COMPLEMENTED SUBSPACES OF L_p DETERMINED BY PARTITIONS AND WEIGHTS

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BY PARTITIONS AND WEIGHTS

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1. Abstract

Norms on a Banach space X determined by partitions and weights were introduced by D. Alspach in 1999. This thesis shows that this new approach unifies many well-known complemented subspaces of L_p developed during last four decades. It is proved that the class of spaces with such norms is stable under sums. We prove that a sequence space X with norm given by finitely many partitions and weights is isomorphic to a subspace of L_p. By introducing the envelope norm, we obtain a necessary condition for a Banach sequence space with norm given by partitions and weights to be isomorphic to a subspace of L_p. Using this we define a space Y_n with norm given by partitions and weights with distance to any subspace of L_p growing with n. This allows us to construct an example of a Banach space with norm given by partitions and weights which is not isomorphic to a subspace of L_p.

2. Introduction

Since the 1960's, understanding the complemented subspaces of L_p has been an interesting topic of research in Banach space theory **[L-P]** and **[L-R]**. Many specialists in this area have used many clever ideas to construct complemented subspaces of L_p. It was shown by Bourgain, Rosenthal and Schechtman in 1979 that up to isomorphism, there are uncountably many complemented subspaces of L_p **[B-S-R]**. In 1999, Dale Alspach proposed a new approach to describe the complemented subspaces of $L_p[0,1], p > 2$. Define for each partition $P = \{N_i\}$ of $\mathbb{N}$ and function $W : \mathbb{N} \to (0,1]$

$$\|(a_i)\|_{P,W} = \left(\sum_i \left(\sum_{j \in N_i} {a_j}^2 w_j^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}}$$

Now suppose that $(P_k, W_k)_{k\in K}$ is a family of pairs of partitions and functions as above. There are two fundamental questions which will be considered in this thesis. What conditions on $(P_k, W_k)_{k\in K}$ imply that

$$\|(a_i)\| = \sup_{k\in K} \|(a_i)\|_{(P_k,W_k)} \tag{2.1}$$

defines a norm on a space of sequences X so that X is isomorphic to a complemented subspace of $L_p[0,1]$? Is every complemented subspace of L_p (other than L_p) isomorphic to a space of this form?

Remark: These spaces have unconditional bases. So an affirmative answer to the second question would include proving that every $\mathcal{L}_p$ space has an unconditional basis. The paper includes five major sections besides this Introduction and a preliminaries section which contains the statement of some known results which will be needed later. Unless otherwise noted we will assume that $p > 2$ through out this thesis. We will also assume that the scalar field is $\mathbb{R}$.

In Section 4, we give a discussion of normalization by the inclusion of discrete partitions. We present well known examples of complemented subspaces of L_p with norm given by partitions and weights. We also prove that the sums of such Banach spaces are stable under these norms, i.e., whose norms are also given by partitions and weights.

In Section 5, we first show that if the norm on a space X is given by finitely many partitions and weights, then X is isomorphic to a subspace of L_p. Then we give the definition of an envelope norm which was suggested to us by Alspach and we prove the existence of the envelope norm generated by a family of partitions and weights. We also give a lower bound on a norm which is necessary for a space to be isomorphic to a subspace of L_p. Finally we show that if a space with norm given by partitions and weights is isomorphic to a subspace of L_p, then its norm is equivalent to the natural envelope norm.

In Section 6, we show that in most cases for single partitions and any weights, the space is isomorphic to a complemented subspace of L_p. For double partitions, we give a sufficient condition for the space to be isomorphic to a complemented subspace of L_p.

In Section 7, we construct an example which demonstrates the difference between a norm given by partitions and weights and the corresponding envelope norm. We also obtain an estimate of the distance between a certain Banach space Y_n with norm given by partitions and weights and ${X_p}^{\otimes n}$. Finally we give an example of a Banach space with norm given by partitions and weights which is not isomorphic to a subspace of

L_p by applying Theorem 5.9.

In Section 8, we construct a diagonal subspace of $(\sum \ell_2)_{\ell_p} \oplus X_p$ which is uncomplemented in the space $(\sum \ell_2)_{\ell_p} \oplus X_p$. However, we are unable to determine whether or not the space is isomorphic to a complemented subspace of L_p. In closing, we list some open questions for further development.

This thesis is a result of discussion between professor Dale Alspach and me for a couple of years in Oklahoma State University. Many ideas were coming from his great knowledge and talent in Banach space theory.

3. Preliminaries

In this section, we state some well known inequalities and define some standard spaces. We also give the definitions of some standard sums of spaces used in studying subspaces of L_p.

3.1. Inequalities. In this subsection, we state some useful inequalities.

3.1.1. **Khintchine's Inequality.** As it is stated in [**L-T-1**, p.66] the inequality of Khintchine has many applications in the study of L_p space theory. Two are of particular interest here. Let $r_n(t) = \operatorname{sign}\sin 2^n \pi t, n = 0, 1, 2, ...$ be the Rademacher functions on $[0, 1]$. For $1 \leq p < \infty$, there exist constants A_p and B_p such that for all scalars (a_n),

$$A_p \left(\sum |a_n|^2\right)^{\frac{1}{2}} \leq \left(\int_0^1 |\sum a_n r_n(t)|^p dt\right)^{\frac{1}{p}} \leq B_p \left(\sum |a_n|^2\right)^{\frac{1}{2}}$$

$A_p = 1$ if $2 \leq p < \infty$ and $B_p = 1$ if $p \leq 2$. An immediate consequence is that $[r_n : n = 0, 1, ...]$ is isomorphic to ℓ_2. A second consequence will be stated in 3.1.3.

Next we state a generalization of Khintchine's Inequality.

3.1.2. **Rosenthal's Inequality.** It is known that any sequence of mean zero independent random variables in L_p is unconditional. Rosenthal's inequality gives more information on these sequences [**R**]. Let $2 < p < \infty$. If $(x_i)_1^n$ are independent mean zero random variables in L_p, then there exists $K_p < \infty$ so that

$$\frac{1}{2} \max\left\{ \left(\sum_{i=1}^n \|x_i\|_p^p\right)^{\frac{1}{p}}, \left(\sum_{i=1}^n \|x_i\|_2^2\right)^{\frac{1}{2}} \right\}$$

$$\leq \left\|\sum_i x_i\right\|_p \leq K_p \max\left\{ \left(\sum_{i=1}^n \|x_i\|_p^p\right)^{\frac{1}{p}}, \left(\sum_{i=1}^n \|x_i\|_2^2\right)^{\frac{1}{2}} \right\}$$

It is shown in **[J-S-Z]** that K_p is of order $p/\ln p$.

3.1.3. **Upper and Lower Estimates in $\mathbf{L_p}$.** The following estimates are well-known and can be found in **[A-O]**. For $(x_i)_1^n \subset L_p$,

$$A_p\left(\sum_1^n \|x_i\|_p^2\right)^{\frac{1}{2}} \leq \left(\int_0^1 \left\|\sum_1^n r_i(t)x_i\right\|_p^p dt\right)^{\frac{1}{p}} \leq \left(\sum_1^n \|x_i\|_p^p\right)^{\frac{1}{p}}$$

if $1 \leq p \leq 2$, and

$$\left(\sum_1^n \|x_i\|_p^p\right)^{\frac{1}{p}} \leq \left(\int_0^1 \left\|\sum_1^n r_i(t)x_i\right\|_p^p dt\right)^{\frac{1}{p}} \leq B_p\left(\sum_1^n \|x_i\|_p^2\right)^{\frac{1}{2}}$$

if $2 < p < \infty$.

Integration against the Rademacher functions yields some useful inequalities for unconditional basic sequences in L_p. If (x_i) is a λ-unconditional normalized basic sequence in L_p, then

$$\lambda^{-1}\left(\sum |a_n|^p\right)^{\frac{1}{p}} \leq \left\|\sum a_n x_n\right\|_p \leq \lambda B_p\left(\sum |a_n|^2\right)^{\frac{1}{2}}$$

if $2 \leq p < \infty$, and

$$\lambda A_p{}^{-1}\left(\sum |a_n|^2\right)^{\frac{1}{2}} \leq \left\|\sum a_n x_n\right\|_p \leq \lambda B_p\left(\sum |a_n|^p\right)^{\frac{1}{p}}$$

if $1 \leq p \leq 2$.

3.2. Sums of Spaces. There are many ways to create Banach space sums of Banach spaces. The reader can find these definitions in many standard Banach theory books, e.g., **[J-L]**.

DEFINITION 3.1. **Direct Sum of Banach Spaces**

Let $X_1, \dots, X_n$ be normed spaces with norms $\|\cdot\|_{X_1}, \dots, \|\cdot\|_{X_n}$. Then the (external) direct sum or direct product of $X_1, \dots, X_n$ is the normed space whose underlying vector space is the vector space direct sum of $X_1, \dots, X_n$ and whose norm is the norm given by the formula

$$\|(x_1, \dots, x_n)\|_2 = \left(\sum_{j=1}^{n} \|x_j\|_{X_j}^2\right)^{\frac{1}{2}}$$

This normed space is denoted by $X_1 \oplus \dots \oplus X_n$ or sometimes $(X_1 \oplus \dots \oplus X_n)_{\ell_2}$.

DEFINITION 3.2. **ℓ_p-Sum**

Let $\{X_n\}$ be a sequence of Banach spaces. Then the ℓ_p-sum of $\{X_n\}$, $(\sum X_n)_{\ell_p}$, is the vector space of sequences (x_n) with $x_n \in X_n$ for all n, and finite norm,

$$\|(x_n)\|_{(\sum X_n)_{\ell_p}} = (\sum \|x_n\|_{X_n}^p)^{\frac{1}{p}}$$

A special case of this is the following well known example of a complemented subspace of L_p.

Suppose we have a sequence of Banach spaces X_n each isometric to ℓ_2. Then the ℓ_p-sum of ℓ_2 is the space $\{(x_n) : x_n \in X_n, \|(x_n)\|_{(\sum \ell_2)_{\ell_p}} < \infty\}$.

Rosenthal implicitly introduced another type of sum in **[R]** and these sums were developed further in **[A-2]**.

DEFINITION 3.3. **$(\mathbf{p}, \mathbf{2})$-Sum**

Let $p > 2$. Let (X_n) be a sequence of subspaces of $L_p(\Omega, \mu)$ for some probability measure μ, and let (w_n) be a sequence of real numbers, $0 < w_n \leq 1$. For any sequence (x_n) such that $x_n \in X_n$ for all n, let

$$\|(x_n)\|_{p,2,(w_n)} = \max\left\{\left(\sum \|x_n\|_p^p\right)^{\frac{1}{p}}, \left(\sum \|x_n\|_2^2 w_n^2\right)^{\frac{1}{2}}\right\}$$

and let

$$X = \left(\sum X_n\right)_{(p,2,(w_n))}$$

$$= \{(x_n) : x_n \in X_n \text{ for all } \|(x_n)\|_{p,2,(w_n)} < \infty\}$$

We will say that X is the $(p, 2, (w_n))$-sum of $\{X_n\}$.

Definition 3.4. **Tensor Product in $\mathbf{L_p}$**

For each $k \in \mathbb{N}$, let $I^k = [0,1]^k$. Let $m, n \in \mathbb{N}$. Let $1 \leq p < \infty$ and let X and Y be closed subspaces of $L_p(I^m)$ and $L_p(I^n)$, respectively. Define the tensor product $X \otimes Y$ of X and Y by

$$X \otimes Y = [x(s)y(t) : x \in X, y \in Y, s \in I^m, t \in I^n]_{L_p(I^{m+n})}$$

We will denote the element $x(s)y(t)$ by $x \otimes y$.

If $X = Y$, then we write this as a tensor power $X^{\otimes 2}$. In general, the tensor power $\otimes_{i=1}^n X$ will also be denoted as $X^{\otimes n}$.

3.3. Well-Known Complemented Subspaces of $\mathbf{L_p}$. [R].

Definition 3.5. **The Spaces $X_{p,w}$**

Let $w = (w_n)$ be a sequence of positive scalars. Define $X_{p,w}$ to be the space of sequences $x = (x_n)$, of scalars, for which both $\sum |x_n|^p$ and $\sum |w_n x_n|^2$ are finite. For

$x \in X_{p,w}$, define the norm as

$$||(x_n)||_{X_{p,w}} = \max\left\{\left(\sum |x_n|^p\right)^{\frac{1}{p}}, \left(\sum |w_n x_n|^2\right)^{\frac{1}{2}}\right\}$$

Rosenthal proved that the following condition on the sequence (w_n) was critical in determining the isomorphic type of $X_{p,W}$.

$$\text{For each } \epsilon > 0, \sum_{w_n < \epsilon} w_n^{\frac{2p}{p-2}} = \infty. \qquad (*)$$

If $w = (w_n)$ satisfies $(*)$, then $X_{p,W}$ is a sequence space realization of X_p. Let $\{f_n\}$ be a sequence of independent mean zero random variables in L_p, and let $w = (w_n) = (||f_n||_2/||f_n||_p)$ satisfy $(*)$, then it follows from Rosenthal's inequality that the space $[f_n]_{L_p}$ is a function space realization of X_p.

DEFINITION 3.6. **The space B_p**

Let $\{X_{p,v^{(n)}}\}$ be a sequence of Banach spaces where $v^{(n)} = \left(\frac{1}{n}\right)^{\frac{p-2}{2p}}$. Each $X_{p,v^{(n)}}$ is isomorphic to ℓ_2, but $v^{(n)}$ is chosen so that

$\sup_{n\in\mathbb{N}} d(X_{p,v^{(n)}}, \ell_2) = \infty$, where $d(X_{p,v^{(n)}}, \ell_2)$ is the Banach-Mazur distance between X_n and ℓ_2. Define B_p to be ℓ_p sum of $(X_{p,v^{(n)}})$.

3.4. Other Results. In this subsection, we quote some other useful results which will be cited later in the paper. The first lemma is well-known and easy to prove.

LEMMA 3.7. Let $\{X_j\}_{j=1}^n$, $||\cdot||_j$ be Banach spaces, for $j = 1, 2, ..., n$. Let $|||\cdot|||$ be a norm on $\mathbb{R}^n$ for which the standard basis is unconditional. Define space $(\sum_{j=1}^n X_j)$ with norm $|||\cdot|||$ by $||(x_j)_{j=1}^n|| := |||(||x_j||)_{j=1}^n|||$. Then $\left(\sum_{j=1}^n X_j\right)_{|||\cdot|||} \sim \left(\sum_{j=1}^n X_j\right)_{||\cdot||_p}$ for $1 \le p \le \infty$.

PROPOSITION 3.8. (Tong)[**T**] Let the matrix $A = (\alpha_{i,j})$ represent a bounded linear operator T from a Banach space X into a Banach space Y with unconditional bases $\{x_i\}_{i=1}^{\infty}$ and $\{y_j\}_{j=1}^{\infty}$, respectively. Then the diagonal of A also represents a bounded linear operator D from X into Y. If the unconditional constants of $\{x_i\}_{i=1}^{\infty}$ and $\{y_j\}_{j=1}^{\infty}$ are 1, then $\|D\| \leq \|T\|$.

4. Norms Determined by Partitions and Weights

In this section, we examine some examples of complemented subspaces of L_p in order to motivate the idea of a norm given by partitions and weights. Then we develop the formal definition of a norm given by an admissible family of partitions and weights. Finally we give some results about sums of spaces with these norms.

4.1. Examples. In the following we will see that many well known complemented subspaces of L_p have equivalent norms of the form defined as in the Introduction. Here it is sometimes convenient to take partitions and weights defined on sets other than $\mathbb{N}$. For each example we will have a family of partitions (P_k) of $\mathbb{N}^m$ for some m and weights (W_k) for k in some index set K.

EXAMPLE 4.1. Examples with one partition and weight.

$K = \{1\}$.

(1) If $P = \{\{i\} : i \in \mathbb{N}\}$ and $W = (w_n)$ is any sequence of positive numbers, then $X \sim \ell_p$ since

$$\|(x_n)\| = \|(x_n)\|_{P,W} = \left(\sum_{n=1}^{\infty}(|x_n|^2 w_n^2)^{\frac{p}{2}}\right)^{\frac{1}{p}} = \left(\sum_{n=1}^{\infty}|x_n|^p w_n^p\right)^{\frac{1}{p}}$$

(2) If $P = \{\mathbb{N}\}$ and $W = (w_n)$ is any sequence of positive numbers, then $X \sim \ell_2$ since

$$\|(x_n)\| = \left((\sum_{n=1}^{\infty}|x_n|^2 w_n^2)^{\frac{p}{2}}\right)^{\frac{1}{p}} = \left(\sum_{n=1}^{\infty}|x_n|^2 w_n^2\right)^{\frac{1}{2}}$$

(3) If the index set is $\mathbb{N}\times\mathbb{N}$, the partition $P = \{\{n\}\times\mathbb{N}\}$, and $W = (w_{n,m})$, then $X \sim (\sum \ell_2)_{\ell_p}$ since

$$\|(x_n)\| = \left(\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}|x_{n,m}|^2 w_{n,m}^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}}$$

EXAMPLE 4.2. Examples with two partitions and weights.

$K = \{1, 2\}$

(1) If $P_1 = \{\{n\}\}$ with weight $W_1 = (1)$ and $P_2 = \{\mathbb{N}\}$ with weight $W_2 = (w_n)$, then X is the space X_{p,W_2}, defined by Rosenthal, with norm $\|(a_i)\| = \max\left\{(\sum |a_n|^p)^{\frac{1}{p}}, (\sum |w_n a_n|^2)^{\frac{1}{2}}\right\}$. Rosenthal, **[R]**, proved the following:

(a) If $\inf\limits_n w_n > 0$, then $X_{p,W_2} \sim \ell_2$.

(b) If $\sum w_n^{\frac{2p}{p-2}} < \infty$, then $X_{p,W_2} \sim \ell_p$.

(c) If there is some $\epsilon > 0$ for which $\{n : w_n \geq \epsilon\}$ and $\{n : w_n < \epsilon\}$ are both infinite and for which $\sum_{w_n < \epsilon} w_n^{\frac{2p}{p-2}} < \infty$, then $X_{p,W_2} \sim \ell_2 \oplus \ell_p$.

(d) If W_2 satisfies $(*)$, then $X_{p,W_2} \sim X_p$.

(2) If $P_1 = \{\{(i,j)\}\}$ with weight $W_1 = (1)$ and $P_2 = \{\{n\} \times \mathbb{N}\}$ with weight $W_2 = (w_{n,m})$ where $w_{n,m} = \left(\frac{1}{n}\right)$ for all n,m, then $X \sim \left(\sum_n X_{p,(\frac{1}{n})}\right)_p$. $X_n = X_{p,(\frac{1}{n})}$ is isomorphic to ℓ_2 and $\sup_{n\in\mathbb{N}} d(X_n, \ell_2) = \infty$, so $X \sim B_p$, as defined by Rosenthal.

EXAMPLE 4.3. An example with four partitions and weights. $K = \{0, 1, 2, 3\}$. Let i represent the first index and j represent the second index in the set $\mathbb{N} \times \mathbb{N}$. Assume the sequences (w_i) and (w'_j) satisfy $(*)$ condition.

$$\text{Let} \quad P_0 = \mathbb{N} \times \mathbb{N} \quad \text{with weight} \quad W_0 = (w_i w'_j) \tag{4.1}$$

$$P_1 = \{\{n\} \times \mathbb{N}\} \quad \text{with weight} \quad W_1 = (w'_j) \tag{4.2}$$

$$P_2 = \{\mathbb{N} \times \{n\}\} \quad \text{with weight} \quad W_2 = (w_i) \tag{4.3}$$

$$P_3 = \{(i,j)\} \quad \text{with weight} \quad W_3 = (1) \tag{4.4}$$

Then this is Schechtman's example, [S], $X \sim X_p \otimes X_p$, with norm

$$\max\left\{\left(\sum_{i,j}|a_{i,j}|^2 w_i^2 w_j'^2\right)^{\frac{1}{2}}, \left(\sum_i\left(\sum_j |a_{i,j}|^2 w_j'^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}},\right.$$

$$\left.\left(\sum_j\left(\sum_i |a_{i,j}|^2 w_i^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}}, \left(\sum_{i,j}|a_{i,j}|^p\right)^{\frac{1}{p}}\right\} \tag{4.5}$$

$$\approx \left\|\sum_{i,j} a_{i,j}(x_i \otimes y_j)\right\|_{L_p(I\times I)}$$

REMARK 4.4. : Case 3 can be generalized by using the index set $\mathbb{N}^n$. If $|K| = 2^n$ and partition and weights are chosen in a manner similar to the above, then $X \sim X_p^{\otimes n}$.

4.2. General Definition of the Norm.

Let A be any countable index set.

DEFINITION 4.5. Let $P = \{N_i\}$ be a partition of A and a function $W : A \to (0, 1]$ be a sequence of weights. Let $x_j \in \mathbb{R}$ for all $j \in A$. Define

$$\|(x_j)_{j\in A}\|_{P,W} = \left(\sum_i\left(\sum_{j\in N_i} {x_j}^2 w_j^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}}$$

.

Suppose that $(P_k, W_k)_{k\in K}$ is a family of pairs of partitions and functions as above. Define a norm on the real valued function on A by $\|(x_i)\| = \sup_{k\in K}\|(x_i)\|_{(P_k,W_k)}$ and let X be the subspace of elements of finite norm.

REMARK 4.6. Because of the nature of this norm, X will have a natural unconditional basis. Thus this approach is limited to complemented subspaces of L_p with unconditional basis. At this time, no complemented subspace of L_p without unconditional basis is known.

PROPOSITION 4.7. Suppose X is given as in Definition 4.5. Then X is a Banach space.

Proof: Let $\mathcal{P} = \{(Q,W)\}$ be a family of pairs of partitions and weights as in Definition 4.5. Define

$$X_{Q,W} = \left\{ (x_b) : \|(x_b)\| = \left(\sum_{q \in Q} \left(\sum_{b \in q} x_b^2 w_b^2 \right)^{\frac{p}{2}} \right)^{\frac{1}{p}} < \infty \right\}$$

where $W = (w_b)$. It is easy to see that $X_{Q,W} \sim \left(\sum_{q \in Q} \ell_2^{|q|} \right)_{\ell_p}$. Since every $X_{Q,W}$ is Banach, then space $\left(\sum_{(Q,W) \in \mathcal{P}} X_{(Q,W)} \right)_{l_\infty(\mathcal{P})}$ is a Banach space. Notice that X is the diagonal space of the space above, i.e.,

$$X = \{ (x_b(Q,W))_{b \in B, (Q,W) \in \mathcal{P}} : \text{ for each } b \in B,$$
$$x_b(Q,W) = x_b(Q',W'), \forall (Q,W), (Q',W') \in \mathcal{P} \} \quad (4.6)$$

and clearly closed. Hence it is a Banach space. □

PROPOSITION 4.8. Suppose X is a Banach space with norm given by one partition and weight. Then $X \sim \ell_p$, $X \sim \ell_2$ $X \sim \ell_2 \oplus \ell_p$, or $X \sim \left(\sum^{\oplus} \ell_2 \right)_{\ell_p}$.

Notice that these are the spaces given in Example 4.1 and their direct sums. The proof is a routine computation after normalization of the basis.

Since normalization of the basis is an important first step to understanding the spaces, we now introduce admissible families of partitions and weights.

DEFINITION 4.9. The partition of A, $\{\{a\} : a \in A\}$, will be called the discrete partition and be denoted as P_d. The partition of A, $\{A\}$, will be called the indiscrete partition and be denoted as P_i.

DEFINITION 4.10. A family of partitions and weights is called admissible if it contains the discrete partition with the trivial weight $(w(a))_{a\in A} = (1)$ and the indiscrete partition with some weight.

The discrete partition is included to force the natural coordinate basis to be normalized. This requirement is not really a restriction because every normalized unconditional basic sequence in L_p has a lower ℓ_p estimate by 3.1.3. The indiscrete partition gives a candidate for a natural ℓ_2 structure on the space X. Because we are concerned with embedding these spaces into L_p, $p > 2$, there always must be some ℓ_2 structure on the space.

Notice that in the previous examples, Rosenthal's space and Schechtman's space have norms given by admissible families of partitions and weights. Each of the other cases can be equivalently renormed using an admissible family of partitions and weights. Unless otherwise noted we will assume from now on that a Banach space X with norm given by partitions and weights is actually given by an admissible family of partitions and weights.

4.3. Sums of Spaces. In this subsection we are going to show some stability results for sums of spaces when the spaces are equipped with these norms.

Let A be a countable index set and let $(X_a)_{a\in A}$ be a family of Banach spaces of functions defined on sets $(B_a)_{a\in A}$ respectively. That is, for each $a \in A$, X_a has a norm given by a family of partitions of B_a and weights on B_a. Let I_a denote the index set of the corresponding family for X_a. For each $i(a) \in I_a$, let $P^{a,i(a)}$ be a partition of B_a and $W^{a,i(a)}$ be a weight function, i.e., $W^{a,i(a)} : B_a \to (0,1]$. For each $a \in A$ and $i(a) \in I_a$, define the norm on X_a with respect to $P^{a,i(a)}, W^{a,i(a)}$ by

$$\|(x_{a,b})_{b\in B_a}\|_{P^{a,i(a)},W^{a,i(a)}} = \left(\sum_{Q\in P^{a,i(a)}}\left(\sum_{b\in Q}(x_{a,b})^2(w^{a,i(a)}(b))^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}}$$

For each a, there will be one distinguished indiscrete partition and weight. We will denote the index of this partition and weight as ( ). Let $P^{a,()} = \{B_a\}$, and $W^{a,()}$ be the associated weight. For each a, define $\|(x_{a,b})_{b\in B_a}\|_2 = \left(\sum_{b\in B_a}(x_{a,b})^2(w^{a,()}(b))^2\right)^{\frac{1}{2}}$. Suppose that for the index set A, we have an associated function $W : A \to (0,1]$. Let $(\sum_{a\in A} X_a)_{p,2,W}$ be defined on $B = \coprod_{a\in A} B_a$ with the norm

$$\|(x_{a,b})_{b\in B_a}\|_2 = \left(\sum_{b\in B_a}(x_{a,b})^2(w^{a,()}(b))^2\right)^{\frac{1}{2}} \tag{4.7}$$

Let $I = \prod_{a\in A} I_a \cup \{(\ )\}$. Let $(i(a)) \in I$. Then there is a natural partition of B and weight on B given by $P_{(i(a))} = \left\{\{a\} \times P : P \in P^{a,i(a)}, a \in A\right\}$ and $W_{(i(a))} = (w_b^{a,i(a)})_{b\in B_a, a\in A}$. We define as a special case the partition and weight for ( ) as $P_{(\ )} = \{\coprod_{a\in A} B_a\}$ and $W_{(\ )} = (W(a)w^{a,(\ )}(b))_{b\in B_a, a\in A}$.

If we expand the definition of the norm we have

$$
\begin{aligned}
&\|(x_{a,b})_{a\in A,b\in B_a}\|_{p,2,W}\\
&= \max\left\{\left(\sum_{a\in A}\|(x_{a,b})_{b\in B_a}\|_{X_a}^p\right)^{\frac{1}{p}}, \left(\sum_{a\in A}\|(x_{a,b})_{b\in B_a}\|_2^2(W(a))^2\right)^{\frac{1}{2}}\right\}\\
&= \max\left\{\left(\sum_{a\in A}\sup_{i(a)\in I_a}\{\|(x_{a,b})_{b\in B_a}\|_{P^{a,i(a)},W^{a,i(a)}}^p\}\right)^{\frac{1}{p}},\right.\\
&\qquad\left.\left(\sum_{a\in A}W(a)^2\sum_{b\in B_a}(w^{a,()}(b))^2|x_{a,b}|^2\right)^{\frac{1}{2}}\right\}
\end{aligned}
$$

Notice that for each $a \in A$, we take a supremum over I_a, then we take summation of those supremums, and finally we take the maximum of two sums. If we consider the index $(i(a))$ which for each a approximates the supremum, it is one element in I. So instead of taking the maximum over each I_a, we can compute the norm for each index in I, and then take supremum of them only once. Hence the norm becomes

$$\|(x_{a,b})_{a\in A,b\in B_a}\|_{p,2,W} = \sup_{(i(a))\in I}\|(x_{a,b})_{b\in B_a}\|_{P_{(i(a))},W_{(i(a))}}$$

This gives us the following result:

PROPOSITION 4.11. Let $(X_a)_{a\in A}$ be a family of Banach spaces each with norm given by partitions and weights. Then the norm of the space $(\sum_a X_a)_{p,2,W}$ can also be expressed as a norm given by partitions and weights. In other words, the class of spaces with norm given by partitions and weights is stable under $p, 2$ sums.

COROLLARY 4.12. Let $(X_a)_{a\in A}$ be a family of Banach spaces with norm given by an admissible family of partitions and weights. Then the norm of the space $(\sum_a X_a)_{\ell_p}$ can also be expressed with partitions and weights.

Proof: We use the same notation as above. Let ϕ be a one-to-one function from $\mathbb{N}$ onto A. Then ϕ enumerates A, i.e., $A = \{\phi(k)\}_{k=1}^{\infty} = \{a_k\}_{k=1}^{\infty}$. We define $W : A \to (0,1]$ by $W(a) = 2^{-\phi^{-1}(a)\frac{p-2}{2p}}$, i.e., $W(a_k) = 2^{-k\frac{p-2}{2p}}$. Then by Holder's inequality, with exponents $\frac{p}{2}$ and $\frac{p}{p-2}$,

$$\left(\sum_{a_k \in A} \|(x_{a_k,b})_{b\in B_{a_k}}\|_2^2 (W(a_k))^2\right)^{\frac{1}{2}}$$

$$\leq \left(\sum_{k=1}^{\infty} 2^{-k}\right)^{\frac{p-2}{p}} \left(\sum_{a_k \in A} \|(x_{a_k,b})_{b\in B_{a_k}}\|_2^p\right)^{\frac{1}{p}}$$

$$\leq \left(\sum_{a_k \in A} \|(x_{a_k,b})_{b\in B_{a_k}}\|_{X_{a_k}}^p\right)^{\frac{1}{p}}$$

$$= \|(x_{a_k,b})_{b\in B_{a_k}, a_k \in A}\|_{(\sum X_{a_k})_p}$$

By Proposition 4.11, we have

$$\left(\sum_{a \in A} \|(x_{a,b})_{b\in B_a}\|_{X_a}^p\right)^{\frac{1}{p}}$$

$$= \|(x_{a,b})_{a\in A, b\in B_a}\|_{p,2,W}$$

$$= \max_{(i(a))\in I} \|(x_{a,b})_{b\in B_a}\|_{P^{a,i(a)}, W^{a,i(a)}}$$

□

5. Embedding into L_p

In this section, we first show that any sequence space X is isomorphic to a subspace of L_p if its norm is given by finitely many partitions and weights. Then we give the definition of an envelope norm which was suggested to us by Alspach. We prove the existence of the envelope norm generated by a family of partitions and weights. We also give a lower bound on a norm which is necessary for a space to be isomorphic to a subspace of L_p. Finally we show that if a space with norm given by partitions and weights is isomorphic to a subspace of L_p, then its norm is equivalent to the natural envelope norm.

5.1. Finitely Many Partitions and Weights.

THEOREM 5.1. Any sequence space X with norm given by finitely many partitions and weights is isomorphic to a subspace of L_p.

Proof: Let X be the sequence space with partitions and weights $(P_n, W_n)_{n=1}^N$. Let X_n be the space of sequences with norm given by one partition and weight (P_n, W_n), $1 \leq n \leq N$. By Lemma 3.7,

$$\left(\sum_{n=1}^{N} X_n\right)_{\ell_\infty} \sim \left(\sum_{n=1}^{N} X_n\right)_{\ell_p}$$

Take an isometric embedding from X into $\left(\sum_{n=1}^{N} X_n\right)_{\ell_\infty}$ by $x \mapsto (x)_{n=1}^N$. Since X_n, $n = 1, 2, ..., N$ is isomorphic to a complemented subspace of L_p by 4.8, then $\left(\sum_{n=1}^{N} X_n\right)_{\ell_p}$ is isomorphic to a complemented subspace of L_p. Hence X is isomorphic to a subspace of L_p. □

5.2. Envelope Norms.

DEFINITION 5.2. Let $X = \{(a_b)_{b\in B}\}$ be a Banach space defined on a countable set B with norm given by a set of partitions and weights $\mathcal{P} = \{(P^i, W^i) : i \in K\}$. $|||\cdot||| = \sup_{i\in K} \|\cdot\|_{P^i,W^i}$ is an *envelope norm* if and only if for any partition Q of B, and any function $q \to (P^{i(q)}, W^{i(q)}) \in \mathcal{P}$ for all $q \in Q$, the partition and weight (P_0, W_0) belongs to $\mathcal{P}$ where $P_0 = \{K : K = q \cap K_{i(q)} \neq \varnothing$ for some $q \in Q,$ some $K_{i(q)} \in P^{i(q)}\}$ and
$W_0 = (w_b^{i(q)})_{b\in q, q\in Q}$ where $W^{i(q)} = (w_b^{i(q)})_{b\in B}$.
In this case we will say that $\mathcal{P}$ satisfies the *envelope property.*

EXAMPLE 5.3. Let X_p be the Rosenthal's space with norm

$$\|(a_i)\| = \max\left\{\left(\sum |a_n|^p\right)^{\frac{1}{p}}, \left(\sum |w_n a_n|^2\right)^{\frac{1}{2}}\right\}$$

where (w_n) satisfies (*) condition.

Let $P_1 = \{\{n\}\}$ with weight $W_1 = (1)$ and $P_2 = \{\mathbb{N}\}$ with weight $W_2 = (w_n)$. Then $\mathcal{P} = \{(P_1, W_1), (P_2, W_2)\}$ defines the norm on X_p. It is easy to see that $\mathcal{P}$ does not have the envelope property. To get a family of partitions and weights which has the envelope property we need to add all of the possible combinations of the given two. Let $\mathcal{Q}$ be the set of all partitions on $\mathbb{N}$. Let $Q \in \mathcal{Q}$ and $T : Q \to \mathcal{P}$. Define

$$P(Q,T) = \{K : K = \{n\} \text{ if } n \in q \text{ and } T(q) = (P_1, W_1) \text{ for some } q \in Q\}$$
$$\cup \{K : K = q \text{ if } T(q) = (P_2, W_2) \text{ for some } q \in Q\} \text{ and}$$

$$W(Q,T) = (w(n))_{n\in q, q\subset \mathbb{N}} \text{ where } w(n) = 1 \text{ if } n \in q, T(q) = (P_1, W_1) \text{ and}$$
$$w(n) = w_n \text{ if } n \in q,\ T(q) = (P_2, W_2).$$

Then an envelope norm is defined by $\sup_{(P,W)\in\tilde{\mathcal{P}}} \|\cdot\|_{(P,W)}$ where

$\tilde{\mathcal{P}} = \{(P(Q,T), W(Q,T)) : Q \in \mathcal{Q}, T : Q \to \mathcal{P}\}$.

In the next section we will see that this procedure can be generalized.

5.3. Existence and Minimality of the Envelope Norm. In this subsection we show that there is a natural envelope norm associated to each norm given by partitions and weights.

PROPOSITION 5.4. Suppose X is a Banach space defined on a countable set B with norm given by a family $\mathcal{P}$ of partitions and weights. Then there exists a natural family of partitions and weights $\tilde{\mathcal{P}}$, (defined below), such that $|||\cdot||| = \sup_{(P,W)\in\tilde{\mathcal{P}}} \|\cdot\|_{P,W}$ is an envelope norm.

Proof: Let $\mathcal{Q}$ be the set of all partitions on B. Let $\mathcal{P} = \{(P_i, W_i) : i \in K\}$ be the given family of partitions and weights for X. Let $Q \in \mathcal{Q}$. Let T be a map from Q into $\mathcal{P}$ defined by $T(q) = (P^{i(q)}, W^{i(q)})$ for all $q \in Q$. Define $P(Q,T) = \{K : K = q \cap p \neq \varnothing, q \in Q, p \in P^{i(q)}\}$ and $W(Q,T) = (w^{i(q)}(b))_{b\in q, q\in Q}$ where $W^{i(q)} = (w^{i(q)}(b))_{b\in B}$. Let $\tilde{\mathcal{P}} = \{(P(Q,T), W(Q,T)) : Q \in \mathcal{Q},\ T : Q \to \mathcal{P}\}$. Define a norm on X as $|||(x_i)||| = \sup_{(P,W)\in\tilde{\mathcal{P}}} \|(x_i)\|_{P,W}$. We claim that $|||\cdot|||$ is an envelope norm.

Let $\bar{Q}$ be any partition of B. Let S be any map from $\bar{Q}$ into $\tilde{\mathcal{P}}$, i.e., $S(\bar{q}) = (P(Q_{\bar{q}}, T_{\bar{q}}), W(Q_{\bar{q}}, T_{\bar{q}}))$ for all $\bar{q} \in \bar{Q}$. For any $\bar{q} \in \bar{Q}$, let $T_{\bar{q}}(q_0) = (P^{i(\bar{q},q_0)}, W^{i(\bar{q},q_0)})$ for all $q_0 \in Q_{\bar{q}}$ and let $\overline{\overline{Q}} = \{\overline{\overline{q}} \neq \varnothing : \overline{\overline{q}} = q_0 \cap \bar{q}, \bar{q} \in \bar{Q}, q_0 \in Q_{\bar{q}}\}$. Because $\bar{Q}$ and $Q_{\bar{q}}$ are partitions, $\overline{\overline{q}}$ uniquely determines $\bar{q} \in \bar{Q}$ and $q_0 \in Q_{\bar{q}}$ such that $\overline{\overline{q}} = q_0 \cap \bar{q}$. From Definition 5.2 we have $P_0 = \{\bar{K} \neq \varnothing : \bar{K} = \bar{q} \cap \bar{K}_{\bar{q}}, \bar{q} \in \bar{Q}, \bar{K}_{\bar{q}} \in P(Q_{\bar{q}}, T_{\bar{q}})\}$, which is exactly what the definition above gives for the partition of B determined by $\bar{Q}$ and

S, $P(\bar{Q}, S)$. Thus

$$\begin{aligned} P_0 &= \bar{P}(\bar{Q}, S) \\ &= \{\bar{K} : \bar{K} = \bar{q} \cap \bar{K}_{\bar{q}} \neq \emptyset, \bar{q} \in \bar{Q}, \bar{K}_{\bar{q}} \in P(Q_{\bar{q}}, T_{\bar{q}})\} \quad (5.1) \\ &= \{\bar{K} : \bar{K} = \bar{q} \cap (q_0 \cap p) \neq \emptyset, \bar{q} \in \bar{Q}, q_0 \in Q_{\bar{q}}, p \in P^{i(\bar{q}, q_0)}\} \quad (5.2) \\ &= \{\bar{K} : \bar{K} = (\bar{q} \cap q_0) \cap p \neq \emptyset, \bar{q} \cap q_0 \in \overline{\overline{Q}}, p \in P^{i(\bar{q}, q_0)}\} \quad (5.3) \\ &= \{\bar{K} : \bar{K} = \overline{\overline{q}} \cap p \neq \emptyset, \overline{\overline{q}} = \bar{q} \cap q_0 \in \overline{\overline{Q}}, p \in P^{i(\bar{q}, q_0)}\} \quad (5.4) \end{aligned}$$

where (5.2) is given by the definition of $P(Q_{\bar{q}}, T_{\bar{q}})$, (5.3) by the definition of $\overline{\overline{Q}}$, and (5.4) by the uniqueness of $\bar{q}$ and q_0.

Define $\overline{\overline{T}} : \overline{\overline{Q}} \to \mathcal{P}$ by $\overline{\overline{T}}(\overline{\overline{q}}) = (P^{i(\bar{q}, q_0)}, W^{i(\bar{q}, q_0)})$ where $\overline{\overline{q}} = \bar{q} \cap q_0$, $q_0 \in Q_{\bar{q}}$, $\bar{q} \in \bar{Q}$. Then we have shown that $\bar{P}(\bar{Q}, S) = P(\overline{\overline{Q}}, \overline{\overline{T}})$.

Because $T_{\bar{q}}(q) = \left(P^{i(\bar{q}, q)}, W^{i(\bar{q}, q)}\right) = \left(P^{i(\bar{q}, q)}, (w^{i(\bar{q}, q)}(b))_{b \in B}\right)$, then
$S(\bar{q}) = (P(Q_{\bar{q}}, T_{\bar{q}}), W(Q_{\bar{q}}, T_{\bar{q}})) = \left(P(Q_{\bar{q}}, T_{\bar{q}}), (w^{i(\bar{q}, q)}(b))_{b \in q, q \in Q_{\bar{q}}}\right)$.
Suppose $W_0 = (w_b)_{b \in B}$. If $\bar{q} \in \bar{Q}$ and $b \in \bar{q}$, then as in Definition 5.2, $w_b = w^{i(\bar{q}, q_0)}(b)$ where $b \in q_0$ and $q_0 \in Q_{\bar{q}}$. Hence for $b \in \overline{\overline{q}} = \bar{q} \cap q_0$, $w^{i(\bar{q}, q_0)}(b)$ is also the choice specified by $\overline{\overline{T}}(\overline{\overline{q}})$. Hence $W_0 = W(\overline{\overline{Q}}, \overline{\overline{T}})$. So $(P_0, W_0) \in \tilde{\mathcal{P}}$. □

COROLLARY 5.5. If X has a norm defined by a finite number of partitions and weights, then there is an equivalent envelope norm on X.

Proof: Suppose $\mathcal{P} = \{(P^i, W^i) : i = 1, 2, ..., n\}$ is the family of partitions and weights on a set B which defines the norm on X. If Q is any partition of B and T is any map from Q into $\mathcal{P}$, then $\|\cdot\|_{P(Q,T), W(Q,T)} \leq n \max_i \|\cdot\|_{(P^i, W^i)}$, where $P(Q,T)$

and $W(Q,T)$ are defined as in Proposition 5.4. Thus $\|\cdot\|_X \le |||\cdot||| \le n\|\cdot\|_X$, where $|||\cdot|||$ is the envelope norm in Proposition 5.4. □

PROPOSITION 5.6. Suppose $\mathcal{P}$ is a family of partitions and weights on B. Then $\tilde{\mathcal{P}}$ as in Proposition 5.4 is the minimal family of partitions and weights on B containing $\mathcal{P}$ and satisfying the envelope property.

Proof: $\mathcal{P} \subset \tilde{\mathcal{P}}$ is clear. Indeed let $(P,W) \in \mathcal{P}$. Choose $Q = P$ and $T : P \to \mathcal{P}$ by $T(p) = (P,W)$ for all $p \in P$. Then $(P,W) = (P(P,T), W(P,T)) \in \tilde{\mathcal{P}}$. So $\mathcal{P} \subset \tilde{\mathcal{P}}$. $\tilde{\mathcal{P}}$ satisfies the envelope property by Proposition 5.4.

Now we prove that $\tilde{\mathcal{P}}$ is the minimal one. Suppose $\mathcal{R} \supset \mathcal{P}$ is a family of partitions and weights such that $\|\cdot\|_{\mathcal{R}}$ is an envelope norm. Let $(P(Q,T), W(Q,T)) \in \tilde{\mathcal{P}}$ where Q is any partition of B and $T : Q \to \mathcal{P}$. Since $\mathcal{R} \supset \mathcal{P}$, then we can also consider T as a map into $\mathcal{R}$. By the definition of the envelope norm, $(P_0, W_0) \in \mathcal{R}$,

$P_0 = \{K : K = q \cap p, q \in Q, p \in P^{i(q)}\}$ and $W_0 = (w^{i(q)}(b))_{b \in q, q \in Q}$

where $W^{i(q)} = (w^{i(q)}(b))_{b \in B}$, where $T(q) = (P^{i(q)}, W^{i(q)})$ for all $q \in Q$.

But by the definition of $P(Q,T)$ and $W(Q,T)$, $P_0 = P(Q,T)$ and $W_0 = W(Q,T)$. So $\tilde{\mathcal{P}} \subset \mathcal{R}$. Hence $\tilde{\mathcal{P}}$ is minimal. □

COROLLARY 5.7. Let $\mathcal{P}$ be a non empty family of partitions and weights on B. Let $(\mathcal{P}_\lambda)_{\lambda \in \Lambda}$ be any chain of families of partitions and weights on B such that each $\mathcal{P}_\lambda$ contains $\mathcal{P}$ and satisfies envelope property. Then $\cap_{\lambda \in \Lambda} \mathcal{P}_\lambda$ satisfies the envelope property.

Proof: By Proposition 5.6, $\widetilde{\cap_{\lambda\in\Lambda}\mathcal{P}_\lambda}$ satisfies the envelope property and is minimal containing $\cap_{\lambda\in\Lambda}\mathcal{P}_\lambda$. Therefore $\mathcal{P}_{\lambda'} \supset \widetilde{\cap_{\lambda\in\Lambda}\mathcal{P}_\lambda}$ for all $\lambda' \in \Lambda$. Hence $\cap_{\lambda'\in\Lambda}\mathcal{P}_{\lambda'} \supset \widetilde{\cap_{\lambda\in\Lambda}\mathcal{P}_\lambda} \supset \cap_{\lambda\in\Lambda}\mathcal{P}_\lambda$, i.e., $\cap_{\lambda\in\Lambda}\mathcal{P}_\lambda = \widetilde{\cap_{\lambda\in\Lambda}\mathcal{P}_\lambda}$. □

5.4. Envelope Norms and Embeddings into L_p. In this section we show that the envelope norm is related to a property of subspaces of L_p with unconditional basis.

Let X be a Banach space defined on B with a norm given by partitions and weights. Let ϕ be a one-to-one map from $\mathbb{N}$ onto B such that $x_n = e_{\phi(n)}$ where $(e_b)_{b\in B}$ is the natural unit vector basis of X. Then (x_n) is an unconditional basis for X. Hence for any $x \in X$, $x = \sum_{n=1}^{\infty} a_n x_n$ for some (a_n). Let Q be any partition of B. Let $\{F_k\}_{k=1}^{\infty}$ be the corresponding partition of $\mathbb{N}$, i.e., $\phi(F_k) = q$ for some $q \in Q$. Then

$$x = \sum_{k=1}^{\infty} \sum_{n\in F_k} a_n x_n = \sum_{k=1}^{\infty} z_k = \sum_{q\in Q} z'_q$$

where

$$\begin{aligned} z_k &= \sum_{n\in F_k} a_n x_n = \sum_{n\in F_k} a_n e_{\phi(n)} \\ &= \sum_{b\in q=\phi(F_k)} a_{\phi^{-1}(b)} e_b = z'_{\phi(F_k)} = z'_q \end{aligned}$$

Since (x_n) is an unconditional basis and (z_k) (hence (z'_q)) is a block of (x_n), (z_k) is an unconditional basic sequence with unconditional constant 1.

In the lemma below we use the notation introduced in Proposition 5.4.

LEMMA 5.8. Let X be a Banach space defined on B. Let $\mathcal{P} = \{(P^i, W^i) : i \in K\}$ be a family of partitions and weights on B. If X is isomorphic to a subspace of L_p, then there exists a constant C, depending only on the Banach Mazur distance to a subspace of L_p, such that for any partition Q of B and any map $T : Q \to \mathcal{P}$, $\|x\| \geq C\|x\|_{(P(Q,T),W(Q,T))}$ where $T(q) = (P^{i(q)}, W^{i(q)})$.

Proof: Let $\phi : \mathbb{N} \to B$ as above and $T : Q \to \mathcal{P}$ such that $T(q) = (P^{i(q)}, W^{i(q)})$. If X is isomorphic to a subspace of L_p, with isomorphism R, then (Rz_k) (hence $(Rz_q'))$ is block of (Rx_n) which is an unconditional basic sequence in L_p. So

$$\|x\|_X = \|\sum_{k=1}^{\infty} z_k\| \tag{5.5}$$

$$\geq \|R\|^{-1}\|\sum_k Rz_k\|_{L_p} \tag{5.6}$$

$$\geq \|R\|^{-1}\lambda^{-1}\left(\sum_k \|Rz_k\|_p^p\right)^{\frac{1}{p}} \tag{5.7}$$

$$\geq \|R\|^{-1}\lambda^{-1}\left(\sum_k \frac{\|z_k\|_X^p}{\|R^{-1}\|^p}\right)^{\frac{1}{p}} \tag{5.8}$$

$$= \frac{\lambda^{-1}}{\|R\|\|R^{-1}\|}\left(\sum_{q\in Q} \|z_q\|_X^p\right)^{\frac{1}{p}} \tag{5.9}$$

$$\geq C\left(\sum_{q\in Q}\sum_{r\in P^{i(q)}}\left(\sum_{\phi(n)\in r\cap q} |a_n|^2(w_{\phi(n)}^{i(q)})^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}} \tag{5.10}$$

$$= C\left(\sum_{\bar{q}\in P(Q,T)}\left(\sum_{\phi(n)\in\bar{q}} |a_n|^2(w_{\phi(n)}^{i(q)})^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}} \tag{5.11}$$

$$= C\|x\|_{P(Q,T),W(Q,T)} \tag{5.12}$$

where (5.6) and (5.8) hold since R is an isomorphism. (5.7) is given by 3.1.3. (5.10) is true since $z_q = \sum_{n\in F_k, \phi(F_k)=q} a_n x_n$, and $\|z'_q\|_X \geq \|z'_q\|_{P^{i(q)},W^{i(q)}}$. In (5.10), q is the unique element of Q such that $\bar{q} \subset q$. □

THEOREM 5.9. Suppose X has a norm given by a family $\mathcal{P}$ of partitions and weights and X is isomorphic to a subspace of L_p. Then there is an envelope norm $|||\cdot|||$ such that $|||\cdot||| \sim \|\cdot\|_X$

Proof: If we take a supremum over all the choices of Q and T in Lemma 5.8, we have $\|x\|_X \geq C|||x|||$, where $|||\cdot|||$ is the envelope norm defined by $\tilde{\mathcal{P}}$ in Proposition 5.4. On the other hand, since $\mathcal{P} \subset \tilde{\mathcal{P}}$, we get $\|x\|_X \leq |||x|||$. Hence $\|x\|_X \sim |||x|||$ □

REMARK 5.10. : Because the natural basis of a space with norm given by partitions and weights is 1 unconditional, the unconditional constant of the image of any block basis under an isomorphism R is at most $\|R\|\|R^{-1}\|$.

6. Classification

In this section, we assume the family of partitions and weights is admissible. We use the phrase "admissible single partition" to present that the family has only one partition besides the discrete partition and the indiscrete partition. In order to understand the computations, we separate the discussion of two regular partitions and the discrete partition in Subsection 6.2. which will help us to complete the proof of Proposition 6.2.

6.1. Admissible Single Partition. Suppose we have index set $\mathbb{N}$.

DEFINITION 6.1. Let (P_d, W_d) be the discrete partition with constant weight $W_d = (1)$. Let $(P^{()}, W^{()})$ be the indiscrete partition and weight. Let (P, W) be a nontrivial partition and weight with $P = \{N_j\}$ and $W = (w_j)$. We define X to be the sequence space such that

$$X = \{(x_i) : \|(x_i)\| < \infty\}$$

where

$$\|(x_i)\| = \left(\max\left\{\sum_k x_k^p, \sum_i \left(\sum_{j\in N_i} x_j^2 w_j^2\right)^{\frac{p}{2}}, \left(\sum_i {x_i}^2 {w_i^{()}}^2\right)^{\frac{p}{2}}\right\}\right)^{\frac{1}{p}}$$

Assume the notations are the same as in Definition 6.1. Below we will discuss the following cases which depend on the behavior of the weights for the indiscrete partition.

PROPOSITION 6.2.

(1) If $\inf_i w_i^{()} \geq \delta > 0$, then for any nontrivial partition and weight (P, W), $X \sim \ell_2$.

(2) Suppose $\sum_i (w_i^{()})^{\frac{2p}{p-2}} < \infty$. Let $P = \{N_i : i \in \mathbb{N}\}$. Let $|N_i|$ be the cardinality of N_i. Let $I = \{i : |N_i| = \infty\}$. Then

$$X \sim \left(\sum_{i\in I} X_{p,W_i}\right)_{\ell_p} \oplus \left(\sum_{i\notin I} X_{p,W_i}^{|N_i|}\right)_{\ell_p}$$

where $W_i = (w_j)_{j\in N_i}$. Hence X is isomorphic to a complemented subspace of L_p.

(3) If we combine cases (1) and (2), i.e., there is some $\delta > 0$, such that $\{i : w_i \geq \delta\}$ and $\{i : w_i \leq \delta\}$ are infinite and $\sum_{w_i<\delta} w_i^{\frac{2p}{p-2}} < \infty$, we also get spaces which are isomorphic to complemented subspaces of L_p.

Proof of Proposition 6.2:

(1) Since $\inf_i w_i^{()} \geq \delta > 0$, then

$$\delta \left(\sum_i x_i^2\right)^{\frac{1}{2}} \leq \|(x_i)\|_X \leq \left(\sum_i x_i^2\right)^{\frac{1}{2}}$$

since $w_i \leq 1$ for all i and $\|\cdot\|_{\ell_p} \leq \|\cdot\|_{\ell_2}$.

(2) Since $\sum_i (w_i^{()})^{\frac{2p}{p-2}} < \infty$, then we can apply Holder's inequality

$$\left(\sum x_j^2 (w_j^{()})^2\right)^{\frac{1}{2}} \leq \left(\sum x_j^p\right)^{\frac{1}{p}} \left(\sum (w_j^{()})^{\frac{2p}{p-2}}\right)^{\frac{p-2}{p}}$$

Thus

$$\|(x_i)\|_X \sim \left(\max\left\{\sum_k x_k^p, \sum_i \left(\sum_{j\in N_i} {x_j}^2 {w_j}^2\right)^{\frac{p}{2}}\right\}\right)^{\frac{1}{p}}$$

Hence

$$X \sim \left(\sum_{i\in I} X_{p,W_i}\right)_{\ell_p} \oplus \left(\sum_{i\notin I} X_{p,W_i}^{|N_i|}\right)_{\ell_p}$$

LEMMA 6.3. If $\mathbb{N} \setminus I$ is infinite, $\left(\sum_{i\notin I} X_{P,W_i}^{|N_i|}\right)_{\ell_p} \sim \ell_p$

Proof of Lemma 6.3: Since $|N_i| < \infty$ for all $i \notin I$ and $X_{p,W}$ is isomorphic to a complemented subspace of L_p for any sequence W with constants independent of W , then $X_{p,W_i}^{|N_i|}$ is uniformly isomorphic to a complemented subspace of ℓ_p. This implies $\left(\sum_i X_{p,W_i}^{|N_i|}\right)_{\ell_p} \overset{c}{\hookrightarrow} \left(\sum \ell_p\right)_{\ell_p}$. Since $(\sum \ell_p)_{\ell_p} \sim \ell_p$, then $\left(\sum_i X_{p,W_i}^{|N_i|}\right)_{\ell_p} \overset{c}{\hookrightarrow} \ell_p$. Since every infinite dimensional complemented subspace of ℓ_p is isomorphic to ℓ_p, **[P]**, then

$$\left(\sum_i X_{p,W_i}^{|N_i|}\right)_{\ell_p} \sim \ell_p$$

□

The rest of the proof of part (2) is a messy computation based on splitting the argument into several cases of the isomorphic type of the ℓ_p sum of X_{p,W_i} for $i \in I$. The results are as follows:

(a) If $|I| < \infty$, then X is isomorphic to one of ℓ_p, X_p, ℓ_2, or $\ell_2 \oplus \ell_p$.

(b) If $|I| = \infty$, then X is isomorphic to one of ℓ_p, X_p, $\ell_2 \oplus \ell_p$, B_p, $(\sum \ell_2)_{\ell_p}$, $(\sum \ell_2)_{\ell_p} \oplus X_p$, $B_p \oplus X_p$, or $(\sum X_p)_{\ell_p}$.

(3) For the proof of (3), X is a direct sum of one of the spaces from (2) and ℓ_2 from (1). □

6.2. Double Partitions. In this subsection, we deal with two comparable partitions. We will see that various conditions on weights produce various complemented subspaces of L_p. We give a sufficient condition for a space to be complemented in L_p. Consequently we get a partial result for the case omitted fron Proposition 6.2.

First we introduce some terminology.

DEFINITION 6.4. Let $P_1 = \{N_i\}$ and $P_2 = \{K_l\}$ be partitions of the natural numbers $\mathbb{N}$. Let $W = \{w_m\}$ and $W' = \{w'_n\}$ be two sequences of weights. We define X to be a sequence space such that

$$X = \{(a_j) : \|(a_j)\| < \infty\}$$

where

$$\|(a_j)\| = \max\left\{ \left(\sum_i \left(\sum_{j\in N_i} {w_j}^2 {a_j}^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}}, \left(\sum_l \left(\sum_{j\in K_l} {w'_j}^2 {a_j}^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}}, \left(\sum_j a_j^p\right)^{\frac{1}{p}} \right\} \quad (6.1)$$

DEFINITION 6.5. We say that P_1 is a refinement of P_2 if for each i, there is an l, such that $N_i \subset K_l$. We denote this as $P_1 \succcurlyeq P_2$. P_1 and P_2 are said to be comparable if $P_1 \succcurlyeq P_2$ or $P_2 \succcurlyeq P_1$.

We assume that we have two comparable partitions in addition to the discrete partition. Without loss of generality we assume that $P_1 \succcurlyeq P_2$. Notice that the case of single admissible partition is of this type with P_2 being the indiscrete partition. Let $\omega_j = \max\{w_j, w'_j\}$ for all j.

Let $F_i = \{j \in N_i : w_j = \omega_j\}$. Then the norm of (a_i) can be written in the following form:

$$\|(a_j)\|_X = \max\left\{\left(\sum_i \left(\sum_{j\in N_i} {w_j}^2 {a_j}^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}}, \left(\sum_l \left(\sum_{j\in K_l} {w_j'}^2 {a_j}^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}}, \left(\sum_j a_j^p\right)^{\frac{1}{p}}\right\} \tag{6.2}$$

$$= \max\left\{\left(\sum_l \sum_{i:N_i\subset K_l} \left(\sum_{j\in N_i} {w_j}^2 {a_j}^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}}, \left(\sum_l \left(\sum_{j\in K_l} {w_j'}^2 {a_j}^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}}, \left(\sum_j a_j^p\right)^{\frac{1}{p}}\right\} \tag{6.3}$$

$$\approx \left[\sum_l \max\left\{\sum_{i:N_i\subset K_l} \left(\sum_{j\in N_i} {w_j}^2 {a_j}^2\right)^{\frac{p}{2}}, \left(\sum_{j\in K_l} {w_j'}^2 {a_j}^2\right)^{\frac{p}{2}}, \sum_j a_j^p\right\}\right]^{\frac{1}{p}} \tag{6.4}$$

$$= \left[\sum_l \max\left\{S_l^p, {S_l'}^p, {S_l''}^p\right\}\right]^{\frac{1}{p}} = \left[\sum_l \max\left\{S_l, S_l', S_l''\right\}^p\right]^{\frac{1}{p}} \tag{6.5}$$

where for all l

$$S_l = \left(\sum_{i:N_i\subset K_l} \left(\sum_{j\in F_i} a_j^2\omega_j^2 + \sum_{j\in N_i\setminus F_i} w_j^2 a_j^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}}$$

$$S_l' = \left(\sum_{i:N_i\subset K_l} \left(\sum_{j\in F_i} {w_j'}^2 a_j^2 + \sum_{j\in N_i\setminus F_i} \omega_j^2 a_j^2\right)\right)^{\frac{1}{2}}$$

$$S_l'' = \left(\sum_{i:N_i\subset K_l} \sum_{j\in F_i} a_j^p + \sum_{j\in N_i\setminus F_i} a_j^p\right)^{\frac{1}{p}}$$

Let

$$S_{l1}=\left(\sum_{i:N_i\subset K_l}\left(\sum_{j\in F_i}a_j^2\omega_j^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}} \qquad S_{l2}=\left(\sum_{i:N_i\subset K_l}\left(\sum_{j\in N_i\setminus F_i}w_j^2a_j^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}}$$

$$S'_{l1}=\left(\sum_{i:N_i\subset K_l}\sum_{j\in F_i}a_j^2{w'_j}^2\right)^{\frac{1}{2}} \qquad S'_{l2}=\left(\sum_{i:N_i\subset K_l}\sum_{j\in N_i\setminus F_i}\omega_j^2a_j^2\right)^{\frac{1}{2}}$$

$$S''_{l1}=\left(\sum_{i:N_i\subset K_l}\sum_{j\in F_i}a_j^p\right)^{\frac{1}{p}} \qquad S''_{l2}=\left(\sum_{i:N_i\subset K_l}\sum_{j\in N_i\setminus F_i}a_j^p\right)^{\frac{1}{p}}$$

Since the basis is unconditional,

$$\begin{aligned}\|(a_j)\| &\approx \left[\sum_l \max\left\{S_{l1}+S_{l2}, S'_{l1}+S'_{l2}, S''_{l1}+S''_{l2}\right\}^p\right]^{\frac{1}{p}}\\ &\approx \left[\sum_l\left(\max\left\{S_{l1},S'_{l1},S''_{l1}\right\}+\max\left\{S_{l2},S'_{l2},S''_{l2}\right\}\right)^p\right]^{\frac{1}{p}}\\ &\approx \left(\sum_l \max\left\{S_{l1},S'_{l1},S''_{l1}\right\}^p\right)^{\frac{1}{p}}+\left(\sum_l \max\left\{S_{l2},S'_{l2},S''_{l2}\right\}^p\right)^{\frac{1}{p}}\end{aligned}$$

Let X_1 and X_2 refer to the subspace corresponding to the first and second term, respectively, in the above norm expression.

We consider X_2 first.

PROPOSITION 6.6. For any refinement $P_1=\{N_i\}$ of P_2 and any weights, X_2 is isomorphic to a complemented subspace of L_p.

Proof:

$$\left[\sum_l \max\left\{\sum_{i:N_i\subset K_l}\left(\sum_{j\in N_i\setminus F_i} w_j^2 a_j^2\right)^{\frac{p}{2}},\right.\right.$$

$$\left.\left.\left(\sum_{i:N_i\subset K_l}\sum_{j\in N_i\setminus F_i}\omega_j^2 a_j^2\right)^{\frac{p}{2}}, \sum_{i:N_i\subset K_l}\sum_{j\in N_i\setminus F_i} a_j^p\right\}\right]^{\frac{1}{p}}$$

$$= \left[\sum_l\left(\max\left\{\left(\sum_{i:N_i\subset K_l}\left(\sum_{j\in N_i\setminus F_i} w_j^2 a_j^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}},\right.\right.\right.$$

$$\left.\left.\left.\left(\sum_{i:N_i\subset K_l}\sum_{j\in N_i\setminus F_i}\omega_j^2 a_j^2\right)^{\frac{1}{2}}, \left(\sum_{i:N_i\subset K_l}\sum_{j\in N_i\setminus F_i} a_j^p\right)^{\frac{1}{p}}\right\}\right)^p\right]^{\frac{1}{p}}$$

$$= \left\|\left(\left\|(a_j)_{j\in K_l\setminus\cup F_i}\right\|_{X_{p,(\omega_j)_{j\in K_l\setminus\cup F_i}}}\right)_l\right\|_{\ell_p}$$

The last equivalence follows from the fact that the ℓ_2 norm dominates the ℓ_p norm and $w_j \leq \omega_j$. Hence we have

$$X_2 = [e_j : j \in \cup_l K_l \setminus (\cup F_i)] \sim \left(\sum_l X_{p,(\omega_j)_{j\in K_l\setminus\cup F_i}}\right)_{\ell_p}$$

Since for each l, $X_{p,(\omega_j)_{j\in K_l\setminus\cup F_i}}$ is isomorphic to a complemented subspace of L_p with constant independent of l, X_2 is isomorphic to a complemented subspace of L_p.

Depending on the cardinality of $K_l \setminus \cup F_i$ and the weights (ω_j), we get the same list of spaces as in Proposition 6.2 (1)or(2). □

We are unable to resolve completely the situation for X_1. We will prove a sufficient condition for X_1 to be isomorphic to a complemented subspace of L_p. We need some preparation first.

Recall that X_1 is the subspace associated with the norm

$$\left(\sum_l \max\left\{S_{l1}, S'_{l1}, S''_{l1}\right\}^p\right)^{\frac{1}{p}}$$

where

$$S_{l1} = \left(\sum_{i:N_i\subset K_l}\left(\sum_{j\in F_i}\omega_j^2 a_j^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}}$$

$$S'_{l1} = \left(\sum_{i:N_i\subset K_l}\sum_{j\in F_i} {a_j}^2 {w'_j}^2\right)^{\frac{1}{2}}$$

$$S''_{l1} = \left(\sum_{i:N_i\subset K_l}\sum_{j\in F_i} {a_j}^p\right)^{\frac{1}{p}}$$

Let F_i be defined as before. For each i, we split F_i into two parts, subsets G_i and $F_i \setminus G_i$. Then we express norm in the following form:

$$\left[\sum_l \max\left\{\sum_{i:F_i\subset K_l}\left(\sum_{m\in G_i}\omega_m^2 a_m^2 + \sum_{m\in F_i\setminus G_i}\omega_m^2 a_m^2\right)^{\frac{p}{2}},\right.\right.$$

$$\left(\sum_{i:F_i\subset K_l}\left(\sum_{m\in G_i} {w'_m}^2 a_m^2 + \sum_{m\in F_i\setminus G_i} {w'_m}^2 a_m^2\right)\right)^{\frac{p}{2}},$$

$$\left.\left.\sum_{i:F_i\subset K_l}\left(\sum_{m\in G_i} a_m^p + \sum_{m\in F_i\setminus G_i} a_m^p\right)\right\}\right]^{\frac{1}{p}} \tag{6.8}$$

$$\approx \left[\sum_l \max\left\{\sum_{i:F_i\subset K_l}\left(\sum_{m\in G_i}\omega_m^2 a_m^2\right)^{\frac{p}{2}},\right.\right.$$

$$\left.\left.\left(\sum_{i:F_i\subset K_l}\sum_{m\in G_i} {w'_m}^2 a_m^2\right)^{\frac{p}{2}}, \sum_{i:F_i\subset K_l}\sum_{m\in G_i} a_m^p\right\}\right]^{\frac{1}{p}} \tag{6.9}$$

$$+\left[\sum_l \max\left\{\sum_{i:F_i\subset K_l}\left(\sum_{m\in F_i\setminus G_i}\omega_m^2 a_m^2\right)^{\frac{p}{2}},\right.\right.$$

$$\left.\left.\left(\sum_{i:F_i\subset K_l}\sum_{m\in F_i\setminus G_i} {w'_m}^2 {a_m}^2\right)^{\frac{p}{2}}, \sum_{i:F_i\subset K_l}\sum_{m\in F_i\setminus G_i} a_m^p\right\}\right]^{\frac{1}{p}} \qquad (6.10)$$

We call X_1^1 and X_1^2 the subspaces associated to (6.9) and (6.10), respectively. Clearly $X_1 = X_1^1 \oplus X_1^2$.

PROPOSITION 6.7. Suppose there exists a constant $M \geq 1$, such that for all i, $\sum_{m\in G_i} \omega_m^{\frac{2p}{p-2}} \leq M$. Then for any weights X_1^1 is isomorphic to a complemented subspace of L_p.

Proof:

$$\left[\sum_l \max\left\{\sum_{i:G_i\subset K_l}\left(\sum_{m\in G_i}\omega_m^2 a_m^2\right)^{\frac{p}{2}},\right.\right.$$

$$\left.\left.\left(\sum_{i:G_i\subset K_l}\sum_{m\in G_i} {w'_m}^2\, {a_m}^2\right)^{\frac{p}{2}}, \sum_{i:G_i\subset K_l}\sum_{m\in G_i} a_m^p\right\}\right]^{\frac{1}{p}} \qquad (6.11)$$

$$\leq \left[\sum_l \max\left\{\sum_{i:G_i\subset K_l}\left(\left(\sum_{m\in G_i}{\omega_m}^{\frac{2p}{p-2}}\right)^{\frac{p-2}{p}}\left(\sum_{m\in G_i}{a_m}^p\right)^{\frac{2}{p}}\right)^{\frac{p}{2}},\right.\right.$$

$$\left.\left.\left(\sum_{i:G_i\subset K_l}\sum_{m\in G_i} {w'_m}^2 {a_m}^2\right)^{\frac{p}{2}}, \sum_{i:G_i\subset K_l}\sum_{m\in G_i} a_m^p\right\}\right]^{\frac{1}{p}} \qquad (6.12)$$

$$\leq \left[\sum_l \max\left\{\sum_{i:G_i\subset K_l} M^{\frac{p-2}{2}}\sum_{m\in G_i}{a_m}^p,\right.\right.$$

$$\left.\left.\left(\sum_{i:G_i\subset K_l}\sum_{m\in G_i} {w'_m}^2 {a_m}^2\right)^{\frac{p}{2}}, \sum_{i:G_i\subset K_l}\sum_{m\in G_i} a_m^p\right\}\right]^{\frac{1}{p}} \qquad (6.13)$$

$$\overset{M^{\frac{p-2}{2p}}}{\approx} \left(\sum_l \|(a_m)\|^p_{X_{p,({w'_m}^{(l)})}}\right)^{\frac{1}{p}} \qquad (6.14)$$

where $(w_m'^{(l)})$ is the subsequence of (w_m') such that $m \in \cup G_i$, such that $G_i \subset K_l$.

On the other hand,

$$\Bigg[\sum_l \max\Bigg\{\sum_{i:G_i\subset K_l}\Big(\sum_{m\in G_i}\omega_m^2 a_m^2\Big)^{\frac{p}{2}},$$

$$\Big(\sum_{i:G_i\subset K_l}\sum_{m\in G_i} w_m'^{\,2} a_m^2\Big)^{\frac{p}{2}}, \sum_{i:G_i\subset K_l}\sum_{m\in G_i} a_m^p\Bigg\}\Bigg]^{\frac{1}{p}} \tag{6.15}$$

$$\geq \Bigg[\sum_l \max\Bigg\{\sum_{i:G_i\subset K_l}\sum_{m\in G_i}\omega_m^p a_m^p,$$

$$\Big(\sum_{i:G_i\subset K_l}\sum_{m\in G_i} w_m'^{\,2} a_m{}^2\Big)^{\frac{p}{2}}, \sum_{i:G_i\subset K_l}\sum_{m\in G_i} a_m^p\Bigg\}\Bigg]^{\frac{1}{p}} \tag{6.16}$$

$$= \Big(\sum_l \|(a_m)\|^p_{X_{p,(w_m'^{(l)})}}\Big)^{\frac{1}{p}} \tag{6.17}$$

Therefore, we have $\|(a_m)\|_{X_1^1} \overset{M^{\frac{p-2}{2p}}}{\approx} \Big(\sum_l \|(a_m)\|^p_{X_{p,(w_m'^{(l)})}}\Big)^{\frac{1}{p}}$.

In order to understand X_1^1, it is sufficient to understand what the spaces $\Big(\sum_l X_{p,(w_m'^{(l)})}\Big)_{\ell_p}$ are. According to various weights, one obtains various spaces isomorphic to $(\sum \ell_2)_{\ell_p}$, X_p, B_p, ℓ_p or ℓ_p direct sum of combination of them depending on the cardinality of the power set of the partition. It is unnecessary to actually determine the isomorphic type of the space since each $X_{p,(w_m'^{(l)})}$ is isomorphic to a complemented subspaces of L_p with constant which do not depend on $(w_m'^{(l)})$. Hence we have proved that X_1^1 is isomorphic to a complemented subspace of L_p. □

We need one more proposition before we prove a sufficient condition for X_1^2 to be isomorphic to a complemented subspace of L_p.

PROPOSITION 6.8. Let K be an infinite subset of $\mathbb{N}$. Let $\{F_i\}$ be a sequence of disjoint infinite subsets of K. Let $W = (w_i)$ and $\mathcal{W} = (\omega_i)$, with $\omega_i \geq w_i$ for all i, be sequences of weights. Let Y be the space of all sequences (b_i) with finite norm defined by

$$\|(b_i)\|_Y = \max\left\{ \left(\sum_{i:F_i \subset K} \left(\sum_{m\in F_i} \omega_m^2 b_m^2 \right)^{\frac{p}{2}} \right)^{\frac{1}{p}}, \left(\sum_{i:F_i\subset K} \sum_{m\in F_i} w_m^2 b_m^2 \right)^{\frac{1}{2}}, \left(\sum_{i:F_i\subset K} \sum_{m\in F_i} b_m^p \right)^{\frac{1}{p}} \right\} \quad (6.18)$$

Let $\overline{w_i} = \sup_{m\in F_i} \frac{w_m}{\omega_m}$. If there exists a constant C such that

$$\|(b_i)\|_Y \approx \max\left\{ \left(\sum_{i:F_i \subset K} \left(\sum_{m\in F_i} \omega_m^2 b_m^2 \right)^{\frac{p}{2}} \right)^{\frac{1}{p}}, C\left(\sum_{i:F_i\subset K} \overline{w_i}^2 \sum_{m\in F_i} \omega_m^2 b_m^2 \right)^{\frac{1}{2}}, \left(\sum_{i:F_i\subset K} \sum_{m\in F_i} b_m^p \right)^{\frac{1}{p}} \right\} \quad (6.19)$$

then Y is isomorphic to a complemented subspace of L_p.

Proof: Since Y is the span of a subsequence of the basis of a version of $\left(\sum X_{p,(\omega_m)}\right)_{p,2,(\overline{w_i})}$, then it is isomorphic to a complemented subspace of L_p, by **[F]**, **[A-2]**. □

COROLLARY 6.9. Let K, W, $\mathcal{W}$, $\{F_i\}$ and Y be the same as in the Proposition 6.8. Let $\varepsilon > 0$, define $F_i' = \left\{ m \in F_i : \frac{w_m}{\omega_m} \geq \varepsilon \right\}$. Let $\overline{\overline{w_i}} = \sup_{m\in F_i'} \frac{w_m}{\omega_m}$. If there exists a constant C'_ε such that

$$\|(b_i)\|_Y \approx \max\left\{\left(\sum_{i:F_i\subset K}\left(\sum_{m\in F_i}\omega_m^2 b_m^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}},\right.$$

$$\left. C'_\varepsilon\left(\sum_{i:F'_i\subset K}\overline{\overline{w_i}}^{2}\sum_{m\in F'_i}\omega_m^2 b_m^2\right)^{\frac{1}{2}}, \left(\sum_{i:F_i\subset K}\sum_{m\in F_i} b_m^p\right)^{\frac{1}{p}}\right\} \tag{6.20}$$

then Y is isomorphic to a complemented subspace of L_p.

Proof: Observe that the right hand side of (6.20) is equivalent to

$$\max\left\{\left(\sum_{i:F_i\subset K}\left(\sum_{m\in F_i\setminus F'_i}\omega_m^2 b_m^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}}, \sum_{i:F_i\subset K}\sum_{m\in F_i\setminus F'_i} b_m^p\right\}$$

$$+\max\left\{\left(\sum_{i:F_i\subset K}\left(\sum_{m\in F'_i}\omega_m^2 b_m^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}}, C'_\varepsilon\left(\sum_{i:F'_i\subset K}\overline{\overline{w_i}}^{2}\sum_{m\in F'_i}\omega_m^2 b_m^2\right)^{\frac{1}{2}},\right.$$

$$\left.\left(\sum_{i:F_i\subset K}\sum_{m\in F'_i} b_m^p\right)^{\frac{1}{p}}\right\} \tag{6.21}$$

Now apply Proposition 6.6 and Proposition 6.8. □

We have following result for the subspace X_1^2

PROPOSITION 6.10. Suppose there exist an M and $(G_i), G_i \subset F_i$ such that for all i, $\sum_{m\in G_i} \omega_m^{\frac{2p}{p-2}} \leq M$. If for all sequences (m_i) with $m_i \in F_i \setminus G_i$, for all i, $\frac{w_{m_i}}{\omega_{m_i}}$ fails $(*)$, then X_1^2 is isomorphic to $(\sum \ell_2)_{\ell_p} \oplus Y$, where Y is a complemented subspace of L_p. Hence X_1^2 is isomorphic to a complemented subspace of L_p.

Lemma 6.11 will help us to prove the Proposition 6.10.

LEMMA 6.11. Suppose G_i, F_i for all i and M are the same as in Proposition 6.10 . If for all (m_i) with $m_i \in F_i \setminus G_i$, for all i, $\frac{w_{m_i}}{\omega_{m_i}}$ fails $(*)$, then for any $C > 0$, there exist i_0 and $\varepsilon > 0$, such that for all (m_i), $m_i \in F_i \setminus G_i$,

$$\sum_{i>i_0, w_{m_i} < \varepsilon \omega_{m_i}} \left(\frac{w_{m_i}}{\omega_{m_i}}\right)^{\frac{2p}{p-2}} \leq C$$

Proof: Let $\mu_{m_i} = \frac{w_{m_i}}{\omega_{m_i}}$, for all i.

Suppose not, then there exists C_0, such that for any i_0 and $\varepsilon > 0$, there exists (m_i) such that

$$\sum_{i \geq i_0, \mu_{m_i} < \varepsilon} \mu_{m_i}{}^{\frac{2p}{p-2}} > C_0$$

Take $i_0 = 1$ and $\varepsilon_0 = 1$. Find a subsequence (m_i^1) such that

$$\sum_{i \geq 1, \mu_{m_i^1} < \varepsilon_0} \mu_{m_i^1}{}^{\frac{2p}{p-2}} > C_0$$

Hence there exists j_1, such that

$$\sum_{1 \leq i \leq j_1, \mu_{m_i^1} < \varepsilon_0} \mu_{m_i^1}{}^{\frac{2p}{p-2}} > C_0$$

Let $i_1 = j_1 + 1$ and $\varepsilon_1 = \frac{1}{2}$. Find subsequence (m_i^2) such that

$$\sum_{i \geq i_1, \mu_{m_i^2} < \varepsilon_1} \mu_{m_i^2}{}^{\frac{2p}{p-2}} > C_0$$

Hence there exists j_2, such that

$$\sum_{j_1 < i \leq j_2, \mu_{m_i^1} < \varepsilon_1} \mu_{m_i^2}{}^{\frac{2p}{p-2}} > C_0$$

Inductively we get a sequence of subsequences of $(\mu_{m_i^k})$ and (j_k) such that

$$\sum_{j_{k-1} < i \leq j_k, \mu_{m_i^k} < \varepsilon_k} \mu_{m_i^k}{}^{\frac{2p}{p-2}} > C_0$$

Now we construct a sequence by using blocks. Let $\mu_{m_i} = \mu_{m_i^k}$ if $i_{k-1} \leq i \leq i_k$, then (μ_{m_i}) satisfies $(*)$. Contradicting our assumption. $\square$

Proof of Proposition 6.10: By Lemma 6.11 with $C = 1$, we find a uniform ε and i_0 so that $\sum_{\mu_{m_i} < \varepsilon \omega_{m_i}} \left(\frac{w_{m_i}}{\omega_{m_i}} \right)^{\frac{2p}{p-2}} < 1$ for all sequences $(m_i)_{i=i_0}^{\infty}$.

Let

$$E_i = \{m : w_m < \varepsilon \omega_m, m \in F_i \setminus G_i\}$$

$$E_i^c = \{m : w_m \geq \varepsilon \omega_m, m \in F_i \setminus G_i\}$$

Let

$$\overline{w_i} = \sup_{m \in E_i} \frac{w_m}{\omega_m}$$

$$\overline{\overline{w_i}} = \sup_{m \in E_i^c} \frac{w_m}{\omega_m}$$

Then

$$\left(\sum_{i: F_i \subset K_l} \sum_{m \in F_i \setminus G_i} {w_m}^2 {a_m}^2 \right)^{\frac{1}{2}}$$

$$= \left(\sum_{i: F_i \subset K_l} \sum_{m \in E_i} {w_m}^2 {a_m}^2 + \sum_{i: F_i \subset K_l} \sum_{m \in E_i^c} {w_m}^2 {a_m}^2 \right)^{\frac{1}{2}}$$

$$\approx \left(\sum_{i: F_i \subset K_l} \sum_{m \in E_i} {w_m}^2 {a_m}^2 \right)^{\frac{1}{2}} + \left(\sum_{i: F_i \subset K_l} \sum_{m \in E_i^c} {w_m}^2 {a_m}^2 \right)^{\frac{1}{2}}$$

Let $\delta < 1$ be a given number. By the definition of $\overline{w_i}$, we can choose a sequence (w_{m_i}) for $m_i \in E_i$ such that

$$\overline{w_i}^{\frac{2p}{p-2}} \leq \left(\frac{w_{m_i}}{\omega_{m_i}} \right)^{\frac{2p}{p-2}} + \frac{\delta}{2^{i+1}}$$

Then by the choice of ϵ and i_0, $\sum_i \overline{w_i}^{\frac{2p}{p-2}} < \sum_i \left(\frac{w_{m_i}}{\omega_{m_i}}\right)^{\frac{2p}{p-2}} + \delta \leq 2 + i_0 < \infty$.

Hence

$$\left(\sum_{i:F_i\subset K_l}\sum_{m\in E_i} {w_{i,m}}^2 {a_{i,m}}^2\right)^{\frac{1}{2}} \leq \left(\sum_{i:F_i\subset K_l} \overline{w_i}^2 \sum_{m\in E_i} \omega_m^2 a_m^2\right)^{\frac{1}{2}}$$

$$\leq \left(\sum_i \overline{w_i}^{\frac{2p}{p-2}}\right)^{\frac{p-2}{p}} \left(\sum_{i:F_i\subset K_l}\left(\sum_{m\in E_i}\omega_m^2 a_m^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}}$$

$$\leq (2+i_0)^{\frac{p-2}{p}} \left(\sum_{i:F_i\subset K_l}\left(\sum_{m\in E_i}\omega_m^2 a_m^2\right)^{\frac{p}{2}}\right)^{\frac{1}{p}}$$

Hence we see immediately that the subspace with basis $\{e_{i,m} : i : F_i \subset K_i, m \in E_i\}$ is isomorphic to $(\sum X_{p,(\omega_m)_{m\in E_i}})_{\ell_p}$.

For the second summand, we notice that $\overline{\overline{w_i}} \leq 1$, we have

$$\varepsilon \left(\sum_{i:F_i\subset K_l} \overline{\overline{w_i}}^2 \sum_{m\in E_i^c} \omega_m^2 a_m^2\right)^{\frac{1}{2}} \leq \left(\sum_{i:F_i\subset K_l}\sum_{m\in E_i^c} w_m^2 a_m^2\right)^{\frac{1}{2}} \tag{6.22}$$

$$\leq \left(\sum_{i:F_i\subset K_l} \overline{\overline{w_i}}^2 \sum_{m\in E_i^c} \omega_m^2 a_m^2\right)^{\frac{1}{2}} \tag{6.23}$$

By Proposition 6.8, we know that this second subspace is isomorphic to a complemented subspace of L_p. Hence we have proved X_1^2 is isomorphic to a complemented subspace of L_p which is Proposition 6.10.

From Propositions 6.7 and 6.10 , we obtain a sufficient condition for X_1 to be isomorphic to a complemented subspace of L_p.

THEOREM 6.12. If there exist an M and (G_i),$G_i \subset F_i$, $\sum_{m \in G_i} \omega_m^{\frac{2p}{p-2}} \leq M$ such that for all sequences (m_i), $m_i \in F_i \setminus G_i$, (ω_{m_i}) fails $(*)$, then X_1 is isomorphic to a complemented subspace of L_p.

Since $X \sim X_1 \oplus X_2$, we obtain following theorem immediately.

THEOREM 6.13. If there exist an M and (G_i),$G_i \subset F_i$, $\sum_{m \in G_i} \omega_m^{\frac{2p}{p-2}} \leq M$ such that for all sequences (m_i), $m_i \in F_i \setminus G_i$, (ω_{m_i}) fails $(*)$, then X is isomorphic to a complemented subspace of L_p.

In order to understand the basic principle of classification of complemented subspaces of L_p, Dale Alspach gave following conjecture: Let X be a space of sequences (a_i). Let K, W and $\{F_i\}$ be the same as in Corollary 6.9. We define a norm on X by

$$\|(a_i)\|_X = \max\left\{ \left(\sum_{i:F_i \subset K} \left(\sum_{m \in F_i} \omega_m^2 a_m^2 \right)^{\frac{p}{2}} \right)^{\frac{1}{p}}, \right.$$

$$\left. \left(\sum_{i:F_i \subset K} \sum_{m \in F_i} w_m^2 a_m^2 \right)^{\frac{1}{2}}, \left(\sum_m a_m^p \right)^{\frac{1}{p}} \right\} \tag{6.24}$$

Then X is isomorphic to a complemented subspace of L_p if and only if there exist $\overline{w_i}$'s and a constant C such that

$$\|(a_i)\|_X \approx \max\left\{ \left(\sum_{i:F_i \subset K} \left(\sum_{m \in F_i} a_m^2 \right)^{\frac{p}{2}} \right)^{\frac{1}{p}}, \right.$$

$$\left. C \left(\sum_{i:F_i' \subset K} \overline{w_i}^2 \sum_{m \in F_i'} w_m^2 a_m^2 \right)^{\frac{1}{2}}, \left(\sum_m a_m^p \right)^{\frac{1}{p}} \right\} \tag{6.25}$$

The sufficient part can be proved by applying the argument above.

The neccessary part is still an open question. See further discussion in Section 8.

6.3. Application to Admissible Single Partitions. In this subsection, we discuss the case which is not included in Proposition 6.2, i.e., the case that $(w_j^{()})$ satisfies $(*)$.

PROPOSITION 6.14. Suppose we have all assumptions as in Proposition 6.2. Suppose $W^{()}$ satisfies $(*)$. Then

(1) If $\inf_j w_j \geq \alpha > 0$, then $X \sim X_1$ as in Subsection 6.2.
(2) If $\sum_j w_j^{\frac{2p}{p-2}} < \infty$, then $X \sim X_p$.
(3) If there is some $\delta > 0$, such that $\{i : w_i \geq \delta\}$ and $\{i : w_i \leq \delta\}$ are infinite and $\sum_{w_i<\delta} w_i^{\frac{2p}{p-2}} < \infty$, then $X \sim X_1 \oplus X_p$.

Proof:

(1) If $\inf_j w_j \geq \alpha > 0$, then

$$\begin{aligned} \|(x_i)\|^p &\sim \max\left\{\sum_i x_i^p, \sum_i \left(\sum_{j\in N_i} x_j^2\right)^{\frac{p}{2}}, \left(\sum_i x_i^2 (w_i^{()})^2\right)^{\frac{p}{2}}\right\} \\ &\sim \max\left\{\sum_i \left(\sum_{j\in N_i} x_j^2\right)^{\frac{p}{2}}, \left(\sum_i x_i^2 (w_i^{()})^2\right)^{\frac{p}{2}}\right\} \\ &\sim X_1 \end{aligned}$$

(2) If $\sum_j w_j^{\frac{2p}{p-2}} = M < \infty$, then by Holder's inequality,

$$\sum_{j\in N_i} x_j^2 w_j^2 \leq M^{\frac{p-2}{p}} \sum_{j\in N_i} x_j^p$$

So

$$
\begin{aligned}
\|(x_i)\|^p &\sim \max\left\{\sum_i x_i^p, \sum_i \left(\sum_{j\in N_i} x_j^2\right)^{\frac{p}{2}}, \left(\sum_i x_i^2 (w_i^{()})^2\right)^{\frac{p}{2}}\right\} \\
&\sim \max\left\{\sum_i x_i^p, \left(\sum_i x_i^2 (w_i^{()})^2\right)^{\frac{p}{2}}\right\} \\
&\sim X_p
\end{aligned}
$$

(3) By combining (1) and (2), $X \sim X_1 \oplus X_p$.

□

7. Distance Between $\mathbf{Y_n}$ and $\mathbf{X_p^{\otimes n}}$ Spaces

In this section, we construct an example which demonstrates the difference between a norm given by partitions and weights and the corresponding envelope norm. We also obtain an estimate of the distance between a certain Banach space Y_n, isomorphic to $X_p^{\otimes n}$, with norm given by partitions and weights, and any subspace of L_p. Finally we give an example of a Banach space which is not isomorphic to a subspace of L_p by applying Theorem 5.9.

7.1. Construction of $\mathbf{Y_n}$. We will let Y_n be a Banach space with norm given by partitions and weights which has essentially the same form as the norm on the sequence space realization of $(X_p)^{\otimes n}$ introduced by Schechtman in 1975 [**S**]. First we will estimate the distance between Y_n and Y_n with the associated envelope norm for the case $n = 3$. Then for any $n \in \mathbb{N}$ we can easily extend the argument to Y_n with the original norm given by partitions and weights and Y_n with the corresponding envelope norm. Consequently we prove that not every sequence space with norm given by partitions and weights is isomorphic to a subspace of L_p and the envelope norm on the sequence space realization of $(X_p)^{\otimes n}$ may be a better choice for some purposes.

EXAMPLE 7.1. We will define Y_3 on $\mathbb{N}^2 \times \mathbb{N}^2 \times \mathbb{N}^2$. Let $(w_i)_{i=1}^{\infty}$ be a sequence of weights such that $w_i \to 0$ as $i \to \infty$. Let $\mathbf{i_1}, \mathbf{i_2}$,and $\mathbf{i_3}$ represent indexes for the first, second and third pair of coordinates, respectively. Define weights on $\mathbb{N}^2$ by $w_{\mathbf{i}} = w_{(m,n)} = w_m$ where $\mathbf{i} = (m, n)$ for all $m, n \in \mathbb{N}$. Let $(e_{\mathbf{i_1},\mathbf{i_2},\mathbf{i_3}})_{\mathbf{i_1},\mathbf{i_2},\mathbf{i_3} \in \mathbb{N}^2}$ be the natural unit vector basis of Y_3. The partitions of $\mathbb{N}^2 \times \mathbb{N}^2 \times \mathbb{N}^2$ and corresponding weights are given as follows,

$$P_0 = \{\mathbb{N}^2 \times \mathbb{N}^2 \times \mathbb{N}^2\} \qquad W_0 = (w_{\mathbf{i}_1} w_{\mathbf{i}_2} w_{\mathbf{i}_3})$$

$$P_1 = \{\{(m,n)\} \times \mathbb{N}^2 \times \mathbb{N}^2 : m,n \in \mathbb{N}\} \qquad W_1 = (w_{\mathbf{i}_2} w_{\mathbf{i}_3})$$

$$P_2 = \{\mathbb{N}^2 \times \{(n,m)\} \times \mathbb{N}^2 : m,n \in \mathbb{N}\} \qquad W_2 = (w_{\mathbf{i}_1} w_{\mathbf{i}_3})$$

$$P_3 = \{\mathbb{N}^2 \times \mathbb{N}^2 \times \{(m,n)\} : m,n \in \mathbb{N}\} \qquad W_3 = (w_{\mathbf{i}_1} w_{\mathbf{i}_2})$$

$$P_4 = \{\{(m,n)\} \times \{(s,t)\} \times \mathbb{N}^2 : m,n,s,t \in \mathbb{N}\} \qquad W_4 = (w_{\mathbf{i}_3})$$

$$P_5 = \{\mathbb{N}^2 \times \{(m,n)\} \times \{(s,t)\} : m,n,s,t \in \mathbb{N}\} \qquad W_5 = (w_{\mathbf{i}_1})$$

$$P_6 = \{\{(m,n)\} \times \mathbb{N}^2 \times \{(s,t)\} : m,n,s,t \in \mathbb{N}\} \qquad W_6 = (w_{\mathbf{i}_2})$$

$$P_7 = \{(l,m,n,s,t,u)\} \text{ for } l,m,n,s,t,u \in \mathbb{N} \qquad W_7 = (1)$$

Then the norm on Y_3 can be calculated by

$$\left\| \sum_{\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3} a_{\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3} e_{\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3} \right\|_{Y_3} = \max_{I \subset \{1,2,3\}} \left\{ \left(\sum_{\mathbf{i}_k : k \in I} \left(\sum_{\mathbf{i}_l : l \in I^c} |a_{\mathbf{i}_1,\mathbf{i}_2,\mathbf{i}_3}|^2 \prod_{l \in I^c} (w_{\mathbf{i}_l})^2 \right)^{\frac{p}{2}} \right)^{\frac{1}{p}} \right\}$$

$$
\begin{aligned}
&= \max\Bigg\{ \left(\sum_{i_1,i_2,i_3} |a_{i_1,i_2,i_3}|^2 (w_{i_1})^2 (w_{i_2})^2 (w_{i_3})^2 \right)^{\frac{1}{2}}, \\
&\quad \left(\sum_{i_1} \left(\sum_{i_2,i_3} |a_{i_1,i_2,i_3}|^2 (w_{i_2})^2 (w_{i_3})^2 \right)^{\frac{p}{2}} \right)^{\frac{1}{p}}, \\
&\quad \left(\sum_{i_2} \left(\sum_{i_1,i_3} |a_{i_1,i_2,i_3}|^2 (w_{i_1})^2 (w_{i_3})^2 \right)^{\frac{p}{2}} \right)^{\frac{1}{p}}, \\
&\quad \left(\sum_{i_3} \left(\sum_{i_1,i_2} |a_{i_1,i_2,i_3}|^2 (w_{i_1})^2 (w_{i_2})^2 \right)^{\frac{p}{2}} \right)^{\frac{1}{p}}, \\
&\quad \left(\sum_{i_1,i_2} \left(\sum_{i_3} |a_{i_1,i_2,i_3}|^2 (w_{i_3})^2 \right)^{\frac{p}{2}} \right)^{\frac{1}{p}}, \\
&\quad \left(\sum_{i_2,i_3} \left(\sum_{i_1} |a_{i_1,i_2,i_3}|^2 (w_{i_1})^2 \right)^{\frac{p}{2}} \right)^{\frac{1}{p}}, \\
&\quad \left(\sum_{i_1,i_3} \left(\sum_{i_2} |a_{i_1,i_2,i_3}|^2 (w_{i_2})^2 \right)^{\frac{p}{2}} \right)^{\frac{1}{p}}, \\
&\quad \left(\sum_{i_1,i_2,i_3} |a_{i_1,i_2,i_3}|^p \right)^{\frac{1}{p}} \Bigg\} \\
&= \max\{S_i\}_{i=0}^{7}
\end{aligned}
$$

where S_i for $i = 0, 1, \ldots, 7$, are the sums in the previous expression in the same order.

Since $w_i \to 0$ as $i \to \infty$, then for any given ϵ, $0 < \epsilon \leq 3$,

there exists an N, such that if $n > N$,

$$w_n < \left(\frac{\epsilon}{3}\right)^{\frac{1}{2}} \leq \left(\frac{\epsilon}{3}\right)^{\frac{1}{p}}$$

Let $n_1, n_2, n_3 > N$. Choose integers K_1, K_2, K_3, such that

$$w_{n_1} K_1^{\frac{1}{2}-\frac{1}{p}} > \left(\frac{3}{\epsilon}\right)^{\frac{1}{p}} \geq 1$$

$$w_{n_2} K_2^{\frac{1}{2}-\frac{1}{p}} > \left(\frac{3}{\epsilon}\right)^{\frac{1}{p}} \geq 1$$

$$w_{n_3} K_3^{\frac{1}{2}-\frac{1}{p}} > \left(\frac{3}{\epsilon}\right)^{\frac{1}{p}} \geq 1$$

Now take three blocks with constant coefficients as follows

K_1 – block with coefficient $(w_{n_1})^{-1} K_1^{-\frac{1}{2}}$ and support

$(n_1, 1, n_2, 1, n_3, 1)$

$(n_1, 2, n_2, 1, n_3, 1)$

$\vdots$

$(n_1, K_1, n_2, 1, n_3, 1)$

K_2 – block with coefficient $(w_{n_2})^{-1} K_2^{-\frac{1}{2}}$ and support

$(n_1, K_1 + 1, n_2, 2, n_3, 2)$

$(n_1, K_1 + 1, n_2, 3, n_3, 2)$

$\vdots$

$(n_1, K_1 + 1, n_2, K_2 + 1, n_3, 2)$

K_3 – block with coefficient $(w_{n_3})^{-1} K_3^{-\frac{1}{2}}$ and support

$(n_1, K_1 + 2, n_2, K_2 + 2, n_3, 3)$

$(n_1, K_1 + 2, n_2, K_2 + 2, n_3, 4)$

$\vdots$

$(n_1, K_1 + 2, n_2, K_2 + 2, n_3, K_3 + 2)$

Now we estimate the eight sums to get an estimate of the norm of the element

$$\sum_{\mathbf{i_1}=(n_1,1)}^{(n_1,K_1)} w_{n_1}^{-1} K_1^{-\frac{1}{2}} e_{\mathbf{i_1},n_2,1,n_3,1} + \sum_{\mathbf{i_2}=(n_2,2)}^{(n_2,K_2+1)} w_{n_2}^{-1} K_2^{-\frac{1}{2}} e_{n_1,K_1+1,\mathbf{i_2},n_3,2}$$
$$+ \sum_{\mathbf{i_3}=(n_3,3)}^{(n_3,K_3+2)} w_{n_3}^{-1} K_3^{-\frac{1}{2}} e_{n_1,K_1+2,n_2,K_2+2,\mathbf{i_3}}. \tag{7.1}$$

$$S_0 = \big[(w_{n_1})^{-2} K_1^{-1} (w_{n_1})^2 (w_{n_2})^2 (w_{n_3})^2 K_1$$
$$+ (w_{n_2})^{-2} K_2^{-1} (w_{n_1})^2 (w_{n_2})^2 (w_{n_3})^2 K_2$$
$$+ (w_{n_3})^{-2} K_3^{-1} (w_{n_1})^2 (w_{n_2})^2 (w_{n_3})^2 K_3\big]^{\frac{1}{2}}$$
$$= \big[(w_{n_2})^2 (w_{n_3})^2 + (w_{n_1})^2 (w_{n_3})^2 + (w_{n_1})^2 (w_{n_2})^2\big]^{\frac{1}{2}} < \epsilon^{\frac{1}{2}}.$$

$$S_1 = [((w_{n_1})^{-2} K_1^{-1} (w_{n_2})^2 (w_{n_3})^2)^{\frac{p}{2}} K_1 + ((w_{n_2})^{-2} K_2^{-1} (w_{n_2})^2 (w_{n_3})^2 K_2)^{\frac{p}{2}}$$
$$+ ((w_{n_3})^{-2} K_3^{-1} (w_{n_2})^2 (w_{n_3})^2 K_3)^{\frac{p}{2}}]^{\frac{1}{p}}$$
$$= [(w_{n_1} K_1^{\frac{1}{2}-\frac{1}{p}})^{-p} (w_{n_2})^p (w_{n_3})^p + (w_{n_3})^p + (w_{n_2})^p]^{\frac{1}{p}} < \epsilon^{\frac{1}{p}}.$$

Similarly we have $S_2 < \epsilon^{\frac{1}{p}}$ and $S_3 < \epsilon^{\frac{1}{p}}$.

$$S_4 = \big(K_1((w_{n_1})^{-2} K_1^{-1} (w_{n_3})^2)^{\frac{p}{2}} + K_2((w_{n_2})^{-2} K_2^{-1} (w_{n_3})^2)^{\frac{p}{2}}$$
$$+ ((w_{n_3})^{-2} K_3^{-1} (w_{n_3})^2 K_3)^{\frac{p}{2}}\big)^{\frac{1}{p}}$$
$$= \big((w_{n_1} K_1^{\frac{1}{2}-\frac{1}{p}})^{-p} (w_{n_3})^p + (w_{n_2} K_2^{\frac{1}{2}-\frac{1}{p}})^{-p} (w_{n_3})^p + 1\big)^{\frac{1}{p}} < (\epsilon + 1)^{\frac{1}{p}}.$$

Similarly we have $S_5 < (\epsilon+1)^{\frac{1}{p}}$ and $S_6 < (\epsilon+1)^{\frac{1}{p}}$.

$$S_7 = ((w_{n_1}K_1^{\frac{1}{2}-\frac{1}{p}})^{-p} + (w_{n_2}K_2^{\frac{1}{2}-\frac{1}{p}})^{-p} + (w_{n_3}K_3^{\frac{1}{2}-\frac{1}{p}})^{-p})^{\frac{1}{p}} < \epsilon^{\frac{1}{p}}.$$

Since ϵ can be arbitrary small, if we take maximum of these eight sums, the norm will be as close to 1 as we want.

Now let us look at the envelope norm of this element (7.1).

Let Q be a partition of $\mathbb{N}^2 \times \mathbb{N}^2 \times \mathbb{N}^2$ such that the support of each of the above three blocks is an element of Q. (The other sets in the partition do not matter.) Let $\mathcal{P}$ be the given family of weights and partitions, i.e., $\mathcal{P} = \{(P_i, W_i) : i = 0, 1, ..., 7\}$. Let $T : Q \to \mathcal{P}$ be a map such that $T(\text{supp}K_1 - \text{block}) = (P_6, W_6)$, $T(\text{supp}K_2 - \text{block}) = (P_5, W_5)$, and $T(\text{supp}K_3 - \text{block}) = (P_4, W_4)$. Then the envelope norm of (7.1) can be estimated from below using $P(Q,T)$

$$\begin{aligned} &\left|\left|\left| \sum_{\mathbf{i}_1=(n_1,1)}^{(n_1,K_1)} w_{n_1}^{-1} K_1^{-\frac{1}{2}} e_{\mathbf{i}_1,n_2,1,n_3,1} + \sum_{\mathbf{i}_2=(n_2,2)}^{(n_2,K_2+1)} w_{n_2}^{-1} K_2^{-\frac{1}{2}} e_{n_1,K_1+1,\mathbf{i}_2,n_3,2} \right.\right.\right. \\ &\left.\left.\left. + \sum_{\mathbf{i}_3=(n_3,3)}^{(n_3,K_3+2)} w_{n_3}^{-1} K_3^{-\frac{1}{2}} e_{n_1,K_1+2,n_2,K_2+2,\mathbf{i}_3} \right|\right|\right| \geq \left(\left(\sum_{i_1=(n_1,1)}^{(n_1,K_1)} (w_{n_1}^{-1} K_1^{-\frac{1}{2}})^2 w_{i_1}^2 \right)^{\frac{p}{2}} \right. \\ &\left. + \left(\sum_{i_2=(n_2,2)}^{(n_2,K_2+1)} (w_{n_2}^{-1} K_2^{-\frac{1}{2}})^2 w_{i_2}^2 \right)^{\frac{p}{2}} + \left(\sum_{i_3=(n_3,3)}^{(n_3,K_3+2)} (w_{n_3}^{-1} K_3^{-\frac{1}{2}})^2 w_{i_3}^2 \right)^{\frac{p}{2}} \right)^{\frac{1}{p}} \\ &\geq \Big(\big((w_{n_1})^{-2} K_1^{-1} (w_{n_1})^2 K_1\big)^{\frac{p}{2}} + \big((w_{n_2})^{-2} K_2^{-1} (w_{n_2})^2 K_2\big)^{\frac{p}{2}} \\ &+ \big((w_{n_3})^{-2} K_3^{-1} (w_{n_3})^2 K_3\big)^{\frac{p}{2}} \Big)^{\frac{1}{p}} = 3^{\frac{1}{p}} \end{aligned}$$

Hence the envelope norm on Y_3 is at best $3^{\frac{1}{p}}$ equivalent to the given norm.

REMARK 7.2. This computation can be generalized. We define Y_n for any $n \in \mathbb{N}$ be a Banach space on $\mathbb{N}^2 \times \mathbb{N}^2 \times \ldots \times \mathbb{N}^2$. Let $w_{\mathbf{i}} = w_{s,t} = w_s$ for $t \in \mathbb{N}$ as above. Let $I \subset \{1, 2, \ldots, n\}$. Define
$P_I = \left\{ \prod_{k=1}^{n} A_k : \text{ where } A_k = \mathbb{N}^2, k \notin I;\ A_k = \{(m_k, l_k)\}, k \in I,\ m_k, l_k \in \mathbb{N} \right\}$ and $W_I = \left(\prod_{k \notin I} w_{\mathbf{i}_k} \right)$. For a given sequence (w_i) such that $w_i \to 0$ as $i \to \infty$ and any $0 < \epsilon \leq 1$, there exists N, such that if $m > N$, then

$$w_m < \left(\frac{\epsilon}{n}\right)^{\frac{1}{2}} \leq \left(\frac{\epsilon}{n}\right)^{\frac{1}{p}}$$

Let $m_1, \ldots, m_n > N$. We choose n blocks with size K_l for $l = 1, \ldots, n$ in $(\mathbb{N}^2)^n$ so that

$$w_{m_l} K_l^{\frac{1}{2} - \frac{1}{p}} > \left(\frac{n}{\epsilon}\right)^{\frac{1}{p}}$$

The K_{l+1}-block would be

$$(m_1, K_1 + l, m_2, K_2 + l, \ldots, m_{l+1}, l + 1, \ldots, m_n, l + 1)$$

$$(m_1, K_1 + l, m_2, K_2 + l, \ldots, m_{l+1}, l + 2, \ldots, m_n, l + 1)$$

$$\vdots$$

$$(m_1, K_1 + l, m_2, K_2 + l, \ldots, m_{l+1}, l + K_{l+1}, \ldots, m_n, l + 1)$$

where $0 \leq l \leq n - 1$.

By applying similar arguments to that of Example 7.1 we have that the value of the envelope norm of the sum of these blocks is at least $n^{\frac{1}{p}}$ while the value of the norm given by partitions and weights remains approximately 1.

7.2. Distance Between $\mathbf{Y_n}$ and $\mathbf{X_p^{\otimes n}}$. Let Y_n be the same as above.

THEOREM 7.3. The distance from Y_n to a subspace of L_p goes to ∞ with n, i.e., there is a sequence $(K(n))$, $K(n) \to \infty$, such that for all isomorphisms $T : Y_n \to Z \subset L_p$, $\|T\|\|T^{-1}\| \geq K(n)$.

Proof: If $T : Y_n \to Z \subset L_p$ is an isomorphism, then by Theorem 5.9, the norm of Y_n given by partitions and weights is equivalent to the envelope norm with a constant depending on $\|T\|\|T^{-1}\|$. Since the envelope norm of some element of norm 1 has value at least $n^{\frac{1}{p}}$, then $\|T\|\|T^{-1}\| \geq \lambda^{-1} n^{\frac{1}{p}} \geq \dfrac{n^{\frac{1}{p}}}{\|T\|\|T^{-1}\|}$ □

COROLLARY 7.4. The distance between Y_n and $(X_p)^{\otimes n}$ goes to ∞ with n.

COROLLARY 7.5. $(\sum_n Y_n)_{\ell_p}$ with norm given by partitions and weights is not isomorphic to a subspace of L_p.

8. Further Development and Open Questions

In this section, we choose a basis (e_k) of $(\sum \ell_2)_{\ell_p}$ and a basis (e'_k) of X_p to construct a diagonal subspace of $(\sum \ell_2)_{\ell_p} \oplus X_p$ which is uncomplemented in the space $(\sum \ell_2)_{\ell_p} \oplus X_p$. However, we are unable to determine whether or not the space is isomorphic to a complemented subspace of L_p.

8.1. A Diagonal Subspace of $(\sum \ell_2)_{\ell_p} \oplus X_p$. This example is of the type for the unresolved case of a single (nontrivial) partition in Section 6.

Let (e_k) be the usual basis of $(\sum \ell_2)_{\ell_p}$. Let (e'_k) be the usual basis of $X_{p,w}$, for some sequence w satisfying $(*)$. Let

$$e''_n = \begin{cases} e_{\frac{n}{2}} & \text{if } n \text{ even} \\ e'_{\frac{n+1}{2}} & \text{if } n \text{ odd} \end{cases}$$

Then $\{e''_n\}_{n=1}^{\infty}$ is an unconditional basis of $(\sum \ell_2)_{\ell_p} \oplus X_{p,w}$.

LEMMA 8.1. Let X be a space with unconditional basis (e_k). Let $\mathbf{A} = (\mathbf{T}_{i,j})$ represent a bounded projection T from X onto

$[e_{2k-1} + e_{2k} : k = 1, 2, ...]$ where the $\mathbf{T}_{i,j}$'s are 2×2 matrixes.

$$\mathbf{T}_{i,j} = \begin{pmatrix} a_{i,j} & a_{i,j+1} \\ a_{i+1,j} & a_{i+1,j+1} \end{pmatrix}, \quad \text{for} \quad i, j \in \{2k+1 : k = 0, 1, 2, \dots\}$$

Then

$$\mathbf{T}_{i,i} = \begin{pmatrix} a_{i,i} & 1 - a_{i,i} \\ a_{i,i} & 1 - a_{i,i} \end{pmatrix}, \quad \text{for} \quad i = 2k+1, k = 0, 1, 2, \dots$$

Proof: Because $(e_{2l-1} + e_{2l})_l$ is a basis for $T(X)$, for any k

$$\mathbf{A}e_k = \sum_{l=1}^{\infty} \beta_l^k (e_{2l-1} + e_{2l}),$$

for some real numbers $(\beta_l^k)_l$, i.e.

$$\begin{pmatrix} a_{1,k} \\ a_{2,k} \\ a_{3,k} \\ a_{4,k} \\ \vdots \end{pmatrix} = \begin{pmatrix} \beta_1^k \\ \beta_1^k \\ \beta_2^k \\ \beta_2^k \\ \vdots \end{pmatrix} \quad \text{for} \quad k = 1, 2, \ldots$$

Hence $\mathbf{A} = (a_{i,j})$ with $a_{i+1,j} = a_{i,j}$ for all $i = 2k+1$, $k = 0, 1, 2, \ldots$ and all j.

Since T is a projection on $[e_{2l-1} + e_{2l}]$, then $\mathbf{A}(e_{2l-1} + e_{2l}) = e_{2l-1} + e_{2l}$. So $a_{i,i} + a_{i,i+1} = 1$. We get $a_{i,i+1} = 1 - a_{i,i}$. Hence

$$\mathbf{T}_{i,i} = \begin{pmatrix} a_{i,i} & 1 - a_{i,i} \\ a_{i,i} & 1 - a_{i,i} \end{pmatrix}, \quad \text{for} \quad i = 2k+1, k = 0, 1, 2, \ldots$$

□

For the sake of convenience, let $(e_{i,m})$ be the usual basis of $(\sum \ell_2)_{\ell_p}$, i.e., $[e_{i,m} : m = 1, 2, ...] = \ell_2$, and let $(e'_{i,m})$ be the usual basis of $X_{p,w}$, where $w = (w_{i,m})$. Suppose that T is a bounded projection from $(\sum \ell_2)_{\ell_p} \oplus X_{p,w}$ onto $[e_{i,m} + e'_{i,m}]$. By Lemma 8.1, the diagonal 2×2 block of the matrix representation of T corresponding to $\{e_{i,m}, e'_{i,m}\}$ is of the form

$$\mathbf{T}_{i,i} = \begin{pmatrix} \alpha_{i,m} & 1 - \alpha_{i,m} \\ \alpha_{i,m} & 1 - \alpha_{i,m} \end{pmatrix}$$

Define

$$P: \quad (\sum \ell_2)_{\ell_p} \oplus X_p \longrightarrow [e_{i,m} + e'_{i,m}] \quad \text{by}$$

$$e_{i,m} \longrightarrow \alpha_{i,m}(e_{i,m} + e'_{i,m})$$

$$e'_{i,m} \longrightarrow (1-\alpha_{i,m})(e_{i,m} + e'_{i,m})$$

for all i, m.

LEMMA 8.2. With the notation above, for fixed i, $(w_{i,m})$ is chosen so that $\frac{1}{n}$ appears infinitely many times for all n, then for any $\epsilon > 0$ and any n, $n > \frac{\sqrt{2}\|P\|}{\epsilon}$, there exist infinitely many m, such that $w_{i,m} = \frac{1}{n}$ and $\alpha_{i,m} > 1-\epsilon$.

Proof: For fixed i, $\left\| P \sum_{m=1}^{\infty} \lambda_m e'_{i,m} \right\| \geq \left(\sum_m \lambda_m^2 (1-\alpha_{i,m})^2 \right)^{\frac{1}{2}}$.
Since $\left\| P \sum_{m=1}^{\infty} \lambda_m e'_{i,m} \right\| \leq \|P\| \max \left\{ \left(\sum_m \lambda_m^p \right)^{\frac{1}{p}}, \left(\sum_m \lambda_m^2 w_{i,m}^2 \right)^{\frac{1}{2}} \right\}$,
then $\left(\sum_m \lambda_m^2 (1-\alpha_{i,m})^2 \right)^{\frac{1}{2}} \leq \|P\| \max \left\{ \left(\sum_m \lambda_m^p \right)^{\frac{1}{p}}, \left(\sum_m \lambda_m^2 w_{i,m}^2 \right)^{\frac{1}{2}} \right\}$. For any $\epsilon > 0$, let $F_\epsilon = \{m : 1-\alpha_{i,m} \geq \epsilon\}$. Fix $n > \frac{\sqrt{2}\|P\|}{\epsilon}$ and choose $\delta > 0$ such that $\frac{\delta^2}{\epsilon^2}\|P\|^2 \leq \frac{1}{2}$. Then choose $F_n \subset \{m : w_{i,m} = \frac{1}{n}\}$ such that $\delta \geq |F_n|^{\frac{1}{p}-\frac{1}{2}}$.
If $\lambda_m = 1$ for all $m \in F_n$ and 0 otherwise,

$$\begin{aligned} \epsilon |F_\epsilon \cap F_n|^{\frac{1}{2}} &\leq \left(\sum_{m \in F_n} (1-\alpha_{i,m})^2 \right)^{\frac{1}{2}} \\ &\leq \max \left\{ \|P\| \left(\sum_{m \in F_n} 1 \right)^{\frac{1}{p}}, \frac{\|P\|}{n} \left(\sum_{m \in F_n} 1 \right)^{\frac{1}{2}} \right\} \end{aligned} \tag{8.1}$$

Hence $\epsilon |F_\epsilon \cap F_n|^{\frac{1}{2}} \leq \max\{\delta \|P\| |F_n|^{\frac{1}{2}}, \frac{\epsilon}{2} |F_n|^{\frac{1}{2}}\}$ and thus $|F_\epsilon \cap F_n| \leq \frac{1}{2}|F_n|$. Hence we obtain that there are infinitely many m with $\alpha_{i,m} > 1-\epsilon$ and $w_{i,m} = \frac{1}{n}$. □

PROPOSITION 8.3. If for each i, $(w_{i,m})$ is chosen so that $\frac{1}{n}$ appears infinitely many times for all n, then $[e_{i,m} + e'_{i,m}]$ is uncomplemented in $(\sum l_2)_{l_p} \oplus X_p$.

Proof: Suppose $[e_{i,m} + e'_{i,m}]$ is complemented in $(\sum \ell_2)_{\ell_p} \oplus X_p$. Let T be the bounded projection. By Lemma 8.1 , the diagonal operator of T has the same form as P defined before Lemma 8.2. By Tong's Lemma[**T**], P is a bounded diagonal projection with $||P|| \leq ||T||$. Let (n_i) be a sequence of natural numbers such that $n_i \geq 4||P||$ for all i and each integer larger than $4||P||$ occurs infinitely often. By the assumption on $(w_{i,m})$ and Lemma 8.2 , for each i, there exists m_i, $\alpha_{i,m_i} \geq \frac{1}{2}$ with $w_{i,m_i} = \frac{1}{n_i}$. Hence we get e_{i,m_i} so that $[(e_{i,m_i})_{i=1}^{\infty}] \sim \ell_p$ and $(w_{i,m_i})_{i=1}^{\infty}$ satisfies $(*)$. Then $\alpha_{i,m_i}(e_{i,m_i}+e'_{i,m_i})$ is equivalent to the unconditional basis of $X_{p,(\frac{1}{n_i})}$. This implies that a basis of ℓ_p is boundedly mapped to the basis of $X_{p,(\frac{1}{n_i})}$. This is impossible. Hence T doesn't exist. □

8.2. Open Questions.

QUESTION 8.4. Is the space $[e_{i,m_i} + e'_{i,m_i} : n \in \mathbb{N}]$ isomorphic to a complemented subspace of L_p?

QUESTION 8.5. Are the spaces $\mathcal{R}_p^{\alpha}$, $\alpha < \omega_1$, defined by Bourgain, Rosenthal, and Schechtman isomorphic to spaces with norms given by partitions and weights? Each of the spaces $\mathcal{R}_p^{\omega \cdot n}$, $n = 1, 2, ...$ has such an equivalent norm, but we do not know if the constants of equivalence depend on n.

These questions are all related to the main motivating questions for this thesis that were stated in the Introduction. One technical question which also may be of interest is the following.

QUESTION 8.6. Suppose that (f_n) is a sequence of mean zero independent random variables in L_p, what are the best constants in the following inequalities

$$C \sup \left(\sum_i \left(\sum_{n \in N_i} \|f_n\|_2^2 \right)^{\frac{p}{2}} \right)^{\frac{1}{p}} \leq \left\| \sum_n f_n \right\|_p \leq C' \sup \left(\sum_i \left(\sum_{n \in N_i} \|f_n\|_2^2 \right)^{\frac{p}{2}} \right)^{\frac{1}{p}}$$

where the supremum is over all partitions $\{N_i\}$ of $\mathbb{N}$.

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