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PROPERTIES AND DISTRIBUTION FUNCTIONS OF POLAR FLUIDS BY
PERTURBATION THEORY

The University of Oklahoma

PH.D.

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THE UNIVERSITY OF OKLAHOMA

GRADUATE COLLEGE

PROPERTIES AND DISTRIBUTION FUNCTIONS OF POLAR
FLUIDS BY PERTURBATION THEORY

A DISSERTATION

SUBMITTED TO THE GRADUATE FACULTY

in partial fulfillment of the requirements for the
degree of

DOCTORATE IN PHILOSOPHY

By

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Norman, Oklahoma

1980

PROPERTIES AND DISTRIBUTION FUNCTIONS OF POLAR
FLUIDY BY PERTURBATION THEORY
A DISSERTATION
APPROVED FOR THE
DEPARTMENT OF PHYSICS & ASTRONOMY

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ABSTRACT

It is proposed that perturbation theories and computer simulation be collectively applied to the study of static and dynamic properties of liquids and dense fluids composed of highly nonspherical and/or polar molecules. Perturbation theories rooted in classical statistical mechanics have, in recent years, proven successful in predicting the equilibrium properties of atomic fluids and molecular fluids composed of weakly anisotropic molecules. We propose to develop and test analogous theories for the thermodynamic and transport properties of fluids containing highly anisotropic molecules. New theories for equilibrium properties will be based on a first order expansion of the angular pair correlation function $g(r\omega_1\omega_2)$, or, equivalently, the function $y(r\omega_1\omega_2)$. Expansions about spherical and nonspherical reference fluids will be explored for static properties. A major strength of the proposed work is the concurrent use of computer simulation to guide and evaluate development of the theories. The investigation includes two different temperatures; $T^*=1.294$ and $T^*=1.277$, with two quadrupole moments of $Q^*=1$ and $Q^*=0.5$, respectively. The density is assumed to be $\rho^*=0.85$.

CHAPTER I

INTRODUCTION

Recently there have been several extensions to the polar system of the theoretical techniques that proved useful in predicting the thermodynamics of simpler fluids. Computational work (1-4) has allowed the evaluation of these theories for uncomplicated models consisting of dipoles embedded in otherwise spherical molecules.

The most surprising success among the theories was a special version of the thermodynamic perturbation theory (5), the so-called Padé approximation due to Stell, Rasaiah and others (6-8). Especially in the last decade there has been great enthusiasm among the laboratory experimentalists, statistical mechanical theorists and those who simulate fluids on the computer because of new developments in the study of the liquid and dense fluid states of matter with the equilibrium properties of "simple" fluids composed of atoms (Ar, Kr) or nearly spherical (CH_4 , O_2) molecules.

The most important of these developments was the theoretical development of the perturbation concept which occurred in equilibrium statistical mechanics. This theory is based on the idea that fluids are related to each other and they differ by small amounts of their properties.

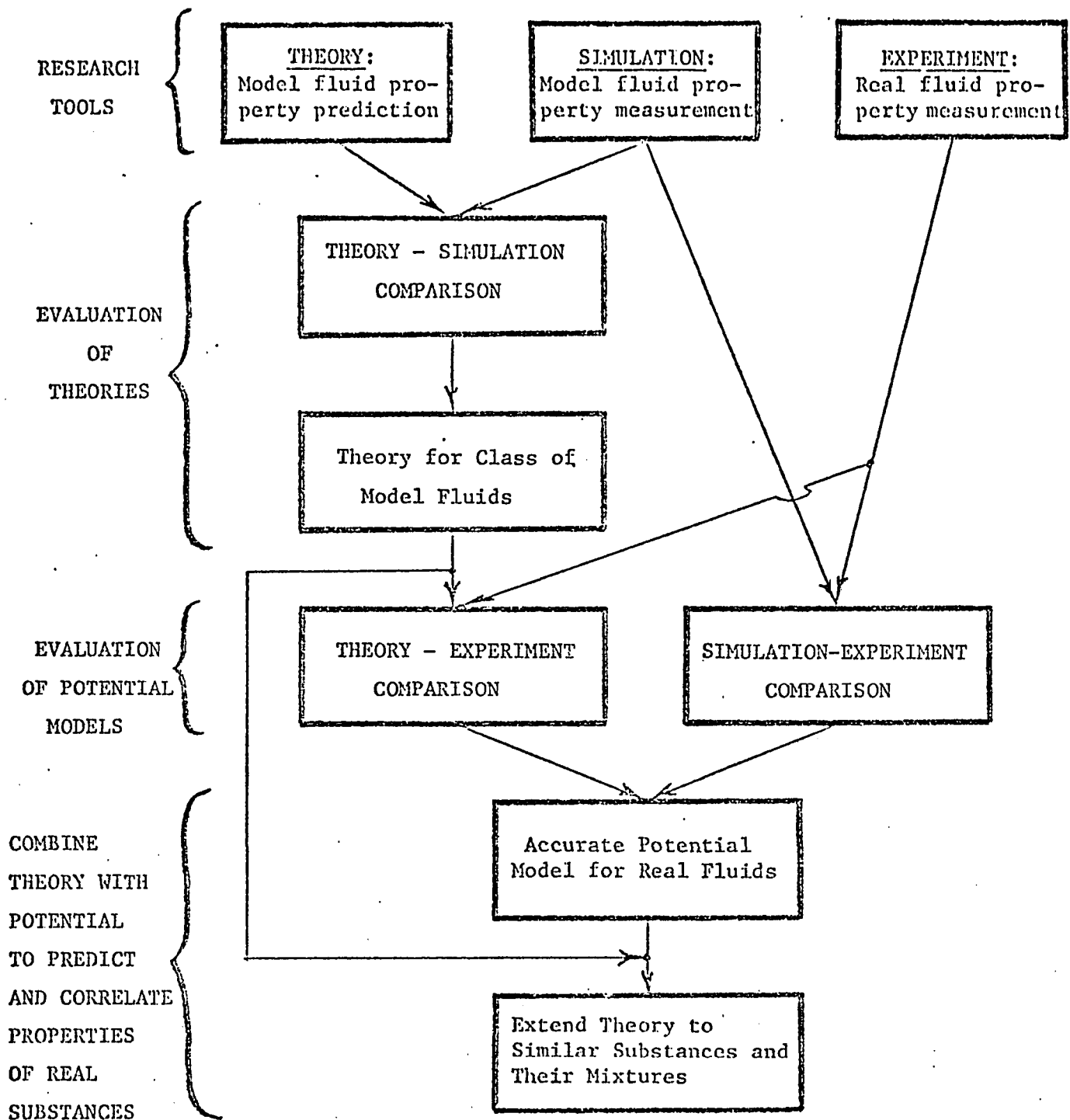
So if we know the properties of a simple fluid, then we can predict the properties of new fluids by treating the differences as perturbed parts. Now this perturbed part has to be calculated successfully. And this "reference" substance can be compared with the results obtained from simulation methods embodied in Monte Carlo and molecular dynamics (9-10) which are available to us.

Figure 1 shows the importance of simulation in providing data for model fluids. Thus, the accuracy of our predictions can be found out by comparing theoretical predictions with simulation results, and by comparing the simulation results with laboratory measurements we can learn about the accuracy of models with real fluids of interest.

The test would be on static properties of the class of model fluids composed of highly nonspherical and/or strongly polar molecules. The reasons can be explained from several viewpoints. From the theoretical point of view, there are presently very few satisfactory theories for predicting the static properties of such fluids. Not very much work has been done on developing perturbation expansions about nonspherical reference fluids which would be applicable to highly anisotropic molecules. There have been a number of thermodynamic perturbation theories developed in the last decade, but most of them are expansions of the Helmholtz free energy or the property of interest directly. But our approach to the equilibrium properties of molecular fluids will be via

second order perturbation theory for the angular pair correlation function $g(r_{12}, \omega_1, \omega_2)$. Since the expansion of $g(r_{12}, \omega_1, \omega_2)$ is equivalent to the free energy expansion (11) and $g(r_{12}, \omega_1, \omega_2)$ can be obtained by computer simulation via a spherical harmonic expansion in the molecular orientations ω_1 and ω_2 (11,13), other equilibrium properties may be obtained from standard formulas in statistical mechanics (14). The success of perturbation theories developed to date has been limited to atomic fluids or to those containing weakly anisotropic molecules for which a spherical reference fluid can be used.

Figure 1: STRATEGY FOR STUDY OF LIQUIDS AND MIXTURES



CHAPTER II

LITERATURE REVIEW

Although the perturbation expansion of the angular pair correlation function $g(r_{12}, \omega_1, \omega_2)$ provides an alternate route to macroscopic thermodynamic properties (e.g., free energy, pressure, internal energy) of polar fluids having moderately strong multipolar strengths, expansion of $g(r_{12}, \omega_1, \omega_2)$ has received much less attention than the free energy expansion.

Among those theories perhaps one of the best is the Padé extension to third order of the Pople expansion of the free energy due to Stell et al. (14,15); the idea here also is to find the perturbed part of the fluid of interest from the reference system.

$$v(x_i, x_j) = v_0(r) + \sum_k \lambda_k \omega_k(x_i, x_j); \quad \lambda_k \geq 0 \quad (2-1)$$

where $v(x_i, x_j)$ is the orientational part of the pair potential for two molecules and $v_0(r)$ is spherically symmetric reference system), $\{\lambda_k\}$ are related to the dipole moment μ , the quadrupole moment Q , and the octupole moment ϕ From recent evidence of computer simulation, the Padé expansion for the free energy in a quadrupole fluid up to moderate values of the quadrupole moment $Q(\epsilon\sigma^5)^{-1/2}$ of $O(1)$ (16) is promising. The Padé expansion for dipoles contains the correct low and high dipolar strength limits and Padé must be (almost)

correctly interpolating at moderate dipolar strengths, but the Pade' theory breaks down at high anisotropy in anisotropic overlap fluids (17,18).

One of the early expansions of $g(r, \omega_1, \omega_2)$ was done by Gubbins and Gray. They wrote the first order expansion which is valid for a Pople reference and multipolar anisotropics (19).

$$g(r_{12}, \omega_1, \omega_2) = g_o(r) (1 - \text{EXP}(\beta u_a(r, \omega_1, \omega_2))) \quad (2-2)$$

where $\beta = 1/KT$, u_a the reference fluid radial distribution function. This equation also breaks down for moderately strong anisotropics in that nonphysical, negative values of $g(r_{12}, \omega_1, \omega_2)$ are obtained for certain pair orientations.

Perram and White also gave a first order theory of $g(r_{12}, \omega_1, \omega_2)$ by using a reference potential of the log-exp mean type on u_a . They only tested the theory for the center pair correlation function $g_c(r)$ (20). Henderson and Gray assumed the potential to consist of two parts: $u_a + u_o$, where u_a was quadrupole-quadrupole interaction and u_o was a reference potential and then they suggested a perturbation theory involving an expansion of the direct correlation function $C(r, \omega_1, \omega_2)$ (22) in the form of $C_a = C_1 + C_2 + C_3 + \dots$ where C_n includes all terms in C_a of order $(u_a)^n$.

They considered two possible closures of the Ornstein-Zernicke equation

$$C_a = -g_o u_a + h_o (h_a - C_a) \quad (2-3)$$

where

$$h_o = g_o - 1$$

(a mean field and a generalized mean field approximation) to obtain $g_o(r, \omega_1, \omega_2)$ from $C(r, \omega_1, \omega_2)$. Anderson, Weeks and Chandler proposed the "blip function theory" for the calculation of radial distribution function and thermodynamic properties of dense fluids. It is written in a form which yields explicit expressions for both the angular and radial correlation of dense fluids of linear molecules (21).

Steel and Sandler compared this approximation with a variety of correlation functions obtained by numerically solving the extended Percus-Yevick equation for non-spherical molecules. They found that agreement is generally good at high densities, but becomes worse at lower densities because of the omission of the effects of attractive interactions in the blip function theory. Also they extended the blip function formula to $g_c(r)$. In addition, Gubbin et al. (25) and Sung Chandler (26) applied Schofield's temperature expansion for $g(r)$ (27) to the first order and made approximations for higher order correlation functions. Although these theories are largely successful for predicting $g_c(r)$ for moderately strong anisotropy, they are, for the most part, first order theories. Hence, they would not be expected to work for strongly anisotropic fluids. More importantly, knowledge of $g_c(r)$ is not sufficient to enable calculation of all thermodynamic properties.

CHAPTER III

SPHERICAL HARMONIC EXPANSION

If we look at OZ equation

$$h(r_{12}, \omega_1, \omega_2) = c(r_{12}, \omega_1, \omega_2) + \rho \int dr_3 \langle c(r_1, \omega_1, \omega_3) h(r_{32}, \omega_3, \omega_2) \rangle_{\omega_3} \quad (3-1)$$

we record the analysis of (3-1) in spherical harmonics. We restrict considerations to linear molecules so that $\omega = (\theta, \phi)$ denotes the orientation of the symmetry axis. Consider any function $F(r, \omega_1, \omega_2)$ that is invariant under simultaneous rotation of r, ω_1 and ω_2 . We first Fourier transform the space variable according to

$$F(k, \omega_1, \omega_2) = \int dr \exp(ikr) f(r, \omega_1, \omega_2) \quad (3-2)$$

then the rotational invariance implies that $F(r, \omega_1, \omega_2)$ and $F(k, \omega_1, \omega_2)$ can be expanded in spherical harmonics

$$F(r, \omega_1, \omega_2) = \sum_{l_1 l_2 l} F(l_1 l_2 l, r) \psi_{l_1 l_2 l}(\omega_1, \omega_2, \omega_r) \quad (3-3)$$

$$F(k, \omega_1, \omega_2) = \sum_{l_1 l_2 l} F(l_1 l_2 l, k) \psi_{l_1 l_2 l}(\omega_1, \omega_2, \omega_k) \quad (3-4)$$

where r and k are the magnitudes of $\vec{r}$ and $\vec{k}$ and ω_r, ω_k denotes the orientation of r and k here

$$\psi_{l_1 l_2 l}(\omega_1, \omega_2, \omega) = \sum_{m_1 m_2 m} C(l_1 l_2 l, m_1 m_2 m) \cdot Y_{l_1}^{m_1}(\omega_1) Y_{l_2}^{m_2}(\omega_2) Y_l^{m*}(\omega) \quad (3-5)$$

where $Y_l^m(\theta, \phi)$ are spherical harmonics and given by

$$Y_1^m(\omega) = (-1)^m \left(\frac{2l+1}{4\pi}\right) \cdot \frac{(l-m)!}{(l+m)!} P_1^m(c_0 \theta) e^{im\phi} \quad (3-6)$$

and the Legendre polynomial $P_1^m(x)$ is

$$P_1^m(x) = \frac{1}{2^{l-1} l!} (1-x^2)^{m/2} \frac{d^{m+1}}{dx^{m+1}} (x^2-1)^l$$

here $-l \leq m \leq l; \quad l \geq 0$

(3-7)

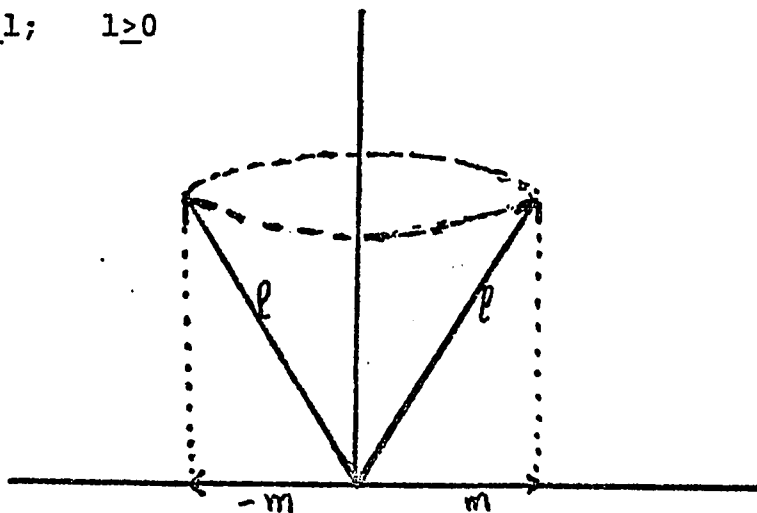


Figure 4

(l is the normal angular momentum and m is the azimuthal angular momentum), and $C(l_1 l_2 l, m_1, m_2, m)$ is Clebsch Gordon coefficient. The properties of $C(l_1 l_2 l, m_1, m_2, m)$ are as follows:

$$\sum_l C(l_1 l_2 l, m_1, m-m_1, m) C(l_1 l_2 l', m_1', m'-m_1', m') = \delta_{m_1 m_1'} \delta_{m m'} \quad (3-8)$$

$$\sum_m C(l_1 l_2 l, m_1, m-m_1, m) C(l_1 l_2, l', m_1, m-m_1, m) = \delta_{l l'} \quad (3-9)$$

and

$$m = m_1 + m_2 \quad (3-10)$$

and where δ_{nk} is Kronocker delta function and is

$$\delta_{nk} = \begin{cases} 1 & n=k \\ 0 & n \neq k \end{cases} \quad (3-11)$$

Also here l_1, l_2, l must satisfy the triangle relation $\Delta(l_1, l_2, l)$ which we can write as

$$|l_1 - l_2| \leq l \leq (l_1 + l_2) \quad (3-12)$$

Figure 4 shows the l and m . Further, parity considerations for linear molecules shows that $F(l_1, l_2, r) = 0$ unless $l_1 + l_2 + l$ is even. l_1 and l_2 must be drawn from the same set of values, as the symmetry of these functions under interchange of identical molecules implies

$$F(l_1, l_2, l, r) = (-1)^{l_1 + l_2} F(l_2, l_1, l, r) \quad (3-13)$$

CHAPTER IV

SPHERICAL HARMONIC EXPANSION OF THE ANGULAR PAIR CORRELATION FUNCTION

For fluid composed of axially symmetric molecules the angular pair correlation function $g(r, \omega_1, \omega_2)$ can be expanded in terms of product of spherical harmonics of molecular orientation. Rotational invariance of $g(r, \omega_1, \omega_2)$ implies that it depends on only (r, ω_1, ω_2) where $\omega_i = (\phi_i, \theta_i)$ denotes the orientation of i^{th} molecule with respect to the intermolecular axis.

In this coordinate system, the harmonic series of g is

$$g(r, \omega'_1, \omega'_2) = 4\pi \sum_{l_1} \sum_{l_2} \sum_{m_1} g(l_1 l_2 m_1 r) Y_{l_1}^{m_1}(\omega'_1) Y_{l_2}^{\bar{m}_1}(\omega'_2) \quad (4-1)$$

($\bar{m} = -m$). Only m_i and $\bar{m}_i$ in any terms since the rotational invariance also implies that the ϕ_i can appear only as $\phi'_1 - \phi'_2$. Putting the Polar axis in (3-3) in the direction of r so that $\omega_i = \omega'_i$ and subsequently comparing to (4-1) we find

$$g(l_1 l_2 l, r) = 4\pi \left(\frac{4\pi}{2l+1} \right)^{1/2} \sum_m C(l_1 l_2 l, m \bar{m} 0) g(l_1 l_2 m, r) \quad (4-2)$$

so the desired harmonics in the space fixed axis can be evaluated from knowledge of the body fixed axis $g(l_1 l_2 m, r)$. Conversely, the $g(l_1 l_2 m, r)$ may be found from the $g(l_1 l_2 l, r)$ by

$$g(l_1 l_2 m, r) = (4\pi)^{-3/2} \sum_l (2l+1)^{1/2} g(l_1 l_2 l, r) C(l_1 l_2 l, m \bar{m} 0) \quad (4-3)$$

For homonuclear diatomic $(0^2, N^2)l_1$ and l_2 are both even. The validity of equations (4-3) and (4-2) can be proven by using equation (4-1) and substituting (4-2). In (4-3), we get

$$g(l_1 l_2 m_1 r) = \sum_l \left(\frac{2}{4\pi} \right)^{\frac{1}{2}} C(l_1 l_2 l, m_1 \bar{m}_1 0) \left(\frac{4\pi}{2l+1} \right)^{\frac{1}{2}} \left[\sum_{\bar{m}} C(l_1 l_2 l m \bar{m}) \cdot \right.$$

$$\left. (g(l_1 l_2 m r)) \right] = \sum_l \sum_{\bar{m}} C(l_1 l_2 l, m_1 \bar{m}_1 0) C(l_1 l_2 l, m \bar{m} 0)$$

$$g(l_1 l_2 m r) = \sum_{\bar{m}} g(l_1 l_2 m r)_{\bar{m}} C(l_1 l_2 m_1 \bar{m}_1 0) \cdot$$

$$C(l_1 l_2 l, m \bar{m} 0)$$

$$= \sum_{\bar{m}} g(l_1 l_2 m r) \delta_{m m_1} \quad (4-4)$$

With the help of orthogonality condition of spherical harmonic functions, equation (4-1) can be inverted to

$$g(l_1 l_2 m r) = 4\pi \int d\omega_1' d\omega_2' Y_{l_1}^{m_1*}(\omega_2) g(r\omega_1' \omega_2') \quad (4-5)$$

This integral can be evaluated by Monte Carlo techniques by taking an average of $Y_{l_1}^{m_1*}(\omega_1') Y_{l_2}^{\bar{m}_1*}(\omega_2)$ over the shell between r and $r+\Delta r$ in particular

$$g(l_1 l_2 m_1 r) = 4\pi g_{000}(r) \langle Y_{l_1}^{m_1}(\omega_1) Y_{l_2}^{\bar{m}_1}(\omega_2) \rangle$$

where g_{000} is the center pair correlation function g_c and $\langle \dots \rangle$ represents an ensemble average over a spherical shell of radius r and thickness Δr . Several workers have used this approach in evaluating $g(r, \omega_1, \omega_2)$ in order to study local structure in fluids composed of linear molecules (6, 7, 32, 33).

Typically, about thirty $g(l_1 l_2 m, r)$ coefficients are evaluated and used to approximate the infinite series of equation (4-1). Convergence of the series appears to be satisfactory for this number of terms.

Figure 2 shows the $g(l_1 l_2 m, r)$ coefficient obtained from a molecular dynamics simulation of a Lennard-Jones plus quadrupole fluid. Figure 3 shows a three dimensional surface of the $g(r, \omega_1, \omega_2)$ function obtained by recombining the expansion of equation (4-1) with the molecular dynamics determined $g(l_1 l_2 m, r)$. In particular, Figure 3 demonstrates that the quadrupolar anisotropy forces molecular pairs into the "tee" orientation rather than parallel or related orientations in the shell of nearest neighbors.

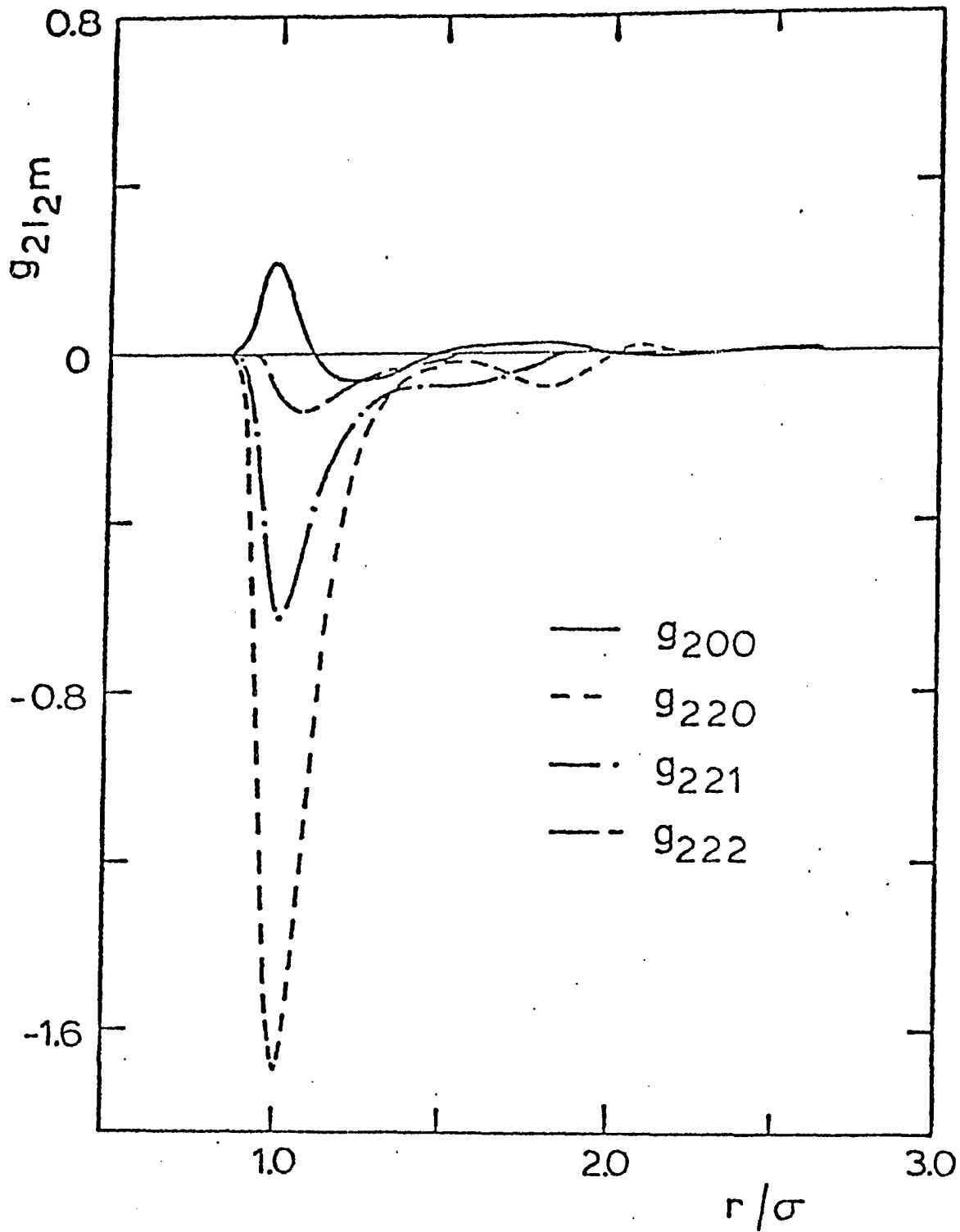


Figure 2: The $g_{2\ell 2m}$ coefficients in the Spherical Harmonic Expansion for the Angular Pair Correlation Function for a Lennard-Jones plus Quadrupole Fluid. $KTe^{-1} = 1.277$, $\rho\sigma^3 = .85$, $Q(\epsilon\sigma^5)^{-\frac{1}{2}} = 1$. [2]

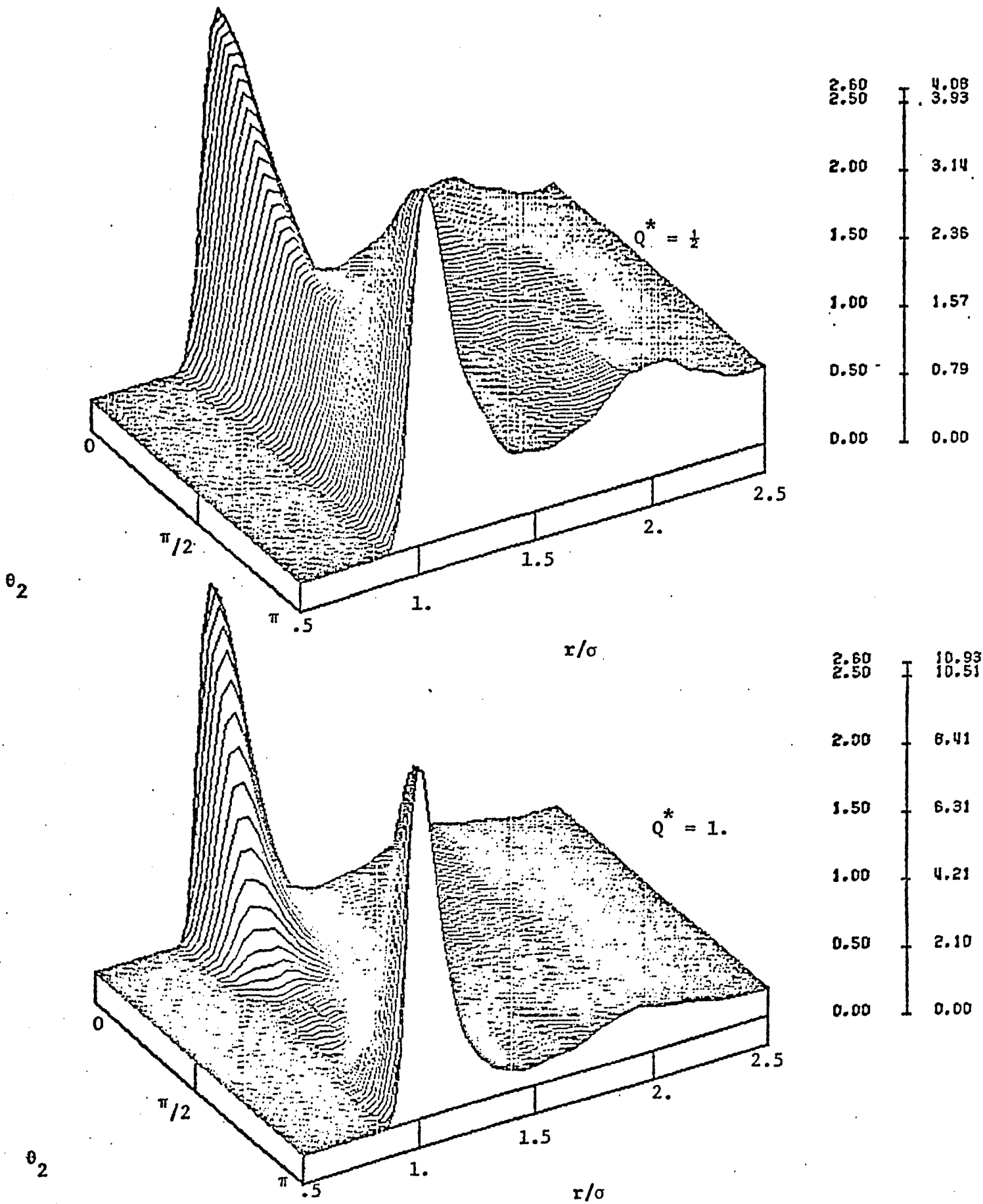


Figure 3 : Effect of the Quadrupole Moment on a Subspace of the Angular Pair Correlation Function for the fluid of Figure 1. Here $\theta_1 = 90^\circ$, ϕ undefined [7].

CHAPTER V

THEORY

Our basic approach is derived from the functional derivation formalism of Percus (126) and Lebowitz (124,125) for inhomogeneous systems. In this approach, the inhomogeneity arises due to the "turning on" of an external potential $U(r)$. With proper selection of the generating function, the Percus-Yevick (PY), Hypernetted Chain (HNC) and Yvon-Born-Green (YGB) internal equations were shown to be first order functional Taylor expansions in terms of the singlet density n_1 . We first introduce the definitions of the important quantities. The partition function of the inhomogeneous system in a grand canonical ensemble is

$$\Xi(u) = \sum_{n \geq 0} (N!_1)^{-1} Z^n \int d\{n\} \exp\{-\beta V_n(\{n\}) - \beta \sum_i u(i)\} \quad (5-1)$$

the singlet density $n_1(1/u)$ is

$$n_1\left(\frac{1}{u}\right) \equiv \sum_{n \geq 1} \frac{Z^n}{(n-1)!} \int d\{n-1\} \exp\{-\beta V_n(\{n\}) - \beta \sum_i u(i)\} \quad (5-2)$$

Here $\{n\}$ is the set of molecular configurations, $\{1, 2, 3, 4, \dots, \dots, N\}$, each number i represents the position r_i of the center of mass (com) and orientation (Euler angles)

ω_i of the i^{th} molecule, β is reciprocal temperature $(KT)^{-1}$, K is the Boltzmann constant, Z being the activity

$\exp(\beta\mu)/\lambda^3$, μ the chemical potential, λ the deBroglie thermal wave length, V_N the N body potential assumed to be pairwise additive.

$$V\{N\} = \sum_{i,j}^N u(i,j) \quad (5-3)$$

where u is the full pair potential, which is separated into a reference potential U_0 and a perturbation term u_p

$$u(12) = u_0(12) + u_p(12) \quad (5-4)$$

where $(12) = (r_{12}, \omega_1, \omega_2)$.

We will talk about the potential in a separate section. To derive the perturbation equation in general, we adopt the generating functional of the PY type, i.e., $\eta_1(\frac{1}{u})\exp(\beta u(1))$. We note that from functional calculus, on Ξ we have the following relations

$$\frac{\partial \ln \Xi(u)}{\partial \exp -\beta u(1)} = (\exp(\beta u(1))) \eta_1(\frac{1}{u}) \quad (5-5)$$

$$\frac{\partial^2 \ln \Xi(u)}{\partial \exp -\beta u(1) \partial \exp -\beta u(2)} = \exp(\beta u(1)) \exp(\beta u(2)) \cdot \quad (5-6)$$

$$[\eta_2(\frac{12}{u}) - \eta_1(\frac{1}{u}) \eta_1(\frac{2}{u})]$$

$$\frac{\partial^3 \ln \Xi(u)}{\partial \exp -\beta u(1) \partial \exp(-\beta u(2)) \partial \exp(-\beta u(3))} = \exp \beta u(1) \exp \beta u(2) \cdot$$

$$\begin{aligned} & \exp \beta u(3) \left[\eta_3(\frac{123}{u}) - \eta_2(\frac{12}{u}) - \eta_1(\frac{3}{u}) - \eta_2(\frac{23}{u}) \eta_1(\frac{1}{u}) \right. \\ & \left. - \eta_2(\frac{13}{u}) \eta_1(\frac{2}{u}) + 2 \eta_1(\frac{1}{u}) \eta_1(\frac{2}{u}) \eta_1(\frac{3}{u}) \right] \end{aligned} \quad (5-7)$$

Consider $\eta_1 \exp(\beta u)$ as a functional of $\exp(-\beta u)$ where η_1 is the singlet density and expand to the second order in a functional Taylor series.

$$\begin{aligned}
 \eta_1 \left(\frac{1}{u} \right) e^{\beta u^0(1)} &= \eta_1^0 \left(\frac{1}{u} \right) e^{\beta u^0(1)} + \int d_3 e^{\beta u^0(1)} e^{\beta u^0(3)} \left[\eta_2 \left(\frac{13}{u^0} \right) \right. \\
 &\quad \left. - \eta_1 \left(\frac{1}{u^0} \right) \eta_1 \left(\frac{3}{u^0} \right) \right] [e^{-\beta u(3)} - e^{-\beta u^0(3)}] + \frac{1}{2} \int d_3 d_4 e^{\beta u^0(1)} \cdot \\
 &\quad e^{\beta u^0(3)} e^{\beta u^0(4)} \left[\eta_3 \left(\frac{134}{u^0} \right) - \eta_2 \left(\frac{13}{u^0} \right) \eta_1 \left(\frac{4}{u^0} \right) + 2 \eta_1 \left(\frac{1}{u^0} \right) \cdot \right. \\
 &\quad \left. \eta_1 \left(\frac{3}{u^0} \right) \eta_1 \left(\frac{4}{u^0} \right) - \eta_2 \left(\frac{14}{u^0} \right) \eta_1 \left(\frac{3}{u^0} \right) - \eta_2 \left(\frac{34}{u^0} \right) \eta_1 \left(\frac{1}{u^0} \right) \right] \cdot \\
 &\quad [e^{-\beta u(3)} - e^{-\beta u^0(3)}] [e^{-\beta u(4)} - e^{-\beta u^0(4)}] \quad (5-8)
 \end{aligned}$$

This can be reduced to

$$\begin{aligned}
 Y(12) &= Y^0(12) \left\{ 1 + \rho \int d_3 \left[\frac{g_3^0(132)}{g^0(12)} - g^0(32) [e^{-\beta u(32)} - 1] \right] \right. \quad (5-9) \\
 &\quad + \left(\frac{1}{2} \right) \rho^2 \int d_3 d_4 \left[\frac{g_4^0(1342)}{g^0(12)} - g_3^0(342) - \frac{g_3^0(132) g^0(42)}{g^0(12)} \right. \\
 &\quad \left. \left. - \frac{g_3^0(142) g^0(32)}{g^0(12)} + 2 g^0(32) g^0(42) \right] [e^{-\beta u(32)} - 1] \right\}
 \end{aligned}$$

where we have introduced the Y-distribution function $Y(12)$ $g(12) \exp \beta u(12)$. The superscript o represents the reference potential u_o .

To make efficient numerical calculation, the higher order correlation functions g_3 , g_4 must be approximated by the two body expressions. The Kirkwood superposition has been intensively used

$$g_3(123) = g(12)g(13)$$

$$g_4(1234) = g_3(123)g_3(134)g_3(124)g_3(234)/(g(12)$$

$$g(13)g(14)g(23)g(24)g(34)) \quad (5-11)$$

Jackson and Feenberg (131) gave a "convolution approximation"

$$g_3^{(c)}(123) = 1 + h(12) + h(12)h(23) + \rho \int d4 h(14)h(24)h(34)$$

$$+ h(23) + h(23)h(31) + h(31) + h(3)h(12) \quad (5-11)$$

For convenience, we develop the perturbation equation with Kirkwood approximation and to the first order terms only.

A similar procedure can be used for the convolution approximation and to the second order terms, substituting the $Y(123)=Y(12)Y(23)Y(13)$ into

$$Y(12) = Y^0(12)\{1 + \rho \int d3 \dots\dots\dots\} \quad (5-12)$$

By keeping only the first order perturbation, we have

$$Y(12) = Y^0(12)\{1 + \rho \Omega^{-1} \int d3 h(13)g^0(32)f(32)\} \quad (5-13)$$

$$\text{where } f(32) = \exp(-\beta u_p(32)) - 1. \quad (5-14)$$

Ω is the angle normalization constant, e.g., $\Omega=8\pi^2$ for the complete Euler angles of linear molecules.

Figure 5 depicts the geometric arrangement of molecules 1, 2, and 3 for a fluid. We have translational invariance; the change of origin to r_i gives Figure 5.2.

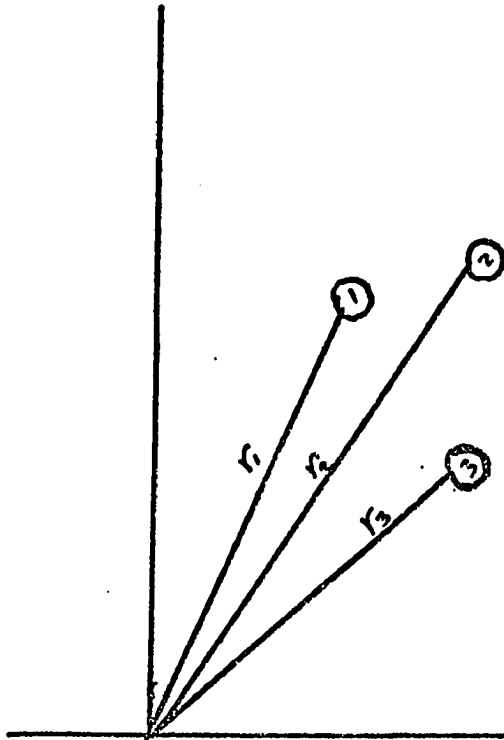


Figure 5.1

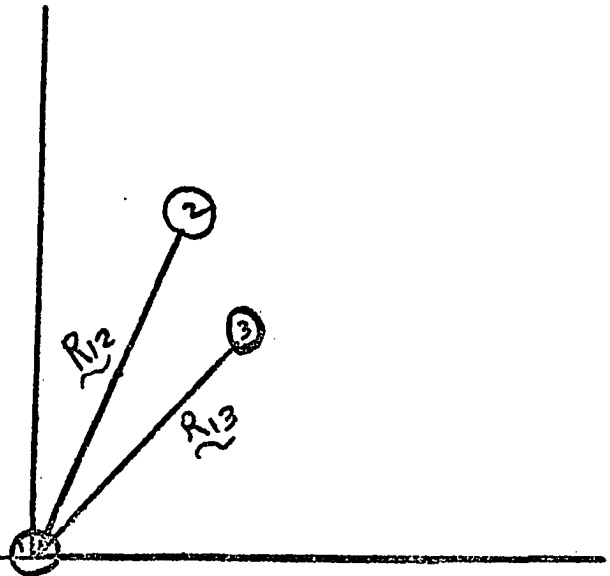


Figure 5.2

Equation (5-13) now can be written

$$Y(R_{12}\omega_1\omega_2) = Y^0(R_{12}\omega_1\omega_2) \{1 + \rho \int dR_{13} < h(R_{13}\omega_1\omega_3) g^0(R_{32}\omega_3\omega_2) f(R_{13} \omega_3 \omega_2 > \omega_3) \} \quad (5-15)$$

where $< \dots >$ is the angle average over ω_3 . To evaluate this equation, we expand the three functions Y/Y_0 , h and $g^0 F$ into spherical harmonics

$$\frac{Y(R_{12}\omega_1\omega_2)}{Y_0(R_{12})} = \sum_{\bar{l}_1} \sum_{\bar{m}_1} \sum_{\bar{n}_1} E_Y(\bar{l}_1 \bar{l}_2 \bar{l} \bar{n}_1 \bar{n}_2 R_{12}) C(\bar{l}_1 \bar{l}_2 \bar{l} \bar{m}_1 \bar{m}_2 \bar{m}) \cdot$$

$$D_{m_1 n_1}^{l_1 *}(\omega_1) D_{m_2 m_2}^{l_2 *}(\omega_2) Y_l^{m*}(\Omega_{12}) \quad (5-16)$$

$$h(R_{13} \omega_1 \omega_3) = \sum_{\substack{l_1 \\ l_3 \\ l}} \sum_{\substack{m_1 \\ m_3 \\ m}} \sum_{n_1} E_Y(l_1 l_3 l, n_1 n_3 R_{13}) C(l_1 l_3 l, m_1 m_3 m)$$

$$D_{m_1 n_1}^{l_1}(\omega_1) D_{m_3 n_3}^{l_3}(\omega_3) Y_l^m(\Omega_{13}) \quad (5-17)$$

$$g^0(R_{23} \omega_2 \omega_3) f(R_{23} \omega_2 \omega_3) = \sum_{l_1} \sum_{l_2} \sum_{l_3} E_F(l_1' l_2' l_3', n_1' n_2' n_3' R_{32})$$

$$\sum_{\substack{l_2' \\ l_1'}} \sum_{\substack{m_2' \\ m_1'}} \sum_{n_2'} Y_l^m(\Omega_{13})$$

$$C(l_3' l_2' l_1', m_3' m_2' m_1') D_{m_3' n_3'}^{l_3'}(\omega_3) D_{m_2' n_2'}^{l_2'}(\omega_2) Y_l^m(\Omega_{32}) \quad (5-18)$$

Here E are the coefficients of expansion, $C(l_1 l_2 l m_1 m_2 m)$ the Clebsch-Gordon coefficients, D the rotational matrices (see Rose, M. E. "Elementary Theory of Angular Momentum", John Wiley, N.Y., 1957) and Y_l^m the spherical harmonic function.

CHAPTER VI

PROPERTIES OF SPHERICAL HARMONIC COEFFICIENTS

Spherical harmonic coefficients $g(l_1 l_2 l r)$ not only give structural information via equation (4-1) but also are related to numerous equilibrium fluid properties via standard formulae in statistical mechanics (8). Thus, we derive the configurational internal energy in terms of the $g(l_1 l_2 l, r)$ coefficients (17).

A number of these results are summarized in Table 1. In general, the relation in Table 1 involves infinite sums over the indices. However, in the special case of fluids which interact with a Lennard-Jones plus a multipotential, the relations reduce to sum over finite number of terms. For example, Table 2 gives the thermodynamic properties which are the specific equations for the properties of Table 1 for the Lennard-Jones plus quadrupole fluids.

In Table 2 the term $J_n^{l_1 l_2 m}$ represents one dimensional integral over the coefficients $g(l_1 l_2 l, r)$.

$$J_n^{l_1 l_2 m} = \int_0^\infty dr r^{-(n-2)} g(l_1 l_2 m, r) \quad (6-1)$$

The results in Table 2 have been tested by computer simulation and found to give highly satisfactory agreement (7,17).

TABLE 1

Examples of Expressions for Equilibrium Properties in Terms of Coefficients $u_{\ell_1 \ell_2 m}(r)$ and $g_{\ell_1 \ell_2 m}(r)$ for Spherical Harmonic Expansions of the Intermolecular Potential $u(r\omega_1\omega_2)$ and Angular Pair Correlation Function $g(r\omega_1\omega_2)$

Configurational Internal Energy

$$U^C/N = \frac{1}{2} \rho \sum_{\ell_1 \ell_2 m} \int d\underline{r} u_{\ell_1 \ell_2 m}(r) g_{\ell_1 \ell_2 m}(r)$$

Pressure

$$P(\rho kT)^{-1} = 1 - \rho (6kT)^{-1} \sum_{\ell_1 \ell_2 m} \int d\underline{r} r \frac{du_{\ell_1 \ell_2 m}(r)}{dr} g_{\ell_1 \ell_2 m}(r)$$

Isothermal Compressibility

$$\chi = (\rho kT)^{-1} + (kT)^{-1} \int d\underline{r} (g_{000}(r) - 1)$$

Mean Squared Torque

$$\langle \tau^2 \rangle = -\rho kT \sum_{\ell_1 \ell_2 m} \ell_1(\ell_1+1) \int d\underline{r} g_{\ell_1 \ell_2 m}(r) u_{\ell_1 \ell_2 m}(r)$$

Kirkwood Angular Correlation Parameters

$$G_L = (2L+1)^{-1} \rho \sum_m (-1)^m \int d\underline{r} g_{LLm}(r)$$

G_0 , G_1 , G_2 are related to the isothermal compressibility, dielectric constant, and Kerr constant, respectively.

TABLE 2

Examples of Expressions for Equilibrium Properties of a Lennard-Jones plus Quadrupole Fluid in Terms of the $J_n^{l_1 l_2 m}$ Integrals of Equation (4)

Configurational Internal Energy

$$U^c/N = 8\pi\rho^* \epsilon [J_{12}^{ooo} - J_6^{ooo}] + U_{QQ}$$

$$U_{QQ} = \frac{12\pi}{5} \rho^* Q^{*2} \epsilon [J_5^{220} + \frac{4}{3} J_5^{221} + \frac{1}{3} J_5^{222}]$$

$$\text{where } \rho^* = \rho\sigma^3 \text{ and } Q^* = Q(\epsilon\sigma^5)^{-\frac{1}{2}}$$

Pressure

$$P(\rho kT)^{-1} = 1 - 16\pi\rho^* (kT)^{-1} [J_6^{ooo} - 2J_{12}^{ooo}] - \frac{5}{3}(kT)^{-1} U_{QQ}$$

Mean Squared Torque

$$\langle \tau^2 \rangle = -12\epsilon T U_{QQ}$$

Kirkwood Angular Correlation Functions

$$G_L = \frac{4\pi\rho^*}{2L+1} \sum_m (-1)^m J_o^{LLm}$$

CHAPTER VII

POTENTIAL MODELS

Recently a new approach to the potential model has been developed. The potentials consist of two parts: $u = u_o + u_a$. The reference system is one in which the molecules interact with an isotropic pair potential $u_o(R_{12})$ defined by

$$u_o(R_{12}) = \langle u(R_{12}\omega_1\omega_2) \rangle_{\omega_1\omega_2} \quad (7-1)$$

where $u(R_{12}, \omega_1, \omega_2)$ is the pair potential for the actual system.

This pair potential is $U(R_{12}\omega_1\omega_2) = U(R_{12}, \theta_1, \theta_2, \phi_{12})$ and the one which has been chosen for study here

$$u(R_{12}, \theta_1, \theta_2, \phi_{12}) = 4\epsilon \left[\left(\frac{\sigma}{R_{12}} \right)^{12} - \left(\frac{\sigma}{R_{12}} \right)^6 \right] + u_a \quad (7-2)$$

with the angle dependent part of the potential defined so that $\langle u_a \rangle_{\omega_1\omega_2} = 0$; thus for equation (7-1), u_o is the Lennard-Jones (12,6) potential. In general, there are several choices for u_a available. The choice depends on the system being investigated. These several parts are:

(1) Dipole-dipole part

$$u_{DD}(12) = - \frac{\mu^2}{R_{12}^2} [2C_1C_2 - S_1S_2C_{12}] \quad (7-3)$$

(2) Dipole-quadrupole part

$$u_{DQ}(12) = \left(\frac{3}{2} \frac{Q \mu}{R_{12}^4}\right) [C_1(3C_2^2-1) - C_2(3C_1^2-1) - 2(C_2 - C_1) S_1 S_2 C_{12}] \quad (7-4)$$

(3) Quadrupole-quadrupole interaction

$$u_{QQ}(12) = \left(\frac{3}{4} \frac{Q^2}{R_{12}^5}\right) [1 - 5C_1^2 - 5C_2^2 - 15C_1^2 C_2^2 + 2(S_1 S_2 C_{12} - 4C_1 C_2)^2] \quad (7-5)$$

(4) Anisotropic overlap interaction

$$u_a = 4E \left(\frac{\delta}{R_{12}}\right)^{12} (3C_1^2 + 3C_2^2 - 2) \quad (7-6)$$

In these equations, $S_1 = \sin \theta_1$, $C_1 = \cos \theta_1$, $C_{12} = \cos(\phi_1 - \phi_2)$ so $C_{12} = \cos \phi_{12}$. μ and Q are dipole and quadrupole moments, respectively, and δ is a dimensionless overlap constant ($-0.25 < \delta < 0.5$), $\theta_1, \theta_2, \phi_{12}$ are the usual orientation angles, referred to the line between molecular centers as polar axis. Figure 7.1 shows the orientation angles in the body fixed coordinate.

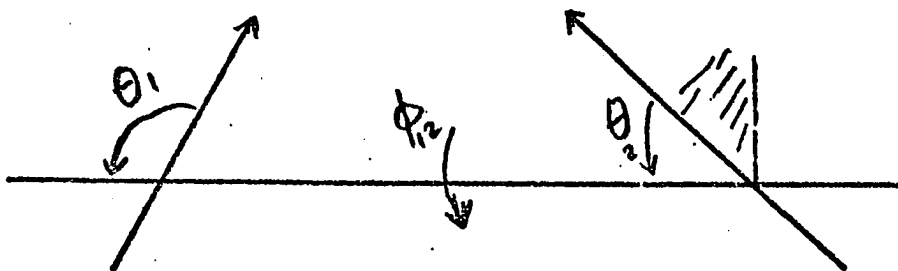
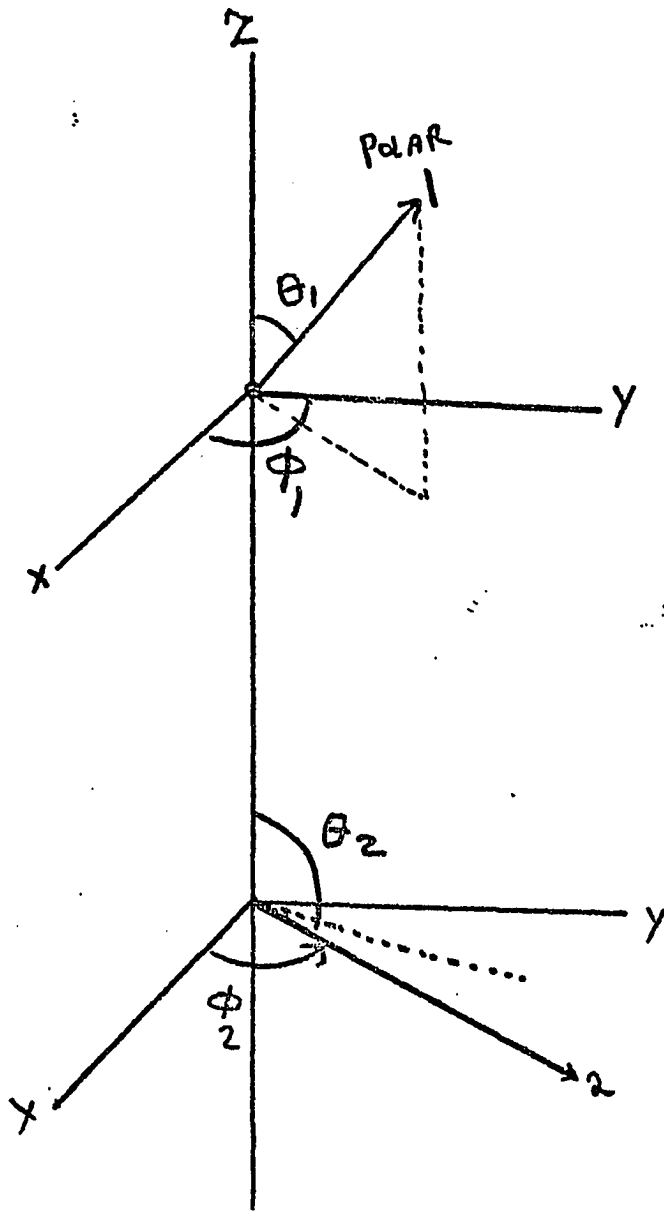


Figure 7.1: The Orientation Angles in the Body Fixed Coordinate

Potential Model

 θ_1, ϕ_1 θ_2, ϕ_2

$$\phi_{12} = \phi_1 - \phi_2$$

Figure 7.2

In our study, the perturbing potential part is the (quadrupole-quadrupole) interaction in the body fixed coordinate. We can expand this potential as

$$u_{QQ}(12) = 4\pi \sum_{m=-2}^2 (-1)^{(2+m)} x^{22m} Y_2^m(\omega_1) Y_2^{m*}(\omega_2) \quad (7-7)$$

where

$$x^{22-2} = x^{222} = \frac{1}{6} \cdot \frac{6}{5} \frac{Q^2}{R^5} = \frac{1}{5} \frac{Q^2}{R^5}$$

$$x^{22-1} = x^{221} = \frac{2}{3} x^{220}$$

$$x^{220} = \frac{6}{5} \frac{Q^2}{R^5} \quad (7-8)$$

And in the space fixed coordinate, we can expand

$$u_{QQ}(12) = \sum_l \sum_{m_1=-2}^2 \sum_{m_2=-2}^2 f(22l, R) C(22l, m_1 m_2 m) Y_2^{m_1}(\omega_1) Y_2^{m_2}(\omega_2) \cdot Y_1^m(\Omega) \quad (7-9)$$

where $l_1 + l_2 + l = \text{even}$ and l has the value of 0, 2, 4.

$$f(2, 2, l, R) = \frac{4\pi}{2} \frac{1}{l+1} \sum_{m_1=-2}^2 C(2 \ 2 \ l, m_1 m_1, 0) 4\pi x^{22m} (-1)^{2+m} \quad (7-10)$$

Figure 7.2 shows $\omega_1 = (\theta_1, \phi_1)$ and $\omega_2 = (\theta_1, \phi_{12})$.

Figure 7.3 shows the effect of the quadrupole moment on a subspace of angular pair correlation function.

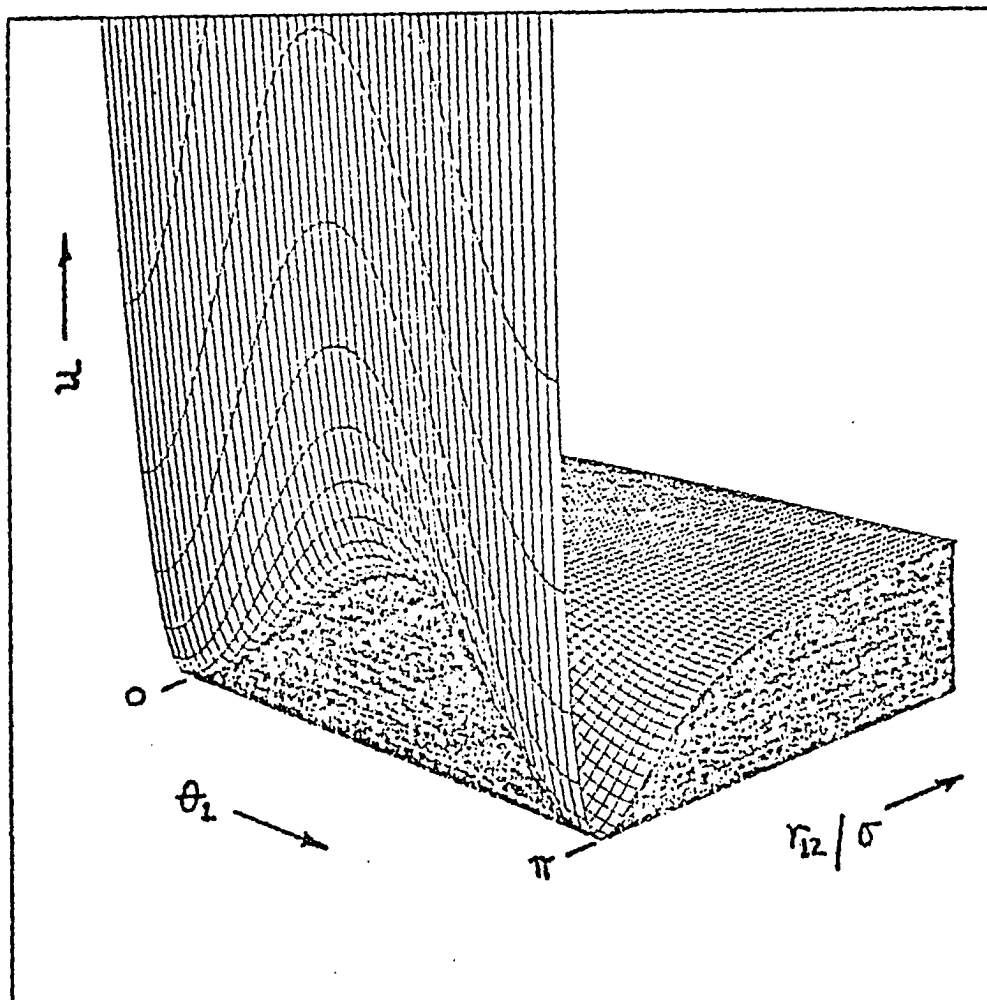


Figure 7.3 Surface of intermolecular pair potential $u(r_{12}^0 \theta_2 \phi)$ for linear molecule modeled by Lennard-Jones plus quadrupole interaction. $Q/(\epsilon \sigma^5)^{1/2} = 1$. θ_2 is fixed at $\pi/2$ and ϕ is undefined.

CHAPTER VIII

CALCULATION

As we mentioned before, we can write

$$\frac{Y(12)}{Y^0(12)} = 1 + \rho \Omega^{-1} \int d^3h(13) g^0(32) f(32) \quad (8-1)$$

or

$$\frac{Y(12)}{Y^0(12)} = 1 + \rho \int dR_{13} \Omega_{13} \langle h g^0 f \rangle_{\omega_3} R_{13}^2 \quad (8-2)$$

Now we can expand; in the space fixed spherical harmonic

$$\begin{aligned} \frac{Y}{Y^0} &= \sum_{\bar{l}_1} \sum_{\bar{m}_1} \sum_{\bar{n}_1} E_Y(\bar{l}_1 \bar{l}_2 \bar{l}, n_1, n_2, R_{12}) C(\bar{l}_1 \bar{l}_2 \bar{l}, \bar{m}_1 \bar{m}_2 \bar{m}) \cdot \\ &\quad \bar{l}_2 \bar{m}_2 \bar{n}_2 \quad n_1 = n_2 = 0 \\ &\quad \bar{l} \bar{m} \\ Y_{\bar{l}_1}^{\bar{m}_1^*}(\omega_1) Y_{\bar{l}_2}^{\bar{m}_2^*}(\omega_2) Y_{\bar{l}}^{\bar{m}}(\Omega_{12}) \end{aligned} \quad (8-3)$$

Expanding h (total correlation function)

$$\begin{aligned} h &= \sum_{\bar{l}_1} \sum_{\bar{m}_1} \sum_{\bar{n}_1} E_h(\bar{l}_1 \bar{l}_3 \bar{l}, n_1, n_3, R_{13}) C(\bar{l}_1 \bar{l}_3 \bar{l}, \bar{m}_1 \bar{m}_3 \bar{m}) \cdot \\ &\quad \bar{l}_3 \bar{m}_3 \bar{n}_3 \quad n_1 = n_3 = 0 \\ &\quad \bar{l} \bar{m} \\ Y_{\bar{l}_1}^{\bar{m}_1^*}(\omega_1) Y_{\bar{l}_3}^{\bar{m}_3^*}(\omega_3) Y_{\bar{l}}^{\bar{m}}(\Omega_{13}) \end{aligned} \quad (8-4)$$

Also, we expand f in the space fixed spherical harmonic

$$f = \sum_{l'_3} \sum_{m'_3} \sum_{n'_3} E_f(l'_3, l'_2, l', n_3, n_2, R_{32}) C(l'_3 l'_2 l, m'_3 m'_2 m') \cdot$$

$$\sum_{l'_2} \sum_{m'_2} \sum_{n'_2}$$

$$\sum_{l'_1} \sum_{m'_1}$$

$$Y_{l'_3}^{m'_3*}(\omega_3) Y_{l'_2}^{m'_2*}(\omega_2) Y_{l'_1}^{m'_1}(\Omega_{32}) \quad (8-5)$$

Furthermore, we can write

$$\langle h g^0 f \rangle_{\omega_3} = \sum \frac{(-1)^{m'_1}}{4\pi} \delta_{l_3 l'_3} \delta_{m_3 \bar{m}_3} E_h E_f(l_1 l_3 l, m_1 m_3 m)$$

$$C(l_3 l_2 l, m_3 m'_2 m') Y_{l_1}^{m'_1*}(\omega_1) (Y_{l_2}^{m'_2}(\omega_2) Y_{l_1}^{m'_1}(\Omega) g^0 \quad (8-6)$$

The terms are non zero only when $l_3 = l'_3$, $m_3 = m'_3$ and since

$$\frac{1}{\Omega} \int d\omega_3 Y_{l_3}^{m'_3*}(\omega_3) Y_{l'_3}^{m'_3*}(\omega_3) = \frac{1}{\Omega} \int d\omega_3 Y_{l_3}^{m'_3*}(-1)^{m'_3} Y_{l'_3}^{m'_3}$$

$$= \frac{1}{4\pi} (-1)^{m_3} \delta_{l_3 l'_3} \delta_{m_3 \bar{m}_3} \quad *8-7)$$

$$Y_l^{m*}(\theta, \phi) = (-1)^m Y_l^{\bar{m}}(\theta, \phi)$$

Substituting in

$$\frac{Y}{Y^0} = 1 + \frac{1}{4\pi} \int dR_{13} d\Omega_{13} R_{13}^2 \sum_{l_1} \sum_{l_3} \sum_{m_1} E_h(l_1 l_3 l, R_{13}) C(l_1 l_3 l, m_1 m_3 m) \cdot$$

$$\sum_{l_3} \sum_{l_1} \sum_{m'_1}$$

$$\sum_{l'_1} \sum_{l'_3} \sum_{m'_3}$$

$$Y_{11}^{m_1^*}(\omega_1) Y_{\ell}^m(\Omega_{13}) g_O f(l_3 l_3' l', R_{32}) C(l_3 l_2' l_1', -m_3 m_2' m_1') \cdot \quad (8-9)$$

$$Y_{12}^{m_2^*}(\omega_2) Y_{11}^{m_1'}(\Omega_{32}) (-1)^{m_3}$$

CHAPTER IX

MODEL

In the space fixed coordinate, the three molecules 1, 2, 3 can be shown as in Figure 9.1 with the polar coordinate of R_1, R_2, R_3 and $(\theta_1, \phi_1), (\theta_2, \phi_2),$

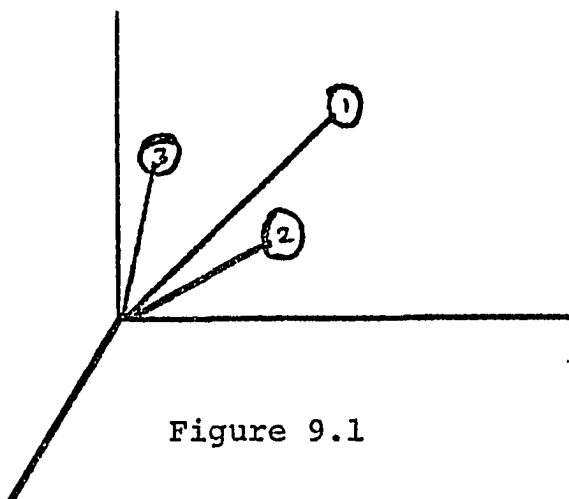


Figure 9.1

or we can simply fix our coordinates such that the origin is on molecule 1 and one axis passes through molecule 2. So Figure 9.1 can be simplified as Figure 9.2.

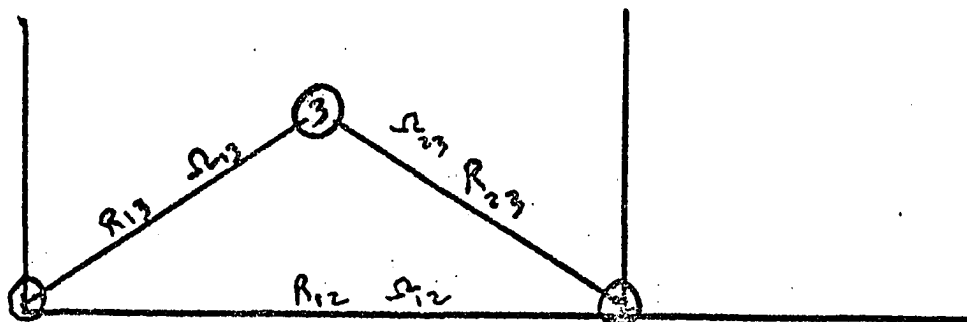


Figure 9.2

Since molecule 1 is always on the origin, new coordinate would be (R_{13}, Ω_{13}) and also (R_{12}, Ω_{12}) . Furthermore, (R_{12}, R_{13}) remains constant and $\Omega_{12}, R_{23}, \Omega_{23}$ would be the only variable factors. In other words, we have to calculate R_{23}, Ω_{23} by knowing Ω_{12} and the constants. Let us call $\Omega_{12} = (\theta_2, \phi_2)$ and $\Omega_{23} = (\theta_3, \phi_3)$. Let us represent R_{23}, R_{13}, R_{12} by vector components so

$$\begin{aligned} R_{23} &= (v_1, v_2, v_3) \\ R_{13} &= (w_1, w_2, w_3) \\ R_{12} &= (u_1, u_2, u_3) \\ (w_1 - u_1) &= (v_1) \\ R_{23} &= (w_2 - u_2) = (v_2) \\ (w_3 - u_3) &= (v_3) \end{aligned} \tag{9-1}$$

but from the geometry we have

$$\begin{aligned} w_1 &= R_{13} \sin \theta_{13} \cos \phi_{13} \\ w_2 &= R_{13} \sin \theta_{13} \sin \phi_{13} \\ w_3 &= R_{13} \cos \theta_{13} \end{aligned} \tag{9-3}$$

also, by similarity, one can write

$$\begin{aligned} u_1 &= R_{12} \sin \theta_{12} \cos \phi_{12} \\ u_2 &= R_{12} \sin \theta_{12} \sin \phi_{12} \\ u_3 &= R_{12} \cos \theta_{12} \end{aligned} \tag{9-4}$$

and

$$v_1 = R_{23} \sin \theta_{23} \cos \phi_{23}$$

$$v_2 = R_{23} \sin \theta_{23} \sin \phi_{23}$$

$$v_3 = R_{23} \cos \theta_{23} \quad (9-5)$$

From this last set of equations we can have

$$R_{23} = \frac{w_1 - u_1}{\sin \theta_{23} \phi_{23}} = \frac{w_3 - u_3}{\cos \theta_{23}} = \frac{R_{13} \cos \theta_{13} - R_{12} \cos \theta_{12}}{\cos \theta_{23}}$$

$$\cos \theta_{23} = \frac{w_3 - u_3}{R_{23}} \quad (9-6)$$

and

$$R_{23}^2 = (w_1 - u_1)^2 + (w_2 - u_2)^2 + (w_3 - u_3)^2 \quad (9-7)$$

$$\cos \phi_{23} = \frac{w_1 - u_1}{R_{23} \sin \theta_{23}} \text{ or } \sin \phi_{23} = \frac{w_2 - u_2}{R_{23} \sin \theta_{23}} \quad (9-8)$$

$$R_{13}^2 = w_1^2 + w_2^2 + w_3^2$$

$$\cos \theta_{13} = \frac{w_2}{R_{13}}$$

$$\sin \theta_{13} = \frac{(w_1^2 + w_2^2)^{\frac{1}{2}}}{R_{13}}$$

$$\cos \phi_{13} = \frac{w_1}{(w_1^2 + w_2^2)^{\frac{1}{2}}} \quad (9-9)$$

$$\sin \theta_{13} \cos \theta_{13} = \frac{w_1}{R_{13}}$$

$$\sin \phi_{13} = \frac{w_2}{(w_1^2 + w_2^2)^{\frac{1}{2}}}$$

$$\text{so } \sin \theta_{13} \sin \theta_{13} = \frac{w_1}{R_{13}}$$

$$\text{but } R_{32}^2 = (w_1 - u_1)^2 + (w_2 - u_2)^2 + (w_3 - u_3)^2 \quad (9-10)$$

and also

$$\cos \theta_{12} = \frac{u_3}{R_{12}} \quad (9-11)$$

$$\sin \theta_{12} = \frac{(u_1^2 + u_2^2)^{\frac{1}{2}}}{R_{12}} \quad (9-12)$$

$$\cos \phi_{12} = \frac{u_1}{(u_1^2 + u_2^2)^{\frac{1}{2}}} \quad (9-13)$$

$$\sin \phi_{12} = \frac{u_2}{(u_1^2 + u_2^2)^{\frac{1}{2}}} \quad (9-14)$$

$$\sin \theta_{12} \cos \theta_{12} = \frac{u_2}{R_{12}} \quad (9-15)$$

$$\sin \theta_{12} \sin \phi_{12} = \frac{u_2}{R_{12}} \quad (9-16)$$

For Ω_{23}

$$\cos \theta_{23} = \frac{w_3 - u_3}{R_{23}} \quad (9-17)$$

$$\sin \theta_{23} = \frac{[(w_1 - u_1)^2 + (w_2 - u_2)^2]^{\frac{1}{2}}}{R_{23}} \quad (9-18)$$

$$\cos \phi_{23} = \frac{w_1 - u_1}{[(w_1 - u_1)^2 + (w_2 - u_2)^2]^{\frac{1}{2}}} \quad (9-19)$$

$$\sin \phi_{23} = \frac{w_2 - u_2}{[(w_1 - u_1)^2 + (w_2 - u_2)^2]^{\frac{1}{2}}} \quad (9-20)$$

And finally,

$$\sin \theta_{23} \cos \phi_{23} = \frac{w_1 - u_1}{R_{23}}$$

$$\sin \theta_{23} \sin \phi_{23} = \frac{w_1 - u_1}{R_{23}} \quad (9-21)$$

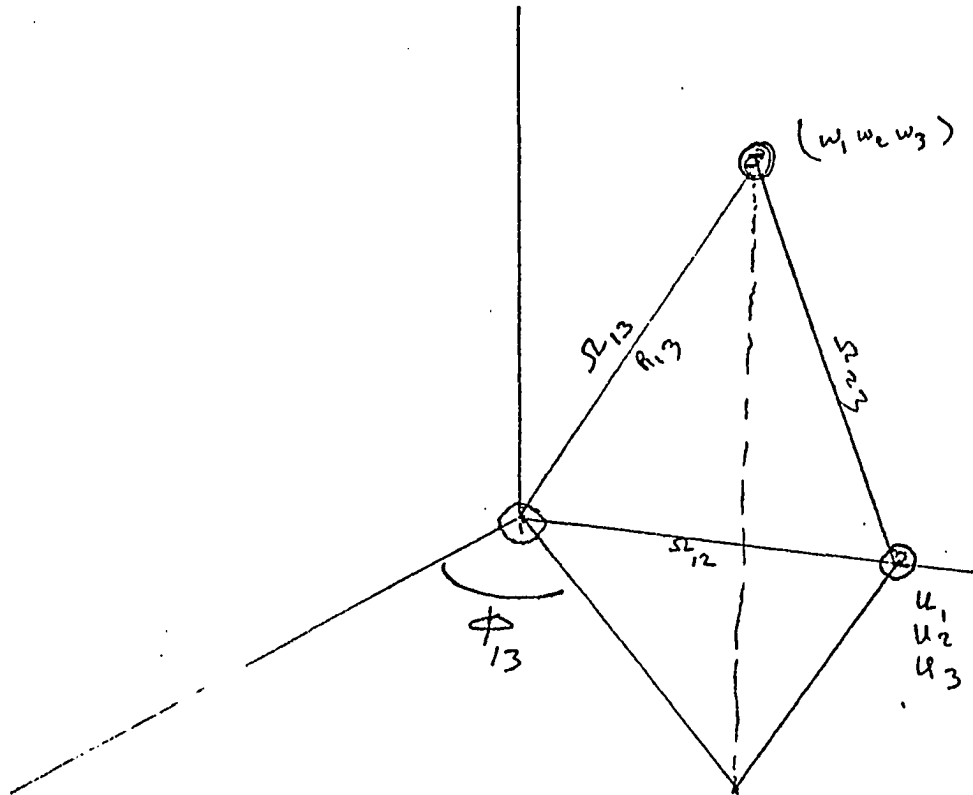


Fig 9-3

CHAPTER X

PROCEDURE

First we had to find $F = \exp(-\beta u) - 1$, so called Mayer function. The spherical harmonic expansion of F in the space fixed coordinate is

$$F(r, \omega_1, \omega_2) = \sum_{\substack{l_1 \\ l_2 \\ l}} \sum_{\substack{m_1 \\ m_2 \\ m}} E_f(l_1 l_2 l, r) C(l_1 l_2 l, m_1 m_2 m) Y_{l_1}^{m_1}(\omega_1) Y_{l_2}^{m_2}(\omega_2) Y_l^{m*}(\Omega) \quad (10-1)$$

where $E_f(l_1 l_2 l, r)$ are the expansion coefficients and by orthogonality condition of $C(l_1 l_2 l, m_1 m_2 m)$ which has been mentioned before, and also because of the orthogonality of spherical harmonics function

$$\int d\omega_1 Y_{l_1}^{m_1}(\omega_1) Y_{l_1'}^{m_1'*}(\omega_1) = \delta_{l_1 l_1'} \delta_{m_1 m_1'} \quad (10-2)$$

we can write

$$E_f(l_1 l_2 l, r) = \int_0^\pi d\theta_1 \int_0^\pi d\theta_2 \int_0^\pi d\theta_3 \int_0^{2\pi} d\phi_1 \int_0^{2\pi} d\phi_2 \int_0^{2\pi} d\phi_3 F(r, \omega_1, \omega_2) \cdot C(l_1 l_2 l, m_1 m_2 m) Y_{l_1}^{m_1}(\omega_1) Y_{l_2}^{m_2*}(\omega_2) Y_l^m(\Omega) \quad (10-3)$$

This is a six fold integral.

Integration

Since this multidimensional integral cannot be evaluated by analytical means, numerical integration processes must be used. The methods generally employed are based on the product of Simpson's rule or the product of Gaussian formulae. While these formulae can be applied to give the value of a multiple integral correct to any accuracy, they are not the most economical formulae in the sense that a large number of integrand function evaluations is needed. Since these evaluations are the most time-consuming part of these computations, economy is an important factor. Integration formulae which are exact for multinomials over the entire region of integration are necessarily more economical than the product formulae and can be applied to the same accuracy by subdividing the integration region.

The construction of a satisfactory multidimensional non-product formula for a given hyper-region is a matter of some difficulty and is in fact a problem in the theory of algebraic equations.

The required formula must have positive weights associated with it in order to avoid loss of accuracy by cancellation and all the integration points must lie entirely within the integration region. Watts has given a general third degree formula valid for the hypercube of side 2 and centered on the origin. Let $\Sigma_n(k)$ denote the point $(\sigma_1^{(k)}, \sigma_2^{(k)}, \dots, \sigma_n^{(k)})$ of the n -dimensional Euclidean space, σ^n , for $k=1, 2, \dots, 2n$

and

$$\sigma_{2S-1}^{(k)} = (2/3)^{\frac{1}{2}} \cos (2S-1) \frac{k\pi}{n}$$

$$\sigma_{2S}^{(k)} = (2/3)^{\frac{1}{2}} \sin (2S-1) \frac{k\pi}{n} \quad (10-4)$$

where $S = 1, 2, \dots \dots n/2$, and when n is odd $\sigma_n^{(k)} = (-1)^k / \sqrt{3}$.

If S_n be the hypercube of side 2 and the center at the origin an integral over S_n may be evaluated by the formula.

The weight for each point is $1/2n$. We also used shifting methods to get a more accurate answer. This method could cut the computer time, except that in this 6-fold integral $f = \exp(-\beta u) - 1$, where u itself must be expanded in the space fixed coordinate and again is 6-fold integral.

With shifting twice within 6 dimensions, we had to calculate one point $(n+2)*(n+2)$ which is 64 summations and still the accuracy was poor. So what we decided was as follows. We calculated these coefficients of expansion in the body fixed, then transferred them to the space fixed coordinate by transformation formula (4-2). The spherical harmonic expansion of the Mayer function in the body fixed coordinate is

$$F(r_{12}, \omega_1, \omega_2) = 4\pi \sum_{\substack{l_1 \\ l_2 \\ m_1}} E_f(l_1 l_2 m_1 r_{12}) Y_{l_1}^{m_1}(\omega_1) Y_{l_2}^{m_1}(\omega_2) \quad (10-5)$$

The $g(l_1 l_2 m, r_{12})$ are the expansion coefficients to be determined. For molecules with a plane of symmetry perpendicular to the molecular axis, the $E_f(l_1 l_2 m_1 r_{12})$ are zero unless

both l_1 and l_2 are even (homonuclear diatomics e.g.). For heteronuclear diatomics $E_f(l_1, l_2, m, r)$ both odd and even values contribute. Multiplying both sides of (10-5) by $Y_{l_1}^{m*}(\omega_1)$ $Y_{l_2}^{\bar{m}*}(\omega_2)$ where * indicates complex conjugate and integrating over ω_1 and ω_2

$$\int d\omega_1 d\omega_2 g(r_{12}, \omega_1, \omega_2) Y_{l_1}^{m*}(\omega_1) Y_{l_2}^{\bar{m}*}(\omega_2) = 4\pi \sum_{l_1 l_2 m} E_g(l_1, l_2, m, r_{12}) \quad (10-6)$$

$$\int d\omega_1 Y_{l_1}^m(\omega_1) Y_{l_1}^{m*}(\omega_1) \int d\omega_2 Y_{l_2}^{\bar{m}}(\omega_2) Y_{l_2}^{\bar{m}*}(\omega_2)$$

$$\text{but } \int d\omega Y_l^m(\omega) Y_{l'}^{m'*}(\omega) = \delta_{ll'} \delta_{mm'}$$

$$E_g(l_1, l_2, m, r_{12}) = \frac{1}{4\pi} \int d\omega_1 d\omega_2 Y_{l_2}^{\bar{m}*}(\omega_2) g(r_{12}, \omega_1, \omega_2) \quad (10-7)$$

Dividing both sides of (10-7) by $\int d\omega_1 d\omega_2 g(r_{12}, \omega_1, \omega_2)$

$$\frac{E_g(l_1, l_2, m, r_{12})}{\int d\omega_1 d\omega_2 g(r_{12}, \omega_1, \omega_2)} = \frac{1}{4\pi} \frac{\int d\omega_1 d\omega_2 Y_{l_1}^{m*}(\omega_1) Y_{l_2}^{\bar{m}*}(\omega_2) g(r_{12}, \omega_1, \omega_2)}{\int d\omega_1 d\omega_2 g(r_{12}, \omega_1, \omega_2)} \quad (10-8)$$

where $\bar{m}_1 = -m_1$

and

$$Y_l^m(\omega) = Y_l^m(\theta, \phi) \quad (10-9)$$

We can write the formula for $g(r_{12}, \omega_1, \omega_2)$ with the same form of expansion coefficient so we have equation (10-7) in terms of

$$g(r_{12}, \omega_1, \omega_2) = 4\pi \sum_{l_1 l_2 m} g(l_1, l_2, m, r_{12}) Y_{l_1}^m(\omega_1) Y_{l_2}^{\bar{m}}(\omega_2) \quad (10-10)$$

The average of a property $X(r_{12}, \omega_1, \omega_2)$ in a spherical shell of radius between r_{12} and $r_{12}+dr_{12}$ is defined by

$$\langle X(r_{12}, \omega_1, \omega_2) \rangle_{\text{shell}} = \frac{\int d\omega_1 d\omega_2 X(r_{12}, \omega_1, \omega_2) g(r_{12}, \omega_1, \omega_1)}{\int d\omega_1 d\omega_2 g(r_{12}, \omega_1, \omega_2)} \quad (10-11)$$

Further, from (10-7) we have

$$g_{000}(r_{12}) = \frac{1}{(4\pi)} \int d\omega_1 d\omega_2 g(r_{12}, \omega_1, \omega_2)$$

Combine (10-10) and (10-11) to give

$$Y(l_1 l_2 m, r_{12}) = 4\pi g_{000}(r_{12}) \langle Y_{l_1}^{m_1*}(\omega_1) Y_{l_2}^{\bar{m}_1*}(\omega_2) \rangle_{\text{shell}} \quad (10-12)$$

Note this $g_{000}(r)$ is the same $g_0(r)$ expression for all $g(l_1, l_2 m, r_{12})$ having l_1 and l_2 even up to $\{l_1, l_2, m\} = \{4, 4, 4\}$ and the additional terms with l_1 and l_2 even, $m = 0$ to $\{l_1, l_2, m\} = \{8, 0, 0\}$ are tabulated in Appendix F of Reference 1.

We had used the data of $g_{000}(r)$ from Verlet where r_{12} goes from 0.7 up to 3.2. (We found the g_{000} for higher r_{12} by extrapolation.)

Calculation of $g_0 F$

As mentioned before, to calculate the expansion coefficients of the F in the body fixed coordinate, we used Simpson's rule to solve multiple integral. Since $\phi_2 = 0$

$$E_F(l_1 l_2 m, r_{12}) = \int d\theta_1 \int d\theta_2 \int d\theta_1 F(r_{12}, \omega_1, \omega_2) Y_{l_1}^{m_1}(\omega_1) Y_{l_2}^{\bar{m}_1}(\omega_2) \quad (10-13)$$

$$\omega_1 = (\theta_1, \phi_1), \quad \omega_2 = (\theta_2, 0)$$

$$\text{and } F(r_{12}, \omega_1, \omega_2) = \exp(-\beta u(r_{12}, \omega_1, \omega_2)) - 1$$

The potential U has been discussed before. $\beta = 1/kT$ where k is the Boltzmann constant. This evaluation has been carried out for two different conditions: $T^* = 1.277$ and $Q = 0.5$; $T^* = 1.294$ and $Q = 1$. Q is the quadrupole moment discussed in the potential chapter.

The $Y_l^m(\omega)$ spherical harmonic functions have been calculated from $l = 0$ to $l = 4$ and in each calculation it has been fit in.

By finding the coefficients of $E_f(l_1 l_2 m_1 r_{12})$ and using the conversion from body to spaced fixed coordinates, we could find the $E_f(l_1 l_2 l, r_{12})$ where $|l_1 - l_2| \leq l \leq (l_1 + l_2)$ and later we were able to find $g^0(r_{12}) * E_f(l_1 l_2 l r_{12})$ for each r_{12} from 0.7 to 3.2 with $dr = 0.04$ when g_0 is at reduced density of 0.85. Pages 55 to 67 show some results for E_f .

After having reduced the $g^0 E_f(l_1 l_2 l, r_{12})$ for reduced density of 0.85 and two different temperatures and quadrupole moments, we are able to calculate the $E_f(l_1 l_2 l, r_{12})$ and the space fixed spherical harmonic expansion coefficients of direct correlation function $h(r_{12}, \omega_1, \omega_2)$. Simulation results are given by Dr. Haile in his dissertation (7) for $l_1 = 0$ to 4, $l_2 = 0$ to 4 and m going from negative of l_1 or l_2 up l_1 or l_2 depending on which one is smaller.

Now since

$$g(l_1 l_2 \ m \ R_{12}) = h(l_1 l_2 \ m \ R_{12}) - 1$$

by transferring the body fixed to space fixed coordinates, we were able to calculate all $E_g(l_1 l_2 l, r_{12})$ from $l_1 = 0$ to 4 and $l_2 = 0$ to 4 for r starting from 0.7 up to 3.2 with $dr = 0.025$. The only difference here is the dr in both coefficients of $E_f(l_1 l_2 l, r_{12})$ and $E_h(l_1 l_2 l \ r_{12})$ which we took care of by changing our dr from 0.04 to 0.025. In Table 1.2, some values for $g(l_1 l_2 \ m \ r_{12})$ and $g(l_1 l_2 l \ r_{12})$, $E_h(l_1 l_2 l \ r_{12})$ are given.

CHAPTER XI

CONVOLUTION PART B

We can arrange the formula for Y and write

$$\left(\frac{Y(12)}{Y_0(n)} - 1\right) = \sum_{\substack{\bar{l}_1 \\ l_2}} \sum_{\substack{\bar{m}_1 \\ \bar{m}_2 \\ m}} E_Y(\bar{l}_1 \bar{l}_2, \bar{l}, R_{12}) C(\bar{l}_1 \bar{l}_2, \bar{l}, \bar{m}_1, \bar{m}_2 m) \cdot$$

$$Y_{\bar{l}_1}^{\bar{m}_1}(\omega_1) Y_{\bar{l}_2}^{\bar{m}_2}(\omega_2) Y_l^{m*}(\Omega_{12}) \quad (11-1)$$

Also for h (13), we had

$$h(13) = \sum_{\substack{l_1 \\ l_3 \\ l}} \sum_{\substack{m_1 \\ m_3 \\ m}} E_h(l_1 l_2 l, R_{13}) C(l_1 l_3 l, m_1 m_3 m) Y_{l_1}^{m_1}(\omega_1) Y_{l_3}^{m_3}(\omega_2) \cdot$$

$$Y_l^m(\Omega_{13}) \quad (11-2)$$

For $g^{\circ}F(23)$, we can write

$$g^{\circ}F(23) = \sum_{\substack{l'_3 \\ l'_2 \\ l'}} \sum_{\substack{m'_3 \\ m'_2 \\ m'}} E_f(l'_3 l'_2 l', R_{32}) C(l'_3 l'_2, m'_3 m'_2 m') \quad (11-3)$$

$$Y_{l'_3}^{m'_3}(\omega_3) Y_{l'_2}^{m'_2}(\omega_2) Y_{l'}^{m'*}(\Omega_{32})$$

Now we substitute for $g^{\circ}f(23)$ and h(13) to get $\langle h^*(13) g^{\circ}f(23) \rangle_{\omega_3}$.

By definition of $\langle \dots \rangle_{\omega_3}$ and using the orthogonality condition for both spherical harmonic functions and Clebsch Gordon coefficients, we have

$$\langle h^*(13)g^0 f(23) \rangle_{\omega_3} = \int d\omega_3 \sum_{\substack{l_1 \\ l_3 \\ l}} \sum_{\substack{m_1 \\ m_3 \\ m}} E_h(l_1 l_3 l, R_{13}) C(l_1 l_3 l, m_1 m_3 m)$$

$$Y_{l_1}^{m_1*}(\omega_1) Y_{l_3}^{m_3*}(\omega_3) Y_{l_1}^m(\Omega_{13}) \sum_{\substack{l'_3 \\ l'_2 \\ l'}} \sum_{\substack{m'_3 \\ m'_2 \\ m'}} E_f(l'_3 l'_2 l', R_{32}) \quad (11-4)$$

$$C(l'_3 l'_2 l' m'_3 m'_2 m'_1) Y_{l'_3}^{m'_3}(\omega_3) Y_{l'_2}^{m'_2}(\omega_2) Y_{l'}^{m'_1*}(\Omega_{32})$$

Since

$$\int d\omega_3 Y_{l'_3}^{m'_3}(\omega_3) Y_{l_3}^{m_3*}(\omega_3) = \frac{1}{4\pi} \delta_{l'_3 l_3} \delta_{m'_3 m_3} \quad (11-5)$$

The only non-zero term will be when $l'_3 = l_3$ and $m'_3 = m_3$ and the average will be

$$\langle h^*(13)g^0 f(23) \rangle_{\omega_3} = \frac{1}{4\pi} \sum_{\substack{l_1 \\ l_3 \\ l}} \sum_{\substack{m_1 \\ m_3 \\ m}} \sum_{\substack{l'_2 \\ l'}} \sum_{\substack{m'_2 \\ m'}} E_h(l_1 l_3 l, R_{12}) \cdot$$

$$E_f(l_3 l'_2 l' R_{12}) C(l_3 l'_2 l, m_3 m'_2 m) C(l_1 l'_2 l' m_1 m'_2 m) \quad (11-6)$$

$$Y_{l_1}^{m_1*}(\omega_1) Y_{l_1}^m(\Omega_{13}) Y_{l'_2}^{m'_2}(\omega_2) Y_{l'}^{m'_1*}(\Omega_{32})$$

Furthermore, we also had for

$$\frac{Y}{Y_0} = 1 + \iint dR_{13} d\Omega_{13} \langle h^* g^0 f \rangle_{\omega_3} R_{13}^2$$

or by substitution for

$$\langle h^* g^0 f \rangle_{\omega_3}$$

$$\frac{Y(12)}{Y_0} - 1 = \frac{\rho}{4\pi} \int dR_{13} R_{13}^2 d\theta_{13} \sin \theta_{13} d\phi_{13} \sum_{l_2'} \sum_{l_1} \sum_{m_2'} \sum_{m'} \\ l_1' \quad l_3 \quad m \quad m^3 \\ l \quad m$$

$$E_h(l_1 l_3 l, R_{13}) C(l_1 l_3 l m_1 m_3 m) Y_{l_1}^{m_1*}(\omega_1) Y_{l_2'}^{m_2'}(\omega_2) Y_l^m(\Omega_{12}) Y_{l_1}^{m_1'*}(\Omega_{32}) \cdot$$

$$C(l_3 l_2' l' m_3 m_2' m') \quad (11-7)$$

From this equation one can have

$$\int d\omega_1 Y_{L_1}^{M_1}(\omega_1) \left(\frac{Y(12)}{Y_0(12)} - 1 \right) = \sum_{\bar{L}_2} \sum_{\bar{m}_2} E_Y(L_1 \bar{L}_2 \bar{L}, R_{12}) \cdot$$

$$C(L_1 \bar{L}_2 \bar{L} \quad \underline{M}_1 \bar{m}_2 \bar{m}) Y_{\bar{L}_2}^{\bar{m}_2}(\omega_2) Y_{\bar{L}}^{\bar{m}^*}(\Omega_{12}) (-1)^{M_1} \quad (11-8)$$

$$\bar{L}_1 = L_1$$

$$\bar{m}_1 = \underline{M}_1$$

where $\underline{M}_1 = -M_1$.

So one can substitute for the left hand side of (11-8) and integrate over $d\omega_2$. The non-zero terms occur when

$$l_1 = L_1$$

$$m_1 = M_1$$

$$\frac{\rho}{4\pi} \int dR_{13} R_{13}^2 d\theta_{13} \sin \theta_{13} d\phi \sum_{l'_1} \sum_{l_3} \sum_{m_3} \sum_{m'_2} E_h(L_1 l_3 l, R_{13}) \cdot$$

$$l'_2 \quad l \quad m \quad m'$$

$$C(L_1 l_3 M_1 m_3 m) Y_{l'_2}^{m'_2}(\omega_2) Y_l^m(\Omega_{12}) Y_{l'}^{m'}(\Omega_{32}) \quad (11-9)$$

Again we can multiply equation (11-8) by $Y_{L_2}^{m*}(\omega_2)$ and integrate over $d\omega_2$ to have

$$\int d\omega_2 Y_{L_2}^{M_2*}(\omega_2) \int d\omega_1 Y_{L_1}^{M_1}(\omega_1) \left(\frac{Y}{Y_0} - 1 \right) = \quad (11-10)$$

$$\sum_l \sum_m E_Y(L_1 L_2 \bar{l}, R_{12}) C(L_1 L_2 \bar{l}, \underline{M_1} M_2 \bar{m}) Y_{\bar{l}}^{\bar{m}*}(\Omega_{12}) (-1)^{M_1}$$

which gave us $\bar{l}_2 = L_2$ and $\bar{m}_2 = M_2$. Substituting in (11-7) we have on the R.H.S.

$$\frac{\rho}{4\pi} \int dR_{13} R_{13} \sin \theta_{13} d\phi_{13} \sum_{l_3} \sum_{m_3} \sum_{l'} \sum_{m'} E_h(L_1 l_3 l, R_{13}) Y_l^m(\Omega_{13}) \cdot$$

$$Y_{l'}^{m'*}(\Omega_{32}) E_f(l_3 L_2 l', R_{32}) C(l_3 L_2 l', m_3 M_2 M') \quad (11-11)$$

where $l'_2 = L_2$

$$m'_2 = M_2$$

Again on the R.H.S. of equation (11-10) we have

$$(-1)^{M_1} \sum_{\bar{L}} \sum_{\bar{L}} E_Y(L_1 L_2 \bar{L}, R_{12}) C(L_1 L_2 \bar{L}, \underline{M}_1 M_2 \bar{m}) Y_{\bar{L}}^{\bar{m}*}(\Omega_{12}) \quad (11-12)$$

When we multiply both sides of equation (11-12) by $Y_L^M(\Omega_{12})$ and integrate over Ω_{12} , we have

$$\int d\Omega_{12} Y_L^M(\Omega_{12}) \int d\omega_2 Y_{L_2}^{M_2}(\omega_2) \int d\omega_1 Y_{L_1}^{M_1}(\omega_1) \left(\frac{Y}{Y_1} - 1 \right) =$$

$$(-1)^{M_1} E_Y(L_1 L_2 L, R_{12}) C(L_1 L_2 L, \underline{M}_1 M_2 M) \quad (11-13)$$

where here again $\bar{L} = L$ and $\bar{m} = M$. And from equation (11-11) the R.H.S. gives

$$\frac{\rho}{4\pi} \int dR_{13} R_{13} d\theta_{13} \sin \theta_{13} d\phi_{13} d\theta_{12} \sin \theta_{12} d\phi_{12} \cdot$$

$$\sum_{l_3} \sum_{m_3} \sum_{l'} \sum_{m'} E_h(L_1 l_3 l, R_{13}) E_f(l_3 L_2 l', R_{32}) \cdot$$

$$l \quad m$$

$$C(L_1 l_3 l, M_1 m_3 m) C(l_3 L_2 l', m_3 M_2 m') Y_L^m(\Omega_{13}) Y_{l'}^{m'}(\Omega_{32}) \cdot$$

$$Y_L^M(\Omega_{12}) \quad (11-14)$$

Finally, the result would be

$$(-1)^{M_1} E_Y(L_1 L_2 L, R_{12}) C(L_1 L_2 L, M_1 M_2 M) =$$

$$\frac{\rho}{\pi} \int dR_{13} R_{13} d\theta_{13} \sin \theta_{13} d\phi_{13} d\theta_{12} \sin \theta_{12} d\phi_{12} \cdot$$

$$\sum_{l_3} \sum_{m_3} \sum_{l'} \sum_{m'} E_h(L_1 l_3 l, R_{13}) E_f(l_2 L_3 l', R_{32}) \cdot$$

$$C(L_1 l_3 l, M_1 m_3 m) C(l_3 L_2 l', m_3 M_2 m') Y_{\ell}^m(\Omega_{13}) \cdot$$

$$Y_{l'}^{m'*}(\Omega_{32}) Y_L^M(\Omega_{12})$$

This is a 5 fold integral which must be computed by computer.

$$\text{RHS} = \sum_{\bar{l}} \sum_{\bar{m}} E_Y(L_1 L_2 \bar{l} R_{12}) C(L_1 L_2 \bar{l} M_1 M_2 \bar{m})$$

$$(-1)^{M_1} Y_{\bar{l}}^{\bar{m}*}(\Omega_{12}) = \sum_m \sum_{\bar{l}} E_Y(L_1 L_2 \bar{l} R_{12}) C(L_1 L \bar{l} M_1 M_2 \bar{m}) \cdot$$

$$(-1)^{M_1} \left(\frac{2l+1}{4\pi}\right)^{\frac{1}{2}} \left(\frac{(\bar{l}-\bar{m})!}{(\bar{l}+\bar{m})!}\right)^{\frac{1}{2}} e^{iM\phi} P_{\bar{l}}^{\bar{m}}(\chi = \cos \theta)$$

$$= \frac{\rho}{4\pi} \int dR_{13} R_{13} d\theta_{13} \sin \theta_{13} d\phi_{13} \sum_{l_3} \sum_{m_3} E_h(L_1 l_2 l R_{13}) \cdot$$

$$\sum_{l_1} \sum_{m_1} \sum_{l} \sum_{m}$$

$$C(L_1 l_3 l M_1 m_3 m) E_f(l_3 L_2 l' R_{32}) C(l_3 L_2 l' m_3 M_2 m') \cdot$$

$$Y_{\bar{l}}^{\bar{m}}(\Omega_{13}) Y_{l'}^{m'*}(\Omega_{32}) (-1)^{M_1}$$

In the special case when

$$\phi_{12} = \frac{\pi}{2}, \quad \phi_{12} = 0$$

$$\theta_{12} = \frac{\pi}{2}, \quad \theta_{12} = 0$$

$$E_Y(l_1 l_2 m_1 R_{12}) = \frac{1}{4\pi} \sum_1 \left(\frac{2l+1}{4\pi} \right)^{\frac{1}{2}} E_Y(l_1 l_2 1, R_{12}) \cdot$$

$$C(l_1 l_2 1 m_1 \bar{m}_1 0)$$

$$\text{since } Y_0^0 = \left(\frac{1}{4\pi} \right)^{\frac{1}{2}}$$

$$Y_1^{-1} \left(\frac{\pi}{2}, \frac{\pi}{2} \right) = - \left(\frac{3}{8\pi} \right)^{\frac{1}{2}}$$

$$Y_1^m(\theta, \phi) = \left(\frac{2l+1}{4\pi} \right)^{\frac{1}{2}} \left(\frac{(l-m)!}{(l+m)!} \right)^{\frac{1}{2}} e^{im\phi} P_1^m(x = \cos \theta)$$

where

$$P_1^m(x) = (-1)^m \sin \theta)^m \frac{1}{2^{l+1} l!} \left(\frac{d}{dx} \right)^{l+m} (x^2-1)^l$$

In the case when

$$\begin{cases} \phi_{12} = 0 \\ \theta_{12} = 0 \end{cases}$$

$$Y_{\bar{l}}^{\bar{m}}(\theta_{12}, \phi_{12}) = Y_{\bar{l}}^{\bar{m}}(0, 0) = \left(\frac{2\bar{l}+1}{4\pi} \right)^{\frac{1}{2}} \frac{(\bar{l}-\bar{m})!}{(\bar{l}+\bar{m})!} \cdot 1 \cdot 0$$

for $\bar{m} \neq 0$

$$\text{For } \phi_{12} = 0 \quad \bar{m} = 0 \quad \theta_{12} = 0$$

$$M_1 = -M_2, C(L_1 L_2 \bar{M}_1 M_2 0) = C(L_1 L_2 \bar{M}_1 M_1 0)$$

But $P_0(x) = 1,$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2-1) = 1 \quad \text{as } x = 1$$

$$P_3(x) = \text{-----}$$

So

$$\text{RHS} \rightarrow E_Y(l_1 l_2^{\ell} R_{12}) + E_Y(l_1 l_2 m_1 R_{12})$$

$$\sum_m \bar{m} = -\bar{l} \rightarrow \bar{\ell} = 0 \text{ only}$$

RHS = $4\pi E_Y(L_1 L_2 M_1 r_{12})$ for $\phi_{12} = 0$ and $\theta_{12} = 0$, depending on which value of M was chosen.

CHAPTER XII

FOURIER TRANSFORMATION

In the convolution chapter we saw a five fold integral is existing which takes lot of computer time. So we decided to use Fourier transformation in order to cut the time.

Assume

$$B(l_2) = \frac{Y(l_2)}{Y_0(r_{12})} - 1 = \sum_{l'} \sum_{m'} E_Y(l' l'' l, r_{12}) C(l' l'' m' m) \cdot$$

$$Y_{l'}^{m'}(\omega_1) Y_{l''}^{m''}(\omega_2) Y_{l_1}^{m*}(\omega_{12}) \quad (12-1)$$

Also

$$B(l_2) = \frac{Y(l_2)}{Y_0(r_{12})} - 1 = \rho \int d^3 h(l_3) g_O f(32)$$

$$= \frac{\rho}{4\pi} \int d^3 x_3 h(l_3) = g_O f(32) \quad (12-2)$$

And

$$\tilde{B}(k) = \sum_{l_1 l_2 l} \sum_{m_1 m_2 m} \tilde{E}_Y(l_1 l_2 l, k) C(l_1 l_2 l, m_1 m_2 m) Y_{l_1}^{m_1}(\theta_k \phi_k) Y_{l_2}^{m_2}(\omega_1) Y_{l_2}^{m_2}(\omega_2)$$

$$\tilde{E}_Y(l_1 l_2 l, k) = i^l (4\pi) \int_0^\infty dr r^2 J_l(kr) E_Y(l_1 l_2 l, r) \quad (12-3)$$

where $J_l(kr)$ is the spherical Bessel function and $\tilde{B}(k)$ is the Hankel transformation of $B(l_2)$.

Now

$$\tilde{\Sigma E}_Y C Y_{1_1}^{m_1} Y_{1_2}^{m_2} Y_1^{m*} = \frac{\rho}{4\pi} \{ (\tilde{\Sigma E}_h (l'_1 l'_2 l' k) C(l'_1 l'_2 l', m'_1, m'_2 m') \cdot$$

$$Y_{1_1}^{m'_1}(\omega_1) Y_{1_2}^{m'_2}(\omega_2) Y_{1'}^{m'^*} (\Sigma E_f (l''_1 l''_2 l'', k) C(l''_1 l''_2 l'', k, m''_1 m''_2 m) \cdot$$

$$Y_{1_1}^{m''_1}(\omega_3) Y_{1_2}^{m''_2}(\omega_2) Y_{1''}^{m''} \} \text{ implies } \begin{matrix} m'_1 = m''_1 \\ l'_1 = l''_1 \end{matrix}$$

also

$$\int_{\ell} ik \cdot (R_{13} - R_{32}) dR_{12} = \int dR_{12} \frac{ik \cdot R_{13}}{1} - \frac{ik R_{32}}{1}$$

and

$$\int dR_{13} dR_{12} h(13) g^{\text{of}}(32) \frac{ik \cdot R_{13}}{1} - \frac{ik R_{32}}{1} =$$

$$\int dR_{13} h(13) \frac{ik \cdot R_{13}}{1} \int dR_{32} g^{\text{of}}(32) \frac{-ik \cdot R_{32}}{1}$$

$$= \tilde{\Sigma E}_h (l'_1 l'_2 l', k) C(l'_1 l'_2 l', m'_1 m'_2 m) Y_{1_1}^{m'_1}(\omega_1) Y_{1_2}^{m'_2}(\omega_3) Y_{1_1}^{m'^*}(\theta_k \phi_k) \cdot$$

$$\tilde{\Sigma E}_f (l''_1 l''_2 l'', k) C(l''_1 l''_2 l'', m''_1 m''_2 m'') \cdot$$

$$Y_{1_1}^{m''_1}(\omega_3) Y_{1_2}^{m''_2}(\omega_2) Y_{1_2}^{m''}(\theta_k \phi_k)$$

Also

$$\tilde{E}_Y^{1, \lambda}(m, n, l, k) C(m, n, l, \nu \lambda) Y_n^m(\omega_1) Y_\nu^n(\omega_2) Y_l^1(k) =$$

$$= \frac{4\pi}{\rho} \{ \tilde{E}_{n_1 \nu_1}^{1, \lambda} E_h(m_1 n_1 l_1, k) C(m_1 n_1 l_1, \mu_1 \nu_1 \lambda_1) Y_{n_1}^{m_1}(\omega_1) \cdot$$

$$\tilde{E}_{n_2 \nu_2}^{1, \lambda} E_h^*(m_1 n_2 l_2, k) C(n_1 n_2 l_2, \nu_1 \nu_2 \lambda_2) Y_{n_2}^{m_2}(\omega_2) \cdot$$

$$\tilde{Y}_{l_2}^{1, \lambda}(k) \} \text{ implies } m_1 = m, n_1 = n$$

$$n_2 = n, -\nu_2 = \nu$$

Since

$$\tilde{Y}_{\nu_2}^{n_2}(\omega_2) = (-1)^{\nu_2 - \nu_2} Y_{\nu_2}^{n_2}(\omega_2)$$

then

$$\tilde{E}_Y^{1, \lambda}(m, n, l, k) C(m, n, l, \nu \lambda) Y_l^1(k) =$$

$$\frac{4\pi}{\rho} \{ \tilde{E}_h^{n_1}(m, n_1 l_1 k) C(n, m_1 l_1 \mu, \nu_1 \lambda_1) Y_{l_1}^{n_1} \tilde{E}_{\nu_1}^{*}(n_1 n, l_2, k) \cdot$$

$$C(n_1 n, l_2, \nu_1 - \nu_1 \lambda_2) (-1)^{\nu_1} Y_{l_2}^{n_2} \}$$

If we look at $Y_l^1(\theta, \phi, k)$, let $\theta_k = 0$, then $Y_l^1 \neq 0$ if $\lambda = 0$.

$$Y_0^1 = \left(\frac{2}{2l+1} \right)^{\frac{1}{2}} \frac{1}{l} \left(\frac{d}{dx} \right)_l (x^2 - 1)^l$$

where $\frac{1}{l} \left(\frac{d}{dx} \right)_l (x - 1)^l = 1$, for $l = 1, 2, 3$,

so

$$Y_1^0 = \left(\frac{2}{4\pi}\right)^{\frac{1}{2}}$$

Then

$$Y_1^\lambda(\theta_k, \phi_k) = Y_1^\lambda(0, \phi_k) = \left(\frac{2}{4\pi}\right)^{\frac{1}{2}} \text{ when } \lambda = 0$$

$$\sum \tilde{E}_Y(m, n, l, k) C(m, n, l, \mu, \nu, 0) \left(\frac{2}{4\pi}\right)^{\frac{1}{2}} = 4\pi \tilde{E}_Y(m, n, \mu, k)$$

where $\tilde{E}_Y(m, n, \mu, k)$ is in body-fixed coordinates.

$$4\pi \tilde{E}_Y(m, n, \mu, k) = \frac{\rho}{4\pi} \sum_{\mathbf{h}} \tilde{E}_{\mathbf{h}}(m, n, \mu, k) \cdot$$

$$\{\tilde{E}_{\mathbf{f}}(n_1, n, \mu, k)^* (-1)^\nu\} (4\pi)^2$$

or

$$\tilde{E}_Y(m, n, \mu, k) = \rho \sum_{\mathbf{h}} \tilde{E}_{\mathbf{h}}(m, n, \mu, k) \tilde{E}_{\mathbf{f}}^*(n_1, n, \nu, k) (-1)^\nu$$

so

$$\tilde{E}_Y(l, l', m, k) = \rho \sum_{\mathbf{l}} \tilde{E}_{\mathbf{h}}(l, l'', m, k) (-1)^m E_{\mathbf{f}}(l'', l', m, k)$$

where

$$\tilde{E}_{\mathbf{f}}(m, n, l, k) = i^l (4\pi) \int_0^\infty dr r^2 J_l(kr) E_{\mathbf{f}}(m, n, l, r)$$

$$\tilde{E}_{\mathbf{f}}^*(m, n, l, k) = (-1)^l i^l (4\pi) \int_0^\infty dr r^2 J_l(kr) E_{\mathbf{f}}^*(m, n, l, r)$$

and

$$\tilde{E}_{\mathbf{f}}^*(m, n, l, k) = (-1)^l \tilde{E}_{\mathbf{f}}(m, n, l, k)$$

CHAPTER XIII

COMPARISON

In order to compare our results with previous work, we chose Dr. Haile's (18) since we have $E_Y(l_1 l_2 m r)$ and Dr. Haile's data are $E_g(l_1 l_2 m r)$, so we transferred our data as follows.

Since

$$\frac{Y(12)}{Y^O(r_{12})} - 1 = 4\pi \sum \sum E_Y(1 \ 1' \ m \ r) Y_1^m(\omega_1) \bar{Y}_{1'}^m(\omega_2) \quad (12-1)$$

or

$$\frac{g(12) e^{-\beta u_a}}{g^O(r_{12})} - 1 = 4\pi \sum \sum E_Y(1 \ 1' \ m \ r) Y_1^m(\omega_1) \bar{Y}_{1'}^m(\omega_2) \quad (12-2)$$

and

$$g(12) = g^O(r_{12}) (4\pi \sum \sum E_Y(1 \ 1' \ m \ r) Y_1^m(\omega_1) \bar{Y}_{1'}^m(\omega_2) e^{-\beta u_a} + g^O(r_{12}) e^{\beta u_a}$$

Expanding $e^{\beta u_a}$ in spherical harmonic, we have

$$g(12) = g^O(r_{12}) (4\pi \sum \sum E_Y(1 \ 1' \ m \ r) Y_1^m(\omega_1) \bar{Y}_{1'}^m(\omega_2)) (4\pi \sum \sum E_\mu(L \ L' \ M \ r) Y_L^M(\omega_1) \bar{Y}_{L'}^M(\omega_2)) + g^O(r_{12}) e^{\beta u_a}$$

Expand the left hand side

$$4\pi \sum \sum E_g(\lambda \lambda' \ \mu \ r) Y_\lambda^\mu(\omega_1) \bar{Y}_{\lambda'}^\mu(\omega_2) = \text{L.H.S.}$$

Multiply both sides by $Y_\Lambda^{k*} Y_{\Lambda'}^{k*}$ and integrate over $d\omega_1$ and $d\omega_2$ we have

$$E_g(\Lambda \Lambda' k r) = g^0(r) (4\pi) \sum_{\substack{L \\ L'}} \sum_{\substack{M \\ M'}} \sum_{\substack{l \\ l'}} \sum_m E_\mu(L L' M r).$$

$$E_Y(l l' m r) \int d\omega_1 \int d\omega_2 Y_L^M(\omega_1) Y_{L'}^M(\omega_2) Y_l^m(\omega_1) Y_{l'}^{\bar{m}}(\omega_2).$$

$$Y_\Lambda^{k*}(\omega_1) Y_{\Lambda'}^{k*}(\omega_2) + g^0 E_\mu(\Lambda \Lambda' k r)$$

$$E_g(\Lambda \Lambda' k r) = g^0(r) \sum_{\substack{L \\ L'}} \sum_{\substack{M \\ M'}} \sum_{\substack{l \\ l'}} \sum_m E_\mu(L L' M \Lambda) E_Y(l l' m r).$$

$$\left[\frac{(2L+1)(2L'+1)(2l'+1)(2l+1)}{(2\Lambda+1)(2\Lambda'+1)} \right]^{\frac{1}{2}} C(L l \Lambda, M, m, k).$$

$$C(L l \Lambda, 0 0 0) C(L' l' \Lambda', M' m' k) C(L' l' \Lambda', 0 0 0)$$

$$+ g^0(r) E_\mu(\Lambda \Lambda' k r)$$

Here $E_\mu(l l' m r) = E_f(l l' m r)$

if $l l' m \neq 0 0 0$

and if $l l' m = 0 0 0$

$$E_f(0 0 0 r) = E_\mu(0 0 0 r) - 1.$$

In the coming pages we show our results for $E_f(\ell_1 \ell_2 m r)$, $E_Y(\ell_1 \ell_2 m r)$ and $E_g(\ell_1 \ell_2 m r)$ at different temperatures and quadrupole moments. Conclusions are given on page 150.

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	3	T=1.2770C	0.5T000	
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2 Q=5
3 T=1.27700 0.50000
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EF(L)	LP	(M)	AT	L	G	LP	ME	D	MP
1	1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2	2
3	3	3	3	3	3	3	3	3	3
4	4	4	4	4	4	4	4	4	4
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28	28	28	28	28	28	28	28	28	28
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49	49	49	49	49	49	49	49	49	49
50	50	50	50	50	50	50	50	50	50

DATA BETWEEN/END

2025 RELEASE UNDER E.O. 14176

62

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$\alpha = 1$

$T = 1.25400$ $I = 00000$

$KCI = 51$ $NC2 = 151$ $N = 101$ $HAPC = 10000E-01$ $10300E-01$ $10000F-01$

21

T=1.0000
KCI= .51, NCN = .151, N ICI, HAPC=.1999E-01 .1999E-01 .1999E-01
I

[illegible]

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DATA FILE NAME

$$T = 1.277$$
[illegible]

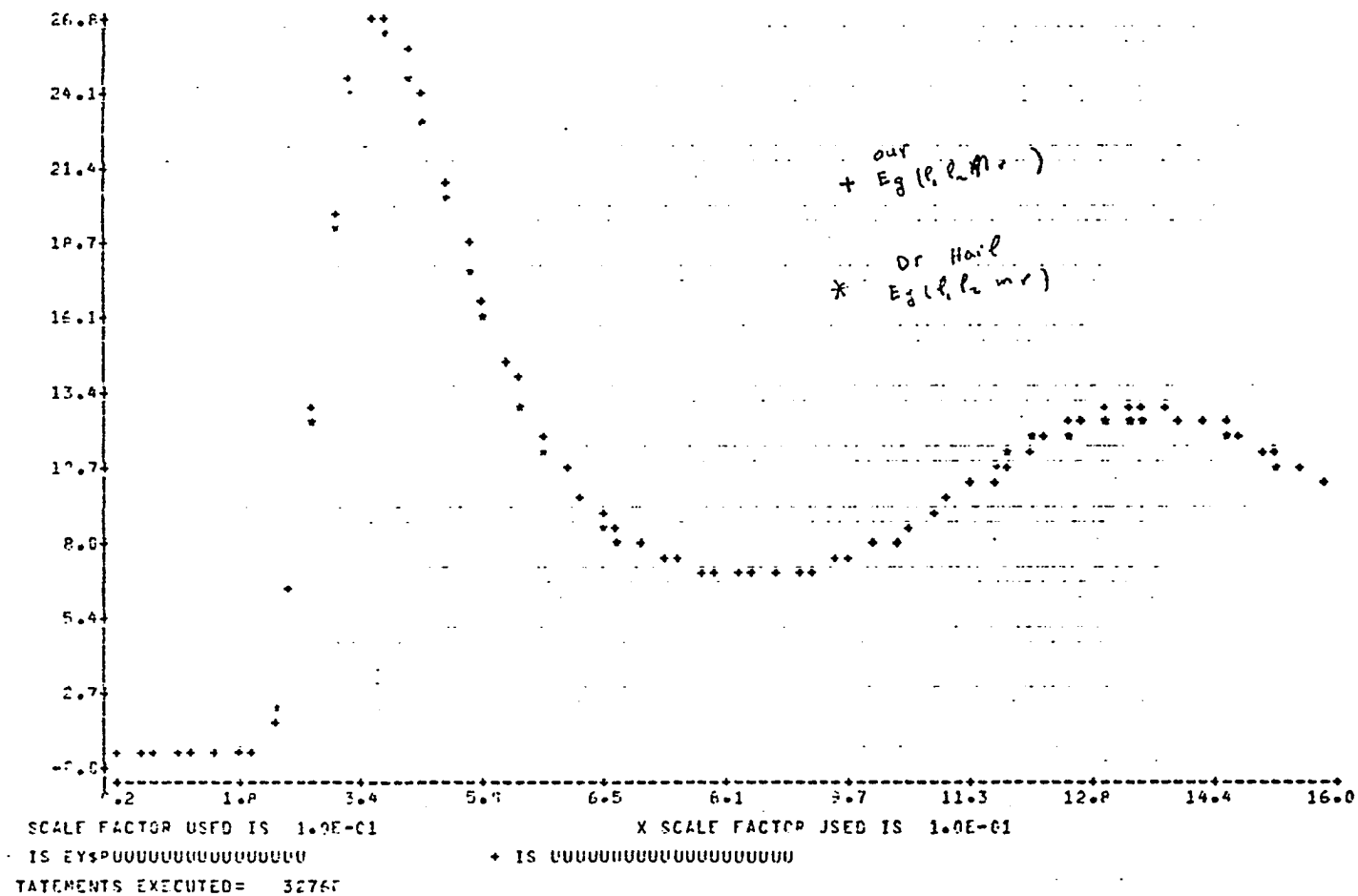
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1111111111111111111111

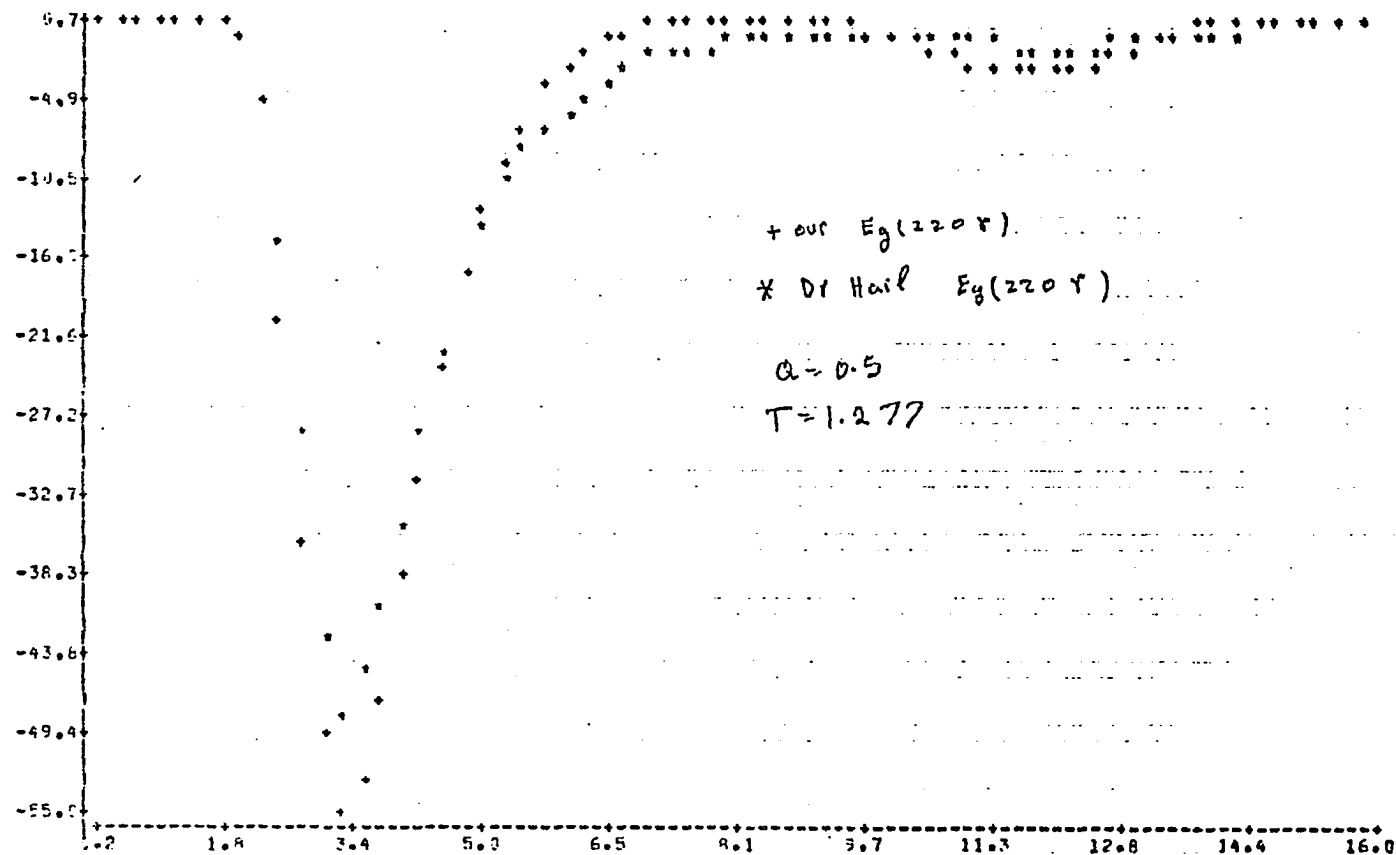
[illegible][illegible][illegible][illegible][illegible][illegible][illegible][illegible]

space fixed
 $E_f(\ell, \ell_2, \ell)$
 $a \sim 1$
 $T = 1.294$

[illegible][illegible]

SP fixed coordinate
 $E_f(l, \lambda, \rho, r)$
 $a = 1$
 $T = 1.294$





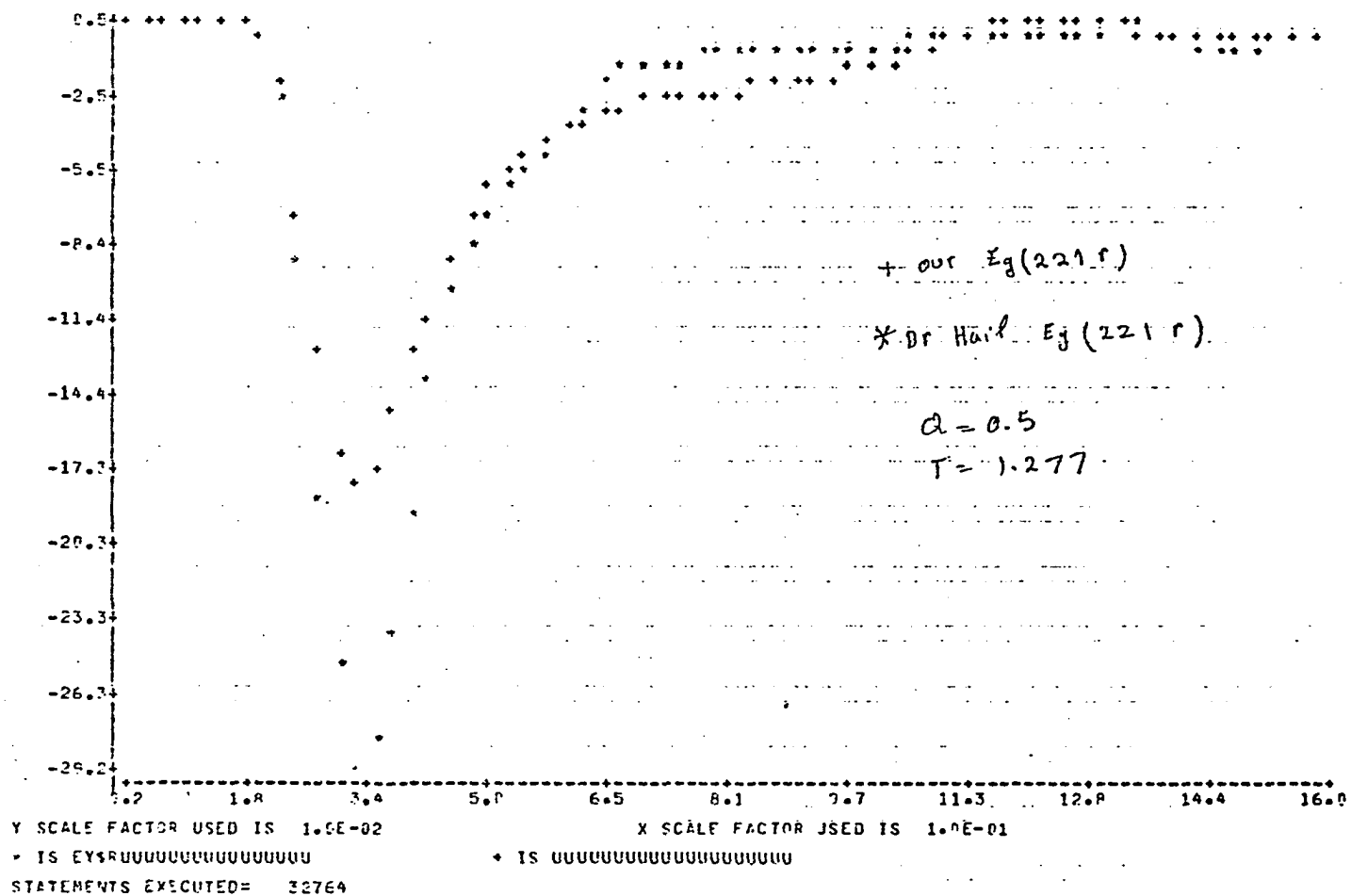
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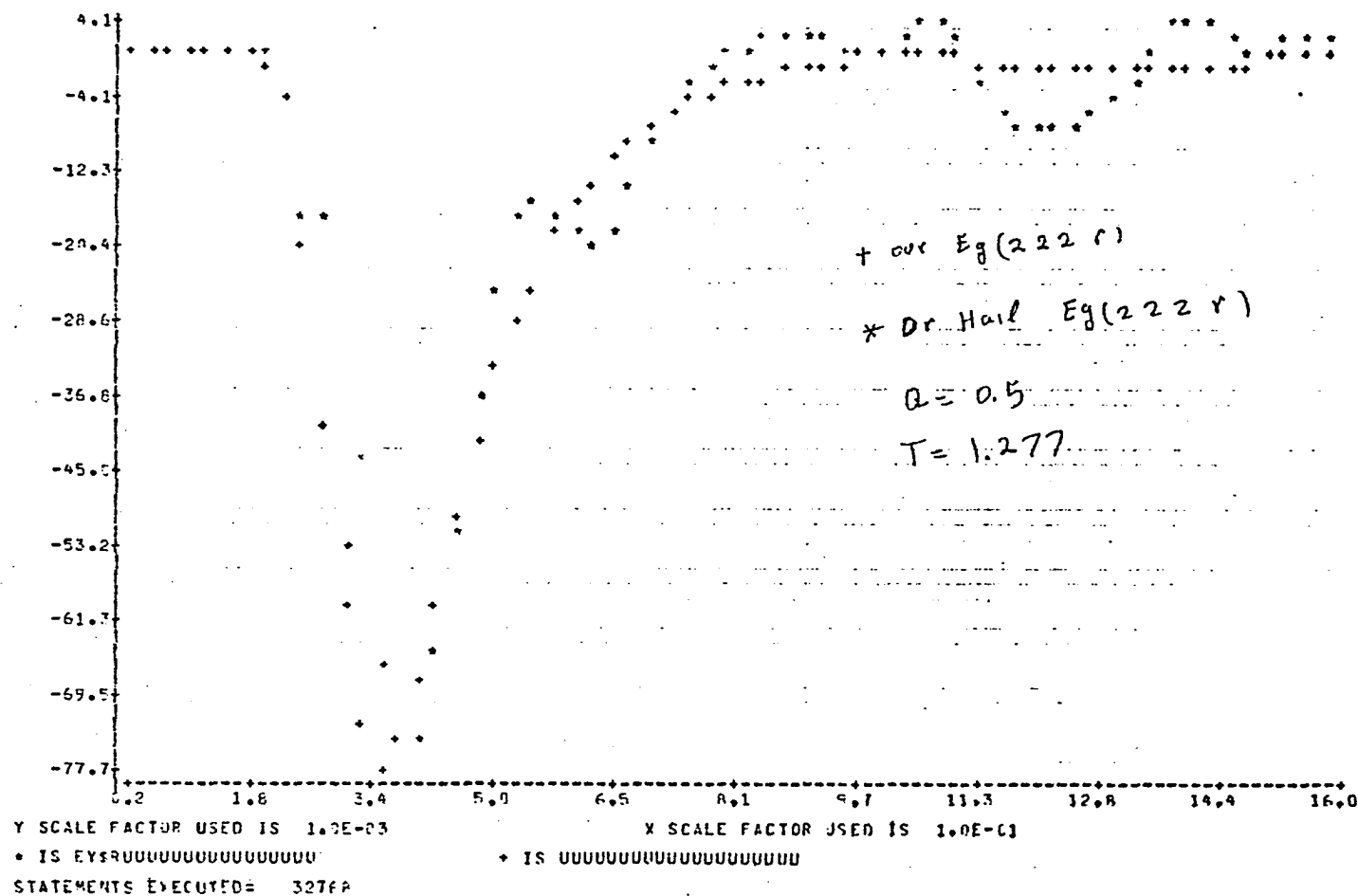
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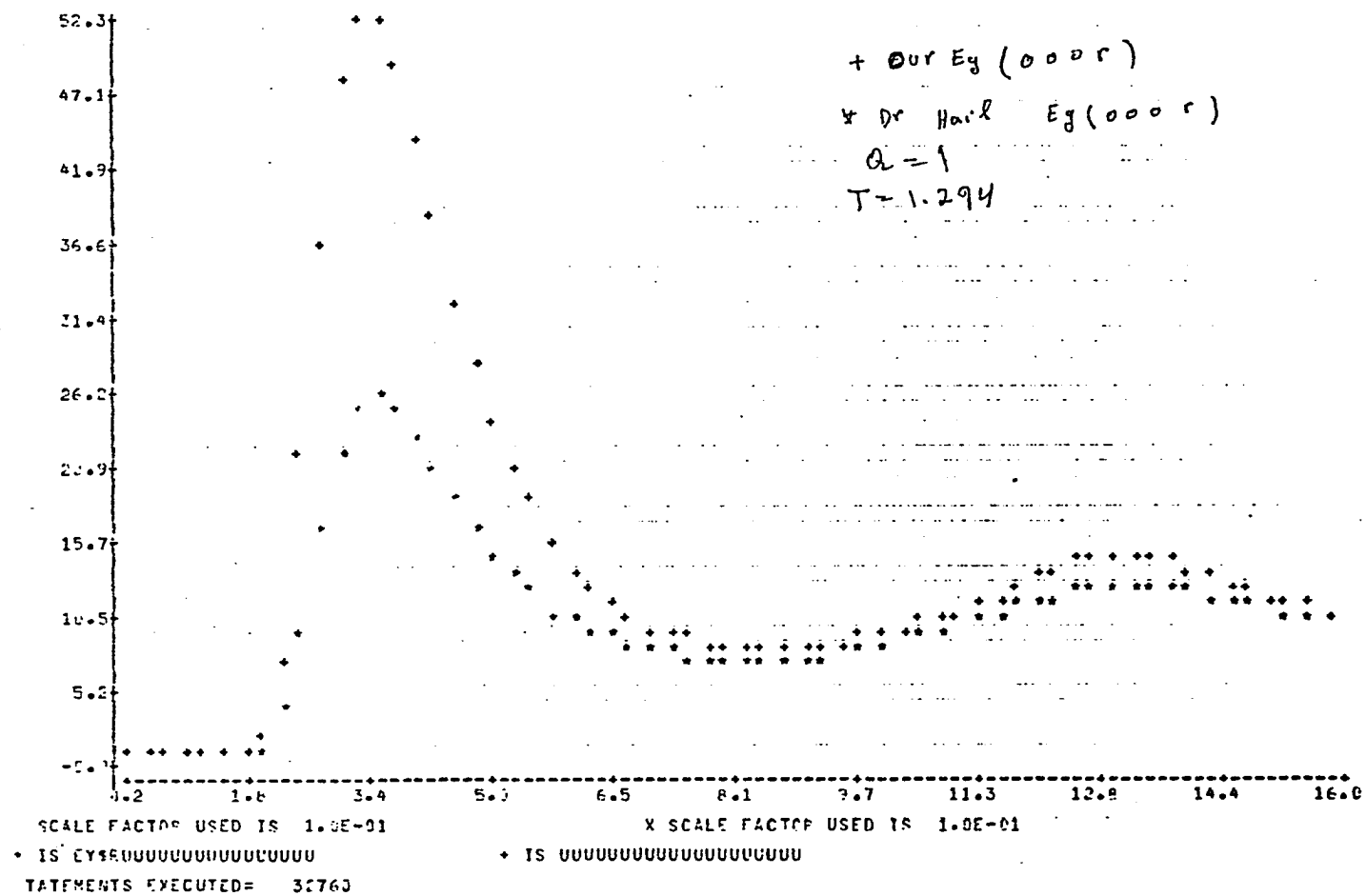
IS EY\$UUUUUUUUUUUUUUUUUUUUUU

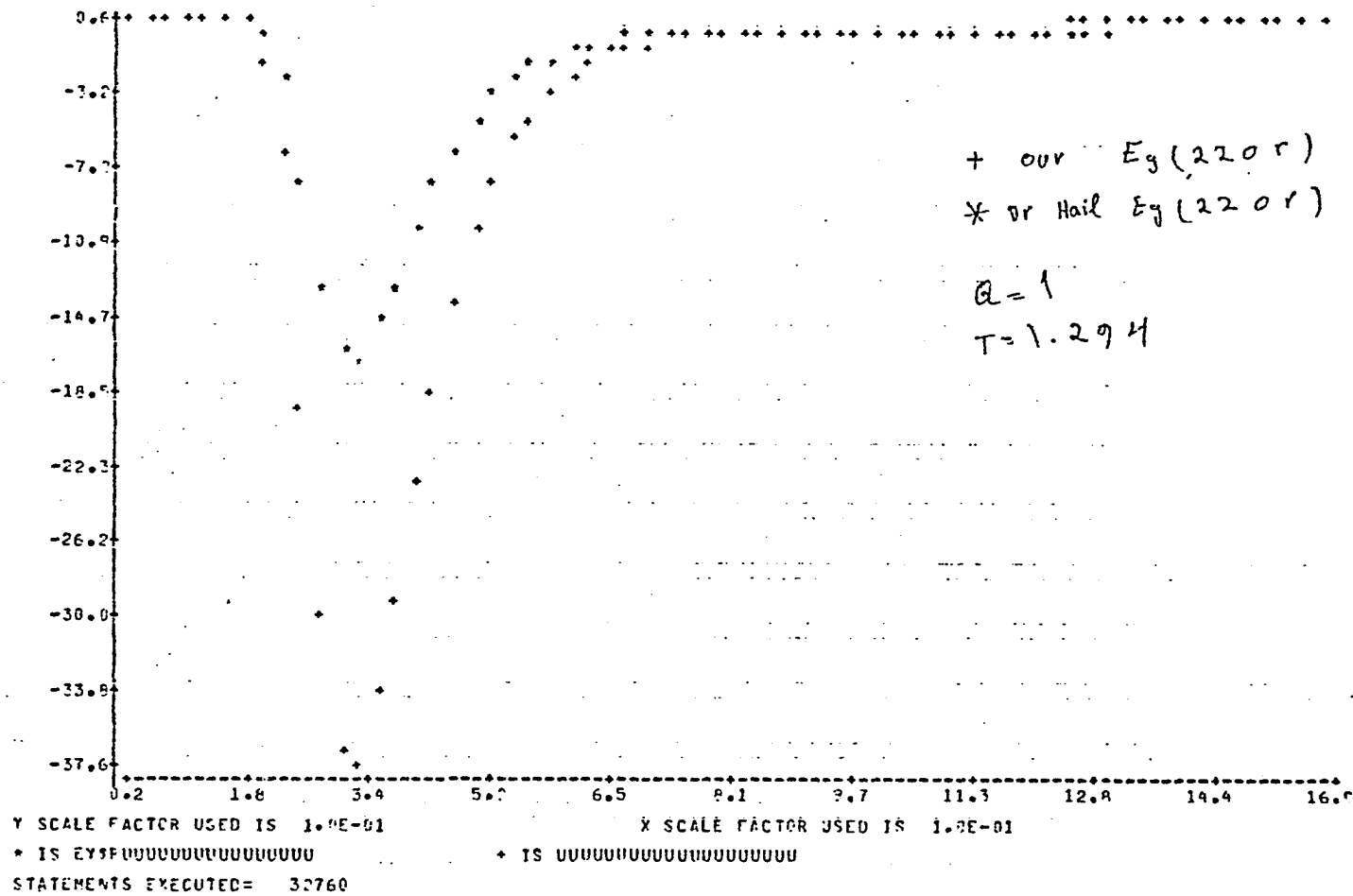
+ IS UUUUUUUUUUUUUUUUUUUUUUU

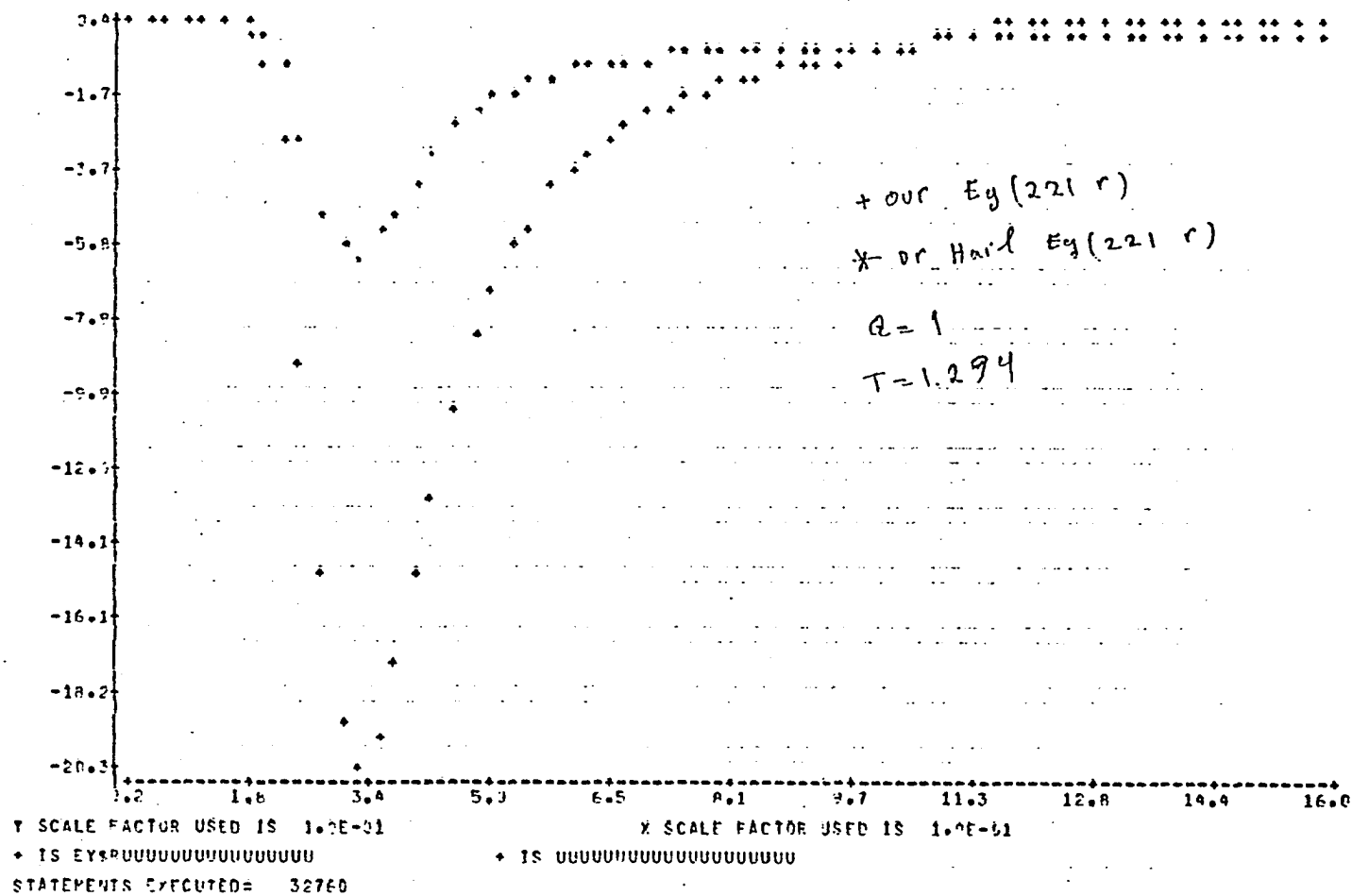
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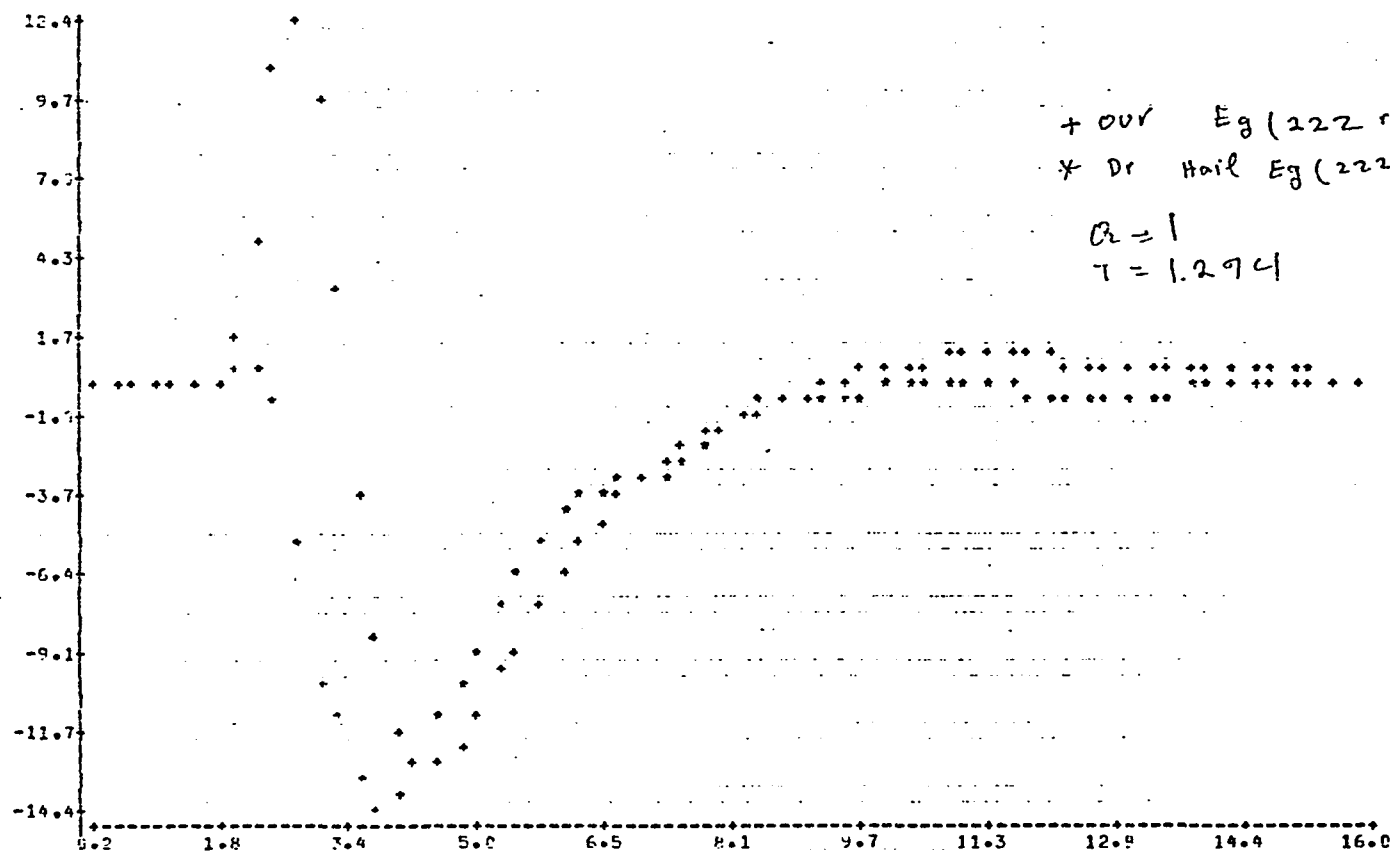












+ OVR Eg(222 r)
 * Dr Hail Eg(222 r)

$$C_2 = 1$$

$$T = 1.274$$

Y SCALE FACTOR USED IS 1.0E-02

X SCALE FACTOR USED IS 1.0E-01

* IS EY9UUUUUUUUUUUUUUUUUUUU

* IS UUUUUUUUUUUUUUUUUUUUUUU

STATEMENTS EXECUTED= 32764

FORTAN IV G1 RELEASE 2.0

MAIN

DATE = 87213

03/14/12

PAGE 0001

0001 CALL COMPA
0002 STOP
0003 END


```

0059 DO 472 I=1,100
0060 X(I) = (I-19.)+0.025
0061 472 Y(I,1) = EH(I)
0062 CALL PRPLOT (X, Y, 100, 6, 90, 6, LARL)
0063 4001 CONTINUE
0064 RETURN
0065 END
    
```

DATA UC-2275 T4

```

0001 SUBROUTINE HAILE (NPOINT)
0002 COMMON /CEYALL/ YNPUT(28,100)
0003 COMMON /CEH/ EH(100)
0004 COMMON /CEG/ G(100)
0005 DIMENSION CU(7,7), EY(7,7)
0006 COMMON /CLA/LA, LAP, MA, MB
0007 FORMAT (F10.5)
0008
0009      6
0010      61
0011      MAP=MA
0012      CU(1,1:7)=YNPUT(1,NPOINT)
0013      EY(1,1:7)=YNPUT(15,NPOINT)
0014      CU(2,1:7)=YNPUT(2,NPOINT)
0015      EY(2,1:7)=YNPUT(16,NPOINT)
0016      CU(3,1:7)=YNPUT(3,NPOINT)
0017      EY(3,1:7)=YNPUT(17,NPOINT)
0018      CU(4,1:7)=YNPUT(4,NPOINT)
0019      EY(4,1:7)=YNPUT(18,NPOINT)
0020      CU(5,1:7)=YNPUT(5,NPOINT)
0021      EY(5,1:7)=YNPUT(19,NPOINT)
0022      CU(6,1:7)=YNPUT(6,NPOINT)
0023      EY(6,1:7)=YNPUT(20,NPOINT)
0024      CU(7,1:7)=YNPUT(7,NPOINT)
0025      EY(7,1:7)=YNPUT(21,NPOINT)
0026      CU(1,8:14)=YNPUT(8,NPOINT)
0027      EY(1,8:14)=YNPUT(22,NPOINT)
0028      CU(2,8:14)=YNPUT(9,NPOINT)
0029      EY(2,8:14)=YNPUT(23,NPOINT)
0030      CU(3,8:14)=YNPUT(10,NPOINT)
0031      EY(3,8:14)=YNPUT(24,NPOINT)
0032      CU(4,8:14)=YNPUT(11,NPOINT)
0033      EY(4,8:14)=YNPUT(25,NPOINT)
0034      CU(5,8:14)=YNPUT(12,NPOINT)
0035      EY(5,8:14)=YNPUT(26,NPOINT)
0036      CU(6,8:14)=YNPUT(13,NPOINT)
0037      EY(6,8:14)=YNPUT(27,NPOINT)
0038      CU(7,8:14)=YNPUT(14,NPOINT)
0039      EY(7,8:14)=YNPUT(28,NPOINT)
0040      SUM=0.
0041      DO 4 NLP=1,5,2
0042      LP=NLP-1
0043      DO 4 NLXP=1,5,2
0044      LXP=NLXP-1
0045      SIZE=1.
0046      DO 41 NL=NLP,5,2
0047      L=NL-1
0048      DO 42 NLX=NLXP,5,2
0049      LX=NLX-1
0050      LAGEUR=(2*L+1)*(2*LX+1)*(2*LP+1)*(2*LXP+1)
0051      A=LAGEUR
0052      A=A/(2.*LA+1)/(2.*LAP+1)
0053      Q=SQRT(A)
0054      PMAX=MIN0(NL,NLP)
0055      HXMAX=MIN0(NLX,NLXP)
0056      PMAX=2*PMAX-1
0057      HXMAX=2*HXMAX-1
0058      DO 40 NH=1, HXMAX

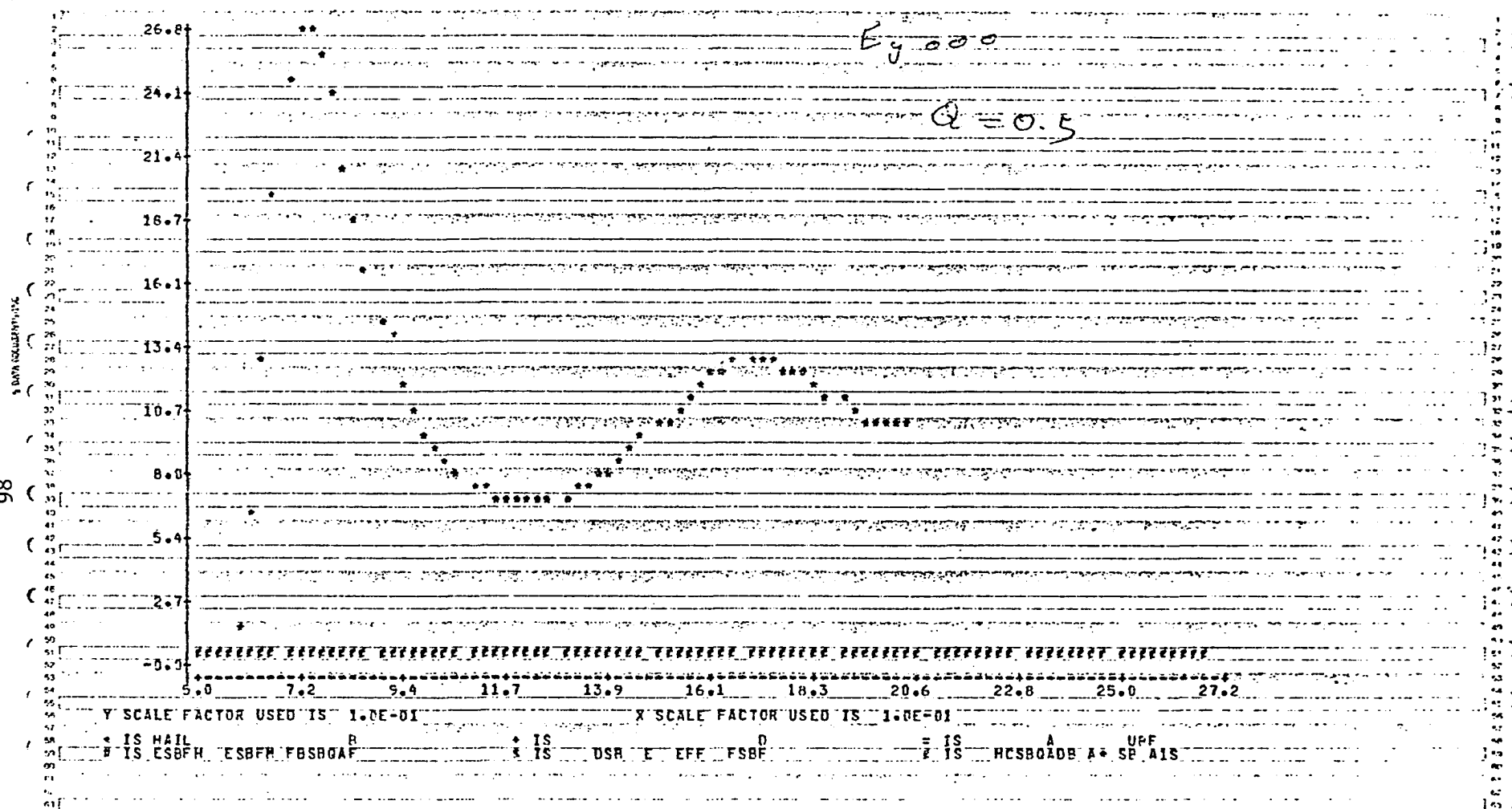
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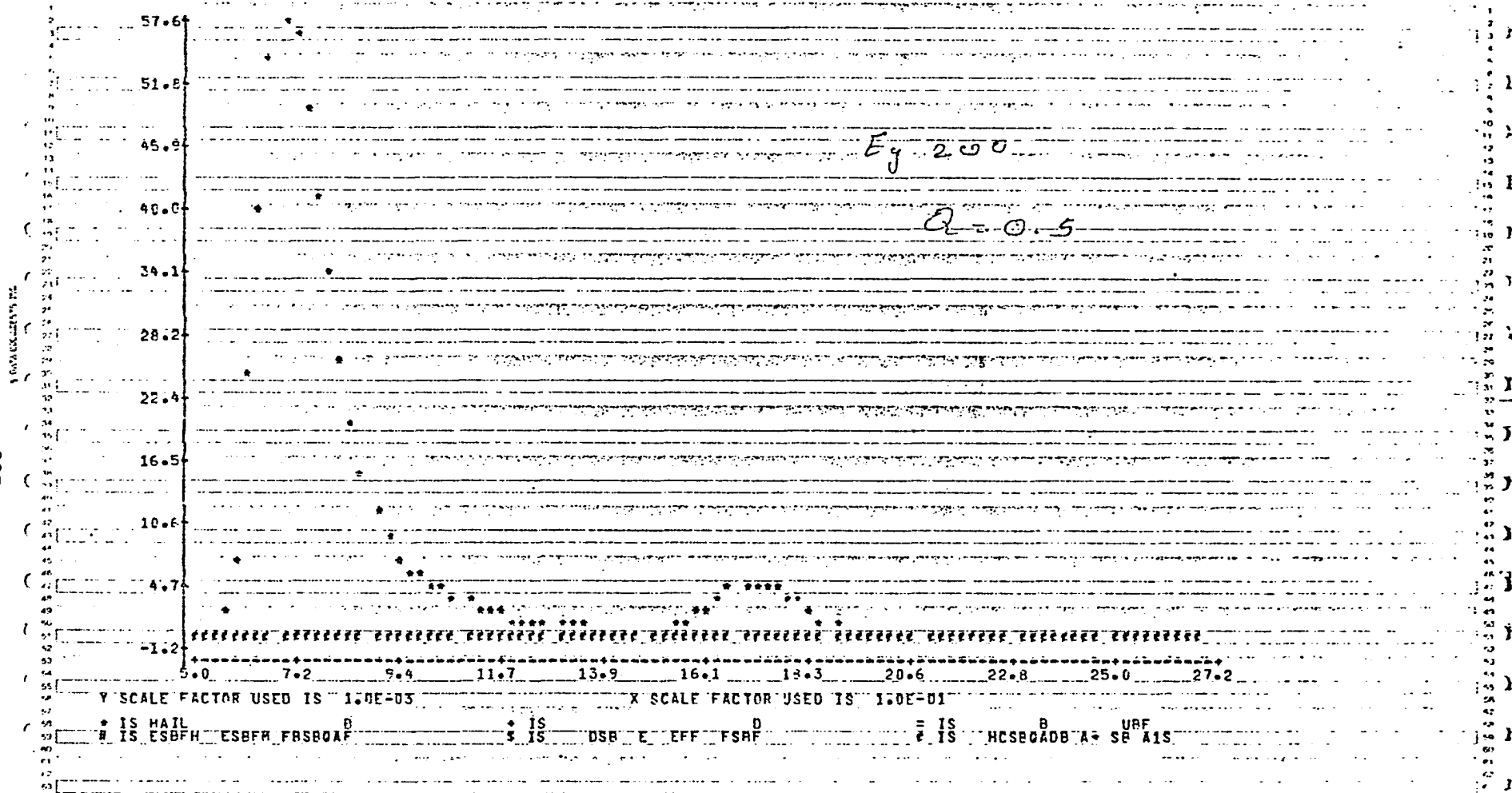
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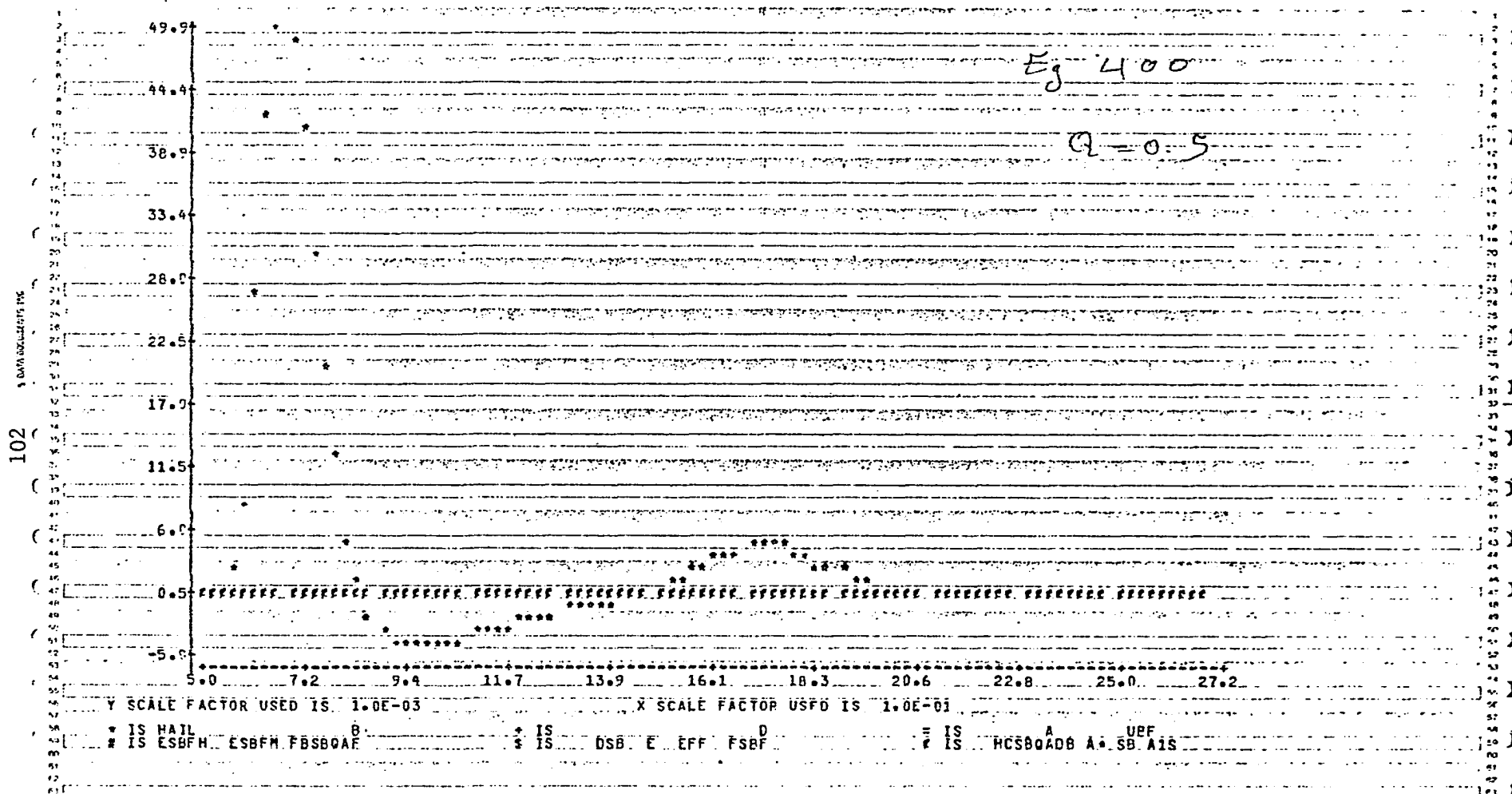
000001      FUNCTION FACT(N)
000002      FACT=1.
000003      IF (N .LE. 0) GO TO 7
000004      U=1.
000005      DO 4, I=1,N
000006      U=U*I
000007      FACT=U
000008      7      RETURN
000009      END
    
```

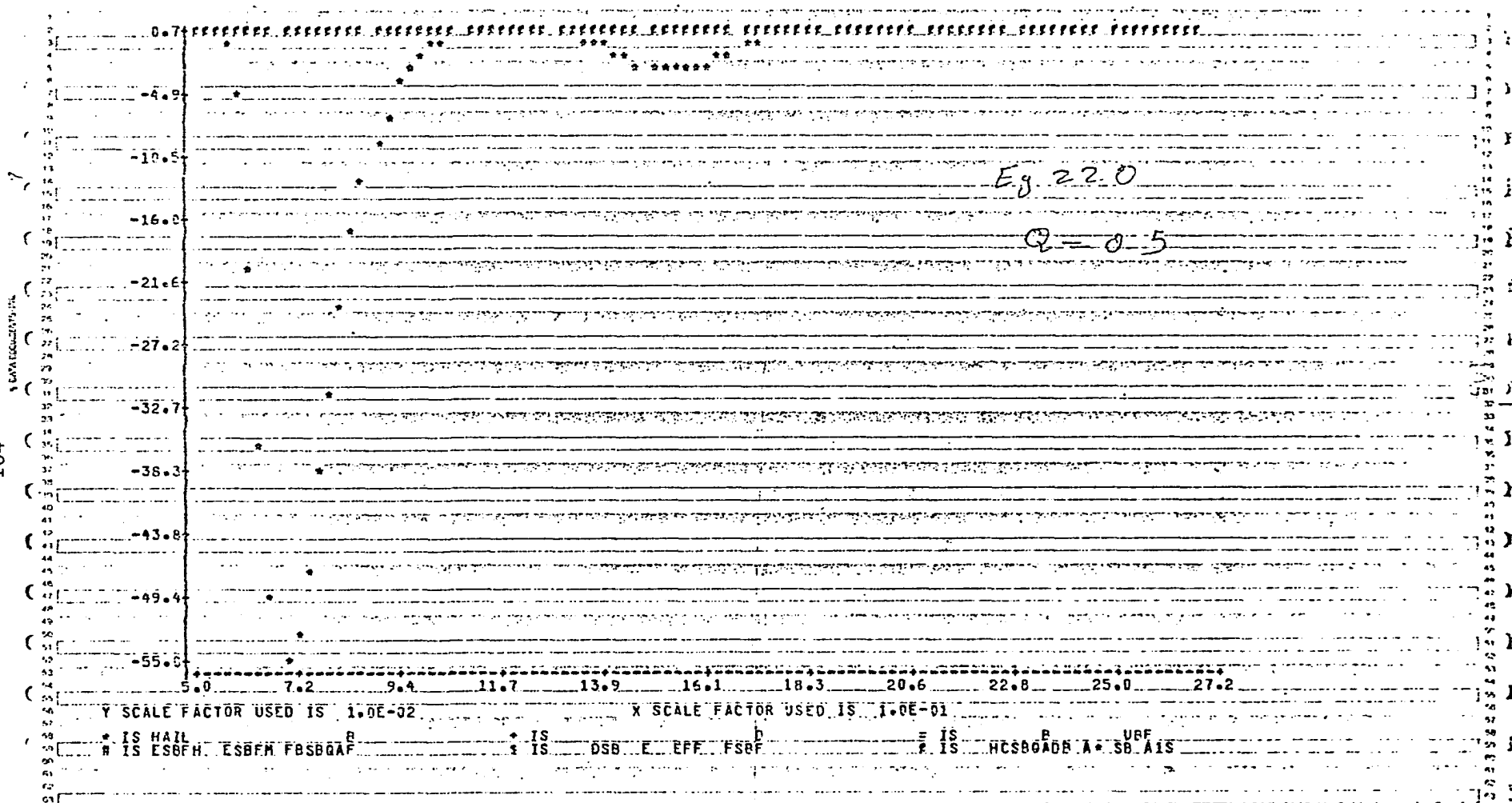
98

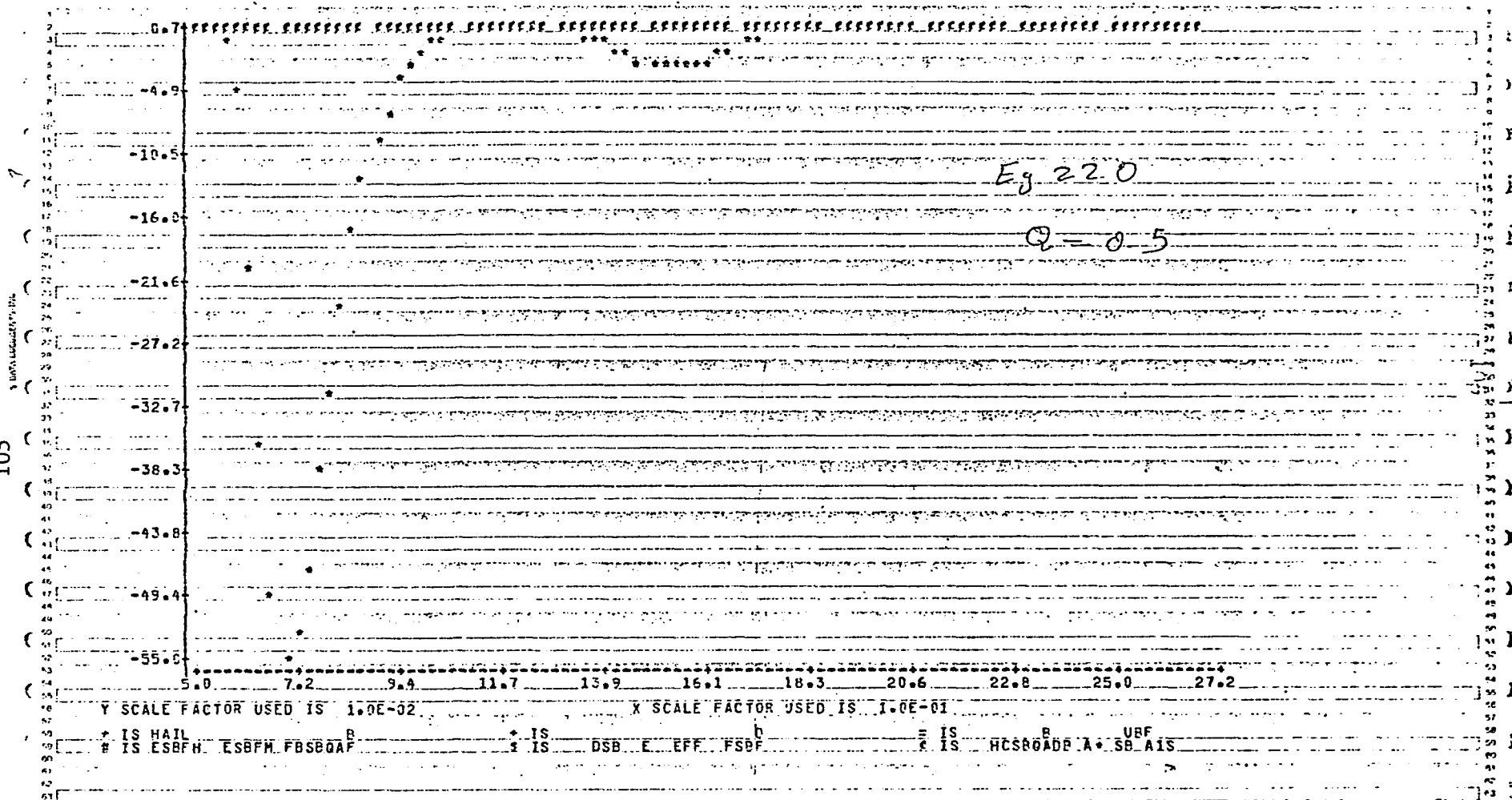


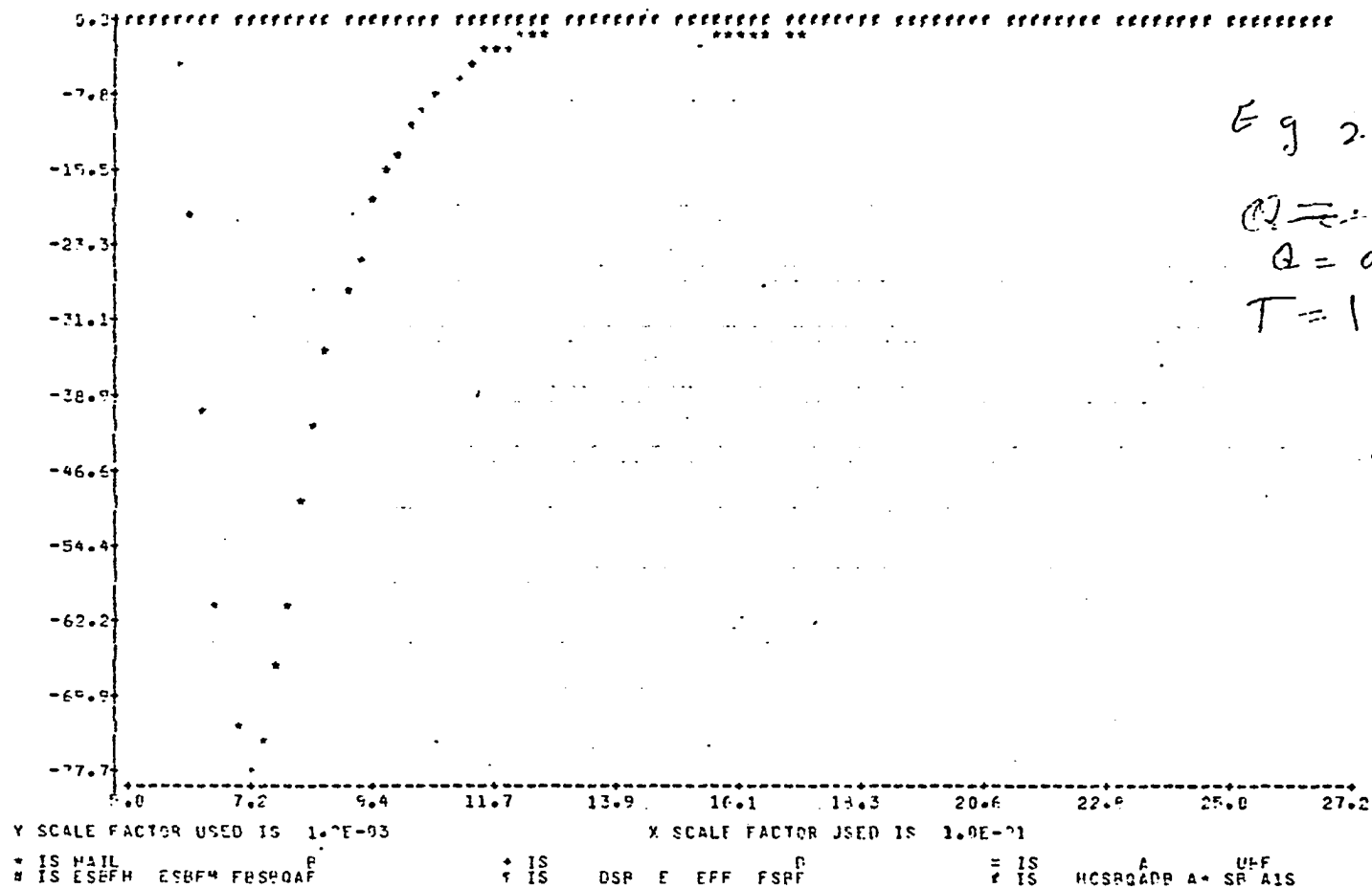
100



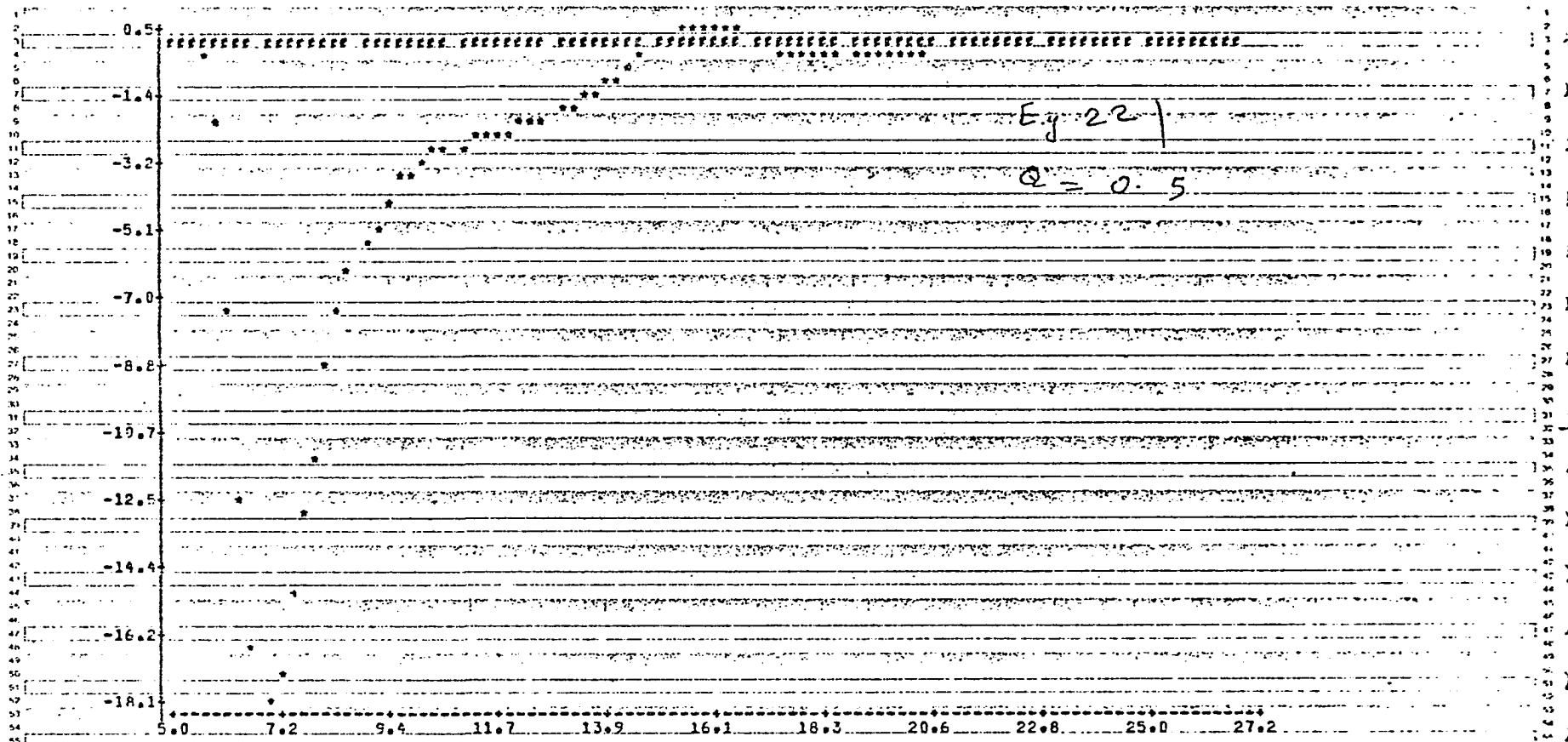








109



Y SCALE FACTOR USED IS $1.0E-02$

X SCALE FACTOR USED IS $1.0E-01$

1 IS MAIL

2 IS ESBFH ESBFH FBSBOAF

3 IS

DSB E EFF FSBF

4 IS

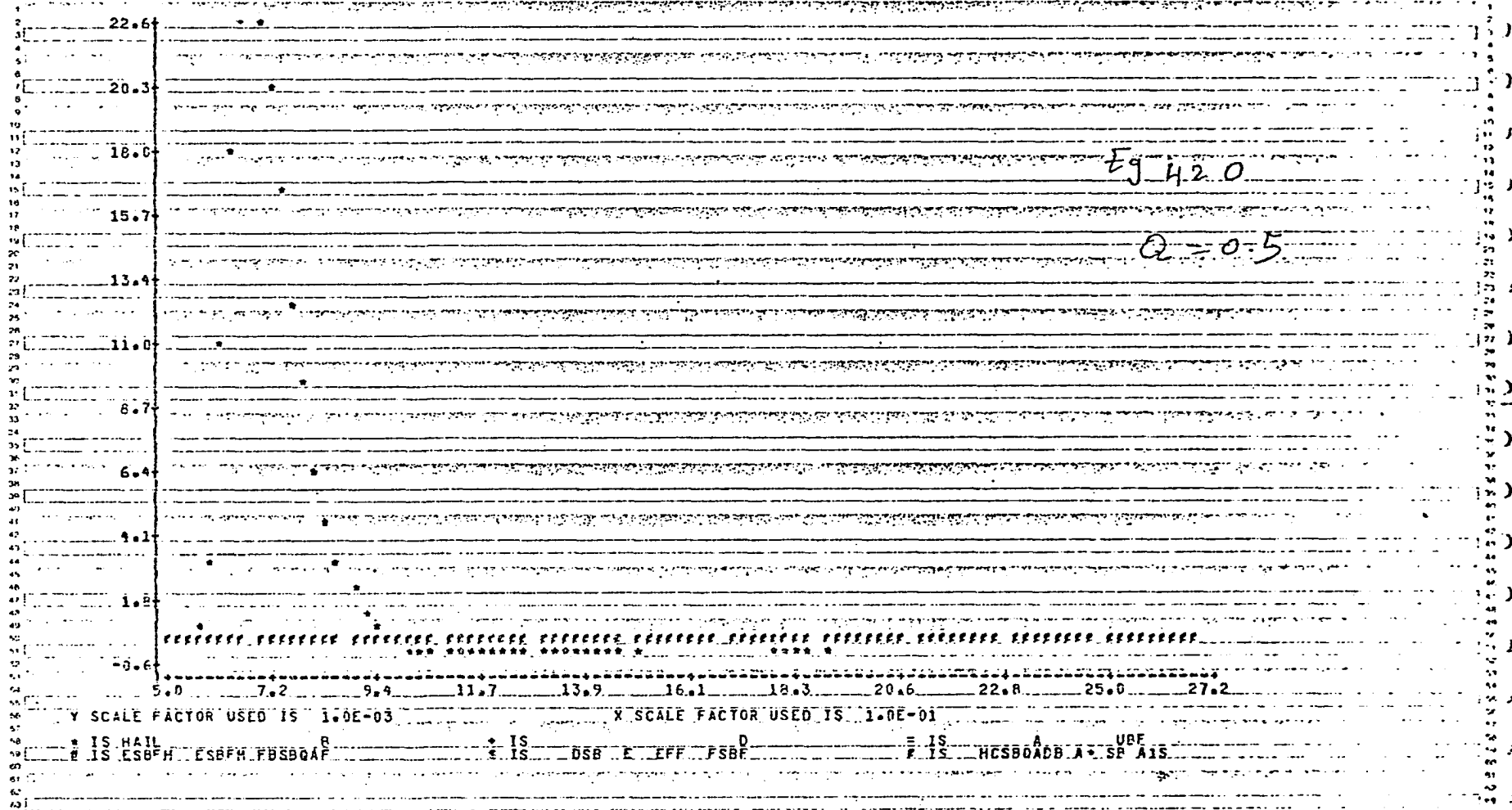
HCSBOADP A SB A1S

5 IS

UBF

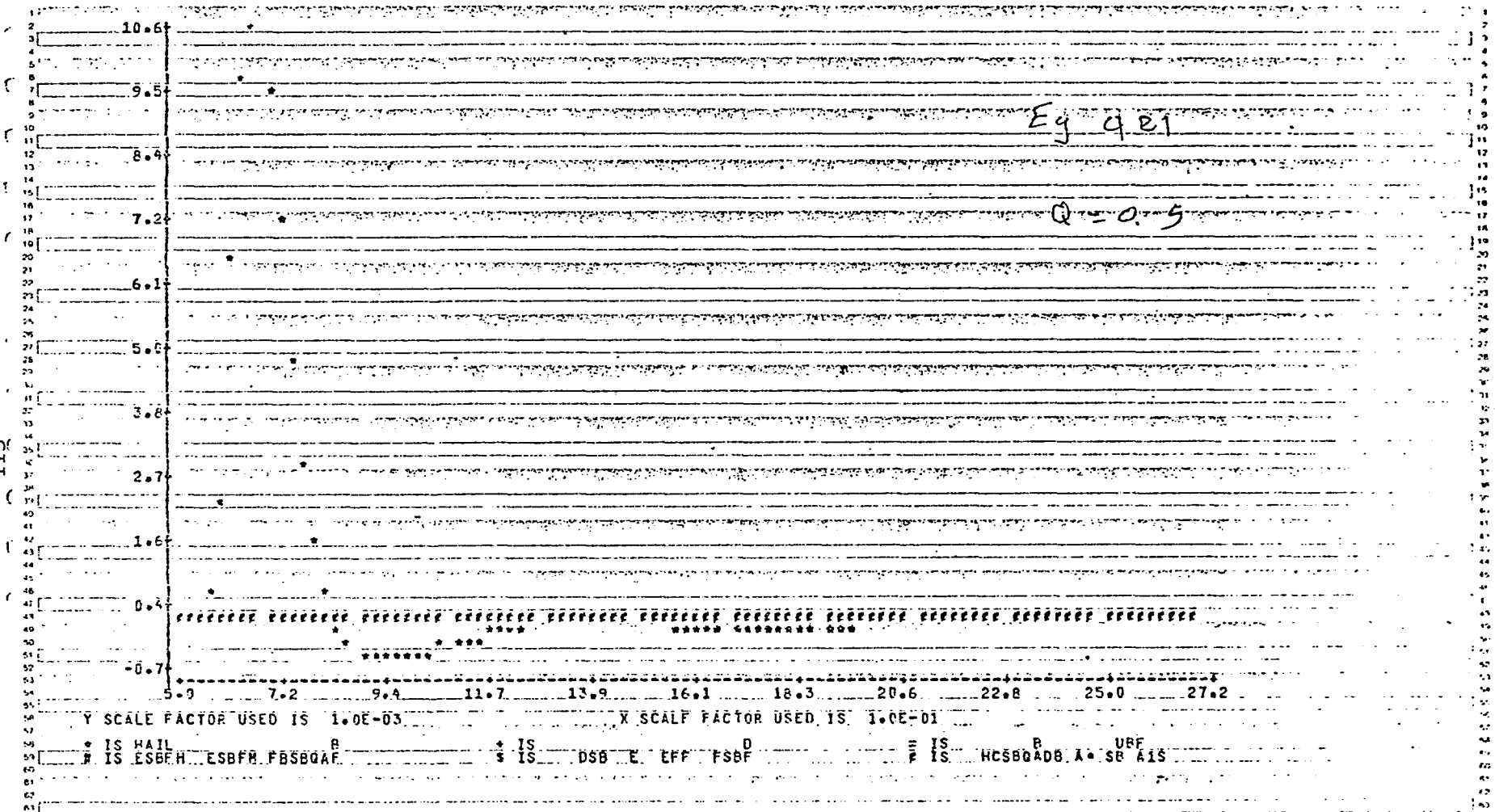
111

DATA GCM-155-15C



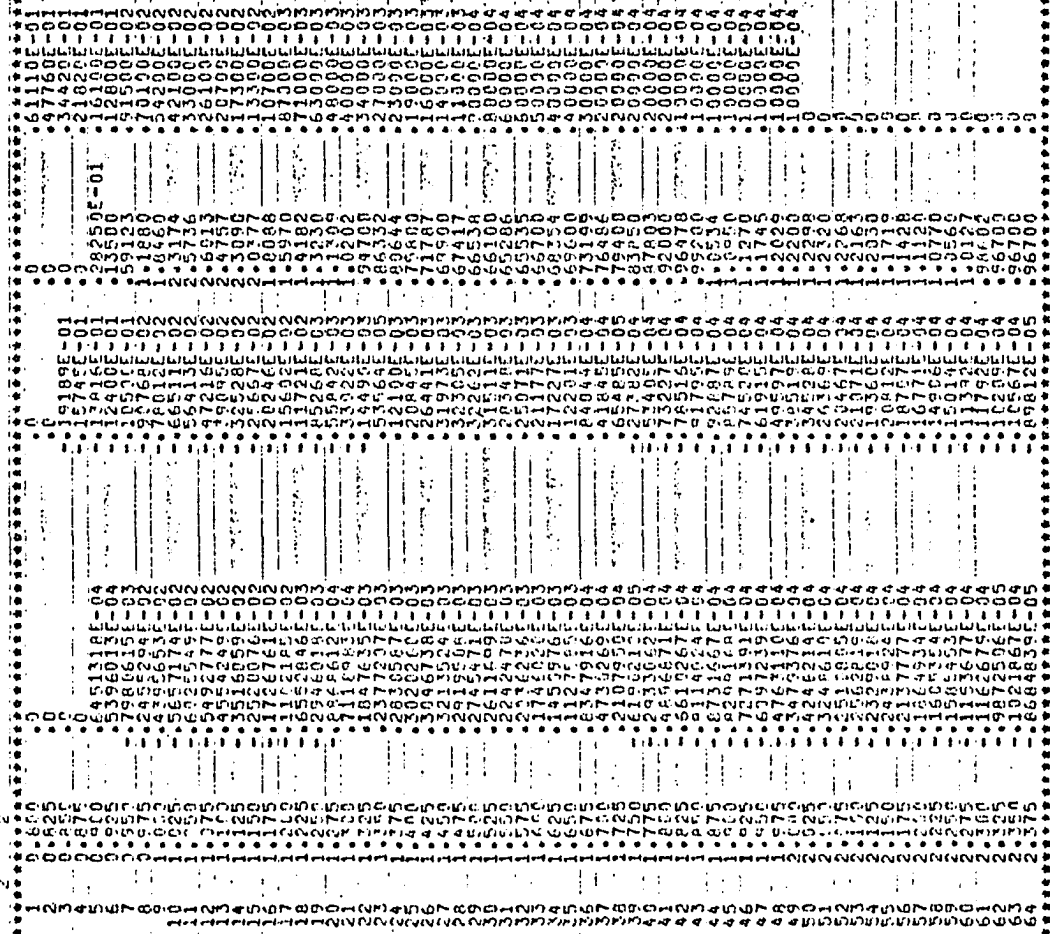
DATA DOCUMENTS, INC.

113

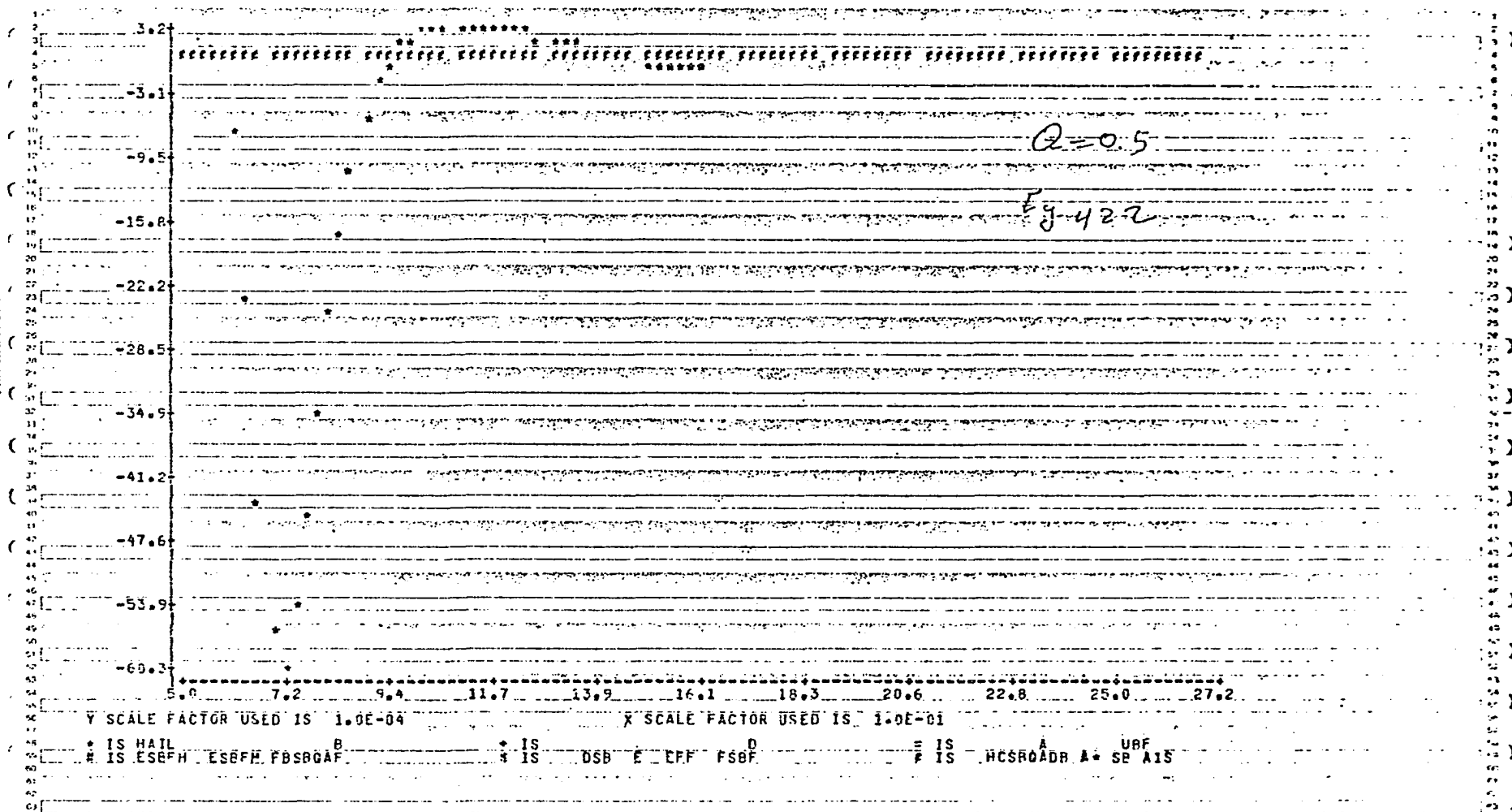


$Q = 0.5$

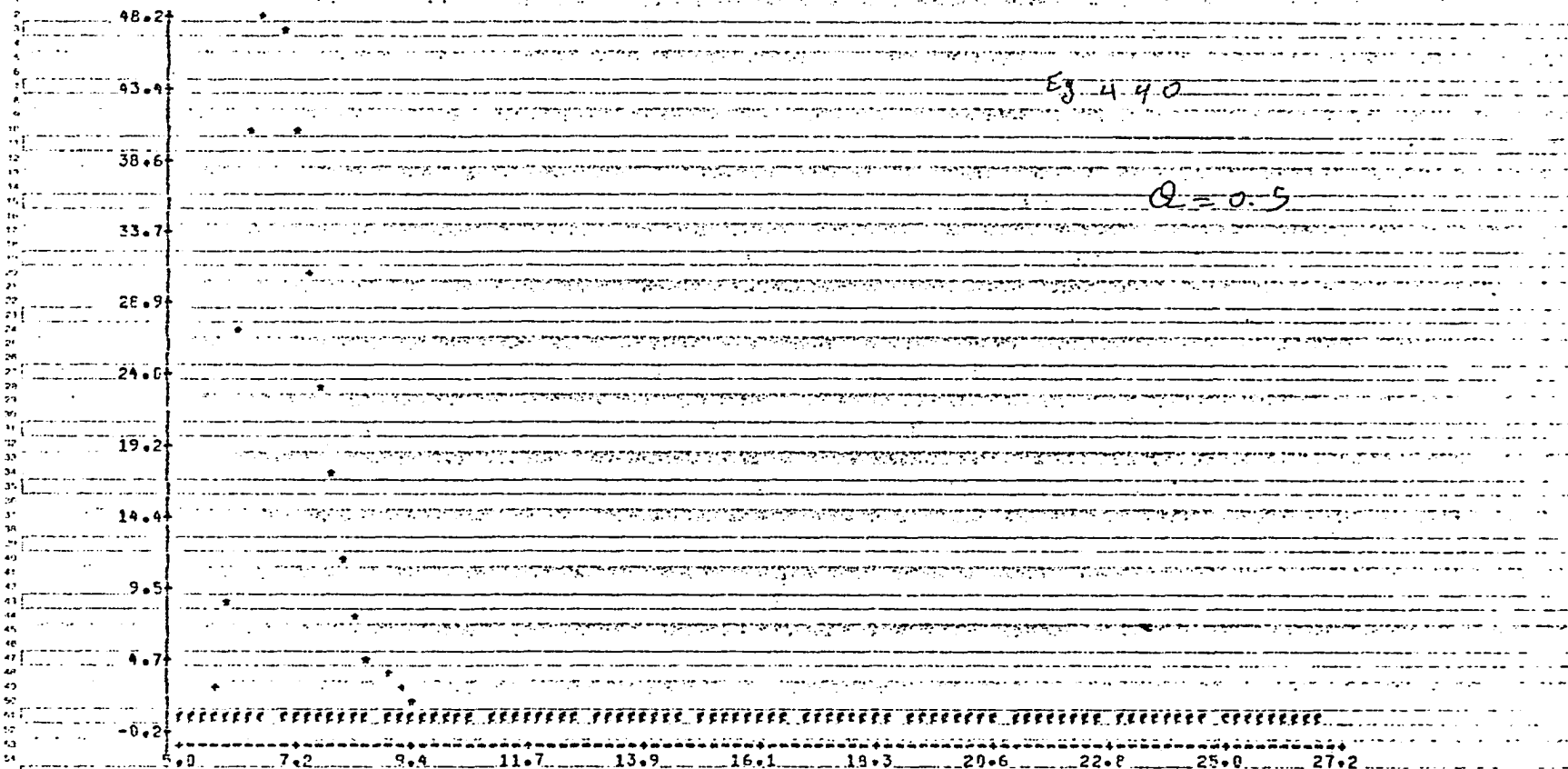
Fig



Fig



53



Y SCALE FACTOR USED IS 1.0E-03

X SCALE FACTOR USED IS 1.0E-01

 * IS HAIL
 * IS ESBFH ESBFH FBSBQAF

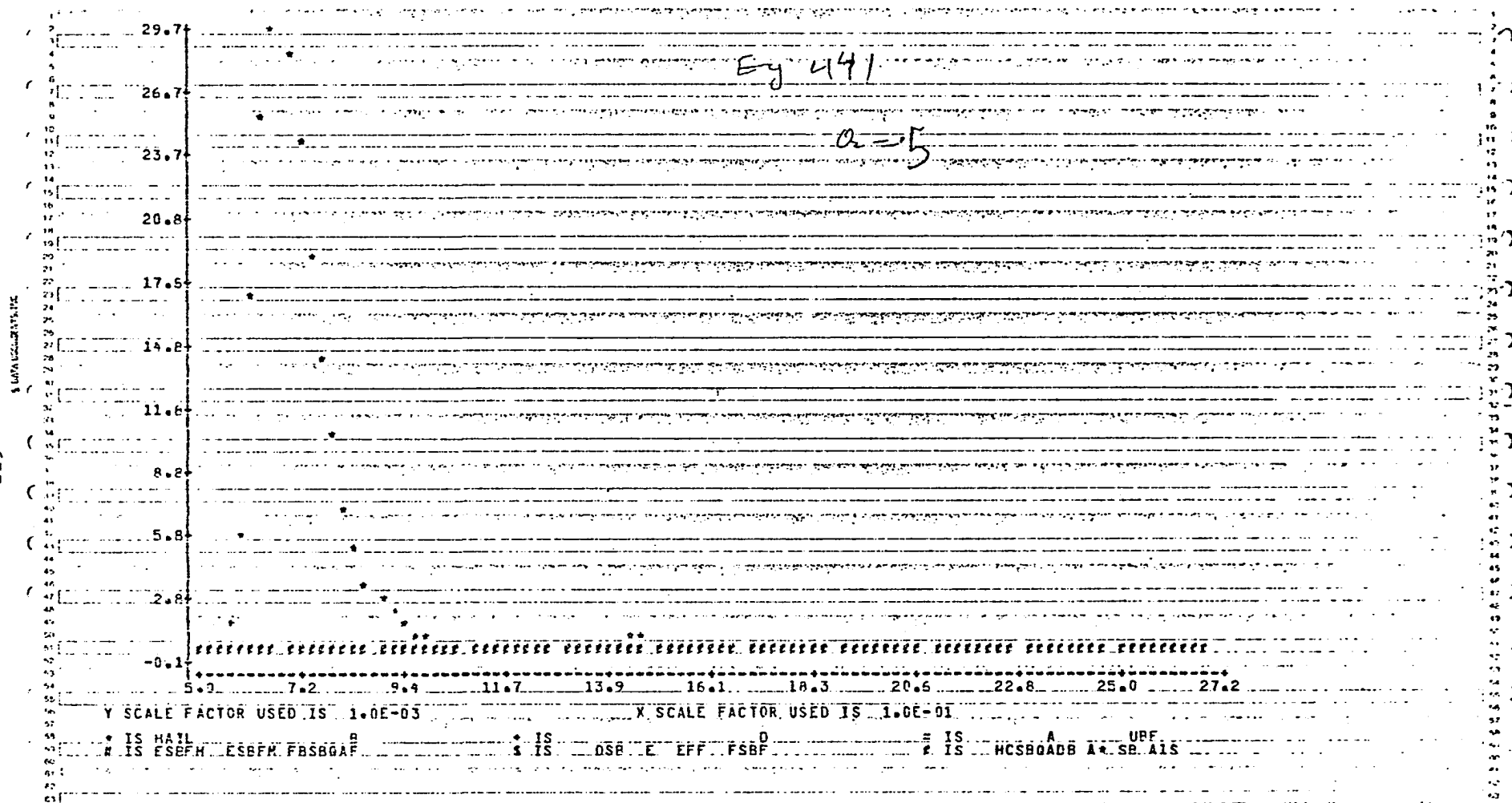
 * IS
 * IS OSR E EFF FSRF

 * IS
 * IS HCSBQADH AF S9 A13

Fig.

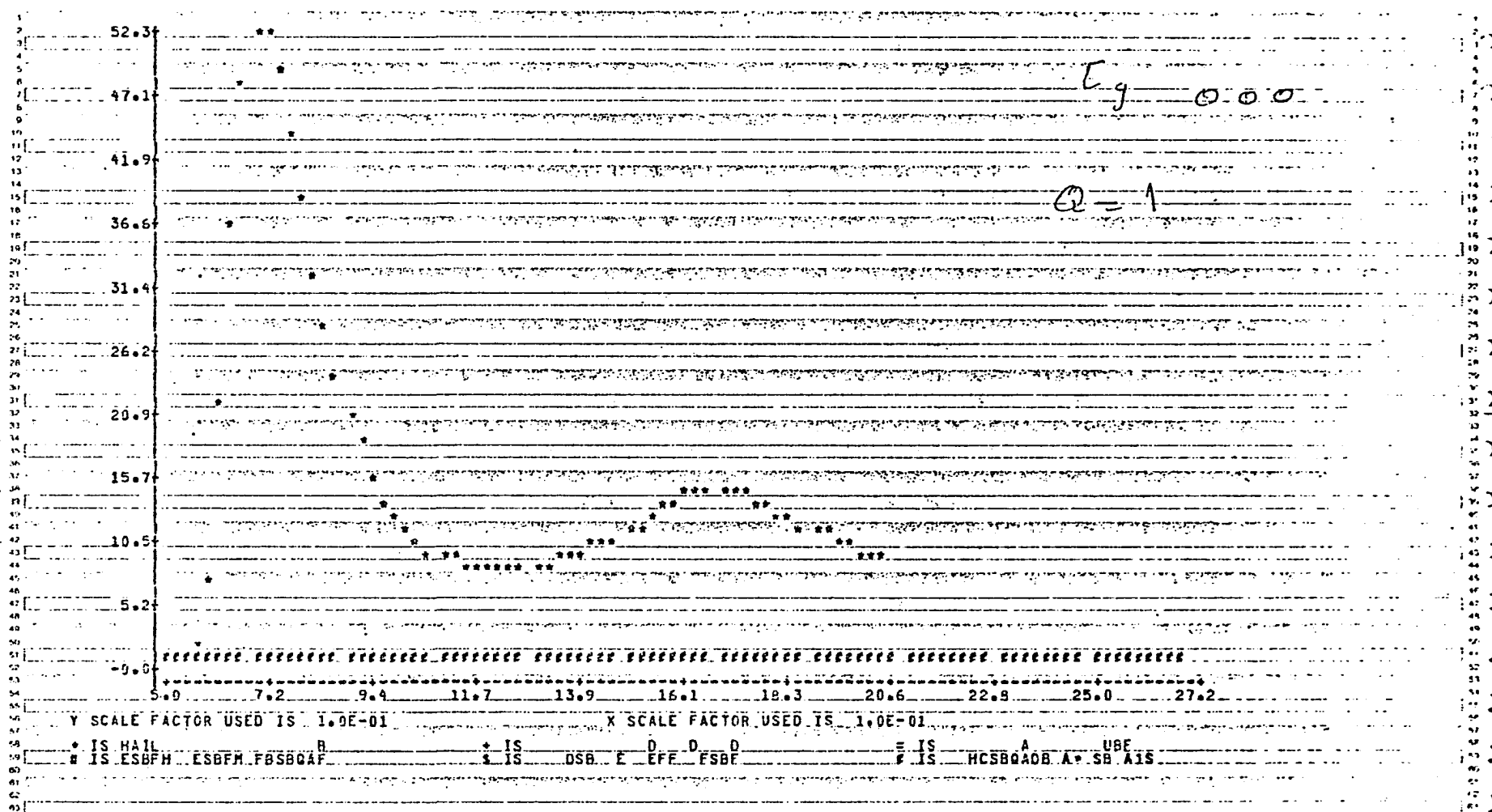
2-0.5

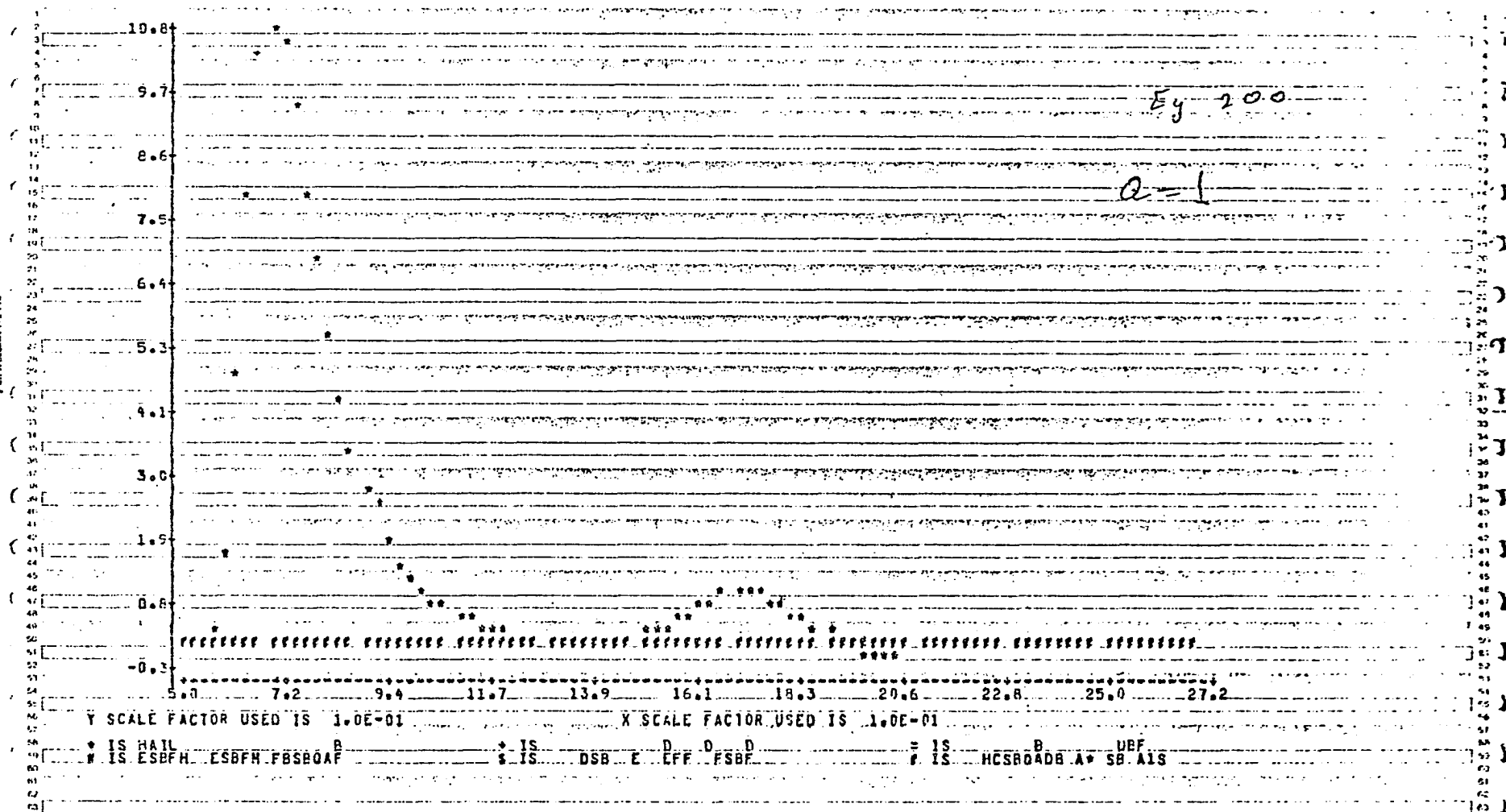
[illegible]



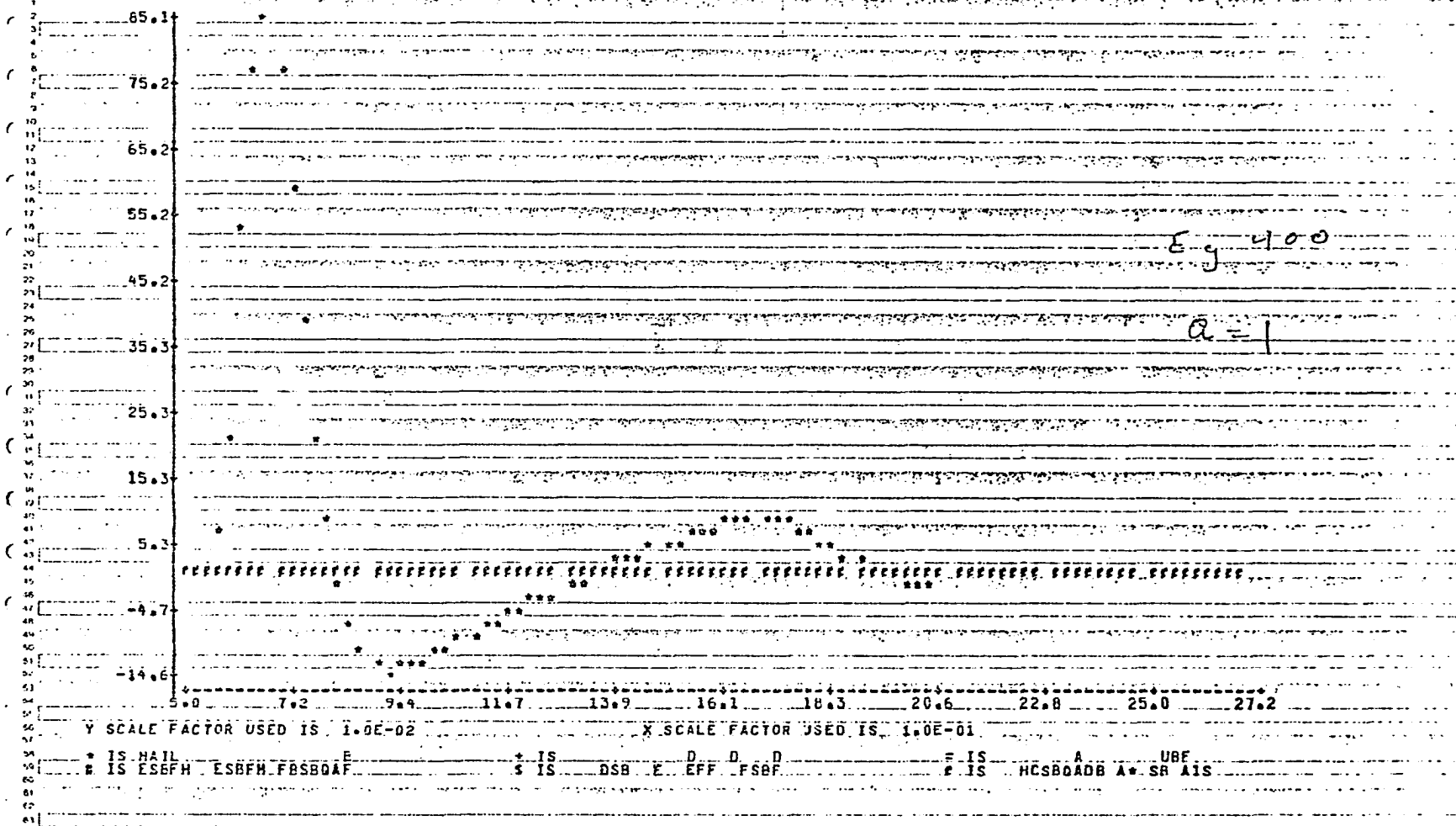
[illegible]

DATA COLLECTED BY

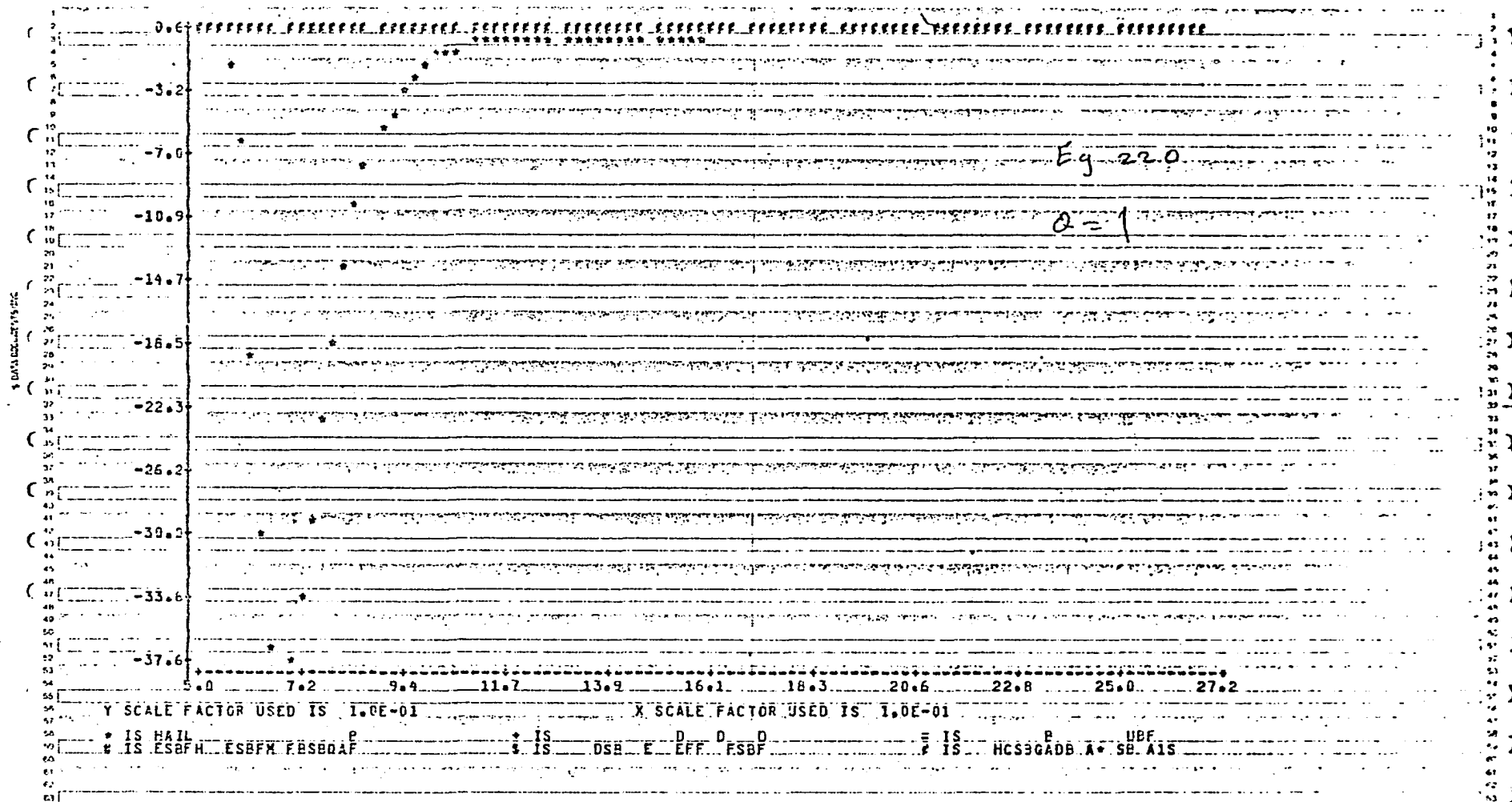




DATA ELEMENTS INC



[illegible]



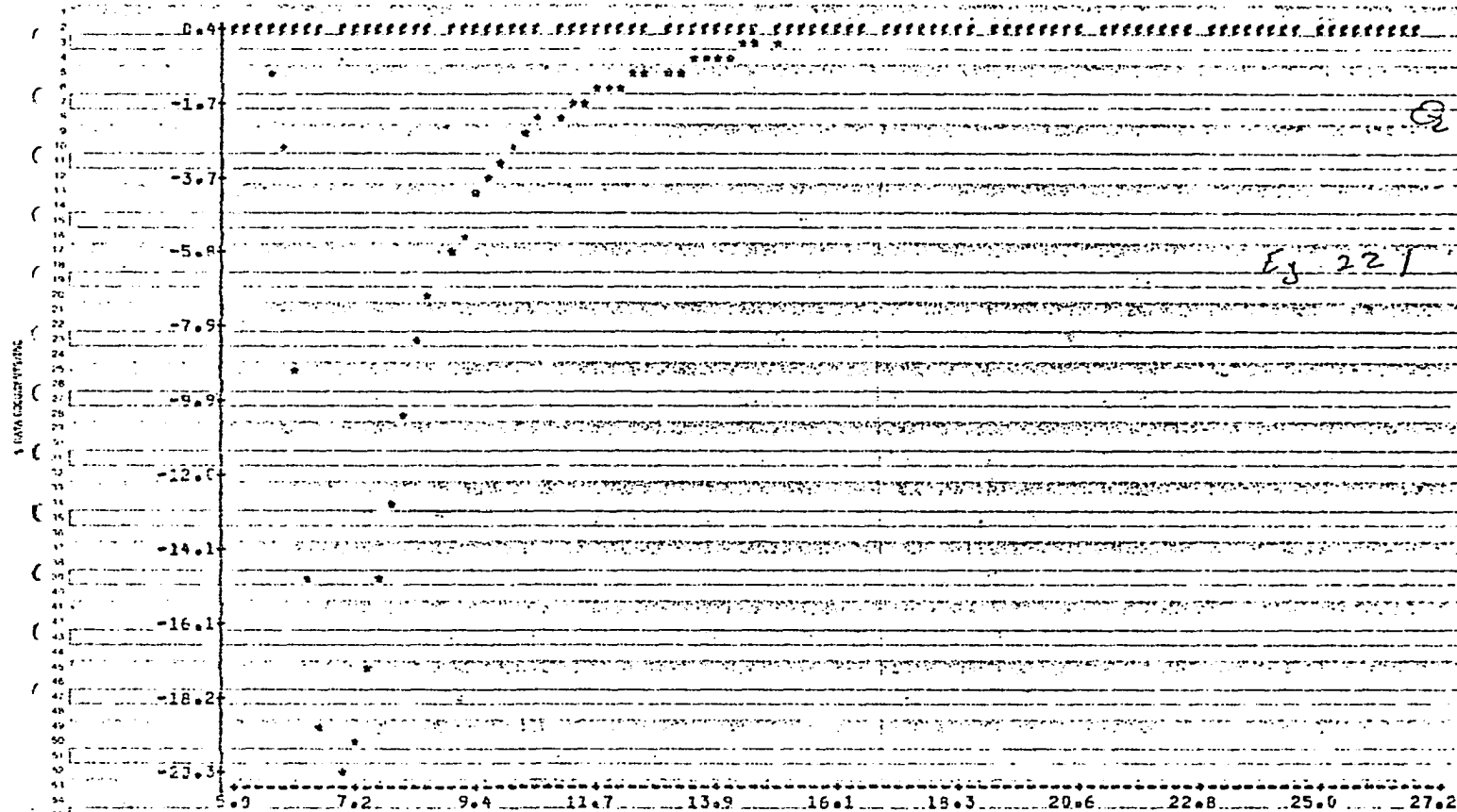
45.

2

10-01,

-01-

123

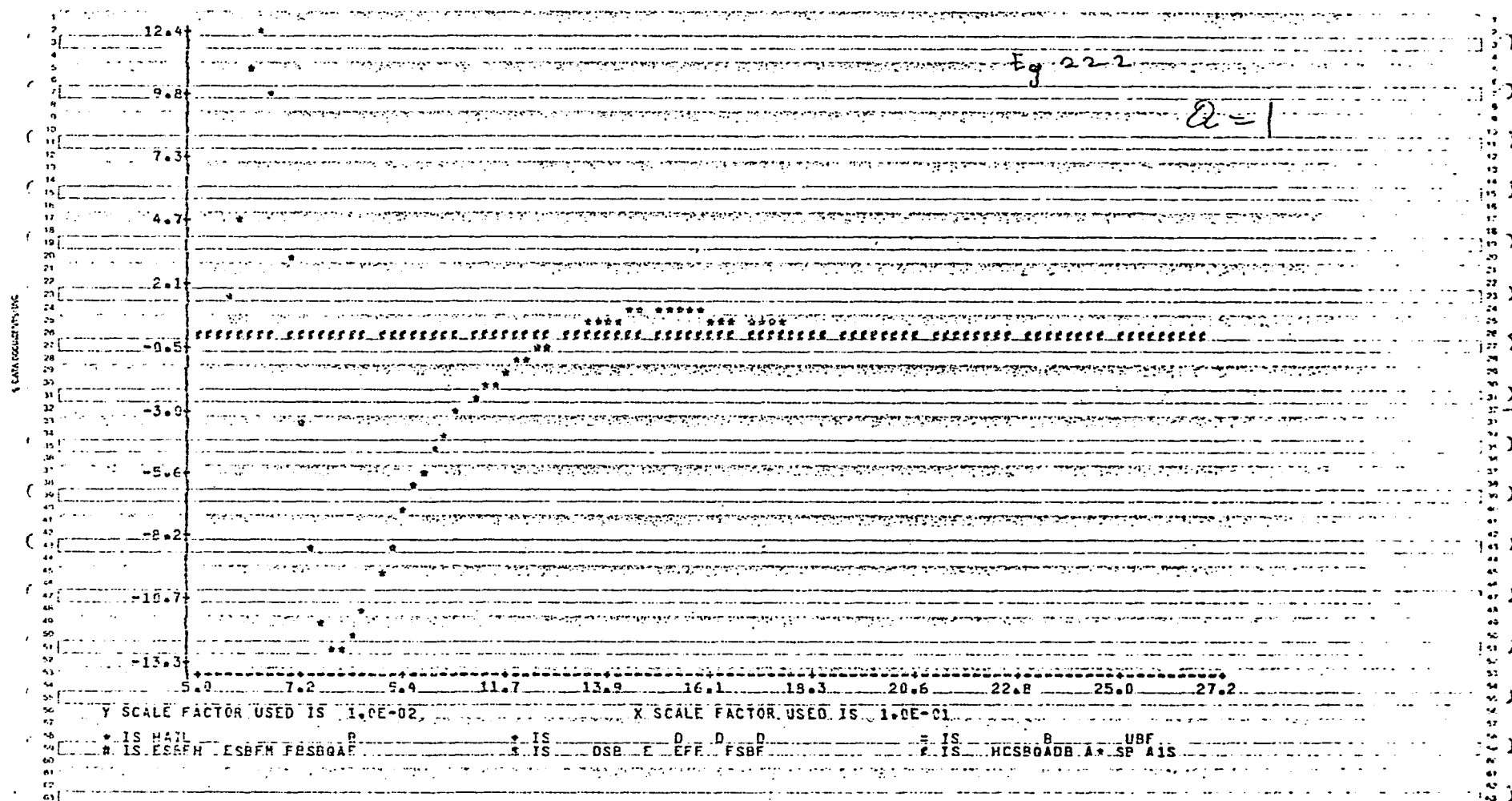


53

1-01

53 2

$\alpha = 1$

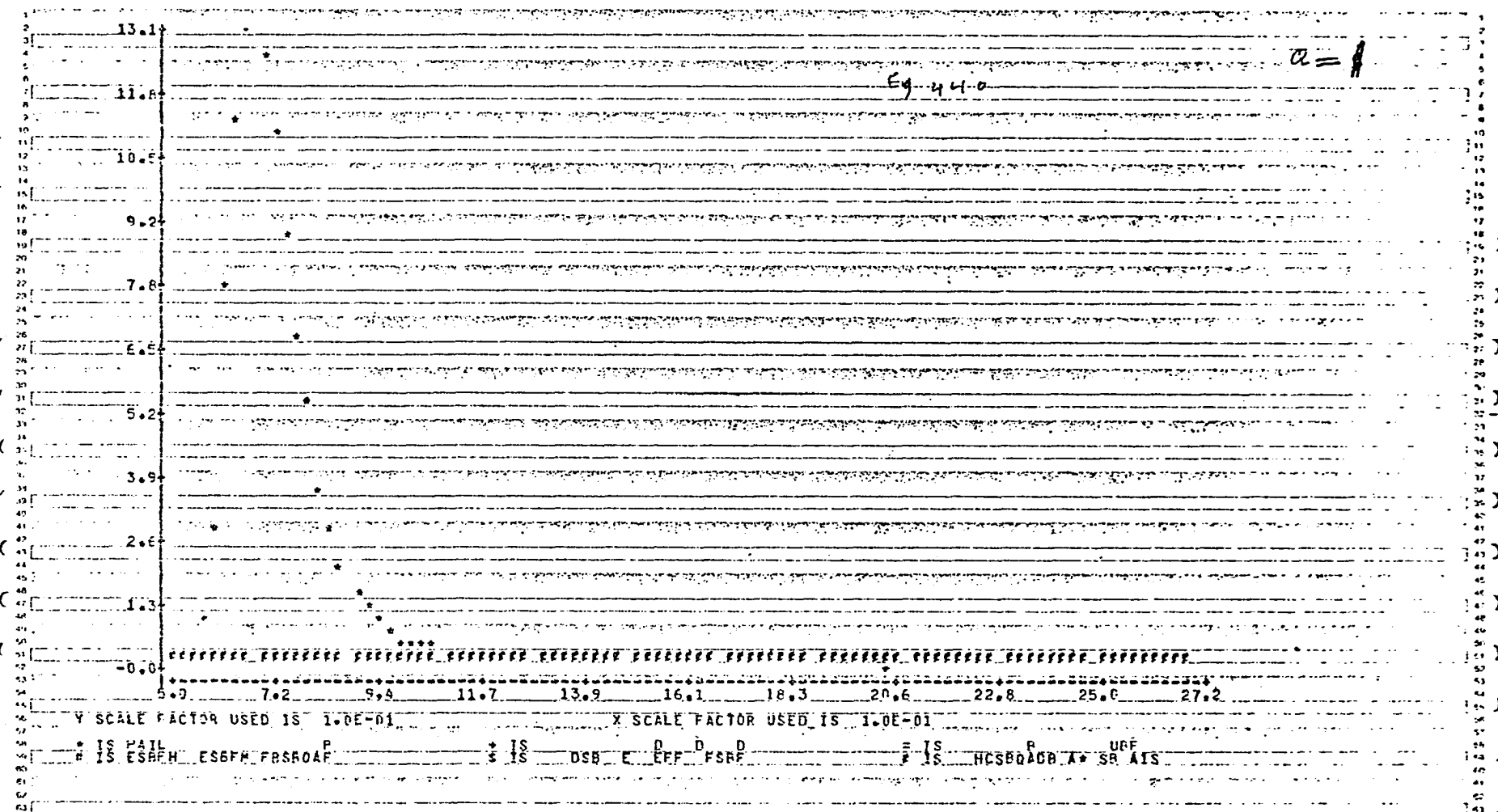


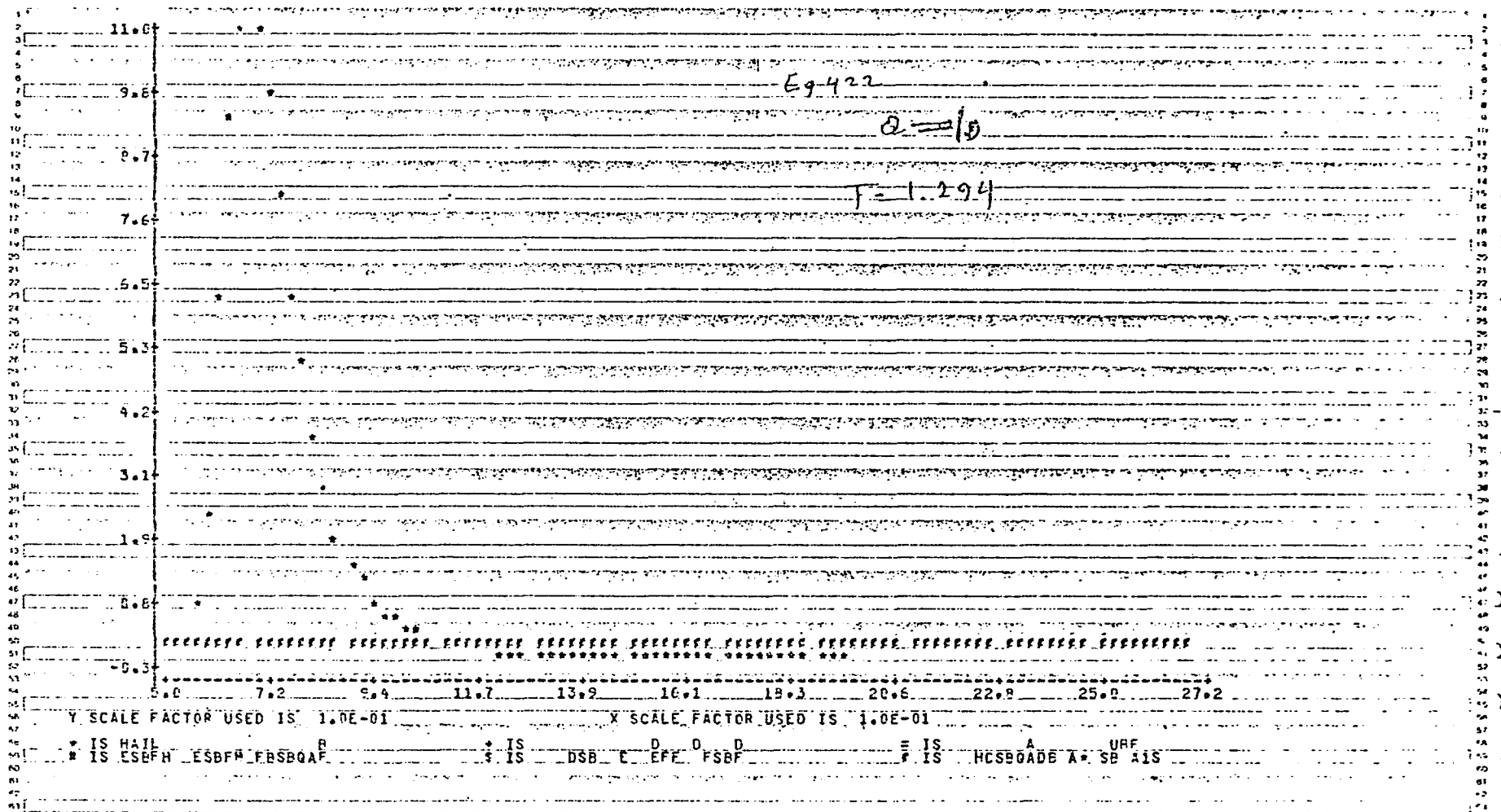
1629 J

67CE-01

EX-101

101





Q=0.4

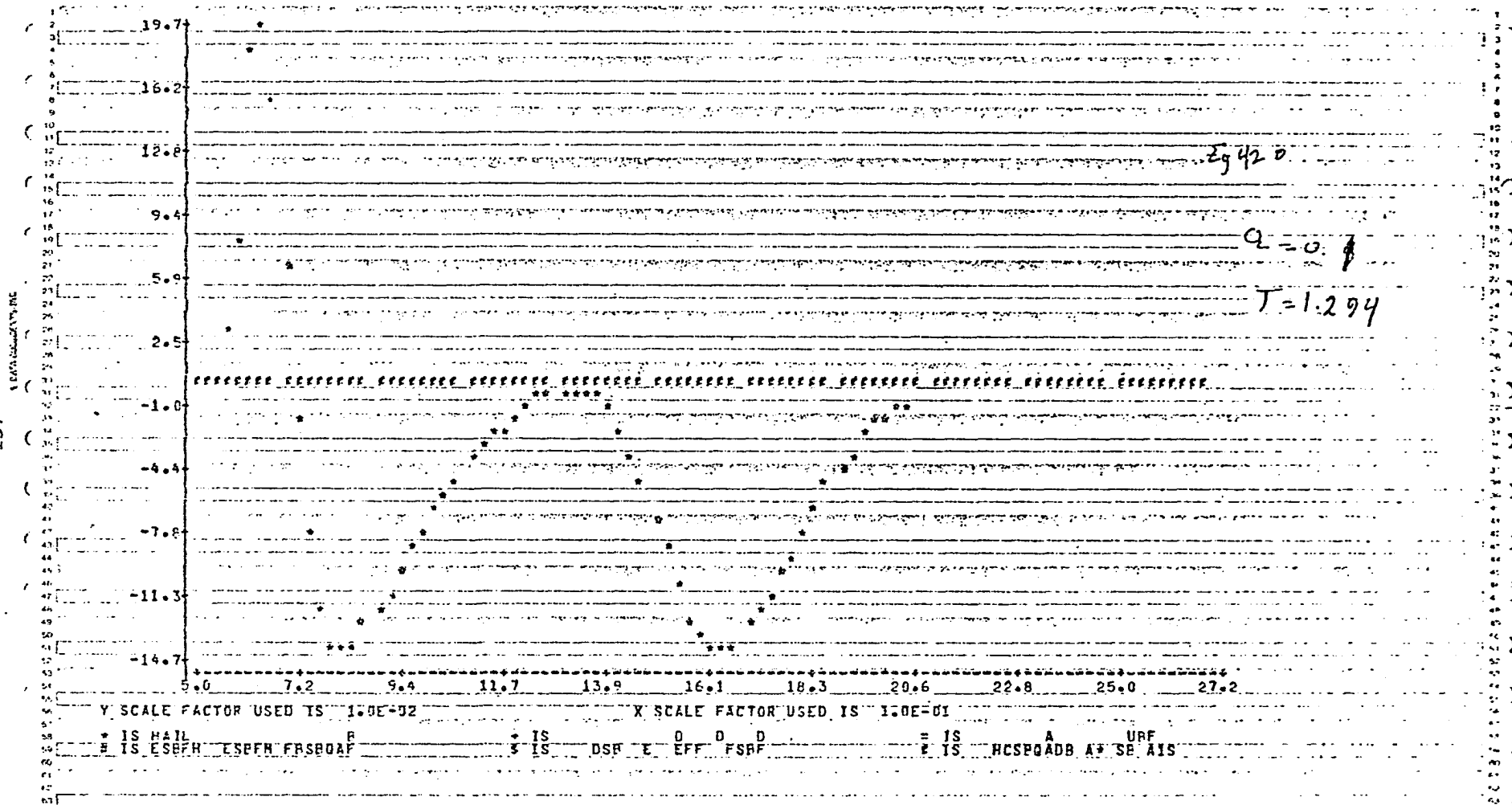
Q=1

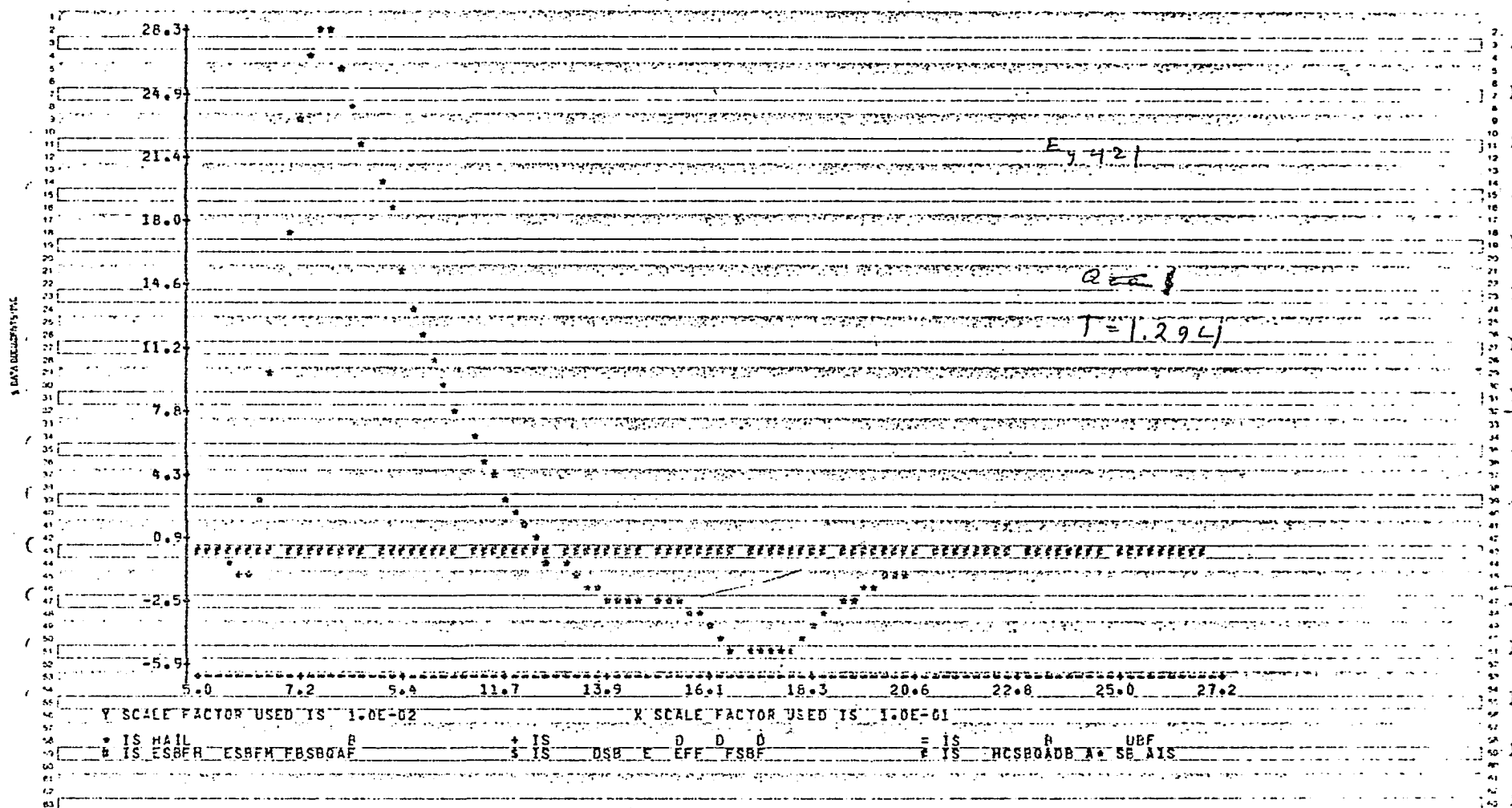
T=1294

Eg

E-101

62





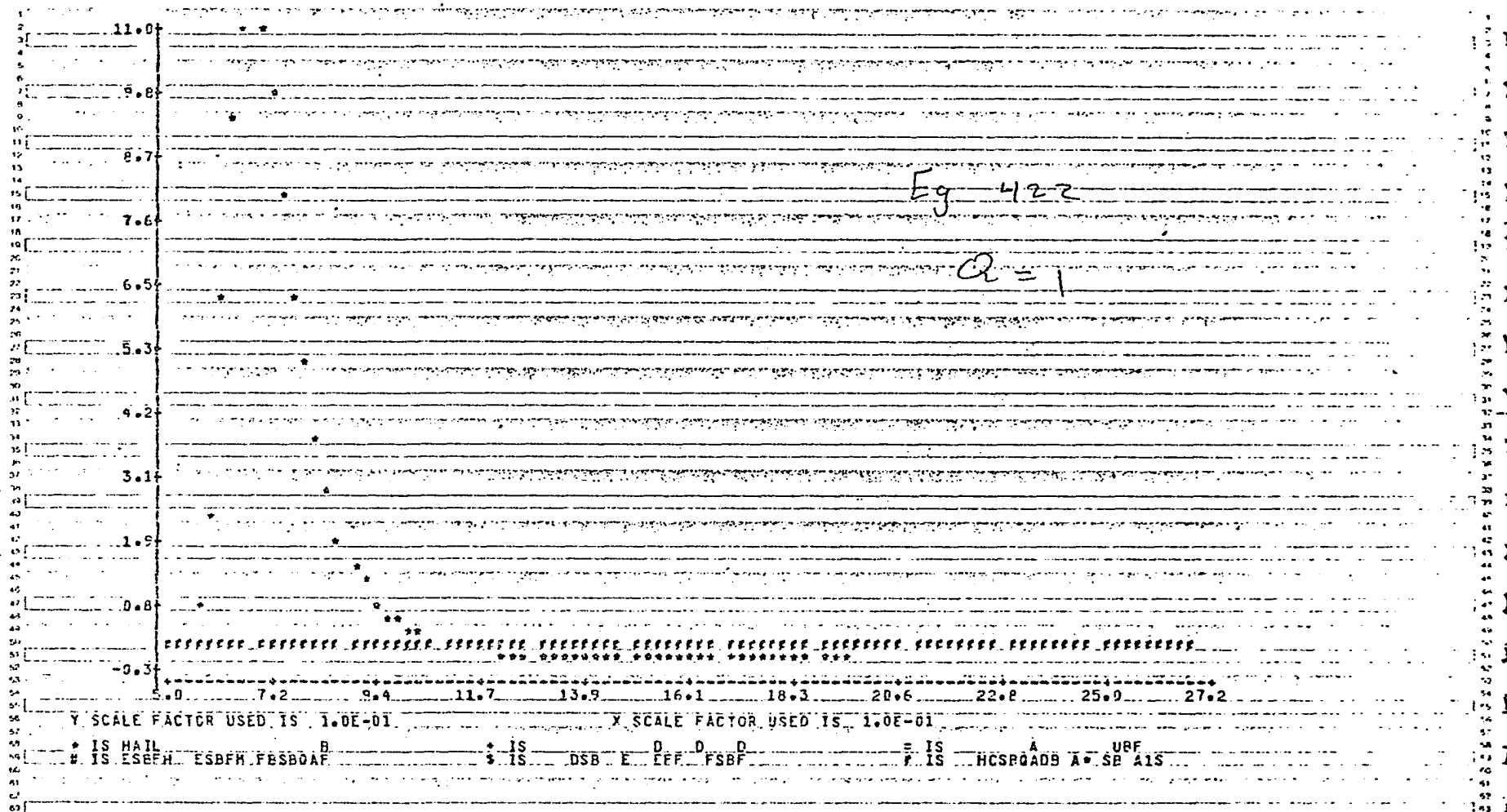
2

2

2

$$Q = 1$$
[illegible]

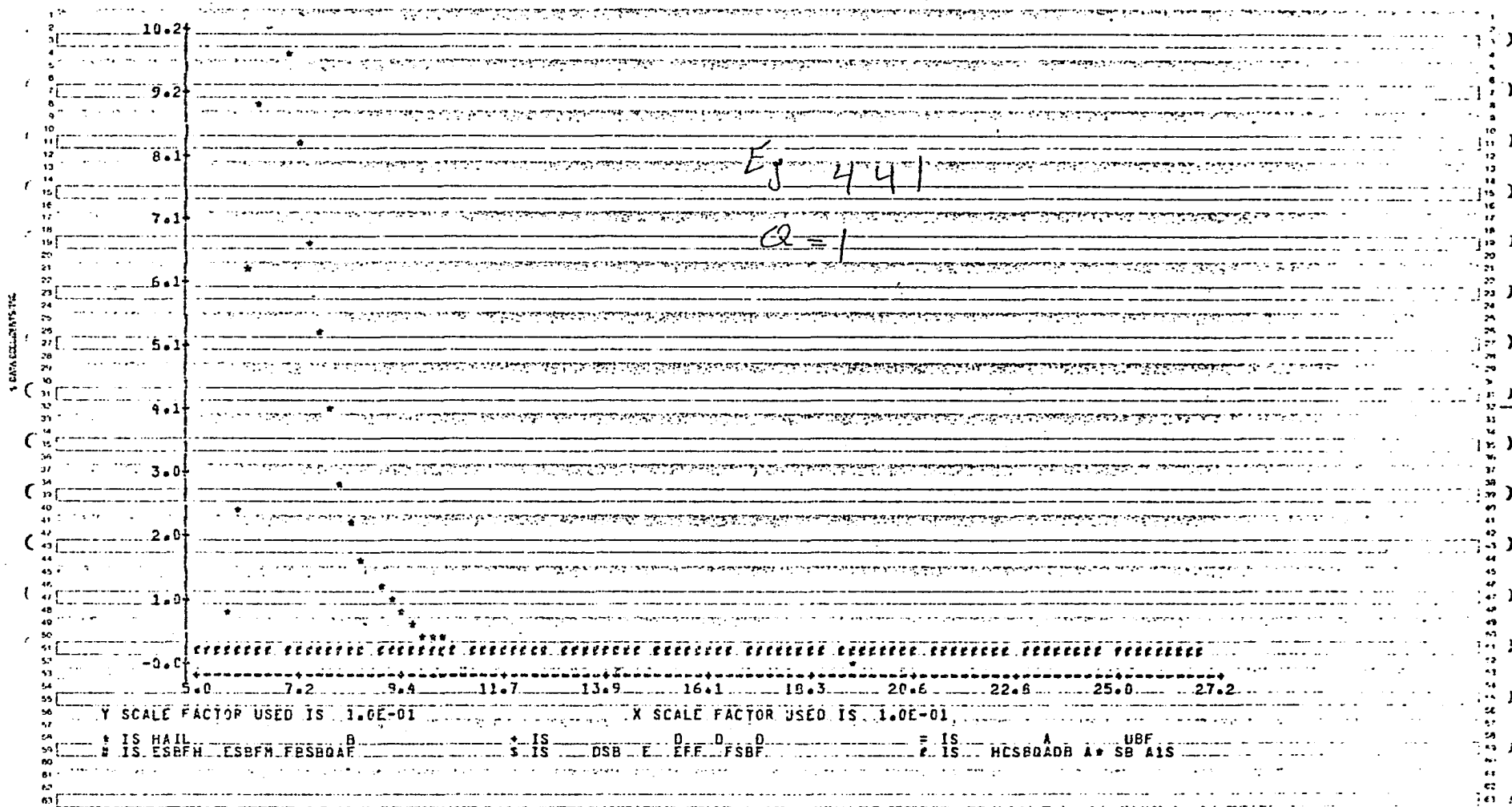
DATA DISPLAYS IN



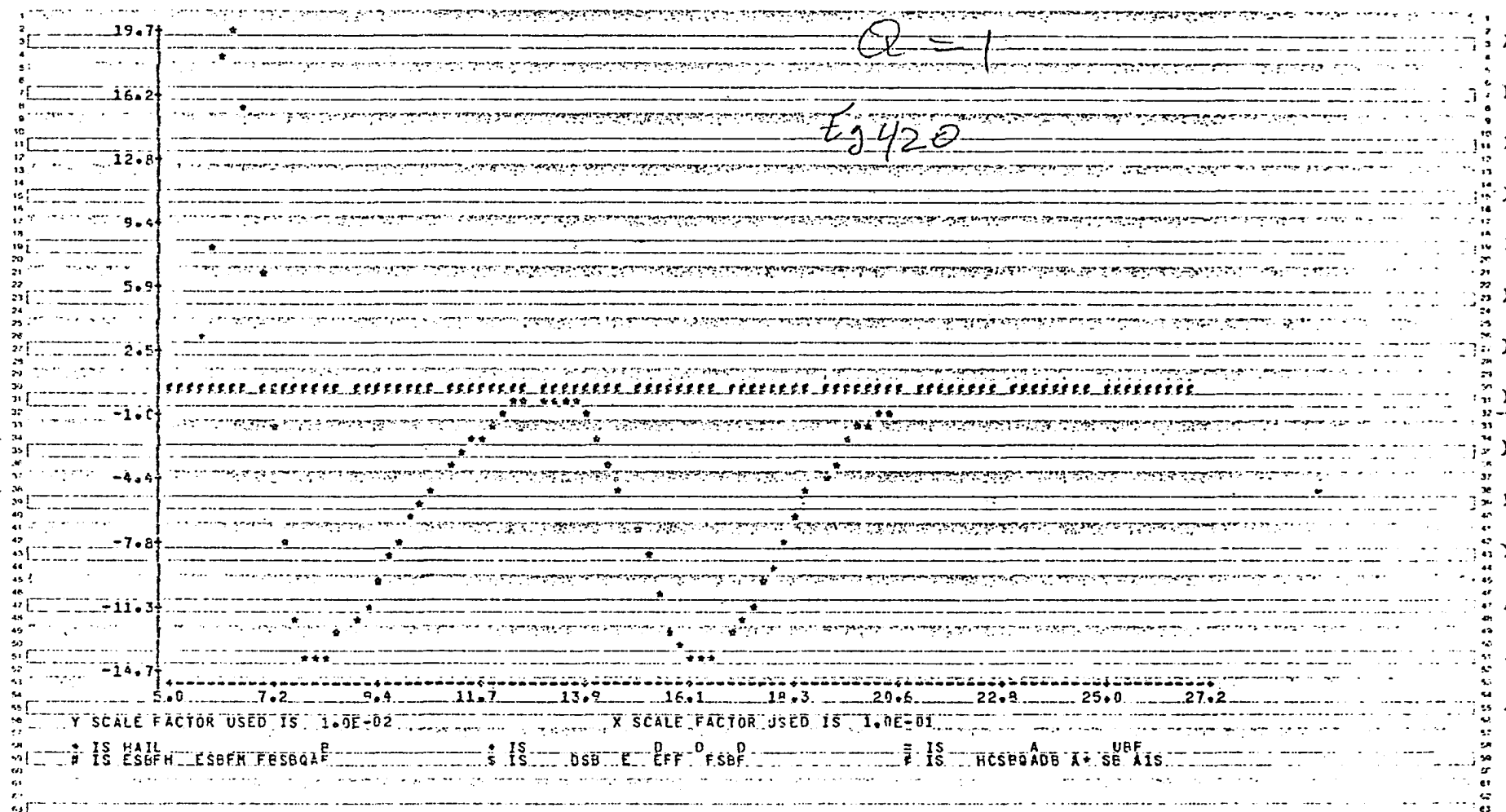
Eg

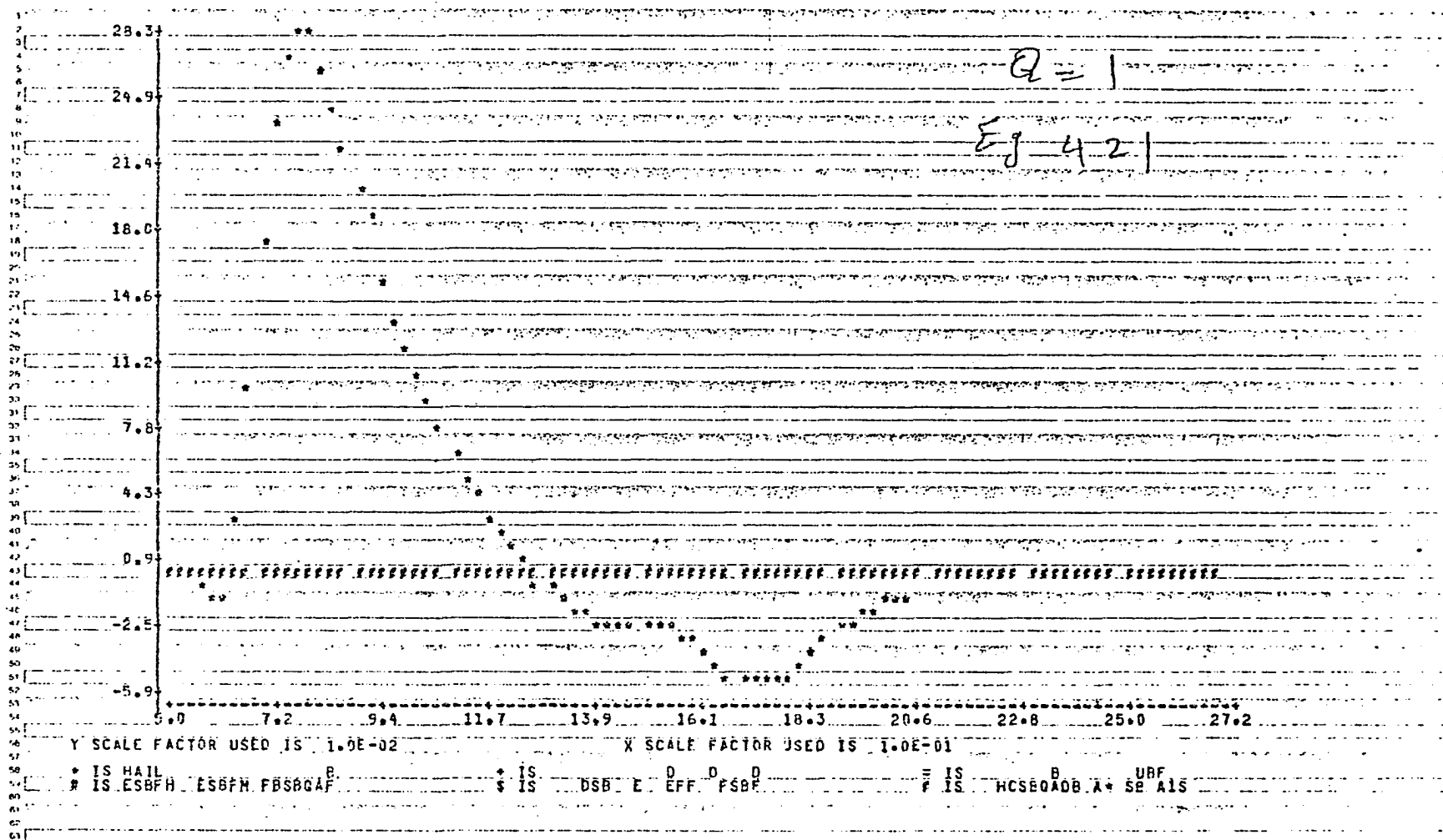
Q-1

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2	0	0	0	24.526
3	0	0	0	3.5693
4	0	0	0	3.2777
5	0	0	0	1.9464
6	0	0	0	3.3918
7	0	0	0	8.3048
8	0	0	0	5.7787
9	0	0	0	4.1674
10	0	0	0	3.8489
11	0	0	0	2.2978
12	0	0	0	1.7592
13	0	0	0	1.3656
14	0	0	0	1.1268
15	0	0	0	8.5249
16	0	0	0	8.8277
17	0	0	0	5.5111
18	0	0	0	4.4722
19	0	0	0	3.9669
20	0	0	0	2.9999
21	0	0	0	2.4722
22	0	0	0	1.2266
23	0	0	0	1.7022
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25	0	0	0	1.1899
26	0	0	0	9.9588
27	0	0	0	9.0188
28	0	0	0	7.1188
29	0	0	0	6.0388
30	0	0	0	5.2988
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33	0	0	0	3.1988
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35	0	0	0	2.5088
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38	0	0	0	1.7588
39	0	0	0	1.5588
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44	0	0	0	8.0388
45	0	0	0	7.7888
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47	0	0	0	5.8688
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62	0	0	0	0.0388
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71	0	0	0	0.5888
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76	0	0	0	0.0388
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94	0	0	0	0.0388
95	0	0	0	0.5888
96	0	0	0	0.0388
97	0	0	0	0.5888
98	0	0	0	0.0388
99	0	0	0	0.5888
100	0	0	0	0.0388



WJ

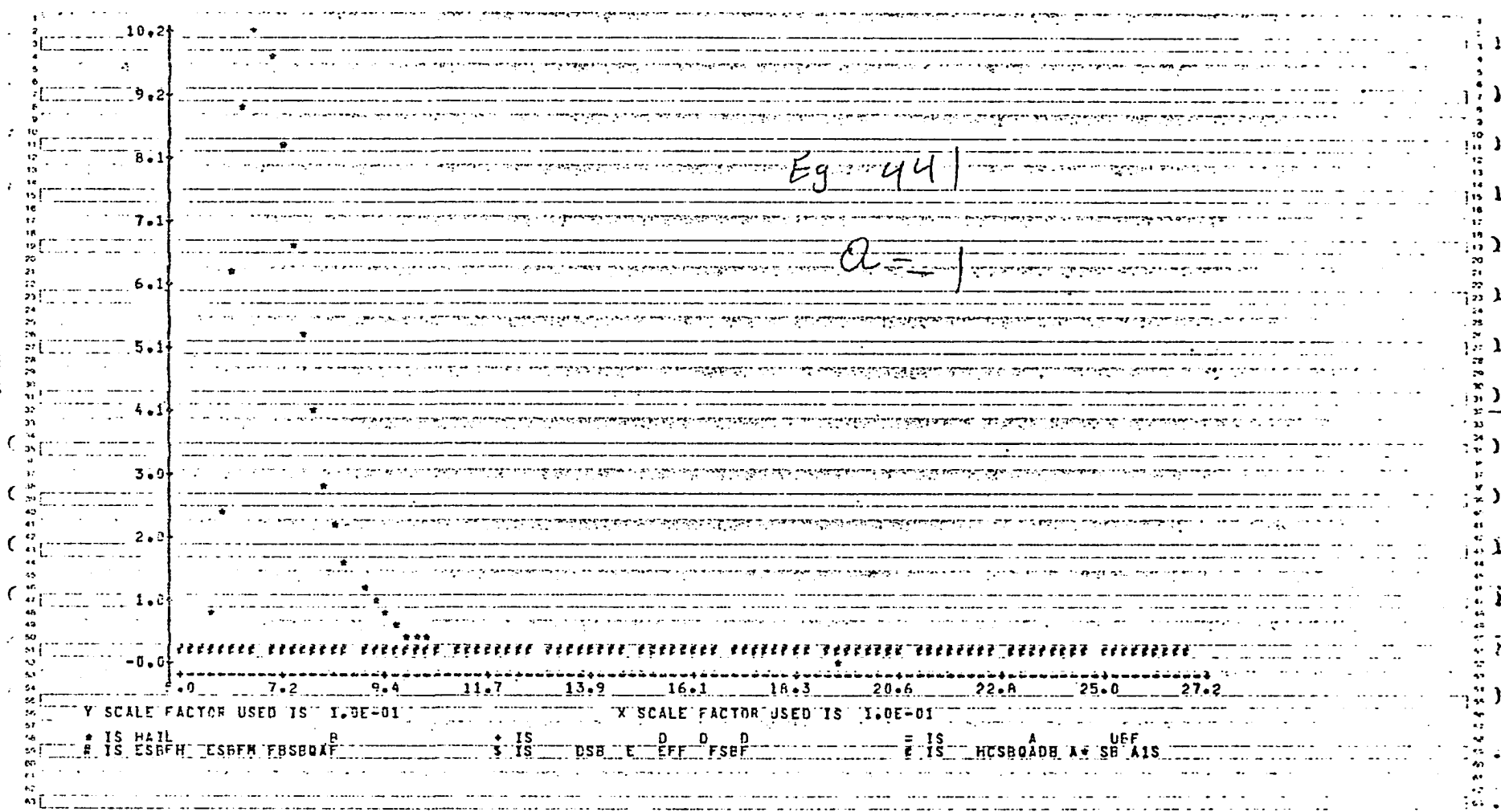




Eg

Q-1

DATA DOCUMENT 148			
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2	0	0	0
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6	0	0	0
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8	0	0	0
9	0	0	0
10	0	0	0
11	0	0	0
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92	0	0	0
93	0	0	0
94	0	0	0
95	0	0	0
96	0	0	0
97	0	0	0
98	0	0	0
99	0	0	0
100	0	0	0



Y SCALE FACTOR USED IS 1.0E-01

X SCALE FACTOR USED IS 1.0E-01

* IS HAIL
 # IS ESBFH ESBFH FBSBQAF
 + IS
 \$ IS DSB E EFF FSBF
 = IS
 E IS HCSBQADB A SB A1S
 A OFF

CONCLUSION

In this research we were looking at the first order perturbation. We looked at two cases of liquid states, one at weak quadrupole moment and the other at high. For low Q we got reasonable results. For high Q the first order perturbation appears to be inadequate.

We looked at Eg 220 and Eg 222 compared with simulation. The comparison in each case was fairly good, Eg 221 did not come as close; our results show more magnification than the simulation does.

Our theory can predict the SPH coefficients, at least qualitatively. The Lennard-Jones reference was found to be good for a low Q ($Q \leq .5$) polar fluid, for high quadrupole fluid we need to use the second order theory or other references.

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