TAN, RICHARD BENG-TOK

BRAUER GROUPS OF H-DIMODULE ALGEBRAS AND TRUNCATED POWER SERIES HOPE ALGEBRAS

The University of Oklahoma PH.D. 1980

THE UNIVERSITY OF OKLAHOMA

GRADUATE COLLEGE

BRAUER GROUPS OF H-DIMODULE ALGEBRAS AND TRUNCATED POWER SERIES HOPF ALGEBRAS

A DISSERTATION

SUBMITTED TO THE GRADUATE FACULTY

in partial fulfillment of the requirements for the

degree of

DOCTOR OF PHILOSOPHY

BY

RICHARD BENG-TOK TAN

February 1980

BRAUER GROUPS OF H-DIMODULE ALGEBRAS
AND TRUNCATED POWER SERIES HOPF ALGEBRAS

APPROVED BY

Robert A. Morris

Harold V. Huneke

Leonard R. Rubin

John C. Cooper

DISSERTATION COMMITTEE

BRAUER GROUPS OF H-DIMODULE ALGEBRAS

AND TRUNCATED POWER SERIES HOPF ALGEBRAS

BY: RICHARD B. TAN

MAJOR PROFESSOR: ROBERT A. MORRIS

The Brauer group of H-dimodule algebras, $BD(R,H)$, consists of equivalence classes of H-Azumaya algebras; where H is a Hopf algebra over a commutative ring R, and an H-dimodule algebra A is defined to be H-Azumaya if certain maps (analogous to the usual map $A \otimes A \rightarrow \mathrm{End}(A)$) are isomorphisms. It is shown that if R is a separably closed field of characteristic p and H is a truncated power series Hopf algebra then a necessary and sufficient condition for an H-Azumaya algebra A to be R-Azumaya (the usual Azumaya R-algebra) is that it be semisimple. An example is given to show that semisimplicity is necessary for this to be true.

$BD_0(R,H)$ is the subset of $BD(R,H)$ consisting of only those H-Azumaya algebras that are already R-Azumaya. If each element $[A]$ in $BD_0(R,H)$ has the property that $A \cong \mathrm{End}(V)$ as an H-module algebra for some finitely generated projective H-module V, then $BD_0(R,H)$ is a subgroup of $BD(R,H)$. For the truncated power series Hopf algebra $\alpha_p = k[x]/(x^p)$, with x primitive, $BD_0(R,\alpha_p) = R^*$ when R is a perfect field of characteristic p and has trivial Brauer group.

ACKNOWLEDGEMENTS

Quite a number of people have helped with the completion of this thesis. I would like to express my gratitude to all of them.

I owe a special debt to my advisor, Professor Robert Morris, who not only suggested the problem but also offered encouragement during the many seemingly wrong tracks and dead ends. His faith in certain theorems made them come out as they are. I am especially grateful for his patience and understanding during the period when I was not doing any mathematics.

Many thanks to Professors Andy Magid, Harold Huneke, Leonard Rubin and John Thompson, who served on my committee and offered constructive criticisms and suggestions for improving the presentation. Professor Harold Huneke, who as the graduate counselor, also spent considerable time arranging the administrative end of completing the thesis.

Special thanks to Professor Moss Sweedler, the proofs in Example I.16 are due to him. Maryann Grade and especially Sydney Henderson braved through the thesis correcting numerous grammatical errors. I have made some amendments since then and probably introduced more errors.

My thanks to Audle Allison for being such a great friend and for the spiritual guidance throughout; to T. K. Pan, who showed concern and care throughout the whole project; and to my family, who maintained cheerfulness all these years wondering why I am still in school.

Finally, special thanks to my typist Linda Fannin, for doing a wonderful job of typing with very short notice.

CONTENTS

INTRODUCTION

The Brauer group, $B(k)$, is the abelian group of similarity classes of central simple algebras over a field k. It is a significant invariant of the field k and holds an important place in the theory of algebras over a field. Its definition was generalized to an arbitrary commutative ring with identity by Azumaya [A] and Auslander and Goldman [AG].

In 1963 while looking for invariants in the theory of quadratic forms over a field, Wall [W] found that the graded Brauer group of 2-graded algebras gives a better connection to quadratic-form theory than the ordinary Brauer group. This theory was then generalized to the commutative ring case by Bass [B] and Small [S_1, S_2].

Wall considered gradings by the group of order 2, so a natural generalization was to extend this to other groups. Knus [K] and Childs, Garfinkel and Orzech [CGO] considered algebras graded by an arbitrary finite abelian group G. They introduced a Brauer group of graded Azumaya algebras, $B_\phi(R,G)$, for G a finite abelian group, R a commutative ring with units group $U(R)$, and $\phi : G \times G \longrightarrow U(R)$ a fixed bimultiplicative map. When $G = C_2$, the cyclic group of order 2, and ϕ is nontrivial, $B_\phi(R,G)$ is the Brauer-Wall group introduced by Wall.

Another generalization of the Brauer group was developed by Fröhlich and Wall [FW]. This is the equivariant Brauer group which involves algebras on which a group acts.

In [L_1], Long introduced $BD(R,G)$, the Brauer group of algebras graded by an arbitrary group G and also acted on by G so that the action preserves the grading. In [L_2], he further extended this to

$BD(R,H)$ by replacing G with an arbitrary commutative and cocommutative Hopf algebra H over a ring R. This coincides with the former Brauer group in [L_1] if we let $H = R[G]$, the group algebra of G over R. Long then computed $BD(R,G)$ for a cyclic group of prime order over an algebraically closed field.

Orzech [O] and Beattie [Be] then extended Long's computations by relaxing the conditions that R be an algebraically closed field, and that G be cyclic of prime order.

None of the above works consider H-Azumaya algebras for any Hopf algebra H other than the group algebra. In this paper we extend the study of H-Azumaya algebras into domains where H is not necessarily the group algebra.

$BD(R,H)$, the Brauer group of H-dimodule algebras, consists of equivalence classes of H-Azumaya algebras; where an H-dimodule algebra A is defined to be H-Azumaya if certain maps (analogous to the usual map $A \otimes A^0 \longrightarrow \mathrm{End}(A)$) are isomorphisms. $BD_0(R,H)$ is a subset of $BD(R,H)$ consisting of only those H-Azumaya algebras that are already R-Azumaya (the usual Azumaya algebras). Long in [L_2] has shown that $BD(K,H) = BD_0(K,H)$ for $H = K[C_p]$, where C_p is the cyclic group of order p and k is an algebraically closed field of characteristic p. This implies that any H-Azumaya algebra is already R-Azumaya. In Chapter I we introduce the truncated power series Hopf algebras, which include group algebras as special cases. We then show that not all H-Azumaya algebras are R-Azumaya when H is a truncated power series Hopf algebra, and we give a necessary and sufficient condition for this to be true. Long's result is then a special case of our theorem.

$BD_0(R,H)$ is not necessarily a subgroup of $BD(R,H)$. In Chapter II we give a sufficient condition for $BD_0(R,H)$ to be a group and extend Long's computation of $BD_0(R,H)$ to $H = \alpha_p$, a truncated power series Hopf algebra not equal to a group algebra. We find that for an appropriate field $BD_0(R,\alpha_p) = R^*$, the multiplicative group of R. Finally, we list some open questions raised by the work in this paper.

CHAPTER 0

PRELIMINARIES

This chapter contains basic definitions and results of Hopf algebras and H-dimodule algebras, leading to the definition of BD(R,H), the Brauer group of H-dimodule algebras.

1. General Conventions

Throughout this paper, R is a commutative ring with identity. Each $\otimes$, Hom, etc. is taken over R, and each map is R-linear unless otherwise stated. All algebras and modules are understood to be R-algebras and R-modules.

B(R) is the usual Brauer group of R. Central separable algebras are called R-Azumaya algebras. H denotes a Hopf algebra over R and C_n is the cyclic group of order n.

2. Coalgebras, Bialgebras and Hopf Algebras

Definition 0.1.

A coalgebra C over R is an R-module C together with the maps:

coproduct or diagonalization $\Delta : C \to C \otimes C$,

counit or augmentation $\varepsilon : C \to R$,

so that the following diagrams commute:

Coassociativity

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"] & C \otimes C \arrow[d, "\Delta\otimes I"] \\ C \otimes C \arrow[r, "I\otimes\Delta"] & C \otimes C \otimes C \end{tikzcd}$$

and

Left and Right Counit

$$\begin{tikzcd} R \otimes C \simeq & C & \simeq C \otimes R \\ & C \otimes C \arrow[ul, "\varepsilon\otimes I"] \arrow[u, leftarrow, "\Delta"] \arrow[ur, "I\otimes\varepsilon"'] & \end{tikzcd}$$

We introduce the Sweedler Σ notation, which is an important tool for coalgebra manipulation. We write $\Delta(h) = \sum_{(h)} h_1 \otimes h_2$, and note that Σ is a dummy, serving merely to remind us that $\Delta(h)$ is a sum of elements of the form $h_1 \otimes h_2$. The symbols h_1 and h_2 do not denote particular elements, but are merely placeholders.

Σ notation can be used to convey precisely the information in commutative diagrams. Thus, $\Delta : C \to C \otimes C$ is coassociative iff $\sum_{(c)} \Delta(c_1) \otimes c_2 = \sum_{(c)} c_1 \otimes \Delta(c_2)$ for all $c \in C$, and $\varepsilon : C \to R$ is a counit iff $\sum_{(c)} \varepsilon(c_1)c_2 = c = \sum_{(c)} c_1\varepsilon(c_2)$.

Definition 0.2.

A bialgebra B is an R-module that is simultaneously an R-algebra under

multiplication	$\cdot_B : B \otimes B \to B$,
unit	$1_B : R \to B$,

and a coalgebra under Δ and ε, such that Δ and ε are algebra maps.

Definition 0.3.

A Hopf algebra H is a bialgebra H equipped with a map, the antipode, $S : H \to H$ such that $\sum_{(h)} h_1 S(h_2) = \varepsilon(h) = \sum_{(h)} S(h_1)h_2$ for all $h \in H$.

Definition 0.4.

A Hopf algebra is commutative if the multiplication is commutative and cocommutative if the diagonalization is commutative, i.e. $\sum_{(h)} h_1 \otimes h_2 = \sum_{(h)} h_2 \otimes h_1$.

Remark 0.5.

If the Hopf algebra H is finitely generated projective as an R-module, then so is its dual $H^* = \mathrm{Hom}(H,R)$. The structure maps of H^* are given as follows:

$$\cdot_{H^*} : H^* \otimes H^* \simeq (H \otimes H)^* \xrightarrow{\Delta^*} H^*$$

$$1_{H^*} : R \simeq R^* \xrightarrow{\varepsilon^*} H^*$$

$$\Delta_{H^*} : H^* \xrightarrow{\cdot^*} (H \otimes H)^* \simeq H^* \otimes H^*$$

$$\varepsilon_{H^*} : H^* \xrightarrow{1^*} R^* \longrightarrow R$$

$$S_{H^*} : H^* \xrightarrow{S^*} H^*$$

where, if $f : X \to Y$ is a morphism of R-modules then $f^* : Y^* \to X^*$ is the map given by $\langle f^*(y^*), x\rangle = \langle y^*, f(x)\rangle$. Thus, for example, $\langle h_1^* \cdot h_2^*, h\rangle = \langle h_1^* \otimes h_2^*, \Delta(h)\rangle$ and $\langle \Delta(h^*), h_1 \otimes h_2\rangle = \langle h^*, h_1 \cdot h_2\rangle$. For more details of Hopf algebras and Σ notation formalisms, see [Sw_1, Sw_2].

3. Examples of Hopf algebras

We give two important examples of Hopf algebras.

Example 0.6.

A classical example of a Hopf algebra is the group algebra, $H=R[G]$, where G is any group. This has the coalgebra structure given by $\Delta(g) = g \otimes g$, $\varepsilon(g) = 1$ and $S(g) = g^{-1}$ for $g \in G$.

An element g in an arbitrary Hopf algebra which satisfies $\Delta(g) = g \otimes g$ and $\varepsilon(g) = 1$ is said to be group-like.

An important special group algebra for us will be the following:

Example 0.7.

$H = R[C_p]$, where R is a ring of characteristic p. As algebras, $R[C_p] \cong R[x]/(x^p-1) \cong R[y]/(y^p)$, where $y = x-1$. The coalgebra structure maps are defined by $\Delta x = x\otimes x$ and hence $\Delta y = y\otimes y + y\otimes 1 + 1\otimes y$; $\varepsilon(x) = 1$ and hence $\varepsilon(y) = 0$, with $S(x) = x^{-1}$. The dual $H^* = k[z]/(z^p-z)$, where $\langle z^i, x^j\rangle = j^i$, has the coalgebra structures given by $\Delta z = z\otimes 1 + 1\otimes z$, $\varepsilon(z) = 0$ and $S(z) = -z$. These structures coincide with the dual structures given in Remark 0.5. In an arbitrary Hopf algebra, an element z that satisfies $\Delta z = z\otimes g + g\otimes z$ (with g group-like) and $\varepsilon(z) = 0$ is said to be primitive with respect to g or g-primitive. If $g = 1$ we simply say z is primitive.

Of prime importance to us is the following:

Example 0.8.

$H = \alpha_p$. This is the Hopf algebra $H = R[x]/(x^p)$ with R a ring of characteristic p. H has the coalgebra structure given by $\Delta x = x\otimes 1 + 1\otimes x$, $\varepsilon(x) = 0$ and $S(x) = -x$. The dual $H^* = R[z]/(z^p)$, where $\langle z,x\rangle = 1$ and $\langle z^i, x^j\rangle = j!\delta_{i,j}$, has the coalgebra structure given by $\Delta z = z\otimes 1 + 1\otimes z$, $\varepsilon(z) = 0$ and $S(z) = -z$. Thus $H \cong H^*$ as a Hopf algebra. We note that $\langle 1, z, \ldots, z^{p-1}\rangle$ is the dual basis to $\langle 1, x, \ldots, \frac{1}{(p-1)!} x^{p-1}\rangle$.

See [P_2, Beispiele 4.1] for more details of the above examples.

4. H-modules, H-comodules and H-dimodules

Definition 0.9.

An R-module M is a (left) H-module if there is an R-linear map $\rightharpoonup_M : H \otimes M \longrightarrow M$, (written simply $\rightharpoonup$ if the meaning is clear) such that the following diagrams commute.

(i)
$$\begin{array}{ccc} H \otimes H \otimes M & \xrightarrow{\cdot \otimes I} & H \otimes M \\ {\scriptstyle I \otimes \rightharpoonup}\downarrow & & \downarrow{\scriptstyle \rightharpoonup} \\ H \otimes M & \xrightarrow{\rightharpoonup} & M \end{array}$$

(ii)
$$\begin{array}{ccc} R \otimes M & \xrightarrow{1_H \otimes I} & H \otimes M \\ & \searrow & \downarrow \\ & & M \end{array}$$

We say that H acts on M by $\rightharpoonup$.

Note that (i) says $(g \cdot h) \rightharpoonup m = g \rightharpoonup (h \rightharpoonup m)$ and (ii) $1_H(r) \rightharpoonup m = rm$ for all $g, h \in H$, $m \in M$ and $r \in R$. Thus the diagrammatic definition above coincides with the elementary definition of a module. The virtue of this approach is that it leads readily into the definition of one of the principal objects of study, the comodules, by dualizing or reversing the arrows in the above diagrams.

Example 0.10.

Any R-module M can be made into an H-module trivially by defining $h \rightharpoonup m = \varepsilon(h)m$ and $1_H(r)m = rm$.

If M and N are H-modules then $M \otimes N$ and Hom(M,N) are also H-modules with H-actions given by $h \rightharpoonup (m \otimes n) = \sum_{(h)} (h_1 \rightharpoonup m) \otimes (h_2 \rightharpoonup n)$, and $(h \rightharpoonup f)(m) = \sum_{(h)} h_1 \rightharpoonup_N [f(S(h_2) \rightharpoonup_M m]$ respectively, where $h \in H$, $m \in M$, $n \in N$ and $f \in \mathrm{Hom}(M,N)$ [L_2, Proposition 1.4].

We now turn to the dual concept of H-comodules.

Definition 0.11.

An R-module M is a (right) H-comodule if there is a map $\chi_M : M \longrightarrow M \otimes H$ (written simple χ if there is no confusion) so that the following diagrams commute.

(i)
$$\begin{array}{ccc} M & \xrightarrow{\chi} & M \otimes H \\ \chi\downarrow & & \downarrow \chi\otimes I \\ M\otimes H & \xrightarrow{I\otimes\Delta} & M\otimes H\otimes H \end{array}$$

(ii)
$$\begin{array}{ccc} M & \xrightarrow{\chi} & M \otimes H \\ & \searrow & \downarrow I\otimes\varepsilon \\ & & M\otimes R \end{array}$$

We write $\chi(m) = \sum_{(m)} m_0 \otimes m_1$, with $m_0 \in M$, $m_1 \in H$. Thus condition (ii) says $\sum_{(m)} m_0 \varepsilon(m_1) = m$. χ is called the <u>coaction</u> of H on M.

<u>Example</u> 0.12.

We can make any R-module M into an H-comodule by defining $\chi(m) = m \otimes 1$. This is the <u>trivial</u> H-comodule structure on M.

Recall that if H is a finitely generated projective Hopf algebra then H* is also a Hopf algebra. It is often more convenient to study the H*-module structure rather than the H-comodule structure. This is done as follows: for $h^* \in H^*$ and $m \in M$, where M is an H-comodule with $\chi(m) = \sum_{(m)} m_0 \otimes m_1$, define the H*-action on M (denoted by the downward $\rightharpoonup$) by $h^* \rightharpoonup m = \sum_{(m)} m_0 \langle h^*, m_1 \rangle$. This makes M into an H*-module. Conversely, if M is an H*-module, then M can be made into an H-comodule [P_1, Proposition 1]. Furthermore, if M and N are H-comodules then $M \otimes N$ and Hom(M,N) are H*-modules and hence H-comodules, provided H is finitely generated projective.

<u>Definition</u> 0.13.

Let H be a commutative and cocommutative Hopf algebra. An R-module M is an <u>H-dimodule</u> if M is an H-module and an H-comodule such that the following diagram commutes.

$$\begin{tikzcd} H \otimes M \arrow[r, "\rightharpoonup"] \arrow[d, "I\otimes\chi"] & M \arrow[d, "\chi"] \\ H \otimes M \otimes H \arrow[r, "\rightharpoonup\otimes I"] & M \otimes H \end{tikzcd}$$

In symbols this says:

$$\sum_{(h \rightharpoonup m)} (h \rightharpoonup m)_0 \otimes (h \rightharpoonup m)_1 = \sum_{(m)} (h \rightharpoonup m_0)\, m_1 .$$

Let H be also finitely generated projective. Then an H-dimodule M is both an H-module and an H*-module. The commutative diagram above then becomes

$$\begin{tikzcd} H \otimes M \arrow[r, "\rightharpoonup"] & M \\ H^* \otimes H \otimes M \arrow[u, "f"] \arrow[r, "I\otimes\rightharpoonup"] & H^* \otimes M \arrow[u, "\rightharpoondown"'] \end{tikzcd}$$

where $f : H^*\otimes H\otimes M \xrightarrow{T\otimes I} H\otimes H^*\otimes M \xrightarrow{I\otimes\rightharpoondown} H\otimes M$ with $T : H\otimes H^* \longrightarrow H^*\otimes H$, the Twist map, i.e. $T(h\otimes h^*) = h^*\otimes h$.

In symbols this says $h \rightharpoonup (h^* \rightharpoondown m) = h^* \rightharpoondown (h \rightharpoonup m)$ for any $h \in H$ and $h^* \in H^*$, i.e. the actions $\rightharpoonup$ and $\rightharpoondown$ commute. Thus, when H is finitely generated projective, M is an H-dimodule iff it is an $H \otimes H^*$-module iff it is an H-H*-bimodule.

5. <u>H-module, H-comodule and H-dimodule algebras</u>

<u>Definition</u> 0.14.

A map $f : M \longrightarrow N$, where M and N are H-modules, is an <u>H-module map</u> if the following diagram commutes.

$$\begin{array}{ccc} H \otimes M & \xrightarrow{I\otimes f} & H \otimes N \\ \downarrow{\rightharpoonup_M} & & \downarrow{\rightharpoonup_N} \\ M & \xrightarrow{f} & N \end{array}$$

In symbols, this says $h \rightharpoonup_N f(m) = f(h \rightharpoonup_M m)$.

We record this standard definition here in diagrammatic form, in order to readily obtain its dual form subsequently.

<u>Definition</u> 0.15.

A is an <u>H-module algebra</u> if A is an R-algebra and an H-module so that the structure maps $1_A : R \longrightarrow A$ and $\cdot_A : A\otimes A \longrightarrow A$ are H-module maps.

These conditions say that $h\rightharpoonup 1_A = \varepsilon(h)\cdot 1_A$ and $h\rightharpoonup (a\cdot b) = \sum_{(h)} (h_1\rightharpoonup a)\cdot(h_2\rightharpoonup b)$ for $a, b\in A$ and $h\in H$.

Note that if $H = R[G]$, G is a group, and A is an H-module algebra, then the above conditions say that $g \rightharpoonup 1_A = 1_A$ and $g \rightharpoonup (a\cdot b) = (g \rightharpoonup a)\cdot(g \rightharpoonup b)$ for $g\in G$; thus g acts on A as an automorphism.

<u>Example</u> 0.16.

If M is an H-module then End(M) is an H-module algebra [L_2, Proposition 1.8].

Let H be cocommutative. If A and B are H-module algebras then so are $A\otimes B$ and A^0, the opposite algebra to A [L_2, Proposition 1.7].

We also have the following dual concepts.

<u>Definition</u> 0.16.

If M and N are H-comodules and $f\in\mathrm{Hom}(M,N)$ then f is said to be an <u>H-comodule map</u> if the following diagram commutes.

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\chi_M"] & N \arrow[d, "\chi_N"] \\ M \otimes H \arrow[r, "f\otimes I"] & N \otimes H \end{tikzcd}$$

This says that $\sum_{(m)} f(m_0)\, m_1 = \sum_{(f(m))} f(m)_0 \otimes f(m)_1$.

Definition 0.17.

A is an H-comodule algebra if it is an R-algebra and an H-comodule so that the structure maps $1_A : R \longrightarrow A$ and $\cdot_A : A\otimes A \longrightarrow A$ are H-comodule maps.

In symbols these conditions become $\chi(1_A) = 1_A \otimes 1_H$ and $\chi(ab) = \sum_{(a)(b)} a_0b_0 \otimes a_1b_1$.

Example 0.18.

Any Hopf algebra H is an H-comodule algebra, by letting $\chi = \Delta$, the diagonal map.

Let H be commutative. If A and B are H-comodule algebras then so are $A\otimes B$ and A^0.

If H is finitely generated projective, and M,N are H-comodules and hence H*-modules, then a map $f : M \longrightarrow N$ is an H-comodule map iff it is an H*-module map [L_2, Proposition 2.3, (iii)]. Furthermore, A is a right H-comodule algebra iff it is a left H*-module algebra.

Definition 0.19.

Let M and N be H-dimodules. Then a map $f : M \longrightarrow N$ is an H-dimodule map if it is both an H-module map and H-comodule map.

Definition 0.20

Let H be commutative and cocommutative. An R-algebra A is an

<u>H-dimodule algebra</u> if it is simultaneously an H-dimodule, H-module algebra and H-comodule algebra.

<u>Example</u> 0.21.

If M is an H-dimodule then End(M) is an H-dimodule algebra [L_2, Proposition 3.6].

If A and B are H-dimodule algebras, so are A^0 and $A\otimes B$ [L_2, Theorem 3.3].

6. <u>The Brauer Group of H-dimodule algebras</u>

Let A and B be H-dimodule algebras. Then $A\otimes B$ is an H-dimodule algebra with the usual structures. We can put a new multiplication on $A\otimes B$ as follows:

$$A\otimes B\otimes A\otimes B \xrightarrow{I\otimes\chi_B\otimes I\otimes I} A\otimes B\otimes H\otimes A\otimes B \xrightarrow{I\otimes I\otimes \longrightarrow_A \otimes I} A\otimes B\otimes A\otimes B \xrightarrow{I\otimes T\otimes I}$$
$$A\otimes A\otimes B\otimes B \xrightarrow{\cdot_A \otimes \cdot_B} A\otimes B.$$

<u>Definition</u> 0.22.

The <u>Smash product</u> of A and B, denoted by A#B, is defined to be the H-dimodule $A\otimes B$ with the above multiplication. In symbols,

$$(a\#b)\cdot(c\#d) \equiv \sum_{(b)} a\cdot(b_1 \longrightarrow c)\#b_0\cdot d.$$

Note that the multiplication in A#B depends only on the comodule structure on B and the module structure of A (and the algebra structure on A and B).

With the structure described above A#B is an H-dimodule algebra with the H-action inherited from the R-module $A\otimes B$.

<u>Definition</u> 0.23.

Let A be an H-dimodule algebra. Then $\overline{A}$ is defined as the H-dimodule

A with multiplication given by $\overline{a}\cdot\overline{b} = \overline{\sum_{(a)} (a_1 \rightharpoonup b)\cdot a_0}$ with H-action inherited from A.

$\overline{A}$ is an H-dimodule algebra [L_2, Proposition 3.5].

Definition 0.24.

Let A be an H-dimodule algebra, finitely generated projective and faithful over R. We define two maps

$$F : A\#\overline{A} \longrightarrow \mathrm{End}(A)$$

and

$$G : \overline{A}\#A \longrightarrow \mathrm{End}(A)^0,$$

by

$$F\,(a\#\overline{b})(c) \equiv \sum_{(b)} a\cdot(b_1 \rightharpoonup c)\cdot b_0$$

and

$$G\,(\overline{a}\#b)(c) \equiv \sum_{(c)} (c_1 \rightharpoonup a)\cdot c_0\cdot b.$$

F and G are H-dimodule algebra maps [L_2, Proposition 4.1].

We now come to the concept of H-Azumaya algebra.

Definition 0.25.

If A is an H-dimodule algebra which is finitely generated projective and faithful over R, such that F and G as defined above are isomorphisms, then A is said to be H-Azumaya.

Example 0.26.

(i) If M is an H-dimodule finitely generated projective and faithful over R, then End(M) is an H-Azumaya algebra.

(ii) If A and B are H-Azumaya so are A#B and $\overline{A}$ [L_2, Theorem 4.3].

We are now ready to define the concept of the Brauer group of H-dimodule algebras.

Definition 0.27.

Let A and B be H-Azumaya. Then A and B are Brauer Equivalent, denoted by $A \sim B$, if there exist H-dimodules M,N so that $A \# \mathrm{End}(M) \approx B \# \mathrm{End}(N)$ as H-dimodule algebras.

$\sim$ is an equivalence relation which respects the operation #.

Definition 0.28.

The Brauer group of H-dimodule algebras, BD(R,H), is the group of Brauer equivalence classes with multiplication induced by # and inverse induced by — .

Remark

By giving trivial H-module and H-comodule structures to an R-algebra A, we can make A into an H-dimodule algebra. We then have $\overline{A} = A^0$, $A \# B = A \otimes B$, etc. and B(R) can be embedded as a subgroup of BD(R,H).

If we put the trivial H-comodule structure on an H-module algebra A, then A becomes an H-dimodule algebra. The equivalence classes of such H-Azumaya algebras form a group BM(R,H), which can be embedded in BD(R,H). Similarly, we can form a subgroup BC(R,H) of BD(R,H), consisting of equivalence classes of H-Azumaya algebras with trivial H-module structures. Furthermore, B(R) can also be embedded in both BM(R,H) and BC(R,H).

Let H = R[G], where G is a finite abelian group. Then an H-dimodule is a G-graded module with a grade-preserving G-action. Furthermore, let ϕ be a bimultiplicative map, $\phi : G \times G \to U(R)$, where U(R) is the group of units of R. We can then define the group $B_\phi(R,G)$, (see [K] and [CGO]), and embed it as a subgroup of

BD(R,R[G] for all ϕ [L_1, Theorem 1.6]. B(R) is also a subgroup of $B_\phi(R,G)$. Childs in [C] has shown that under certain conditions, $BD(R,R[G]) \simeq B_\phi(R,G\times G)$ for some ϕ.

CHAPTER I

H-AZUMAYA ALGEBRAS OF TRUNCATED POWER SERIES HOPF ALGEBRAS

Elements of $BD(R,H)$, the H-Azumaya algebras, are different types of objects than the ordinary R-Azumaya algebras. However, under certain conditions, H-Azumaya algebras are R-Azumaya. This in no way implies that $BD(R,H) = B(R)$, since the equivalence classes and multiplications are not the same. Furthermore, there may be R-Azumaya algebras that have non-trivial H-actions or H-coactions, so they would not be in the image of $B(R)$ embedded in $BD(R,H)$. In order to study when an H-Azumaya algebra is R-Azumaya the appropriate object to consider is not $B(R)$ but $BD_0(R,H)$, the principal object of study in this paper.

Throughout this chapter, we assume that H is a commutative and cocommutative Hopf algebra, finitely generated projective and faithful over R.

We begin with the definition of $BD_0(R,H)$.

1. $BD_0(R,H)$

Definition 1.1.

$BD_0(R,H) = \{[A]\varepsilon BD(R,H) : A \text{ is R-Azumaya}\}$.

Thus $BD_0(R,H)$ consists of classes of H-Azumaya algebras such that there is at least one R-Azumaya algebra in each class. We now show that this means every member of the equivalence class must also be R-Azumaya.

Proposition 1.2.

Let $[A]\varepsilon BD_0(R,H)$ and $B\varepsilon[A]$. Then B is R-Azumaya.

Proof

$B\varepsilon[A]$ implies $B \sim A$, and by definition there exist H-dimodules M and N such that $A \# End(M) \simeq B \# End(N)$ as H-dimodule algebras. By Long's Proposition 3.10 [L_2], we know that if C is an H-dimodule algebra and V an H-dimodule then $C \# End(V) \simeq C \otimes End(V)$ as H-dimodule algebras. So we have $B \# End(N) \simeq B \otimes End(N) \simeq A \otimes End(N)$ as H-dimodule algebras. Now, $A \otimes End(N)$ is R-Azumaya since both A and End(N) are. Thus $B \otimes End(N)$ is also R-Azumaya and so must be B, since tensor components of R-Azumaya algebra must also be R-Azumaya [OS, Ex. 2.15(a)].//

Long in [L_2] has shown that $BD(K,H) = BD_0(K,H)$ for $H = K[C_p]$ and K an algebraically closed field of characteristic p. This implies that any H-Azumaya algebra is already R-Azumaya. This is not true in general, but under extra conditions it is the case for truncated power series Hopf algebras.

2. Truncated Power Series Hopf Algebras

Definition 1.3.

A truncated power series Hopf algebra is the algebra $H = R[x_1, \ldots, x_n]/(x_1^{s_1}, \ldots, x_n^{s_n})$, where s_i is a positive integer for each i, with some coalgebra structure imposed on it.

As any H-comodule is equivalent to an H*-module, it is essential to study the structure of H* for a truncated power series Hopf algebra. We first need a definition.

Definition 1.4.

A sequence of elements of a coalgebra over R, $\{d_0, d_1, \ldots, d_n\}$, is a sequence of divided powers of length n if $\Delta d_i = \sum_{j=0}^{i} d_j \otimes d_{i-j}$ and $\varepsilon(d_i) = \delta_{0,i}$ for all i.

We note that in the above definition, if R is a field and $d_0 \neq 0$, the augmentation condition follows from $(\varepsilon \otimes I)\Delta = I$.

Proposition 1.5.

If H is a truncated power series Hopf algebra then H* is generated as an algebra by sequences of divided powers.

Proof

See [MP, Theorem 5 and Lemma 6].//

Remark

We note that if a truncated power series Hopf algebra is an algebra over a field of characteristic p, then $s_i = p^{t_i}$ for each i. Further, both $K[C_p]$ and α_p are truncated power series Hopf algebras. Their duals are generated as algebras by $\{1,z\}$, a sequence of divided powers of length 2, since $\Delta z = 1 \otimes z + z \otimes 1$ (See Examples 0.7 and 0.8).

Definition 1.6.

Let A and C be algebras over R. Then a sequence of maps $\{d_0, d_1, \ldots, d_n\}$ from A to C is a higher derivation from A to C if for $0 \leq j \leq n$,

(i) $d_j(1) = \delta_{0,j} \cdot 1_C$

(ii) $d_i(ab) = \sum_{j=0}^{i} d_j(a) \cdot d_{i-j}(b)$.

n is called the degree of the higher derivation.

Thus, a derivation is just a higher derivation of degree 1, $\{d_0 = 1, d\}$, so that $d(1) = 0$ and $d(ab) = a \cdot d(b) + d(a) \cdot b$.

Proposition 1.7.

Let $\{d_0, d_1, \ldots, d_n\}$ be a sequence of divided powers in a Hopf algebra H. Then $\{d_0, d_1, \ldots, d_n\}$ is a higher derivation on any H-module algebra A into itself.

Proof

By definition of H-module algebra, we have $h \rightharpoonup (ab) = \sum_{(h)} (h_1 \rightharpoonup a)(h_2 \rightharpoonup b)$ for all $a, b \in A$ with $\Delta h = \sum_{(h)} h_1 \otimes h_2$. But $\Delta d_i = \sum_{j=0}^{i} d_j \otimes d_{i-j}$, so we have $d_i(ab) = d_i \rightharpoonup (ab) = \sum_{j=0}^{i} (d_j \rightharpoonup a) \cdot (d_{i-j} \rightharpoonup b)$. //

3. When is an H-Azumaya algebra R-Azumaya?

We answer the above question for truncated power series Hopf algebras in this section. Some preliminaries are required.

Proposition 1.8.

Let A be an H-comodule. If $h^* \rightharpoonup a = 0$ for all $h^* \in 1^{\perp} = \{h^* \in H^* : \langle h^*, 1 \rangle = 0\}$, then $\chi(a) = a \otimes 1$.

Proof

Let $\langle 1, h_1, \ldots, h_m \rangle$ be a projective basis for H. Also let $\chi(a) = a_0 \otimes 1 + a_1 \otimes h_1 + \ldots + a_m \otimes h_m$. Then, by definition,

$h^* \rightharpoonup a = a_0 \langle h^*, 1 \rangle + a_1 \langle h^*, h_1 \rangle + \ldots + a_m \langle h^*, h_m \rangle$.

Set $h^* = h_i^*$, the dual to h_i, for $i > 0$. Then $h^* \rightharpoonup a = a_i \langle h^*, h_i \rangle = a_i$.

Now, $h^* \in 1$, so by assumption $a_i = 0$ for $i > 0$. Thus $\chi(a) = a_0 \otimes 1$. By counitary property, we have

$$\begin{aligned} a &= a_0\varepsilon(1) + a_1\varepsilon(h_1) + \ldots + a_m\varepsilon(h_m) \\ &= a_0\varepsilon(1) = a_0, \end{aligned}$$

so $\chi(a) = a \otimes 1$.//

Proposition 1.9.

Let $A = A_0 \oplus A_1 \oplus \ldots \oplus A_r$, where each A_i is an R-algebra generated by a central orthogonal idempotent e_i. Suppose d is a derivation on A, i.e. $d(ab) = ad(b) + d(a)b$ for all $a, b \in A$. Then $d(e_i) = 0$ for all i.

Proof

Let $d(e_i) = \sum_{j=1}^{r} a_{ij}e_j$, with $a_{ij} \in A_j$. Now,

$$\begin{aligned} d(e_i) &= d(e_ie_i) \qquad \text{since } e_i \text{ is idempotent,} \\ &= e_id(e_i) + d(e_i)e_i \\ &= e_i(\sum_{j=1}^{r} a_{ij}e_j) + (\sum_{j=1}^{r} a_{ij}e_j)e_i \\ &= a_{ii}e_i + a_{ii}e_i \qquad \text{as } e_i \text{ is central orthogonal,} \\ &= (a_{ii} + a_{ii})e_i. \end{aligned}$$

This implies that $a_{ij} = 0$ for all $j \neq i$. Furthermore, $a_{ii} + a_{ii} = a_{ii}$, so $a_{ii} = 0$ also. Thus $d(e_i) = 0$ for all i.//

Proposition 1.10.

Let A be as in Proposition 1.9. Further, let $\{d_0 = 1, d_1, \ldots, d_n\}$ be a sequence of higher derivations on A. Then $d_j(e_i) = 0$ for all i if $j > 0$.

<u>Proof</u>

We proceed by induction on j. If $j = 1$ then d_1 is just a derivation so $d_1(e_i) = 0$ by Proposition 1.9. Suppose $d_j(e_i) = 0$ for $j > 0$. Then
$$d_{j+1}(e_i) = d_{j+1}(e_ie_i)$$
$$= \sum_{r=0}^{j+1} d_r(e_i)d_{r-j}(e_i)$$
$$= e_id_{j+1}(e_i) + d_{j+1}(e_i)e_i,$$
by the induction hypothesis. The same argument leading to Proposition 1.9 now shows $d_{j+1}(e_i) = 0$.//

<u>Theorem</u> 1.11.

Let R be a separably closed field of characteristic p and H a truncated power series Hopf algebra. Suppose A is a semisimple H-Azumaya algebra. Then A is R-Azumaya.

<u>Proof</u>

A is a semisimple algebra over a separably closed field R, so that $A = A_0 \oplus \ldots \oplus A_r$ where each A_i is an R-algebra generated by a central orthogonal idempotent e_i. By Proposition 1.5, H* is generated as an algebra by sequences of divided powers. Each sequence is a higher derivation on A by Proposition 1.7, and Proposition 1.10 now implies that $d_j(e_i) = 0$ for all $d_j \neq 1$. It then follows from Proposition 1.8 that $\chi(e_i) = e_i \otimes 1$ for each i. Now, A is H-Azumaya, so that both maps $F : A \# \bar{A} \to \mathrm{End}(A)$ and $G : \bar{A} \# A \to \mathrm{End}(A)^0$ in Definition 0.24 are isomorphisms. We have

$$F_{1\#\bar{c}_i}(c) = 1_A \cdot (1_H \rightharpoonup c)\bar{e}_i \text{ for all } c \in A$$
$$= c\bar{e}_i$$

and $F_{e_i\#\bar{1}}(c) = e_i\cdot(1\rightharpoonup c)1$ for all $c\in A$

$= e_i c$

$= ce_i$ since e_i is central.

Thus, $F_{1\#\bar{e}_i} = F_{e_i\#\bar{1}}$, so that $1\#\bar{e}_i = e_i\#\bar{1}$ since F is an isomorphism. Hence $e_i = \lambda\cdot 1_A$ for some $\lambda\in R$. Being an idempotent e_i must be 1_A for all i. So there could only be one matrix ring $A \simeq A_0$, and $A \simeq R_m$ which is R-Azumaya.//

4. Group Algebras

We now apply Theorem 1.11 to group algebras and obtain Long's result as a special case.

The assumption that A be semisimple is necessary in general, but for group algebras we obtain this as a consequence of the following:

Proposition 1.12.

If R is a field and H = R[G], then any H-Azumaya algebra is semisimple.

Proof

This is Long's Theorem 1.9 [L1].//

Theorem 1.13.

Let $G = C_{s_1}\oplus\ldots\oplus C_{s_n}$, where $s_i = p^{t_i}$ with t_i a positive integer. Let H = R[G] with R a separably closed field of characteristic p. Then $BD(R,H) = BD_0(R,H)$.

Proof

$$H = R[G] \simeq R[x_1, \ldots, x_n]/(x_1^{s_1}-1, \ldots, x_n^{s_n}-1)$$

$$\simeq R[y_1, \ldots y_n]/(y_1^{s_1}, \ldots, y_n^{s_n})$$

where $y_i = x_i-1$.

Thus H is a truncated power series Hopf algebra. Proposition 1.12 guarantees that the hypothesis of Theorem 1.11 is satisfied.//

The above theorem generalizes Long's Proposition 5.2 [L2] for $G = C_p$ and Beattie's Lemma 4.3 [Be] where $G = C_p n$.

We remark here that the above theorem does not imply that $BD(R,H) = B(R)$, since the equivalence classes and multiplication in $BD(R,H)$ are not the ordinary ones in $B(R)$. In fact, Long has computed that $BD_0(R,H) = C_{p-1}$ for $H = R[C_p]$ and R is an algebraically closed field of characteristic p [L2, Theorem 5.8], and it is well known that $B(R) = 0$.

5. Example

We now give an example to show that semisimplicity is necessary in the hypothesis of Theorem 1.11.

Example 1.14.

Let R be a field of characteristic p and $H = \alpha_p$, as in Example 0.8. Also, let the algebra be $A = R[a]/(a^p)$. Make A into an H-dimodule by defining the action on A by $x \rightharpoonup a^n = na^{n-1}$ and the coaction by $\chi(a^n) = (a\otimes 1 + 1\otimes x)^n$. Then $x \rightharpoonup$ behaves like a derivative, so that the usual differential rule, $(x^n \cdot x^m) \rightharpoonup a^r = x^n \rightharpoonup (x^m \rightharpoonup a^r)$ holds; hence A is an H-module. The coaction $\chi(a) = a\otimes 1 + 1\otimes x$ induces an H^*-action by $z \rightarrow a = a\langle z,1\rangle + 1\langle z,x\rangle = 1$. This extends to $z \rightarrow a^n = na^{n-1}$, another derivative, and by similar argument A is an H^*-module, hence an H-comodule. Both the actions $x \rightharpoonup$ and $z \rightarrow$ commute so A is an H-dimodule. For A to be an H-module algebra, we check the two conditions of Definition 0.15.

Recall that $\varepsilon(x) = 0$ and $\Delta x = x\otimes 1 + 1\otimes x$, so that $x \rightharpoonup 1_A = 0 = \varepsilon(x)\cdot 1_A$ and $x \rightharpoonup (a^n \cdot a^m) = x \rightharpoonup a^{n+m}$

$$= (n+m)x^{n+m-1}$$

$$= (x \rightharpoonup a^n)(1 \rightharpoonup a^m) + (1 \rightharpoonup a^n)(x \rightharpoonup a^m).$$

A is therefore an H-module algebra, and similarly an H*-module algebra, hence an H-comodule algebra.

To see that A is H-Azumaya we need the following propositions.

Proposition 1.15.

Let A be a finitely generated projective algebra over a connected ring R with rank r and projective basis $\langle a_0, \ldots, a_{r-1}\rangle$. Then $\mathrm{End}(A) = A_\ell F_0 \oplus \ldots \oplus A_\ell F_{r-1}$, where $A_\ell = \{a_\ell \in \mathrm{End}(A) : a \in A$ and $a_\ell(x) = ax$ for $x \in A\}$ consists of left multiplications and $F_i \in \mathrm{End}(A)$ such that $F_i(a_n) = \delta_{i,n}$.

Proof

This is a known and easy linear algebra computation. We only note here that if $f \in \mathrm{End}(A)$ then $f = \sum_{i=0}^{r-1} f(a_n)_\ell \circ F_i$. //

Proposition 1.16.

Let $A = R[a]/(a^p)$ and $x \rightharpoonup a^n = na^{n-1}$ as in Example 1.14. Then the F_i's in Proposition 1.15 can be generated by $x \rightharpoonup$.

Proof

We construct our F_i's inductively, starting with F_{p-1}. Define $F_{p-1} \equiv (x^{p-1} \rightharpoonup\ )/(p-1)!$.

Then $F_{p-1}(a^n) = (x^{p-1} \rightharpoonup a^n)/(p-1)!$

$$= \begin{matrix} 1 & \text{if } n = p-1 \\ 0 & \text{otherwise} \end{matrix}$$

$$= \delta_{n,p-1}.$$

Now, $(x^{p-2} \rightharpoonup a^n)/(p-2)! = 0$ if $n < p-2$

$= 1$ if $n = p-2$

$= (p-1)a$ if $n = p-1$.

So we define $F_{p-2} \equiv (x^{p-2} \rightharpoonup \ )/(p-2)! - (p-1)a_\ell \circ F_{p-1}$. Then $F_{p-2}(a^n) = \delta_{n,p-2}$. Continue the process to get $F_{p-3}, \ldots, F_0$.//

The above two propositions say that End(A) is generated as an algebra by left multiplications, a_ℓ, and a derivation, $x \rightharpoonup$. We use this result to show that F and G are isomorphisms, hence A is H-Azumaya.

Proposition 1.17.

$F : A \# \bar{A} \to \mathrm{End}(A)$ is an isomorphism.

Proof

Recall that $F_{a\#\bar{b}}(c) = \sum_{(b)} a\cdot(b_1 \rightharpoonup c)b_0$. We check F on algebra generators $a\#\bar{1}$ and $1\#\bar{a}$.

$F_{a\#\bar{1}}(c) = a\cdot(1 \rightharpoonup c)\cdot 1 = ac$, for all $c \in A$. Thus,

$F_{a\#\bar{1}} = a_\ell$, the left multiplication. Now,

$$F_{1\#\bar{a}}(c) = 1\cdot(1 \rightharpoonup c)\cdot a + 1\cdot(x \rightharpoonup c)\cdot 1$$

$$= ca + x \rightharpoonup c$$

$$= ac + x \rightharpoonup c \quad \text{since A is commutative.}$$

Therefore $F_{1\#\bar{a}} = a_\ell + x \rightharpoonup$, so that

$$F_{1\#\bar{a} - a\#\bar{1}} = a_\ell + x \rightharpoonup - a_\ell = x \rightharpoonup .$$

But a_ℓ and $x \rightharpoonup$ generate End(A), hence the image of F contains generators of End(A). Thus F is surjective and by counting dimensions, F must be an isomorphism.//

Proposition 1.18.

$G : \overline{A} \# A \to \mathrm{End}(A)^0$ is an isomorphism.

Proof

Recall that $G_{\overline{a}\#b}(c) \equiv \sum_{(c)} (c_1 \rightharpoonup a)\cdot c_0 \cdot b$. We again check G on generators $\overline{1}\#a$ and $\overline{a}\#1$.

$$\begin{aligned} G_{\overline{1}\#a}(c) &= \sum_{(c)} (c_1 \rightharpoonup 1)\cdot c_0 \cdot a \\ &= \sum_{(c)} \varepsilon(c_1)\cdot c_0 \cdot a \quad \text{since } c_1 \rightharpoonup 1 = \varepsilon(c_1)\cdot 1_A \\ &= c\cdot a \quad \text{by counitary property} \\ &= ac. \end{aligned}$$

Thus $G_{\overline{1}\#a} = a_\ell$, the left multiplication. Now $G_{\overline{a}\#1}(c) = \sum_{(c)} (c_1 \rightharpoonup a)\cdot c_0 \cdot 1$. We look at the module generators $\langle 1, a, \ldots, a^{p-1}\rangle$. So let $c = a^n$, $0 \leq n < p$. Then

$$\chi(c) = (a^n) = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} \otimes x^i, \text{ so that}$$

$$\begin{aligned} G_{\overline{a}\#1}(a^n) &= \sum_{i=0}^{n} \binom{n}{i} (x^i \rightharpoonup a)\cdot a^{n-i}\cdot 1 \\ &= (1 \rightharpoonup a)\cdot a^n + \binom{n}{1} (x \rightharpoonup a) a^{n-1} \qquad \text{as } x^i \rightharpoonup a = 0 \text{ for } i > 1. \\ &= a\cdot a^n + n\cdot 1 \cdot a^{n-1} \\ &= a_\ell(a^n) + x \rightharpoonup a^n. \end{aligned}$$

Thus $G_{\overline{a}\#1} = a_\ell + x \rightharpoonup$, and so

$G_{\overline{a}\#1} - G_{\overline{1}\#a} = a_\ell + x \rightharpoonup - a_\ell = x \rightharpoonup = G_{\overline{a}\#1-\overline{1}\#a}$.

As before, G is an isomorphism.//

We have shown that A is an H-Azumaya algebra. This H-Azumaya algebra is commutative and local, thus cannot be R-Azumaya nor semi-simple. In general, then, $BD(R,H) \neq BD_0(R,H)$ for truncated power series Hopf algebras.

As a final theorem we have the following corollary to Theorem 1.11.

Theorem 1.19.

Let R be a separably closed field of characteristic p and H a truncated power series Hopf algebra. Let A be an H-Azumaya algebra. Then a necessary and sufficient condition for A to be R-Azumaya is that it be semisimple.

Proof

Theorem 1.11 provides one condition. For the other, suppose A is R-Azumaya. Then A is central simple, hence semisimple.//

CHAPTER II

A COMPUTATION OF $BD_0(R,H)$

In this chapter we compute $BD_0(R,H)$ for $H = \alpha_p$. Throughout H denotes a commutative, cocommutative Hopf algebra, finitely generated projective and faithful over R. All R-modules are also finitely generated projective and faithful over R.

1. When $BD_0(R,H)$ is a subgroup.

$BD_0(R,H)$, the subset of $BD(R,H)$ consisting of only R-Azumaya algebras, in general is not a subgroup of $BD(R,H)$. Orzech has given the following example:

Example 2.1 [O, Example 2.10].

Let R be a ring such that 2 is a unit, and $H = R[G]$ where $G = C_2 \times C_2$, the Klein four group. Let $A = M_2(R)$, the ring of 2×2 matrices over R. Then with appropriate action and coaction, A is H-Azumaya and in fact R-Azumaya, so A is in $BD_0(R,H)$. Its inverse $\overline{A}$, however, is commutative and thus not central, so that $\overline{A}$ is not in $BD_0(R,H)$. Hence $BD_0(R,H)$ is not a subgroup of $BD(R,H)$.

However, we do have the following result.

Theorem 2.2.

Suppose each element $[A]$ in $BD_0(R,H)$ is such that $A \simeq End(V)$ as an H-module algebra for some finitely generated projective H-module V. Then $BD_0(R,H)$ is a subgroup of $BD(R,H)$.

Remark

We mean by the above hypothesis that A is an H-module algebra and that there exists an H-module V such that $A \simeq End(V)$, where the structure on V induces the original structure on A. The hypothesis also implies that $A \simeq End(V)$ as an R-algebra for some R-module V, i.e. $B(R) = 0$. However, $B(R) = 0$ is not a sufficient condition. When $B(R) = 0$, we do have $A \simeq End(V)$ for some R-module V if $[A] \varepsilon B(R)$; and we can put a trivial H-action on A to make End(V) an H-module algebra, but not all [A] in $BD_0(R,H)$ are trivial H-modules.

We need the following proposition.

Proposition 2.3.

Let M be an H-module and B an H-dimodule algebra. Then $End(M) \# B \simeq End(M) \otimes B$ as an H-module algebra.

Proof

This is just Long's Proposition 3.7 [L_2] specialized down to an H-module. The first part of his proof does not require M to be an H-dimodule so that the same proof goes through. We only record here the maps for later reference. The map $\phi : End(M) \# B \rightarrow End(M) \otimes B$ given by $\phi(f\#b) = \sum_{(b)} f\cdot f_{b_1} \otimes b$ is an isomorphism of H-module algebras, where $f_h \varepsilon End(M)$ is defined by $f_h(m) = h \rightharpoonup m$. The inverse of ϕ is given by $\phi^{-1} : End(M) \otimes B \rightarrow End(M) \# B$, $\phi^{-1}(f\otimes b) = \sum_{(b)} f\cdot f_{S(b_1)} \# b_0$.//

Proof of Theorem 2.2:

(i) Let [A] and [B] be in $BD_0(R,H)$. Then by assumption, $A \simeq End(M)$ and $B \simeq End(N)$ as H-module algebras for some H-modules M and N. Recall that the multiplication in $A \# B$ depends only on the comodule structure on B, the module structure on A and the algebra structures on A and B.

Thus $A \# B \simeq \mathrm{End}(M) \# B$ as an H-module algebra

$\simeq \mathrm{End}(M) \otimes B$ by Proposition 2.3

$\simeq \mathrm{End}(M) \otimes \mathrm{End}(N)$ by assumption on B

$\simeq \mathrm{End}(M \otimes N)$,

which is an R-Azumaya algebra.

Therefore $[A \# B] \varepsilon BD_0(R,H)$.

(ii) Let $[A] \varepsilon BD_0(R,H)$ and $A \simeq \mathrm{End}(M)$ as an H-module algebra. Then, by definition of an H-Azumaya algebra we have $\mathrm{End}(A) \simeq A \# \overline{A}$ as an H-dimodule algebra. Thus, in particular $\mathrm{End}(A) \simeq A \# \overline{A}$ as an H-module algebra

$\simeq \mathrm{End}(M) \otimes \overline{A}$ by

Proposition 2.3. Hence $\mathrm{End}(M) \otimes \overline{A}$ is R-Azumaya and so each tensor component must also be R-Azumaya. $BD_0(R,H)$ is therefore closed under # and —, thus a subgroup of BD(R,H).//

2. $BD_0(R,H)$ for $H = \alpha_p$

We now show that for $H = \alpha_p$, $BD_0(R,H)$ is indeed a subgroup of BD(R,H) for an appropriate R. This is done by showing that for each $[A] \varepsilon BD_0(R,H)$, $A \simeq \mathrm{End}(V)$ as an H-module algebra for some H-module V. We then apply Theorem 2.2 to get the result.

For the rest of this chapter $H = \alpha_p$, and R is a ring of characteristic p. x and z are the generators for $H = \alpha_p$ and H^* respectively, as in Example 0.8.

We first need to develop some machinery.

Definition 2.4.

A derivation d on algebra A is Inner if there is a b in A such that $d(a) = ba - ab$ for all a in A.

Proposition 2.5

If an algebra A is separable then any derivation on A is inner.

Proof

See [OS, Proposition 4.11].//

Proposition 2.6

Let $[A] \varepsilon BD_0(R,\alpha_p)$. Then there are non-zero elements c and u in A such that $x \rightharpoonup a = ua - au$ and $z \rightharpoondown a = ca - ac$ for all a in A. Furthermore, $z \rightharpoondown u = cu - uc = \alpha 1_A$ and $c^p = \alpha_c$, $u^p = \alpha_u$ where α, α_c and α_u are in R.

Proof

The generators x and z are primitives, so they act on A as derivations, by Proposition 1.6. By the above proposition, x and z are inner, so there exist c and u in A such that $x \rightharpoonup a = ua - au$ and $z \rightharpoondown a = ca - ac$ for all a in A.

A is an H-dimodule, so the actions commute, i.e., $z \rightharpoondown (x \rightharpoonup a) = x \rightharpoonup (z \rightharpoondown a)$ for all a in A; or $z \rightharpoondown (ua-au) = x \rightharpoonup (ca-ac)$. Therefore $c(ua-au) - (ua-au)c = u(ca-ac) - (ca-ac)u$, so that $(cu-uc)a = a(cu-uc)$ for all a in A. Thus cu-uc must be in the center of A, i.e., $cu-uc = \alpha \varepsilon R$, as claimed.

Now, by a straightforward but tedious induction we have $x^s \rightharpoonup a = \sum_{r=0}^{s} (-1)^r \binom{s}{r} u^{s-r} a u^r$, so that

$$x^p \rightharpoonup a = \sum_{r=0}^{p} (-1)^r \binom{p}{r} u^{p-r} a u^r = u^p a - a u^p = 0, \text{ since } x^p = 0.$$

Therefore, $u^p a = a u^p$ for all a in A, hence u^p is in the center of A and thus $u^p = \alpha_u \varepsilon R$. A similar computation holds for c, so that $c^p = c_\alpha \varepsilon R$.//

The elements u and c in A in the above proposition are hardly unique. In fact, there are as many u's and c's as there are ring elements, as the next proposition shows.

Proposition 2.7

Let a map $d : A \to A$ be given by $d(a) = ua - au$ for all a in A, where A is a central R-algebra and $u \varepsilon A$. Then $d(a) = wa - aw$ for some w in A iff $w = \alpha + u$ for some α in R.

Proof

Suppose there is another w in A such that $d(a) = ua - au = wa - aw$ for all a in A. Then $(w - u)a = a(w - u)$, so $(w - u)$ is a central element and $w - u = \alpha$ for some α in R. Thus $w = \alpha + u$.

Conversely, if $w = \alpha + u$ for $\alpha \varepsilon R$, then $u = w - \alpha$ so that

$$\begin{aligned} d(a) &= ua - au \\ &= (w - \alpha)a - a(w - \alpha) \\ &= wa - aw \quad \text{for all a in A.}// \end{aligned}$$

We now introduce some notations.

Definition 2.8

Let $[A] \varepsilon BD_0(R, \alpha_p)$. Set $U_A \equiv \{u \varepsilon A : x \rightharpoonup a = ua \text{ for all a in A}\}$ and $C_A \equiv \{c \varepsilon A : z \rightharpoonup a = ca - ac \text{ for all a in A}\}$.

Even though there are many possible choices for u and c, $z \rightharpoonup u = cu - uc$ is always the same for $u \varepsilon U_A$ and $c \varepsilon C_A$. Since this will assume a significant role later on we record this observation as the following:

Proposition 2.9

$z \rightharpoonup u = cu - uc = \alpha \cdot 1_A$ for $u \varepsilon U_A$ and $c \varepsilon C_A$.

Proof

By Proposition 2.6, there is at least one $\bar{u}$ and a $\bar{c}$ such that $z \rightharpoonup \bar{u} = \bar{c}\bar{u} - \bar{u}\bar{c} = \alpha \cdot 1_A$. Let $u \varepsilon U_A$ and $c \varepsilon C_A$. Then from Proposition 2.7 $u = \lambda + \bar{u}$ and $c = \tau + \bar{c}$ for some λ and τ in R. We have

$$\begin{aligned} z \rightharpoonup u &= cu - uc \\ &= (\tau + \bar{c})(\lambda + \bar{u}) - (\lambda + \bar{u})(\tau + \bar{c}) \\ &= \bar{c}\bar{u} - \bar{u}\bar{c} \\ &= \alpha \cdot 1_A . // \end{aligned}$$

We sharpen Proposition 2.6 slightly for our need.

Proposition 2.10

Let $[A] \varepsilon BD_0(R, \alpha_p)$ where R is a ring of characteristic p such that every p^{th}-root of each of its elements is in R. Then there are non-zero u and c in A such that $x \rightharpoonup a = ua - au$ and $z \rightharpoonup a = ca - ac$, where $c^p = 0 = u^p$.

Proof

Proposition 2.6 already provides us with u and c such that $u^p = \alpha_u$ and $c^p = \alpha_c$. If $\alpha_u = 0 = \alpha_c$, we are done. If not, then pick $\bar{u} = u + \lambda$ where $\lambda = (-\alpha_u)^{1/p}$. Then $u \varepsilon U_A$. Now $\bar{u}^p = (u + \lambda)^p = u^p + \lambda^p = \alpha_u - \alpha_u = 0$, as required. Similarly, we can pick $\bar{c} = c + \tau$ where $\tau = (-\alpha_c)^{1/p}$.//

We will use the notation $R = R^{1/p}$ to denote that R contains every p^{th}-root of each of its elements.

We are now ready to make A into End(V) as an H-module algebra.

Proposition 2.11

Let $[A] \varepsilon BD_0(R,H)$, $H = \alpha_p$, $B(R) = 0$ and R a ring of characteristic p such that $R = R^{1/p}$. Then $A \simeq End(V)$ as an H-module algebra for some finitely generated H-module V.

Proof

The ordinary Brauer group $B(R) = 0$ implies $A \simeq End(V)$ as an R-algebra for some R-module V. We now make V into an H-module. Define $x \rightharpoonup v \equiv u(v)$ where $u \in U_A$, $v \in V$ and x is the generator of H. Now, extend this over H by $x^i \rightharpoonup v \equiv u^i(v)$. Note that by Proposition 2.10 we can pick u so that $u^p = 0 = x^p$. Also, define $\alpha \rightharpoonup v \equiv \alpha v$ for $\alpha \in R$.

It is easily seen that this gives V an H-action.

We now show that the above construction induces a structure on $A \simeq End(V)$ that coincides with the original structure on A. Recall from section 0.4 that if V is an H-module then End(V) is also an H-module by the following action,

$$(h \rightharpoonup a)(v) = \sum_{(h)} h_1 \rightharpoonup [a(S(h_2) \rightharpoonup v)]$$

for $a \in End(V)$, $h \in H$ and $v \in V$.

$$\begin{aligned}
\text{Thus, } (x \rightharpoonup a)(v) &\equiv x \rightharpoonup [a(S(1) \rightharpoonup v] + 1 \rightharpoonup [a(S(x) \rightharpoonup v)] \\
&= x \rightharpoonup [a(1 \rightharpoonup v)] + 1 \rightharpoonup [a(-x \rightharpoonup v)] \\
&= x \rightharpoonup a(v) - a(x \rightharpoonup v) \\
&= u[a(v)] - a[u(v)], \text{ by construction} \\
&= (ua - au)(v) \\
&= (x \rightharpoonup a)(v), \text{ the original structure.//}
\end{aligned}$$

Corollary 2.12

Suppose we have the same assumptions as Proposition 2.11. Then $A \simeq End(v)$ as an H-comodule algebra for some H-comodule V.

Proof

This is equivalent to giving V an H*-action. So, define $z \rightharpoondown v \equiv c(v)$, $\alpha \rightharpoondown v \equiv \alpha v$, where $c \varepsilon C_A$, $v \varepsilon V$ and $\alpha \varepsilon R$. The same argument as Proposition 2.11 now yields the original H-comodule structure on A, because α_p is self dual.//

Proposition 2.13

Suppose we have the same assumptions as Proposition 2.11. Then $A \approx End(V)$ as an H-dimodule algebra for some H-dimodule V iff $cu = uc$, where $u \varepsilon U_A$ and $c \varepsilon C_A$.

Proof

By Proposition 2.11 and Corollary 2.12, V is an H-module and H*-module (hence H-comodule). For V to be an H-dimodule, we need to show that the actions x and z do commute.

$$\begin{aligned} \text{Now, } x \rightharpoonup (z \rightharpoondown v) &= x \rightharpoonup c(v) \\ &= u[c(v)] \\ &= uc(v), \\ \text{and } \quad z \rightharpoondown (x \rightharpoonup v) &= z \rightharpoondown u(v) \\ &= cu(v). \end{aligned}$$

Thus V is an H-dimodule iff $uc = cu$.//

Theorem 2.14

Let $H = \alpha_p$ and $B(R) = 0$ where R is a ring of characteristic p and $R = R^{1/p}$. Then $BD_0(R,H)$ is a subgroup of $BD(R,H)$.

Proof

By Proposition 2.11, if $[A] \varepsilon BD_0(R,H)$ then $A \approx End(V)$ as an H-module algebra for H-module V. Theorem 2.2 now yields the conclusion.//

3. Grade of $A \otimes B$.

For each $[A] \in BD_0(R,\alpha_p)$, Proposition 2.9 provides us with a unique $\alpha \in R$ such that $z \rightharpoondown u = cu - uc = \alpha \cdot 1_A$. This unique α is called the Grade of A.

We now show that the grade of A is never 1. Two auxiliary results are first needed.

Proposition 2.15

Let $[A] \in BD_0(R,\alpha_p)$, $x \rightharpoonup a = ua - au$ and $z \rightharpoondown a = ca - ac$ for all a in A. Then

(i) $x^k \rightharpoonup u^n = 0$ and $z^k \rightharpoondown c^n = 0$, $0 < k < p$ for all n.

(ii) $x^k \rightharpoonup c = 0$ and $z^k \rightharpoondown u = 0$ if $k > 1$.

(iii) $x \rightharpoonup c^n = uc^n - c^n u = -n\alpha c^{n-1}$,

$z \rightharpoondown u^n = cu^n - u^n c = n\alpha n^{n-1}$ for all n.

(iv) $x^k \rightharpoonup u^i c^j = u^i (x^k \rightharpoonup c^j)$

$z^k \rightharpoondown u^i c^j = (z^k \rightharpoondown u^i)\, c^j$

(v) $z^k \rightharpoondown (x^m \rightharpoonup u^i c^j) = (z^k \rightharpoondown u^i)(x^m \rightharpoonup c^j)$.

Proof

(i) and (ii) are just trivial consequences of $x \rightharpoonup u = 0$, $z \rightharpoondown c = 0$ and $cu - uc = \alpha$. (iii) follows because x and z are both primitives so they behave like derivations.

(iv) $$x^k \rightharpoonup u^i c^j = \sum_{r=0}^{k} (-1)^r \binom{k}{r} u^{k-r} \cdot u^i c^j u^r$$

$$= u^i \cdot \sum_{r=0}^{k} (-1)^r \binom{k}{r} u^{k-r} \cdot c^j u^r$$

$$= u^i (x^k \rightharpoonup c^i).$$

$z^k \rightharpoondown$ is similarly easy.

(v) $z^k \rightharpoondown (x^m \rightharpoonup u^i c^j) = z^k \rightharpoondown [u^i \cdot (x^m \rightharpoonup c^j)]$, from part (iv).
Now $x \rightharpoonup c^j = -j\alpha c^{j-1}$, so $x^m \rightharpoonup c^j$ contains only c's and no u's.
Thus $z^k \rightharpoondown [u^i \cdot (x^m \rightharpoonup c^j)] = (z^k \rightharpoondown u^i)(x^m \rightharpoonup c^j)$, by part (iv) again.//

Proposition 2.16

Let $u \varepsilon U_A$ and $c \varepsilon C_A$. Then $\chi(c) = c \otimes 1$ and $\chi(u) = u \otimes 1 + \alpha \otimes x$, where $\chi : A \to A \otimes H$ is the comodule map and α is the grade of A.

Proof

By Proposition 2.15 (i), $z^k \rightharpoondown c = 0$ for $k > 0$. Hence $h^* \rightharpoondown c = 0$ for all $h^* \varepsilon 1^{\perp}$, the augmentation ideal. Now Proposition 1.8 gives $\chi(c) = c \otimes 1$. For the second part, let $\chi(u) = u_0 \otimes 1 + \ldots + u_{p-1} \otimes x^p$. Then $h^* \rightharpoondown u = \langle h^*, 1 \rangle u_0 + \ldots + \langle h^*, x^{p-1} \rangle u_{p-1}$ for $h^* \varepsilon H^*$, so that $z^k \rightharpoondown u = u_k$. Proposition 2.15 (ii) gives $z^k \rightharpoondown u = 0$ for $k > 1$, so that $u_k = 0$ for $k > 1$. Now, $u_1 = z \rightharpoondown u = cu - uc = \alpha$ and $u_0 = 1 \rightharpoondown u = u$. Thus $\chi(u) = u \otimes 1 + \alpha \otimes x$, as claimed.//

Proposition 2.17

Let $[A] \varepsilon BD_0(R, \alpha_p)$. Then the grade of A is never 1.

Proof

We need to show that $z \rightharpoondown u = cu - uc \neq 1$. A is H-Azumaya, so by definition, $F : A \# \overline{A} \to \mathrm{End}(A)$ given by $F_{a\#b}(f) = \sum_{(b)} a \cdot (b_1 \rightharpoonup f) \cdot b_0$, for a, b, $f \varepsilon A$, is an isomorphism of H-dimodule algebras. Now,

$$\begin{aligned} F_{1\#\overline{u}}(f) &= 1(1 \rightharpoonup f)u + 1(x \rightharpoonup f)\alpha \quad \text{by Proposition 2.16,} \\ &= fu + \alpha(x \rightharpoonup f) \\ &= uf - (x \rightharpoonup f) + \alpha(x \rightharpoonup f) \quad \text{since } x \rightharpoonup f = uf - fu \\ &= uf + (\alpha - 1)(x \rightharpoonup f), \end{aligned}$$

so that $(\alpha - 1)(x \rightharpoonup f) = F_{1\#\bar{u}}(f) - uf$. But $F_{u\#\bar{1}}(f) = u(1 \rightharpoonup f)1 = uf$, hence $(\alpha - 1)(x \rightharpoonup f) = F_{1\#\bar{u}}(f) - F_{u\#\bar{1}}(f)$. Suppose $\alpha = 1$. Then $0 = F_{1\#\bar{u}}(f) - F_{u\#\bar{1}}(f)$, so $u\#\bar{1} = 1\#\bar{u}$, since F is an isomorphism. Thus $u = \lambda \cdot 1_A$ for some $\lambda \varepsilon R$, by the Lemma of faithfully flat descent. But then

$$\begin{aligned} 1 = \alpha = z \rightharpoonup u &= cu - uc \\ &= c\lambda - \lambda c \\ &= 0, \end{aligned}$$

a contradiction. Hence $\alpha \neq 1$.//

We proceed on the computation of the grade of $A \otimes B$.

Proposition 2.18

Let $[A], [B] \varepsilon BD_0(R, \alpha_p)$. Suppose $\bar{u} \varepsilon U_{A \otimes B}$. Then $\bar{u} = u \otimes 1 + 1 \otimes w$ where $u \varepsilon U_A$ and $w \varepsilon U_B$.

Proof

Recall from section 0.4 that

$$h \rightharpoonup (a \otimes b) = \sum_{(h)} (h_1 \rightharpoonup a) \otimes (h_2 \rightharpoonup b), \text{ so that}$$

$$\begin{aligned} x \rightharpoonup (a \otimes b) &= (x \rightharpoonup a) \otimes (1 \rightharpoonup b) + (1 \rightharpoonup a) \otimes (x \rightharpoonup b), \ a \varepsilon A \text{ and } b \varepsilon B \\ &= (ua - au) \otimes b + a \otimes (wb - bw), \ u \varepsilon U_A \text{ and } w \varepsilon U_B \\ &= ua \otimes b - au \otimes b + a \otimes wb - a \otimes bw \\ &= (u \otimes 1 + 1 \otimes w)(a \otimes b) - (a \otimes b)(u \otimes 1 + 1 \otimes w) \\ &= \bar{u}(a \otimes b) - (a \otimes b)\bar{u}. \end{aligned}$$

Thus $\bar{u} = u \otimes 1 + 1 \otimes w$ is in $U_{A \otimes B}$.//

Proposition 2.19

Let $[A], [B] \varepsilon BD_0(R, \alpha_p)$ where A is of grade α and B is of grade β. Then $A \otimes B$ is of grade $\alpha + \beta$.

Proof

We have only to compute $z \rightharpoonup \bar{u}$, for $\bar{u}\varepsilon U_{A\otimes B}$. Now, $\bar{u} = u\otimes 1 + 1\otimes w$ from the previous proposition and $\chi(u) = u\otimes 1 + \alpha\otimes x$ and $\chi(w) = w\otimes 1 + \beta\otimes x$ from Proposition 2.16. Also, recall that $\chi(a\otimes b) = \sum_{(a)(b)} (a_0\otimes b_0)\ a_1 b_1$, $a_0\varepsilon A$, $b_0\varepsilon B$, a_1, $b_1\varepsilon H$. Thus $\chi(\bar{u}) = (u\otimes 1)\otimes 1 + (\alpha\otimes 1)\otimes x + (1\otimes w)\otimes 1 + (1\otimes\beta)\otimes x$, so that $z \rightharpoonup \bar{u} = \alpha\otimes 1 + 1\otimes\beta = (\alpha + \beta)(1\otimes 1)$

$$= (\alpha + \beta)\cdot 1_{A\otimes B}.//$$

4. Grade of A # B

The next few results examine the grade of an H-Azumaya algebra in $BD_0(R,\alpha_p)$ under various algebra homomorphisms.

Proposition 2.20

Let $[A]$, $[B]\varepsilon BD_0(R,\alpha_p)$. Suppose $\phi : A \rightarrow B$ is an epimorphism of H-module algebras. Then $\phi(u)\varepsilon U_B$ if $u\varepsilon U_A$.

Proof

Since ϕ is an epimorphism, any element in B can be written as $\phi(a)$ for some a in A.

So $x \rightharpoonup \phi(a) = \phi(x \rightharpoonup a)$, ($\phi$ is an H-module map)

$= \phi(ua - au)$

$= \phi(u)\phi(a) - \phi(a)\phi(u)$.

Thus $\phi(u)\varepsilon U_B$.//

Corollary 2.21

Let $[A]$, $[B]\varepsilon BD_0(R,\alpha_p)$. Suppose $\phi : A \rightarrow B$ is an emimorphism of H*-module algebras. Then $\phi(c)\varepsilon C_B$ if $c\varepsilon C_A$.

Proof

Same as Proposition 2.20.//

Proposition 2.22

Let $[A]$, $[B]\varepsilon BD_0(R,\alpha_p)$. Suppose $\phi : A \to B$ is an isomorphism of H-module and H*-module algebras. Then A and B must be of the same grade.

Proof

Suppose A is of grade α, i.e. $z \rightharpoonup u = \alpha\cdot 1_A$ for $u\varepsilon U_A$. Let $w\varepsilon U_B$. Then

$$\begin{aligned} z \rightharpoonup w &= z \rightharpoonup \phi(u) \quad \text{by Proposition 2.20} \\ &= \phi(z \rightharpoonup u) \quad (\phi \text{ is H*-module map}) \\ &= \phi(\alpha\cdot 1_A) \\ &= \alpha\cdot 1_B. \end{aligned}$$

Thus ϕ preserves grade.//

We now compute the grade of A # B.

Proposition 2.23

Let $[A]$, $[B]\varepsilon BD_0(R,\alpha_p)$, $B(R) = 0$ and $R = R^{1/p}$. Suppose $\overline{w}\varepsilon U_{A\#B}$. Then $\overline{w} = (1 - \beta)\cdot(u\#1) + 1\#w$, where $u\varepsilon U_A$, $w\varepsilon U_B$ and β is the grade of B.

Proof

$A \cong \mathrm{End}(V)$ as an H-module algebra for some H-module V, by Proposition 2.11. Furthermore, from Proposition 2.3, there is an isomorphism $\phi : \mathrm{End}(V) \# B \to \mathrm{End}(V) \otimes B$ of H-module algebras, and the inverse of ϕ, $\phi^{-1} = \psi$, is given by $\psi(f\otimes b) = \sum_{(b)} f\cdot f_{S(b_1)}\# b_0$, where f, $f_{S(b_1)}\varepsilon\mathrm{End}(V)$ and $\chi(b) = \sum_{(b)} b_0\otimes b_1$.

Let $\overline{w} \varepsilon U_{A\#B}$. Then $\overline{w} = \chi(\overline{u})$ for some $\overline{u} \varepsilon U_{A\otimes B}$, since ψ is an H-module algebra isomorphism (Proposition 2.20). Now, by Proposition 2.18 $\overline{u} = u\otimes 1 + 1\otimes w$, where $u \varepsilon U_A$ and $w \varepsilon U_B$.

So, $\overline{w} = \psi(\overline{u}) = \psi(u\otimes 1 + 1\otimes w)$

$$= u\cdot f_{S(1)}\#1 + 1\cdot f_{S(1)}\#w + 1\cdot f_{S(x)}\#\beta,$$

where $f_h \varepsilon \mathrm{End}(V) \cong A$ is defined by $f_h(v) = h \rightharpoonup v$ for $v \varepsilon V$. Now, $f_{S(1)}(v) = S(1) \rightharpoonup v = 1 \rightharpoonup v$, so that $f_{S(1)} = 1$. Also, $f_{S(x)}(v) = S(x) \rightharpoonup v = -x \rightharpoonup v = -u(v)$, (by Proposition 2.11), so $f_{S(x)} = -u$. Thus,

$$\begin{aligned}\overline{w} = \psi(\overline{u}) &= u\cdot 1\#1 + 1\cdot 1\#w + 1\cdot(-u)\#\beta \\ &= u\#1 + 1\#w - \beta(u\#1). \\ &= (1-\beta)(u\#1) + 1\#w. //\end{aligned}$$

Proposition 2.24

Let $[A]$, $[B] \varepsilon BD_0(R,\alpha_p)$, $B(R) = 0$ and $R = R^{1/p}$. Suppose A is of grade α and B is of grade β. Then A # B is of grade $\alpha + \beta - \alpha\beta$.

Proof

We compute $z \rightharpoonup \overline{u}$ for $\overline{u} \varepsilon U_{A\#B}$. By the previous proposition, $\overline{u} = (1-\beta)(u\#1) + 1\#w$ for $u \varepsilon U_A$ and $w \varepsilon U_B$. Now, $A \# B = A \otimes B$ as H-modules, and $\chi : A \# B \rightarrow A \# B \otimes H$ depends only on the R-module structure of A # B.

So, $\chi(\overline{u}) = (1-\beta)\chi(u\otimes 1) + \chi(1\otimes w)$

$$= (1-\beta)(u\otimes 1\otimes 1 + \alpha\otimes 1\otimes x) + 1\otimes w\otimes 1 + 1\otimes\beta\otimes x.$$

Hence,
$$\begin{aligned} z \rightharpoonup \overline{u} &= (1-\beta)(\alpha\otimes 1) + 1\otimes\beta \\ &= (\alpha - \alpha\beta + \beta)(1\otimes 1) \\ &= (\alpha - \alpha\beta + \beta)(1\#1).\end{aligned}$$

Thus A # B is of grade $\alpha + \beta - \alpha\beta$.//

Remark

We note that $A \otimes B$ and $A \# B$ have different grades, so that Proposition 2.22 does not apply here. This is due to the fact that $\phi : \mathrm{End}(V) \# B \to \mathrm{End}(V) \otimes B$ (where $A \simeq \mathrm{End}(V)$), as given in Proposition 2.3, is only an H-module algebra map. It does not preserve the H-comodule structure, hence cannot be an H*-module map.

5. $BD_0(R,\alpha_p)$.

We are now ready to characterize $BD_0(R,\alpha_p)$. Following Long, we say that A is of Type $1 - \alpha$ if $[A] \in BD_0(R,\alpha_p)$ is of grade α.

Theorem 2.25

Let $R = R^{1/p}$ and $B(R) = 0$. Then the Type map, $\tau : BD_0(R,\alpha_p) \to R^*$, given by $\tau([A]) = 1 - \alpha$, where α is the grade of A and R^* is the multiplicative group of R, is a group monomorphism.

Proof

We have seen that the grade of A is never 1, so the type of A is never 0 and we are in the right range.

(i) τ is well-defined, for let $[A] \sim [B]$ in $BD_0(R,\alpha_p)$. Then by definition, there exist H-dimodules M and N such that $A \# \mathrm{End}(M) \simeq B \# \mathrm{End}(N)$ as H-dimodule algebras. By Proposition 2.22, they must have the same grade. Now, M and N are H-dimodules, so it follows from Proposition 2.13 that End(M) and End(N) must be of grade 0. Thus $A \# \mathrm{End}(M)$ has grade $\alpha + 0 - \alpha \cdot 0 = \alpha$ and $B \# \mathrm{End}(N)$ has grade $\beta + 0 - \beta \cdot 0 = \beta$. Since they must be the same we have $\alpha = \beta$ and so $\tau([A]) = \tau([B])$.

(ii) τ is a group homomorphism, since

$$\begin{aligned}\tau([A]\#[B]) &= \tau([A\#B]) \\ &= 1-(\alpha + \beta - \alpha\beta) \\ &= (1-\alpha)\cdot(1-\beta) \\ &= \tau([A])\cdot\tau([B]).\end{aligned}$$

(iii) τ is a monomorphism, for suppose $\tau([A]) = 1$. Then the grade of A must be 0, by the definition of τ; so that $z \rightharpoonup u = cu - uc = 0$ for $u \in U_A$ and $c \in C_A$. It follows from Proposition 2.13 again that we must have $A \simeq \mathrm{End}(V)$ as an H-dimodule algebra for some H-dimodule V. Let M be any H-dimodule, then as H-dimodule algebras, we have

$$\begin{aligned}A \# \mathrm{End}(M) &\simeq \mathrm{End}(V) \# \mathrm{End}(M) \\ &\simeq \mathrm{End}(V \otimes M)\end{aligned}$$ (by Long Cor. 3.8 [L_2]). This means that $[A] \sim [0]$ in $BD_0(R, \alpha_p)$.//

The rest of this section is devoted to showing that τ is in fact onto, by constructing an appropriate H-Azumaya algebra for each type in R^*.

Construction 2.26

Let $\alpha \in R$, where R is a connected ring. We set $V = R[y]/(y^p)$. Define $u : V \to V$ by $u(v) = y \cdot v$, the left multiplication. Then $u \in \mathrm{End}(V)$. Also, define $c : V \to V$ by $c(y^n) = n\alpha y^{n-1}$, for $v = y^n$, i.e. a derivation and a left α-multiplication. Then $c \in \mathrm{End}(V)$ too. Now, let $A_\alpha = \mathrm{End}(V)$, so that A_α is R-Azumaya.

We now make A_α into an H-dimodule algebra.

Lemma 2.27

If c and u are as defined in Construction 2.26, then $cu - uc = \alpha$.

Proof

Let $v = y^n$, $0 \leq n < p$. Then

$$\begin{aligned}(cu - uc)(v) &= cu(y^n) - uc(y^n)\\ &= c(y^{n+1}) - u(n\alpha y^{n-1})\\ &= (n+1)\,\alpha y^n - n\alpha y^n\\ &= \alpha\cdot y^n\\ &= \alpha\cdot v,\end{aligned}$$

so that $cu - uc = \alpha$ on v and by linearity, on all of V.//

Lemma 2.28

A_α is an H-module and an H-comodule.

Proof

Define the action, $\rightharpoonup : H \otimes A_\alpha \to A_\alpha$ by $x \rightharpoonup a = ua - au$. This makes A into an H-module. Note that $x^s \rightharpoonup a = \sum_{r=0}^{s} (-1)^r \binom{s}{r} u^{s-r} a u^r$ for $0 \leq s < p$. Also, define $\rightharpoondown : H^* \otimes A_\alpha \to A_\alpha$ by $z \rightharpoondown a = ca - ac$. Then A_α is also an H^*-module, so an H-comodule.//

If we now define $\rightharpoonup_V : H \otimes V \to V$ and $\rightharpoondown_V : H^* \otimes V \to V$ by $x \rightharpoonup_V v = u(v)$ and $z \rightharpoondown_V v = c(v)$ respectively, then by a similar argument as in Proposition 2.11, V is an H-module and H-comodule. The above definitions on V also induce structures on $A_\alpha = \mathrm{End}(V)$ that coincide with the original structures. Furthermore, V is an H-dimodule iff $\alpha = 0$ by Proposition 2.13.

Proposition 2.29

A_α is an H-dimodule.

Proof

We only need to show that the actions $x \rightharpoonup$ and $z \rightharpoondown$ commute. Now,

$$x \rightharpoonup z \rightharpoondown a - z \rightharpoondown x \rightharpoonup a$$
$$= x \rightharpoonup (ca - ac) - z \rightharpoondown (ua - au)$$
$$= u(ca - ac) - (ca - ac)\,u - c(ua - au) + (ua - au)c$$
$$= (uc - cu)a + a(cu - uc)$$
$$= (-\alpha)a + a(\alpha) \quad \text{since } cu - uc = \alpha$$
$$= 0, \text{ so } x \rightharpoonup z \rightharpoondown a = z \rightharpoondown x \rightharpoonup a.//$$

Lemma 2.30

A_α is an H-module algebra.

Proof

We need to show

(i) $h \rightharpoonup 1_A = \varepsilon(h)\cdot 1_A$ for all $h \varepsilon H$.

Now, $x^s \rightharpoonup 1_A = \sum_{r=0}^{s} (-1)^r \binom{s}{r} u^{s-r}\cdot 1 \cdot u^r$, if $s \neq 0$,

$$= u^s \cdot \sum_{r=0}^{s} (-1)^r \binom{s}{r}$$
$$= u^s \cdot 0 = 0.$$

Also, $\varepsilon(x^s) = 0$ if $s \neq 0$.

If $s = 0$ then $x^s \rightharpoonup 1_A = 1 \rightharpoonup 1_A = 1_A = \varepsilon(1)\cdot 1_A$;

so $x^s \rightharpoonup 1_A = \varepsilon(x^s)\cdot 1_A$ for $0 \leq s < p$.

(ii) $h \rightharpoonup (ab) = \sum_{(h)} (h_1 \rightharpoonup a)(h_2 \rightharpoonup b)$ for a, b in A.

Now, $x \rightharpoonup (ab) = uab - abu$

and $\sum_{(x)} (x_1 \rightharpoonup a)(x_2 \rightharpoonup b) = (1 \rightharpoonup a)(x \rightharpoonup b) + (x \rightharpoonup a)(1 \rightharpoonup b)$

$$= a(ub - bu) + (ua - au)b$$
$$= -abu + uab.//$$

Corollary 2.31

A_α is an H-comodule algebra.

Proof

This is equivalent to checking if A is an H*-module algebra and a similar argument as above will work.//

Proposition 2.32

A_α is an H-dimodule algebra.

Proof

A_α is already an H-dimodule and the previous two lemmas make it into an H-module and an H-comodule algebra.//

We remark that V is an H-dimodule iff $\alpha = 0$. But A_α is an H-dimodule algebra regardless of α.

We now turn to the structures of A_α and $\mathrm{End}(A_\alpha)$.

Proposition 2.33

(i) $A_\alpha = \mathrm{End}(V)$ is generated as an R-algebra by c and u if $\alpha \neq 0$, where c and u are as in Construction 2.26.

(ii) $\mathrm{End}(A_\alpha)$ is generated as an R-algebra by $x\rightharpoonup$, $z\rightharpoondown$, c_ℓ and u_ℓ if $\alpha \neq 0$, where $c_\ell(a) = ca$ and $u_\ell(a) = ua$ (for $a\varepsilon A$) are the left multiplications.

Proof

We only sketch a proof here. Recall from Chapter I, Proposition 1.15, that if A is an algebra with dimension r, then $\mathrm{End}(A) = A_\ell F_0 \oplus . . . \oplus A_\ell F_{r-1}$, where A_ℓ is the subalgebra generated by left-multiplication and $F_i\varepsilon\mathrm{End}(A)$ such that $F_i(a_n) = \delta_{i,n}$ for a basis element a_n. Furthermore, Proposition 1.15 shows that for that particular A, F_i's are generated by the derivation $x\rightharpoonup$. So End(A) is generated by left-multiplications and derivation. Now, for A_α = End(V), u behaves as a left-multiplication and c as an α-derivation, by construction. So, if $\alpha \neq 0$, by a similar proof as Proposition 1.16, c and u do generate End(V). A similar but more lengthy proof also works for End(A_α), except there the left-multiplications are generated by c_ℓ and u_ℓ, and the derivations are generated by $x\rightharpoonup$ and $z\rightharpoondown$ .//

Finally, we show that A_α is in fact H-Azumaya. It is already an H-dimodule algebra, so that we have only to show that the maps $F : A_\alpha \# \overline{A}_\alpha \to \mathrm{End}(A_\alpha)$ and $G : \overline{A}_\alpha \# A_\alpha \to \mathrm{End}(A_\alpha)^0$ are isomorphisms of H-dimodule algebras.

<u>Proposition</u> 2.34

$F : A_\alpha \# \overline{A}_\alpha \to \mathrm{End}(A_\alpha)$ given by $F_{a\#\overline{b}}(f) = \sum_{(b)} a\cdot(b_1 \rightharpoonup f)\cdot b_0$ where a, b, b_0, $f \varepsilon A_\alpha$ and b_1 εH, is an isomorphism of H-dimodule algebras, if $\alpha \neq 0$, $\alpha \neq 1$ and R is a connected ring.

<u>Proof</u>

F is already an H-dimodule algebra map by Long's Proposition 4.1 [L_2], so we have only to show it is onto. It is sufficient to use a dimension argument, since R is connected.

(i) $F_{a\#\overline{1}}(f) = a\cdot(1 \rightharpoonup f)\cdot 1$ where a, $f \varepsilon A$,

$= a\cdot f$

$= a_\ell(f)$,

so $F_{a\#\overline{1}} = a_\ell$, and we can generate all the left-multiplications.

(ii) We now show that $x\rightharpoonup$ and $z\rightharpoondown$ can also be generated.

$$F_{1\#\overline{u}}(f) = 1\cdot(1\rightharpoonup f)\cdot u + 1\cdot(x\rightharpoonup f)\alpha \quad \text{since } \chi(u) = u\otimes 1 + \alpha\otimes x$$

$$= fu + \alpha(x\rightharpoonup f)$$

$$= uf - x\rightharpoonup f + \alpha(x\rightharpoonup f) \quad \text{since } x\rightharpoonup f = uf - fu,$$

$$= uf + (\alpha - 1)(x\rightharpoonup f).$$

Thus, $(\alpha - 1)(x\rightharpoonup f) = F_{1\#\overline{u}}(f) - F_{u\#\overline{1}}(f)$, since $F_{u\#\overline{1}}(f) = uf$ by Part (i) above. Now $\alpha \neq 1$, so $x\rightharpoonup f$ can be generated.

Finally, $F_{1\#\overline{c}}(f) = 1(1\text{—}\, f)c$ since $\chi(c) = c\otimes 1$

$$= f\cdot c,$$

so $z\rightharpoondown f = cf - fc$

$$= F_{c\#\overline{1}}(f) - F_{1\#\overline{c}}(f).$$

Now, u_ℓ, c_ℓ, $x\rightharpoonup$ and $z\rightharpoondown$ are the generators of $\mathrm{End}(A_\alpha)$, by Proposition 2.33. We have generated them so F must be onto, hence an isomorphism.//

Proposition 2.35

Suppose R is a connected ring and $\alpha \neq 0$, $\alpha \neq 1$. Then $G : \overline{A}_\alpha \# A_\alpha \to \mathrm{End}(A_\alpha)^0$ given by $G_{\overline{a}\#b}(f) = \sum_{(f)} (f_1\rightharpoonup a)\cdot f_0\cdot b$ is an isomorphism of H-dimodule algebras.

Proof

By a similar but longer and more tedious computation as in Proposition 2.34, we can show that $G_{u\#\overline{1}}(f) = u_\ell(f)$, $G_{c\#\overline{1}} - {}_{1\#\overline{c}}(f) = (1-\alpha)(z\rightharpoondown f)$, $G_{u\#\overline{1}} - {}_{1\#\overline{u}}(f) = x\rightharpoonup f$ and $c_\ell(f) = z\rightharpoondown f + G_{1\#\overline{c}}(f)$. Thus, we can generate all the generators of $\mathrm{End}(A_\alpha)$ so that G is onto and hence an isomorphism.//

We can now characterize $BD_0(R,\alpha_p)$ completely.

Theorem 2.36

Let R be a field of characteristic p such that $R = R^{1/p}$ and $B(R) = 0$. Then the type map $\tau : BD_0(R,\alpha_p) \to R^*$ given by $\tau([A]) = 1 - \alpha$, α the grade of A, is a group isomorphism.

Proof

τ is already a group monomorphism, by Theorem 2.25. If $\alpha \neq 0$ and $\alpha \neq 1$, then the previous construction yields an H-Azumaya algebra of the appropriate type. If $\alpha = 0$ then by Lemma 2.27 $cu - uc = 0$, so $V = R[y]/(y^p)$ is an H-dimodule by the remark after Lemma 2.28. But then $A_\alpha = \mathrm{End}(V)$ must be H-Azumaya, by Long's Theorem 4.3, [L_2]. Thus τ is onto.//

6. Concluding Remarks

In this last section, we give a few directions in which the results in this chapter could be pursued. We first recast Theorem 2.36 into a form that we can generalize. Some preliminary results are needed.

Proposition 2.37

Let $H = \alpha_p$. Then the only group-like element in H is 1 and primitives are of the form αx with $\alpha \varepsilon R$.

Proof

(i) 1 is group-like since $\Delta 1 = 1 \otimes 1$. If $n > 0$, then $\Delta(x^n) = \sum_{i=0}^{n} x^i \otimes x^{n-i}$ so x^n cannot be group-like. It remains to check $\alpha \varepsilon R$. But $\Delta\alpha = \Delta(\alpha \cdot 1) = \alpha(\Delta 1)$

$$= \alpha(1 \otimes 1) \neq \alpha \otimes \alpha, \text{ if } \alpha \neq 1,$$

so 1 is the only group-like.

(ii) Any x^n with $n > 1$ cannot be a primitive. That leaves αx with $\alpha \in R$. Now,

$$\begin{aligned}\Delta(\alpha x) &= \alpha(\Delta x) \\ &= \alpha(1 \otimes x + x \otimes 1) \\ &= 1 \otimes \alpha x + \alpha x \otimes 1,\end{aligned}$$

so that αx is a primitive.//

Definition 2.38

Let C and D be coalgebras. Then a coalgebra map $f : C \to D$ is a map that satisfies $(f \otimes f)\Delta_C = \Delta_D f$ and $\varepsilon_C = \varepsilon_D f$.

We note that this is just a dual definition of an algebra map.

Definition 2.39

Let A and B be bialgebras. Then a bialgebra map $f : A \to B$ is a map that is both an algebra and a coalgebra map.

Proposition 2.40

Let $f : C \to C$ be a coalgebra map. Then f carries group-like to group-like and primitive to primitive.

Proof

This is a standard and easy computation.//

We are now ready to characterize R^*.

Proposition 2.41

$R^* \cong \text{Autbialg}(\alpha_p)$ as a group, where $\text{Autbialg}(\alpha_p)$ denotes the group of bialgebra automorphisms of α_p.

Proof

Let $\phi \varepsilon \mathrm{Autbialg}(\alpha_p)$. Then ϕ must preserve group-likes and primitives, by the previous proposition. The only group-like is 1 and the primitives are αx, $\alpha \varepsilon R$. We look at the generator x of α_p. x is primitive so $\phi(x) = \alpha x$, $\alpha \neq 0$ (since ϕ is 1-1). Thus each ϕ is associated with a unique α and we write $\phi = \phi_\alpha$. Define $\tau : R^* \to \mathrm{Autbialg}(\alpha_p)$ by $\tau(\alpha) = \phi_\alpha$. Then $\tau(\alpha\beta) = \phi_{\alpha\beta} = \phi_\alpha \cdot \phi_\beta = \tau(\alpha) \cdot \tau(\beta)$, so τ is a group homomorphism. If $\tau(\alpha) = \mathrm{Id}$ then $\alpha = 1$, thus τ is 1-1. It is clearly onto, hence τ is a group isomorphism.//

We restate Theorem 2.36 in its new form.

Theorem 2.42

Let R be a connected ring of characteristic p such that $R = R^{1/p}$ and $B(R) = 0$. Then $BD_0(R,\alpha_p) \simeq \mathrm{Autbialg}(\alpha_p)$ as a group.

Remark 2.43

A very nice result would be to generalize the above theorem to any truncated power series Hopf algebra H and get $BD_0(R,H) \simeq \mathrm{Autbialg}(H)$. To this end, there are such objects as inner higher derivations (see [RS]), that reduce down to the ordinary inner derivations, but they are much harder to deal with. We also have to show first that $BD_0(R,H)$ is actually a subgroup.

Remark 2.44

Orzech in [O, Theorem 4.4] has shown that $BD_0(R,G) \simeq B(R) \oplus \mathrm{Aut}(G)$, if G is a finite abelian group of exponent m and either G is cyclic of prime order p or [G:1] is a unit in R, where R is a connected and separably closed ring of characteristic p that contains a primitive m-th root of 1 satisfying $\mathrm{Pic}_m(R) = 0$ and $H^2(G,U(R)) = 0$.

Now, $\mathrm{Aut}(G) \simeq \mathrm{Autbialg}(R[G])$, so when $B(R) = 0$ this is the same result for $H = R[G]$ that we have obtained for $H = \alpha_p$ in Theorem 2.42. We could pursue this direction by relaxing the condition $B(R) = 0$ and show that $BD_0(R,H) \simeq B(R) \oplus \mathrm{Autbialg}(H)$ for $H = \alpha_p$ and may be a truncated power series Hopf algebra for an appropriate R.

Remark 2.45

Let $H = \alpha_p$. Then Example 1.14 shows that there is an H-Azumaya algebra that is not R-Azumaya, so that $BD(R,H) \neq BD_0(R,H)$. It would be nice to compute $BD(R,H)$ for $H = \alpha_p$ and some specific R. In this direction Long in [L_1, Theorem 2.7] has shown that if R is a separably closed field of characteristic not dividing n then $BD(R,C_n) \simeq D$, where D is the Dihedral group of order 2(n-1).

REFERENCES

[A] G. Azumaya, "On Maximally central algebras", Nagoya Math. J. 2 (1951), 119-150.

[AG] M. Auslander and O. Goldman, "The Brauer group of a commutative ring", Trans. Amer. Math. Soc. 97 (1960), 367-409.

[Ba] M. Bass, Lectures on Topics In Algebraic k-Theory, Tata Institute of Fundamental Research, Bombay, 1967.

[Be] M. Beattie, "Brauer Groups of H-module and H-dimodule algebras", Ph.D. dissertation, Queen's University, October 1976.

[C] L. N. Childs, "The Brauer group of graded algebras II: graded Galois extensions", Trans. Amer. Math. Soc. 204 (1975), 137-160.

[CGO] ____________, G. Garfinkel and M. Orzech, "The Brauer group of graded Azumaya algebras", Trans. Amer. Math. Soc. 175 (1973), 299-326.

[FW] A. Fröhlich and C. T. C. Wall, "Generalizations of the Brauer group", to appear (cited in [L_1,L_2]).

[K] M. A. Knus, "Algebras graded by a group", in Category Theory, Homology Theory and their Applications II, Lecture Notes in Mathematics 92, 117-133, Springer-Verlag, Berlin, 1969.

[L_1] F. W. Long, "A generalization of the Brauer group of graded algebras", Proc. London Math. Soc. (3) 29 (1974), 237-256.

[L_2] ____________, "The Brauer group of dimodule algebras", J. Algebra 30 (1974), 559-601.

[MP] R. A. Morris and B. Pareigis, "On curves on formal groups", Trans. Amer. Math. Soc. 233 (1977), 309-319.

[O] M. Orzech, "On the Brauer group of modules having a grading and an action", Canad. J. Math. 28-3 (1976), 533-552.

[OS] ____________ and C. Small, The Brauer group of Commutative rings, Lecture Notes in Pure and Applied Mathematics, No. 11, Marcel Dekker, New York, 1975.

[P_1] B. Pareigis, "When Hopf algebras are Frobenius algebras". J. Algebra 18 (1971), 588-596.

[P_2] ____________, Endliche Hopf-algebren, Vorlesungs-Ausarbeitung, 1973, Mathematisches Institut der Universität München.

[RS] A. Roy and R. Sridharan, "Higher derivations and central simple algebras", Nagoya Math. J. 32 (1968), 21-30.

[S_1] C. Small, "The Brauer-Wall group of a commutative ring", Trans. Amer. Math. Soc. 156 (1971), 455-491.

[S_2] ____________, "The group of quadratic extensions", J. Pure Appl. Alg. 2 (1972), 83-105.

[Sw_1] M. E. Sweedler, Hopf Algebras, Benjamin, New York, 1969.

[Sw_2] ____________, "Hopf algebras and combinatorics", Umbral calculus conference, University of Oklahoma, May 1978.

[W] C. T. C. Wall, "Graded Brauer groups", J. Reine Angew. Math. 213 (1964), 187-199.