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DORAISWAMY, IDA GRACE PANCHARATNAM

*SEMI *SIMPLICITY OF *MODULES OVER *SIMPLE *RINGS

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*SEMI *SIMPLICITY OF *MODULES OVER *SIMPLE *RINGS

APPROVED BY

Andy Magid

Leonard R. Rubin

Harold Huneke

Roger Alperin

Kevin A. Grasse

DISSERTATION COMMITTEE

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INTRODUCTION

This thesis is a study of the structure of a *simple *ring and its category of *modules.

We state the definitions needed for a description of the thesis.

k is an algebraically closed field. G is a linear algebraic group over k. A *ring is a commutative Noetherian k-algebra which is a rational G-module such that G acts by k-algebra automorphisms, [M]. A *module M over a *ring is an R-module and a rational G-module such that $g(rm) = (gr)(gm)$ for g in G, r in R and m in M, [M]. An *ideal of a *ring is an ideal which is a sub*module of R, [M]. A *ring R is *simple if the only *ideals of R are zero and R, [M].

We define an R-*module M to be *simple if the only G-stable R-sub*modules of M are the zero *module and the *module itself. An R-*module M is *semi*simple if it is a direct sum of *simple R-sub*modules of itself. A *ring R is *semi*simple if it is *semi-*simple as an R-*module.

In the first chapter we establish the category of R-*modules and $R\text{-}^{*}$module morphisms. In order for an R-module M to be an $R\text{-}^{*}$module, we need to define a rational G-action on M. With an appropriate definition of G-action we prove that the direct sum of a family of $R\text{-}^{*}$modules is an $R\text{-}^{*}$module, the tensor product $M \otimes_R N$ of $R\text{-}^{*}$modules M and N is an $R\text{-}^{*}$module, $\mathrm{Hom}_R(M,N)$ is an $R\text{-}^{*}$module if M is a finitely generated $R\text{-}^{*}$module and M/N is an $R\text{-}^{*}$module if N is an $R\text{-sub}^{*}$module of M. By considering only the morphisms that preserve G-action we establish a subcategory which is an abelian category. Since direct limit exists in this category, every $R\text{-}^{*}$module is the direct limit of its family of finitely generated $R\text{-sub}^{*}$modules. These are useful results for the development of the theory.

The second chapter contains the theory of *semi*simplicity. Three equivalent conditions for defining *semi*simplicity are established. The conditions are similar to those of semisimplicity in the category of R-modules. The techniques for proving the equivalence are also similar except for establishing that every $R\text{-}^{*}$module M contains a *simple sub*module if every sub*module N of M is a direct summand. This is because not every principal submodule of M is a sub*module.

We also examine the properties of *modules over *simple *rings when certain restrictions are imposed on the algebraic group. The main theorem is that when G is a connected linear algebraic group and R is a *simple *ring then every finitely generated $R\text{-}^{*}$module M is R-torsion-free. Thus, a *simple *ring is an integral domain if G is a connected linear algebraic group. We establish this result by proving that every associated prime of M is G-stable and therefore an *ideal.

R being *simple every associated prime reduces to the zero *ideal proving that M is torsion-free. If R is a finitely generated k-algebra that is *simple then M is R-projective. This is shown by proving that $M_{\mathcal{M}}$ is $R_{\mathcal{M}}$-free for every maximal ideal $\mathcal{M}$ of M.

If further G is a connected linearly reductive algebraic group and R a *simple *ring then every finitely generated R-*module is *semi*simple and, therefore, R-projective. Under the same conditions for G every nonzero R-*module is *semi*simple and, therefore, R-projective.

If G is a linear algebraic group then $k[G]$ with appropriate G-action defined on it is a *simple *ring. In Chapter Three, we examine $k[SL_n]$-*modules with SL_n-action for $n \geq 2$ when k is algebraically closed and the characteristic is zero. We prove that every *simple $k[SL_n]$-*module is R-isomorphic to $k[SL_n]$. This isomorphism preserves SL_n-action as well. Consequently, every $k[SL_n]$-*module is R-isomorphic to either $R^{(n)}$, $n < \infty$, or $R^{(\chi)}$, the isomorphism preserving SL_n-action as well. We call such an isomorphism (R-G)-isomorphism. We establish the isomorphism by the following sequence of arguments. It is a fact that for any linear algebraic group G, if $G \to GL(V)$ is a faithful representation of G in V then $k[G] = k[V + V^*]$, V^* being the dual of V, [M*]. If W is a *simple G-module then W is the homomorphic image of $\bigoplus_{i=1}^{n} (V \oplus V^*)^{\otimes_k d_i}$ for $d_i > 0$, the homomorphism being that of G-modules. In particular, if $G = SL_n$ then $k[SL_n] = k[GL_n]/(1 - D)$ where $k[GL_n] = k[x_{11}, x_{12}, \ldots, x_{nn}, 1/D]$ and

$D = D(x_{11},\ldots,x_{nn})$ is the determinant form. We define R_n to be an SL_n-module isomorphic to $\langle x_{11},x_{22},\ldots,x_{n1}\rangle$. Then W is the homomorphic image of $\bigoplus_{i=1}^{m} (R_n \oplus R_n^*)^{\otimes_k d_i}$, the homomorphism being that of SL_n-modules. $R \otimes_k R_n \underset{RG}{\cong} R^n$. This leads to the isomorphism $M \underset{RG}{\cong} R$ if M is *simple. Therefore every $k[SL_n]$-*module is (R-G)-isomorphic to $R^{(n)}$ or $R^{(X)}$. The existence of such an isomorphism follows from a more general theorem [CPM]. But we construct an explicit form of the isomorphism in this thesis.

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CHAPTER I

THE CATEGORY OF R-*MODULES

Starting with the basic definitions of this theory, this chapter establishes the category of R-*modules and R-*module *homomorphisms. Properties of R-*modules and R-*module *homomorphisms needed for the development of this theory are demonstrated. That this category contains a subcategory which is Abelian is also shown.

Throughout this thesis k is an algebraically closed field and G a linear algebraic group over k. For any algebraic set V, $k[V]$ denotes its coordinate ring. If R is a commutative ring then Mod_R is the category of R-modules and R-module homomorphisms. I is an indexing set.

Definition 1.1 A finite dimensional vector space V over k with G-action is a G-module if the induced homomorphism $G \to GL(V)$ is a homomorphism of algebraic groups over k. [M]

That is, if $\varphi : G \to GL(V)$ is the induced homomorphism then φ is a homomorphism of groups and is a k-morphism. k-morphism means that if $f \in k[GL(V)]$ then $f \circ \varphi \in k[G]$. [F]

Definition 1.2 A vector space W over k with G-action is a rational G-module if W is a union of finite dimensional G-modules in the above sense. [M]

Definition 1.3 A *ring R is a commutative Noetherian k-algebra which is a rational G-module such that G acts by k-algebra automorphisms. [M]

G acts rationally on R in the following sense. If $v \in R$ and $S_v = \langle gv \mid g \in G\rangle$, the vector space spanned by gv for all $g \in G$, then S_v is finite dimensional over k and the induced homomorphism $G \to GL(S_v)$ is a homomorphism of algebraic groups and $R = \bigcup_{v \in R} S_v$.

Definition 1.4 (Notation) $\langle gv \mid g \in G\rangle$ denotes the vector space generated over k by gv, for all $g \in G$. $\langle v_1, v_2, \ldots, v_n\rangle$ denotes the vector space generated over k by $v_1, v_2, \ldots, v_n$.

Definition 1.5 A *module M over a *ring R is an R-module and a rational G-module such that $g(rm) = (gr)(gm)$, for all $g \in G$, $r \in R$, $m \in M$. [M]

A *module M over a *ring R is said to be an R-*module.

A *module M over a *ring R is a rational G-module in the following sense. If $m \in M$ and $V_m = \langle gm \mid g \in G\rangle$ then V_m is finite dimensional over k and the induced homomorphism $G \to GL(V_m)$ is a homomorphism of algebraic groups. Moreover $M = \bigcup_{m \in M} V_m$.

Definition 1.6 An (R-G)-*module homomorphism of *modules over a *ring R is an R-module homomorphism preserving G-action.

Definition 1.7 (Notation) If $g \in G$ and M an R-*module then g_M denotes the G-action of g on M.

Definition 1.8 For each pair (M,N) of R-*modules define a set, ${}^*\mathrm{Hom}(M,N)$, of morphisms of M into N where ${}^*\mathrm{Hom}(M,N) = \{f \in \mathrm{Hom}_R(M,N) \mid \langle g_N f g_M^{-1} \mid g \in G\rangle$ is finite dimensional over $k\}$. f is said to be a *homomorphism if $f \in {}^*\mathrm{Hom}(M,N)$.

Proposition 1.9 The category whose objects are R-*modules and whose morphisms are *homomorphisms of R-*modules, as defined in 1.8, is a category and denote it by ${}^*\mathrm{Mod}_R$.

Proof: We first define composition of *homomorphisms, establishing that the composite map so defined is a *homomorphism.

For each triple (M,N,L) of R-*modules, define a map ${}^*\mathrm{Hom}(M,N) \times {}^*\mathrm{Hom}(N,L) \to {}^*\mathrm{Hom}(M,L)$ by $(u,v) = v \cdot u$ where $u \in {}^*\mathrm{Hom}(M,N)$, $v \in {}^*\mathrm{Hom}(N,L)$ and $\cdot$ is the usual composition of maps. Denote $v \cdot u$ by vu. By the definition of a *homomorphism, $\langle g_N u g_M^{-1} \mid g \in G\rangle$ and $\langle g_L v g_N^{-1} \mid g \in G\rangle$ are finite dimensional vector spaces over k. Let $\langle g_N u g_M^{-1} \mid g \in G\rangle = \langle f_1, f_2, \ldots, f_m \mid f_i \in {}^*\mathrm{Hom}(M,N)\rangle$ and $\langle g_L v g_N^{-1} \mid g \in G\rangle = \langle h_1, h_2, \ldots, h_n \mid h_i \in {}^*\mathrm{Hom}(N,L)\rangle$. Then $\langle g_L v u g_N^{-1} \mid g \in G\rangle \subseteq \langle h_i f_j \mid 1 \le i \le n,\ 1 \le j \le m\rangle$ which is finite dimensional over k. Moreover $vu \in \mathrm{Hom}_R(M,L)$. Therefore $vu \in {}^*\mathrm{Hom}(M,L)$.

(i) Let M_1, M_2, M_3, M_4 be R-*modules and $f_1 \in {}^*\mathrm{Hom}(M_1,M_2)$, $f_2 \in {}^*\mathrm{Hom}(M_2,M_3)$, $f_3 \in {}^*\mathrm{Hom}(M_3,M_4)$. $f_3(f_2 f_1) = (f_3 f_2) f_1$ as R-module homomorphisms. Moreover, $f_3(f_2 f_1), (f_3 f_2) f_1 \in {}^*\mathrm{Hom}(M_1,M_4)$.

Thus composition of *homorphisms is associative.

(ii) For any R-*module M, let 1_M be the identity map of M onto M. Then $1_M \in \mathrm{Hom}_R(M,M)$. Moreover, $\langle g_M 1_M g_M^{-1} \mid g \in G\rangle \cong k$ and therefore finite dimensional over k. Thus, $1_M \in {}^*\mathrm{Hom}(M,M)$. Let $f \in {}^*\mathrm{Hom}(M,N)$ and $h \in {}^*\mathrm{Hom}(N,M)$ for any R-*module N. $f1_M = f$ and $1_M h = h$ as R-module homomorphisms. Also, $f1_M, f \in {}^*\mathrm{Hom}(M,N)$ and $1_M h, h \in {}^*\mathrm{Hom}(N,M)$. Therefore, $f1_M = f$ and $1_M h = h$ as R-*module *homomorphisms. Thus, 1_M is a left identity in ${}^*\mathrm{Hom}(M,N)$ and a right identity in ${}^*\mathrm{Hom}(N,M)$.

(iii) Let the pairs (M_1,N_1) and (M_2,N_2) of R-*modules be distinct. If $f \in {}^*\mathrm{Hom}(M_1,N_1) \cap {}^*\mathrm{Hom}(M_2,N_2)$ with $f \neq 0$, then $f \in \mathrm{Hom}_R(M_1,N_1) \cap \mathrm{Hom}_R(M_2,N_2)$. This implies that $M_1 = M_2$ and $N_1 = N_2$.

(i), (ii) and (iii) establish the proposition.

For an R-module M to be an R-*module, M should be a rational G-module and the G-action on M should satisfy the condition $g(rm) = (gr)(gm)$ for all $g \in G$, $r \in R$ and $m \in M$. We show that these two conditions are satisfied whenever it is necessary to establish that an R-module is an R-*module.

<u>Proposition 1.10</u> If $\{M_i \mid 1 \le i \le n\}$ is a finite family of R-*modules, then $\bigoplus_i M_i$ is an R-*module.

<u>Proof:</u> $M = \bigoplus_i M_i$ is an R-module.

(i) Let G act on M as follows: $g(r((m_i)_i)) = g((rm_i)_i) = (g(rm_i))_i = ((gr)(gm_i))_i = (gr)((gm_i)_i) = (gr)(g((m_i)_i))$ for all $g \in G$, $r \in R$, $m_i \in M_i$, $1 \le i \le n$.

(ii) Let $x = (m_i)_i \in M$ with $m_i \in M_i$. For each m_i $\exists$ a G-stable finite dimensional subspace V_{m_i} of M_i, over k, such that $m_i \in V_{m_i}$ and the homomorphism $\mu_i : G \to GL(V_{m_i})$ is a homomorphism of algebraic groups over k. Let $V_x = \bigoplus_i V_{m_i}$, $1 \le i \le n$. V_x is finite dimensional over k and is G-stable. The induced homomorphism $\mu : G \to GL(V_x)$ is defined by $\mu(g) = (\mu_i(g))_i$ for all $g \in G$. If $g_1, g_2 \in G$ then $\mu(g_1 g_2) = (\mu_i(g_1 g_2))_i = (\mu_i(g_1)\mu_i(g_2))_i = ((\mu_i(g_1))_i)((\mu_i(g_2))_i) = \mu(g_1)\mu(g_2)$. This proves that μ is a homomorphism of groups.

Now it is sufficient to show that if $\varphi \in k[GL(V_x)]$ then $\varphi \cdot \mu \in k[G]$. The homomorphism $\mu : G \to GL(V_x)$ can be factored as $G \xrightarrow{\lambda} \bigoplus_j GL(V_{m_j}) \xrightarrow{i} GL(V_x)$ where λ is defined by $\lambda(g) = (\mu_j(g))_j$ for all $g \in G$ and i is the inclusion map. Therefore, $i\lambda = \mu$. $k[\bigoplus_j GL(V_{m_j})] = \bigotimes_k k[GL(V_{m_j})]$. If $\varphi \in k[\bigoplus_j GL(V_{m_j})]$ then $\varphi = \sum_\ell (\otimes f_{\ell j})$ where $f_{\ell j} \in k[GL(V_{m_j})]$ for all j. But $\lambda : G \to \bigoplus_j GL(V_{m_j})$ and $\varphi \in \bigotimes_k k[GL(V_{m_j})]$. Therefore, $\varphi\lambda = \sum_\ell(\sum_j f_{\ell j}\mu_j) \in k[G]$ since $f_{\ell j}\mu_j \in k[G]$ for all ℓ, j. This proves that $\mu : G \to GL(V_x)$ is a k-morphism and therefore $\mu : G \to GL(V_x)$ is a homomorphism of algebraic groups.

If $x \in M$ then V_x is a G-module and $M = \bigcup_{x \in M} V_x$. That is, M is a rational G-module. (i) and (ii) establish that $M \in {}^*\mathrm{Mod}_R$. This completes the proof of Proposition 1.10.

<u>Proposition 1.11</u> If $\{M_i \mid i \in I\}$ is an infinite family of R-*modules then $\bigoplus_{i \in I} M_i$ is an R-*module.

Proof: $\oplus M_i$ is an R-module. If $x = (x_i)_i \in M$ where $M = \underset{i}{\oplus} M_i$ then all but a finite number of x_i terms are zero. A G-action defined on M as in Prop. 1.10 satisfies the required condition for G-action. $V_x = \oplus V_{x_i}$ where all but a finite number of G-modules V_{x_i} are zero modules. This forces V_x to be finite dimensional over k. V_x is also G-stable. Therefore the argument that the homomorphism μ : $G \to GL(V_x)$ is a k-morphism is the same as that in Prop. 1.10. $M = \underset{x \in M}{\cup} V_x$. Therefore M is an $R\text{-}^*$module. This completes the proof of Prop. 1.11.

Proposition 1.12 Let M be a finitely generated $R\text{-}^*$module and N an $R\text{-}^*$module. If a G-action on $\mathrm{Hom}_R(M,N)$ is defined by $g \circ f = gfg^{-1}$ for all $g \in G$, $f \in \mathrm{Hom}_R(M,N)$ then $\mathrm{Hom}_R(M,N)$ is an $R\text{-}^*$module. Moreover, $\mathrm{Hom}_R(M,N) = {}^*\mathrm{Hom}(M,N)$.

Proof: $\mathrm{Hom}_R(M,N)$ is an R-module. If $f \in \mathrm{Hom}_R(M,N)$ let $V_f = \langle gfg^{-1} \mid g \in G\rangle$. It is sufficient to prove that V_f is finite dimensional over k and the homomorphism $G \to GL(V_f)$ is a k-morphism.

Let M be generated by $m_1,\ldots,m_s$, with $m_i \in M$. Then $V_{m_i} = \langle gm_i \mid g \in G\rangle$ is finite dimensional over k. Let $V_{m_i} = \langle m_i, g_{2i}m_i,\ldots,g_{pi}m_i\rangle$ with $g_{ji} \in G$, $1 \le i \le s$ and $2 \le j \le p$. So also let $V_{f(m_i)} = \langle f(m_i), h_{2i}f(m_i),\ldots,h_{qi}f(m_i)\rangle$ with $h_{ji} \in G$, $1 \le i \le s$ and $2 \le j \le q$. $V_{f(g_{ji}m_i)} = \langle f(g_{ji}m_i), d_{2i}f(g_{ji}m_i), \ldots, d_{ri}f(g_{ji}m_i)\rangle$ with $d_{\ell i} \in G$, $1 \le i \le s$, $2 \le j \le p$ and $2 \le \ell \le r$. We now prove that $\langle gfg^{-1} \mid g \in G\rangle$ is contained in the span of f, $h_{n_i}f$, $d_{\ell i}f_{ji}$ with $1 \le i \le s$, $2 \le n \le q$, $2 \le \ell \le r$

and $2 \leq j \leq p$ and therefore is finite dimensional over k .

If $g \in G$ then $gfg^{-1}(m_i) = \sum_{j=2}^{p} \lambda_{ji}(g^{-1})gf(g_{ji}m_i)$ with $\lambda_{ji} \in k[G]$ since M is a rational G-module. $\sum_{j=2}^{p} \lambda_{ji}(g^{-1})gf(g_{ji}m_i) = \sum_{j=2}^{p} \lambda_{ji}(g^{-1})(\sum_{\ell=2}^{r} \mu_{\ell i}(g)d_{\ell i}f(g_{ji}m_i))$ with $\mu_{\ell i} \in k[G]$ since N is a rational G-module and $f(g_{\ell i}m_i) \in N$ for all i,j . Therefore $gfg^{-1}(m_i) = \sum_{j=2}^{p} \sum_{\ell=2}^{r} \lambda_{ji}(g^{-1})\mu_{\ell i}(g)d_{\ell i}f(g_{ji}m_i)$ with $\lambda_{ji},\mu_{\ell i}$ $k[G]$, $1 \leq i \leq s$, $2 \leq j \leq p$, $2 \leq \ell \leq r$. Thus, V_f is contained in the span of f , $h_{n_i}f$, $d_{\ell i}fg_{ji}$ for all i,j,ℓ,n and the induced homomorphism $G \to GL(V_f)$ is a k-morphism. Moreover, $\mathrm{Hom}_R(M,N) = \bigcup_{f \in \mathrm{Hom}_R(M,N)} V_f$ and therefore a rational G-module.

Thus, $\mathrm{Hom}_R(M,N)$ is an R-*module. Moreover, $^*\mathrm{Hom}(M,N) \subseteq \mathrm{Hom}_R(M,N)$. If $f \in \mathrm{Hom}_R(M,N)$ then $V_f = \langle gfg^{-1} \mid g \in G\rangle$ is finite dimensional over k . Therefore, $f \in {}^*\mathrm{Hom}(M,N)$. That is, $\mathrm{Hom}_R(M,N) \subseteq {}^*\mathrm{Hom}(M,N)$. Then $^*\mathrm{Hom}(M,N) = \mathrm{Hom}_R(M,N)$. This completes the proof of Prop. 1.12.

<u>Proposition 1.13</u> Let M,N be R-*modules. If a G-action on $^*\mathrm{Hom}(M,N)$ is defined by $g \circ f = gfg^{-1}$ for all $g \in G$, $f \in {}^*\mathrm{Hom}(M,N)$ then $^*\mathrm{Hom}(M,N)$ is an R-*module.

<u>Proof</u>: We first prove that $^*\mathrm{Hom}(M,N)$ is closed under addition and $R\,{}^*\mathrm{Hom}(M,N) \subseteq {}^*\mathrm{Hom}(M,N)$, thus establishing that $^*\mathrm{Hom}(M,N)$ is an R-module.

(i) Let $f_1,f_2 \in {}^*\mathrm{Hom}(M,N)$.

$<g(f_1 + f_2)g^{-1} \mid g \in G> \subseteq <gf_1g^{-1} \mid g \in G> + <gf_2g^{-1} \mid g \in G>$ which is finite dimensional over k. Therefore, $f_1 + f_2 \in {}^*\mathrm{Hom}(M,N)$.

(ii) Let $r \in R$, $f \in {}^*\mathrm{Hom}(M,N)$. R is a rational G-module. Therefore, let $<gr \mid g \in G> = <r_1,r_2,\ldots,r_m>$. $<gfg^{-1} \mid g \in G>$ is finite dimensional over k by the definition of ${}^*\mathrm{Hom}(M,N)$. Let $<gfg^{-1} \mid g \in G> = <f_1,f_2,\ldots,f_m>$. Then $<g(rf)g^{-1} \mid g \in G> = <(gr)(gfg^{-1}) \mid g \in G> \subseteq <r_if_j \mid 1 \le i \le n , 1 \le j \le m>$ which is finite dimensional over k. Therefore, $rf \in {}^*\mathrm{Hom}(M,N)$.

(i) and (ii) imply that ${}^*\mathrm{Hom}(M,N)$ is an R-module.

$V_f = <gfg^{-1} \mid g \in G>$ for $f \in {}^*\mathrm{Hom}(M,N)$ is finite dimensional over k. Therefore the argument in Prop. 1.12 can be modified to prove that the induced homomorphism $G \to GL(V_f)$ is a k-morphism. Moreover, ${}^*\mathrm{Hom}(M,N) = \bigcup_{f \in {}^*\mathrm{Hom}(M,N)} V_f$. Then ${}^*\mathrm{Hom}(M,N)$ is a rational G-module and therefore an R-*module. This completes the proof of Proposition 1.13.

<u>Proposition 1.14</u> Let M and M' be R-*modules. Then $M \otimes_R M'$ is an R-*module.

<u>Proof:</u> $M \otimes_R M'$ is an R-module.

(i) A G-action on $M \otimes_R M'$ is defined by $g(m \otimes m') = (gm) \otimes (gm')$ for all $m \in M$, $m' \in M'$ and $g \in G$. If $r \in R$, $m \in M$, $g \in G$ then $g(r(m \otimes m')) = g(rm \otimes m') = g(rm \otimes m') = g(rm) \otimes gm' = ((gr)(gm)) \otimes gm' = (gr)(gm \otimes gm') = (gr)g(m \otimes m')$. This satisfies the requirement for G-action.

(ii) Any element in $M \otimes_k M'$ is of the form $\sum_{i=1}^{n} m_i \otimes m_i'$ with $m_i \in M$ and $m_i' \in M'$ for all i. Let $V_{m_i} = \langle gm_i \mid g \in G \rangle$ and $V_{m_i'} = \langle gm_i' \mid g \in G \rangle$. Since M and M' are R-*modules, V_{m_i} and $V_{m_i'}$ are G-modules. Let $\{\alpha_1,\ldots,\alpha_n\}$ and $\{\beta_1,\beta_2,\ldots,\beta_m\}$ be k-bases for V_{m_i} and $V_{m_i'}$, respectively. Then $\{\alpha_i \otimes \beta_j \mid 1 \le i \le n\ ,\ 1 \le j \le m\}$ is a k-basis for $V_{m_i} \otimes_k V_{m_i'}$. If $g \in G$, then $g\alpha_i = \sum_{j=1}^{n} a_{ij}(g)\alpha_j$ with $a_{ij} \in k[G]$ for all j. $g\beta_j = \sum_{\ell=1}^{m} b_{j\ell}(g)\beta_\ell$ with $b_{j\ell} \in k[G]$ for all ℓ. $g(\alpha_i \otimes \beta_j) = (g\alpha_i) \otimes (g\beta_j) = (\sum_{j=1}^{n} a_{ij}(g)\alpha_j) \otimes (\sum_{\ell=1}^{m} b_{j\ell}(g)\beta_\ell) = \sum_{j=1}^{n}\sum_{\ell=1}^{m} (a_{ij}(g)\alpha_j \otimes b_{j\ell}(g)\beta_\ell) = \sum_{j=1}^{n}\sum_{\ell=1}^{m} (a_{ij}(g) \otimes b_{j\ell}(g))(\alpha_j \otimes \beta_\ell)$ with $a_{ij} \otimes b_{j\ell} \in k[G] \otimes_k k[G] = k[G \times G]$. This establishes that the induced homomorphism $G \to GL(V_{m_i} \otimes_k V_{m_i'})$ is a k-morphism. Therefore, $V_{m_i} \otimes_k V_{m_i'}$ is a rational G-module. So also is $\sum_i V_{m_i} \otimes_k V_{m_i'}$. Let $S_x = \sum_i V_{m_i} \otimes_k V_{m_i'}$. Then $M \otimes_k M' = \bigcup_{x \in M \otimes_k M'} S_x$ is a rational G-module. $M \otimes_k M' \to M \otimes_R M'$ is a G-module surjection. So by Lemma 1.15, $M \otimes_R M'$ is a rational G-module.

(i) and (ii) imply that $M \otimes_R M'$ is an R-*module. This completes the proof of Prop. 1.14.

<u>Lemma 1.15</u> Let W and V be finite dimensional G-modules such that $W \subseteq V$. Then V/W is a G-module.

Proof: Let $x_1, x_2, \ldots, x_m$ be a basis of W and $x_1, x_2, \ldots, x_m, x_{m+1}, \ldots, x_n$ a basis of V. The homomorphism $\mu : G \to GL(V)$ induced by the G-action on V is a k-morphism. That is, if $g \in G$ then

$$gx_i = \sum_{j=1}^{n} \alpha_{ji}(g)x_j \quad \text{with} \quad \alpha_{ji} \in k[G] \, .$$

(i) V/W is a finite dimensional vector space over k with $x_{m+1} + W, \ldots, x_n + W$ as basis.

(ii) Let $y_i = x_i + W$, $m + 1 \leq i \leq n$. A G-action on V/W is defined as $gy_i = gx_i + W$ for $m + 1 \leq i \leq n$ and $gx_i + W = \sum_{j=1}^{n} \alpha_{ji}(g)y_j$. This implies that the homomorphism $\mu : G \to GL(V/W)$ is a k-morphism.

(i) and (ii) establish that V/W is a G-module. This completes the proof of Lemma 1.15.

Proposition 1.16 If N is an R-sub*module of an R-*module M, then M/N is an R-*module.

Proof: M/N is an R-module.

(i) A G-action on M/N is defined as follows. If $g \in G$ and $\bar{x}$ is the canonical image of $x \in M$ in M/N then let $g\bar{x} = \overline{gx}$. If $r \in R$ then $g(r\bar{x}) = g(\overline{rx}) = \overline{g(rx)} = \overline{(gr)(gx)} = (gr)\overline{gx} = (gr)g\bar{x}$.

(ii) If $x \in M$ then let $V_x = \langle gx \mid g \in G \rangle$ and $V_x = (V_x + N)/N = V_x/V_x \cap N \subseteq M/N$. The induced homomorphism $\mu : G \to GL(V_x)$ is a k-morphism and $V_x \cap N$ is a G-submodule of V_x. By Lemma 1.15, $V_{\bar{x}}$ is a G-module. $M/N = \bigcup_{\bar{x} \in M/N} V_{\bar{x}}$ is a rational G-module.

(i) and (ii) establish that M/N is an R-*module. This completes the proof of Prop. 1.16.

Next we show that the category whose objects are R-*modules and whose morphisms are (R-G)-*module homomorphisms as defined in 1.6 (written as (R-G)-homomorphism) is a category. It is a subcategory of $^*\mathrm{Mod}_R$. We denote it by Mod(R-G) . We also establish some properties of Mod(R-G) that make Mod(R-G) an abelian category. We prove that the direct limit of a direct system of R-*modules over a directed set exist in this category. Finally we prove that every R-*module is the direct limit of its family of finitely generated sub*modules.

Definition 1.17 If M and M' are R-*modules then let $\mathrm{Hom}_{RG}(M,M')$ denote the set of (R-G)-homomorphisms from M to M' .

Proposition 1.18 The category whose objects are R-*modules and whose morphisms are (R-G)-homomorphisms is a subcategory of $^*\mathrm{Mod}_R$.

Proof: (R-G)-homomorphisms of R-*modules are R-*module *homomorphisms. Therefore for every pair of objects (M,N) of $^*\mathrm{Mod}_R$, $\mathrm{Hom}_{RG}(M,N) \subseteq {}^*\mathrm{Hom}(M,N)$. Moreover, for any R-*module M , the identity morphism 1_M , defined in Prop. 1.9, preserves G-action and, therefore, is an (R-G)-homomorphism. If M,N,L are R-*modules and $f : M \to N$, $g : N \to L$ are (R-G)-homomorphisms then the composite map gf preserves G-action and therefore is an (R-G)-homomorphism. Thus the category whose objects are R-*modules and whose morphisms are (R-G)-homomorphisms is a subcategory of $^*\mathrm{Mod}_R$. This completes the proof of Prop. 1.18.

Proposition 1.19 If $A \in \mathrm{Mod(R\text{-}G)}$ then $^*\mathrm{Hom}(A,\_)$ and $A \underset{R}{\otimes} \_\_$ are functors from Mod(R-G) to Mod(R-G) . The ordered pair $^*\mathrm{Hom}(A,\_)$, $A \underset{R}{\otimes} \_\_$ is an adjoint pair. That is, if $B,C \in \mathrm{Mod(R\text{-}G)}$ there exists an isomorphism $\Phi : \mathrm{Hom}_{RG}(B, {}^*\mathrm{Hom}(A,C)) \to \mathrm{Hom}_{RG}(A \underset{R}{\otimes} B, C)$ which is natural in each variable.

<u>Proof</u>: (i) Let $B \in \mathrm{Mod}(R\text{-}G)$. Then ${}^*\mathrm{Hom}(A,B) \in \mathrm{Mod}(R\text{-}G)$ by Prop. 1.13.

(ii) Let $f : B \to C$ be in $\mathrm{Mod}(R\text{-}G)$. Then ${}^*\mathrm{Hom}(A,\_)(f) :$ ${}^*\mathrm{Hom}(A,B) \to {}^*\mathrm{Hom}(A,C)$ is defined by $\varphi \mapsto f \circ \varphi$ for all $\varphi \in$ ${}^*\mathrm{Hom}(A,B)$. $f \in \mathrm{Hom}_{RG}(B,C) \subseteq {}^*\mathrm{Hom}(B,C)$ and $\varphi \in {}^*\mathrm{Hom}(A,B)$. Therefore, $f \circ \varphi \in {}^*\mathrm{Hom}(A,C)$. If $r \in R$ then $r\varphi \mapsto f \circ r\varphi$. $f \circ r\varphi(x) =$ $f(r\varphi(x)) = rf(\varphi(x)) = rf \circ \varphi(x)$ for all $x \in A$. Therefore ${}^*\mathrm{Hom}(A,\_)(f) \in \mathrm{Hom}_R({}^*\mathrm{Hom}(A,B),{}^*\mathrm{Hom}(A,C))$. If $g \in G$ then $g \circ \varphi \mapsto f \circ (g \circ \varphi) = f \circ (g\varphi g^{-1}) = gf\varphi g^{-1}$ since $f \in \mathrm{Hom}_{RG}(B,C)$. Therefore, $f \circ (g \circ \varphi) = g \circ ({}^*\mathrm{Hom}(A,\_)(f)(\varphi))$. That is, ${}^*\mathrm{Hom}(A,\_)(f) : {}^*\mathrm{Hom}(A,B) \to {}^*\mathrm{Hom}(A,C)$ is in $\mathrm{Mod}(R\text{-}G)$.

(iii) Let $M \xrightarrow{h} N \xrightarrow{\varphi} P$ be in $\mathrm{Mod}(R\text{-}G)$. Then ${}^*\mathrm{Hom}(A,\_)(\varphi h) :$ ${}^*\mathrm{Hom}(A,M) \to {}^*\mathrm{Hom}(A,P)$ is defined by $f \mapsto \varphi hf$ and $\varphi hf :$ $({}^*\mathrm{Hom}(A,\_)(\varphi) \circ {}^*\mathrm{Hom}(A,\_)(h))(f)$.

(iv) ${}^*\mathrm{Hom}(A,\_)(1_A) : {}^*\mathrm{Hom}(A,A) \to {}^*\mathrm{Hom}(A,A)$ is defined by $\varphi \mapsto$ $1_A\varphi = \varphi$. This implies that ${}^*\mathrm{Hom}(A,\_)(1_A) = 1_{{}^*\mathrm{Hom}(A,A)}$.

(i), (ii), (iii), (iv) prove that ${}^*\mathrm{Hom}(A,\_)$ is a functor from $\mathrm{Mod}(R\text{-}G)$ to $\mathrm{Mod}(R\text{-}G)$.

Next we establish that $A \otimes_R \_$ is a functor from $\mathrm{Mod}(R\text{-}G)$ to $\mathrm{Mod}(R\text{-}G)$.

$A \otimes_R \_$ is a functor from $\mathrm{Mod}(R\text{-}G)$ to the category of abelian groups. Therefore it is sufficient to prove that if $B \in \mathrm{Mod}(R\text{-}G)$ then $A \otimes_R B$ is in $\mathrm{Mod}(R\text{-}G)$ and if $f : B \to C$ is in $\mathrm{Mod}(R\text{-}G)$ then $(A \otimes_R \_)(f) : A \otimes_R B \to A \otimes_R C$ defined by $\sum_i (x_i \otimes y_i) \mapsto \sum_i (x_i \otimes f(y_i))$

is in Mod(R-G) .

(i) By Prop. 1.14, if $B \in$ Mod(R-G) then $A \otimes_R B$ is in Mod(R-G) .

(ii) $(A \otimes_R \_)(f)(r(\Sigma x_i \otimes y_i)) = \Sigma_i (rx_i \otimes f(y_i)) = r(\Sigma_i x_i \otimes f(y_i)) =$ $r(A \otimes \_)(f)(\Sigma_i (x_i \otimes y_i))$. Therefore $(A \otimes_R \_)(f) \in \mathrm{Hom}_R(A \otimes_R B , A \otimes_R C)$.

If $g \in G$, then $g(\Sigma_i (x_i \otimes y_i)) = \Sigma_i (gx_i \otimes gy_i)$.

$(A \otimes_R \_)(f)(\Sigma_i (gx_i \otimes gy_i)) = \Sigma_i (gx_i \otimes f(gy_i)) = \Sigma_i (gx_i \otimes gf(y_i))$ since $f \in \mathrm{Hom}_{RG}(B,C)$. Then $(A \otimes_R \_)(f)(\Sigma_i (gx_i \otimes gy_i)) = \Sigma_i g(x_i \otimes f(y_i))$.

This implies $(A \otimes_R \_)(f) \in \mathrm{Hom}_{RG}(A \otimes_R B , A \otimes_R C)$. Therefore $(A \otimes \_)(f) : A \otimes_R B \to A \otimes_R C$ is in Mod(R-G) .

(i) and (ii) imply that $A \otimes_R \_$ is a functor from Mod(R-G) to Mod(R-G) .

Now we prove the second assertion. Define Φ : $\mathrm{Hom}_{RG}(B, {}^*\mathrm{Hom}(A,C)) \to \mathrm{Hom}_{RG}(A \otimes_R B,C)$ as follows. If $f \in$ $\mathrm{Hom}_{RG}(B,\mathrm{Hom}^*(A,C))$ then let f_b denote $f(b)$ for $b \in B$. Then $f_b : A \to C$ is an R-*module *homomorphism. Define $\Phi(f) : A \otimes_R B \to C$ by $\Sigma_i a_i \otimes b_i \to \Sigma_i f_{b_i}(a_i)$ for $a_i \in A$, $b_i \in B$. This map is the same as the one constructed for proving the adjointness of the pair $\mathrm{Hom}_R(A,C)$, $A \otimes_R \_$ in Mod_R . Therefore, $\Phi(f) \in \mathrm{Hom}_R(A \otimes_R B,C)$. It is sufficient to prove that $\Phi(f)$ preserves G-action. Suppose $g \in G$. Then $\Phi(f)(g(a \otimes b)) = \Phi(f)(ga \otimes gb) = f_{gb}(ga)$. $f \in$ $\mathrm{Hom}_{RG}(B, {}^*\mathrm{Hom}_R(A,C))$ and therefore f preserves G-action. Therefore $f(gb) = g \circ f(b) = g \circ f_b$. $g \circ f_b = gf_bg^{-1}$ and

$f_{gb}(ga) = gf_b g^{-1}(ga) = gf_b(a) = g\Phi(f)(a \otimes b)$.

Now define $\Psi : \mathrm{Hom}_{RG}(A \underset{R}{\otimes} B, C) \to \mathrm{Hom}_{RG}(B, {}^*\mathrm{Hom}(A,C))$ as follows. If $(f : A \underset{R}{\otimes} B \to C) \in \mathrm{Hom}_{RG}(A \underset{R}{\otimes} B, C)$ then let $\Psi(f) : B \to {}^*\mathrm{Hom}(A,C)$ be defined by $b \mapsto (f_b : A \to C)$ where $f_b(a) = f(a \otimes b)$ for all $a \in A$, $b \in B$. This is essentially the same map as the one for proving the adjointness of the pair $\mathrm{Hom}_R(A,C)$, $A \underset{R}{\otimes} \_$ in Mod_R . Therefore, it is sufficient to prove that (i) $\Psi(f)(b) \in {}^*\mathrm{Hom}(A,C)$ for all $b \in B$ and (ii) $\Psi(f)$ preserves G-action.

(i) $b \in B$ and B is an R-*module. Therefore, let $\langle gbg^{-1} \mid g \in G\rangle = \langle b_1, b_2, \ldots, b_n \mid b_i \in B\rangle$. Then $\Psi(f)(b) = f_b$. $gf_b g^{-1} : A \to C$ is defined by $a \mapsto gf_b(g^{-1}a) = gf(g^{-1}a \otimes b) = f(a \otimes gb) = \sum_i \lambda_i f(a \otimes b_i) = \sum_i \lambda_i f_{b_i}(a)$ with $a \in A$, $\lambda_i \in k$ for all i . That is, $\langle gf_b g^{-1} \mid g \in G\rangle \subseteq \langle f_{b_1}, f_{b_2}, \ldots, f_{b_n}\rangle$ and therefore finite dimensional over k .

(ii) $\Psi : \mathrm{Hom}_{RG}(A \underset{R}{\otimes} B, C) \to \mathrm{Hom}_{RG}(B, {}^*\mathrm{Hom}(A,C))$. If $f \in \mathrm{Hom}_{RG}(A \underset{R}{\otimes} B, C)$ and $g \in G$ then $\Psi(g \circ f) = \Psi(gfg^{-1}) = \Psi(gg^{-1}f)$ since f preserves G-action. Therefore, $\Psi(g \circ f) = \Psi(f)$. On the other hand, $g \circ \Psi(f) = g\Psi(f)g^{-1}$ and $g\Psi(f)g^{-1} : B \to {}^*\mathrm{Hom}(A,C)$ where $b \mapsto (g \circ f_{g^{-1}b} : A \to C)$ But $g \circ f_{g^{-1}b}(a) = gf_{g^{-1}b}(g^{-1}a) = gfg^{-1}(a \otimes b) = a \otimes b$. Therefore, $\Psi(g \circ f) = g \circ \Psi(f)$.

That Φ and Ψ are inverse to each other and the isomorphism $\mathrm{Hom}_{RG}(B, {}^*\mathrm{Hom}(A,C)) \cong \mathrm{Hom}_{RG}(A \underset{R}{\otimes} B, C)$ is natural in each variable follows from the isomorphism $\mathrm{Hom}_R(B, \mathrm{Hom}(A,C)) \cong \mathrm{Hom}_R(A \underset{R}{\otimes} B, C)$ and the fact that it is natural in each variable. This completes the proof of

Prop. 1.19.

Remark 1.20 The left exactness of $^*\mathrm{Hom}$ follows from that of Hom . So also the right exactness of $A \otimes_R \_$.

Proposition 1.21 If M and M' are R-*modules then $\mathrm{Hom}_{RG}(M,M')$ is an abelian group under the usual addition of morphisms.

Proof: It is sufficient to prove that $\mathrm{Hom}_{RG}(M,M')$ is a subgroup of the abelian group $\mathrm{Hom}_R(M,M')$.

Suppose $\varphi \in \mathrm{Hom}_{RG}(M,M')$. $(-\varphi)$ preserves G-action. If $\varphi,\psi \in \mathrm{Hom}_{RG}(M,M')$ then $(\psi - \varphi)$ preserves G-action. Thus, $(-\varphi)$, $(\varphi - \psi)$ are in $\mathrm{Hom}_{RG}(M,M')$. This completes the proof of Prop. 1.21.

Proposition 1.22 Composition of morphisms is bilinear in Mod(R-G) . That is, given R-*modules M,N,P and (R-G)-homomorphisms $M \underset{f'}{\overset{f}{\rightrightarrows}} N$, $N \underset{g'}{\overset{g}{\rightrightarrows}} P$, the distributive laws $(g + g') \circ f = g \circ f + g' \circ f$ and $g \circ (f + f') = g \circ f + g \circ f'$ are satisfied.

Proof: The distributive laws are satisfied in Mod_R . But $(g + g')f$, gf , gf' , $gf + gf'$, $g(f + f')$ are (R-G)-homomorphisms. Therefore, the above equalities are true in Mod(R-G) also.

Proposition 1.23 Mod(R-G) has a zero object such that for each object $A \in \mathrm{Mod}(R\text{-}G)$ there is a unique homomorphism $0 \to A$ and a unique morphism $A \to 0$.

Proof: The zero object 0 of Mod_R is an object of Mod(R-G) since the zero module is an R-*module. If M is an R-*module then each set $\mathrm{Hom}_{RG}(0,M)$ and $\mathrm{Hom}_{RG}(M,0)$ has exactly one element, the inclusion map and the zero map, respectively, for if they have more than one element then 0 cannot be the zero object in Mod_R . This completes the proof of Prop. 1.23.

Proposition 1.24 For every pair of objects M,N in Mod(R-G) there is a diagram in the category $M \underset{i_1}{\overset{p_1}{\rightleftarrows}} C \underset{i_2}{\overset{p_2}{\rightleftarrows}} N$ with $p_1 \circ i_1 = 1_M$, $p_2 \circ i_2 = 1_N$ and $(i_1 \circ p_1) + (i_2 \circ p_2) = 1_C$.

Proof: M,N are in Mod_R. Therefore there is a diagram in Mod_R, $M \underset{i_1}{\overset{p_1}{\rightleftarrows}} C \underset{i_2}{\overset{p_2}{\rightleftarrows}} N$, where $C = M \oplus N$, p_1 and p_2 are projection maps, i_1 and i_2 are inclusion maps satisfying the above equalities. But $M \oplus N$ is an R-*module (by 1.10). Projection and inclusion maps preserve G-action. Therefore, $M \underset{i_1}{\overset{p_1}{\rightleftarrows}} M \oplus N \underset{i_2}{\overset{p_2}{\rightleftarrows}} N$ is the required diagram in Mod(R-G). This completes the proof of Prop. 1.24.

Propositions 1.21, 1.22, 1.23 and 1.24 establish that Mod(R-G) is an additive category.

Proposition 1.25 If M and M' are R-*modules and $f : M \to M'$ is and(R-G)-homomorphism then the kernel object and the cokernel object in Mod_R are R-*modules. We denote them by ker f and coker f, respectively.

Proof: $\ker f = \{m \in M \mid f(m) = 0\}$ is an R-module

(i) G-action on ker f is the same as that on M. Therefore, $g(rm) = (gr)(gm)$ for all $g \in G$, $r \in R$, $m \in \ker f$.

(ii) Let $m \in \ker f$ and $V_m = \langle gm \mid g \in G\rangle$. $f(gm) = gf(m) = 0$ for all $g \in G$. Therefore, $V_m \subseteq \ker f$ and $\ker f = \bigcup_{m \in \ker f} V_m$. V_m is a rational G-module since M is an R-*module. Thus, ker f is a rational G-module and therefore an R-*module.

coker $f = M'/f(M)$ is an R-module.

(i) G-action on $f(M)$ is the same as that on M' .

(ii) Any element in $f(M)$ is of the form $f(m)$ with $m \in M$. $V_{f(m)} = \langle gf(m) \mid g \in G \rangle \subseteq f(M)$. Thus, $f(M) = \bigcup_{m \in M} V_{f(m)}$. $V_{f(m)}$ is a rational G-module since M' is an R-*module. Therefore, $f(M)$ is a rational G-module.

(i) and (ii) establish that $f(M)$ is an R-sub*module of M' . By Prop. 1.16, coker f is an R-*module. This completes the proof of Prop. 1.25.

<u>Proposition 1.26</u> If M and M' are R-*modules then every (R-G)-homomorphism $f : M \to M'$ has a kernel and a cokernel.

<u>Proof:</u> By Prop. 1.25, ker f and coker f are R-*modules.

(i) Consider the (R-G)-homomorphism $i : \ker f \to M$ which is the inclusion map. Then $f \circ i$ is the zero map. If $h : P \to M$ is any (R-G)-homomorphism such that $f \circ h$ is the zero map then there is a unique R-module homomorphism $h' : P \to \ker f$ with $h = i \circ h'$. That is, we have the following commutative diagram in Mod_R .

$$\begin{tikzcd}
& P \arrow[dl, "h'"'] \arrow[d, "h"] & \\
\ker f \arrow[r, tail] & M \arrow[r] & M'
\end{tikzcd}$$

If $g \in G$, $x \in P$ then $i \circ h'(gx) = h(gx) = gh(x) = gh'(x)$ since i is the inclusion map. Therefore, $h'(gx) = gh'(x)$. That is, h' is an (R-G)-homomorphism. (h' is unique in Mod(R-G) since it is unique in Mod_R .) Therefore, $\ker f \overset{i}{\rightarrowtail} M$ is the kernel of $f : M \to M'$ in Mod(R-G) .

(ii) Consider the canonical map $M \xrightarrow{\eta} M/\ker f$. η preserves G-action. $\eta \circ f$ is the zero map. If $h : M \to P$ is any (R-G)-homomorphism then there is a unique R-module homomorphism $h' : M/\ker f \to P$ with $h = h' \circ \eta$. That is, we have the following commutative diagram in Mod_R .

$$\begin{tikzcd} M \arrow[r, "\eta"] \arrow[d, "h"'] & M/\ker f \arrow[dl, dashed, "h'"] \\ P & \end{tikzcd}$$

If $g \in G$ and $\bar{m} \in M/\ker f$ where $\bar{m} = \eta(m)$ with $m \in M$ then $h'(\bar{m}) = h(m)$. $h'(g\bar{m}) = h'(\overline{gm}) = h(gm) = gh(m) = gh'(\bar{m})$. h' preserves G-action. h' is unique in Mod(R-G) since it is unique in Mod_R . Therefore, $\eta : M \to M/\ker f$ is the cokernel of $f : M \to M'$ in Mod(R-G) . This completes the proof of Prop. 1.26.

Remark 1.27 The above proposition implies that if $f : M \to M'$ is an (R-G)-homomorphism then the kernel of f in Mod(R-G) is the kernel of f in Mod_R and the cokernel of f in Mod(R-G) is the cokernel of f in Mod_R .

Proposition 1.28 If $f : M \to M'$ is an (R-G)-homomorphism whose kernel is 0 , then f is the kernel of its cokernel. If $f : M \to M'$ is an (R-G)-homomorphism whose cokernel is 0 , then f is the cokernel of its kernel. An (R-G)-homomorphism whose kernel and cokernel are 0 is an (R-G)-isomorphism.

Proof: Mod_R is an abelian category. Therefore the above statement is true in Mod_R . By Remark 1.27, the proposition is true in Mod(R-G) . This completes the proof of Prop. 1.28.

Propositions 1.25, 1.26, 1.28 establish that Mod(R-G) is an abelian category.

A direct system $\{M,\pi\}$ over the directed set I, in Mod(R-G), means that to each $i \in I$, there is an R-*Module M_i and to each pair $(i,j) \in I \times I$ with $i < j$, there is an (R-G)-homomorphism $\pi_i^j : M_i \to M_j$ such that for all $i \in I$, π_i^i is the identity map and for $i < j < \ell$ in I, $\pi_j^\ell \circ \pi_i^j = \pi_i^\ell$.

<u>Proposition 1.29</u> Let $\{M,\pi\}$ be a direct system over I in Mod(R-G). For each pair $(j,\ell) \in I \times I$ with $j < \ell$ and each $m_j \in M_j$, the element $\pi_j^\ell m_j - m_j$ is an element of $\bigoplus_i M_i$. Then there exists a smallest R-sub*module N of $\bigoplus_i M_i$ containing such elements as $\pi_j^\ell m_j - m_j$ for all pairs $(j,\ell) \in I \times I$ with $j < \ell$. Moreover the quotient module $\bigoplus_i M_i/N$ is an R-*module.

<u>Proof</u>: If N is an R-sub*module of $\bigoplus_i M_i$ then $\bigoplus_i M_i/N$ is an R-*module by Prop. 1.16. Therefore it is sufficient to prove that such an N exists. We prove this by using Zorn's Lemma. Let A be the collection of all R-sub*modules of $\bigoplus_i M_i$ containing the set of elements described in the proposition. This collection is nonempty since $\bigoplus_i M_i \in A$. A can be partially ordered by $\supseteq$. If $\{N_i\}$ is any chain in A then $\cap N_i$ is an R-*module in A and is minimal for $\{N_i\}$. Therefore, A has a minimal element N. This completes the proof of Proposition 1.29.

We now establish that the above quotient $\bigoplus_i M_i/N$ is the appropriate categorical definition of direct limit in the category

Mod(R-G) .

<u>Proposition 1.30</u> Let $\{M,\pi\}$ be a direct system over the directed set I as in Prop. 1.29. Let $\underset{i}{\oplus} M_i/N$ be the $R\text{-}^*$module described in Prop. 1.29. If P is an $R\text{-}^*$module and for each $i \in I$ there are (R-G)-homomorphisms $f_i : M_i \to P$ such that $f_j \pi_i^j = f_i$ for all $i < j$, then there is a unique (R-G)-homomorphism $\Phi : \underset{i}{\oplus} M_i/N \to P$ such that $\Phi(x + N) = f_i(x)$ if $x \in M_i \subseteq \underset{i}{\oplus} M_i$.

<u>Proof:</u> $\eta : \underset{i}{\oplus} M_i \to \underset{i}{\oplus} M_i/N$ is the cokernel of $N \to \underset{i}{\oplus} M_i$. The set $\{f_i \mid i \in I\}$ induces $f : \underset{i}{\oplus} M_i \to P$ with $f(N) = 0$. Therefore there exists a $\Phi : \underset{i}{\oplus} M_i/N \to P$ satisfying the above conditions. This completes the proof of Prop. 1.30.

<u>Proposition 1.31</u> Let P be an $R\text{-}^*$module. Then P is the direct limit of the family of finitely generated R-sub*modules of P .

<u>Proof:</u> Let $A = \{M_i \mid i \in I\}$ be the collection of finitely generated R-sub*modules of P . $A \neq \phi$. For if $p \in P$, $p \neq 0$, then $RV_p \in A$. Let $M_i, M_j \in A$ and let $M_i = \langle m_1, \ldots, m_s \rangle$ and $M_j = \langle n_1, \ldots, n_r \rangle$ as $R\text{-}^*$modules. $V_{m_i} = \langle gm_i \mid g \in G \rangle$ is a finitely generated vector space over k . Let $V_{m_i} = \langle m_{i_1}, m_{i_2}, \ldots, m_{i_{\ell_i}} \rangle$, $1 \le i \le s$ and $V_{n_i} = \langle n_{i_1}, \ldots, n_{i_{k_i}} \rangle$, $1 \le i \le r$ as k-vector spaces. Let $V = \langle m_{11}, \ldots, m_{1s_1}, \ldots, m_{s\ell_s}, n_{11}, \ldots, n_{rk_r} \rangle$ be a finitely generated vector space over k , which is G-stable. This implies that RV is a finitely generated $R\text{-}^*$module. Moreover, $M_i + M_j \subseteq RV \in A$. That is, there exists a $k_0 \in I$ $\ni RV = M_{k_0}$ and $M_i + M_j \subseteq M_{k_0}$. Define $i \le j$ if $M_i \subseteq M_j$ and let

$\mu_{ij} : M_i \to M_j$ be the embedding of M_i in M_j . By this definition I is a directed set. Then $\varinjlim M_i = \sum_i M_i = \bigcup_i M_i \subseteq P$. If $p \in P$ then $RV_p \in A$. This implies that $P \subseteq \bigcup M_i$. Therefore $P = \varinjlim M_i$. This completes the proof of Proposition 1.31.

CHAPTER II

*SEMI*SIMPLICITY

Starting with the definitions of a *simple *ring and a *simple R-*module, we define a *semi*simple R-*module. We show that if G is a connected linear algebraic group and R is a *simple *ring then every finitely generated R-*module is R-torsion free and R-projective. Thus a *simple *ring is an integral domain if G is a connected linear algebraic group. If further G is a connected, linearly reductive algebraic group and R is a *simple *ring then every R-*module is *semi*simple and, therefore, R-projective.

<u>Notation</u>: $\underset{G}{\cong}$, $\underset{R}{\cong}$, $\underset{RG}{\cong}$ denote G-module isomorphism, R-module isomorphism and R-G isomorphism, respectively.

<u>Definition 2.1</u> An <u>*ideal</u> of a *ring R is an ideal which is a sub-*module of R . [M]

<u>Definition 2.2</u> A <u>*ring</u> R is <u>*simple</u> if the only *ideals of R are the zero *ideal and R . [M]

<u>Definition 2.3</u> An <u>R-*module</u> M is <u>*simple</u> if the only sub*modules of M are the zero *module and M .

Example of a *Simple *Ring 2.4

If G is a linear algebraic group over k then $k[G]$ is a finitely generated k-algebra. This can be made into a *ring by defining a G-action on $k[G]$. If $g \in G$, $f \in k[G]$ then let $g \circ f(h) = f(g^{-1}h)$ for all $h \in G$. If $\Delta : k[G] \to k[G] \otimes_k k[G]$ be comultiplication, then $\Delta(f) = \sum_1^n a_i \otimes b_i$ if $f(xy) = \sum_i a_i(x)b_i(y)$ for all $x,y \in G$. Then $(g \circ f)(x) = f(g^{-1}x) = \sum_i a_i(g^{-1})b_i(x)$. So $g \circ f = \sum_i a_i(g^{-1})b_i$ and $V_f = \langle gf \mid g \in G\rangle \subseteq \langle b_1, b_2, \ldots, b_n\rangle$. Moreover, $G \to GL(V_f)$ is known to be a k-morphism. Suppose I is a nonzero *ideal in $k[G]$. $I \subset k[G]$ implies $V(I) \subset G$. $V(I) \neq 0$ and $V(I)$ is closed in G. If $x \in V(I)$ with $x \neq 0$, $f \in I$ with $f \neq 0$ and $g \in G$ then $f(x) = 0$ and $g \circ f(x) = 0$. Therefore, $f(g^{-1}x) = 0$. This implies that $g^{-1}x \in V(I)$. That is, $Gx \subseteq V(I) \subset G$. But $Gx = G$. Therefore, $V(I) = G$. This implies that $I = 0$, which is a contradiction. Therefore the only *ideals of $k[G]$ are the zero *ideal and $k[G]$.

Remark 2.5 The following proposition leads to the definition of *semi-*simple R-*modules.

Proposition 2.6 Let G be a linear algebraic group over k, R a *ring and M a nonzero R-*module. Then the following conditions on M are equivalent.

(i) M is the sum of a family of *simple sub*modules of M.

(ii) M is the direct sum of a family of *simple sub*modules of M.

(iii) Every sub*module N of M is a direct summand. That is, there exists a sub*module N' of M such that $M = N \oplus N'$.

Proof: (i) $\Rightarrow$ (ii). Let $M = \sum_{i\in I} M_i$ be a sum (not necessarily direct) of *simple sub*modules where I is the indexing set. Let J be a maximal subset of I such that $M' = \sum_{i\in J} M_i$ is a direct sum. If $j \in I$, $M_j \cap M'$ is a sub*module of M_j. But M_j is *simple. Therefore $M_j \cap M' = M_j$ or the zero *module 0. If $M_j \cap M'$ is 0 then $M_j + M'$ is a direct sum contradicting the maximality of J. Therefore, $M_j \subseteq M'$ for all $j \in I$. This implies that $M = \sum_{j\in J} M_j$ which is a direct sum.

(ii) $\Rightarrow$ (iii). Let N be a sub*module of M where $M = \sum_{i\in I} M_i$ is a direct sum of *simple sub*modules of M. Let J be the maximal subset of I such that $M' = N + \sum_{i\in J} M_i$ is a direct sum. By repeating the same argument as above, $M' = M$. That is, N is a direct summand of M.

(iii) $\Rightarrow$ (i). It is sufficient to prove that M contains a *simple sub*module. If M is not *simple, let m be a nonzero element of M. Let $V_m = \langle gm \mid g \in G\rangle$. Then RV_m is a sub*module of M. If RV_m is not *simple, then replace M by RV_m. Now we prove that RV_m contains a *simple sub*module. For this, we use Zorn's Lemma. Let $A = \{M' \subset M \mid m \notin M', M' \text{ is a sub}^*\text{module of } M\}$. Since M is not *simple, let N be a proper sub*module of M. Either $m \in N$ or $m \notin N$. Suppose $m \in N$. By (iii) there exists a nonzero sub*module N' of M such that $M = N \oplus N'$. Then $m \notin N'$ and, therefore, $N' \in A$. This proves that $A \neq \phi$.

Let $\{M_i\}$ be a chain in A. $\cup M_i$ is a sub*module of M and

$m \notin \cup M_i$. Therefore, $\cup M_i \in A$. By Zorn's Lemma A has a maximal element. Let M_1 be the maximal element in A . That is, if M' is a sub*module of M such that M' does not contain m and $M' \supset M_1$, then $M' = M_1$. Let $\bar{N}$ be a nonzero sub*module of M/M_1 where $\bar{N}$ is the canonical image of $N \subseteq M$ in M/M_1 . Since $\bar{N} \neq 0$, $N \neq M$, so $m \in N$. Since $M = RV_m$, $M = N$. Then $\bar{N} = M/M_1$. Therefore, M/M_1 is *simple. By (iii), there exists a sub*module M_2 of M such that $M = M_1 \oplus M_2$. The canonical map $\eta : M \twoheadrightarrow M/M_1$ induces an R-module isomorphism $f : M_2 \twoheadrightarrow M/M_1$. If $g \in G$, $m_2 \in M_2$ then $f(gm_2) = gm_2 + M_1 = g(m_2 + M_1) = gf(m_2)$. That is, f preserves G-action. Therefore, M_2 and M/M_1 are isomorphic as rational G-modules. Therefore, M_2 is *simple. Thus, M contains a *simple sub*module M_2 . This completes the proof of Prop. 2.6.

Definition 2.7 An R-*module satisfying the above three conditions is said to be a *semi*simple R-*module.

Definition 2.8 A *ring R is said to be *semi*simple if R is *semi-*simple as a *module over itself.

Proposition 2.9 Every sub*module and every factor *module of a *semi-*simple *module is *semi*simple.

Proof: (i) Let N be a sub*module of an R-*module M where M is *semi*simple. Let N' be the sum of all *simple sub*modules of N . Since M is *semi*simple, $M = N' \oplus M'$ where M' is a sub*module of M . If $x \in N$, $x \neq 0$ then $x = n' + m'$, $n' \in N'$ and $m' \in M'$. Therefore, $m' = x - n' \in N$ and $N = N' \oplus M' \cap N$, a direct sum. This contradicts the maximality of N' since $M' \cap N$ is a sub*module of M and therefore is either *simple or contains a *simple sub*module. Thus,

$N = N'$.

(ii) Let N be a sub*module of M . M/N is an R-*module. M is *semi*simple. Therefore, $M = N \oplus N'$. N' is the direct sum of *simple sub*modules of M , by (i). The canonical map $\eta : M \twoheadrightarrow M/N$ induces an R-module isomorphism $f : N' \twoheadrightarrow M/N$. If $g \in G$, $n' \in N'$ then $f(gn') = g(n' + N) = gf(n')$. That is, f preserves G-action. Therefore, N' and M/N are isomorphic as rational G-modules. Therefore M/N is the direct sum of *simple sub*modules of M since N' is. This completes the proof of Corollary 2.9.

Lemma 2.10 Let G be a connected linear algebraic group. If R is a *simple *ring and M a finitely generated nonzero R-*module then M is a torsion free R-module and R is an integral domain.

Proof: R is Noetherian. Therefore there are only finitely many associated primes of M . Let $\{M_1, M_2, \dots, M_r\} = \mathrm{ass}(M)$. Let each M_i be the annihilator, $\mathrm{ann}(a_i)$, of $a_i \in M$, $a_i \neq 0$. The zero divisors $Z(R)$ of M is $\bigcup_i M_i$. We prove that each M_i is G-stable and, in fact, if $g \in G$ then $gM_i = M_i$ for all i .

(i) Let $xy \in g(M_i)$, $x \neq 0$, $y \neq 0$. Then $xy = ga$ for some $a \in M_i$. Therefore, $(g^{-1}x)(g^{-1}y) = a$ and is in M_i . Since M_i is a prime ideal either $g^{-1}x \in M_i$ or $g^{-1}y \in M_i$. That is, either $x \in g(M_i)$ or $y \in g(M_i)$ proving that $g(M_i)$ is a prime ideal for all $g \in G$ and for all i .

(ii) $M_i = \mathrm{ann}(a_i)$, $a_i \in M$, $a_i \neq 0$. If $x \in g(M_i)$, $x \neq 0$, $g \in G$ then $x = ga$ for some $a \in M_i$, $a \neq 0$. Then $aa_i = 0$; $\Rightarrow g(aa_i) = 0$; $\Rightarrow (ga)(ga_i) = 0$; $\Rightarrow ga \in \mathrm{ann}(ga_i)$; $\Rightarrow g(M_i) \subseteq \mathrm{ann}(ga_i)$. Conversely, if $r \in \operatorname{ann}(ga_i)$ then $r(ga_i) = 0$; $\Rightarrow (g^{-1}r)a_i = 0$; $\Rightarrow g^{-1}r \in M_i$; $\Rightarrow r \in g(M_i) \Rightarrow \operatorname{ann}(ga_i) \subseteq g(M_i)$; $\Rightarrow g(M_i) = \operatorname{ann}(ga_i)$ for all i, where $ga_i \neq 0$. (i) and (ii) imply that $g(M_i) = M_j$ for some $1 \leq j \leq r$.

(iii) But G is connected and G permutes the finite number of elements $M_1, M_2, \ldots, M_r$. This implies that $g(M_i) = M_i$ for all i . Therefore, M_i is an *ideal of R for all i . But R is *simple. Therefore, $M_i = 0$ for all i . Then M is R-torsion-free.

R is a finitely generated R-*module. Therefore by the above result R is an integral domain.

This completes the proof of Lemma 2.10.

<u>Proposition 2.11</u> Let G be a connected linear algebraic group. If R is a finitely generated k-algebra that is *simple and M a nonzero finitely generated R-*module, then M is R-projective.

<u>Proof</u>: We first establish that if S is a multiplicatively closed subset of R and $S^{-1}M$ is a free $S^{-1}R$-module generated by $\frac{m_1}{1}, \frac{m_2}{1}, \ldots, \frac{m_n}{1}$ with $m_i \in M$, for all i , then there exists an $\alpha_0 \in S$ such that

(a) $F = \sum_i Rm_i$ is a free R-module.

(b) If $S_1 = \{1, \alpha_0, \alpha_0^2, \ldots\}$ then $S_1^{-1}F = S_1^{-1}M$ and are free as $S_1^{-1}R$-modules with $\frac{m_1}{1}, \frac{m_2}{1}, \ldots, \frac{m_n}{1}$ as basis.

(i) R is an integral domain. Let K be its quotient field. An R-module homomorphism $\varphi : M \to K \otimes_R M = (R-0)^{-1}M$ defined as $m \mapsto 1 \otimes m$

for all $m \in M$ is injective. $(R-0)^{-1}M$ is a K-vector space and M is a finitely generated R-module. Therefore, $(R-0)^{-1}M$ is finite dimensional over K. Let $\frac{m_1}{1}, \ldots, \frac{m_n}{1}$ be the K-basis with $m_i \in M$ for all i.

(ii) Let $F = \Sigma Rm_i \subseteq M$. Suppose there exist $r_i \in R$, $1 \le i \le n$ such that $\sum_i r_i m_i = 0$. Then, $\sum_i r_i \frac{m_i}{1} = 0$. But $\sum_i r_i \frac{m_i}{1} \in (R-0)^{-1}M$, a K-vector space with $\frac{m_1}{1}, \ldots, \frac{m_n}{1}$ as basis. Therefore $r_i = 0$, $1 \le i \le n$. Therefore, F is a free R-module with $m_1, m_2, \ldots, m_n$ as basis.

Moreover, $F \subseteq M$ implies $(R-0)^{-1}F \subseteq (R-0)^{-1}M$. If $x \in (R-0)^{-1}M$, $x \neq 0$, then $x = \sum_i \alpha_i \frac{m_i}{1}$ with $\alpha_i \in K$ and $\sum_i \alpha_i \frac{m_i}{1} = \sum_i \frac{\alpha_i}{1} m_i \in (R-0)^{-1}F$. Thus, $(R-0)^{-1}F = (R-0)^{-1}M$. $(R-0)^{-1}M$ has a K-basis $\frac{m_1}{1}, \ldots, \frac{m_n}{1}$. Therefore, $(R-0)^{-1}F$ has $\frac{m_1}{1}, \ldots, \frac{m_n}{1}$ as K-basis. Moreover, $(R-0)^{-1}M/F = 0$. This implies that $\exists\ d_0 \in R-0$ such that $S_0^{-1}M/F = 0$ where $S_0 = \{1, d_0, d_0^2, \ldots\}$. That is, $S_0^{-1}M = S_0^{-1}F$.

(iii) Now we will prove that $S_0^{-1}F$ is a free $S_0^{-1}R$ module.

If $x \in S_0^{-1}F$ then $x = \sum_i \frac{r_i m_i}{d_0^\alpha}$ with $r_i \in R$, $\alpha \in Z^+$ for all i. Therefore, $S_0^{-1}F$ is generated by $\frac{m_1}{1}, \ldots, \frac{m_n}{1}$ over $S_0^{-1}R$.

Suppose there exist $r_1, \ldots, r_n \in R$ such that $\frac{r_1}{d_0^{\alpha_1}} \cdot \frac{m_1}{1} + \cdots + \frac{r_n}{d_0^{\alpha_n}} \cdot \frac{m_n}{1} = 0$ with $\alpha_i \in Z^+$ for all i. This implies $\sum_i \frac{r_i' m_i}{d_0^\alpha} = \frac{0}{d_0^\beta}$

for $\alpha,\beta \in Z^+$, $r_i' \in R$ for all i . Then there exists $d_0^\gamma \in S_0$ such that $d_0^\gamma d_0^\beta(\sum_i r_i'm_i) = 0$. But F is a free $R-^*$module. Therefore, $r_i'd_0^{\gamma+\beta} = 0$ for all i . But R is an integral domain and $d_0^{\gamma+\beta} \neq 0$. Therefore, $r_i' = 0$ for all i . That is, $\frac{r_i}{d_0^{\alpha_i}} = 0$ for all i .

Thus, $S_0^{-1}F$ is free over $S_0^{-1}R$. Since $S_0^{-1}F = S_0^{-1}M$, each is generated by $\frac{m_1}{1}, \frac{m_2}{1}, \ldots, \frac{m_n}{1}$ over $S_0^{-1}R$.

Suppose S is a multiplicatively closed subset of R and $S^{-1}M$ is a free $S^{-1}R$-module generated by $\frac{m_1}{1}, \frac{m_2}{1}, \ldots, \frac{m_\ell}{1}$ with $m_i \in M$, for all i . Then we can replace K by $S^{-1}R$ in (i), (ii) and (iii) thus establishing (a) and (b).

(iv) Let $\mathcal{M}$ be a maximal ideal of R such that $M_{\mathcal{M}}$ is a free $R_{\mathcal{M}}$-module with $\frac{m_1}{1}, \ldots, \frac{m_r}{1}$ as basis where $m_i \in M$ for all i . By what we have established above there exists a $d \in R - \mathcal{M}$ such that if $F = \sum_i Rm_i$, then F is R-free and $S^{-1}F = S^{-1}M$ as $S^{-1}R$-modules where $S = \{1,d,d^2,\ldots\}$. Moreover, $S^{-1}F$ and $S^{-1}M$ are free $S^{-1}R$-modules with $\frac{m_1}{1}, \ldots, \frac{m_r}{1}$ as basis.

Let $\mathcal{M}'$ be any maximal ideal of R such that $d \notin \mathcal{M}'$. This implies that $S \subseteq R - \mathcal{M}'$. $S^{-1}F = S^{-1}M$ implies $(S^{-1}F)_{R-\mathcal{M}'} = (S^{-1}M)_{R-\mathcal{M}'}$. Then $(R - \mathcal{M}')^{-1}F = (R - \mathcal{M}')^{-1}M$ as $(R - \mathcal{M}')^{-1}R$-modules. That is, $F_{\mathcal{M}'} = M_{\mathcal{M}'}$ as $R_{\mathcal{M}'}$-modules. By an argument similar to that in (iii) we can establish that $F_{\mathcal{M}'}$ is a free $R_{\mathcal{M}'}$-module.

(v) Now we will establish that there exists a maximal ideal M_0 of R such that M_{M_0} is a free R_{M_0}-module.

R is a Hilbert ring and is a domain. Therefore the Jacobson radical $J = 0$. This implies that there is a maximal ideal M_0 of R such that $d_0 \notin M_0$ (d_0 defined in (ii)). For otherwise, $d_0 \in J$ which is a contradiction. Then $S_0 \subseteq R - M_0$. But by (iii), $S_0^{-1}F = S_0^{-1}M$. Therefore, by localizing at $R - M_0$, we have $F_{M_0} = M_{M_0}$. By an argument similar to that in (iii) we can establish that F_{M_0} is R_{M_0}-free. Therefore, M_{M_0} is R_{M_0}-free.

(vi) Let $\max(R)$ be the collection of maximal ideals of R. Let $X = \{M \in \max(R) \mid M_M$ is a free R_M-module$\}$. By (v), X is nonempty. If $M \in X$ then, by (iv), there exists a $d \notin M$, such that if M' is a maximal ideal of R not containing d then $M_{M'}$ is a free $R_{M'}$-module.

If $U_s = \{M \in \max(R) \mid s \notin M\}$ then U_s is a basic open set in $\max(R)$ under the Zariski topology on $\max(R)$. Therefore, $M \in X$ implies that $M \in U_d \subseteq X$ for some $d \in R - M$. Thus, X is open in $\max(R)$.

Now we will prove that X is G-stable. If $M \in X$ then M_M is a free R_M-module. Suppose for $g \in G$, $g(M) \subsetneq M'$ where M' is a maximal ideal of R. Then $M \subsetneq g^{-1}(M')$, which is a contradiction. Therefore, $g(M)$ is maximal for all $g \in G$. Now it is sufficient to prove that $M_{g(M)}$ is a free $R_{g(M)}$-module.

Let $\frac{m_1}{1}, \ldots, \frac{m_n}{1}$ be the basis of M_M as free R_M-module with $m_i \in M$ for all i. If $g \in G$ then let N be the $R_{g(M)}$-module generated by $\frac{g(m_1)}{1}, \frac{g(m_2)}{1}, \ldots, \frac{g(m_n)}{1}$. If there exists $\frac{r_i}{s_i} \in (R - g(M))^{-1}R$ for all i such that $\sum_{i=1}^{n} \frac{r_i}{s_i}(g(m_i)) = \frac{0}{s}$ with $s \in R - g(M)$ then $g(\sum_{i=1}^{s} g^{-1}(r_i)g^{-1}(s_i')m_i) = 0$ with $g^{-1}s_i' \in R - M$. This can be written as $\sum_{i=1}^{s} \frac{g^{-1}(r_i)}{q_i}\frac{m_i}{1} = \frac{0}{q}$ with $q_i, q \in R - M$. But M_M is R_M-free. Therefore, $\frac{g^{-1}(r_i)}{q_i} = \frac{0}{p_i}$ with $q_i, p_i \in R - M$ for all i. Then, $p_i g^{-1}(r_i) = 0$ for all i. That is, $g(p_i)r_i = 0$ for all i. But $g(p_i) \neq 0$. Therefore, $r_i = 0$ for all i since R is an integral domain. Thus, N is $R_{g(M)}$-free. It remains to be proved that $M_{g(M)} \subseteq N$.

If $y \in M$, $y \neq 0$ then $y = g(g^{-1}y)$. $g^{-1}(y) = (r_1m_1 + r_2m_2 + \cdots + r_nm_n)$ for some $r_i \in R$ for all i. Therefore, $y = (gr_1)(gm_1) + \cdots + (gr_n)(gm_n)$ with $gr_i \in R$ for all i. Then $\frac{y}{1} \in N$. Thus $M_{g(M)} \subseteq N$ proving $M_{g(M)} = N$. $M_{g(M)}$ is $R_{g(M)}$-free for all $g \in G$. Therefore X is G-stable.

(vii) We now prove that M_M is R_M-free for every $M \in \max(R)$.

By (vi), X is open in $\max(R)$. Therefore, $\max(R) - X$ is closed in $\max(R)$ under Zariski topology and is G-stable. Suppose $\max(R) - X \neq \phi$. $(\max(R), R)$ is an affine algebraic set and $\max(R) - X$ is closed in $\max(R)$. This implies that there exists an ideal I of R such that $\max(R) - X = \{M \in \max(R) \mid M \supseteq I\}$. $\max(R) - X$ is G-stable since X is. Therefore, I is G-stable. This implies that I

is an *ideal. But R is *simple. Therefore, either $I = 0$ or $I = R$. But $I \neq R$. Therefore, $I = 0$. Then $\max(R) - X = \max(R)$. This implies that $X = \phi$ which is a contradiction. Therefore, $\max(R) - X = \phi$. That is, $\max(R) = X$. Therefore, $M_{\mathcal{M}}$ is $R_{\mathcal{M}}$-free for every $\mathcal{M} \in \max(R)$.

(viii) R is Noetherian. M is a finitely generated R-module, therefore is of finite presentation. Moreover, $M_{\mathcal{M}}$ is $R_{\mathcal{M}}$-free for all maximal ideals $\mathcal{M}$ of R. Therefore, M is R-projective. [(K), 3.3.7] This completes the proof of Proposition 2.11.

<u>Corollary 2.11.1</u> Let G be a connected linear algebraic group. If R is a finitely generated k-algebra that is *simple and M a nonzero R-*module then M is R-flat.

<u>Proof</u>: By Prop. 1.31, M is the direct limit of the family of finitely generated R-sub*modules of M. Every nonzero finitely generated R-*module is R-projective and the direct limit of a family of R-projective modules is R-flat. This completes the proof of Cor. 2.11.1.

We quote some definitions and results from Fogarty's Invariant Theory needed for further development of this theory.

<u>Definition 2.12</u> If G is an affine group, we say that G is <u>linearly reductive</u> if every rational G-module is completely reducible. [F, 4.6]

<u>Notation:</u> If M is a rational G-module then $M^G = \{m \in M \mid gm = m, \forall g \in G\}$.

<u>Definition 2.13</u> If M is a rational G-module, then M is said to be <u>G-ergodic</u> if $M^G = (0)$.

<u>From now on we assume that G is a linearly reductive algebraic group.</u>

<u>Lemma 2.14</u> Any rational G-module M contains a unique G-ergodic

submodule M_G . Moreover, $M = M^G \oplus M_G$ and M_G is the unique G-complement of M^G in M . [F - 5.2]

Definition 2.15 Let M be a rational G-module. We denote by P_M *the projection of* M *onto* M^G whose kernel is M_G . P_M is called the *Reynolds operator* of M . [F]

Remark 2.16 From the uniqueness of the Reynolds operator, it follows that if $\eta : M \to M'$ is a G-homomorphism of rational G-modules, then $\eta \circ P_M = P_{M'} \circ \eta$. [F]

Remark 2.17 If M and N are rational G-modules and $\eta : M \to N$ is a G-homomorphism that is onto then the restriction of η to M^G , that is, $\eta : M^G \to N^G$, is onto.

Proof: By the property of Reynolds operator (2.16) we have the following commutative diagram of rational G-modules and G-homomorphisms.

$$\begin{tikzcd}
M \arrow[r, two heads, "P_M"] \arrow[d, "\eta"'] & M^G \arrow[d, "\eta"] \\
N \arrow[r, two heads, "P_N"'] & N^G
\end{tikzcd}$$

$\eta : M \to N$ is onto. Therefore, $P_N \circ \eta$ is onto. This implies that $\eta \circ P_M$ is onto. Thus, $\eta : M^G \to N^G$ is onto.

Lemma 2.18 If R is a *ring, N a finitely generated R-*module and M an R-*module then $\mathrm{Hom}_R(N,M)$ is an R-*module with G-action defined by $g \circ f = gfg^{-1}$ for all $g \in G$, $f \in \mathrm{Hom}_R(N,M)$. (Prop. 1.12) If $\mathrm{Hom}_{RG}(N,M)$ is the collection of (R-G)-homomorphisms of N into M then $\mathrm{Hom}_{RG}(N,M) = \mathrm{Hom}_R(N,M)^G$.

Proof: (i) If $f \in \mathrm{Hom}_R(N,M)^G$ and $g \in G$ then $g \circ f = f$. Therefore, $g \circ f(n) = gf(g^{-1}n) = f(n)$. That is, $f(g^{-1}n) = g^{-1}f(n)$.

Therefore, f preserves G-action which implies that $f \in Hom_{RG}(N,M)$.

(ii) Conversely, if $f \in Hom_{RG}(N,M)$ then $g \circ f(n) = gf(g^{-1}n) = gg^{-1}f(n) = f(n)$ for all $n \in N$, $g \in G$. That is, $f \in Hom_R(N,M)^G$.

(i) and (ii) imply that $Hom_{RG}(N,M) = Hom_R(N,M)^G$. This completes the proof of Lemma 2.18.

<u>Proposition 2.19</u> Let G be a linearly reductive algebraic group, R a finitely generated k-algebra that is *simple, N a finitely generated R-*module and M an R-*module. Let $p \in Hom_R(N,M)$ and $f \in Hom_{RG}(M,N)$ such that $f \circ p = I_N$, the identity map on N . Then there exists an (R-G)-homomorphism $h : N \to M$ such that $f \circ h = I_N$.

<u>Proof</u>: By Prop. 1.12, $Hom_R(N,M)$ and $Hom_R(N,N)$ are R-*modules with appropriate G-action.

(i) Consider the R-module homomorphism $\Phi : Hom_R(N,M) \to Hom_R(N,N)$, $\varphi \xrightarrow{\Phi} f \circ \varphi$ where $\varphi \in Hom_R(N,M)$. If $g \in G$, then $\Phi(g \circ \varphi) = f \circ (g \circ \varphi) = f \circ (g\varphi g^{-1}) = g(f \circ \varphi)g^{-1} = g \circ \Phi(\varphi)$. Therefore, Φ is a G-homomorphism. Let $\psi \in Hom_R(N,N)$ then $p \circ \psi \in Hom_R(N,M)$ and $f \circ (p \circ \psi) = (f \circ p) \circ \psi = \psi$ since $f \circ p = I_N$. That is, $\Phi(p \circ \psi) = \psi$. Therefore, Φ is onto.

(ii) G is a linearly reductive algebraic group. Therefore, $Hom_R(N,M) = Hom_R(N,M)^G \oplus Hom_R(N,M)_G$ and $Hom_R(N,N) = Hom_R(N,N)^G \oplus Hom_R(N,N)_G$. (Lemma 2.14). Moreover, by Remark 2.17, the G-homomorphism $\Phi : Hom_R(N,M)^G \to Hom_R(N,N)^G$ is onto. But $I_N \in Hom_R(N,N)^G$. This implies that there exists $h \in Hom_R(N,M)^G$ such that $\Phi(h) = f \circ h = I_N$. It was proved in Lemma 2.18 that $Hom_{RG}(N,M) = Hom_R(N,M)^G$. Therefore, $h \in Hom_{RG}(N,M)$ as required.

This completes the proof of Proposition 2.19.

Corollary 2.19.1 Let G be a connected linearly reductive algebraic group, R a finitely generated k-algebra that is *simple, M a finitely generated R-*module and N an R-*module. If $\varphi : M \to N$ is an (R-G)-homomorphism which is onto, then there exists an (R-G)-homomorphism $\pi : N \to M$ such that $\varphi \circ \pi = I_N$. Moreover, $M = \ker \varphi \oplus \pi(N)$, direct sum of R-*modules.

Proof: $\varphi : M \to N$ is an R-module homomorphism that is onto. M is a finitely generated R-module. Therefore, N is a finitely generated R-*module. By Prop. 2.11, N is a projective R-module. Therefore, φ splits. That is, there exists an R-module homomorphism $f : N \to M$ such that $\varphi \circ f = I_N$. But φ preserves G-action. Therefore, by Prop. 2.19, there exists an (R-G)-homomorphism π in $\mathrm{Hom}_{RG}(N,M)$ such that $\varphi \circ \pi = I_N$. Then $\pi : N \to M$ is the required (R-G)-homomorphism. Thus, $\ker \varphi \rightarrowtail M \underset{\pi}{\overset{\varphi}{\rightleftarrows}} N$ is a split exact sequence of R-*modules and (R-G)-homomorphisms. Therefore, $M = \ker \varphi \oplus \pi(N)$. This completes the proof of Corollary 2.19.1.

Proposition 2.20 Let G be a connected linearly reductive algebraic group and R a finitely generated k-algebra that is *simple. Then every finitely generated nonzero R-*module M is a direct sum of finitely generated *simple R-sub*modules of M. That is, M is *semi*simple.

Proof: R is a Noetherian ring. M is a finitely generated R-module. Therefore, M is a Noetherian R-*module. Therefore, by Prop. 2.6, it is sufficient to prove that every nonzero sub*module N of M is a direct summand of M.

Consider the exact sequence of R-*modules and (R-G)-homomorphisms

$N \overset{i}{\rightarrowtail} M \xrightarrow{\eta} M/N$ where i is the inclusion map and η is the canonical map $M \twoheadrightarrow M/N$. i and η are (R-G)-homomorphisms. M is a finitely generated R-*module. Therefore, by Corollary 2.19.1, $M = N \oplus \pi(M/N)$ where π is a (R-G)-homomorphism $\pi : M/N \to M$ such that $\eta \circ \pi = I_{M/N}$. Thus, N is a direct summand of M . This completes the proof of Proposition 2.20.

<u>Corollary 2.20.1</u> If G is a connected linearly reductive algebraic group, R a finitely generated k-algebra that is *simple and V a finite dimensional rational G-module, then $R \otimes_k V$ is a finitely generated R-*module and, therefore, *semi*simple.

<u>Proposition 2.21</u> Let G be a connected linearly reductive algebraic group and R a finitely generated k-algebra that is *simple. Then every nonzero R-*module M is the direct sum of finitely generated *simple R-sub*modules of M and, therefore, *semi*simple.

<u>Proof</u>: Let $A = \{N \subseteq M \mid N$ is a direct sum of finitely generated *simple sub*modules of $M\}$. M is a nonzero R-*module. Therefore, M contains a nonzero element m . Let $V_m = \langle gm \mid g \in G\rangle$. Then RV_m is a finitely generated R-*module. Therefore by Prop. 2.20, RV_m is a direct sum of finitely generated *simple sub*modules of RV_m and, therefore, of M . $RV_m \in A$, thus A is nonempty.

Let $\{M_i \mid i \in I\}$ be a chain in A . $\cup M_i \in A$ and contains each M_i . Therefore by Zorn's Lemma A has a maximal element. Let it be M' . If $M' \neq M$, then let $x \in M - M'$. RV_x is a finitely generated sub*module of M and $RV_x \nsubseteq M'$. Let $RV_x = N$ and $M'' = M' \oplus N$. $M' \neq M''$. Therefore, M''/M' is a nonzero R-*module and the natural map $\eta : M'' \to M''/M'$ is an (R-G)-homomorphism that is onto.

M''/M' is a finitely generated R-*module. Therefore by Prop. 2.11, M''/M' is R-projective. Therefore there exists an R-module homomorphism $\pi : M''/M' \to M''$ such that $\eta \circ \pi = I_{M''/M'}$. But by Prop. 2.19, there exists an (R-G)-homomorphism $\pi' : M''/M' \to M''$ such that $\eta \circ \pi' = I_{M''/M'}$. Thus we have a split short exact sequence of R-*modules and (R-G)-homomorphisms

$$\begin{tikzcd} M' \arrow[r, hook, "i"] & M'' \arrow[r, two heads, "\eta"] & M''/M' \arrow[l, bend left, "\pi'"] \end{tikzcd}$$

where i is the inclusion map.

Therefore, $M'' = M' \oplus \pi'(M''/M')$. M''/M' is a finitely generated R-*module. π' is an (R-G)-homomorphism. Therefore, $\pi'(M''/M')$ is a finitely generated sub*module of M and therefore, *semi*simple. Let $\pi'(M''/M') = \bigoplus_i N_i$ where each N_i is a finitely generated *simple sub-*module of M. Then $M'' = M' \oplus (\bigoplus_i N_i)$. This contradicts the maximality of M'. Therefore, $M = M'$. This completes the proof of Proposition 2.21.

<u>Corollary 2.21.1</u> Let G be a connected linearly reductive algebraic group. If R is a finitely generated k-algebra that is *simple, then every nonzero R-*module M is R-projective.

<u>Proof</u>: Every nonzero R-*module M is the direct sum of finitely generated R-sub*modules of M. Every finitely generated R-*module is R-projective by 2.11. Therefore, M is R-projective. This completes the proof of Corollary 2.21.1.

<u>Proposition 2.22</u> Let G be a connected linearly reductive algebraic group. If R is a finitely generated k-algebra that is *simple and M is a nonzero sub*module of $R^{(n)}$, $n < \infty$, then $M = R^{(m)}$ for some m, $1 \le m \le n$.

<u>Proof</u>: Let $\pi_i : R^{(n)} \twoheadrightarrow R$ be the projection map on the i-th

coordinate. Consider the sequence of R-*modules and $(R\text{-}G)$-homomorphisms given by $M \overset{\mu}{\hookrightarrow} R^{(n)} \overset{\pi_i}{\twoheadrightarrow} R$ where μ is the inclusion map. Then $\pi_i \circ \mu$ is an $(R\text{-}G)$-homomorphism of M into R. $\pi_i \circ \mu(M)$ is a sub-*module of R. But R is *simple. Therefore, $\pi_i \circ \mu(M) = 0$ or R.

<u>Case (i)</u> If M is *simple, then $\pi_i \circ \mu : M \to R$ is either the zero map or an $(R\text{-}G)$-isomorphism. But M is nonzero and therefore, $\pi_i \circ \mu$ is nonzero for some i. Then, $M \underset{RG}{\cong} R$.

<u>Case (ii)</u> If M is not *simple, then M is *semi*simple. Therefore, $M = \bigoplus_j M_j$ where M_j is a *simple sub*module of $R^{(n)}$, for all j. By Case (i), $M_j \underset{RG}{\cong} R$ for all j. Therefore, $M \underset{RG}{\cong} R^{(m)}$ for some $m \in Z^+$, $1 \leq m \leq n$ since M is an R-submodule of $R^{(n)}$. This completes the proof of Proposition 2.22.

CHAPTER III

$k[SL_n]$-*MODULES WITH SL_n-ACTION

$SL_n(k)$ is a connected linearly reductive algebraic group if $n \geq 2$ and the characteristic of k is zero. We denote $SL_n(k)$ by either SL_n or G and $k[SL_n]$ by R. If SL_n-action on R is defined by $g \circ f(h) = f(g^{-1}h)$ for all $g,h \in G$, $f \in k[SL_n]$ then $k[SL_n]$ is a *simple *ring by 2.4.

In this chapter we establish that every *simple R-*module is $(R\text{-}G)$-isomorphic to R. Consequently, every R-*module is $(R\text{-}G)$-isomorphic to either $R^{(n)}$, $n > 0$, or $R^{(X)}$.

The existence of, but not the explicit form of, the isomorphism follows from a general theorem of Cline, Parshall and Scott [CPS]. But in this chapter we give the explicit form of the isomorphism for $k[SL_n]$-*modules.

First, we introduce some notations and state the definitions and facts needed for the sequence of results that lead to the final statement. $R = k[SL_n] = k[x_{11}, x_{12}, \ldots, x_{nn}]$ with

$\Gamma_{11}x_{11} + \Gamma_{21}x_{21} + \cdots + \Gamma_{n1}x_{n1} = 1$ where Γ_{i1} is the cofactor of

x_{i1} in the determinant $\begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & & & \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{vmatrix}$ and x_{ij} is the

coordinate function for all i,j .

<u>Definition 3.1</u> Definition of G-action on R . If $g \in SL_n$ then let

$$g = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \vdots & & & \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{bmatrix}$$ with $\alpha_{ij} \in k$ for all i,j . Then

$$g^{-1} = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ \vdots & & & \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}$$ where A_{ij} is the cofactor of α_{ij}

for all i,j .

SL_n-action is defined by $g \circ f(h) = f(g^{-1}h)$ for all $g,h \in SL_n$ and

$f \in k[SL_n]$. Suppose $h = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$ with $a_{ij} \in k$ for all

i,j . Then $g \circ x_{ij}(h) = x_{ij}(g^{-1}h)$ where

$$g^{-1}h = \begin{bmatrix} A_{11} & \cdots & A_{n1} \\ \vdots & & \\ A_{1n} & \cdots & A_{nn} \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} .$$

$x_{ij}(g^{-1}h) = A_{1i}a_{1j} + A_{2i}a_{2j} + \cdots + A_{ni}a_{nj}$. Therefore, $g \circ x_{ij} = A_{1i}x_{1j} + A_{2i}x_{2j} + \cdots + A_{ni}x_{nj}$ for all i,j .

<u>Some facts about G and R . 3.2</u>

1. $\sum_{j=1}^{n} \Gamma_{j1}x_{j1} = 1$. Therefore, $\sum_{j=1}^{n} \Gamma_{j1}x_{j1}$ is a G-invariant element in R .

2. R is a *simple *ring. Therefore every finitely generated R-*module is projective, by 2.11.

3. $V = \langle x_{1j}, x_{2j}, \ldots, x_{nj} \rangle$ is a simple G-module for all $1 \le j \le n$.

   <u>Proof</u>: $V \cong k^{(n)}$ as k-vector spaces. Suppose $u,w \in k^{(n)}$ then there exists a $\sigma \in GL_n$ such that $\sigma u = w$. This implies that there is a $\sigma \in SL_n$ such that $\sigma u = w$. Then $\sigma(\langle u \rangle) = \sigma(\langle w \rangle)$. Suppose W is a G-submodule of V . Then W contains every line in W . Then, $W = V$. That is, V is G-simple.

4. $RV \subseteq R$ and RV is an R-*module. But R is *simple. Therefore, $RV = R$.

<u>Definition 3.3</u> Let $R_n = \langle e_1, e_2, \ldots, e_n \rangle$ with G-action defined by $g^{-1} \circ e_i = \alpha_{i1}e_1 + \alpha_{i2}e_2 + \cdots + \alpha_{in}e_n$ for all i and g is as in 3.1. Then $g \circ e_i = A_{1i}e_1 + A_{2i}e_2 + \cdots + A_{ni}e_n$ for all i .

<u>Remark 3.4</u> If $V = \langle x_{11}, x_{21}, \ldots, x_{n1} \rangle$ then $V \underset{G}{\cong} R_n$ by the G-module

homomorphism $f : V \to R_n$ defined by $x_{i1} \to e_i$ for all i. Thus, we have $R \underset{k}{\otimes} V \underset{RG}{\cong} R \underset{k}{\otimes} R_n$. Next we establish that $R \underset{k}{\otimes} V \underset{RG}{\cong} R^{(n)}$ and therefore $R \underset{k}{\otimes} V \underset{RG}{\cong} R \underset{k}{\otimes} R_n \underset{RG}{\cong} R^{(n)}$.

<u>Isomorphism 3.5</u> $R \underset{k}{\otimes} R_n \underset{RG}{\cong} R^{(n)}$.

<u>Proof:</u> By 3.4, it is sufficient to prove $R \underset{k}{\otimes} V \underset{RG}{\cong} R^{(n)}$.

(i) Consider the exact sequence of R^*-modules and (R-G)-homomorphisms, $K \overset{i}{\rightarrowtail} R \underset{k}{\otimes} V \overset{\varphi}{\twoheadrightarrow} RV = R$ where $K = \ker \varphi$, i is the injection map and φ is defined as $1 \otimes v \mapsto v$ for all $v \in V$. $\sum_{i=1}^{n} \Gamma_{i1} \otimes x_{i1} \in R \underset{k}{\otimes} V$. Then $\varphi(\sum_{i=1}^{n} \Gamma_{i1} \otimes x_{i1}) = \sum_{i=1}^{n} \Gamma_{i1} x_{i1} = 1$. Define an R-homomorphism $\Pi : R \to R \underset{k}{\otimes} V$ by $1 \overset{\pi}{\longmapsto} \sum_{i=1}^{n} \Gamma_{i1} \otimes x_{i1}$. With G-action as defined in 3.1, for $g \in G$, we have

$$g(\Gamma_{11} \otimes x_{11} + \Gamma_{21} \otimes x_{21} + \cdots + \Gamma_{n1} \otimes x_{n1})$$
$$= \Big((A_{11}\operatorname{cof} A_{n1} + A_{12}\operatorname{cof} A_{n2} + \cdots + A_{1n}\operatorname{cof} A_{nn})(\Gamma_{n1} \otimes x_{11})$$
$$+ (A_{11}\operatorname{cof} A_{n-1,1} + A_{12}\operatorname{cof} A_{n-1,2} + \cdots + A_{1n}\operatorname{cof} A_{n-1,n})(\Gamma_{n-1,1} \otimes x_{11})$$
$$+ \cdots + (A_{11}\operatorname{cof} A_{11} + A_{12}\operatorname{cof} A_{12} + \cdots + A_{1n}\operatorname{cof} A_{1n})(\Gamma_{11} \otimes x_{11})\Big)$$
$$+ \cdots + \Big((A_{n1}\operatorname{cof} A_{n1} + A_{n2}\operatorname{cof} A_{n2} + \cdots + A_{nn}\operatorname{cof} A_{nn})(\Gamma_{n1} \otimes x_{n1})$$
$$+ (A_{n1}\operatorname{cof} A_{n-1,1} + A_{n2}\operatorname{cof} A_{n-1,2} + \cdots + A_{nn}\operatorname{cof} A_{n-1,n})(\Gamma_{n-1,1} \otimes x_{n1})$$
$$+ \cdots + (A_{n1}\operatorname{cof} A_{11} + A_{n2}\operatorname{cof} A_{12} + \cdots + A_{nn}\operatorname{cof} A_{1n})(\Gamma_{11} \otimes x_{n1})\Big)$$
$$= \Gamma_{11} \otimes x_{11} + \Gamma_{21} \otimes x_{21} + \cdots + \Gamma_{n1} \otimes x_{n1} ,$$

since $\sum_{j=1}^{n} A_{ij} \operatorname{cof} A_{kj} = 0$ if $i \neq k$ and 1 if $i = k$. That is,

$\sum_{i=1}^{n} \Gamma_{i1} \otimes x_{i1}$ is G-invariant. Moreover, $r \overset{\pi}{\longmapsto} r(\sum_{i=1}^{n} \Gamma_{i1} \otimes x_{i1})$. Thus, φ and π are (R-G)-homomorphisms such that $\varphi \circ \pi = I_R$. Therefore the above short exact sequence splits. Therefore,

$R \otimes_k V \underset{RG}{\cong} K \oplus R$.

(ii) Characterization of the elements of K . If $x \in R \otimes_k V$ then $x - \pi \circ \varphi(x) \in K$. That is, $x - \varphi(x)\pi(1) \in K$. On the other hand, if $x \in K$, then $x = \sum_{i=1}^{n} r_i \otimes x_{i1}$ such that $\varphi(x) = \sum_i r_i x_{i1} = 0$. Then $\varphi(x)\pi(1) = 0$. That is, $x = x - \varphi(x)\pi(1)$. Therefore, $K = \{x - \varphi(x)\pi(1) \mid x \in R \otimes_k V\}$. Let $x = b_1 \otimes x_{11} + b_2 \otimes x_{21} + \cdots + b_n \otimes x_{n1}$ with $b_i \in R$ for all i .

$$x - \varphi(x)\pi(1) = b_1 \otimes x_{11} + b_2 \otimes x_{21} + \cdots + b_n \otimes x_{n1}$$
$$- \sum_i b_i x_{i1}(\Gamma_{11} \otimes x_{11} + \cdots + \Gamma_{n1} \otimes x_{n1})$$
$$= \sum_{i=1}^{n} (-b_i\Gamma_{11}x_{i1} \otimes x_{11} - b_i\Gamma_{21}x_{i1} \otimes x_{21} - \cdots$$
$$+ b_i(1 - \Gamma_{i1})x_{i1} \otimes x_{i1} - \cdots - b_i\Gamma_{n1}x_{i1} \otimes x_{n1}) .$$

Coefficient of $b_i = -\Gamma_{11}x_{i1} \otimes x_{11} - \Gamma_{21}x_{i1} \otimes x_{21} - \cdots + (1 - \Gamma_{i1})x_{i1} \otimes x_{i1} - \cdots - \Gamma_{n1}x_{i1} \otimes x_{n1}$. But from the determinantal properties of $\det(x_{ij})$, $\Gamma_{j1}x_{i1} + \Gamma_{j2}x_{i2} + \cdots + \Gamma_{jn}x_{in} = 0$ for all $j \neq i$ and $\Gamma_{i1}x_{i1} + \Gamma_{i2}x_{i2} + \cdots + \Gamma_{in}x_{in} = 1$. Therefore, the coefficient of b_i can be written as $\sum_{j=2}^{n} x_{ij}(\Gamma_{1j} \otimes x_{11} + \Gamma_{2j} \otimes x_{21} + \cdots + \Gamma_{nj} \otimes x_{n1})$. Let $\alpha_j = \Gamma_{1j} \otimes x_{11} + \Gamma_{2j} \otimes x_{21} + \cdots + \Gamma_{nj} \otimes x_{n1}$ for all $2 \leq j \leq n$. α_j is G-invariant for all j .

$\varphi(\alpha_j) = \sum_{i=1}^{n} \Gamma_{ij}x_{i1} = 0$, $2 \leq j \leq n$, by the property of a determinant.

Therefore, $\alpha_j \in K$ for all j . On the other hand, if $y \in K$, then y can be written as $\sum_{i=1}^{n} b_i(x_{i2}\alpha_2 + x_{i3}\alpha_3 + \cdots + x_{in}\alpha_n)$ with $b_i \in R$, $1 \leq i \leq n$.

(iii) Now define an (R-G)-homomorphism $\Phi : R^{(n-1)} \to K$ as $(0,\ldots,1,0,\ldots,0) \mapsto \alpha_{i+1}$ where 1 is the i-th coordinate in $(0,\ldots,0,1,0,\ldots,0)$. Then $(r_i)_i \mapsto \sum_{i=1}^{n-1} r_i\alpha_{i+1}$ where $(r_i)_i \in R^{(n-1)}$. We will prove that Φ is an (R-G)-isomorphism.

If $(b_i)_i,(b_i')_i \in R^{(n-1)}$ then $(b_i)_i = (b_i')_i \Leftrightarrow b_i = b_i'$ for all i . Then $\sum_{i=1}^{n-1} b_i\alpha_{i+1} = \sum_{i=1}^{n-1} b_i'\alpha_{i+1}$. That is, $\Phi((b_i)) = \Phi((b_i'))$. Thus, Φ is well-defined. Let $y \in K$, then $y = \sum_{i=1}^{n} b_i(x_{i2}\alpha_2 + x_{i3}\alpha_3 + \cdots + x_{in}\alpha_n)$ with $b_i \in R$ for all i . Therefore, $\sum_{i=1}^{n} b_ix_{ij} \overset{\Phi}{\longmapsto} \sum_{2\leq j\leq n} (\sum_{i=1}^{n} b_ix_{ij}\alpha_j)$ proving that Φ is onto. If $(b_i),(b_i') \in R^{(n-1)}$ $\Phi((b_i)) = \Phi((b_i'))$ then $\sum_{i=1}^{n} b_i\alpha_{i+1} = \sum_{i=1}^{n} b_i'\alpha_{i+1}$. That is, $\sum_{j=1}^{n} (\sum_{i=1}^{n-1} (b_i - b_i')\Gamma_{j,i+1}) \otimes x_{j1} = 0$. But $K \subseteq R \otimes_k V$ which is a free R-module generated by $\{1 \otimes x_{j1} \mid 1 \leq j \leq n\}$. Therefore, $\sum_{i=1}^{n-1} (b_i - b_i')\Gamma_{j,i+1} = 0$ for all $1 \leq j \leq n$. Multiplying the n equations successively by x_{j2} , $1 \leq j \leq n$ and adding, we get $\sum_{i=1}^{n-1} (\sum_{j=2}^{n} (b_i - b_i')x_{j2}\Gamma_{ji}) = 0$. Again by the properties of determinants

$(b_1 - b_1')1 = 0$. That is, $b_1 = b_1'$. Similarly, $b_i = b_i'$, $1 \le i \le n-1$. Thus, $(b_i)_i = (b_i')_i$ proving Φ is 1-1.

α_i is G-invariant for all i . Therefore, Φ is an (R-G)-homomorphism. Thus, $K \underset{RG}{\cong} R^{(n-1)}$. Therefore, $R \underset{k}{\otimes} V \underset{RG}{\cong} R^{(n)}$ by combining (i) and (ii). That is, $R \underset{k}{\otimes} R_n \underset{RG}{\cong} R^{(n)}$.

<u>Isomorphism 3.6</u> If A is an R^*-module then $A \underset{R}{\otimes} (R \underset{k}{\otimes} R_n) \underset{RG}{\cong} (A \underset{R}{\otimes} R) \underset{k}{\otimes} R_n$ and therefore, $A \underset{R}{\otimes} (R \underset{k}{\otimes} R_n) \underset{RG}{\cong} A \underset{k}{\otimes} R_n$.

<u>Proof</u>: $A \underset{R}{\otimes} (R \underset{k}{\otimes} R_n) \underset{Z}{\cong} (A \underset{R}{\otimes} R) \underset{k}{\otimes} R_n$. The Z-homomorphism is defined by

$a \underset{R}{\otimes} (r \underset{k}{\otimes} r_n) \overset{\varphi}{\longmapsto} (a \underset{R}{\otimes} r) \underset{k}{\otimes} r_n$ for $a \in A$, $r \in R$ and $r_n \in R_n$. It is sufficient to prove that this is an (R-G)-homomorphism. Let $\alpha \in R$. Then $\alpha(a \underset{R}{\otimes} (r \underset{k}{\otimes} r_n)) = \alpha a \underset{R}{\otimes} (r \underset{k}{\otimes} r_n)$ and $\alpha a \underset{R}{\otimes} (r \underset{k}{\otimes} r_n) \overset{\varphi}{\longmapsto}$ $(\alpha a \underset{R}{\otimes} r) \underset{k}{\otimes} r_n = \alpha((a \underset{R}{\otimes} r) \underset{k}{\otimes} r_n) = \alpha\varphi(a \underset{R}{\otimes} (r \underset{k}{\otimes} r_n))$. Thus, φ is an R-homomorphism. If $g \in G$ then $g \circ (a \underset{R}{\otimes} (r \underset{k}{\otimes} r_n)) = ga \underset{R}{\otimes} g(r \underset{k}{\otimes} r_n) =$ $ga \underset{R}{\otimes} (gr \underset{k}{\otimes} gr_n)$ and $ga \underset{R}{\otimes} (gr \underset{k}{\otimes} gr_n) \overset{\varphi}{\longmapsto} (ga \underset{R}{\otimes} gr) \underset{k}{\otimes} gr_n$ and $(ga \underset{R}{\otimes} gr) \underset{k}{\otimes} gr_n = g(a \underset{R}{\otimes} r) \underset{k}{\otimes} gr_n = g((a \underset{R}{\otimes} r) \underset{k}{\otimes} r_n) = g\varphi(a \underset{R}{\otimes} (r \underset{k}{\otimes} r_n))$. Thus, φ preserves G-action.

<u>Remark 3.7</u> $\underset{R}{\otimes}$ can be replaced by $\underset{k}{\otimes}$ and we get the isomorphism $A \underset{k}{\otimes} (R \underset{k}{\otimes} R_n) \underset{RG}{\cong} (A \underset{k}{\otimes} R) \underset{k}{\otimes} R_n$.

<u>Isomorphism 3.8</u> $R \underset{k}{\otimes} R_n^{\underset{k}{\otimes} d} \underset{RG}{\cong} R^{(n^d)}$, $d \ge 0$.

<u>Proof</u>: The proof is by induction on d .

(i) If $d = 1$ then $R \underset{k}{\otimes} R_n \underset{RG}{\cong} R^{(n)}$ by 3.5.

(ii) <u>Induction Hypothesis</u>: Let $R \underset{k}{\otimes} R_n^{\otimes_k d-1} \underset{RG}{\cong} R^{(n^{d-1})}$, $d > 1$.

In 3.6, replace A by $R \underset{k}{\otimes} R_n^{\otimes_k d-1}$. Then $\left(R \underset{k}{\otimes} R_n^{\otimes_k d-1}\right) \underset{R}{\otimes} (R \underset{k}{\otimes} R_n) \underset{RG}{\cong}$ $\left(R \underset{k}{\otimes} R_n^{\otimes_k d-1}\right) \underset{k}{\otimes} R_n$. By 3.7, $\left(R \underset{k}{\otimes} R_n^{\otimes_k d-1}\right) \underset{k}{\otimes} R_n \underset{RG}{\cong} R \underset{k}{\otimes} R_n^{(d)}$. But

$(R \underset{k}{\otimes} R_n)^{\otimes_k d-1} \underset{R}{\otimes} (R \underset{k}{\otimes} R_n) \underset{RG}{\cong} R^{(n^{d-1})} \underset{R}{\otimes} R^{(n)}$ by induction hypothesis and

by 3.5. Thus, $R \underset{k}{\otimes} R_n^{(d)} \underset{RG}{\cong} R^{(n^{d-1})} \underset{R}{\otimes} R^{(n)} \underset{RG}{\cong} R^{(n^d)}$.

*<u>Remark 3.9</u> $\rho : SL_n \to R_n$ is a faithful representation. Therefore $k[SL_n] = k[R_n + R_n^*]$ where $k[R_n + R_n^*]$ is the k-algebra generated by by $R_n + R_n^*$ over k . R_n^* is the dual of R_n . $R_n = \langle e_1, e_2, \ldots, e_n\rangle$ (Def. 3.3). $R_n^* = \langle e_1^*, e_2^*, \ldots, e_n^*\rangle$ (dual basis). R_n^* is a G-module G-action being defined by $g \circ e_n^*(x) = e_n^*(g^{-1}x)$ for all $g \in G$, $x \in R_n$.

<u>Lemma 3.10</u> If W is any nonzero SL_n-module that is simple, then there exist $i > 0$, $1 \le i < \infty$ and a SL_n-module homomorphism

$$\Phi : \bigoplus_{i=1}^{m} (R_n \oplus R_n^*)^{\otimes_k d_i} \to W \text{ that is onto.}$$

<u>Proof:</u> Choose $\varphi \in W^*$. For each $x \in W$, define $f_x \in k[SL_n]$ by

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* We refer to "Representative functions on discrete groups and solvable arithmetic subgroups" by G. D. Mostow, <u>American Journal</u>, 1970 for the result.

$f_x(h) = \varphi(h^{-1}x)$ for all $h \in SL_n$. Then a map $\hat{\varphi} : W \to k[SL_n]$ can be defined by $x \longmapsto f_x$. Let $g \in SL_n$. Then $gx \overset{\hat{\varphi}}{\longmapsto} f_{gx}$ and $f_{gx}(h) = \varphi(h^{-1}gx)$ and $g \circ f_x(h) = f_x(g^{-1}h) = \varphi(h^{-1}gx)$. Therefore $\hat{\varphi}(gx) = g \circ \hat{\varphi}(x)$. That $\hat{\varphi}$ is an SL_n-module homomorphism is verified easily. Since $\hat{\varphi}$ preserves SL_n-action and W is SL_n-simple, $\hat{\varphi}$ is SL_n-module injection.

Thus it is sufficient to prove the lemma for SL_n-submodules of $k[SL_n]$. Since $k[SL_n] = k[R_n + R_n^*]$, any SL_n-submodule W of $k[SL_n]$ satisfies $W \subseteq \sum_{i=1}^{\ell} (R_n + R_n^*)^{d_i}$ for some $\ell < \infty$. Since SL_n-modules are semisimple, W is a direct summand, hence, a homomorphic image of $\sum_{i=1}^{\ell} (R_n + R_n^*)^{d_i}$. That is, there is $f : \sum_{i=1}^{\ell} (R_n + R_n^*)^{d_i} \to W$ that is onto. Then we have $\bigoplus_{i=1}^{\ell} (R_n \oplus R_n^*)^{\otimes_k d_i} \xrightarrow{\text{onto}} \sum_{i=1}^{\ell} (R_n + R_n^*)^{d_i} \overset{f}{\twoheadrightarrow} W$.

This completes the proof of Lemma 3.10.

<u>Lemma 3.11</u> $R \otimes_k R_n^* \underset{RG}{\cong} R^{(n)}$ implies $R \otimes_k R_n^* \underset{RG}{\cong} R^{(W)}$.

<u>Proof:</u> $R_n^* \cong \mathrm{Hom}_k(R_n, k)$ as k-vector spaces. $\mathrm{Hom}_k(R_n, k)$ is a G-module, G-action being defined by $g \circ f(x) = f(g^{-1}x)$ for all $f \in \mathrm{Hom}_k(R_n, k)$. Then $R_n^* \underset{G}{\cong} \mathrm{Hom}_k(R_n, k)$. Thus $R \otimes_k R_n^* \underset{RG}{\cong} R \otimes_k \mathrm{Hom}_k(R_n, k)$. $R \otimes_k \mathrm{Hom}_k(R_n, k)$ is a free R-module with basis $1 \otimes e_i^*$, $1 \le i \le n$. $R \otimes_k R_n$ is a free R-module with basis $1 \otimes e_i$, $1 \le i \le n$. Therefore we can define $\Phi : R \otimes_k \mathrm{Hom}_k(R_n, k) \to \mathrm{Hom}_R(R \otimes_k R_n, R)$ by $1 \otimes e_i^* \mapsto \widehat{1 \otimes e_i^*}$ where $\widehat{1 \otimes e_i^*} : R \otimes_k R_n \to R$ is defined by $1 \otimes e_j \mapsto e_i^*(e_j)$ for $1 \le j \le n$. This completely defines Φ. That Φ is an R-module homomorphism follows from the fact

$\sum_{i=1}^{n} a_i \otimes e_i^* \to \widehat{\sum_i a_i \otimes e_i^*}$ where $\widehat{\sum a_i \otimes e_i^*} : R \otimes_k R_n \to R$ is defined by $1 \otimes e_j \mapsto a_j$ for $a_j \in R$ and $1 \le j \le n$. If $\widehat{\sum a_i \otimes e_i^*} = \widehat{\sum r_i \otimes e_i^*}$ for $a_i, r_i \in R$, $1 \le i \le n$, then $a_i = r_i$ for all i. Then $\sum_i a_i \otimes e_i^* = \sum_i r_i \otimes e_i^*$. Thus Φ is one-to-one. Let

$h \in \mathrm{Hom}_R(R \otimes_k R_n, R)$. Let $h(1 \otimes e_i) = r_i$, $1 \le i \le n$. Then

$\widehat{\sum r_i \otimes e_i^*} = h$. Thus Φ is onto. Let $g \in G$. Then $g \circ (1 \otimes e_i^*) = 1 \otimes g \circ e_i^*$, $\widehat{g \circ (1 \otimes e_i^*)} = \widehat{1 \otimes g \circ e_i^*}$ where $\widehat{1 \otimes g \circ e_i^*}$ is defined by $1 \otimes e_j \mapsto g \circ e_i^*(e_j) = e_i^*(g^{-1} e_j)$. On the other hand,

$g \circ \widehat{1 \otimes e_i^*}(1 \otimes e_j) = \widehat{1 \otimes e_i^*}(1 \otimes g^{-1} e_j) = e_i^*(g^{-1} e_j)$. Thus Φ preserves G-action. Therefore $R \otimes_k \mathrm{Hom}_k(R_n, k) \underset{RG}{\cong} \mathrm{Hom}_R(R \otimes_k R_n, R)$. But $R \otimes_k R_n \underset{RG}{\cong} R^{(n)}$. Let $\alpha : R \otimes_k R_n \to R^{(n)}$ be the isomorphism. Then $\mathrm{Hom}_R(R \otimes_k R_n, R) \underset{R}{\cong} \mathrm{Hom}_R(R^{(n)}, R)$ and the isomorphism $\Phi : \mathrm{Hom}_R(R \otimes_k R_n, R) \to \mathrm{Hom}_R(R^{(n)}, R)$ is defined by $h \mapsto h\alpha^{-1}$ for all $h \in \mathrm{Hom}_R(R \otimes_k R_n, R)$. Let $g \in G$, then $g \circ h \mapsto (g \circ h)\alpha^{-1}$ where $(g \circ h)\alpha^{-1}(x) = gh(g^{-1}\alpha^{-1}x) = gh\alpha^{-1}(g^{-1}x)$ for all $x \in R^{(n)}$, since α preserves G-action. But $g \circ (h\alpha^{-1})(x) = gh\alpha^{-1}(g^{-1}x)$. That is, Φ preserves G-action. Thus, $\mathrm{Hom}_R(R \otimes_k R_n, R) \underset{RG}{\cong} \mathrm{Hom}_R(R^{(n)}, R)$. But

$\mathrm{Hom}_R(R^{(n)}, R) \underset{R}{\cong} \prod_{i=1}^{n} \mathrm{Hom}_R(R, R) = \bigoplus_{i=1}^{n} \mathrm{Hom}_R(R, R)$ and the isomorphism is defined as follows. Let $\mu_i : R \to R^{(n)}$ be the R-module injection into the i-th coordinate for all $1 \le i \le n$. Then if $h \in \mathrm{Hom}_R(R^{(n)}, R)$,

$h \mapsto \bigoplus_{i=1}^{n} h \circ \mu_i$ under the above isomorphism. Let $g \in G$, then $g \circ h \mapsto \bigoplus_{i=1}^{n} (g \circ h) \circ \mu_i = g(\bigoplus_{i=1}^{n} h \circ \mu_i)$. Thus, $\mathrm{Hom}_R(R^{(n)},R) \underset{RG}{\cong} \bigoplus_{i=1}^{n} \mathrm{Hom}_R(R,R)$. $\mathrm{Hom}_R(R,R) \underset{R}{\cong} R$ and the isomorphism $\lambda : \mathrm{Hom}_R(R,R) \to R$ is defined by $f \mapsto f(1)$. If $g \in G$, then $g \circ f \mapsto (g \circ f)(1) = f(g^{-1}1) = f(1)$. On the other hand, $g \circ (f(1)) = f(1)$ since $f(1) \in k$. That is, λ preserves G-action. Therefore, $\mathrm{Hom}_R(R,R) \underset{RG}{\cong} R$. Thus, $R \otimes_k R_n^* \underset{RG}{\cong} R^{(n)}$. This completes the proof of Lemma 3.11.

<u>Proposition 3.12</u> Every nonzero *simple R-*module is (R-G)-isomorphic to R.

<u>Proof:</u> Let M be a nonzero *simple R-*module. If $m \in M$ and $m \neq 0$, then let $V_m = \langle gm \mid g \in G \rangle$. Since G is linearly reductive, $V_m = \bigoplus_i V_i$ where each V_i is G-simple. Since M is *simple, $M = RV_i$ for each i. We choose one such V_i and let $V_i = V$. Then, $M = RV$ where V is a simple G-submodule of M. The map $\mu : R \otimes_k V \to RV$ defined by $r \otimes v \mapsto rv$ where $r \in R$, $v \in V$ is an (R-G)-homomorphism that is onto. $R \otimes_k V$ is a finitely generated R-*module. Therefore by Cor. 2.20.1, $R \otimes_k V \underset{RG}{\cong} RV + \ker \mu$. V is a simple G-module. By Lemma 3.10, there is a G-module homomorphism $\varphi : \bigoplus_{i=1}^{m} (R_n \oplus R_n^*)^{\otimes_k d_i} \twoheadrightarrow V$ with $m < \infty$, $d_i < \infty$. Then $R \otimes_k \left(\bigoplus_{i=1}^{m} (R_n \oplus R_n^*)^{\otimes_k d_i} \right) \xrightarrow{1 \otimes \varphi} R \otimes_k V$

is an (R-G)-homomorphism that is onto. $R \underset{k}{\otimes} \left[\sum_{i=1}^{m} (R_n \oplus R_n^*)^{\underset{k}{\otimes} d-1} \right]$ is a finitely generated R-*module. Therefore, by Cor. 2.20.1 again,

$$R \underset{k}{\otimes} \left[\sum_{i=1}^{m} (R_n \oplus R_n^*)^{\underset{k}{\otimes} d_i} \right] \underset{RG}{\cong} \ker(1 \otimes \varphi) \oplus R \underset{k}{\otimes} V \ . \quad \text{But}$$

$R \underset{k}{\otimes} R_n^{\underset{k}{\otimes} d} \underset{RG}{\cong} R^{(n^d)}$ and $(R \underset{k}{\otimes} R_n^*)^{\underset{k}{\otimes} d} \underset{RG}{\cong} R^{(n^d)}$. Therefore, by Prop. 2.22, $R \underset{k}{\otimes} V \underset{RG}{\cong} R^{(m_0)}$ for some $m_0 > 0$. The same proposition applied to the (R-G)-isomorphism $R \underset{k}{\otimes} V \underset{RG}{\cong} RV \oplus \ker \mu$ gives $RV \underset{RG}{\cong} R^{(p)}$ for some $p > 0$. But $RV = M$ and M is *simple. Therefore, $M \underset{RG}{\cong} R$. This completes the proof of Proposition 3.12.

<u>Corollary 3.12.1</u> Every nonzero R-*module M is (R-G)-isomorphic to either $R^{(n)}$, $n < \infty$, or $R^{(\chi)}$.

<u>Proof</u>: By Prop. 2.21, M is the direct sum of *simple R-sub*modules of M . Therefore, by Prop. 3.12, $M \underset{RG}{\cong} R^{(n)}$, $n < \infty$, or $M \underset{RG}{\cong} R^{(\chi)}$. This completes the proof of Cor. 3.12.1.

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