ASYMP) GO TO 41
            ZER=COEFF*CSIN(T1)*CSIN(T2)+1.-CCOS(T1)*CCOS(T2)
            GO TO 42
     40     ZER=A*COEFF*SIN(RT2)-COS(RT2)
            GO TO 42
     41     ZER=PST+RST
     42     ZR=REAL(ZER)
            ZI=AIMAG(ZER)
            IF(ZR.EQ.ZERO.AND.ZI.EQ.ZERO) GO TO 7
            IF(ZI) 3,20,4
      3     KSIGN=1
            GO TO 5
     20     KSIGN=2
            GO TO 5
      4     KSIGN=3
      5     IF(ZF) 6,21,8
      6     ISIGN=1
            IF(L-1) 11,11,9
      7     WFN=WN
            WN=WN+DELF
            IF(WN.LT.WNU) GO TO 2
            GO TO 15
     21     ISIGN=2
            IF(L-1) 11,11,9
      8     ISIGN=3
            IF(L-1) 11,11,9
      9     IF(JSIGN-ISIGN) 12,10,12
     10     IF(LSIGN-KSIGN) 13,11,13
     11     OLDWN=WN
            OLDZI=ZI
            OLDZR=ZR
            WN=WN+DELF
            IF(WN.GT.WNU) GO TO 15
            LSIGN=KSIGN
            JSIGN=ISIGN
            L=L+1
            GO TO 2
     12     WFN=(ZR*OLDWN-OLDZR*WN)/(ZR-CLDZR)
            CALL FREQ(OLCWN,WN,WFN)
            GO TO 14
```

```
 13     WFN=(ZI*OLDWN-OLDZI*WN)/(ZI-OLDZI)
        CALL FREQ(OLDWN,WN,WFN)
 14     CALL MDC(WFN,DR)
        ADR=ABS(DR)
        IF(ADR.LT.DMIN) GO TO 58
        WRITE(6,*) I,J,K,WFN,DR,N
        FN(N)=WFN
        D3N(N)=DR
        N=N+1
 58     K=K+1
        IF(WND.EQ.ZERO) GO TO 39
        IF(WN.LT.WNU) GO TO 11
 15     CONTINUE
 16     CONTINUE
        NF=N-1
        PRINT 1
 1      FORMAT(1H1)
        WRITE(6,*) NF
        PRINT 1
        DO 18 I=1,NF
 18     WRITE(6,63) FN(I),D3N(I)
 63     FORMAT(11X,2E20.3)
        PRINT 1
 C
 C      SUMMING THE MODAL DISPLACEMENTS TO DETERMINE THE U3 DISPLACEMENTS
 C      AS A FUNCTION OF TIME
 C
        T=2.0E-8
        FO=3.7037E-4
        DO 61 M=1,200
        U3N=0.
        U3NS=0.
        DO 60 K=1,NF
        DFN=FN(K)
        ARG=DFN*T
        DISP=D3N(K)
        PHI=DISP*SIN(ARG)/DFN
        PHIS=DISP*(1.-COS(ARG))/(DFN*DFN)
        U3N=U3N+FO*PHI
        U3NS=U3NS+FO*PHIS
 60     CONTINUE
        U3(M)=U3N
        U3S(M)=U3NS
 61     T=T+2.0E-8
        PRINT 1
        WRITE(6,62) (U3(M),M=1,200)
 62     FORMAT(11X,5E20.3)
        PRINT 1
        WRITE(6,62) (U3S(M),M=1,200)
        PRINT 1
        STOP
        END
```

```
      SUBROUTINE FREQ(X1.X2.WFN)
      COMMON CL.CT.WND.ASYMP
      COMPLEX DIFL.DIFT.XKL.XKT.RS.PST.RST.T1.T2.COEFF.ZC.A
      A=CMPLX(0..1.)
      C=WFN
      DO 2 I=1.5
      WFL=C*C/(CL*CL)
      WFT=C*C/(CT*CT)
      DIFL=WFL-WND
      DIFT=WFT-WND
      XKL=CSQRT(DIFL)
      XKT=CSQRT(DIFT)
      RS=WND-DIFT
      PST=-4.*WND*XKL*XKT
      RST=-RS*RS
      T1=XKL*XL3
      T2=XKT*XL3
      RT1=CABS(T1)
      RT2=CABS(T2)
      COEFF=.5*((PST/RST)+(RST/PST))
      IF(WND.GT.WFL.AND.WND.LT.WFT.AND.RT1.GT.ASYMP) GO TO 40
      IF(WND.GT.WFT.AND.RT2.GT.ASYMP) GO TO 41
      ZC=COEFF*CSIN(T1)*CSIN(T2)+1.-CCOS(T1)*CCOS(T2)
      GO TO 42
   40 ZC=A*COEFF*SIN(RT2)-COS(RT2)
      GO TO 42
   41 ZC=PST+RST
   42 ZR=REAL(ZC)
      ZI=AIMAG(ZC)
      IF(ZR) 3.4.5
    4 IF(ZI) 3.6.5
    3 X1=C
      GO TO 2
    5 X2=C
    2 C=(X1+X2)/2.
    6 WFN=C
      RETURN
      END
```

```
      SUBROUTINE MDC(WFN.DR)
      COMMON XL1.XL2.XL3.XC1.XC2.XC3.CL.WND.IZERO.CT
      COMMON N1.N2.XK1.XK2.X1.X2.X3
      COMPLEX A1.A2.A3.A4.A5.D.DIFL.DIFT.XKL.XKT.RS.PST.RST.T1.T2.DK.SK
      COMPLEX DKL.SKL.TT1.TT2.D1.C2.D3.D4.D5.D6.D7.D8.D9.D10.C1.C2.C3.C4
      COMPLEX AE1.AE2.AE3.BE1.BE2.BE3.Q1.Q2.CE1.CE2.CE3.E1N.E2N.E3N.EN
      COMPLEX P3NX.P3NXC.SX3.SX4.TX3.TX4
      COMPLEX CS1.CS2.CC1.CC2
C
C     DETERMINATION OF THE MODAL COEFFICIENTS
C
      DIFL=(WFN*WFN/(CL*CL))-WND
      DIFT=(WFN*WFN/(CT*CT))-WND
      XKL=CSQRT(DIFL)
      XKT=CSQRT(DIFT)
      RS=WND-DIFT
      PST=-4.*WND*XKL*XKT
      RST=-RS*RS
      T1=XKL*XL3
      T2=XKT*XL3
      CS1=CSIN(T1)
      CS2=CSIN(T2)
      CC1=CCOS(T1)
      CC2=CCOS(T2)
   23 IF(N1.GT.IZERO.AND.N2.GT.IZERO) GO TO 30
      IF(N1.EQ.IZERO.AND.N2.GT.IZERO) GO TO 31
      IF(N1.GT.IZERO.AND.N2.EQ.IZERO) GO TO 32
      ETA1=0.
      ETA2=0.
      ETA3=4.
      A6=0.
      A5=CMPLX(0..0.)
      A4=CMPLX(0..0.)
      A3=CMPLX(0..0.)
      A2=CMPLX(1..0.)
      A1=CMPLX(0..0.)
      GO TO 34
   30 ETA1=1.
      ETA2=1.
      ETA3=1.
      GO TO 33
   31 ETA1=0.
      ETA2=2.
      ETA3=2.
      GO TO 33
   32 ETA1=2.
      ETA2=0.
      ETA3=2.
      A6=0.
      A5=CMPLX(0..0.)
      A4=CMPLX(1..0.)
      A3=-RST*(CC1-CC2)/(PST*CS1+RST*CS2)
      A2=(XK1*XK1-DIFT)/(2.*XK1*XKL*XKT)
      A1=-2.*XK1*A3/(XK1*XK1-DIFT)
      GO TO 34
   33 A6=1.
      A5=-RST*(CC1-CC2)/(PST*CS1+RST*CS2)
      A4=XK1/XK2
```

```
0056            A3=A4*A5
0057            A2=RS/(2.*XK2*XKL*XKT)
0058            A1=-A2*A5*PST/RST
          C
          C     CALCULATION OF THE GENERALIZED MASS EN.
          C
0059      34    DK=XKL-XKT
0060            SK=XKL+XKT
0061            DKL=T1-T2
0062            SKL=T1+T2
0063            D1=(XL3/2.)+CS1*(CC1/(2.*XKL))
0064            D2=CS1*(CS1/(2.*XKL))
0065            D3=(XL3/2.)-CS1*(CC1/(2.*XKL))
0066            C1=CSIN(DKL)/(2.*DK)
0067            C2=CSIN(SKL)/(2.*SK)
0068            C3=(1.-CCOS(SKL))/(2.*SK)
0069            C4=(1.-CCOS(DKL))/(2.*DK)
0070            D4=C1+C2
0071            D5=C3-C4
0072            D6=C3+C4
0073            D7=C1-C2
0074            D8=(XL3/2.)+CS2*(CC2/(2.*XKT))
0075            D9=CS2*(CS2/(2.*XKT))
0076            D10=(XL3/2.)-CS2*(CC2/(2.*XKT))
0077            AE1=A1*A1*D1+2.*A1*A2*D2+A2*A2*D3
0078            AE2=A1*(A3*D4+A4*D5)+A2*(A3*D6+A4*D7)
0079            AE3=A3*A3*D8+2.*A3*A4*D9+A4*A4*D10
0080            BE1=AE1
0081            BE2=A1*(A5*D4+A6*D5)+A2*(A5*D6+A6*D7)
0082            BE3=A5*A5*D8+2.*A5*A6*D9+A6*A6*D10
0083            Q1=XK1*A4+XK2*A6
0084            Q2=XK1*A3+XK2*A5
0085            CE1=A1*A1*D3+2.*A1*A2*D2+A2*A2*D1
0086            CE2=Q1*(A1*D6-A2*D4)+Q2*(A2*D5-A1*D7)
0087            CE3=Q1*Q1*D8-2.*Q1*Q2*D9+Q2*Q2*D10
0088            E1N=XK1*XK1*AE1+2.*XK1*AE2+AE3
0089            E2N=XK2*XK2*BE1+2.*XK2*BE2+BE3
0090            E3N=DIFL*CE1+(XKL*CE2/XKT)+(CE3/DIFT)
0091            EN=(XL1*XL2/4.)*(ETA1*E1N+ETA2*E2N+ETA3*E3N)
          C
          C     DETERMINATION OF THE U3 MODAL DISPLACEMENT COEFFICIENTS
          C
0092            SX1=XK1*X1
0093            SX2=XK2*X2
0094            SX3=XKL*X3
0095            SX4=XKT*X3
0096            P3NX=COS(SX1)*COS(SX2)*(XKL*(A1*CSIN(SX3)-A2*CCOS(SX3))
               1+(1./XKT)*(Q1*CCOS(SX4)-Q2*CSIN(SX4)))
0097            TX1=XK1*XC1
0098            TX2=XK2*XC2
0099            TX3=XKL*XC3
0100            TX4=XKT*XC3
0101            P3NXC=COS(TX1)*COS(TX2)*(XKL*(A1*CSIN(TX3)-A2*CCOS(TX3))
               1+(1./XKT)*(Q1*CCOS(TX4)-Q2*CSIN(TX4)))
0102            D=P3NXC*P3NX/EN
0103            DF=REAL(D)
0104      36    RETURN
0105            END
```

-0.436E-04	-0.864E-04	-0.128E-03	-0.166E-03	-0.202E-03
-0.234E-03	-0.262E-03	-0.285E-03	-0.303E-03	-0.316E-03
-0.323E-03	-0.323E-03	-0.319E-03	-0.308E-03	-0.292E-03
-0.271E-03	-0.245E-03	-0.215E-03	-0.182E-03	-0.145E-03
-0.106E-03	-0.651E-04	-0.236E-04	0.182E-04	0.593E-04
0.992E-04	0.137E-03	0.172E-03	0.204E-03	0.233E-03
0.257E-03	0.277E-03	0.292E-03	0.302E-03	0.308E-03
0.309E-03	0.305E-03	0.296E-03	0.284E-03	0.268E-03
0.248E-03	0.226E-03	0.202E-03	0.176E-03	0.149E-03
0.122E-03	0.946E-04	0.684E-04	0.437E-04	0.210E-04
0.614E-06	-0.169E-04	-0.313E-04	-0.422E-04	-0.496E-04
-0.532E-04	-0.532E-04	-0.494E-04	-0.422E-04	-0.317E-04
-0.182E-04	-0.210E-05	0.163E-04	0.364E-04	0.579E-04
0.801E-04	0.103E-03	0.125E-03	0.146E-03	0.167E-03
0.185E-03	0.202E-03	0.216E-03	0.227E-03	0.235E-03
0.241E-03	0.242E-03	0.241E-03	0.236E-03	0.228E-03
0.217E-03	0.203E-03	0.187E-03	0.169E-03	0.149E-03
0.127E-03	0.105E-03	0.821E-04	0.592E-04	0.367E-04
0.151E-04	-0.528E-05	-0.241E-04	-0.409E-04	-0.556E-04
-0.678E-04	-0.774E-04	-0.844E-04	-0.885E-04	-0.900E-04
-0.888E-04	-0.850E-04	-0.789E-04	-0.707E-04	-0.606E-04
-0.490E-04	-0.361E-04	-0.224E-04	-0.825E-05	0.606E-05
0.201E-04	0.336E-04	0.462E-04	0.576E-04	0.675E-04
0.757E-04	0.820E-04	0.864E-04	0.886E-04	0.887E-04
0.868E-04	0.828E-04	0.769E-04	0.693E-04	0.602E-04
0.497E-04	0.382E-04	0.259E-04	0.132E-04	0.282E-06
-0.125E-04	-0.248E-04	-0.364E-04	-0.470E-04	-0.565E-04
-0.645E-04	-0.710E-04	-0.758E-04	-0.788E-04	-0.801E-04
-0.796E-04	-0.774E-04	-0.736E-04	-0.683E-04	-0.617E-04
-0.541E-04	-0.456E-04	-0.365E-04	-0.271E-04	-0.177E-04
-0.860E-05	-0.485E-08	0.779E-05	0.145E-04	0.200E-04
0.240E-04	0.264E-04	0.270E-04	0.258E-04	0.227E-04
0.178E-04	0.111E-04	0.260E-05	-0.744E-05	-0.189E-04
-0.315E-04	-0.451E-04	-0.594E-04	-0.741E-04	-0.889E-04
-0.104E-03	-0.118E-03	-0.131E-03	-0.143E-03	-0.154E-03
-0.164E-03	-0.171E-03	-0.177E-03	-0.180E-03	-0.182E-03
-0.181E-03	-0.178E-03	-0.172E-03	-0.165E-03	-0.155E-03
-0.144E-03	-0.131E-03	-0.117E-03	-0.101E-03	-0.844E-04
-0.673E-04	-0.499E-04	-0.325E-04	-0.155E-04	0.926E-06
0.164E-04	0.307E-04	0.436E-04	0.548E-04	0.642E-04

-0.436E-12	-0.174E-11	-0.388E-11	-0.683E-11	-0.105E-10
-0.149E-10	-0.199E-10	-0.254E-10	-0.312E-10	-0.374E-10
-0.438E-10	-0.503E-10	-0.567E-10	-0.630E-10	-0.690E-10
-0.747E-10	-0.798E-10	-0.845E-10	-0.884E-10	-0.917E-10
-0.942E-10	-0.959E-10	-0.968E-10	-0.969E-10	-0.961E-10
-0.945E-10	-0.921E-10	-0.890E-10	-0.853E-10	-0.809E-10
-0.760E-10	-0.706E-10	-0.649E-10	-0.590E-10	-0.529E-10
-0.467E-10	-0.406E-10	-0.345E-10	-0.287E-10	-0.232E-10
-0.180E-10	-0.133E-10	-0.902E-11	-0.524E-11	-0.200E-11
0.708E-12	0.287E-11	0.450E-11	0.561E-11	0.626E-11
0.647E-11	0.630E-11	0.581E-11	0.507E-11	0.415E-11
0.311E-11	0.204E-11	0.101E-11	0.902E-13	-0.654E-12
-0.116E-11	-0.137E-11	-0.123E-11	-0.703E-12	0.238E-12
0.162E-11	0.344E-11	0.572E-11	0.843E-11	0.116E-10
0.151E-10	0.190E-10	0.231E-10	0.276E-10	0.322E-10
0.370E-10	0.418E-10	0.466E-10	0.514E-10	0.561E-10
0.605E-10	0.647E-10	0.686E-10	0.722E-10	0.754E-10
0.781E-10	0.805E-10	0.823E-10	0.837E-10	0.847E-10
0.852E-10	0.853E-10	0.850E-10	0.844E-10	0.834E-10
0.822E-10	0.807E-10	0.791E-10	0.774E-10	0.756E-10
0.738E-10	0.720E-10	0.704E-10	0.689E-10	0.676E-10
0.665E-10	0.656E-10	0.650E-10	0.647E-10	0.647E-10
0.650E-10	0.655E-10	0.663E-10	0.673E-10	0.686E-10
0.700E-10	0.716E-10	0.733E-10	0.751E-10	0.768E-10
0.786E-10	0.803E-10	0.819E-10	0.834E-10	0.847E-10
0.858E-10	0.866E-10	0.873E-10	0.877E-10	0.878E-10
0.877E-10	0.873E-10	0.867E-10	0.859E-10	0.848E-10
0.836E-10	0.823E-10	0.808E-10	0.792E-10	0.776E-10
0.760E-10	0.745E-10	0.730E-10	0.715E-10	0.702E-10
0.691E-10	0.681E-10	0.673E-10	0.666E-10	0.662E-10
0.659E-10	0.658E-10	0.659E-10	0.661E-10	0.665E-10
0.669E-10	0.674E-10	0.680E-10	0.685E-10	0.690E-10
0.694E-10	0.697E-10	0.698E-10	0.698E-10	0.695E-10
0.690E-10	0.682E-10	0.672E-10	0.659E-10	0.642E-10
0.623E-10	0.601E-10	0.576E-10	0.549E-10	0.519E-10
0.487E-10	0.453E-10	0.419E-10	0.383E-10	0.346E-10
0.310E-10	0.274E-10	0.239E-10	0.205E-10	0.173E-10
0.143E-10	0.116E-10	0.911E-11	0.694E-11	0.509E-11
0.357E-11	0.240E-11	0.157E-11	0.109E-11	0.950E-12
0.113E-11	0.160E-11	0.234E-11	0.333E-11	0.453E-11