EQUIVARIANT COHOMOLOGY OF A B–VARIETY

AND BETTI NUMBERS

WITH APPLICATION

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1 Statement of the problem and results

We have studied T-invariant rational equivalence in a B-variety X, i.e a smooth projective variety over $\mathbb{C}$ with a T-action; $T = (\mathbb{C}^*)^{n+1}$ is the algebraic torus, and a finite set of fixed points of T. Our main research goal is to prove the conjecture that the equivariant k-th Chow group $A_k^T(X)$ is isomorphic to the ordinary k-th Chow group $A_k(X)$, find a computational algorithm for $A_k^T(X)$, and apply it to some interesting cases. The previous conjecture has been proved using torus action on families. In a B-variety X we say that two T-invariant k-dimensional subvarieties V, V' are T-invariantly rationally equivalent if there exists a sequence of T-invariant k-dimensional subvarieties $V_0 = V, V_1, ..., V_n = V'$ and a sequence of T-invariant $(k+1)$-dimensional subvarieties $W_1, W_2, ..., W_n$ such that V_{i-1}, V_i are contained in W_i and V_{i-1} is linearly equivalent to V_i in W_i. Theorem 5.5.44 is an interesting new result. It gives a necessary and sufficient condition for two T-invariant subvarieties $D_1, D_2 \subset X$ of dimension k to be T-invariantly rationally equivalent using the weights of the characters $\chi_i(t) = t_i$ where $t \in T$ and T-invariant subvarieties $Z \subset X$ of dimension $k+1$. We have investigated the case where the set of fixed components is a finite set of fixed points.

We have studied the Hilbert scheme $\mathrm{Hilb}\, X$ which is the scheme representing the functor $\mathrm{Hilb}_X : \{\text{Category of schemes}\} \longrightarrow \{\text{Category of sets}\}$ where $\mathrm{Hilb}_X\, U$ is the set of flat families of closed subschemes W of X parametrized by U. If $U' \longrightarrow U$ is any morphism, $W \longmapsto W \times_U U'$ gives a map $\mathrm{Hilb}_X\, U \longrightarrow \mathrm{Hilb}_X\, U'$, which makes Hilb_X a contravariant functor on the category of schemes. In Theorem 5.5.22, we have proved that for any B-variety X any component of the Hilbert scheme $\mathrm{Hilb}\, X$ can be embedded T-equivariantly in $\mathbb{P}(V)$ for some T-representation V. This result was used in the proof of the conjecture that the equivariant k-th Chow group $A_k^T(X)$ is isomorphic to the ordinary k-th Chow group $A_k(X)$. On the other hand, Theorem 5.5.22 was also used to prove Theorem 5.5.2. The Hilbert scheme Hilb X was also useful to understand the limit of fibers of a family contained in X or $\mathbb{C} \times X$. We were interested in the situation $Z \sim 0$ in V where V is a subvariety of the B-variety X, trying to understand why $\lim_{t \longrightarrow 0} t.Z \sim 0$ inside $\lim_{t \longrightarrow 0} t.V$ where t lives in a one parameter subgroup $\mathbb{C}^* \subset T$. Initially, this case was studied using the limit concept and the Hilbert scheme Hilb X. This approach ended with an unsolved question. The limit argument above was used in the proof of the injectivity part of the conjecture above. The successful approach used to prove the above limit argument was based on the following observation in section 10.1 of [10]: Let $\mathcal{G}$ denote an irreducible variety of dimension $m > 0$. The notation "$a \in \mathcal{G}$" will be used to denote a regular, closed point of $\mathcal{G}$(Appendix B.1 [14]). By abuse of notation we will write t in place of $Spec(k(t))$, where $k(t)$ is the residue field of the local ring of $\mathcal{G}$ at the point, and we denote by

$$t : \{t\} \longrightarrow \mathcal{G}$$

the canonical inclusion of $Spec(k(t))$ in $\mathcal{G}$. The assumption that the point is regular means that t is a regular embedding of codimension m. Any $(k+m)-$cycle α on a scheme $\mathcal{Y}$, or more generally any rational equivalence class $\alpha \in A_{k+m}(\mathcal{Y})$ determines a family of k-cycle classes $\alpha_t \in A_k(Y_t)$, for all $t \in \mathcal{G}$, by the formula

$$\alpha_t = t^!(\alpha)$$

where $t^! : A_{k+m}(\mathcal{Y}) \longrightarrow A_k(Y_t)$ is the refined Gysin homomorphism defined from the fiber square

$$\begin{array}{ccc} Y_t & \longrightarrow & \mathcal{Y} \\ \downarrow & & \downarrow \\ \{t\} & \longrightarrow & \mathcal{G} \end{array}$$

by construction of section 6.2 in [17]. If $\alpha = [\mathcal{V}]$ where $\mathcal{V}$ is a subvariety of $\mathcal{Y}$ of pure dimension $k+m$, then $\alpha_t = [\mathcal{V}]_t = \{s(V_t, \mathcal{V})\}_k$ where $V_t = \mathcal{V} \cap Y_t$, and $s(V_t, \mathcal{V})$ is the Segre class of V_t in $\mathcal{V}$. (This follows from proposition 6.1 (a) and the fact that the normal bundle to t in $\mathcal{G}$ is trivial). In particular, if $\mathcal{V} \subset Y_t$, then $[\mathcal{V}]_t = 0$.

In 1987, Ellingsrud and Strømme gave a precise description of the additive structure of the homology of $\mathrm{Hilb}^d(\mathbb{P}^2)$, applying the results of Bialynicki-Birula on the cellular decompositions defined by a torus action to the natural action of the maximal torus of SL(3) on $\mathrm{Hilb}^d(\mathbb{P}^2)$. We say that a scheme X has a cellular decomposition if there is a filtration $X = X_n \supset X_{n-1} \supset ... \supset X_0 \supset X_{-1} = \phi$ by closed subschemes with each $X_i - X_{i-1}$ a disjoint union of schemes U_{ij} isomorphic to affine spaces $\mathbb{A}^{n_{ij}}$. The U_{ij}'s are called the cells of the decomposition. A rather easy consequence of the fact that this action has finitely many fixed points is that the cycle maps between the Chow groups and the homology groups are isomorphisms. In particular there is no odd homology, and the homology groups are all free. They computed the ranks of these groups, i.e the Betti numbers of $\mathrm{Hilb}^d(\mathbb{P}^2)$. In a recent work, Laurent Evain gave a new proof of the result by Ellingsrud and Strømme, namely the main lemma of the computation of the Betti numbers of the Hilbert scheme $\mathrm{Hilb}^d(\mathbb{P}^2)$. Also he described the Bialynicki-Birula cells of $\mathrm{Hilb}^d(\mathbb{P}^2)$ by means of explicit flat families.

Let X be a smooth projective variety with a $\mathbb{C}^*$-action and a finite set of fixed points $\{p_1, ..., p_n\}$ of the action. Then by the Bialynicki-Birula theorem X has a cellular decomposition with cells $X_{p_i} = \{x \in X : \lim_{t \longrightarrow 0} t.x = p_i\}$. Let $\mathcal{H} = Hilb^2(\mathbb{P}^2)$ and let $\mathcal{H}^T$ be the fixpoint locus of the $\mathbb{C}^*$-action on $\mathcal{H}$; then $\mathcal{H}^T = \{(x^2, y), (x, y^2), (x^2, z), (x, z^2), (y^2, z), (y, z^2), (x, y) \cap (x, z), (x, y) \cap (y, z), (y, z) \cap (x, z)\}$. Let $X_p = \{Z \in \mathcal{H} : \lim_{t \longrightarrow 0} t.Z = p\}$ be the cell that corresponds to the fixpoint $p \in \mathcal{H}^T$. Consider the $\mathbb{C}^*$-action on $\mathbb{P}^2$ given by $t.(x_0, x_1, x_2) = (t^a x_0, t^b x_1, t^c x_2)$ such that $a > b > c$. Let $y = \frac{x_1}{x_0}$, $z = \frac{x_2}{x_0}$, and let $A = (y^2, z)$, $B = (y, z^2)$ then A, B

are two fixpoints of $\mathcal{H}$ supported at $p_0 = (1,0,0)$. Now the $\mathbb{C}^*$-action above can be written as follows: $t.(y,z) = (t^{b-a}y, t^{c-a}z)$. Let $\mathcal{Q}_A = \mathbb{C}[y,z]/A$ then the tangent space $\mathcal{T}_A\mathcal{H} = \mathrm{Hom}((y^2,z), \mathbb{C}[y,z]/(y^2,z)) = \mathrm{Hom}(A, \mathcal{Q}_A) = A^{\vee} \otimes \mathcal{Q}_A$. Let $e_1 = y^2$, $e_2 = z$, $e_3 = 1$, and $e_4 = y$ then the set $\{e_i^{\vee} \otimes e_j : i = 1,2,\ j = 3,4\}$ forms a basis for $T_A\mathcal{H}$. Let us apply the $\mathbb{C}^*$-action above on the basis elements and then count the number of positive weights to get the dimension of the cell X_A. First we will compute the weights of the $\mathbb{C}^*$-action on the basis element $e_1^{\vee} \otimes e_3$. Since $t.e_1^{\vee} = t^{-1}.e_1 = t^{-1}.y^2 = t^{-2(b-a)}y^2 = t^{2(a-b)}y^2 = t^{2(a-b)}e_1^{\vee}$, and $t.e_3 = e_3$, it follows that $t.(e_1^{\vee} \otimes e_3) = (t^{2(a-b)}e_1^{\vee}) \otimes e_3 = t^{2(a-b)}(e_1^{\vee} \otimes e_3)$. Using the additive notation, the weight of the $\mathbb{C}^*$-action on basis element $e_1^{\vee} \otimes e_3$ is $2(a-b)$. Similarly $t.(e_1^{\vee} \otimes e_4) = t^{2(a-b)}e_1^{\vee} \otimes t^{b-a}e_4 = t^{2(a-b)+b-a}(e_1^{\vee} \otimes e_4) = t^{a-b}(e_1^{\vee} \otimes e_4)$, $t.(e_2^{\vee} \otimes e_3) = t^{a-c}(e_2^{\vee} \otimes e_3)$, and $t.(e_3^{\vee} \otimes e_4) = t^{b-c}(e_2^{\vee} \otimes e_4)$. Therefore the weights of the $\mathbb{C}^*$-action on the basis above are $2(a-b)$, $a-b$, $a-c$, and $b-c$. Now since $a > b > c$ the number of positive weights is 4. So $X_A \simeq \mathbb{C}^4$. Therefore $\dim X_A = 4$. To calculate the dimension of the cell X_B, let $\mathcal{Q}_B = \mathbb{C}[y,z]/B$ then $T_B\mathcal{H} = \mathrm{Hom}(B, \mathcal{Q}_B) = B^{\vee} \otimes \mathcal{Q}_B$. By similar reasoning one may generate 4 basis elements. Applying the $\mathbb{C}^*$-action on those elements we pick up 4 weights namely, $a-b$, $c-b$, $2(a-c)$, and $a-c$. Now since $a > b > c$ the number of positive weights is 3. So $X_B \simeq \mathbb{C}^3$. Therefore $\dim X_A = 3$.

Recall that the equivariant cohomology of X where X is a topological space with a T-action, $T = (\mathbb{C}^*)^3$ is the algebraic torus acting on X, is defined to be $H_T^*(X) = H^*(X \times_T ET)$ where $H^*(X \times_T ET)$ is the ordinary cohomology of the bundle $X \times_T ET$ over BT with fiber X, and $ET \longrightarrow BT$ is a principal T-bundle. We want to understand how to deform an i^{th} dimensional cell of $\mathrm{Hilb}^2(\mathbb{P}^2)$ to an $(i+1)^{th}$ dimensional cell of $\mathrm{Hilb}^2(\mathbb{P}^2)$. In the above example we deform the 3-dimensional cell X_B to the 4-dimensional cell X_A via $y \longmapsto ay + bx$ where a, b $\in \mathbb{C}^*$. We were able to understand what the closure of every cell is and how those cells are connected using deformations.

The result is a perfectly symmetric picture of $\text{Hilb}^2(\mathbb{P}^2)$. We have done the same investigation for $\text{Hilb}^3(\mathbb{P}^2)$. This is the part that has been completed. The remaining part is to define the isomorphism

$$H_T^*(\text{Hilb}^2(\mathbb{P}^2)) \otimes \mathbb{Q}(\lambda_1, \lambda_2, \lambda_3) \simeq \oplus_{j=1}^9 H_T^*(z_j) \otimes \mathbb{Q}(\lambda_1, \lambda_2, \lambda_3) \tag{1}$$

where $T = (\mathbb{C}^*)^3$ is the algebraic torus acting on $\text{Hilb}^2(\mathbb{P}^2)$, $\{z_j\}_{j=1}^9$ is the fixed point locus of T, and λ_i denote the weight of the character $\rho_i(t_1, t_2, t_3) = t_i$, $i = 1, 2, 3$. For $\mathcal{H} = \text{Hilb}^d(\mathbb{P}^2)$ although we have discovered the nature of the cells and learned its cellular decomposition, we would like to find a computational algorithm for the integral $\int_{\text{Hilb}^2(\mathbb{P}^2)} P$ where P is a polynomial in classes of cells. We would like to compute integrals like

$$\int_{\text{Hilb}^2(\mathbb{P}^2)} [X_{(y,z^2)}].[X_{(x,yz)}]$$

A $(\mathbb{C}^*)^n$-action on a smooth complex projective variety X has an infinite number of 1-parameter subgroups $\psi : \mathbb{C}^* \longrightarrow (\mathbb{C}^*)^n$ where $\psi(t) = (t^{a_1}, ..., t^{a_n})$, $a_i \in \mathbb{Z}$. We want to understand how the cellular decompostion of X depends on a_i, $i = 1, ..., n$. Asume that the $\mathbb{C}^*$-action above has a finite set of fixed points. For example if $a_1 < a_2 < ... < a_n$ we get a finite set of fixed points. The set $\{(a_1, ..., a_n) : a_1 < a_2 < ... < a_n\} \subset \mathbb{Z}^n$ gives us a cellular decomposition while the set $\{(a_1, ..., a_n) : a_1 > a_2 < ... < a_n\} \subset \mathbb{Z}^n$ gives us another cellular decomposition. We have tried to describe the sets in $\mathbb{Z}^n$ which give different cellular decomposition and understand the general structure. For this purpose we studied the toric varieties $\mathbb{P}^n$, $\mathbf{F}_n$ where $\mathbf{F}_n$ denotes the rational ruled surface $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n))$ and the nontoric varieties $\text{Hilb}^2(\mathbb{P}^2)$, $\text{Hilb}^3(\mathbb{P}^2)$ to find answers to these questions.

A computational algorithm for the integral $\int_X P(c_{a_r}(E_r), [Y_i])$ is known by equivariant cohomology where P is a polynomial in chern classes of the vector bundles E_r on the

B-variety X and classes of G-invariant subvarieties $Y_i \subset X$ where $G = \mathbb{C}^* \subset T = (\mathbb{C}^*)^n$ is a 1-parameter subgroup. The simple result is that the localization formula of equivariant cohomology implies the following formula in terms of various weights of the G-action on X

$$\int_X P(c_{a_r}(E_r), [Y_i]) = \sum_{k=1}^{n} \frac{P(g_{a_r}, f_{Y_i})}{\prod_{i=1}^{u} \beta_i^k} \tag{2}$$

where $f_{Y_i}(z)$ is the product of the weights of the normal bundle $\mathcal{N}_{Y_i|X}$ at z if $z \in Y_i$ and zero otherwise, $g_{a_r}(z) = \sigma_{a_r}$(weights of the G-action on $E_r|_z$) where g_{a_r} is the ${a_r}^{th}$ elementary symmetric function. We have computed the integral $\int_X P(c_{a_r}(E_r), [Y_i])$ by replacing $c_{a_r}(E_r)$(resp.$[Y_i]$) by the equivariant chern class $c^G_{a_r}(E_r)$(resp.$[Y_{iG}]$) and pulling back via the inclusion map $i_{pt} : pt \longrightarrow (\mathbb{CP}^\infty)^n$. To see that this gives the integral $\int_X P(c_{a_r}(E_r), [Y_i])$, consider the commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & pt \\ \Big\downarrow{\scriptstyle i_X} & & \Big\downarrow{\scriptstyle i_{pt}} \\ X_G & \longrightarrow & BG \end{array} \tag{3}$$

where i_X is the inclusion map. This implies $i^*_{pt} \circ \int_{X_G} = \int_X \circ i^*_X$. Hence $i^*_{pt} \circ \int_{X_G} P(c^G_{a_r}(E_r), [Y_{iG}]) = \int_X i^*_X P(c^G_{a_r}(E_r), [Y_{iG}]) = \int_X P(c_{a_r}(E_r), [Y_i])$. The key point is that $\int_{X_G}$ is an equivariant integral which can be computed by localization. By localization we mean the procedure of equivariantly restricting a class to each fixed point of the G-action on X and dividing by the equivariant Euler class of the normal bundle of that component. This reduces the calculations of an equivariant integral of a class to the sum over all fixed point components of the equivariant integrals of the restrictions of the class.

2 Localization

2.1 Equivariant cohomology

For the sake of convenience we will use a *smooth* B-variety to denote a projective variety with a torus action and a finite set of fixed points where the torus T is the maximal torus $(\mathbb{C}^*)^{n+1}$ unless it is mentioned otherwise.

Definition 2.1.1. Let X be an n-dimensional B-variety, and let $T = (\mathbb{C}^*)^{n+1}$ be the maximal torus. Fix a system of homogeneous coordinates $x_0, ..., x_n$.

(a) Let ρ_j be the character of T defined by $\rho_i(t_1, ...t_{n+1}) = t_i$. Let $\chi_0, ..., \chi_n$ denote the complex characters of T where $\chi_i = \prod_{j=0}^{n} \rho_j^{n_{ij}}$ and $n_{ij} \in \mathbb{Z}$. Then T acts on X via $t.x_i = \chi_i^{-1}(t)x_i$, and on points $(a_0, ..., a_n)$, this action is given by

$$t.(a_0, ..., a_n) = (\chi_0(t)a_0, ..., \chi_n(t)a_n)$$

(b) Let $\psi \in N = \mathrm{Hom}(\mathbb{C}^*, T)$, an let $\chi \in M = \mathrm{Hom}(T, \mathbb{C}^*)$ then $\chi \circ \psi(t) = t^k$, where $k \in \mathbb{Z}$. We define the dual pairing $<,>: N \otimes M \longrightarrow \mathbb{Z}$ by $< \psi, \chi >= k$. The one parameter subgroup $\psi : \mathbb{C}^* \longrightarrow T$ acts on X via $\psi(t).x_i = t^{-<\psi,\chi_i>}x_i$, and on points $(a_0, ..., a_n)$, this action is given by

$$\psi(t).(a_0, ..., a_n) = (t^{<\psi,\chi_0>}a_0, ..., t^{<\psi,\chi_n>}a_n)$$

(c) If $f : X \longrightarrow \mathbb{P}^1$ is a rational function on X then we define the action of the 1-parameter subgroup $\psi(t)$ on $f(x)$ by

$$\psi(t).f(x) = f(\psi^{-1}(t).x) = f(t^{-<\psi,\chi_0>}x_0, ..., t^{-<\psi,\chi_n>}x_n)$$

Definition 2.1.2. Let Y be a topological space and let $T = (\mathbb{C}^*)^{r+1}$ be an algebraic torus acting on Y. A principal T-bundle $\mathcal{B}$ over Y consists of the data of a topological space $\mathcal{B}$ and a continuous map $f : \mathcal{B} \longrightarrow Y$, together with additional data consisting of an open covering $\{U_i\}$ of Y with homeomorphisms $\rho_i : f^{-1}(U_i) \longrightarrow U_i \times T$, such that for any i we have $\pi_1 \circ \rho_i = f|_{f^{-1}(U_i)}$ where π_1 is the projection map to U_i, and transition functions $g_{ij} : U_i \bigcap U_j \longrightarrow T$ given by $(\rho_i \circ \rho_j^{-1})(y,t) = (y, g_{ij}(y) \cdot t)$ where $(y,t) \in (U_i \bigcap U_j) \times T$.

Remark 2.1.3. Let $\mathcal{B}$ be a principal T-bundle over Y then there exists a T-action $\mathcal{B} \times T \longrightarrow \mathcal{B}$ on $\mathcal{B}$ given by $b.t = \rho_i^{-1}(\rho_i(b).t)$, where $b \in f^{-1}(U_i)$ for some i.

Definition 2.1.4. Let X be a topological space with a T-action $T \times X \longrightarrow X$, and let $\mathcal{B} \longrightarrow Y$ be a principal T-bundle. The fiber bundle $\mathcal{B} \times_T X$ is defined to be

$$\mathcal{B} \times_T X = (\mathcal{B} \times X)/((u,x) \sim (u \cdot t^{-1}, t \cdot x)) \tag{4}$$

for any $x \in X$, $t \in T$, and $u \in \mathcal{B}$.

Definition 2.1.5. Let S be the tautological bundle on $\mathbb{CP}^\infty$ whose sheaf of sections is $\mathcal{O}_{\mathbb{CP}^\infty}(-1)$, and let $BT = (\mathbb{CP}^\infty)^n$. The principal T-bundle ET over BT is defined to be

$$ET = \pi_1^* S \oplus ... \oplus \pi_n^* S \tag{5}$$

where $\pi_i : BT \longrightarrow \mathbb{CP}^\infty$ is the i^{th} projection map. For the definition of a tautological line bundle see [8].

Definition 2.1.6. Let X be a topological space with a T-action, and $X_T = \mathcal{B} \times_T X$ be a fiber bundle over BT with fiber X. The equivariant cohomology of X is defined to be

$$H_T^*(X) = H^*(X_T) \tag{6}$$

where $H^*(X_T)$ is the ordinary cohomology of X_T.

Remark 2.1.7. If $Y \subset X$ is a T-invariant subvariety of X then $Y_T \subset X_T$. For example if $X = \mathbb{P}^2$, $Y = \mathcal{Z}(x_0 = 0)$ then $[(x_0 = 0)_T] \subset \mathbb{P}^2_T$.

Remark 2.1.8. $H_T^*(\{point\}) = H^*(BT)$.

Definition 2.1.9. A representation of an algebraic group G is a vector space V over $\mathbb{C}$ and a map $G \times V \longrightarrow V$ where $(g, v) \longmapsto g \cdot v$ such that:

(a) $g \cdot (av_1 + bv_2) = a(g \cdot v_1) + b(g \cdot v_2)$ where a,b $\in \mathbb{C}$.

(b) $1 \cdot v = v$.

(c) $(g_1 g_2) \cdot v = g_1 \cdot (g_2 \cdot v)$.

Remark 2.1.10. There exists a one-to-one correspondence between the group of characters $M(T)$ and the set of 1-dimensional representations of T given by

$$\chi \longleftrightarrow (t \cdot z = \chi(t)z) \tag{7}$$

Example 2.1.11. Let $V = \mathbb{C}$ be a vector space with an action of $T = (\mathbb{C}^*)^3$ given by

$$t \cdot z = \prod_{i=0}^{2} (t_i)^{m_i} z$$

where $t_i \in T$, $m_i \in \mathbb{Z}$ are fixed integers, and $z \in \mathbb{C}$. As mentioned in remark 2.1.10 this action corresponds to the character $\chi : T \longrightarrow (\mathbb{C})^*$ given by

$$\chi(t_0, t_1, t_2) = \prod_{i=0}^{2} (t_i)^{m_i}$$

Let $M(T)$ be the group of characters of T then $M(T) \simeq \mathbb{Z}^3$ via the group isomorphism

$$(m_0, m_1, m_2) \longmapsto (t \longmapsto \prod_{i=0}^{2} (t_i)^{m_i})$$

Fact 2.1.12. If V is a finite dimensional representaion for $T = (\mathbb{C}^*)^n$ then there exists characters $\chi_1, ..., \chi_n \in M(T)$ such that $V \simeq \oplus_{i=1}^n V_{\chi_i}$. See [2].

Let $\chi \in M(T)$ where $T = (\mathbb{C}^*)^{n+1}$. Define the action $T \times \mathbb{C} \longrightarrow \mathbb{C}$ as follows. $(t, z) \mapsto t.z$ where $t \cdot z = \chi(t)z$, $t \in T$ and $z \in \mathbb{C}$. So by the previous remark this gives a 1-dimensional representation $\mathbb{C}_\chi$. If $L_\chi = (\mathbb{C}_\chi)_T$ is the corresponding line bundle over BT, then the assignment $\chi \mapsto -c_1(L_\chi)$ defines an isomorphism $\phi : M(T) \longrightarrow H^2(BT)$: first we show ϕ is injective. Suppose $-c_1(L_\chi) = 1$, then L_χ is the trivial bundle. So $\chi(t) = 1$. So ϕ is injective. Second we show that ϕ is surjective. So let $\alpha \in H^2(BT) = \mathbb{C}[\lambda_0, ..., \lambda_n]$, then $\alpha = \sum_{i=0}^n a_i\lambda_i$. But $\lambda_i = c_1(\mathcal{O}(\lambda_i)) = c_1(L_{\chi_i})$. It follows $\alpha = -\sum_{i=0}^n a_i(-c_1(L_{\chi_i})) = -\sum_{i=0}^n a_i\phi(\chi_i) = -\sum_{i=0}^n \phi(a_i\chi_i) = \phi(-\sum_{i=0}^n a_i\chi_i)$. So ϕ is surjective.

We call $\phi(\chi)$ the weight of χ. In particular, if $\chi_i \in M(T)$ is defined by $\chi_i(t_0, t_1, ..., t_n) = t_i$, then we let λ_i denote the weight of χ_i, i=0,1,...,n. Thus we get the isomorphism $H_T^*(\{point\}) = H^*(BT) \simeq \mathbb{C}[\lambda_0, ..., \lambda_n]$ (see [8]). We denote the line bundle L_{χ_i} by $\mathcal{O}(-\lambda_i)$, so that $\lambda_i = c_1(\mathcal{O}(\lambda_i))$.

Definition 2.1.13. Let X be a topological space with a T-action. An equivariant vector bundle is a vector bundle E over X such that the action of T on X lifts to an action on E which is linear on fibers. In this situation, E_T is a vector bundle over X_T, and the equivariant Chern classes $c_k^T(E) \in H_T^*(X)$ are defined by the ordinary Chern classes $c_k(E_T)$. If E has rank r then the top Chern class $c_r^T(E)$ is called the equivariant Euler class of E and is denoted by $Euler_T(E) \in H_T^*(X)$.

Example 2.1.14. The diagonal action of $T = (\mathbb{C}^*)^{n+1}$ on $\mathbb{C}^{n+1}$ gives an equivariant vector bundle E over $Y = point$ such that $E_T = \oplus_{i=0}^n \mathcal{O}(\lambda_i)$. Thus $\lambda_0, ..., \lambda_n$ are the weights of this representation.

Remark 2.1.15. Consider the action of $T = (\mathbb{C}^*)^{n+1}$ on $X = \mathbb{P}^n$ given by

$$(t_0, ..., t_n).(x_0, ..., x_n) = (t_0^{-1}x_0, ..., t_n^{-1}x_n) \tag{8}$$

The inverses has been chosen so that $(t_0, ..., t_n)$ acts on the homogeneous form $x_j \in H^0(\mathcal{O}_{\mathbb{P}^n}(1))$ as multiplication by t_j. Note that $\mathbb{P}_T^n$ is the projectivization of the vector bundle $F_T = \oplus_{i=0}^{n}\mathcal{O}(-\lambda_i)$ over BT. Thus $\mathbb{P}_T^n = \mathbb{P}(E_T^*) = \mathbb{P}(F_T)$, where E^* is the dual of the bundle E defined in the previous example, and $\mathbb{P}$ denotes the projectivization. This gives the tautological line bundle $\mathcal{O}_{\mathbb{P}_T^n}(1)$, and we have $p = c_1(\mathcal{O}_{\mathbb{P}_T^n}(1)) \in H_T^*(\mathbb{P}^n)$. Now since p is defined to be the equivariant Chern class $c_1^T(\mathcal{O}_{\mathbb{P}_T^n}(1))$, we refer to p as the equivariant hyperplane class.

Remark 2.1.16. Consider the vector bundle $F_T = \oplus_{i=0}^{n}\mathcal{O}(-\lambda_i)$ on BT in the previous remark. Let $\pi^*(F_T)$ be the pullback of F_T via the map $\pi : \mathbb{P}(F_T) \longrightarrow BT$, then $\pi^*(F_T)$ is a vector bundle on $\mathbb{P}(F_T)$ and it has the subbundle

$$S = \{(z, e) \in \pi^*(F_T) : z \in \mathbb{P}(F_T), e \in (F_T)_{\pi(z)}\}$$

where $(F_T)_{\pi(z)}$ is the fiber of F_T over $\pi(z) \in BT$. The fiber of S over z, denoted by S_z, is defined as follows: For any $x \in BT$ let $(F_T)_x$ be the fiber of the bundle F_T over x. If $z \in \mathbb{P}(F_T)$ where

$$\mathbb{P}(F_T) = \{V \subset (F_T)_x : V \text{ is a linear subspace of } (F_T)_x,\ x \in BT\} = \bigcup_{x \in BT} (\mathbb{P}(F_T))_x$$

we fix a linear subspace $l \subset (F_T)_x$ such that $z \in l$. Define the fiber S_z by $S_z = l$. Now $S = \mathcal{O}_{\mathbb{P}(F_T)}(-1) = \mathcal{O}_{\mathbb{P}_T^n}(-1)$ (because $\mathbb{P}(F_T) = \mathbb{P}(\oplus_{i=0}^{n}\mathcal{O}(-\lambda_i)) = \mathbb{P}(\mathbb{C}_T^{n+1}) = \mathbb{P}_T^n$), and the map $\phi : \pi^*(F_T^*) \longrightarrow S^*$ is surjective where $\pi^*(F_T^*) = \oplus_{i=0}^{n}\mathcal{O}(\lambda_i)$ (see [8]). Note that the induced map on the fibers $\phi_z : (\pi^*(F_T^*))_z \longrightarrow (S^*)_z$ is defined as follows: Since $\pi^*(F_T^*)_z = (F_T^*)_{\pi(z)} = \mathrm{Hom}((F_T)_{\pi(z)}, \mathbb{C})$, and $(S^*)_z = V^* = \mathrm{Hom}(V, \mathbb{C})$, where V is a linear subspace of $(F_T)_{\pi(z)}$. We define the map ϕ_z by $\phi_z(f) = f|_V$, where $f \in \mathrm{Hom}((F_T)_{\pi(z)}, \mathbb{C})$.

Theorem 2.1.17. $[(x_i = 0)_T] = p - \lambda_i$ where $x_i \in H^0(\mathcal{O}_{\mathbb{P}^n}(1))$.

Proof. Let $F_T = \oplus_{i=0}^{n}\mathcal{O}(-\lambda_i)$ be the vector bundle defined above, and let $i : \pi^*\mathcal{O}(\lambda_0) \hookrightarrow \pi^*(F_T^*)$ be the inclusion map. Consider $\psi = \phi \circ i : \pi^*\mathcal{O}(\lambda_0) \longrightarrow \mathcal{O}(p)$ then for any $y \in \mathbb{P}(F_T)$ the induced map on the fibers $\psi_y : (\pi^*\mathcal{O}(\lambda_0))_y \longrightarrow (\mathcal{O}(p))_y$ is defined as follows: let s be a section of $\pi^*\mathcal{O}(\lambda_0)$, and let a_i be a section of $\pi^*\mathcal{O}(-\lambda_0)$, $i = 0, 1, ..., n$. If $y \in \mathbb{P}(F_T)$ then $s(y) \in (\pi^*\mathcal{O}(\lambda_0))_y$ which implies that $\psi_y(s(y))$ lives in $(\mathcal{O}(p))_y = S_y^* = \mathrm{Hom}(S_y, \mathbb{C})$ where $S_y \subset (\pi^* F_T)_y = \oplus_{i=0}^n (\pi^*\mathcal{O}(-\lambda_i))_y$. Define $\psi_y(s(y))$ by

$$\psi_y(s(y))(a_0(y), ..., a_n(y)) = s(y)a_0(y) \tag{9}$$

then $s(y)a_0(y) \in (\pi^*\mathcal{O}(\lambda_0))_y \otimes (\pi^*\mathcal{O}(-\lambda_0))_y = \pi^*((\mathcal{O}(\lambda_0))_y \otimes (\mathcal{O}(-\lambda_0))_y)$ which is isomorphic to $\mathbb{C}$ because $\pi^*((\mathcal{O}(\lambda_0))_y \otimes (\mathcal{O}(-\lambda_0))_y) = \pi^*(\mathcal{O}(0))_y \simeq \pi^*\mathbb{C} \simeq \mathbb{C}$. Now the map ψ_y induces the map $\phi_y : S_y \longrightarrow (\pi^*\mathcal{O}(-\lambda_0))_y$ in $\mathrm{Hom}(S_y, (\pi^*\mathcal{O}(-\lambda_0))_y)$ where $\phi : S \longrightarrow \pi^*\mathcal{O}(-\lambda_0)$ lives in $\mathrm{Hom}(S, \pi^*\mathcal{O}(-\lambda_0))$. But $\mathrm{Hom}(S, \pi^*\mathcal{O}(-\lambda_0)) = \Gamma(S^* \otimes \pi^*\mathcal{O}(-\lambda_0)) = \mathrm{Hom}(S \otimes \pi^*\mathcal{O}(\lambda_0), \mathbb{C})$. Therefore $\mathrm{Hom}(S \otimes \pi^*\mathcal{O}(\lambda_0), \mathbb{C})$ is not empty. Recall $\mathbb{P}(F_T) \simeq \mathbb{P}_T^n$, let $x = (x_0, ..., x_n) \in \mathbb{P}_T^n$. Define the section $x_0 : S_x \otimes (\pi^*\mathcal{O}(\lambda_0))_x \longrightarrow \mathbb{C}$ by

$$x_0((a_0(x), ..., a_n(x)) \otimes d_0(x)) = a_0(x)d_0(x) \tag{10}$$

then $x_0 \in \mathrm{Hom}(S \otimes \pi^*\mathcal{O}(\lambda_0), \mathbb{C}) = \Gamma(S^* \otimes \pi^*\mathcal{O}(-\lambda_0)) = \Gamma(S^* \otimes \mathcal{O}(-\lambda_0))$. But $\Gamma(S^* \otimes \mathcal{O}(-\lambda_0)) = \Gamma(\mathcal{O}(p) \otimes \mathcal{O}(-\lambda_0)) = \Gamma(\mathcal{O}(p - \lambda_0))$. Therefore the equivariant class $[(x_0 = 0)_T] = c_1(\mathcal{O}(p - \lambda_0)) = p - \lambda_0$. Similarly we can define the section $x_i : S_x \otimes (\pi^*\mathcal{O}(\lambda_i))_x \longrightarrow \mathbb{C}$ by

$$x_i((a_0(x), ..., a_n(x)) \otimes d_i(x)) = a_i(x)d_i(x) \tag{11}$$

where $i = 1, ..., n$. It follows $x_i \in \mathrm{Hom}(S \otimes \pi^*\mathcal{O}(\lambda_i), \mathbb{C}) = \Gamma(S^* \otimes \pi^*\mathcal{O}(-\lambda_i))$. But $\Gamma(S^* \otimes \pi^*\mathcal{O}(-\lambda_i)) = \Gamma(S^* \otimes \mathcal{O}(-\lambda_i)) = \Gamma(\mathcal{O}(p) \otimes \mathcal{O}(-\lambda_i)) = \Gamma(\mathcal{O}(p - \lambda_i))$. Thus $[(x_i = 0)_T] = c_1(\mathcal{O}(p - \lambda_i)) = p - \lambda_i$, $i = 1, ..., n$. Thus theorem 2.1.17 is proven.

Let $a_i \in \mathbb{Z}$, $i = 1, ..., n + 1$. Consider the embedding $i : \mathbb{C}^* \hookrightarrow (\mathbb{C}^*)^{n+1}$ of a one parameter subgroup acting on $\mathbb{P}^n$ where i is given by $i(t) = (t^{a_1}, ..., t^{a_{n+1}})$. Let λ be the weight of the character $\chi(t) = t$. We will check that the weights $a_i\lambda$ of the characters t^{a_i} of the given one parameter subgroup are just the pullback of the weights of the characters $\chi_i(t) = t_i$ of $(\mathbb{C}^*)^{n+1}$. Take $n = 2$. Consider the $\mathbb{C}^*$ action on $\mathbb{P}^2$ given by

$$t \cdot (a, b, c) = (ta, t^2b, c)$$

then $t.(a, b, c) = (ta, t^2b, c) = (t, t^2, 1).(a, b, c) = \imath(t)(a, b, c)$, where the inclusion map $\imath : \mathbb{C}^* \longrightarrow (\mathbb{C}^*)^3$ is given by $\imath(t) = (t, t^2, 1)$. We are going to calculate $\imath^*(\lambda_i)$, i=0,1,2. Let $\chi \in M(\mathbb{C}^*)$ where $\chi(t) = t$. Then the map $\imath$ defined above induces the maps $\imath^* : H^2(B(\mathbb{C}^*)^3) \longrightarrow H^2(B(\mathbb{C}^*))$, $\imath_1^* : M((\mathbb{C}^*)^3) \longrightarrow M(\mathbb{C}^*)$ where $\imath_1^*(\rho) = \rho \circ \imath$ for any character $\rho \in M((\mathbb{C}^*)^3)$, and $\imath^*(\lambda_k) = b_k\lambda$ where $b_k \in \mathbb{Z}$. Recall that $H^2(B(\mathbb{C}^*)^n) = \mathbb{Z}[\lambda_1, ..., \lambda_n]$. Also recall that the map $\psi_n : M((\mathbb{C}^*)^n) \longrightarrow H^2(B(\mathbb{C}^*)^n)$ is an isomorphism where $\psi_n(\beta) = -c_1(L_\beta)$, β is a character in $M((\mathbb{C}^*)^n)$. Clearly $\imath^* \circ \psi_2 = \psi_1 \circ \imath_1^*$.

(1) To calculate $\imath^*(\lambda_1)$. Let $\chi_1(t_1, t_2, t_3) = t_1$. Since $\imath^* \circ \psi_2 = \psi_1 \circ \imath_1^*$ then $\imath^* \circ \psi_2(\chi_1) = \psi_1 \circ \imath_1^*(\chi_1)$ so $\imath^*(\psi_2(\chi_1)) = \psi_1(\imath_1^*(\chi_1))$. Therefore $\imath^*(\lambda_1) = \psi_1(\chi)$. But $\psi_1(\chi) = \lambda$. Thus $\imath^*(\lambda_1) = \lambda$.

(2) To calculate $\imath^*(\lambda_2)$. Let $\chi_2(t_1, t_2, t_3) = t_2$. Since $\imath^* \circ \psi_2 = \psi_1 \circ \imath_1^*$ then $\imath^* \circ \psi_2(\chi_2) = \psi_1 \circ \imath_1^*(\chi_2)$ so $\imath^*(\psi_2(\chi_2)) = \psi_1(\imath_1^*(\chi_2))$. Thus using the additive notation we have $\imath^*(\lambda_2) = \psi_1(2\chi) = 2\psi_1(\chi)$. Hence $\imath^*(\lambda_2) = 2\lambda$.

(3) To calculate $\imath^*(\lambda_3)$. Let $\chi_3(t_1, t_2, t_3) = t_3$. Since $\imath^* \circ \psi_2 = \psi_1 \circ \imath_1^*$ then $\imath^* \circ \psi_2(\chi_3) = \psi_1 \circ \imath_1^*(\chi_3)$ so $\imath^*(\psi_2(\chi_3)) = \psi_1(\imath_1^*(\chi_3))$. Thus $\imath^*(\lambda_3) = \psi_1(1) = 0$. Hence $\imath^*(\lambda_1) = 0$.

3 A FORMULA FOR ORDINARY INTEGRALS

The notation of equivariant cohomology and localization theorem that we are about to explain is valid for any compact connected Lie group. We will only state without proof the results that will be using. The main reference here is [1]. For a detailed exposition of this subject we suggest Chapter 9 of [8]. Let $X^T = \cup X_j$ be the decomposition of the fixed point locus into its connected components. X_j is smooth for all j. Let $i_j : X_j \longrightarrow X$ be the inclusion. The normal bundle $\mathcal{N}_j$ of X_j in X is equivariant therefore it has an equivariant Euler class $Euler_T(\mathcal{N}_j)$. We will be using the following form of the localization theorem to calculate equivariant integrals.

Theorem 3.0.18. Let $\alpha \in H^*(X_T) \otimes \mathbb{C}(\lambda_0, ..., \lambda_s)$. Then

$$\int_{X_T} \alpha = \sum_{j=1}^{n} \int_{(X_j)_T} \frac{i_j^*(\alpha)}{Euler_T(\mathcal{N}_j)} \tag{12}$$

Fact 3.0.19. Let X be a smooth projective variety with a torus action such that the fixed point locus $X^T = \{p_i \in X : i = 1, ..., n\}$. There exists a localization map $\varphi_i : H^i(X) \longrightarrow H^i_T(p_i)$ such that if $\omega \in H^*(X)$ then

$$\int_X \omega = \sum_{i=1}^{n} \frac{\varphi_i(\omega)}{\prod \text{ weights of } \mathcal{T}_{p_i} X} \tag{13}$$

Fact 3.0.20. Let Z be a T-invariant submanifold of the B-variety X and let the set $X^T = \{p_j\}_{j=1}^n$ be the fixed point locus of T. Then $[Z] \in H^*(X)$ and

$$\varphi_i([Z]) = \left\{ \begin{array}{cc} \text{product of the weights of } (\mathcal{N}_{Z|X})|_{p_j} & p_j \in Z \\ 0 & p_j \notin Z \end{array} \right\}$$

In this study we will be interested in the case $X = \mathbb{P}^s$. Let $\chi_0, \chi_1, ..., \chi_s$ be characters of the torus $T = (\mathbb{C}^*)^{s+1}$. Clearly a basis for the characters of the torus is given by $\varepsilon_i(t_0, ..., t_s) = t_i$. In terms of this basis let $\chi_i = (a_{ij})$. We will say that the weight of the character ε_i is λ_i where $\lambda_i \in H^*(BT) = \mathbb{C}[\lambda_0, ..., \lambda_n]$. Similarly the weight of the character χ_i is $\sum_j a_{ij}\lambda_j$. Let $\mathcal{O}(\chi_i) = \mathcal{O}(\sum_j a_{ij}\lambda_j)$ be a line bundle over $(\mathbb{CP}^\infty)^{s+1}$. Consider the following action of T on $\mathbb{P}^s$

$$(t_0, ..., t_s).(z_0, ..., z_s) = (\chi_0(t)z_0, ..., \chi_s(t)z_s) \tag{14}$$

Then $\mathbb{P}^s_T = \mathbb{P}(\oplus_i \mathcal{O}(\chi_i))$. Let $p = c_1(\mathcal{O}_{\mathbb{P}^s_T}(1))$. Then

$$H^*_T(\mathbb{P}^s) \simeq \mathbb{C}[\lambda_0, ..., \lambda_s, p] / \prod_i (p - \sum_j a_{ij}\lambda_j) \tag{15}$$

as rings. We will be interested in the case $\chi_i = \varepsilon_i$. For the corresponding T-action we have

$$H_T^*(\mathbb{P}^s) = \mathbb{C}[\lambda_0, ..., \lambda_s, p] / \prod_j (p - \lambda_j) \tag{16}$$

Let us see what the localization theorem says in this case. The locus of the fixed points consists of points p_j for $j = 0, 1, .., s$ where p_j is the point whose j-th coordinate is 1 and all other ones are 0. Let $\phi_j = \prod_{k \neq j}(p - \lambda_k)$ for $j = 0, 1, ..., s$. Then for $\alpha, \beta \in H^*(\mathbb{P}_T^s) \otimes_{\mathbb{C}} \mathbb{C}(\lambda_0, ..., \lambda_s)$ we have

$$\alpha = \beta \Leftrightarrow \int_{\mathbb{P}_T^s} \alpha \cup \phi_k = \int_{\mathbb{P}_T^s} \beta \cup \phi_k \text{ for all k} \tag{17}$$

Also $i_j^*(\phi_j) = \prod_{k \neq j}(\lambda_j - \lambda_k) = Euler_T(\mathcal{N}_j)$. The localization theorem says that for any polynomial $G(p) \in \mathbb{C}(\lambda_0, ..., \lambda_s)[p]$ we have

$$\int_{\mathbb{P}_T^s} G(p) = \sum_{k=1}^{n} \frac{G(\lambda_j)}{\prod_{k \neq j}(\lambda_j - \lambda_k)} \tag{18}$$

Definition 3.0.21. Let X be a B-variety, and Let X^T be the fixed point locus of the torus $T = \mathbb{C}^*$. Then

(a) If $Y \subset X$ is a $T-$invariant submanifold, $\mathcal{N}_{Y|X}$ is the normal bundle of Y in X. Let λ be the weight of the character $\chi(t) = t$. We define the map $f_Y : X^T \longrightarrow \mathbb{Z}$ that corresponds to Y as follows

$$f_Y(z) = \left\{ \begin{array}{cc} \frac{\text{product of the weights of } (\mathcal{N}_{Y|X})|_z}{\lambda^r} & z \in Y \\ 0 & z \notin Y \end{array} \right\}$$

where r is the codimension of Y in X.

(b) Let E be a vector bundle on a smooth projective variety X, we define the function $g_i : X^T \longrightarrow \mathbb{Z}$ as follows: let $\{b_i : b_i \in \mathbb{Z}\}_{i=1}^n$ be the weights of T on $E|_z$. Define $g_i(z) = c_i(\mathcal{O}(b_0) \oplus ... \oplus \mathcal{O}(b_n))$. It follows $g_i(z) := \sigma_i$(weights of T on $E|_z$) where σ_i be the i-th elementary symmetric function. If $X^T = \{p_i\}_{i=1}^n$ then we set $g_i = (g_{i1}, ..., g_{in})$ where $g_{ij} = g_i(p_j)$, $j = 1, ..., n$.

Theorem 3.0.22. Let X be a B-variety, and let $X^T = \{p_i\}_{i=1}^n$ be the fixed point locus of T where T is a one dimensional torus. Let E_r be a vector bundle on X with a T-action, $r = 1, ..., \mu$, and let $Y_i \subset X$ be a a $T-$invariant subvariety, $i = 1, ..., \nu$. For $r = 1, .., \mu$, let $g^T_{a_r} = (g^T_{a_r 1}, ..., g^T_{a_r n})$ be the function on X^T that corresponds to the chern class $c^T_{a_r}(E_r)$, where $g^T_{a_r k} = \sigma_{a_r}$(weights of T on $(E_r|_{p_k})_T$). For $i = 1, .., \nu$, let $f^T_{Y_i} = (f^T_{Y_i 1}, ..., f^T_{Y_i n})$ be the function that corresponds to $[Y_{iT}]$, where $f^T_{Y_i k}$ is equal to the product of the weights of $((\mathcal{N}_{Y_i|X})_{p_k})_T$. Consider the polynomial

$$P(x_1, .., x_\mu, y_1, .., y_\nu) = \sum_{\sum_{k=1}^{\mu} i_k + \sum_{r=1}^{\nu} j_r = u} a_{\{\sum_{k=1}^{\mu} i_k + \sum_{r=1}^{\nu} j_r = u\}} (\prod_{k=1}^{\mu} x_k^{i_k})(\prod_{r=1}^{\nu} y_r^{j_r}) \quad (19)$$

of degree equal to u where u is the dimension of X. Let $P(c_{a_r}(E_r), Y_i)$ denote the polynomial of chern classes and subvarieties $P(c_{a_1}(E_1), ..., c_{a_r}(E_r), [Y_1], ..., [Y_\nu])$, then

$$\int_X P(c_{a_r}(E_r), [Y_i]) = \sum_{k=1}^{n} \frac{P(g_{a_r}, f_{Y_i})}{\prod_{i=1}^{u} \beta_i^k} \quad (20)$$

where dim X $= u$, $\lambda\beta_1^k, ..., \lambda\beta_u^k$ are the weights of the tangent space $T_{p_k}X$ such that $\beta_i^k \in \mathbb{Z}$, $i = 1, ..., u$, $k = 1, ..., n$.

Proof. Suppose that we have a vector bundle E with a T-action over a smooth variety X with a T-action($T = \mathbb{C}^*$). Let $X^T = \{p_1, ..., p_n\}$ be the fixed point locus of T. For each $p_j \in X^T$ the restriction $(E|_{p_j})_T$ decomposes into characters of T, say $\chi_j^1, .., \chi_j^s$. So $(E|_{p_j})_T = \oplus_{k=1}^s (\mathbb{C}_{\chi_j^k})_T$. If $L_{\chi_j^k} = (\mathbb{C}_{\chi_j^k})_T$ then $(E|_{p_j})_T = \oplus_{k=1}^s L_{\chi_j^k}$. Let $\chi(t) = t$, and let λ denote the weight of the character χ. Then using the additive notation for characters (i.e, $(\beta\chi + \gamma\chi)(t) = \chi^\beta(t).\chi^\gamma(t)$) we have $\chi_j^k = \alpha_j^k\chi$ where $\alpha_j^k \in \mathbb{Z}$, $k = 1, ..., s$. But $L_{\chi_j^k} = L_{\chi ... \chi} = L_\chi \otimes ... \otimes L_\chi = \mathcal{O}(-\lambda) \otimes ... \otimes \mathcal{O}(-\lambda) = \mathcal{O}(-\alpha_j^k\lambda)$. It follows (weight of χ_j^k) = (weight of $\alpha_j^k\chi$) $= -c_1(L_{\alpha_j^k\chi}) = -c_1(\mathcal{O}(-\alpha_j^k\lambda)) = \alpha_j^k\lambda$. So the action of T on $(E|p_j)_T$ has weights $\alpha_j^1\lambda, ..., \alpha_j^s\lambda$ and $(E|p_j)_T = \oplus_{k=1}^s \mathcal{O}(-\alpha_j^k\lambda)$. Let $i_j : p_j \longrightarrow X$ be the inclusion map. Then the map i_j induces the map i_{jT}^* : $H_T^*(X) \longrightarrow H_T^*(p_j)$. Therefore

$$i_{jT}^*(c_k^T(E)) = c_k^T(E|_{p_j}) = \sigma_k(\alpha_j^1\lambda, ..., \alpha_j^s\lambda) = \lambda^k\sigma_k(\alpha_j^1, ..., \alpha_j^s) \tag{21}$$

Now let us check that $c_k^T(E|_{p_j}) = \sigma_k(\alpha_j^1\lambda, ..., \alpha_j^s\lambda)$. Consider the fiber bundle

$$(E|_{p_j})_T = \oplus_{k=1}^s \mathcal{O}(-\alpha_j^k\lambda).$$

For simplicity let $V = E|_{p_j}$. Now if $c(V)$ is the total chern class of V then

$$\begin{aligned} c(V_T) = c(\oplus_{k=1}^s \mathcal{O}(-\alpha_j^k\lambda)) &= \prod_{k=1}^s c(\mathcal{O}(-\alpha_j^k\lambda)) \\ &= \prod_{k=1}^s (1 - \alpha_j^k\lambda) \\ &= \sum_{j=0}^n \sigma_j(-\alpha_j^1\lambda, ..., -\alpha_j^s\lambda)) \end{aligned}$$

Let $P(x_1, .., x_\mu, y_1, .., y_\nu) = \sum_{r=1}^\mu \sum_{i=1}^\nu b_{ri} x_r^{m_r} y_i^{n_i}$. Consider the fiber diagram

$$\begin{array}{ccc} X & \longrightarrow & pt \\ \downarrow{\scriptstyle i_X} & & \downarrow{\scriptstyle i_{pt}} \\ X_T & \longrightarrow & BT \end{array} \tag{22}$$

This implies $i_{pt}^* \circ \int_{X_T} = \int_X \circ i_X^*$. It follows that

$$i_{pt}^* \int_{X_T} P(c_{a_r}^T(E_r), [Y_i]_T) = \int_X i_X^* P(c_{a_r}^T(E_r), [Y_i]_T) = \int_X P(c_{a_r}(E_r), [Y_i]) \tag{23}$$

Let us check that $i_X^* c_{a_r}^T(E_r) = c_{a_r}(E_r)$: Consider the commutative diagram

$$\begin{array}{ccc} E_r & \hookrightarrow & (E_r)_T \\ \downarrow & & \downarrow \\ X & \overset{i_X}{\hookrightarrow} & X_T \end{array} \tag{24}$$

This implies $i_X^*(E_r)_T = E_r$. Now $i_X^*(c_{a_r}^T(E_r)) = i_X^*(c_{a_r}(E_r)_T) = c_{a_r}(i_X^*(E_r)_T) = c_{a_r}(E_r)$. Let $\mathcal{N}_k = T_{p_k}X$ then by theorem 3.0.22 we have

$$\int_{X_T} P(c_{a_r}^T(E_r), [Y_i]_T) = \sum_{k=1}^{n} \frac{i_{kT}^* P(c_{a_r}^T(E_r), [Y_i]_T)}{Euler_T(\mathcal{N}_k)} \tag{25}$$

Now since the rank of the bundle $\mathcal{N}_k = dim$ X $= u$, it follows that

$$Euler_T(\mathcal{N}_k) = c_u^T(\mathcal{N}_k) = \sigma_u(\lambda\beta_1^k, ..., \lambda\beta_u^k) = \lambda^u \sigma_u(\beta_1^k, ..., \beta_u^k) = \lambda^u \prod_{t=1}^{u} \beta_t^k \tag{26}$$

Let $\mathcal{I} = \int_X P(c_{a_r}(E_r), [Y_i])$. If $\lambda\alpha_j^1, ..., \lambda\alpha_j^s$ are the weights of the torus T on $(E_r|_{p_j})_T$, $\lambda\delta_{j1}, ..., \lambda\delta_{jq_i}$ are the weights of T on $((\mathcal{N}_{Y_i|X})|_{p_j})_T$ where $q_i = \text{codim}Y_i$. Let $\mathcal{I} = \int_X P(c_{a_r}(E_r), [Y_i])$ then

$$\begin{aligned}
\mathcal{I} &= i_{pt}^* \int_{X_T} P(c_{a_r}^T(E_r), [Y_i]_T) \\
&= i_{pt}^* \sum_{k=1}^{n} \frac{i_{kT}^* P(c_{a_r}^T(E_r), [Y_i]_T)}{\lambda^u \prod_{t=1}^{u} \beta_t^k} \\
&= i_{point}^* \sum_{k=1}^{n} \frac{P(i_{kT}^* c_{a_r}^T(E_r), i_{kT}^*[Y_i]_T)}{\lambda^u \prod_{t=1}^{u} \beta_t^k} \\
&= i_{pt}^* \sum_{k=1}^{n} \frac{P(c_{a_r}(E_r|_{p_k}), i_{kT}^*[Y_i]_T)}{\lambda^u \prod_{t=1}^{u} \beta_t^k} \\
&= i_{pt}^* \sum_{k=1}^{n} \frac{P(\lambda^{a_r}\sigma_{a_r}(\alpha_k^{r_1}, ..., \alpha_k^{r_s}), \text{product of the weights of}(\mathcal{N}_{Y_i|X})_{p_k})}{\lambda^u \prod_{t=1}^{u} \beta_t^k} \\
&= i_{pt}^* \sum_{k=1}^{n} \frac{P(\lambda^{a_r}\sigma_{a_r}(\alpha_k^{r_1}, ..., \alpha_k^{r_s}), \lambda^{q_i} \prod_{l=1}^{q_i} \delta_{kl})}{\lambda^u \prod_{t=1}^{u} \beta_t^k}
\end{aligned}$$

Let $\mathcal{Q} = \frac{P(\lambda^{a_r}\sigma_{a_r}(\alpha_k^{r_1},...,\alpha_k^{r_s}),\lambda^{q_i}\prod_{l=1}^{q_i}\delta_{kl})}{\lambda^u\prod_{t=1}^{u}\beta_t^k}$ and let $a_{\{...\}}$ denote $a_{\{\sum_{k=1}^{\mu} i_k+\sum_{r=1}^{\nu} j_r=u\}}$. It follows that

$$\begin{aligned}
\mathcal{Q} &= \sum_{\sum_{k=1}^{\mu} i_k+\sum_{r=1}^{\nu} j_r=u} a_{\{...\}}\frac{\prod_{k=1}^{\mu}(\lambda^{a_r}\sigma_{a_r}(\alpha_k^{r_1},...,\alpha_k^{r_s}))^{i_k}\bullet\prod_{r=1}^{\nu}(\lambda^{q_i}\prod_{l=1}^{q_i}\delta_{kl})^{j_r}}{\lambda^u\prod_{t=1}^{u}\beta_t^k}\\
&= \sum_{\sum_{k=1}^{\mu} i_k+\sum_{r=1}^{\nu} j_r=u} a_{\{...\}}\frac{\lambda^{\mu a_r}\prod_{k=1}^{\mu}(\sigma_{a_r}(\alpha_k^{r_1},...,\alpha_k^{r_s}))^{i_k}\bullet\lambda^{\nu q_i}\prod_{r=1}^{\nu}(\prod_{l=1}^{q_i}\delta_{kl})^{j_r}}{\lambda^u\prod_{t=1}^{u}\beta_t^k}\\
&= \sum_{\sum_{k=1}^{\mu} i_k+\sum_{r=1}^{\nu} j_r=u} a_{\{...\}}\lambda^{\mu a_r+\nu q_i-u}\frac{\prod_{k=1}^{\mu}(\sigma_{a_r}(\alpha_k^{r_1},...,\alpha_k^{r_s}))^{i_k}\bullet\prod_{r=1}^{\nu}(\prod_{l=1}^{q_i}\delta_{kl})^{j_r}}{\prod_{t=1}^{u}\beta_t^k}
\end{aligned}$$

Therefore

$$\mathcal{I} = i_{pt}^*\sum_{k=1}^{n}\sum_{\sum_{k=1}^{\mu} i_k+\sum_{r=1}^{\nu} j_r=u} a_{\{...\}}\lambda^{\mu a_r+\nu q_i-u}\frac{\prod_{k=1}^{\mu}(\sigma_{a_r}(\alpha_k^{r_1},...,\alpha_k^{r_s}))^{i_k}\bullet\prod_{r=1}^{\nu}(\prod_{l=1}^{q_i}\delta_{kl})^{j_r}}{\prod_{t=1}^{u}\beta_t^k} \tag{27}$$

First note that $\mu a_r + \nu q_i - u$ can not be greater than zero because the degree of the polynomial P is equal to u. It follows that

$$\begin{aligned}
\mathcal{I} &= 0.i_{point}^*\Big(\sum_{k=1}^{n}\sum_{r=1}^{\mu}\sum_{i=1}^{\nu} b_{r_i}\frac{(\sigma_{a_r}(\alpha_k^{r_1},...,\alpha_k^{r_s}))^{m_r}(\prod_{i=1}^{q}\delta_i^k)^{n_i}}{\prod_{t=1}^{u}\beta_t^k}\Big)\\
&= 0.
\end{aligned}$$

If $\mu a_r+\nu q_i-u < 0$, then every term in the polynomial P, which is a product of classes, has dimension less than the dimension of X and greater than zero. But the integral $\int_X$ is the pushforward map π_* where $\pi : X_T \longrightarrow pt$. Now since the pushforward map π_* preserves dimension, it follows that $\int_X P = 0$. Let $\sum_{\{...\}}$ denote $\sum_{\sum_{k=1}^{\mu} i_k+\sum_{r=1}^{\nu} j_r=u}$.

If $\mu a_r + \nu q_i - u = 0$, then

$$\begin{aligned}
\mathcal{I} &= i^*_{point}\Big(\sum_{k=1}^{n}\sum_{\{...\}} a_{\{...\}}\lambda^{\mu a_r+\nu q_i-u}\frac{\prod_{k=1}^{\mu}(\sigma_{a_r}(\alpha_k^{r_1},...,\alpha_k^{r_s}))^{i_k}\bullet\prod_{r=1}^{\nu}(\prod_{l=1}^{q_i}\delta_{kl})^{j_r}}{\prod_{t=1}^{u}\beta_t^k}\Big)\\
&= i^*_{point}\Big(\sum_{k=1}^{n}\sum_{\{...\}} a_{\{...\}}\frac{\prod_{k=1}^{\mu}(\sigma_{a_r}(\alpha_k^{r_1},...,\alpha_k^{r_s}))^{i_k}\bullet\prod_{r=1}^{\nu}(\prod_{l=1}^{q_i}\delta_{kl})^{j_r}}{\prod_{t=1}^{u}\beta_t^k}\Big)\\
&= i^*_{point}\sum_{k=1}^{n}\frac{P(\sigma_{a_r}(\alpha_k^{r_1},...,\alpha_k^{r_s}),\prod_{l=1}^{q_i}\delta_{kl})}{\prod_{t=1}^{u}\beta_t^k}\\
&= \sum_{k=1}^{n} i^*_{point}\Big(\frac{P(\sigma_{a_r}(\alpha_k^{r_1},...,\alpha_k^{r_s}),\prod_{l=1}^{q_i}\delta_{kl})}{\prod_{t=1}^{u}\beta_t^k}\Big)
\end{aligned}$$

Note that $\alpha_k^{r_i} \in \mathbb{Z}$, so $\sigma_{a_r}(\alpha_k^{r_1},...,\alpha_k^{r_s}) \in \mathbb{Z}$. Also $\delta_{kl} \in \mathbb{Z}$ implies $\prod_{l=1}^{q_i}\delta_{kl} \in \mathbb{Z}$. Thus $P(\sigma_{a_r}(\alpha_k^{r_1},...,\alpha_k^{r_s}),\prod_{l=1}^{q_i}\delta_{kl}) \in \mathbb{Z}$. But $\beta_t^k \in \mathbb{Z}$. It follows $\frac{P(\sigma_{a_r}(\alpha_k^{r_1},...,\alpha_k^{r_s}),\prod_{l=1}^{q_i}\delta_{kl})}{\prod_{t=1}^{u}\beta_t^k} \in \mathbb{Q}$. Therefore $i^*_{point}(\frac{P(\sigma_{a_r}(\alpha_k^{r_1},...,\alpha_k^{r_s}),\prod_{l=1}^{q_i}\delta_{kl})}{\prod_{t=1}^{u}\beta_t^k}) = \frac{P(\sigma_{a_r}(\alpha_k^{r_1},...,\alpha_k^{r_s}),\prod_{l=1}^{q_i}\delta_{kl})}{\prod_{t=1}^{u}\beta_t^k}$. It follows that

$$\mathcal{I} = \sum_{k=1}^{n}\frac{P(\sigma_{a_r}(\alpha_k^{r_1},...,\alpha_k^{r_s}),\prod_{l=1}^{q_i}\delta_{kl})}{\prod_{t=1}^{u}\beta_t^k} \tag{28}$$

Now since $\sigma_{a_r}(\alpha_k^{r_1},...,\alpha_k^{r_s}) = \sigma_{a_r}(\text{ weights of T on } E_r|_{p_j}) = g^T_{a_r j}$, $f^T_{Y_i k}$ = product of the weights of $((\mathcal{N}_{Y_i|X})_{p_k})_T$. It follows that

$$\mathcal{I} = \sum_{k=1}^{n}\frac{P(g_{a_r}, f_{Y_i})}{\prod_{t=1}^{u}\beta_t^k} \tag{29}$$

4 Examples

Example 4.0.23. Consider the action of $T = \mathbb{C}^*$ on $\mathbb{P}^3$ given by

$$t.(x_0,x_1,x_2,x_3) = (t^{-a_0}x_0, t^{-a_1}x_1, t^{-a_2}x_2, t^{-a_3}x_3)$$

where all the a_i's are distinct and non zero. Let $L = Z(x_2,x_3)$=$\{(x_0,x_1,x_2,x_3) \in \mathbb{P}^3 : x_2 = x_3 = 0\}$ be an algebraic subvariety of $\mathbb{P}^3$, and let $c_1(\mathcal{O}_{\mathbb{P}^3}(1))$ be the first chern class of the vector bundle $\mathcal{O}_{\mathbb{P}^3}(1)$. In this example we are going to calculate

$$\int_{\mathbb{P}^3} c_1(\mathcal{O}_{\mathbb{P}^3}(1)).L$$

Note that the fix point locus of the T-action on $\mathbb{P}^3 = \{p_0, ..., p_3\}$ where the i^{th} coordinate of p_i is nonzero, all other coordinates being 0. First we need to calculate the weights of the bundle $(\mathcal{N}_{L|_{\mathbb{P}^3}})_{p_0}$. Consider the exact sequence

$$0 \longrightarrow \mathcal{T}L \longrightarrow \mathcal{T}\mathbb{P}^3|_L \longrightarrow \mathcal{N}_{L|_{\mathbb{P}^3}} \longrightarrow 0$$

which induces the exact sequence

$$0 \longrightarrow \mathcal{T}_{p_0}L \longrightarrow \mathcal{T}_{p_0}\mathbb{P}^3|_L \longrightarrow (\mathcal{N}_{L|_{\mathbb{P}^3}})_{p_0} \longrightarrow 0$$

It follows that $(\mathcal{N}_{L|_{\mathbb{P}^3}})_{p_0} = (\mathcal{T}_{p_0}\mathbb{P}^3|_L)/(\mathcal{T}_{p_0}L)$. This is a two dimensional vector space with basis $\{\frac{\partial}{\partial(\frac{x_2}{x_0})}, \frac{\partial}{\partial(\frac{x_3}{x_0})}\}$. We will calculate the weights of the T-action on this basis. Since

$$t.\frac{\partial}{\partial(\frac{x_2}{x_0})} = \frac{\partial}{\partial(\frac{t^{a_2}x_2}{t^{a_0}x_0})} = t^{a_0-a_2}\frac{\partial}{\partial(\frac{x_2}{x_0})} \tag{30}$$

this implies that the weight of the T-action on the basis element $\frac{\partial}{\partial(\frac{x_2}{x_0})}$ is equal to $a_0 - a_2$. Similarly the weight of the T-action on the basis element $\frac{\partial}{\partial(\frac{x_3}{x_0})}$ is equal to $a_0 - a_3$. Therefore the weights of the torus action on the basis above are $a_0 - a_2, a_0 - a_3$. Now by part (a) of definition 3.0.21 above it follows that

$$f_L(p_0) = (a_0 - a_2)(a_0 - a_3) \tag{31}$$

Similarly since $\{\frac{\partial}{\partial(\frac{x_2}{x_1})}, \frac{\partial}{\partial(\frac{x_3}{x_1})}\}$ is a basis for $(\mathcal{N}_{L|_{\mathbb{P}^3}})_{p_1}$ we have

$$f_L(p_1) = (a_1 - a_2)(a_1 - a_3) \tag{32}$$

Note that $f_L(p_2) = 0$ since $p_2 \notin L$. Also $f_L(p_3) = 0$ because $p_3 \notin L$. Let $f_L = (f_L^0, f_L^1, f_L^2, f_L^3)$. Since $f_L^i = (a_i - a_2)(a_i - a_3)$, $i = 0, 1, 2, 3$ we have

$$f_L = ((a_0 - a_2)(a_0 - a_3), (a_1 - a_2)(a_1 - a_3), 0, 0) \tag{33}$$

Now we are going to find the map $g_1 : X^T \longrightarrow \mathbb{Z}$ that corresponds to the chern class $c_1(\mathcal{O}_{\mathbb{P}^3}(1))$. Clearly $g_1(p_k) = \sigma_1(\text{weights of } T \text{ on } (\mathcal{O}_{\mathbb{P}^3}(1))|_{p_k})$ by part (b) of definition 3.0.21. Note that $\Gamma(\mathcal{O}_{\mathbb{P}^3}(1))$ =Span$\{x_i$: $i = 0, 1, 2, 3\}$. So $\Gamma(\mathcal{O}_{\mathbb{P}^3}(1) \mid_{p_0}) =$ Span$\{x_0\}$ and $t.x_0 = t^{a_0}x_0$. It follows that $g_1(p_0) = a_0$. Similarly the weight of the torus action on $(\mathcal{O}_{\mathbb{P}^3}(1))|_{p_i} = a_i$, $i = 1, 2, 3$. Let $g_1 = (g_1^0, g_1^1, g_1^2, g_1^3)$ where $g_1^i = g_1(p_i)$, $i = 0, 1, 2, 3$ then $g_1 = (a_0, a_1, a_2, a_3)$. Let $\mathcal{I} = \int_{\mathbb{P}^3} c_1.L$ then by *theorem* 3.0.22 it follows that

$$\begin{aligned}
\mathcal{I} &= \sum_{i=0}^{3} \frac{f_L^i.g_1^i}{\text{product of the weights of } \mathcal{T}_{p_i}X} \\
&= \frac{a_0(a_0-a_2)(a_0-a_3)}{(a_0-a_1)(a_0-a_2)(a_0-a_3)} + \frac{a_1(a_1-a_2)(a_1-a_3)}{(a_1-a_0)(a_1-a_2)(a_1-a_3)} + 0 \\
&= \frac{a_0}{a_0-a_1} + \frac{a_1}{a_1-a_0} \\
&= 1.
\end{aligned}$$

Example 4.0.24. Consider the action of $T = \mathbb{C}^*$ on $\mathbb{P}^4$ given by

$$t.(x_0, x_1, x_2, x_3, x_4) = (t^{-a_0}x_0, t^{-a_1}x_1, t^{-a_2}x_2, t^{-a_3}x_3, t^{-a_4}x_4)$$

where all the a_i's are distinct and non zero. Let $L = Z(x_2, x_3)$=$\{(x_0, x_1, x_2, x_3, x_4) \in \mathbb{P}^4 : x_2 = x_3 = 0\}$ be an algebraic subvariety of $\mathbb{P}^4$, and let $c_2(\mathcal{T}_{\mathbb{P}^4})$ be the second chern class of the tangent bundle $\mathcal{T}_{\mathbb{P}^4}$. In this example we are going to calculate

$$\int_{\mathbb{P}^4} c_2(\mathcal{T}_{\mathbb{P}^4}).L$$

The fix point locus of the T-action on $\mathbb{P}^4 = \{p_0, ..., p_4\}$ where the i^{th} coordinate of p_i is nonzero, all other coordinates being 0. First we need to calculate the weights of $(\mathcal{N}_{L|_{\mathbb{P}^4}})_{p_0}$. Consider the exact sequence

$$0 \longrightarrow \mathcal{T}L \longrightarrow \mathcal{T}\mathbb{P}^4|_L \longrightarrow \mathcal{N}_{L|_{\mathbb{P}^4}} \longrightarrow 0$$

which induces the exact sequence

$$0 \longrightarrow \mathcal{T}_{p_0}L \longrightarrow \mathcal{T}_{p_0}\mathbb{P}^4|_L \longrightarrow (\mathcal{N}_{L|_{\mathbb{P}^4}})_{p_0} \longrightarrow 0$$

It follows that $(\mathcal{N}_{L|_{\mathbb{P}^4}})_{p_0} = (\mathcal{T}_{p_0}\mathbb{P}^4|_L)/(\mathcal{T}_{p_0}L)$. This is a 2-dimensional vector space with basis $\{\frac{\partial}{\partial(\frac{x_2}{x_0})}, \frac{\partial}{\partial(\frac{x_3}{x_0})}\}$. We will calculate the weights of the T-action on this basis. Since

$$t.\frac{\partial}{\partial(\frac{x_2}{x_0})} = \frac{\partial}{\partial(\frac{t^{a_2}x_2}{t^{a_0}x_0})} = t^{a_0-a_2}\frac{\partial}{\partial(\frac{x_2}{x_0})} \tag{34}$$

this implies that the weight of the T-action on the basis element $\frac{\partial}{\partial(\frac{x_2}{x_0})}$ is equal to $a_0 - a_2$. Similarly the weight of the T-action on the basis element $\frac{\partial}{\partial(\frac{x_3}{x_0})}$ is equal to $a_0 - a_3$. Therefore the weights of the T-action on the basis above are $a_0 - a_2, a_0 - a_3$. Now by part (a) of definition 3.0.21 above it follows that

$$f_L(p_0) = (a_0 - a_2)(a_0 - a_3) \tag{35}$$

Since $\{\frac{\partial}{\partial(\frac{x_2}{x_1})}, \frac{\partial}{\partial(\frac{x_3}{x_1})}\}$ is a basis for $(\mathcal{N}_{L|_{\mathbb{P}^4}})_{p_1}$ it follows that

$$f_L(p_1) = (a_1 - a_2)(a_1 - a_3) \tag{36}$$

Similarly $f_L(p_4) = (a_4 - a_2)(a_4 - a_3)$. Note that $f_L(p_2) = 0$ since $p_2 \notin L$. Also $f_L(p_3) = 0$ because $p_3 \notin L$. Let $f_L = (f_L^0, f_L^1, f_L^2, f_L^3)$. Since f_L^i=$(a_i - a_2)(a_i - a_3)$, $i = 0, 1, 2, 3, 4$. Then

$$f_L = ((a_0 - a_2)(a_0 - a_3), (a_1 - a_2)(a_1 - a_3), 0, 0, (a_4 - a_2)(a_4 - a_3)) \tag{37}$$

Now we are going to find the map $g_2 : X^T \longrightarrow \mathbb{Z}$ that corresponds to the chern class $c_2(\mathcal{T}_{\mathbb{P}^4})$. Clearly, $g_2(p_k)$=σ_2(weights of T on $\mathcal{T}_{p_k}\mathbb{P}^4$) by part (b) of definition 3.0.21. Now let us find $g_2(p_0)$. Consider $(\mathcal{T}\mathbb{P}^4)|_{p_0}$=$\mathcal{T}_{p_0}\mathbb{P}^4$ which is a four dimensional vector space with basis

$$\{\frac{\partial}{\partial(\frac{x_1}{x_0})}, \frac{\partial}{\partial(\frac{x_2}{x_0})}, \frac{\partial}{\partial(\frac{x_3}{x_0})}, \frac{\partial}{\partial(\frac{x_4}{x_0})}\}$$

Since $t.\frac{\partial}{\partial(\frac{x_1}{x_0})} = \frac{\partial}{\partial(\frac{t^{a_1}x_1}{t^{a_0}x_0})} = t^{a_0-a_1}\frac{\partial}{\partial(\frac{x_1}{x_0})}$ then the weight of the T-action on the basis element $\frac{\partial}{\partial(\frac{x_1}{x_0})} = a_0 - a_1$. Similarly the weight of the T-action on the basis element $\frac{\partial}{\partial(\frac{x_2}{x_0})} = a_0 - a_2$. Thus the weight of the T-action on $\mathcal{T}_{p_0}\mathbb{P}^4$ are $a_0 - a_1, a_0 - a_2, a_0 - a_3, a_0 - a_4$. It follows that $g_2(p_0) = \sigma_2(a_0 - a_1, a_0 - a_2, a_0 - a_3, a_0 - a_4)$. Consider the tangent space $(\mathcal{T}_{\mathbb{P}^4})|_{p_1}=\mathcal{T}_{p_1}\mathbb{P}^4$ which is a four dimensional vector space with basis

$$\{\frac{\partial}{\partial(\frac{x_0}{x_1})}, \frac{\partial}{\partial(\frac{x_2}{x_1})}, \frac{\partial}{\partial(\frac{x_3}{x_1})}, \frac{\partial}{\partial(\frac{x_4}{x_1})}\}$$

As above we have $g_2(p_1)=\sigma_2(a_1 - a_0, a_1 - a_2, a_1 - a_3, a_1 - a_4)$. Similarly we have $g_2(p_2)=\sigma_2(a_2 - a_0, ..., a_2 - a_4)$, $g_2(p_3)=\sigma_2(a_3 - a_0, ..., a_3 - a_4)$, and $g_2(p_4) = \sigma_2(a_4 - a_0, ..., a_4 - a_3)$. Let $g_2 = (g_2^0, g_2^1, g_2^2, g_2^3, g_2^4)$ where $g_2^i=g_2(p_i)$, $i = 0, 1, 2, 3, 4$. Let $\mathcal{I} = \int_{\mathbb{P}^4} c_2.L$ then by theorem 3.0.22 it follows that

$$\begin{aligned}
\mathcal{I} &= \sum_{i=0}^{4} \frac{f_L^i.g_2^i}{\text{product of the weights of } \mathcal{T}_{p_i}X} \\
&= \frac{(\sigma_2(a_0 - a_1, a_0 - a_2, a_0 - a_3, a_0 - a_4)).(a_0 - a_2)(a_0 - a_3)}{(a_0 - a_1)(a_0 - a_2)(a_0 - a_3)(a_0 - a_4)} \\
&+ \frac{(\sigma_2(a_1 - a_0, a_1 - a_2, a_1 - a_3, a_1 - a_4)).(a_1 - a_2)(a_1 - a_3)}{(a_1 - a_0)(a_1 - a_2)(a_1 - a_3)((a_1 - a_4))} \\
&+ 0 + 0 \\
&+ \frac{(\sigma_2(a_4 - a_0, a_4 - a_1, a_4 - a_2, a_4 - a_3)).(a_4 - a_2)(a_4 - a_3)}{(a_4 - a_0)(a_4 - a_1)(a_4 - a_2)(a_4 - a_3)} \\
&= 10.
\end{aligned}$$

Example 4.0.25. Consider the action of $T = \mathbb{C}^*$ on $\mathbb{P}^4$ given by

$$t.(x_0, x_1, x_2, x_3, x_4) = (t^{-a_0}x_0, t^{-a_1}x_1, t^{-a_2}x_2, t^{-a_3}x_3, t^{-a_4}x_4) \tag{38}$$

where all the a_i's are distinct and non zero. Let $c_1(\mathcal{T}_{\mathbb{P}^4}), c_3(\mathcal{T}_{\mathbb{P}^4})$ be the first, and third chern classes of the tangent bundle $\mathcal{T}_{\mathbb{P}^4}$. We are going to calculate

$$\int_{\mathbb{P}^4} c_1(\mathcal{T}_{\mathbb{P}^4}).c_3(\mathcal{T}_{\mathbb{P}^4})$$

Note that the fix point locus of the T-action on $\mathbb{P}^4 = \{p_0, ..., p_4\}$ where the i^{th} coordinate of p_i is nonzero, all other coordinates being 0. By part (b) of definition 3.0.21

we have $g_j(p_k) = \sigma_j(\text{weights of T on } \mathcal{T}_{p_k}\mathbb{P}^4)$, $j = 1, 3$, and $k = 0, \ldots 4$. Let us find $g_j(p_0)$, $j = 1, 3$. Consider the tangent space $(\mathcal{T}\mathbb{P}^4)|_{p_0} = \mathcal{T}_{p_0}\mathbb{P}^4$ which has the basis

$$\{\frac{\partial}{\partial(\frac{x_1}{x_0})}, \frac{\partial}{\partial(\frac{x_2}{x_0})}, \frac{\partial}{\partial(\frac{x_3}{x_0})}, \frac{\partial}{\partial(\frac{x_4}{x_0})}\}$$

Since

$$t.\frac{\partial}{\partial(\frac{x_1}{x_0})} = \frac{\partial}{\partial(\frac{t^{-a_1}x_1}{t^{-a_0}x_0})} = t^{-a_0+a_1}\frac{\partial}{\partial(\frac{x_1}{x_0})} \tag{39}$$

then the weight of the T-action on the basis element $\frac{\partial}{\partial(\frac{x_1}{x_0})}$ is equal to $a_0 - a_1$. Similarly the weight of the T-action on the basis element $\frac{\partial}{\partial(\frac{x_2}{x_0})} = a_0 - a_2$. Thus the weights of the T-action on $\mathcal{T}_{p_0}\mathbb{P}^4$ are $a_0 - a_1, a_0 - a_2, a_0 - a_3, a_0 - a_4$. It follows that

$$g_1(p_0) = \sigma_1(a_0 - a_1, \ldots, a_0 - a_4), \ g_3(p_0) = \sigma_3(a_0 - a_1, \ldots, a_0 - a_4) \tag{40}$$

Similarly we have

$$g_1(p_i) = \sigma_1(a_i - a_0, \ldots, a_i - a_4), \ g_3(p_i) = \sigma_3(a_i - a_0, \ldots, a_i - a_4) \tag{41}$$

where $i = 1, 2, 3, 4$. Let $g_1 = (g_1^0, g_1^1, g_1^2, g_1^3, g_1^4)$, $g_3 = (g_3^0, g_3^1, g_3^2, g_3^3, g_3^4)$ where $g_j^i = g_2(p_i)$, $j = 1, 2$, and $i = 0, 1, 2, 3, 4$. Let $\mathcal{I} = \int_{\mathbb{P}^4} c_1.c_3$ then by theorem 3.0.22 it follows that

$$
\begin{aligned}
\mathcal{I} &= \sum_{i=0}^{4} \frac{g_1^i . g_3^i}{\text{product of the weights of } \mathcal{T}_{p_i} X} \\
&= \frac{(\sigma_1(a_0 - a_1, ..., a_0 - a_4)).(\sigma_3(a_0 - a_1, ..., a_0 - a_4))}{(a_0 - a_1)(a_0 - a_2)(a_0 - a_3)(a_0 - a_4)} \\
&+ \frac{(\sigma_1(a_1 - a_0, ..., a_1 - a_4)).(\sigma_3(a_1 - a_0, ..., a_1 - a_4))}{(a_1 - a_0)(a_1 - a_2)(a_1 - a_3)((a_1 - a_4))} \\
&+ \frac{(\sigma_1(a_2 - a_0, ..., a_2 - a_4)).(\sigma_3(a_2 - a_0, ..., a_2 - a_4))}{(a_2 a_0)(a_2 - a_1)(a_2 - a_3)((a_2 - a_4))} \\
&+ \frac{(\sigma_1(a_3 - a_0, ..., a_3 - a_4)).(\sigma_3(a_3 - a_0, ..., a_3 - a_4))}{(a_3 - a_0)(a_3 - a_1)(a_3 - a_2)((a_3 - a_4))} \\
&+ \frac{(\sigma_1(a_4 - a_0, ..., a_4 - a_3)).(\sigma_3(a_4 - a_0, ..., a_4 - a_3))}{(a_4 - a_0)(a_4 - a_1)(-a_4 + a_2)(a_4 - a_3)} \\
&= 50.
\end{aligned}
$$

Another way to calculate $\int_{\mathbb{P}^4} c_1.c_3$ is the following. Consider the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^4} \longrightarrow \mathcal{O}_{\mathbb{P}^4}(1)^5 \longrightarrow \mathcal{T}\mathbb{P}^4 \longrightarrow 0$$

Let $c(\mathcal{O}_{\mathbb{P}^4}(1))$ be the total chern class, and let $h = c_1(\mathcal{O}_{\mathbb{P}^4}(1))$ be the hyperplane class. Using properties of chern classes we have

$$c(\mathcal{O}_{\mathbb{P}^4}(1)^5) = c(\mathcal{O}_{\mathbb{P}^4})\ c(\mathcal{T}\mathbb{P}^4) = 1\ .\ c(\mathcal{T}\mathbb{P}^4) \tag{42}$$

Therefore

$$c(\mathcal{T}\mathbb{P}^4) = c(O_{\mathbb{P}^4}(1)^5) = \prod_{i=1}^{5} c_1(O_{\mathbb{P}^4}(1)) = (1+h)^5 = 1 + 5h + 10h^2 + 10h^3 + 5h^4.$$

But $c(\mathcal{T}\mathbb{P}^4) = \sum_{i=0}^{5} c_i(\mathcal{T}\mathbb{P}^4)$, where $c_i(\mathcal{T}\mathbb{P}^4)$ is the i-th chern class. It follows that $c_1(\mathcal{T}\mathbb{P}^4) = 5h$, $c_3(\mathcal{T}\mathbb{P}^4) = 10h^3$. Therefore

$$\begin{aligned} \int_{\mathbb{P}^4} c_1.c_3 &= \int_{\mathbb{P}^4} 5h.10h^3 \\ &= \int_{\mathbb{P}^4} 50h^4 \\ &= 50 \int_{\mathbb{P}^4} h^4 \\ &= 50\ .1 \\ &= 50. \end{aligned}$$

Example 4.0.26. Consider the action of $T = \mathbb{C}^*$ on $\mathbb{P}^2$ given by

$$t.(x_0, x_1, x_2) = (t^{-a_0}x_0, t^{-a_1}x_1, t^{-a_2}x_2) \tag{43}$$

where all the a_i's are distinct and non zero. The fixed point locus of the torus action on $\mathbb{P}^2 = \{p_0, p_1, p_2\}$ where the i^{th} coordinate of p_i is nonzero, all other coordinates being 0. Recall that $\mathbb{P}^2_T = \mathbb{P}(\oplus_{i=0}^{2} \mathcal{O}_{\mathbb{P}^2_T}(-a_i\lambda))$, where λ is the weight of the character $\chi(t) = t$ which lives in the group of characters $M(\mathbb{C}^*)$. Also recall that

$$p = c_1(\mathcal{O}_{\mathbb{P}^2_T}(1)) \in H^*_T(\mathbb{P}^2) = \mathbb{C}[p, \lambda_0, \lambda_1, \lambda_2]/(\prod_{i=0}^{3} p - \lambda_i) \tag{44}$$

is the equivariant hyperplane class. As before, the map $i_{jT} : (p_j)_T \hookrightarrow \mathbb{P}^2_T$ induces the map $i^*_{jT} : H^*(\mathbb{P}^2_T) \longrightarrow H^*((p_j)_T)$. We are going to calculate

$$\int_{\mathbb{P}^2_T} \lambda.p$$

By theorem 3.0.22 it follows that

$$\int_{\mathbb{P}^2_T} \lambda.p = \sum_{j=0}^{2} \int_{(p_j)_T} \frac{i^*_{jT}(\lambda.p)}{Euler_T(N_j)} = \sum_{j=0}^{2} \frac{i^*_{jT}(\lambda.p)}{Euler_T(N_j)} \tag{45}$$

But $i_{jT}^*(\lambda.p) = i_{jT}^*(\lambda)\ i_{jT}^*(p) = \lambda\ .\ a_j\lambda = a_j\lambda^2, j = 0,1,2$. Thus

$$\int_{\mathbb{P}_T^2} \lambda.p = \sum_{j=0}^{2} \int_{(p_j)_T} \frac{i_{jT}^*(\lambda.p)}{Euler_T(N_j)} = \sum_{j=0}^{2} \frac{i_{jT}^*(\lambda.p)}{Euler_T(N_j)} = \sum_{j=0}^{2} \frac{a_j\ \lambda^2}{Euler_T(N_j)} \tag{46}$$

Recall $[p_{0T}] = [(x_1 = 0)_T].[(x_2 = 0)_T] = (p - \lambda_0)(p - \lambda_1)$, and

$$\begin{aligned}
\mathrm{Euler}_T(N_0 = \mathcal{T}_{p_0}\mathbb{P}^2) :=&\ \mathrm{Euler}_T((\mathcal{O}(p - a_1\lambda) \oplus \mathcal{O}(p - a_2\lambda))|_{(p_0)_T}) \\
=&\ \mathrm{Euler}_T((\mathcal{O}(p - a_1\lambda)|_{(p_0)_T} \oplus \mathcal{O}(p - a_2\lambda)|_{(p_0)_T}) \\
=&\ \mathrm{Euler}_T((\mathcal{O}_{(p_0)_T}(a_0\lambda - a_1\lambda) \oplus \mathcal{O}_{(p_0)_T}(a_0\lambda - a_2\lambda)) \\
=&\ c_1(\mathcal{O}_{(p_0)_T}(a_0\lambda - a_1\lambda))\ .\ c_1(\mathcal{O}_{(p_0)_T}(a_0\lambda - a_2\lambda)) \\
=&\ (a_0\lambda - a_1\lambda)(a_0\lambda - a_2\lambda) \\
=&\ (a_0 - a_1)(a_0 - a_2)\ \lambda^2
\end{aligned}$$

Similarly $Euler_T(N_1) = (a_1 - a_0)(a_1 - a_2)\lambda^2$, $Euler_T(N_2) = (a_2 - a_0)(a_2 - a_1)\ \lambda^2$. Thus

$$\begin{aligned}
\int_{\mathbb{P}_T^2} \lambda.p =&\ \sum_{j=0}^{2} \frac{a_j\ \lambda^2}{Euler_T(N_j)} \\
=&\ \frac{a_0\ \lambda^2}{(a_0 - a_1)(a_0 - a_2)\ \lambda^2} \\
+&\ \frac{a_1\ \lambda^2}{(a_1 - a_0)(a_1 - a_2)\ \lambda^2} \\
+&\ \frac{a_2\ \lambda^2}{(a_2 - a_0)(a_2 - a_1)\ \lambda^2} \\
=&\ \frac{a_0(a_1 - a_2) - a_1(a_0 - a_2) + a_2(a_0 - a_1)}{(a_0 - a_1)(a_0 - a_2)(a_1 - a_2)} \\
=&\ \frac{0}{(a_0 - a_1)(a_0 - a_2)(a_1 - a_2)} \\
=&\ 0.
\end{aligned}$$

Example 4.0.27. First recall that if E is a vector bundle of rank r on X, and $0 < d < r$, there is a Grassmann bundle, denoted $\mathrm{Grass}_d(E)$, of d-planes in E, with a projection map $\pi : \mathrm{Grass}_d(E) \longrightarrow X$, and a universal rank d subbundle $\mathcal{S}$ of π^*E; $\mathcal{S}$ is also called the tautological bundle on $\mathrm{Grass}_d(E)$. The bundle $\mathcal{Q} = (\pi^*E)/\mathcal{S}$ is called the universal quotient bundle, and the sequence below is called the universal exact sequence.

$$0 \longrightarrow \mathcal{S} \longrightarrow \pi^* E \longrightarrow \mathcal{Q} \longrightarrow 0$$

Let $\mathcal{G} = \text{Grass}_2(\mathbb{C}^4)$ be the Grassmann of 2-planes in $\mathbb{C}^4$. If $E = \mathbb{C}^4 \times X$ is the trivial bundle on X then we have the following universal exact sequence

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathbb{C}^4 \times \mathcal{G} \longrightarrow \mathcal{Q} \longrightarrow 0$$

where $\mathcal{S}$ is the tautological bundle on $\mathcal{G}$, and $(\mathbb{C}^4 \times \mathcal{G})/\mathcal{S}$ is the universal quotient bundle on $\mathcal{G}$. Moreover $T_{\mathcal{G}} = \text{Hom}(\mathcal{S}, \mathcal{Q}) = \mathcal{S}^{\vee} \otimes \mathcal{Q}$. Consider the action of $(\mathbb{C}^*)^4$ on $\mathbb{P}^3$ given by

$$(t_1, ..., t_4).(x_0, ..., x_3) = ({t_1}^{-1} x_0, ..., {t_4}^{-1} x_3) \tag{47}$$

This action induces an action on $\mathcal{G}$ with $\binom{4}{2} = 6$ fixed points p_{ij}, $i \in I = \{1, 2, 3\}$, $j \in J = \{2, 3, 4\}$ and $i < j$ where the $p_{ij} = \{(0, a_i, a_j, 0) : a_i, a_j \in \mathbb{C}\}$. Recall fact 3.0.19. Then we have the following special case.

SPECIAL CASE. The following is a special case of fact 3.0.19: If E is a vector bundle over X, and $\omega = c_i(E)$ then we have $\varphi_i(\omega)$=σ_i(weights of E at p_i), where σ_i is the i-th elementry symmetric function.

CALCULATIONS

(a) We are going to calculate $\int_{\mathcal{G}} c_1^4(\mathcal{T}_{\mathcal{G}})$. First we will calculate the weights of $\mathcal{T}_{p_{12}}\mathcal{G}$. Since p_{12} is a 2-dimensional vector subspace of $\mathbb{C}^4$ we let V denote this vector subspace. Then $\mathcal{T}_{p_{12}}\mathcal{G}$=$\mathcal{T}_V\mathcal{G}$=Hom$(V, \mathbb{C}^4/V)$=$V^{\vee} \otimes \mathbb{C}^4/V$. Let $\{e_1, ..., e_4\}$ be the standard basis for $\mathbb{C}^4$ where $e_1 = (1, 0, 0, 0), ..., e_4 = (0, 0, 0, 1)$. Then $\{e_1^{\vee}, e_2^{\vee}\}$ is a basis for the dual space $V^{\vee}$ of V where

$$e_i^{\vee}(e_j) = \left\{ \begin{array}{cc} 1 & i = j \\ 0 & i \neq j \end{array} \right\}$$

and $\{[e_3],[e_4]\}$ is a basis for $\mathbb{C}^4/V$. Let λ_i be the weight of the character $\chi_i(t_1,...,t_4)=t_i$. Now since

$$((t_1,t_2,t_3,t_4).e_1^\vee)(e_1)=e_1^\vee((t_1^{-1},t_2^{-1},t_3^{-1},t_4^{-1}).e_1)=e_1^\vee((t_1,0,0,0))=t_1e_1^\vee(e_1)$$

then the weight of the T-action on $e_1^\vee$ is equal to λ_1. Similarly the weight of the T-action on $e_2^\vee$ is equal to λ_2. Thus the weights of the T-action on the basis $\{e_1^\vee, e_2^\vee\}$ are λ_1,λ_2. A similar argument gives the weights of the T-action for the basis $\{[e_3],[e_4]\}$. Those weights are $-\lambda_3,-\lambda_4$. To see this, first note that

$$(t_1,t_2,t_3,t_4).[e_3]=[(t_1,t_2,t_3,t_4).e_3])=[(0,0,t_3^{-1},0)]=[t_3^{-1}(0,0,1,0)]=t_3^{-1}\ [e_3]$$

Similarly $(t_1,t_2,t_3,t_4).[e_4]=t_4^{-1}\ [e_4]$. Using the additive notation, the weights of the T-action on the basis $\{[e_3],[e_4]\}$ are $-\lambda_3,-\lambda_4$. Since $\{e_1^\vee\otimes[e_3], e_1^\vee\otimes[e_4], e_2^\vee\otimes[e_3], e_2^\vee\otimes[e_4]\}$ is a basis for the vector space $\mathcal{T}_V\mathcal{G}=V^\vee\otimes\mathbb{C}^4/V$. It follows easily that the weights of the T-action on the vector space $\mathcal{T}_V\mathcal{G}$ are $\lambda_1-\lambda_3,\lambda_1-\lambda_4,\lambda_2-\lambda_3,\lambda_2-\lambda_4$ because

$$(t_1,t_2,t_3,t_4).(e_i^\vee\otimes[e_j])=t_i\ e_i^\vee\otimes t_j^{-1}\ [e_j]=t_it_j^{-1}\ e_i^\vee\otimes[e_j] \qquad (48)$$

where $i=1,2,\ j=3,4$. Now using the additive notation the weight of the T-action on the basis element $e_i^\vee\otimes[e_j]$ is equal to $\lambda_i-\lambda_j$. In table 1 (on page 31), we have listed the weights of the T-action on the bundles $\mathcal{T}_\mathcal{G}$, $\mathcal{S}$, and $\mathcal{Q}$ at the point p_{ij}.

Let $\mathcal{G}^T=\{p_{ij}: i\in I,\ j\in J\}$ be the fixed point locus of the torus action on $\mathcal{G}$. We will calculate the following integrals:

(a) Let $\mathcal{I}=\int_\mathcal{G} c_1^4(\mathcal{T}_\mathcal{G})$ then

$$\begin{aligned}\mathcal{I} &= \sum_{p_{ij}\in\mathcal{G}^T}\frac{\sigma_1^4(\text{weights of }\mathcal{T}_{p_{ij}}\mathcal{G})}{Euler_T((\mathcal{N}_\mathcal{G})_{p_{ij}})}\\ &= 512.\end{aligned}$$

(i,j)	weights of $\mathcal{T}_{p_{ij}}G$	wts of S at p_{ij}	wts of $\mathcal{Q}$ at p_{ij}
$(1,2)$	$\lambda_1-\lambda_3, \lambda_1-\lambda_4, \lambda_2-\lambda_3, \lambda_2-\lambda_4$	$-\lambda_1, -\lambda_2$	$-\lambda_3, -\lambda_4$
$(1,3)$	$\lambda_1-\lambda_2, \lambda_1-\lambda_4, \lambda_3-\lambda_2, \lambda_3-\lambda_4$	$-\lambda_1, -\lambda_3$	$-\lambda_2, -\lambda_4$
$(1,4)$	$\lambda_1-\lambda_2, \lambda_1-\lambda_3, \lambda_4-\lambda_2, \lambda_4-\lambda_3$	$-\lambda_1, -\lambda_4$	$-\lambda_2, -\lambda_3$
$(2,3)$	$\lambda_2-\lambda_1, \lambda_2-\lambda_4, \lambda_3-\lambda_1, \lambda_3-\lambda_4$	$-\lambda_2, -\lambda_3$	$-\lambda_1, -\lambda_4$
$(2,4)$	$\lambda_2-\lambda_1, \lambda_2-\lambda_3, \lambda_4-\lambda_1, \lambda_4-\lambda_3$	$-\lambda_2, -\lambda_4$	$-\lambda_1, -\lambda_3$
$(3,4)$	$\lambda_3-\lambda_1, \lambda_3-\lambda_2, \lambda_4-\lambda_1, \lambda_4-\lambda_2$	$-\lambda_3, -\lambda_4$	$-\lambda_1, -\lambda_2$

Table 1: the weights of the T-action on the bundles $\mathcal{T}_{\mathcal{G}}$, $\mathcal{S}$, and $\mathcal{Q}$ at the point p_{ij}

(b) To calculate $\int_{\mathcal{G}} c_1^2(\mathcal{T}_{\mathcal{G}})c_2(\mathcal{T}_{\mathcal{G}})$. Let $\mathcal{I} = \int_{\mathcal{G}} c_1^2(\mathcal{T}_{\mathcal{G}})c_2(\mathcal{T}_{\mathcal{G}})$ then

$$\begin{aligned}\mathcal{I} &= \sum_{p_{ij}\in\mathcal{G}^T} \frac{\sigma_1^2(\text{weights of } \mathcal{T}_{p_{ij}}\mathcal{G}).\sigma_2(\text{weights of } \mathcal{T}_{p_{ij}}\mathcal{G})}{Euler_T((\mathcal{N}_{\mathcal{G}})_{p_{ij}})} \\ &= 224.\end{aligned}$$

(c) Let $\mathcal{I} = \int_{\mathcal{G}} c_1(\mathcal{T}_{\mathcal{G}}).c_3(\mathcal{T}_{\mathcal{G}})$ then

$$\begin{aligned}\mathcal{I} &= \sum_{p_{ij}\in\mathcal{G}^T} \frac{\sigma_1(\text{weights of } \mathcal{T}_{p_{ij}}\mathcal{G}).\sigma_3(\text{weights of } \mathcal{T}_{p_{ij}}\mathcal{G})}{Euler_T((\mathcal{N}_{\mathcal{G}})_{p_{ij}})} \\ &= 48.\end{aligned}$$

Example 4.0.28. Let $Hilb^2\mathbb{P}^2$ be the Hilbert scheme parametrizing finite subschemes of length 2 in the projective plane (see [12]). Consider the $\mathbb{C}^*$-action on $\mathbb{P}^2$ given by

$$t.(x_0, x_1, x_2) = (t^{-a}x_0, t^{-b}x_1, t^{-c}x_2) \tag{49}$$

where a, b, c are non zero integers and $a \neq b$, $b \neq c$, and $a \neq c$. The fixed points are clearly $p_0 = (1,0,0)$, $p_1 = (0,1,0)$, and $p_2 = (0,0,1)$. Let L be the line $x_2 = 0$, and put $F_0 = \{p_0\}$, $F_1 = L - p_0$, and $F_2 = \mathbb{P}^2 - L$. Then $F_i \simeq \mathbb{A}^i$, and they define a cellular decomposition of $\mathbb{P}^2$. The action of $\mathbb{C}^*$ on $\mathbb{P}^2$ induces in a natural way an action of $\mathbb{C}^*$ on $Hilb^2\mathbb{P}^2$. If $Z \subset \mathbb{P}^2$ corresponds to a fix point of this action, clearly the support of Z is contained in the fixpoint set $\{p_0, p_1, p_2\}$ of $\mathbb{C}^*$. Therefore we may write $Z = Z_0 \cup Z_1 \cup Z_2$ where Z_i is supported in p_i and corresponds to a fixed point in $Hilb^2\mathbb{P}^{d_i}$, where d_i = length $\mathcal{O}_{Z_i}$. For any $Z \subset \mathbb{P}^2$ of finite length 2 we can write Z uniquely as a disjoint union $Z = Z_0 \cup Z_1 \cup Z_2$ where each Z_i is a closed subscheme of $\mathbb{P}^2$ supported in F_i. Put $d_i(Z)$ = length $(\mathcal{O}_{Z_i})$. For any triple (d_0, d_1, d_2) of non-negative integers with $d = d_0 + d_1 + d_2$, we define $W(d_0, d_1, d_2)$ to be the locally closed subset of $Hilb^2\mathbb{P}^2$ corresponding to subschemes Z with $d_i(Z) = d_i$ for $i = 0, 1, 2$. Clearly

$$Hilb^2\mathbb{P}^2 = \bigcup_{d_0+d_1+d_2=2} W(d_0, d_1, d_2) \tag{50}$$

Let $\mathcal{H} = Hilb^2\mathbb{P}^2$ and let $\mathcal{H}^T$ be the fixpoint locus of the $\mathbb{C}^*$-action on $\mathcal{H}$. Then $\mathcal{H}^T = \{(x^2, y), (x, y^2), (x^2, z), (x, z^2), (y^2, z), (y, z^2), (x, y) \cap (x, z), (x, y) \cap (y, z), (y, z) \cap (x, z)\}$. Let $X_p = \{Z \in \mathcal{H} : lim_{t \longrightarrow 0} t.Z = p\}$ be the cell that corresponds to the fixpoint $p \in \mathcal{H}^T$. Given any $\mathbb{C}^*$-action on $\mathbb{P}^2$ that respects the cellular decomposition $\{F_0, F_1, F_2\}$ of $\mathbb{P}^2$, then this action induces a cellular decomposition of $Hilb^2\mathbb{P}^2$ and $W(d_0, d_1, d_2)$ is a union of cells from this decomposition. The spaces $W(2,0,0)$, $W(0,2,0)$, and $W(0,0,2)$ are contained in $Hilb^2\mathbb{P}^2$. They are unions of cells from a cellular decomposition of $Hilb^2\mathbb{P}^2$. The cells contained in $W(2,0,0)$ (resp. $W(0,2,0)$, $W(0,0,2)$) are exactly those corresponding to fixpoints supported in p_0 (resp. p_1, p_2).

CASE 1. Consider the $\mathbb{C}^*$-action on $\mathbb{P}^2$ given by $t.(x_0, x_1, x_2) = (t^{-a}x_0, t^{-b}x_1, t^{-c}x_2)$ such that $a > b > c$. Let $y = \frac{x_1}{x_0}$, $z = \frac{x_2}{x_0}$, and let $A = (y^2, z)$, $B = (y, z^2)$ then A, B are two fixpoints of $\mathcal{H}$ supported at the point $p_0 = (1, 0, 0)$. Clearly $W(2,0,0) = X_A \cup X_B$. Now the $\mathbb{C}^*$-action above can be written as follows: $t.(y, z) = (t^{a-b}y, t^{a-c}z)$. Let $\mathcal{Q}_A = \mathbb{C}[x, y]/A$ then the tangent space

$$\mathcal{T}_A\mathcal{H} = \mathrm{Hom}((y^2, z), \mathbb{C}[x, y]/(y^2, z)) = \mathrm{Hom}(A, \mathcal{Q}_A) = A^{\vee} \otimes \mathcal{Q}_A \tag{51}$$

Let $e_1 = y^2$, $e_2 = z$, $e_3 = 1$, $e_4 = y$ then the set $\{e_i^{\vee} \otimes e_j : i = 1, 2,\ j = 3, 4\}$ form a basis for $T_A\mathcal{H}$. Let us apply the $\mathbb{C}^*$-action above on the basis elements and then count the number of positive weights to get the dimension of the cell X_A. First we will compute the weights of the $\mathbb{C}^*$-action on the basis element $e_1^{\vee} \otimes e_3$. Since $t.e_1^{\vee} = t^{-1}.e_1 = t^{-1}.y^2 = t^{-2(a-b)}y^2 = t^{2(b-a)}y^2 = t^{2(b-a)}e_1^{\vee}$, $t.e_3 = e_3$. It follows that $t.(e_1^{\vee} \otimes e_3) = (t^{2(b-a)}e_1^{\vee}) \otimes e_3 = t^{2(b-a)}(e_1^{\vee} \otimes e_3)$. Using the additive notation, the weight of the $\mathbb{C}^*$-action on basis element $e_1^{\vee} \otimes e_3$ is $2(b-a)$. Similarly $t.(e_1^{\vee} \otimes e_4) = t^{2(b-a)}e_1^{\vee} \otimes t^{a-b}e_4 = t^{2(b-a)+a-b}(e_1^{\vee} \otimes e_4) = t^{b-a}(e_1^{\vee} \otimes e_4)$, $t.(e_2^{\vee} \otimes e_3) = t^{c-a}(e_2^{\vee} \otimes e_3)$, and $t.(e_2^{\vee} \otimes e_4) = t^{c-b}(e_2^{\vee} \otimes e_4)$. Therefore the weights of the $\mathbb{C}^*$-action on the basis above are $2(b-a)$, $b-a$, $c-a$, and $c-b$. Now since $a > b > c$ the number of positive weights is 0. According to Bialynicki-Birula theorem (see [12]) $\dim X_A = \dim(T_A\mathcal{H})^+ = 0$ where $(T_A\mathcal{H})^+$ denotes the part of $T_A\mathcal{H}$ where the weights of the $\mathbb{C}^*-$action are positive So $X_A = \{A\}$. To calculate the dimension of the cell X_B, let $\mathcal{Q}_B = \mathbb{C}[x,y]/B$ then $T_B\mathcal{H} = \text{Hom}(B, \mathcal{Q}_B) = B^{\vee} \otimes \mathcal{Q}_B$. Again you have 4 basis elements. Applying the $\mathbb{C}^*$-action on those elements we pickup 4 weights namely, $b-a$, $b-c$, $2(c-a)$, and $c-a$. Now since $a > b > c$ the number of positive weights is 1. Thus $W(2,0,0) = X_A \cup X_B = \{A\} \cup \mathbb{C}^1$. Consider the fixpoint $I = (x,y) \cap (y,z) \in \mathcal{H}^T$ then

$$T_I\mathcal{H} = \text{Hom}((x,y), \mathbb{C}[x,y]/(x,y)) \oplus \text{Hom}((y,z), \mathbb{C}[y,z]/(y,z)) \tag{52}$$

Applying the $\mathbb{C}^*$-action on the basis elements we pickup 4 weights namely, $a-c$, $b-c$, $b-a$, and $c-a$. Now since $a > b > c$ the number of positive weights is 2. It follows that $X_I \simeq \mathbb{C}^2$. So $\dim X_I = 2$. In table 2(on page 34), we have listed all the fixed points of $\mathcal{H} = Hilb^2\mathbb{P}^2$, the weights of the $\mathbb{C}^*$-action on the tangent space at the fixed point, and the cell that corresponds to the fixed point. We let N^+ denote the number of positive weights. Now we can calculate the BETTI numbers of $\mathcal{H}$. Let the BETTI number b_i = number of i-th dimensional cells where $i = 0, 1, 2, 3, 4$. It follows that: $b_0 = 1$, $b_1 = 2$, $b_2 = 3$, $b_3 = 2$, and $b_4 = 1$.

Fixed point p	weights of the $\mathbb{C}^*$-action on $\mathcal{T}_p\mathcal{H}$	N^+	X_p
(x^2, y)	$2(a-c), a-c, b-c, b-a$	3	$\mathbb{C}^3$
(x, y^2)	$2(b-c), b-c, a-c, a-b$	4	$\mathbb{C}^4$
(x^2, z)	$2(a-b), a-b, c-b, c-a$	2	$\mathbb{C}^2$
(x, z^2)	$2(c-b), c-b, a-b, a-c$	2	$\mathbb{C}^2$
(y^2, z)	$2(b-a), b-a, c-a, c-b$	0	$\{(y^2, z)\}$
(y, z^2)	$2(c-a), c-a, b-a, b-c$	1	$\mathbb{C}^1$
$(x, y) \cap (x, z)$	$a-c, b-c, a-b, c-b$	3	$\mathbb{C}^3$
$(x, y) \cap (y, z)$	$a-c, b-c, b-a, c-a$	2	$\mathbb{C}^2$
$(x, z) \cap (y, z)$	$a-b, c-b, b-a, c-a$	1	$\mathbb{C}$

Table 2: the fixed points of $\mathcal{H} = Hilb^2\mathbb{P}^2$, the weights of the $\mathbb{C}^*$-action on the tangent space at the fixed point, and the cell that corresponds to the fixed point.

5 Characterization of T-invariant rational equivalence

Let $T = (\mathbb{C}^*)^{n+1}$ be the algebraic torus. Consider the action of T on $\mathbb{P}^n$ given by

$$t.(x_0, ..., x_n) = (t_0^{-1}x_0, ..., t_n^{-1}x_n)$$

In this section a subvariety Y of $\mathbb{P}^n$ is T-invariant if it is fixed by the torus T, i.e, $t.Y = Y$ where $t \in T$.

5.1 T-invariant linear equivalence in $\mathbb{P}^n$

Definition 5.1.1. Let X be a B-variety. An embedding $\psi : X \hookrightarrow \mathbb{P}^n$ is a T-equivariant embedding if for any subvariety $Z \subset X$ we have $\psi(t.Z) = t.\psi(Z)$ for all $t \in T$.

In section 5.5 we will prove that for any component $\mathrm{Hilb}^P X$ of the Hilbert scheme $\mathrm{Hilb}\,X$ there exists a T-representation V such that $\mathrm{Hilb}^p X$ can be embedded T-equivariantly in $\mathbb{P}(V)$.

In the following definitions we will consider schemes satisfying the following condition: (*) X is noetherian integral separated scheme which is regular in codimension one. See [25].

Definition 5.1.2. A prime divisor on X is a closed integral subscheme Y of codimension one. A Weil divisor is an element of the free abelian group Div X generated by the prime divisors. We write a divisor as $D = \sum n_i Y_i$, where the Y_i are prime divisors, the n_i are integers, and only finitely many n_i are different from zero. If all the $n_i \geq 0$, we say that D is effective. If Y is a prime divisor on X, let $\xi \in Y$ be its generic point. Then the local ring $\mathcal{O}_{\xi,X}$ is a discrete valuation ring with quotient field K, the function field of X. We call the corresponding discrete valuation ν_Y the valuation of Y. Note that since X is separated, Y is uniquely determined by its valuation. Let $f \in K^*$ be any nonzero rational function on X. Then $\nu_Y(f)$ is an integer. If it is positive we say that f has a zero along y, of that order; if it is negative, we say f has a pole along Y, of order $-\nu_Y(f)$.

Lemma 5.1.3. Let X satisfy (*), and let $f \in K^*$ be a nonzero function on X. Then $\nu_Y(f) = 0$ for all except finitely many prime divisors Y. See [25]

Definition 5.1.4. Let X satisfy (*), and let $f \in K^*$. We define the divisor of f, denoted by (f), by $(f) = \sum \nu_Y(f)$ Y, where the sum is taken over all prime divisors of X. By the lemma 5.1.3, this is a finite sum, hence it is a divisor. Any divisor which is equal to the divisor of a function is called a principal divisor.

Definition 5.1.5. Let X satisfy (*). Two divisors D and D' are said to be linearly equivalent, written $D \sim D'$, if $D - D'$ is a principal divisor.

Definition 5.1.6. Let X be a B-variety. We define the action of T on the Weil divisor $D = \sum n_i Y_i$ by setting $t.D = \sum n_i(t.Y_i)$, where $t.Y_i = \{t.y : y \in Y_i\}$.

Remark 5.1.7. Let $X = \mathbb{P}^n$. Consider the fiber diagram

$$\begin{array}{ccc} X & \xrightarrow{\pi_{pt}} & pt \\ \downarrow{\scriptstyle i_X} & & \downarrow{\scriptstyle i_{pt}} \\ X_T & \xrightarrow{\pi} & BT \end{array} \tag{53}$$

The inclusion map $i : X \hookrightarrow X_T$ induces the map $i_X^* : H_T^*(X) \longrightarrow H^*(X)$ defined by $i_X^*([Z_T]) = [Z_T \times_{X_T} X]$ where Z is a subvariety of X. But $Z_T \times_{X_T} X = Z$. So $i_X^*([Z_T]) = [Z]$.

(1) Check $i_X^*(p) = H$. Let E be an equivariant rank r vector bundle over X. Consider the following commutative diagram

$$\begin{array}{ccc} E & \hookrightarrow & E_T \\ \downarrow & & \downarrow \\ X & \xhookrightarrow{i_X} & X_T \end{array} \tag{54}$$

It follows that $i_X^*(E_T) = E$. Note that $i_X^*(c_j^T(E)) = i_X^*(c_j(E_T)) = c_j(i_X^*(E_T)) = c_j(E)$. So $i_X^*(c_j^T(E)) = c_j(E)$. Thus

$$i_X^*(p) = i_X^*(c_1(\mathcal{O}_{\mathbb{P}_T^n}(1))) = i_X^*(c_1^T(\mathcal{O}_{\mathbb{P}^n}(1)) = c_1(\mathcal{O}_{\mathbb{P}^n}(1)) = H \tag{55}$$

(2) Check $i_X^*(\lambda_j) = 0$. First note that $\lambda_j := \pi^*(\lambda_j)$, $i = 0, 1, 2, ..., n$. Also note that the commutative diagram in (1) above implies $i_X^* \circ \pi^* = \pi_{pt}^* \circ i_{pt}^*$. Now $i_X^*(\lambda_j) = i_X^*(\pi^*(\lambda_j)) = \pi_{pt}^*(i_{pt}^*(\lambda_j)) = \pi_{pt}^*(0) = 0$ because $i_{pt}^*(\lambda_j) \in H^2(pt) = 0$.

I will use i_j^* instead of i_{jT}^* for simplicity where $i_{jT}^* : H_T^*(X) \longrightarrow H_T^*(p_j)$ is the map induced by the equivariant inclusion $i_{jT} : (p_j)_T \hookrightarrow X_T$.

Theorem 5.1.8. Let $X = \mathbb{P}^n$ equipped with a T-action and a finite set of fixed points $\{p_j\}_{j=0}^n$ where the $n+1$ fixed points p_j, ordered as usual, so that the j-th coordinate of p_j is nonzero, all other coordinates being zero. Let $H_T^*(X)$ be the equivariant cohomology of $\mathbb{P}^n$, and let $D_i \subset X$ be a T-invariant subvariety of codimension one, $i = 1, 2$. Then

(a) $D_1 \sim D_2 \Leftrightarrow [D_{1T}] - [D_{2T}] \in Span\{\lambda_i : i = 0, 1, 2, ..., n\}$, where $Span\{\lambda_i : i = 0, 1, 2, ..., n\} = \{\sum_{i=0}^n c_i\lambda_i : c_i \in \mathbb{Z}\}$

(b) Let $i_j{}^* : H_T^*(X) \longrightarrow H_T^*(p_j)$ be the map induced by the equivariant inclusion $i_{jT} : (p_j)_T \hookrightarrow X_T$. If $[D_T] \in H_T^2(X)$ is any equivariant class such that there exists $a_i \in \mathbb{Z}$, $i = 0, 1, 2, ..., n$ with $i_j^*([D_T]) = \sum_{i=0}^n a_i\lambda_i$ for all j, then $[D_T] = \sum_{i=0}^n a_i\lambda_i$.

Remark 5.1.9. (a),(b) implies $D_1 \sim D_2 \Leftrightarrow$ there exists $a_i \in \mathbb{Z}$, $i = 0, 1, 2, ..., n$ such that $i_j^*([D_{1T}] - [D_{2T}]) = \sum_{i=0}^n a_i\lambda_i$, $\forall j = 0, 1, 2, ..., n$.

Proofs.

(a) ($\Rightarrow$): Suppose that $D_1 \sim D_2$. Since the equivariant class $[D_{1T} - D_{2T}] \in H_T^2(X)$ then

$$[D_{1T} - D_{2T}] = a.p + \sum_{i=0}^{n} c_i\lambda_i \tag{56}$$

Consider the map $i : X \hookrightarrow X_T$ which induces the map $i_X^* : H_T^*(X) \longrightarrow H^*(X)$ where $i_X^*([Z_T]) = [Z]$, Z is a subvariety of X. Applying i_X^* to equation (56) above we get

$$i_X^*([D_{1T} - D_{2T}]) = i_X^*(a.p + \sum_{i=0}^{n} c_i\lambda_i) = a.i_X^*(p) + \sum_{i=0}^{n} c_i.i_X^*(\lambda_i) \tag{57}$$

It follows from remark 5.1.7 above that

$$i_X^*([(D_1 - D_2)_T]) = i_X^*([D_{1T} - D_{2T}]) = a.H + \sum_{i=0}^{n} c_i.0 = aH \tag{58}$$

But $D_1 \sim D_2$ implies $[D_1 - D_2] = 0$. Thus $0 = [D_1 - D_2] = aH$ implies $a = 0$. Hence $[D_{1T} - D_{2T}] = \sum_{i=0}^n c_i\lambda_i \in \text{Span}\{\lambda_i : i = 0, 1, 2, ..., n\}$.

($\Leftarrow$):Suppose $[D_{1T} - D_{2T}] \in Span\{\lambda_i : i = 0, 1, 2, ..., n\}$ then there exists $c_i \in \mathbb{Z}$ such that $[D_{1T} - D_{2T}] = \sum_{i=0}^{n} c_i\lambda_i$. It follows

$$i_X^*([(D_1 - D_2)_T]) = i_X^*([D_{1T} - D_{2T}]) = i_X^*(\sum_{i=0}^{n} c_i\lambda_i) = \sum_{i=0}^{n} c_i.i_X^*(\lambda_i) = \sum_{i=0}^{n} c_i.0 = 0$$

which implies $[D_1 - D_2] = 0$. Thus $D_1 \sim D_2$.

(b) Consider the equivariant class $[D_T] \in H_T^2(X)$ such that there exists $a_i \in \mathbb{Z}$ with $i_j^*([D_T]) = \sum_{i=0}^{n} a_i\lambda_i$, $\forall j = 0, 1, 2, ..., n$. Since $[D_T] \in H_T^*(X)$ it follows that $[D_T] = a.p + \sum_{i=0}^{n} c_i\lambda_i$ for some $a, c_i \in \mathbb{Z}$, $i = 0, 1, 2, ..., n$. Applying the map i_j^*, $j = 0, 1, 2, ..., n$ we get

$$i_j^*([D_T]) = i_j^*(a.p + \sum_{i=0}^{n} c_i\lambda_i) = a.i_j^*(p) + \sum_{i=0}^{n} c_i.i_j^*(\lambda_i) = a.\lambda_j + \sum_{i=0}^{n} c_i.\lambda_i \quad (59)$$

But $i_j^*([D_T]) = \sum_{i=0}^{n} a_i\lambda_i$. It follows $\sum_{i=0}^{n} a_i.\lambda_i = a.\lambda_j + \sum_{i=0}^{n} c_i.\lambda_i$, $\forall j$. So $\sum_{i=0}^{n}(a_i - c_i).\lambda_i = a.\lambda_j$, $j = 0, 1, 2, ..., n$ implies $a\lambda_0 = a\lambda_1 = ... = a\lambda_n$ which implies $a = 0$. Thus $[D_T] = a.p + \sum_{i=0}^{n} c_i\lambda_i = 0.p + \sum_{i=0}^{n} c_i\lambda_i = \sum_{i=0}^{n} c_i\lambda_i$ where $c_i \in \mathbb{Z}$, $i = 0, 1, 2, ..., n$.

5.2 T-INVARIANT LINEAR EQUIVALENCE IN A B-VARIETY

As in section 2.1 we will use a B-variety to denote a projective variety with a torus action and a finite set of fixed points.

Remark 5.2.1. Let $\mathcal{F}$ be a sheaf on a topological space X, $\pi : X_T \longrightarrow BT$ be a continuous map of spaces. Consider an injective resolution of $\mathcal{F}$

$$0 \to \mathcal{F} \to \mathcal{F}^0 \to \mathcal{F}^1 \to \mathcal{F}^2 \to ...$$

where $\mathcal{F}^i$ is a sheaf on X, $i \geq 0$. We then have a long exact sequence of sheaves

$$\pi_*\mathcal{F}^{i-1} \xrightarrow{\delta_{i-1}} \pi_*\mathcal{F}^{i} \xrightarrow{\delta_{i}} \pi_*\mathcal{F}^{i+1}$$

The i-th direct image sheaf $\mathcal{R}^i\pi_*\mathcal{F} = \ker\delta_i/\operatorname{im}\delta_{i-1}$. If $\mathcal{F} = \mathbb{C}$ then $\mathcal{R}^i\pi_*\mathbb{C}$ is a sheaf on BT. Furthermore, $\mathcal{R}^i\pi_*\mathcal{F}$ is the sheaf associated to the presheaf

$$V \longmapsto H^i(\pi^{-1}(V), \mathcal{F}\mid_{\pi^{-1}(V)}) = H^i(\pi^{-1}(V), \mathbb{C})$$

on BT.

Definition 5.2.2. A *spectral* sequence is a sequence $\{\mathcal{E}_r, d_r\}(r \geq 0)$ of bigraded groups

$$\oplus_{p,q\geq 0}\mathcal{E}_r^{p,q}$$

together with differentials

$$d_r : \mathcal{E}_r^{p,q} \longrightarrow \mathcal{E}_r^{p+r,q-r+1},\ d_r^2 = 0,$$

such that

$$H^*(\mathcal{E}_r) = \mathcal{E}_{r+1}.$$

Notation. $\{\mathcal{E}_\infty^{p,q}\}_{p,q\geq 0}$ this notation means the following:

(1) For $k >> 0$, $\mathcal{E}_k$ degenerates i.e. we have

$$\mathcal{E}_k^{p,q} = \mathcal{E}_{k+1}^{p,q} = \mathcal{E}_{k+2}^{p,q} = ...$$

Where all $d_k = 0$. We say $\mathcal{E}_\infty^{p,q} = \mathcal{E}_k^{p,q}$.

(2) Suppose $\mathcal{E}_k$ degenerates when $k = n$. Then there exists a filtration $(F^p\mathcal{E}_\infty, d)$ of $\mathcal{E}_\infty$ whose graded quotients are $\mathcal{E}_\infty^{p,q}$ where $p + q = n$ i.e. $\operatorname{Gr}\mathcal{E}_\infty = \oplus_{p\geq 0}\operatorname{Gr}^p\mathcal{E}_\infty$ where $\operatorname{Gr}^p\mathcal{E}_\infty = F^p\mathcal{E}_\infty/F^{p+1}\mathcal{E}_\infty = \mathcal{E}_\infty^{p,q}$, $p + q = n$.

Definition 5.2.3. Suppose we are given topological spaces X, Y with a continuous map $f : X \longrightarrow Y$ and a sheaf $\mathcal{F}$ over X. The $q-th$ direct image sheaf is the sheaf $\mathcal{R}^q f_*(\mathcal{F})$ on Y associated to the presheaf

$$U \longrightarrow H^q(f^{-1}(U), \mathcal{F}).$$

The *Leray* spectral sequence, is a spectral sequence $\{\mathcal{E}_r\}$ with

$$\left\{ \begin{array}{c} \mathcal{E}_\infty \Longrightarrow H^*(X, \mathcal{F}), \\ \mathcal{E}_2^{p,q} = H^p(Y, \mathcal{R}^q f_*(\mathcal{F})) \end{array} \right\}.$$

Example 5.2.4. Consider the map $\pi : \mathbb{P}^n_T \longrightarrow BT$ then the sheaf $\mathcal{R}^1\pi_*\mathbb{C} = 0$ because the fiber $(\mathcal{R}^1\pi_*\mathbb{C})_p = H^1(\pi^{-1}(p), \mathbb{C}) = H^1(\mathbb{P}^n, \mathbb{C}) = 0$, $p \in BT$. Thus $H^j(BT, \mathcal{R}^1\pi_*\mathbb{C}) = 0$, $j \geq 0$.

Remark 5.2.5. Consider the map $\pi : X_T \longrightarrow BT$ then the *Leray* spectral sequence $\{\mathcal{E}_r\}$ degenerates i.e., $\mathcal{E}_\infty^{p,q} = \mathcal{E}_2^{p,q}$, $p, q \geq 0$ and for any sheaf $\mathcal{F}$ on X_T there exists a filtration of $H^2(\mathcal{F})$

$$H^2(\mathcal{F}) = F^0H^2(\mathcal{F}) \supset F^1H^2(\mathcal{F}) \supset F^2H^2(\mathcal{F}) \supset ... \supset F^nH^2(\mathcal{F}) \supset F^{n+1}H^2(\mathcal{F}) = 0$$

such that $F^pH^2(\mathcal{F})/F^{p+1}H^2(\mathcal{F}) \simeq \mathcal{E}_\infty^{p,2-p}$. For simplicity I will use F^p to denote for $F^pH^2(\mathcal{F})$. Now if $\mathcal{F} = \mathbb{C}$ then

$$F^0/F^1 = \mathcal{E}_\infty^{0,2} = H^0(BT, \mathcal{R}^2\pi_*\mathbb{C}) \tag{60}$$

$$F^1/F^2 = \mathcal{E}_\infty^{1,1} = H^1(BT, \mathcal{R}^1\pi_*\mathbb{C}) = 0 \tag{61}$$

Because $\mathcal{R}^1\pi_*\mathbb{C} = 0$. But $F^1 \neq 0$ implies $F^1 = F^2$.

$$F^2/F^3 = \mathcal{E}_\infty^{2,0} = H^2(BT, \mathcal{R}^0\pi_*\mathbb{C}) = H^2(BT, \mathbb{C}) \tag{62}$$

$$F^3/F^4 = \mathcal{E}_\infty^{3,-1} = 0 \tag{63}$$

Thus $F^3 = 0$ and by a similar argument $F^4 = \mathcal{F}^5 = ... = 0$.

Lemma 5.2.6. Let X be a B-variety, and let $\mathbb{C}$ be the constant sheaf on X_T. Consider the map $\pi : X_T \longrightarrow BT$. Then $\pi_*\mathbb{C} = \mathbb{C}$.

Proof. Let $\mathcal{X} = X_T$, $\mathcal{Y} = BT$. Consider the constant sheaf $\mathbb{C}_{\mathcal{X}}$ on $\mathcal{X}$, and the constant sheaf $\mathbb{C}_{\mathcal{Y}}$ on $\mathcal{Y}$. We need to define a morphism of sheaves $\psi : \mathbb{C}_{\mathcal{Y}} \longrightarrow \pi_*\mathbb{C}_{\mathcal{X}}$. So let $\mathcal{U}$ be an open subset of $\mathcal{Y}$. Define a morphism of abelian groups $\psi(\mathcal{U}) : \mathbb{C}_{\mathcal{Y}}(\mathcal{U}) \longrightarrow \pi_*\mathbb{C}_{\mathcal{X}}(\mathcal{U})$ as follows: For simplicity we will use ψ to denote for $\psi(\mathcal{U})$. Let $\mathbb{C}_{\mathcal{Y}}(\mathcal{U}) = \{f : \mathcal{U} \longrightarrow \mathbb{C}$ where f is a constant map $\}$, $\pi_*\mathbb{C}_{\mathcal{X}}(\mathcal{U}) = \mathbb{C}_{\mathcal{X}}(\pi^{-1}(\mathcal{U})) = \{g : \pi^{-1}(\mathcal{U}) \longrightarrow \mathbb{C}$ where g is a constant map $\}$. We define $\psi : \mathbb{C}_{\mathcal{Y}}(\mathcal{U}) \longrightarrow \pi_*\mathbb{C}_{\mathcal{X}}(\mathcal{U})$ by $\psi(f) = f \circ \pi$ for any $f \in \mathbb{C}_{\mathcal{Y}}(\mathcal{U})$. Clearly ψ is well-defined.

claim: ψ is an isomorphism. By abuse of notation we define $\psi^{-1} : \pi_*\mathbb{C}_{\mathcal{X}}(\mathcal{U}) \longrightarrow \mathbb{C}_{\mathcal{Y}}(\mathcal{U})$, and again for simplicity we will use ψ^{-1} to denote for $\psi(\mathcal{U})^{-1}$, by $\psi^{-1}(g) = h$ where $h(p) = g(\pi^{-1}(p))$ for any $p \in \mathcal{U}$. This definition makes sense because g is a regular function on the fiber $\pi^{-1}(p) = X$, which is a connected projective B-variety, so g is a constant function on X. It remains to check that $\psi^{-1} \circ \psi = \mathrm{Id}_{\mathbb{C}_{\mathcal{Y}}(\mathcal{U})}$, and $\psi \circ \psi^{-1} = \mathrm{Id}_{\pi_*\mathbb{C}_{\mathcal{X}}(\mathcal{U})}$. *First.* let $f \in \mathbb{C}_{\mathcal{Y}}(\mathcal{U})$ then $(\psi^{-1} \circ \psi)(f) = \psi^{-1}((\psi)(f)) = \psi^{-1}(f \circ \pi)$. If $p \in \mathcal{U}$ then $(\psi^{-1}(f \circ \pi))(p) = (f \circ \pi)(\pi^{-1}(p)) = f(p)$. It follows $\psi^{-1}(f \circ \pi) = f$. Thus $\psi^{-1} \circ \psi = \mathrm{Id}_{\mathbb{C}_{\mathcal{Y}}(\mathcal{U})}$. *Second.* Let $g \in \pi_*\mathbb{C}_{\mathcal{X}}(\mathcal{U})$, and let $\delta = (\psi^{-1})(g)$ then $(\psi \circ \psi^{-1})(g) = \psi(\psi^{-1}(g)) = \psi(\delta) = \delta \circ \pi$. Now if $s \in \pi^{-1}(\mathcal{U})$ then $(\delta \circ \pi)(s) = \delta(\pi(s)) = (\psi^{-1}(g))(\pi(s)) = g(\pi^{-1}(\pi(s))) = g(s)$ because g is a constant map. It follows $(\delta \circ \pi)(s) = g(s)$ for each $s \in \pi^{-1}(\mathcal{U})$. Thus $(\psi \circ \psi^{-1})(g) = g$. Hence $\psi \circ \psi^{-1} = \mathrm{Id}_{\pi_*\mathbb{C}_{\mathcal{X}}(\mathcal{U})}$.

Definition 5.2.7. Let U be an open subset of the topological space X and let $\tilde{\mathcal{G}}$ be the sheaf associated with the presheaf $\mathcal{G}$ on X. Then $\tilde{\mathcal{G}}(U)$ is defined as the set of functions s from U to the union $\cup_{p \in U}\mathcal{G}_p$ of stalks of $\mathcal{G}$ over points of U, such that

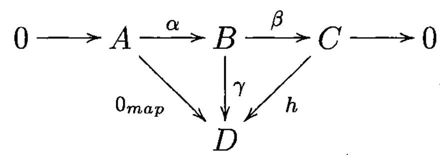
(1) for each $p \in U$, $s(p) \in \mathcal{G}_p$, and

(2) for each $p \in U$, there is a neighborhood V of p, contained in U, and an element

$t \in \mathcal{G}(V)$, such that for all $q \in V$, the germ t_q of t at q is equal to $s(q)$.

i.e, $\tilde{\mathcal{G}}(\mathcal{U}) = \{(s(p))_{p \in U}$: $s(p) \in \mathcal{G}_p$ and for each $p \in U$, there is a neighborhood V of p, contained in U, and an element $t \in \mathcal{G}(V)$, such that for all $q \in V$, the germ t_q of t at q is equal to $s(q)\}$.

Remark 5.2.8. Consider the commutative diagram of group homomorphisms with the horizontal row exact



(64)

Claim. If $\gamma\alpha = 0_{map}$ then there exists a group homomorphism $h : C \longrightarrow D$ such that $h\beta = t\gamma$. Proof. Let $c \in C$. Since β is surjective there exists $b \in B$ such that $\beta(b) = c$. Define $h(c) = \gamma(b)$. Now if $b' \in B$ such that $\beta(b') = c$ then $\beta(b' - b) = \beta(b') - \beta(b) = 0$. But the horizontal row in diagram (66) is exact. So there exists $a \in A$ such that $\alpha(a) = b' - b$. So $\gamma\alpha(a) = \gamma(b' - b)$. But $\gamma\alpha = 0_{map}$. It follows $\gamma(b' - b) = 0$. Thus $\gamma(b) = \gamma(b')$. Hence $h\beta = \gamma$.

Fact 5.2.9. Let V be a T-invariant open subset of the B-variety X. Let $\gamma : ET \longrightarrow BT$ be a principal T-bundle. Let $f : V \longrightarrow \mathbb{C}$ be a rational function such that $t.f = \rho(t)f$ where ρ is the character with weight $\sum_{i=0}^{n} a_i\lambda_i$. We define the sheaf $\mathcal{O}((f_T))$ as follows: Let $g \in \Gamma(\gamma^*(\mathbb{C}_\rho)_T)$. Then $g(a) \in (\gamma^*(\mathbb{C}_\rho)_T)_a = ((\mathbb{C}_\rho)_T)_{\gamma(a)} = \mathbb{C}_\rho$. So $g(at^{-1}) = t.g(a) = \rho(t)g(a)$. Define $s([a, v]) = g(a)f(v)$ where $[a, v]$ is a class in $V_T = ET \times_T V$. We check that s is well-defined: $s([at^{-1}, tv]) = g(at^{-1})f(tv) = \rho(t)g(a)\rho^{-1}(t)f(v) = g(a)f(v) = s([a, v])$. Let $f^{-1}(\infty)$ be the divisor of poles of f. Define $\mathcal{O}((f_T))(V_T)$ to be the ring $\mathcal{O}((f_T))(V_T) := \{s : (V - f^{-1}(\infty))_T \longrightarrow \mathbb{C} : s([a, v]) = g(a)f(v)$ where $g \in \Gamma(\gamma^*(\mathbb{C}_\rho)_T)\}$.

Fact 5.2.10. Let X be a B-variety. Then the k-dimensional vector space $H^2(X, \mathbb{C})$ over $\mathbb{C}$ is generated by the set $\{[D_1], ..., [D_k] : D_i \subset X$ is a $\mathbb{C}^*$-invariant subvariety of codimension $1\}$.

Proof. Let X be an n-dimensional B-variety, i.e a smooth projective variety with a T-action and a finite set of fixed points $\{x_0, ..., x_n\}$. Then by the Bialynicki-Birula theorem (see [12]) X has a cellular decomposition with cells $X_i = \{x \in X : \lim\limits_{t \longrightarrow 0} t.x = x_i\}$. By part (ii) of proposition 1.5 in [12], $H^2(X, \mathbb{C})$ is generated by the classes of the closure of the $(n-1)$-dimensional cells (note that $H^2(X, \mathbb{C}) = H_{2(n-1)}(X, \mathbb{C})$). It follows that the k-dimensional vector space $H^2(X, \mathbb{C})$ is generated by the set $\{[\overline{X_{n-1,1}}], ..., [\overline{X_{n-1,k}}]\}$ where $\overline{X_{n-1,j}}$ is the closure of the $(n-1)$-dimensional cell $X_{n-1,j}$. It remains to check that $X_{n-1,j}$ is $\mathbb{C}^*$-invariant. Let $x \in X_{n-1,j}$ we show that $t'.x \in X_{n-1,j}$ for any $t' \in \mathbb{C}^*$. So we need to check that $\lim\limits_{t \longrightarrow 0} t.(t'.x) = x_j$. But $\lim\limits_{t \longrightarrow 0} t.(t'.x) = \lim\limits_{t \longrightarrow 0} (t.t').x = \lim\limits_{t'.t \longrightarrow 0} (t.t').x$. Let $r = t.t'$ then $\lim\limits_{t'.t \longrightarrow 0} (t.t').x = \lim\limits_{r \longrightarrow 0} r.x = x_j$ because $x \in X_{n-1,j}$. So $x \in X_{n-1,j}$ implies $t'.x \in X_{n-1,j}$ for any $t' \in \mathbb{C}^*$. Therefore $X_{n-1,j}$ is $\mathbb{C}^*$-invariant. So $\overline{X_{n-1,j}}$ is $\mathbb{C}^*$-invariant.

Lemma 5.2.11. Let X be a B-variety, and let $\mathbb{C}$ be the constant sheaf on X_T. Consider the map $\pi : X_T \longrightarrow BT$. Then $\mathcal{R}^2\pi_*\mathbb{C} = \mathbb{C}^k$ where k is the dimension of $H^2(X, \mathbb{C})$.

Proof. Fix a T-invariant divisor $D \subseteq X$. Then $\hat{\mathcal{F}} = \mathcal{R}^2\pi_*\mathbb{C}$ is the sheaf associated to the presheaf $\mathcal{F}$ where $\mathcal{F}(U) = H^2(\pi^{-1}(U), \mathbb{C})$, U open in BT. Now consider the equivariant divisor $D_T \subset X_T$. Let $\mathcal{L}_{D_T}$ be the line bundle on X_T associated to the divisor D_T, then $c_1(\mathcal{L}_{D_T} |_{\pi^{-1}(U)}) \in H^2(\pi^{-1}(U), \mathbb{C}) = \mathcal{F}(U)$. Let s_U=$c_1(\mathcal{L}_{D_T} |_{\pi^{-1}(U)})$. Let V be an open subset of U, consider the restriction map $\rho^U_V : \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$ which is defined as follows: Let $i : \pi^{-1}(V) \hookrightarrow \pi^{-1}(U)$ be the inclusion map. Define $\rho^U_V(s_U) = i^*s_U$ where

$$i^*s_U = i^*c_1(\mathcal{L}_{D_T} |_{\pi^{-1}(U)}) = c_1(i^*\mathcal{L}_{D_T} |_{\pi^{-1}(U)}) = c_1(\mathcal{L}_{D_T} |_{\pi^{-1}(V)}) = s_V$$

So $\rho_V^U(s_U) = s_V$. Therefore we get a global section $\hat{\mathcal{D}} \in \mathcal{F}(BT) = H^2(X_T, \mathbb{C})$. Consider the exact sequence

$$0 \to H^2(BT, \mathbb{C}) \hookrightarrow^{\pi^*} H^2(X_T, \mathbb{C}) \to^{\psi} H^0(BT, \mathcal{R}^2\pi_*\mathbb{C}) \to 0 \tag{65}$$

Let D be a T-invariant divisor in X. Define the map $\phi : H^2(X, \mathbb{C}) \longrightarrow H^0(BT, \mathcal{R}^2\pi_*\mathbb{C})$ by $\phi([D]) = \tilde{\mathcal{D}}$ where $\tilde{\mathcal{D}} = \psi(\hat{D})$.

(*) ϕ is an isomorphism:

First we show that ϕ is injective. Consider the exact sequence

$$0 \to H^2(BT, \mathbb{C}) \hookrightarrow^{\pi^*} H^2(X_T, \mathbb{C}) \to^{\psi} H^0(BT, \mathcal{R}^2\pi_*\mathbb{C}) \to 0 \tag{66}$$

So $\ker \psi = \operatorname{im} \pi^* \subset \mathbb{C}[\lambda_0, ..., \lambda_n]$ $(H^2(BT, \mathbb{C}) \subset H^*(BT, \mathbb{C}) = \mathbb{C}[\lambda_0, ..., \lambda_n])$. Now Suppose that $\phi([D]) = 0$ where D is a T-invariant divisor in X. Then $\phi([D]) = \tilde{D} = \psi(\hat{D}) = 0$. So $\hat{D} \in \ker \psi = \operatorname{im} \pi^* \subset \mathbb{C}[\lambda_0, ..., \lambda_n]$. So $\hat{D} = \sum_{i=0}^n a_i \lambda_i$ where $a_i \in \mathbb{C}$. So $\hat{D} - \sum_{i=0}^n a_i \lambda_i = 0 \in \mathcal{F}(BT) = H^2(X_T, \mathbb{C})$. It follows that $\rho_U^{BT}(\hat{D} - \sum_{i=0}^n a_i \lambda_i) = 0$ where U is an open subset of BT and $\rho_U^{BT} : \mathcal{F}(BT) \longrightarrow \mathcal{F}(U)$. So $\rho_U^{BT}(\hat{D}) - \rho_U^{BT}(\sum_{i=0}^n a_i \lambda_i) = 0$. So $c_1(\mathcal{L}_{D_T} |_{\pi^{-1}(U)}) - \sum_{i=0}^n a_i \lambda_i = 0)$. Let $j : \pi^{-1}(U) \longrightarrow \pi^{-1}(BT)$ be the inclusion map. Then $j^*(c_1(\mathcal{L}_{D_T}) - \sum_{i=0}^n a_i \lambda_i) = 0$ (note that $j^*(\sum_{i=0}^n a_i \lambda_i) = \sum_{i=0}^n a_i \lambda_i$ because j^* is a $\mathbb{C}[\lambda_0, ..., \lambda_n]$-module homomorphism). But $c_1(\mathcal{L}_{D_T}) - \sum_{i=0}^n a_i \lambda_i = c_1(\mathcal{O}(D_T)) - c_1(\mathcal{O}(\sum_{i=0}^n a_i \lambda_i)) = c_1(\mathcal{O}(D_T) \otimes \mathcal{O}(-\sum_{i=0}^n a_i \lambda_i)) = c_1(\mathcal{O}(D_T - \sum_{i=0}^n a_i \lambda_i))$. So $j^*(c_1(\mathcal{O}(D_T - \sum_{i=0}^n a_i \lambda_i))) = 0$. But j^* is the restriction map. It follows $D_T - \sum_{i=0}^n a_i \lambda_i$ is linearly equivalent to zero. So $[D_T] - \sum_{i=0}^n a_i \lambda_i$ is rationally equivalent to zero. So $i_X^*([D_T] - \sum_{i=0}^n a_i \lambda_i) = 0$. But $i_X^*([D_T]) = [D]$ and $i_X^*(\lambda_i) = 0$. Therefore $[D] = 0$. So ϕ is injective.

Second we check that ϕ is surjective. Note that the surjectivity of the map ϕ follows immediately from diagram below because it is commutative. So $\phi \circ i_X^* = \psi$. But ψ is surjective so ϕ is surjective.

$$\begin{array}{ccc} H^2(X_T, \mathbb{C}) & \xrightarrow{\psi} & H^0(BT, \mathcal{R}^2\pi_*\mathbb{C}) \\ \downarrow{\scriptstyle i_X^*} & \nearrow{\scriptstyle \phi} & \\ H^2(X, \mathbb{C}) & & \end{array} \tag{67}$$

To check that the above diagram is commutative it is enough to check that $\psi([F_T] - \widehat{i_X^*[F_T]}) = 0$ where F is a T-invariant subvariety of X of codimension one (because $\psi([F_T] - \widehat{i_X^*[F_T]}) = \psi([F_T]) - \psi(\widehat{i_X^*[F_T]}) = \psi([F_T]) - \phi \circ i_X^*([F_T]))$. Let $\{U_i\}_i$ be an open cover for BT. Since $\rho_{U_i}^{BT}([\hat{F}]) = c_1(\mathcal{L}_{F_T} \mid_{\pi^{-1}(U_i)}) = i^*_{\pi^{-1}(U_i)}(c_1(\mathcal{L}_{F_T}) = i^*_{\pi^{-1}(U_i)}[F_T] = \rho_{U_i}^{BT}[F_T]$. So $\rho_{U_i}^{BT}([\hat{F}] - [F_T]) = 0$ for all i. It follows from a sheaf axiom that $[\hat{F}] - [F_T] = 0$. So $\psi([F_T] - \widehat{i_X^*[F_T]}) = \psi(0) = 0$.

Third we will check that ϕ is well-defined. Suppose $[D] = [D']$ i.e $D \sim D'$ where D and D' are T-invariant divisors in X. Then there exists a rational function $f : X \longrightarrow \mathbb{C}$ such that $D - D' = (f)$.

STEP 1: there exists a character ρ such that $t.f = \rho(t)f$.
Let $t \in T$ then $t.f$ and f have the same zeros and poles since $D - D'$ is T-invariant. So $\frac{t.f}{f}$ has no zeros or poles in X where X is a compact set. Therefore $\frac{t.f}{f} = c_t$ where c_t is a constant. Let us check that c_t is a character. First recall that a character $\chi : T \longrightarrow \mathbb{C}^*$ satisfies the property $\chi(t.t') = \chi(t)\chi(t')$ where $t,\ t' \in T$. Also recall that $(t.t').x = t.(t'.x)$ where $x \in X$. So it is enough to check that $c_{t.t'} = c_t c_{t'}$. First note that $t.f(x) = c_t f(x)$ implies $f(t^{-1}x) = c_t f(x)$. Write $f(t^{-1}x) = (f \circ t^{-1})(x)$

where t^{-1} defines the bijection on X given by $x \longmapsto t^{-1}x$. So $(f \circ t^{-1})(x) = c_t f(x)$. Now $c_{t.t'} f(x) = (t.t').f(x) = f((t.t')^{-1}.x) = (f \circ (t.t')^{-1})(x) = (f \circ (t^{-1}t'^{-1}))(x)$. But $(f \circ (t^{-1}t'^{-1}))(x) = (f \circ t^{-1})(t'^{-1}x)$. It follows $c_{t.t'} f(x) = (f \circ t^{-1})(t'^{-1}x) = (c_t f)(t'^{-1}x) = c_t f(t'^{-1}x) = c_t c_{t'} f(x)$. Therefore $c_{t.t'} = c_t c_{t'}$. Thus step 1 is done.

STEP 2: $\mathcal{O}((f_T)) \simeq \pi^*(\mathbb{C}_\rho)_T$

PROOF. Let V be a T-invariant open subset of the B-variety X. Define a morphism of abelian groups $\varphi(V_T) : \pi^*(\mathbb{C}_\rho)_T(V_T) \longrightarrow \mathcal{O}((f_T))(V_T)$ as follows: for simplicity we use φ to denote $\varphi(V_T)$. If $\beta \in \pi^*(\mathbb{C}_\rho)_T(V_T)$ then $\beta([a,v]) := \{a\} \times g_\beta(a)$ where the function $g_\beta : ET \longrightarrow \mathbb{C}$. So $t.g_\beta(a) = \rho(t)g_\beta(a)$ where $a \in ET$. Define $\varphi(\beta) = g_\beta(a)f(v)$ where $a \in ET$, $v \in V$. Then $\varphi(\beta) \in \mathcal{O}((f_T))(V_T)$ by fact 5.2.9. To check that φ is an isomorphism, we define the morphism of abelian groups $\mu(V_T) : \mathcal{O}((f_T))(V_T) \longrightarrow \pi^*(\mathbb{C}_\rho)_T(V_T)$ as follows: for simplicity let μ denote $\mu(V_T)$. If $s \in \mathcal{O}((f_T))(V_T)$ then by fact 5.2.9 $s([a,v]) = \sum_{i=1}^k g_i(a)f(v)$ where $g_i \in \Gamma(\gamma^*(\mathbb{C}_\rho)_T)$. We define $\mu(s) = \alpha$ where $\alpha([a,v]) = \{a\} \times \sum_{i=1}^k g_i(a)$. Now $\alpha \in \pi^*(\mathbb{C}_\rho)_T(V_T)$ because $\alpha([a,v]) \in (\pi^*(\mathbb{C}_\rho)_T)_{[a,v]} = ((\mathbb{C}_\rho)_T)_a = \{a\} \times \mathbb{C}_\rho$ and $g_i(a) \in \mathbb{C}_\rho$ (since $g_i \in \Gamma(\gamma^*(\mathbb{C}_\rho)_T)$). Clearly $\mu \circ \varphi = Id_{\pi^*(\mathbb{C}_\rho)_T(V_T)}$ and $\varphi \circ \mu = Id_{\mathcal{O}((f_T))(V_T)}$. Thus $\mathcal{O}((f_T)) \simeq \pi^*(\mathbb{C}_\rho)_T$. Hence step 2 is done.

Note that $\pi^*(\mathbb{C}_\rho)_T = \pi^*\mathcal{O}(-\sum_{i=0}^n a_i\lambda_i)$ where $\sum_{i=0}^n a_i\lambda_i$ is the weight of the character ρ. But $\pi^*\mathcal{O}(\sum_{i=0}^n a_i\lambda_i) = \mathcal{O}(\sum_{i=0}^n a_i\pi^*\lambda_i) = \mathcal{O}(\sum_{i=0}^n a_i\lambda_i)$ since $\lambda_i := \pi^*\lambda_i$. So $\pi^*(\mathbb{C}_\rho)_T = \mathcal{O}(-\sum_{i=0}^n a_i\lambda_i)$. Now let us check that ϕ is well-defined. Suppose $[D] = [D']$, i.e $D \sim D'$ where D and D' are T-invariant divisors in X. Then there exists a rational function $f : X \longrightarrow \mathbb{C}$ such that $D - D' = (f)$. Now $\tilde{\mathcal{D}} - \tilde{D'} = \psi(\hat{D}) - \psi(\hat{D'}) = \psi(\hat{D} - \hat{D'})$.

STEP 3: $\hat{D} - \hat{D'} = -\sum_{i=0}^n a_i\pi^*\lambda_i$. Therefore

$$\tilde{\mathcal{D}} - \tilde{D'} = \psi(-\sum_{i=0}^{n} a_i \pi^* \lambda_i) = -\sum_{i=0}^{n} a_i (\psi \circ \pi^*)(\lambda_i) = 0$$

because $\psi \circ \pi^* = 0_{map}$ since the sequence (66) is exact. Therefore ϕ is well-defined. Now we prove $\hat{D} - \hat{D'} = -\sum_{i=0}^{n} a_i \pi^* \lambda_i$. Let U_i be an open cover for BT. Then $(\hat{D} - \hat{D'}) |_{U_i} = s_{U_i} - s'_{U_i} = c_1(\mathcal{L}_{D_T} |_{\pi^{-1}(U_i)}) - c_1(\mathcal{L}_{D'_T} |_{\pi^{-1}(U_i)}) = i^*_{\pi^{-1}(U_i)}(c_1(\mathcal{L}_{D_T}) - c_1(\mathcal{L}_{D'_T}))$ where the map $i_{\pi^{-1}(U_i)} : \pi^{-1}(U_i) \longrightarrow X_T$ is the inclusion map. Let $\mathcal{O}(D_T)$ denote $\mathcal{L}_{D_T}$ then $(\hat{D} - \hat{D'}) |_{U_i} = i^*_{\pi^{-1}(U_i)}(c_1(\mathcal{O}(D_T)) - c_1(\mathcal{O}(D'_T))) = i^*_{\pi^{-1}(U_i)}(c_1(\mathcal{O}(D_T \otimes \mathcal{O}(D'^*_T)) = i^*_{\pi^{-1}(U_i)} c_1(D_T - D'_T) = i^*_{\pi^{-1}(U_i)} c_1(\mathcal{O}(f_T))$. But $\mathcal{O}((f_T)) \simeq \pi^*(\mathbb{C}_\rho)_T = \mathcal{O}(-\sum_{i=0}^{n} a_i \lambda_i)$. It follows

$$(\hat{D} - \hat{D'}) |_{U_i} = i^*_{\pi^{-1}(U_i)}(c_1(\mathcal{O}(-\sum_{i=0}^{n} a_i \lambda_i))) = i^*_{\pi^{-1}(U_i)}(-\sum_{i=0}^{n} a_i \lambda_i) = -\sum_{i=0}^{n} a_i \lambda_i$$

since $i^*_{\pi^{-1}(U_i)} : H^2(\pi^{-1}(U_i), \mathbb{C}) \longrightarrow H^2(X_T, \mathbb{C})$ is a $\mathbb{C}[\lambda_0, ..., \lambda_n]-$module homomorphism. So $(\hat{D} - \hat{D'} + \sum_{i=0}^{n} a_i \lambda_i) |_{U_i} = 0$ for all i. It follows from a sheaf axiom $\hat{D} - \hat{D'} + \sum_{i=0}^{n} a_i \lambda_i = 0$. So $\hat{D} - \hat{D'} = -\sum_{i=0}^{n} a_i \lambda_i$. Thus step 3 is done.

I will use i_j^* instead of i_{jT}^* for simplicity where $i_{jT}^* : H_T^*(X) \longrightarrow H_T^*(p_j)$ is the map induced by the equivariant inclusion $i_{jT} : (p_j)_T \hookrightarrow X_T$.

Theorem 5.2.12. Let X be a B-variety and let $D_i \subset X$ be a $T-$*invariant* subvariety of codimension one, $i = 1, 2$. Then

(a) $D_1 \sim D_2 \Leftrightarrow [D_{1T} - D_{2T}] \in Span\{\lambda_i : i = 0, 1, ..., n\}$, where $Span\{\lambda_i : i = 0, 1, ..., n\} = \{\sum_{i=0}^{n} c_i \lambda_i : c_i \in \mathbb{Z}\}$

(b) Let $p_j \in X^T$, $i_j{}^* : H_T^*(X) \longrightarrow H_T^*(p_j)$ be the map induced by the equivariant inclusion $i_{jT} : p_{jT} \hookrightarrow X_T$. If $[D_T] \in H_T^2(X)$ is any equivariant divisor such that there exists $a_i \in \mathbb{Z}$, $i = 0, 1, 2$ with $i_j^*([D_T]) = \sum_{i=0}^{n} a_i \lambda_i$, $\forall j = 0, 1, ..., n$, then $[D_T] = \sum_{i=0}^{n} a_i \lambda_i$.

Proof.

(a) ($\Rightarrow$): Suppose that $D_1 \sim D_2$ then $[D_1 - D_2] = 0$. But $i_X^*([(D_1 - D_2)_T]) = [D_1 - D_2] = 0$ implies $[(D_1 - D_2)_T] \in \ker i_X^*$.

CLAIM. $\ker i_X^* = \text{Span}\{\lambda_i : i = 0, 1, ..., n\}$. Consider the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\pi_{pt}} & pt \\ \downarrow{\scriptstyle i_X} & & \downarrow{\scriptstyle i_{pt}} \\ X_T & \xrightarrow{\pi} & BT \end{array} \tag{68}$$

Which gives the following commutative diagram

$$\begin{array}{ccc} H^*(BT) & \xrightarrow{i_{pt}^*} & H^*(pt) \\ \downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle \pi_{pt}^*} \\ H^*(X_T) & \xrightarrow{i_X^*} & H^*(X) \end{array} \tag{69}$$

Note that $H^*(BT) = \mathbb{C}[\lambda_0, \lambda_1, ..., \lambda_n] \subset H^*(X_T)$ where $\lambda_i := \pi^*(\lambda_i)$, $i = 0, ..., n$. Since the above diagram is commutative, $i_X^* \circ \pi^*(\lambda_i) = \pi_{pt}^* \circ i_{pt}^*(\lambda_i) = 0$, $i = 0, ..., n$ which implies $i_X^*(\lambda_i) = 0$, $i = 0, ..., n$. Thus $\ker i_X^*$ contains $\text{span}\{\lambda_i : i = 0, ...n\}$. It remains to show that $\ker i_X^* \subset span\{\lambda_i : i = 0, ..., n\}$.

$$H^2(\mathcal{F}) = F^0 \supset F^1 = F^2 \tag{70}$$

This gives rise to a short exact cohomology sequence

$$0 \to F^2 \hookrightarrow F^0 \to F^0/F^2 \to 0 \tag{71}$$

But $F^3 = 0$ (see remark 5.2.5 above) implies $F^2 = F^2/F^3 = H^2(BT, \mathbb{C})$, and $F^0/F^2 = F^0/F^1 = H^0(BT, \mathcal{R}^2\pi_*\mathbb{C})$. By lemma 5.2.11 $\mathcal{R}^2\pi_*\mathbb{C} = \mathbb{C}^k$. It follows that

$$F^0/F^2 = H^0(BT, \mathbb{C}^k) = \oplus_{i=1}^k H^0(BT, \mathbb{C}) = \oplus_{i=1}^k \mathbb{C} = \mathbb{C}^k$$

where $H^0(BT, \mathbb{C}) = \mathbb{C}$ because BT is connected. Also, by Lemma 5.2.6 $\pi_*\mathbb{C} = \mathbb{C}$. Thus our short exact sequence becomes

$$0 \to H^2(BT, \mathbb{C}) \hookrightarrow H^2(X_T, \mathbb{C}) \to H^0(BT, \mathbb{C}^k) \to 0 \tag{72}$$

Recall from Lemma 5.2.11 the isomorphism $\phi : H^2(X, \mathbb{C}) \longrightarrow H^0(BT, \mathbb{C}^k)$ defined by $\phi([D]) = \tilde{D}$ where D is a T-invariant divisor in X, and $\tilde{D}$ is the global section associated with D. Let $\rho = \phi^{-1}$ then $\rho \circ \psi = i_X^*$. Since $\mathcal{R}^2\pi_*\mathbb{C} \simeq \mathbb{C}^k$ it follows that $H^0(BT, \mathcal{R}^2\pi_*\mathbb{C}) = H^0(BT, \mathbb{C}^k) = \oplus_{i=1}^k H^0(BT, \mathbb{C})$. But $H^0(BT, \mathbb{C}) = \mathbb{C}$ because BT is connected. So $H^0(BT, \mathcal{R}^2\pi_*\mathbb{C}) = \mathbb{C}^k$. But $H^2(X, \mathbb{C})$ is a k-dimensional space. It follows ρ is an isomorphism of k-dimensional spaces. Also note that $i_X^* \circ \pi^* = 0_{map}$, where 0_{map} is the zero map. Clearly $H^2(BT, \mathbb{C})$ is a subset of $H^*(BT, \mathbb{C}) = \mathbb{C}[\lambda_0, ..., \lambda_n]$. If c is in $H^2(BT, \mathbb{C})$, then $c = \sum_{i=1}^k a_i \lambda_i$, $a_i \in \mathbb{C}$. Since $i_X^* \circ \pi^* = \pi_{pt}^* \circ i_{pt}^*$ it follows that $(i_X^* \circ \pi^*)(c) = (\pi_{pt}^* \circ i_{pt}^*)(c)$. But $(\pi_{pt}^* \circ i_{pt}^*)(\sum_{i=1}^k a_i \lambda_i) = \sum_{i=1}^k a_i \pi_{pt}^*(i_{pt}^*(\lambda_i)) = \sum_{i=1}^k a_i \pi_{pt}^*(0) = 0$. Thus $i_X^* \circ \pi^* = 0_{map}$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^2(BT, \mathbb{C}) & \xrightarrow{\pi^*} & H^2(X_T, \mathbb{C}) & \xrightarrow{\psi} & H^0(BT, \mathbb{C}^k) & \longrightarrow & 0 \\ & & & \searrow^{0_{map}} & \downarrow{\scriptstyle i_X^*} & \swarrow_{\rho} & & & \\ & & & & H^2(X, \mathbb{C}) & & & & \end{array} \tag{73}$$

Now we show that ρ is injective $\Leftrightarrow \ker i_X^* = H^2(BT, \mathbb{C})$ (see the diagram above).

($\Leftarrow$): Suppose $\rho(s) = 0$ where $s \in H^0(BT, \mathbb{C}^k)$. We will show that $s = 0$. Since ψ is surjective there exists $w \in H^2(X_T, \mathbb{C})$ such that $\psi(w) = s$. It follows $\rho(\psi(w)) = \rho(s) = 0$. But $\rho \circ \psi = i_X^*$ implies $i_X^*(w) = 0$ which implies $w \in keri_X^* = H^2(BT, \mathbb{C}) \subseteq H^*(BT, \mathbb{C}) = \mathbb{C}[\lambda_0, ..., \lambda_n]$. It follows $w = \sum_{i=1}^m b_i\lambda_i$,$b_i \in \mathbb{C}$. Now $w = \sum_{i=1}^m b_i\lambda_i := \sum b_i\pi^*(\lambda_i)$. It follows $s = \psi(w) = \psi(\sum_{i=1}^m b_i\pi^*(\lambda_i)) = \sum_{i=1}^m b_i\psi(\pi^*(\lambda_i)) = \sum_{i=1}^m b_i.0 = 0$ (because $\ker\psi = \operatorname{im}\pi^*$ since our sequence above is exact). Thus $s = 0$. Hence ρ is injective.

($\Rightarrow$): Suppose ρ is injective. Let $\alpha \in \ker i_X^*$ then $i_X^*(\alpha) = 0$. It follows $0 = i_X^*(\alpha)$ $=(\rho \circ \psi)(\alpha)$$= \rho(\psi(\alpha))$. But ρ is injective implies $\psi(\alpha) = 0$ which implies $\alpha \in \ker\psi = \operatorname{im}\pi^*$. Thus $\alpha \in H^2(BT, \mathbb{C})$. Hence $\ker i_X^* \subset H^2(BT, \mathbb{C})$. It remains to show that $\ker i_X^*$ contains $H^2(BT, \mathbb{C})$. So, let $\beta \in H^2(BT, \mathbb{C})$. By part (2) of remark 5.1.8 we have $i_X^* \circ \pi^* = 0_{map}$. It follows $\operatorname{im}\pi_X^* \subset \ker i_X^*$. Thus β lives in $\ker i_X^*$. Hence $\ker i_X^* \supset H^2(BT, \mathbb{C})$.

But ρ is an isomorphism. It follows $\ker i_X^* = H^2(BT, \mathbb{C}) \subseteq H^*(BT, \mathbb{C}) = \mathbb{C}[\lambda_0, ..., \lambda_n]$. Thus we have $\ker i_X^* \subseteq Span\{\lambda_i : i = 0, ..., n\}$. But $\ker i_X^* \supset Span\{\lambda_i : i = 0, ..., n\}$. Therefore $\ker i_X^* = Span\{\lambda_i : i = 0, ..., n\}$. Thus laim 1 above is proven . Hence we are done with the proof of Theorem 5.2.12 part (a)($\Rightarrow$).

Now let us prove the other direction of part (a) of Theorem 5.2.12.

(a)($\Leftarrow$): Suppose $[D_{1T} - D_{2T}] \in Span\{\lambda_i : i = 0, 1, ..., n\}$ then there exists $c_i \in \mathbb{Z}$ such that $[D_{1T} - D_{2T}] = \sum_{i=0}^2 c_i\lambda_i$. Consider the inclusion map $i_X : X \hookrightarrow X_T$ where $i_X^*([Z]_T) = [Z]$. Consider diagram (69). Note that $\lambda_i := \pi^*(\lambda_i)$. It follows $[D_{1T} - D_{2T}] = \sum_{i=0}^2 c_i.\pi^*(\lambda_i)$ which implies that $i_X^*([D_{1T} - D_{2T}]) = i^*(\sum_{i=0}^2 c_i.\pi^*(\lambda_i)) = \sum_{i=0}^2 c_i.i_X^*(\pi_X^*(\lambda_i)) = \sum_{i=0}^2 c_i.\pi_{pt}^*(i_{pt}^*(\lambda_i))$. But $i_{pt}^*(\lambda_i) = 0$. It follows $i_X^*([D_{1T} - D_{2T}]) = \sum_{i=0}^2 c_i.\pi_{pt}^*(0) = \sum_{i=0}^2 c_i.0 = 0$. Thus $[D_1 - D_2] = i_X^*([(D_1 - D_2)_T]) = i_X^*([D_{1T} - D_{2T}]) = 0$. Hence $D_1 \sim D_2$.

(b) Let $\mathbb{C}(\lambda) = \mathbb{C}(\lambda_0, ..., \lambda_n)$ be the field of fractions of $\mathbb{C}[\lambda_0, ..., \lambda_n]$. Consider the map $\phi : H_T^*(X) \otimes_{\mathbb{C}[\lambda]} \mathbb{C}(\lambda) \longrightarrow \oplus_{j=0}^n H_T^*(p_j) \otimes_{\mathbb{C}[\lambda]} \mathbb{C}(\lambda)$ where $\phi(\alpha \otimes f(\lambda)) = (i_j^*(\alpha).f(\lambda))_{j=0}^n$. Let $[D_T] \in H_T^2(X) \subset H_T^2(X) \otimes \mathbb{C}(\lambda)$ such that for each j we have $i_j^*([D_T]) = l(\lambda)$ where $l(\lambda) = \sum_{i=0}^n a_i\lambda_i$. Then $\phi(l(\lambda) \otimes 1) = \phi(1 \otimes l(\lambda)) = (i_j^*(1).l(\lambda))_{j=0}^n = (l(\lambda))_{j=0}^n = (i_j^*([D_T]))_{j=0}^n$. But $\phi([D_T] \otimes 1) = (i_j^*([D_T])_{j=0}^n$. It follows $\phi(l(\lambda) \otimes 1) = \phi([D_T] \otimes 1)$. But ϕ is injective implies $l(\lambda) \otimes 1 = [D_T] \otimes 1$ which implies $(l(\lambda) - [D_T]) \otimes 1 = 0$. Thus $[D_T] = l(\lambda)$.

5.3 T-INVARIANT RATIONAL EQUIVALENCE IN $\mathbb{P}^2$

Definition 5.3.1. Let X be a projective variety. Two k-dimensional subvarieties V and V' are said to be rationally equivalent, written $V \sim V'$, if there exists a sequence of k-dimensional subvarieties $V_0 = V, V_1, ..., V_n = V'$ and a sequence of $(k+1)$-dimensional subvarieties $W_1, ..., W_n$ such that V_{i-1}, V_i are contained in W_i, and $V_{i-1} \sim V_i$ in W_i , $i = 1, ..., n$.

Definition 5.3.2. Let Z be a B-variety. A T-invariant k-dimensional subvariety V of Z is T-invariantly rationally equivalent to a T-invariant k-dimensional subvariety V' of Z, written $V \overset{T}{\sim} V'$, if there exists a sequence of T-invariant k-dimensional subvarieties $V_0 = V, V_1, ..., V_n = V'$ and a sequence of T-invariant $(k+1)$-dimensional subvarieties $W_1, ..., W_n$ such that V_{i-1}, V_i are contained in W_i, $i = 1, ..., n$ and $V_{i-1} \sim V_i$ in W_i , $i = 1, ..., n$.

Definition 5.3.3. If $f : X \longrightarrow Y$ is a regular embedding of codimension d, and $g : Y' \longrightarrow Y$ is an arbitrary morphism such that Y' is a smooth variety. Form the fiber square i.e. $X' = X \times_Y Y' = g^{-1}(X)$. We define the *refined Gysin homomorphism* $f_! : A_k(X') \longrightarrow A_{k-d}(Y')$ by the formula $f_![V] = f'_*[V].[X']$ where f' is the morphism in the fiber square. See 6.2 in [17].

Example 5.3.4. Let X be a projective variety with an action of $T = (\mathbb{C}^*)^m$, and a finite set of fixed points, and let $Z \subset X$ be a T-invariant subvariety. Consider the fiber square below where i is a regular embedding of codimension d, i_T is the

$$\begin{array}{ccc} Z_T & \overset{i_T}{\hookrightarrow} & X_T \\ \downarrow & & \downarrow \\ Z & \overset{i}{\hookrightarrow} & X \end{array}$$

inclusion map induced by i. Then $i_! : A_k(Z_T) \longrightarrow A_{k-d}(X_T)$ is given by the formula $i_![V] = i_{T*}[V].[Z_T]$.

Conjecture 5.3.5. Let X be an n-dimensional projective variety with a torus action $T = (\mathbb{C}^*)^{m+1}$ and a finite set of fixed points $X^T = \{q_j\}_{j=0}^k$. Consider the equivariant maps $i_{jX}^* : H_T^*(X) \longrightarrow H_T^*(q_j)$, $i_{jZ}^* : H_T^*(Z) \longrightarrow H_T^*(q_j)$, where Z is a T-invariant subvariety of X, and $q_j \in Z$ for some j. Let $D \subset Z$ be a T-invariant subvariety. If $i : Z \hookrightarrow X$ is the inclusion map we have

$$(i_{jX}^* \circ i_!)([D_T]) = (i_{jZ}^*)([D_T]).(i_{jX}^*)([Z_T]).$$

Example 5.3.6. Consider the action of $T = (\mathbb{C}^*)^3$ on $X = \mathbb{P}^2$. Let $D = \mathcal{Z}(x_0, x_1)$, and let $Z = \mathcal{Z}(x_0)$. Let $i : Z \hookrightarrow X$ be the inclusion map then

$$\begin{aligned} i_{jX}^* \circ i_![D_T] = &\quad i_{jX}^* i_!(p - \lambda_1) = i_{jX}^*([Z_T].(p - \lambda_1)) = i_{jX}^*((p - \lambda_0)(p - \lambda_1)) \\ = &\quad (\lambda_j - \lambda_0)(\lambda_j - \lambda_1) \end{aligned}$$

But $i_{jZ}^*[D_T].i_{jX}^*[Z_T] = i_{jZ}^*(p - \lambda_1).i_{jX}^*(p - \lambda_0) = (\lambda_j - \lambda_0)(\lambda_j - \lambda_1)$. Thus

$$(i_{jX}^* \circ i_!)([D_T]) = (i_{jZ}^*)([D_T]).(i_{jX}^*)([Z_T]).$$

Theorem 5.3.7. Let $X = \mathbb{P}^2$ equipped with a torus action $T = (\mathbb{C}^*)^3$ and a finite set of fixed points $X^T = \{p_j\}_{j=0}^2$ of the torus action, where the three fixed points p_j, ordered as usual, so that the $j-th$ coordinate of p_j is nonzero, all other coordinates being zero. Let $Z \subset X$ be a $T-$invariant subvariety of codimension one, $D_i \subset Z$ be a $T-$invariant subvariety of codimension two in X, $i = 1, 2$. Then

(1) If Z is irreducible then

$$D_1 \sim D_2 \text{ in } Z \Leftrightarrow [D_{1T} - D_{2T}] \in Span\{\lambda_i.[Z_T] : i = 0, 1, 2\}$$

where $Span\{\lambda_i.[Z_T] : i = 0,1,2\} = \{\sum_{i=0}^{2} c_i\lambda_i.[Z_T] : c_i \in \mathbb{Z}\}$

(2) If $D_1 \sim D_2$ in Z where Z is an irreducible subvariety of X then there exists an $l(\lambda) = \sum_{i=0}^{2} a_i\lambda_i$, $a_i \in \mathbb{Z}$, $i = 0,1,2$ such that $i_j{}^*([D_{1T} - D_{2T}]) = l(\lambda).i_j{}^*([Z_T])$, $j = 0,1,2$, where $i_j{}^* : H_T^*(X) \longrightarrow H_T^*(p_j)$, $j = 0,1,2$ is induced by the equivariant inclusion $i : Z_j \hookrightarrow X$, $j = 0,1,2$.

(3) If $D_1 \overset{T}{\sim} D_2$ and $D_i \subset \cup_{k=0}^{m} Z_k$, $i = 1,2$ is a codimension 2 subvariety of X such that $D_i = \sum_{k=0}^{2} a_{ik}[p_k]$, $i = 1,2$, $\sum_{k=0}^{2} a_{1k} = \sum_{k=0}^{2} a_{2k}$, and $[Z_k] = [x_k = 0]$ where x_k is a coordinate of $\mathbb{P}^2$, $k = 0,1,2$. Then there exists an $l_k(\lambda) = \sum_{i=0}^{2} b_{ik}\lambda_i$, $b_{ik} \in \mathbb{Z}$, $i = 0,1,2$, $k = 0,1,2$ such that $i_j{}^*([D_{1T} - D_{2T}]) = \sum_{k=0}^{2} l_k(\lambda).i_j{}^*([Z_{kT}])$, $j = 0,1,2$.

Proofs. Part(1) is a special case of part(1) of theorem 5.4.1 which will be proven in the next section. To prove part(2), Suppose that $Z \subset \mathbb{P}^2$ is an irreducible, $T-$invariant subvariety of codimension 1 then the cycle $[Z] = [x_i = 0]$ for some i, $i = 0,1,2$. So the equivariant cycle $[Z_T] = p - \lambda_i$, for some i, $i = 0,1,2$. Since $D_i \subset Z$, $i = 1,2$ is a $T-$invariant subvariety of codimension 2 in $\mathbb{P}^2$, and $D_1 \sim D_2$ in Z then $[(D_1 - D_2)_T] = \sum_{r=0}^{2} a_r[p_{rT}]$ such that $\sum_{r=0}^{2} a_r = 0$. Clearly $[D_1 - D_2] = \sum_{r=0}^{2} a_r[p_r]$ is a class on Z implies $a_r = 0$, for some r. Now if $a_0 = a_1 = 0$ then $a_2 = 0$. So $[D_1 - D_2] = 0$. Take $l(\lambda) = 0$. Similarly if $a_0 = a_2 = 0$ or $a_1 = a_2 = 0$ then $[D_1 - D_2] = 0$. Take $l(\lambda) = 0$, so $i_j{}^*([D_{1T} - D_{2T}]) = l(\lambda).i_j{}^*([Z_T])$ is satisfied, $j = 0,1,2$. Suppose $a_2 = 0$ such that $a_2 \neq a_0$, and $a_2 \neq a_1$ then $a_1 = -a_0$, and $[Z_T] = p - \lambda_2$. Now

$$\begin{aligned}
i_j{}^*([D_{1T} - D_{2T}]) &= i_j{}^*(a_0[(p_0 - p_1)_T]) = a_0.i_j{}^*([(p_0 - p_1)_T]) \\
&= a_0.i_j{}^*([p_{0T} - p_{1T}]) = a_0.i_j{}^*((p - \lambda_1)(p - \lambda_2) - (p - \lambda_0)(p - \lambda_2)) \\
&= a_0.i_j{}^*((\lambda_0 - \lambda_1)(p - \lambda_2)) = a_0.(\lambda_0 - \lambda_1)i_j{}^*(p - \lambda_2) \\
&= a_0.(\lambda_0 - \lambda_1)i_j{}^*([Z_T])
\end{aligned}$$

Take $l(\lambda) = a_0.(\lambda_0 - \lambda_1)$ then $i_j{}^*([D_{1T} - D_{2T}]) = l(\lambda).i_j{}^*([Z_T])$, $j = 0, 1, 2$. Similarly, If $a_1 = 0$ such that $a_1 \neq a_0$, and $a_1 \neq a_2$ then take $l(\lambda) = a_0.(\lambda_0 - \lambda_2)$. Finally, if $a_0 = 0$ such that $a_0 \neq a_1$, and $a_0 \neq a_2$ then take $l(\lambda) = a_1.(\lambda_1 - \lambda_2)$.

To prove part(3), Suppose $D_1 \overset{T}{\sim} D_2$ in $\mathbb{P}^2$ then $D_{1T} - D_{2T} = \sum_{i=0}^{2} a_i p_{iT}$, $a_i \in \mathbb{Z}$, $i = 0, 1, 2$ such that $\sum_{i=0}^{2} a_i = 0$. Now since $a_2 = -a_0 - a_1$ then

$$i_j{}^*([D_{1T} - D_{2T}]) = i_j{}^*(\sum_{i=0}^{2} a_i[p_{iT}]) = i_j{}^*(a_0[(p_0 - p_2)_T] + a_1([(p_1 - p_2)_T]).$$

But $[(p_0 - p_2)_T] = (\lambda_0 - \lambda_2)(p - \lambda_1)$, and $[(p_1 - p_2)_T] = (\lambda_1 - \lambda_2)(p - \lambda_0)$. So

$$\begin{aligned} i_j{}^*([D_{1T} - D_{2T}]) &= i_j{}^*(a_0(\lambda_0 - \lambda_2)(p - \lambda_1) + a_1(\lambda_1 - \lambda_2)(p - \lambda_0)) \\ &= a_0(\lambda_0 - \lambda_2)i_j{}^*(p - \lambda_1) + a_1(\lambda_1 - \lambda_2)i_j{}^*(p - \lambda_0) \end{aligned}$$

Let $l_0(\lambda) = a_1(\lambda_1 - \lambda_2)$, $l_1(\lambda) = a_0(\lambda_0 - \lambda_2)$, $l_2(\lambda) = 0$, and let the equivariant class $[Z_{iT}] = [(x_i = 0)_T] = p - \lambda_i$, $i = 0, 1, 2$ then $i_j{}^*([D_{1T} - D_{2T}]) = \sum_{k=0}^{2} l_k(\lambda).i_j{}^*([Z_{kT}])$, $j = 0, 1, 2$.

5.4 T-INVARIANT RATIONAL EQUIVALENCE IN A B-VARIETY

Theorem 5.4.1. Let X be an n-dimensional B-variety where $X^T = \{q_j\}_{j=0}^{m}$. let $Z \subset X$ be a $T-$invariant subvariety of dimension $k + 1 < n$, and let $D_i \subset Z$ be a $T-$invariant subvariety of dimension k, $i = 1, 2$. Then

(1) If Z is irreducible then

$$D_1 \sim D_2 \text{ in } Z \Leftrightarrow [D_{1T} - D_{2T}] \in Span\{\lambda_i.[Z_T] : i = 0, 1, ..., n\}$$

where $Span\{\lambda_i.[Z_T] : i = 0, 1, ..., n\} = \{\sum_{i=0}^{n} c_i \lambda_i.[Z_T] : c_i \in \mathbb{Z}\}$.

(2) If $D_1 \sim D_2$ in Z where Z is an irreducible subvariety of X then there exists an $l(\lambda) = \sum_{i=0}^{n} a_i\lambda_i$, $a_i \in \mathbb{Z}$, $i = 0, ..., n$ such that $i_{jX}^*([D_{1T} - D_{2T}]) = l(\lambda).i_{jX}^*([Z_T])$, $j = 0, ..., m$.

(3) If $D_1 \overset{\mathrm{T}}{\sim} D_2$ and $D_i \subset \cup_{k=1}^{m} Z_k$, $i = 1, 2$ is a subvariety of dimension r such that $D_i = \sum_{k=1}^{m} D_{ik}$, and $D_{1k} \sim D_{2k}$ in Z_k where $Z_k \subset X$ is an irreducible T-invariant subvariety of dimension $r+1$ and $D_{ik} \subset Z_k$ is a T-invariant subvariety of dimension r. Then there exists an $l_k = \sum_{i=0}^{n} a_{ik}\lambda_i$ where $a_{ik} \in \mathbb{Z}$, $i = 0, ..., n$, $k = 1, ..., m$ such that

$$i_{jX}^*([D_{1T} - D_{2T}]) = \sum_{k=1}^{m} l_k.i_{jX}^*([Z_{kT}])$$

where $j = 0, ..., n$.

Proofs.

(1) Proof. ($\Longrightarrow$) Suppose that $D_1 \sim D_2$ in Z where D_i, Z are T-invariant then $D_1 - D_2 = (f)$ where $f : Z \longrightarrow \mathbb{P}^1$ is a rational function on Z. As in the proof of lemma 5.2.11 there exists a character ρ such that $t.f = \rho(t)f$. Recall the map $\pi : X_T \longrightarrow BT$ where X_T is a fiber bundle over BT. Let V be a T-invariant open subset of the B-variety X. Recall $V_T = ET \times_T V$ and let $[a, x]$ denote a class in V_T. We define the sheaf $\mathcal{O}(D_{jT})$, $j = 1, 2$ as follows: $\mathcal{O}(D_{jT})(V_T) := \{\sum_{i=1}^{d} h_i(x)g_i(a) : h_i(x)$ is a meromorphic function on V, and $g_i(a)$ is a continuous function on ET such that $(h_i) + D_j \geq 0$ and $h_i(tx)g_i(at^{-1}) = h_i(x)g_i(a)\}$. Claim: $\mathcal{O}(D_{1T}) \simeq \mathcal{O}(D_{2T}) \otimes \pi^*(\mathbb{C}_{\rho^{-1}})_T$. Let φ denote $\varphi(V_T)$. We define the morphism $\varphi : \mathcal{O}(D_{2T})(V_T) \otimes \pi^*(\mathbb{C}_{\rho^{-1}})_T(V_T) \longrightarrow \mathcal{O}(D_{1T})(V_T)$ by $\varphi(h(x)g(a) \otimes k(x)l(a)) = k(x)h(x)f^{-1}(x)g(a)l(a)$. First we check $(khf^{-1}) + D_1 \geq 0$. Clearly $(khf^{-1}) + D_1 = (k) + (h) + (f^{-1}) + D_1 = (k) + (h) + D_2$ (note that $(f^{-1}) = D_2 - D_1$). But $(h) + D_2 \geq 0$ (because $h(x)g(a)$ lives in $\mathcal{O}(D_{2T})(V_T)$). Also $(k) \geq 0$ (because $k(x)l(a)$ is a section so it has to be defined for all $a \in ET$, and for all $v \in V$, i.e k, l can not have poles). Second we check that $k(tx)h(tx)f^{-1}(tx)g(at^{-1})l(at^{-1}) = k(x)h(x)f^{-1}(x)g(a)l(a)$.

Clearly $k(tx)h(tx)f^{-1}(tx)g(at^{-1})l(at^{-1}) = k(x)l(a)\rho^{-1}(t)h(x)g(a)\rho(t)f^{-1}(x) = k(x)h(x)f^{-1}(x)g(a)l(a)$ (note that $f^{-1}(t^{-1}x) = t.f^{-1}(x) = \rho^{-1}(t)f^{-1}(x)$ implies $f^{-1}(tx) = \rho(t)f^{-1}(x)$). Therefore $k(x)h(x)f^{-1}(x)g(a)l(a) \in \mathcal{O}(D_{1T})(V_T)$. It is easy to check that φ is an isomorphism.

Let $\sum_{i=0}^{n} a_i\lambda_i$ be the weight of the character ρ. Then $c_1(\pi^*(\mathbb{C}_\rho)_T) = \pi^*c_1((\mathbb{C}_\rho)_T)$. But $\pi^*c_1((\mathbb{C}_\rho)_T) = \pi^*c_1(\mathcal{O}(-\sum_{i=0}^{n} a_i\lambda_i)) = \pi^*(-\sum_{i=0}^{n} a_i\lambda_i) = -\sum_{i=0}^{n} a_i\pi^*\lambda_i = -\sum_{i=0}^{n} a_i\lambda_i$ (note that $\pi^*\lambda_i := \lambda_i$). Now $\mathcal{O}(D_{1T}) \simeq \mathcal{O}(D_{2T}) \otimes \pi^*(\mathbb{C}_\rho)_T$ and $\mathcal{O}((f_T)) \simeq \pi^*(\mathbb{C}_\rho)_T$ (see lemma 5.2.11) implies $[D_{1T} - D_{2T}] = (f_T)$. Also $\mathcal{O}((f_T)) \simeq \pi^*(\mathbb{C}_\rho)_T$ (see lemma 5.2.11) implies $(f_T) = c_1(\pi^*(\mathbb{C}_\rho)_T\mid_{Z_T})$. So the equivariant cycle $[D_{1T} - D_{2T}] = (f_T) = c_1(\pi^*(\mathbb{C}_\rho)_T|_{Z_T}) = c_1(\pi^*(\mathbb{C}_\rho)_T).[Z_T] = (-\sum_{i=0}^{n} a_i\lambda_i).[Z_T]$ which lives in Span $\{\lambda_i.[Z_T] : i = 0,1,...,n\}$. Therefore $[D_{1T} - D_{2T}]$ belongs to Span $\{\lambda_i.[Z_T] : i = 0,1,...,n\}$.

($\Longleftarrow$) Suppose that $[D_{1T} - D_{2T}] = \sum_{i=1}^{m} a_i\lambda_i.[Z_T]$ where $a_i \in \mathbb{Z}$, $i = 0,...,n$. Recall the map $i_X^* : H_T^*(X) \longrightarrow H^*(X)$ where $i_X^*([W_T]) = [W]$. It follows $[D_1 - D_2] = i_X^*([D_{1T} - D_{2T}]) = i_X^*(\sum_{i=1}^{m} a_i\lambda_i.[Z_T]) = \sum_{i=1}^{m} a_i i_X^*(\lambda_i).i_X^*([Z_T])$. But $i_X^*(\lambda_i) = 0$ implies $[D_1 - D_2] = 0$. Thus $D_1 \sim D_2 \subset Z$.

(2) Suppose $D_1 \sim D_2$ in Z then by part(1) of this theorem, we have $[D_{1T} - D_{2T}]$ in $Span\{\lambda_i.[Z_T] : i = 0,1,...,n\}$, i.e, $[D_{1T} - D_{2T}] = \sum_{i=0}^{n} a_i\lambda_i.[Z_T]$. Let $l(\lambda) = \sum_{i=0}^{n} a_i\lambda_i$ then $i_{jX}^*([D_{1T} - D_{2T}]) = i_{jX}^*(\sum_{i=0}^{n} a_i\lambda_i.[Z_T]) = \sum_{i=0}^{n} a_i\lambda_i.i_{jX}^*([Z_T]) = l(\lambda).i_{jX}^*([Z_T])$.

$$
(3)\quad \begin{aligned} i_{jX}^*([D_{1T} - D_{2T}]) &= i_{jX}^*(\textstyle\sum_{k=1}^{m}[(D_{1k} - D_{2k})_T]) \\ &= \textstyle\sum_{k=1}^{m} i_{jX}^*([(D_{1k} - D_{2k})_T]) \end{aligned}
$$

Using part(2) there exsists an $l_k(\lambda) = \sum_{i=0}^{n} b_{ik}\lambda_i$, where $b_{ik} \in \mathbb{Z}$, $k = 1,...,m$ such that $i_{jX}^*([(D_{1k} - D_{2k})_T]) = l_k(\lambda).i_{jX}^*([Z_{kT}])$, where $j = 0,...,n$. Therefore $i_{jX}^*([D_{1T} - D_{2T}]) = \sum_{k=1}^{m} l_k(\lambda).i_{jX}^*([Z_{kT}])$, $j = 0,...,n$.

5.5 MAIN THEOREM

Definition 5.5.1. Let $\psi \in N = \text{Hom}(\mathbb{C}^*, T)$, and let $\chi \in M = \text{Hom}(T, \mathbb{C}^*)$ then $\chi \circ \psi(t) = t^k$, where $k \in \mathbb{Z}$. We define the dual pairing $<,>: N \otimes M \longrightarrow \mathbb{Z}$ by $< \psi, \chi >= k$.

Definition 5.5.2. A family of closed subschemes of a given scheme Y over a base B is a closed subscheme $X \subset B \times Y$, together with the restriction to X of the projection map $B \times Y \longrightarrow B$; the fibers of X over $b \in B$ are the naturally closed subchemes of the fibers Y_b of $B \times Y$ over B.

Definition 5.5.3. A family of schemes $\psi : X \longrightarrow B$ is flat if for every point $x \in X$ the local ring $\mathcal{O}_{X,x}$, regarded as a $\mathcal{O}_{B,\psi(x)}-$module via the map of local rings $\Psi : \mathcal{O}_{B,\psi(x)} \longrightarrow \mathcal{O}_{X,x}$ is flat.

Remark 5.5.4. As a reference for the notion of the limit of a 1-parameter familly of schemes see II.3.4 of [18].

Definition 5.5.5. Let $\mathcal{D} \subset X \times \mathbb{C}^*$ be a subscheme over $\mathbb{C}^*$where X is a B-variety. Let $\mathcal{D}_t$ denote the fiber of $\mathcal{D}$ over $t \in \mathbb{C}^*$. We define $\lim\limits_{t \longrightarrow 0} \mathcal{D}_t$ by $\lim\limits_{t \longrightarrow 0} \mathcal{D}_t := (\overline{\mathcal{D}})_0$ where $\overline{\mathcal{D}}$ is the closure of $\mathcal{D}$ in $X \times \mathbb{C}$.

Fact 5.5.6. Saying the functor $\underline{\text{Hilb}}_X$ is representable is the same thing as saying that there exists a universal family, that is a scheme H and a subscheme $C \subseteq X \times H$ flat over H such that given a subscheme $Y \subset X \times B$ flat over the scheme B there exists a unique morphism $h : B \longrightarrow H$ such that $Y = (B \times_H C)_h$ where $(B \times_H C)_h$ is the fiber product via h and B is any base scheme. In this case H denotes the Hilbert scheme Hilb X.

Fact 5.5.7. $\overline{(\mathbb{C}^* \times_H C)_f} = (\mathbb{C} \times_H C)_{f'}$ where $\overline{(\mathbb{C}^* \times_H C)_f}$ is the closure of $(\mathbb{C}^* \times_H C)_f$ in $(\mathbb{C} \times_H C)_{f'}$ and $f' : \mathbb{C} \longrightarrow H$ is the unique morphism extending $f : \mathbb{C}^* \longrightarrow H$. The existence and uniqueness of f' follows from Proposition 6.2 in Chapter one in [25] since the Hilbert scheme H is projective. Note that the closure of $(\mathbb{C}^* \times_H C)_f$ in $(\mathbb{C} \times_H C)_{f'}$

is equal to the closure of $(\mathbb{C}^* \times_H C)_f$ in $\mathbb{C}\times X$ because $(\mathbb{C}^* \times_H C)_f \subset (\mathbb{C}\times_H C)_{f'}$ and $(\mathbb{C}\times_H C)_{f'} \subseteq \mathbb{C}\times X$ is a closed subscheme being the pullback of the closed subscheme $C \subset H \times X$ as shown in the diagram below.

$$\begin{array}{ccccc} (\mathbb{C}^* \times_H C)_f & \hookrightarrow & (\mathbb{C} \times_H C)_{f'} & \longrightarrow & C \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{C}^* & \hookrightarrow & \mathbb{C} & \xrightarrow{f'} & H \end{array} \tag{74}$$

Form the fiber square

$$\begin{array}{ccc} (\mathbb{C}^* \times_H C)_f & \overset{h}{\hookrightarrow} & (\mathbb{C} \times_H C)_{f'} \\ \downarrow & & \downarrow \\ \mathbb{C}^* & \hookrightarrow & \mathbb{C} \end{array} \tag{75}$$

We will show that h is a dominant morphism. Therefore $\overline{(\mathbb{C}^* \times_H C)_f} = (\mathbb{C} \times_H C)_{f'}$. So let $Y = (\mathbb{C}^* \times_H C)_f$ and let $E = (\mathbb{C} \times_H C)_{f'}$. We need to show that $h : Y \hookrightarrow E$ is a dominant morphism. Consider the affine schemes $U = Spec\mathrm{A}$, $W = SpecA_b$, where A is a Noetherian ring and $b \in A$ is a nonzero divisor. We will show that $h' : SpecA_b \hookrightarrow SpecA$ is a dominant morphism. Let $\eta_i \in U_i$ be the generic point for U_i where U_i is an irreducible component of U. It is enough to show that $\eta_i \in W$. Therefore $\overline{\eta_i} \subset \overline{W}$. But $\overline{\eta_i} = U_i$. So $U_i \subset \overline{W}$ for all i. So $U \subset \overline{W}$. Therefore $U = \overline{W}$. It follows h' is dominant.

Now let us show that $\eta_i \in W$. Recall that a generic point of $U = Spec\mathrm{A}$ corresponds to a minimal prime ideal in the ring A. It is enough to show that if P is a minimal prime ideal in A and $b \in A$ is a nonzero divisor then P lives in A_b. Now to prove the previous statement we need to check that if P is a minimal prime ideal in A and $b \in A$ is a nonzero divisor then $b \notin P$. Suppose $b \in P$. Let $\beta : A \longrightarrow A_P$ be the ring homomorphism defined by $\beta(x) = x/1$. Let $Q = S^{-1}P$ where $S = A - P$, then Q is an ideal in A_P. But A_P is a zero dimensional ring (because p is a minimal prime) and Noetherian (because A is Noetherian) so it is Artinian. It follows there exists a positive integer n such that $Q^n = 0$. Now since $b \in P$ then $b/1 \in S^{-1}P = Q$. So $b^n/1 \in Q^n = 0$. So there exists $s \in A - P$ such that $sb^n = 0$. But b is a nonzero divisor. Contradiction.

Definition 5.5.8. Let $i : D \hookrightarrow X$ be a closed imbedding where X is a B-variety X. Let the map $\varphi : \mathbb{C}^* \times X \longrightarrow X$ be given by $\varphi(t,x) = t^{-1}.x$. Consider the fiber square

$$\begin{array}{ccc} \mathcal{D} & \hookrightarrow & \mathbb{C}^* \times X \\ \downarrow & & \downarrow \\ D & \hookrightarrow & X \end{array} \tag{76}$$

Since $\underline{\text{Hilb}}_X$ is a representable then as in fact 5.5.3 there exists a universal family– that is, a scheme H and a subscheme $C \subseteq X \times H$ flat over H –such that $\mathcal{D} = (\mathbb{C}^* \times_H C)_g$ for a unique morphism $g : \mathbb{C}^* \longrightarrow H$. We define $\lim_{t \longrightarrow 0} \mathcal{D}_t$ by $\lim_{t \longrightarrow 0} \mathcal{D}_t := (\overline{\mathcal{D}})_0$.

Remark 5.5.9. The scheme $\mathcal{D} = (\mathbb{C}^* \times_H C)_f$ in the definition above is flat over $\mathbb{C}^*$, i.e $\mathcal{D}$ is a flat family. For simplicity we let $\mathbb{C}^* \times_H C$ denote $(\mathbb{C}^* \times_H C)_f$. Consider the fiber square

$$\begin{array}{ccc} \mathbb{C}^* \times_H C & \longrightarrow & C \\ \downarrow & & \downarrow \alpha \\ \mathbb{C}^* & \xrightarrow{f} & H \end{array} \tag{77}$$

Let $z = (t,x) \in \mathcal{D}$ such that $f(t) = \alpha(x) = s \in H$. Then $\mathcal{O}_{\mathcal{D},z} = \mathcal{O}_{\mathbb{C}^* \times_H C,(t,x)} = \mathcal{O}_{\mathbb{C}^*,t} \otimes_{\mathcal{O}_{H,s}} \mathcal{O}_{C,x}$. Now we verify $\mathcal{O}_{\mathcal{D},z}$ is a flat $\mathcal{O}_{\mathbb{C}^*,t}$–module. Let $E \longrightarrow F$ be an $\mathcal{O}_{\mathbb{C}^*,t}$–module monomorphism. Consider the tensored sequence below

$$0 \longrightarrow \mathcal{O}_{\mathcal{D},z} \otimes_{\mathcal{O}_{\mathbb{C}^*,t}} E \longrightarrow \mathcal{O}_{\mathcal{D},z} \otimes_{\mathcal{O}_{\mathbb{C}^*,t}} F \tag{78}$$

We will check that (78) is a monomorphism. But $\mathcal{O}_{\mathcal{D},z} \otimes_{\mathcal{O}_{\mathbb{C}^*,t}} E$ is equal to the tensor product $\mathcal{O}_{\mathbb{C}^*,t} \otimes_{\mathcal{O}_{H,s}} \mathcal{O}_{C,x} \otimes_{\mathcal{O}_{\mathbb{C}^*,t}} E = \mathcal{O}_{C,x} \otimes_{\mathcal{O}_{H,s}} E$. Similarly $\mathcal{O}_{\mathcal{D},z} \otimes_{\mathcal{O}_{\mathbb{C}^*,t}} F = \mathcal{O}_{C,x} \otimes_{\mathcal{O}_{H,s}} F$. So we get the sequence

$$0 \longrightarrow \mathcal{O}_{C,x} \otimes_{\mathcal{O}_{H,s}} E \longrightarrow \mathcal{O}_{C,x} \otimes_{\mathcal{O}_{H,s}} F \tag{79}$$

which is a monomorphism because the universal family C is flat over H (by definition). Therefore (78) is a monomorphism.

Notation. We will use the script letter $\mathcal{V}$ to denote the scheme over $\mathbb{C}^*$ with fibers $t.V$, the script letter $\mathcal{E}$ to denote a scheme over $\mathbb{C}$ with fibers $t.E$, and so on as defined in definition 5.5.5 unless otherwise specified.

Definition 5.5.10. Let X be a B-variety, and let $\mathbb{C}^*$ be a 1-parameter subgroup of T. Fix $x \in X$. Let $g : \mathbb{C}^* \longrightarrow X$ be given by $g(t) = t.x$. Consider the imbedding $\mathbb{C}^* \longrightarrow \mathbb{P}^1$ $(t \longmapsto (1,t))$. Since X is projective there exists a unique liftiting $g' : \mathbb{P}^1 \longrightarrow X$. We define $\lim_{t \longrightarrow 0} t.x = g'(1,0)$.

Definition 5.5.11. Let E be a subvariety of a B-variety X, and let Hilb be the component of the Hilbert scheme Hilb$X$ containing E. Let $\mathcal{E} \subset \mathbb{C}^* \times X$ be a flat family over $\mathbb{C}^*$ with fibers $t.E$. Define $F : \mathbb{C}^* \longrightarrow$ Hilb by $F(t) = t.E$. Since Hilb is projective there exists a unique lifting $F' : \mathbb{C} \longrightarrow$ Hilb. We define $\lim_{t \longrightarrow 0} t.E = F'(0)$.

Proposition 5.5.12. The Hilbert definition of the limit (Definition 5.5.11) and the closure definition of the limit (Definition 5.5.5) agree.

Proof. Let $V \subset X$ be a subvariety of the B-variety X, and let $\mathcal{V} \subseteq \mathbb{C}^* \times X$ be the scheme with fibers $\mathcal{V}_t = t.V$. Recall that the functor $\underline{\text{Hilb}}_X$ is representable by the scheme $H = \mathrm{Hilb}\, X$, i.e there exists an isomorphism $\alpha : \underline{\mathrm{Hilb}}_X \longrightarrow \mathrm{Hom}(\ ,H)$. Consider the commutative diagram below

$$\begin{array}{ccc} \underline{\mathrm{Hilb}}_X\mathbb{C}^* & \xrightarrow{\alpha_{\mathbb{C}^*}} & \mathrm{Hom}(\mathbb{C}^*, H) \\ \downarrow & & \downarrow \\ \underline{\mathrm{Hilb}}_X Spec(k(t)) & \xrightarrow{\alpha_t} & \mathrm{Hom}(Spec(k(t)), H) \end{array} \tag{80}$$

where α_t denote $\alpha_{Spec(k(t))}$. Let $f : Spec(k(t)) \longrightarrow \mathbb{C}^*$ be a morphism of schemes, then $h(f) : \underline{\mathrm{Hilb}}_X\mathbb{C}^* \longrightarrow \underline{\mathrm{Hilb}}_X Spec(\mathrm{k(t)})$ is defined as follows: if $\mathcal{Y} \subseteq \mathbb{C}^* \times X$ is a closed subscheme which is flat over $\mathbb{C}^*$. We define $h(f)(\mathcal{Y}) = \mathcal{Y}_t$ where $\mathcal{Y}_t$ is the fiber of $\mathcal{Y}$ over $t \in \mathbb{C}^*$. Clearly since $\mathcal{Y}$ is flat over $\mathbb{C}^*$ then $\mathcal{Y}_t$ is flat over $Spec(\mathrm{k(t)})$. For simplicity I will use α to denote $\alpha_{\mathbb{C}^*}$. Recall $\mathcal{V} \subseteq \mathbb{C}^* \times X$ is flat over $\mathbb{C}^*$. Let $g = \alpha(\mathcal{V})$ and let $H(B)$ be the set of closed points of the Hilbert scheme $H = \mathrm{Hilb}\, X$ where B is any base scheme. Now since the set $H(\mathbb{C}^*)$ is identified with the set of morphisms $\mathrm{Hom}(Spec(k(t)), H)$ and $\mathrm{Hom}(Spec(k(t)), H)$ is identified with the set of fibers of flat families in $\underline{\mathrm{Hilb}}_X\mathbb{C}^*$. It follows from the commutativity of diagram (80) that $g(t) = \mathcal{V}_t = t.V$. By proposition I.6.8 in [25], there exists a unique morphism $g' : \mathbb{C} \longrightarrow H$ extending g. Let $C \subseteq X \times H$ be a universal family (fact 5.5.6). Form the fiber square

$$\begin{array}{ccc} \mathbb{C} \times_H C & \longrightarrow & C \\ \downarrow & & \downarrow \\ \mathbb{C} & \xrightarrow{g'} & H \end{array} \tag{81}$$

Let $\mathcal{V}' = \mathbb{C} \times_H C$. But $\overline{\mathcal{V}} = \mathcal{V}'$ (fact 5.5.7). So $(\overline{\mathcal{V}})_0 = (\mathcal{V}')_0$. Note that $(\mathcal{V}')_0 = g'(0)$ follows from representability: consider the bijection of sets $\alpha_{\mathbb{C}} : \underline{\mathrm{Hilb}}_X\mathbb{C} \longrightarrow \mathrm{Hom}(\mathbb{C}, H)$, then $\alpha_{\mathbb{C}}(\mathcal{V}') = g'$. Now since the set $H(\mathbb{C})$ is identified with the set of morphisms $\mathrm{Hom}(Spec(k(t)), H)$ and $\mathrm{Hom}(Spec(k(t)), H)$ is identified with the set of fibers of flat families in $\underline{\text{Hilb}}_X\mathbb{C}$. It follows from the commutativity of diagram (82) below that $g'(t) = (\mathcal{V}')_t$

$$\begin{array}{ccc} \underline{\text{Hilb}}_X\mathbb{C} & \xrightarrow{\alpha_{\mathbb{C}}} & \text{Hom}(\mathbb{C}, H) \\ \downarrow & & \downarrow \\ \underline{\text{Hilb}}_X Spec(k(t)) & \xrightarrow{\alpha_t} & \text{Hom}(Spec(k(t)), H) \end{array} \qquad (82)$$

So $g'(0) = (\mathcal{V}')_0$. Therefore $(\overline{\mathcal{V}})_0 = (\mathcal{V}')_0 = g'(0)$. Thus the Hilbert definition of the limit $(\lim\limits_{t\longrightarrow 0} t.V = g'(0))$ and the closure definition of the limit $(\lim\limits_{t\longrightarrow 0} t.V = (\overline{\mathcal{V}})_0)$ agree. Thus the proposition is proven.

The following key lemma which will be used later in proving several results is due to Strømme.

Lemma 5.5.13. Let X be a B-variety and let V be an $(n+1)-$dimensional representation for the torus T acting on X. Then by Fact 2.1.12 there exist characters $\chi_0,\dots,\chi_n$ such that $V = \oplus_{i=0}^{n} V_{\chi_i}$. Let $\overline{x} \in \mathbb{P}(V)$. Choose a 1-parameter subgroup $\psi : \mathbb{C}^* \hookrightarrow T$ such that $\forall i \neq j$ we have $\chi_i - \chi_j \notin H_\psi = \{\chi :< \psi, \chi >= 0\}$ where H_ψ is the orthogonal complement of ψ in M. Then $\lim\limits_{t\longrightarrow 0} \psi(t).\overline{x}$ is T-invariant.

Proof. Let $\overline{x} = (x_0, ..., x_n) \in \mathbb{P}(V)$, we calculate $\psi(t).\overline{x}$ by

$$\psi(t).\overline{x} = (t^{<\psi,\chi_0>}x_0, ..., t^{<\psi,\chi_n>}x_n)$$

But $\chi_i - \chi_j \notin H_\psi$ implies $< \psi, \chi_i - \chi_j > \neq 0$ where $i \neq j$ which implies $< \psi, \chi_i > \neq < \psi, \chi_j >$ where $i \neq j$. Therefore $t^{<\psi,\chi_i>} \neq t^{<\psi,\chi_j>}$ for each i, j where $i \neq j$. Let $m = \min \{< \psi, \chi_i > \in \mathbb{Z} : i = 0, ..., n\}$ then this minimum is unique. For if both $< \psi, \chi_k >$, and $< \psi, \chi_l >$ is a minimum of the set $\{< \psi, \chi_i > \in \mathbb{Z} : i = 0, ..., n\}$ then $< \psi, \chi_k - \chi_l > = < \psi, \chi_k > - < \psi, \chi_l > = 0$ which implies $\chi_k - \chi_l \in H_\psi$. Contradiction.

Let $m = <\psi, \chi_k>$ then

$$\begin{aligned}
\psi(t).\overline{x} &= \overline{(t^{<\psi,\chi_0>}x_0, ..., t^{<\psi,\chi_{k-1}>}x_{k-1}, t^m x_k, t^{<\psi,\chi_{k+1}>}x_{k+1}, ..., t^{<\psi,\chi_n>}x_n)} \\
&= \overline{t^{-m}(t^{<\psi,\chi_0>}x_0, ..., t^{<\psi,\chi_{k-1}>}x_{k-1}, t^m x_k, t^{<\psi,\chi_{k+1}>}x_{k+1}, ..., t^{<\psi,\chi_n>}x_n)} \\
&= \overline{(t^{<\psi,\chi_0>-m}x_0, ..., t^{<\psi,\chi_{k-1}>-m}x_{k-1}, x_k, t^{<\psi,\chi_{k+1}>-m}x_{k+1}, ..., t^{<\psi,\chi_n>-m}x_n)}
\end{aligned}$$

But $m = \min \{<\psi, \chi_i> \in \mathbb{Z} : i = 0, ..., n\}$ implies $(<\psi, \chi_i> -m) > 0$ for all $i \neq k$ which implies $\lim_{t \longrightarrow 0} t^{<\psi,\chi_i>-m}x_i = 0$. Therefore we have $\lim_{t \longrightarrow 0} \psi(t).\overline{x} = (0, ..., 0, x_k, 0, ..., 0) = (0, ..., 0, 1, 0, ..., 0) = p_k$ which is T-invariant.

Lemma 5.5.14. Let $\varphi : S \longrightarrow N$ be a graded homomorphism of graded rings (preserving degrees) such that $\varphi_d : S_d \longrightarrow N_d$ is an isomorphism for all $d \geq d_0$, where d_0 is an integer. Then $f : projN \longrightarrow projS$ is an isomorphism.

Proof. Let $\{g_\alpha\}$ be a set of generators for N_+ where $N_+ = \bigoplus_{d>0} N_d$(See II.2 in [25]). Then $\cup_\alpha D_N(g_\alpha) = \cup_\alpha \{x \in ProjN : g_\alpha \notin x\} = ProjN$ (since every prime in Proj N must omit some g_α). If $y \in ProjN$ then $g_\alpha \notin y$ for some α otherwise $g_\alpha \in y$ for each α so $N_+ \subseteq y$ contradiction because $y \not\supseteq N_+$ (since $y \in ProjN$). Therefore $y \in D_N(g_\alpha)$ for some α. So $y \in \cup_\alpha D_N(g_\alpha)$. But $\cup_\alpha D_N(g_\alpha) \subseteq ProjN$. It follows $ProjN = \cup_\alpha D_N(g_\alpha)$. Note that $g_\alpha \notin x$ iff ${g_\alpha}^{d_0} \notin x$ for x prime. So we can replace the set of g_α's by elements of $N_{\geq d_0}$ and still have a cover of ProjN by distinguished open sets. Our strategy is as follows. We first show that $f|_{D_N(g_\alpha)} : D_N(g_\alpha) \longrightarrow D_S(\varphi^{-1}(g_\alpha))$ is an isomorphism for each α and then show that the open sets $D_S(\varphi^{-1}(g_\alpha))$ cover ProjS. Then showing that f is injective completes the proof. Let $g = g_\alpha$ be one of our g_α's. By Proposition 2.5 in [25], $D_N(g) \simeq SpecN_{(g)}$ where $N_{(g)}$ is the subring of elements of degree 0 in the localized ring $N_{(g)}$. So $f|_{D_N(g_\alpha)}$ is a morphism of schemes where $f|_{D_N(g_\alpha)} : SpecN_{(g)} \longrightarrow SpecS_{(\varphi^{-1}(g))}$. This map is induced by the map $\overline{\varphi} : S_{(\varphi^{-1}(g))} \longrightarrow N_{(g)}$ where $\overline{\varphi}$ is the localization of the ring homomorphism $\varphi : S \longrightarrow N$. So we just need to verify that $\overline{\varphi} : S_{(\varphi^{-1}(g))} \longrightarrow N_{(g)}$

is an isomorphism. Recall $\overline{\varphi}$ is defined by $\overline{\varphi}(s/h) = \varphi(s)/\varphi(h)$ where $h \notin (\varphi^{-1}(g))$ implies $\varphi(h) \notin (g)$. Suppose that $\overline{\varphi}(a/b) = 0$. Then $0 = \overline{\varphi}(a\varphi^{-1}(g))/(b\varphi^{-1}(g)) = \varphi(a\varphi^{-1}(g))/\varphi(b\varphi^{-1}(g))$. So $\varphi(a\varphi^{-1}(g))/\varphi(b\varphi^{-1}(g)) = 0$ in $N_{(g)}$ implies that there exists an integer n such that $g^n\varphi(a\varphi^{-1}(g)) = 0$ in N. So $a\varphi^{-1}(g)^{n+1} = 0$ in S since φ is an isomorphism in high enough degree (To see that simply apply φ^{-1} to both sides you get $\varphi^{-1}(g^n\varphi(a\varphi^{-1}(g))) = 0$ in S. But φ is an isomorphism in high enough degree implies $\varphi^{-1}(g^n)\varphi^{-1}(\varphi(a\varphi^{-1}(g)))) = 0$ in S which implies $\varphi^{-1}(g^n)a\varphi^{-1}(g) = 0$. So $a\varphi^{-1}(g^{n+1}) = 0$ because φ^{-1} is a ring homorphism in high enough degree. So $a\varphi^{-1}(g)^{n+1} = 0$ in S). Thus $a = 0$ in $S_{(\varphi^{-1}(g))}$, so $\frac{a}{b} = 0$ in $S_{(\varphi^{-1}(g))}$. This shows that $\overline{\varphi}$ is injective. To see that $\overline{\varphi}$ is surjective. Let $a/g^n \in N_{(g)}$ then $\varphi^{-1}(ag)/\varphi^{-1}(g^{n+1})$ is a well-defined element of $S_{(\varphi^{-1}(g))}$ and $\overline{\varphi}(\varphi^{-1}(ag)/\varphi^{-1}(g^{n+1})) = ag/g^{n+1} = a/g^n$, which shows that $\overline{\varphi}$ is surjective. Next we verify $\cup_\alpha D_S(\varphi^{-1}(g_\alpha)) = ProjS$. Clearly $\cup_\alpha D_S(\varphi^{-1}(g_\alpha)) \subseteq ProjS$ because $D_S(\varphi^{-1}(g_\alpha)) \subseteq ProjS$ for each α. To show that $\cup_\alpha D_S(\varphi^{-1}(g_\alpha)) \supseteq ProjS$. Let $x \in ProjS$. Suppose that $x \notin \cup_\alpha D_S(\varphi^{-1}(g_\alpha))$ then $x \notin D_S(\varphi^{-1}(g_\alpha))$ for all α. Then $\varphi^{-1}(g_\alpha) \in x$ for each α, so, since we may assume that $\{g_\alpha\}$ generates $N_{\geq d_0}$ we have $\varphi^{-1}(N_{\geq d_0}) \subseteq x$. But φ is an isomorphism in high enough degree implies $S_{\geq d_0} = \varphi^{-1}(N_{\geq d_0}) \subseteq x$. So $S_{\geq d_0} \subseteq x$ and x is prime. So $S_+ \subseteq x$, a contradiction since $x \in ProjS$. Now we verify that the induced map $f : ProjN \longrightarrow ProjS$ is injective. Let $p, q \in ProjN$ and suppose that $f(p) = f(q)$. Then $\varphi^{-1}(p) = \varphi^{-1}(q)$. But φ_d is an isomorphism for $d \geq d_0$ implies $p \cap N_d = q \cap N_d$ (This is true because $\varphi^{-1}(p) = \varphi^{-1}(q)$ implies $\varphi_d^{-1}(p) = \varphi_d^{-1}(q)$ so $\varphi_d^{-1}(p) \cap S_d = \varphi_d^{-1}(q) \cap S_d$ where $S_d = \varphi^{-1}(N_d)$ for $d \geq d_0$. So $\varphi_d^{-1}(p) \cap \varphi^{-1}(N_d) = \varphi_d^{-1}(q) \cap \varphi^{-1}(N_d)$. Thus $\varphi_d^{-1}(p \cap N_d) = \varphi_d^{-1}(q \cap N_d)$. But φ_d is an isomorphism for $d \geq d_0$ implies $p \cap N_d = q \cap N_d$ for $d \geq d_0$). So if $a \in p$ homogeneous then $a^d \in p \cap N_d$ so $a^d \in q \cap N_d$ so $a^d \in q$ so $a \in q$ because q is prime. Thus $p \subseteq q$. Likewise $a \in q$ implies $a \in p$. Thus $p = q$ so f is injective.

Lemma 5.5.15. (a) Let $\varphi : S \longrightarrow N$ be a surjective homomorphism of graded rings, preserving degrees. If $U = \{p \in ProjN : p \not\supseteq \varphi(S_+)\}$ then $U = ProjN$.

(b) The morphism $f : ProjN \longrightarrow ProjS$ is a closed immersion.

(c) If $I \subset S$ is a homogeneous ideal, take $N = S/I$ and let Y be the closed subscheme of $X = ProjS$ defined as the image of the closed immersion $ProjS/I \longrightarrow X$. Then different homogeneous ideals can give rise to the same closed subscheme, i.e, If d_0 is an integer and $I' = \oplus_{d \geq d_0} I_d$ then I and I' determine the same closed subscheme.

Proof.

(a) We know that $U \subseteq ProjN$. We need to show that $U \supseteq ProjN$. Let $q \in ProjN$ then $q \not\supseteq N_+$. Since φ is graded and surjective, $\varphi(S_+) = N_+$. Thus $q \in ProjN$ and $q \not\supseteq \varphi(S_+)$. It follows $q \in U$.

(b) Let $f : ProjN \longrightarrow ProjS$ be a morphism. Since φ is surjective then by the first isomorphism theorem $N \simeq S/Ker\varphi$. So $f(ProjN) = f(Proj(S/Ker\varphi)) = V(Ker\varphi)$ where $V(Ker\varphi)$ is defined in [25]) is a closed subset of $ProjS$. This follows from the fact that there is a one to one correspondence between homogeneous ideals of $S/Ker\varphi$ and homogeneous ideals of S which contains $Ker\varphi$. So $f : ProjN \longrightarrow ProjS$ is a homeomorphism of $ProjN$ onto the closed subscheme $V(Ker\varphi)$ of $ProjS$. Now let us check that the map of structure sheaves $\mathcal{O}_{ProjS} \longrightarrow f_*\mathcal{O}_{ProjN}$ is surjective. Let $y \in V(Ker\varphi)$ it is enough to check that the map on the stalk $\mathcal{O}_{ProjS,y} \longrightarrow (f_*\mathcal{O}_{ProjN})_y$ is surjective. But since the map $f : ProjN \longrightarrow V(Ker\varphi)$ is a bijection it follows $y = f(x)$ for some $x \in ProjN$. So $(f_*\mathcal{O}_{ProjN})_y = \mathcal{O}_{ProjN,f^{-1}(y)} = \mathcal{O}_{ProjN,x}$. Recall that $\varphi : S \longrightarrow N$ induces $f : ProjN \longrightarrow ProjS$ i.e, $f(p) = \varphi^{-1}(p)$ for any $p \in ProjN$. It follows that the map on the stalk is the map $\mathcal{O}_{ProjS,\varphi^{-1}(x)} \longrightarrow \mathcal{O}_{ProjN,x}$. By Proposition 2.5 page 76 in [25] we have $\mathcal{O}_{ProjN,x} = N_{(x)}$, and $\mathcal{O}_{ProjN,\varphi^{-1}(x)} = S_{(\varphi^{-1}(x))}$. So the map on the stalk corresponding to the point $x \in ProjN$ is the map

$$S_{(\varphi^{-1}(x))} \longrightarrow N_{(x)}$$

which is surjective because $\varphi : S \longrightarrow N$ is surjective. Thus the induced map on the sheaves is surjective.

(c) Let $\varphi : S/I' \longrightarrow S/I$ be the natural projection homomorphism. This map makes sense because S/I is a quotient of S/I'. Indeed $S/I = (S/I')/\oplus_{o \leq d < d_0} I_d$. The map φ is a graded homomorphism of graded rings such that $\varphi_d : (S/I')_d \longrightarrow (S/I)_d$ is the identity map for $d \geq d_0$. By lemma 5.5.14 φ induces an isomorphism $f : ProjS/I \longrightarrow ProjS/I'$. But the map $ProjS/I \longrightarrow ProjS$ (resp. $ProjS/I' \longrightarrow ProjS$) is a closed immersion implies $ProjS/I$ (resp. $ProjS/I'$) is isomorphic to a closed subscheme V (resp.V') of ProjS. It follows that $V \simeq V'$ because $ProjS \simeq ProjS/I'$. Thus I and I' give rise to the same closed subscheme.

Fact 5.5.16. For any subscheme $C \subseteq \mathbb{P}^r$ there exists an integer m such that the Hilbert function $h(n)$ of C is equal to the Hilbert polynomial $p(n)$ for $n \geq m$. See [22].

Fact 5.5.17. There exists an integer m, which is the same integer m in fact 5.5.16, such that the homogeneous ideal of C is generated by its m-th graded piece. See [22].

Definition 5.5.18. A numerical polynomial is a polynomial $P(z) \in \mathbb{Q}[z]$ such that $P(n) \in \mathbb{Z}$ for all $n \gg 0$, $n \in \mathbb{Z}$.

Definition 5.5.19. An algebraic family of closed subschemes of a scheme X, parametrized by $\mathcal{T}$, is a closed subscheme $Z \subset X \times \mathcal{T}$.

Definition 5.5.20. Let $\underline{\text{Hilb}}_X(\mathcal{T})$ be the set of flat families of closed subschemes Z of X parametrized by $\mathcal{T}$. If $\mathcal{T}' \longrightarrow \mathcal{T}$ is any morphism, $Z \mapsto Z \times_{\mathcal{T}} \mathcal{T}'$ gives a map $\underline{\text{Hilb}}_X(\mathcal{T}) \longrightarrow \underline{\text{Hilb}}_X(\mathcal{T}')$, which makes $\underline{\text{Hilb}}_X$ a contravariant functor on the category of schemes. The scheme $\text{Hilb}\, X$ representing the functor $\underline{\text{Hilb}}_X$ is called the *Hilbert scheme* of X.

Definition 5.5.21. If P is a numerical polynomial, let $\underline{\text{Hilb}}^P{}_X$ be the open and closed subfunctor of $\underline{\text{Hilb}}_X$ given by flat families with Hilbert polynomial P in all geometric fibers (See [34]). The scheme representing this functor $\underline{\text{Hilb}}^P{}_X$ is denoted by Hilb^P_X.

Let X be a B-variety. Fix an imbedding of X as a closed subset of $\mathbb{P}^r$ for some r. Then we have the following theorem

Theorem 5.5.22. Let X be a B-variety and let P be a numerical polynomial. Let $\text{Hilb}^P X$ be a component of the Hilbert scheme $\text{Hilb}\, X$ then there exists a T-representation V such that $\text{Hilb}^P X$ can be embedded T-equivariantly in $\mathbb{P}(V)$.

Proof. Recall $\underline{\text{Hilb}}_X(\mathcal{A})$ is the set of flat families of closed subschemes Z of X parametrized by $\mathcal{A}$. Now since the definition of $\underline{\text{Hilb}}_X(\mathcal{A})$ is independent of the embedding of X in $\mathbb{P}^r$ then we can replace X by $\mathbb{P}^r$. Recall $\text{Hilb}^P \mathbb{P}^r$ is the scheme representing the subfunctor $\underline{\text{Hilb}}^P_{\mathbb{P}^r}$ where $\underline{\text{Hilb}}^P_{\mathbb{P}^r}$ is a subfunctor of the contravariant functor $\underline{\text{Hilb}}_{\mathbb{P}^r}$ defined as follows: given a scheme C we let $\underline{\text{Hilb}}^P_{\mathbb{P}^r} C$ be the set of flat families (of closed subschemes E of X parametrized by C) with Hilbert polynomial P in all geometric fibers (see [34]). We will show that there exists a T-representation V such that $\text{Hilb}^P \mathbb{P}^r$ can be embedded T-equivariantly in $\mathbb{P}(V)$. By virtue of fact 5.5.16, for any subscheme C the subspace

$$\Lambda_C = H^0(\mathbb{P}^r, I_C(m)) \subset H^0(\mathbb{P}^r, \mathcal{O}(m))$$

of polynomials of degree m in $\mathbb{P}^r$ vanishing on C has codimension exactly $P(m)$. By fact 5.5.17, the subscheme C is determined by the subspace Λ_C. Thus we can associate to C the point

$$\Lambda_C \in \text{Grass}(N - P(m), N)$$

where $N = \dim H^0(\mathbb{P}^r, \mathcal{O}(m)) = \binom{m+r}{r}$. Let $k = N - P(m)$ then the locus of points in $\text{Grass}(k, N)$ arising in this way coincides with the Hilbert scheme $\text{Hilb}^P \mathbb{P}^r$ set-theoretically. The above description does not give the scheme structure. Clearly

$\text{Hilb}^P\mathbb{P}^r$ can be embedded in Grass(k, N) via the embedding

$$C \longmapsto \Lambda_C = H^0(\mathbb{P}^r, I_C(m))$$

Let $W = H^0(\mathbb{P}^r, \mathcal{O}(m))$ then Grass(k, N) is embedded in $\mathbb{P}(\wedge^k W)$ via the Plucker embedding as follows: Let $U = \text{Span } \{u_i \in W : i = 1, ..., k\} \in \text{Grass}(k, N)$. Define the embedding $\psi : \text{Grass}(k, N) \longrightarrow \mathbb{P}(\wedge^k W)$ by $\psi(U) = [u_1 \wedge u_2 \wedge ... \wedge u_k]$. For more details see Lecture 6 in [23]. Given a T-action on $\mathbb{P}^r$, this action will induce an action on $W = H^0(\mathbb{P}^r, \mathcal{O}(m))$ which will induce an action on $V = \wedge^k W$ given by

$$t.(w_1 \wedge w_2 \wedge ... \wedge w_k) = t.w_1 \wedge t.w_2 \wedge ... \wedge t.w_k$$

Thus $\text{Hilb}^P \mathbb{P}^r$ is embeded in $\mathbb{P}(\wedge^k W)$. Let $\phi : \text{Hilb}^P\mathbb{P}^r \longrightarrow \text{Grass}(k, N)$ be defined by $\phi(Y) = H^0(\mathbb{P}^r, I_Y(m))$ Claim: $\psi \circ \phi : \text{Hilb}^P\mathbb{P}^r \hookrightarrow \mathbb{P}(V)$ is a T-equivariant embedding, i.e $\psi \circ \phi(t.Z) = t.(\psi \circ \phi(Z))$. So if $Y \in \text{Hilb}^P\mathbb{P}^r$ then $\phi(Y) = H^0(I_Y(m)) =$ Span $\{g_1, ..., g_k : \deg g_i = m\}$. So $\psi \circ \phi(Y) = [g_1 \wedge g_2 \wedge ... \wedge g_k]$. Therefore $t.(\psi \circ \phi(Y)) = t.[g_1 \wedge g_2 \wedge ... \wedge g_k] = [t.g_1 \wedge t.g_2 \wedge ... \wedge t.g_k]$ where $t.g_i(x) = g_i(t^{-1}.x)$. Now let $Z \in \text{Hilb}^P\mathbb{P}^r$ then the homogeneous ideal of Z, $I(Z) = (f_1, ..., f_i)$ where f_j is a homogeneous polynomial. By fact 5.5.17, $I(Z)$ is generated by its m-th graded piece $I(Z)_m = \text{Span } \{h_1, ..., h_s\}$ where $\deg h_j = m$. Now we have two homogeneous ideals that give rise to Z, namely $(f_1, ..., f_i)$ and $(h_1, ..., h_s)$. By part (c) of lemma 5.5.15 different homogeneous ideals can give rise to the same closed subscheme so we have $Z = \mathcal{Z}(f_1, ..., f_i) = \mathcal{Z}(h_1, ..., h_s)$. So $t.Z = t.\mathcal{Z}(h_1, ..., h_s) = \mathcal{Z}(t.h_1, ..., t.h_s)$. Now $\psi \circ \phi(t.Z) = \psi \circ \phi(\mathcal{Z}(t.h_1, ..., t.h_s)) = \psi(\text{ Span } \{t.h_1, ..., t.h_s\}) = [t.h_1 \wedge ... \wedge t.h_s]$. But $[t.h_1 \wedge ... \wedge t.h_s] = t.[h_1 \wedge ... \wedge h_s] = t.(\psi \circ \phi(Z))$. Thus $\psi \circ \phi(t.Z) = \text{t}.(\psi \circ \phi(Z))$.

Fact 5.5.23. The scheme $\mathcal{C} = \{(W, S) : W, S \in \text{Hilb}\, X \text{ and } W \subset S\}$ is a closed subscheme of $\text{Hilb}\, X \times \text{Hilb}\, X$. See [29].

Notation The field of rational functions on a variety X is denoted by $R(X)$; the non-zero elements of this field form the multiplicative group $R(X)^*$.

Definition 5.5.24. Let X be a projective variety. A k-cycle on X is a finite formal sum $\sum n_i[V_i]$ where the V_i are k-dimensional subvarieties of X, and the n_i are integers. The group of k-cycles on X, denoted $Z_k(X)$, is the free abelian group on the k-dimensional subvarieties of X; to a subvariety V of X corresponds $[V]$ in $Z_k(X)$.

Definition 5.5.25. Let X be an algebraic sheme. For any $(k+1)-$dimensional subvariety W of X, and any $f \in R(W)^*$, define the k-cycle $[(f)]$ on X by

$$[(f)] = \sum \operatorname{ord}_V(f)[V],$$

the sum over all codimension one subvarieties V of W; here $\operatorname{ord}_V(f)[V]$ is the order of the function in $R(X)^*$. (see [17]).

Definition 5.5.26. Let X be a variety, V a subvariety of codimension one. The local ring $A = \mathcal{O}_{V,X}$ is a one-dimensional local domain. For $r \in A$ we define $\operatorname{ord}_V(r) = \ell_A(A/(r))$ where ℓ_A denotes the length of the A-module in parentheses.

Definition 5.5.27. Let V be an irreducible $(k+1)$-dimensional subvariety of X. A k-cycle β is linearly equivalent to zero in V if there exists $g \in R(V)^*$ such that $\beta = [(g)]$.

Definition 5.5.28. A k-cycle α is rationally equivalent to zero, written $\alpha \sim 0$, if there exists a finite number of $(k+1)-$dimensional subvarieties W_i of X, and $f_i \in R(W_i)^*$, such that $\alpha = \sum[(f_i)]$.

Definition 5.5.29. The k-cycles rationally equivalent to zero form a subgroup $R_k(X)$ of $Z_k(X)$. The group of k-cycles modulo rational equivalence on X is the factor group $A_k(X) = Z_k(X)/R_k(X)$. Define $Z_*(X)$(resp. $A_*(X)$) to be the direct sum of $Z_k(X)$(resp. $A_k(X)$) for $k = 0, ..., \dim X$. A cycle (resp. cycle class) on X is an element of $Z_*(X)$(resp. $A_*(X)$).

Remark 5.5.30. Let X be a B-variety. Two subvarieties V, V' are rationally equivalent iff the corresponding cycles are rationally equivalent. This follows immediately from the definition.

Definition 5.5.31. Let $\mathcal{D} \subseteq \mathbb{C} \times X$ be a family over $\mathbb{C}$, the fiber of $\mathcal{D}$ over $t \in \mathbb{C}$ is denoted $\mathcal{D}_t$. Let $\mathcal{D}^* = \mathcal{D} - \mathcal{D}_0$. We define $\lim_{t \longrightarrow 0} \mathcal{D}_t$ by setting $\lim_{t \longrightarrow 0} \mathcal{D}_t = (\overline{\mathcal{D}^*})_0$ where $(\overline{\mathcal{D}^*})_0$ is the fiber of the family $\overline{\mathcal{D}^*}$ over zero.

Example 5.5.32. Consider the scheme $\mathcal{D} =$ Spec $\mathbb{C}[t,x]/(tx)$ over $\mathbb{C}$. Then $\overline{\mathcal{D}^*} =$ Spec $\mathbb{C}[t]$. So $\overline{\mathcal{D}^*} \neq \mathcal{D}$.

Definition 5.5.33. Let X be a B-variety. For any k-cycle $\alpha = \sum n_V[V]$ on X, the support of α, written $|\alpha|$, is the union of subvarieties V with non-zero coefficient in α. If $t \in T$

(a) We define t.[V] by setting t.[V]=[t.V].

(b) We define t.α by setting t.$\alpha = \sum n_V t.[V]$.

(c) Let $\mathcal{V} \subseteq \mathbb{C}^* \times X$ be a subscheme over $\mathbb{C}^*$. We define $\lim_{t \longrightarrow 0} t.\alpha = \sum n_V[(\overline{\mathcal{V}})_0]$.

Remark 5.5.34. Let $f \in R(V)^*$ be an irreducible function where V is a subvariety of the B-variety X. Let $\mathcal{Z}(f)$ be the zero locus of f which is a subvariety of X, i.e $\mathcal{Z}(f) = \{x \in V : f(x) = 0\}$. Then $t.\mathcal{Z}(f) = \mathcal{Z}(t.f)$ where $t.f(x) = f(t^{-1}.x)$, $\mathcal{Z}(t.f) = \{x \in t.V : (t.f)(x) = 0\}$, and $t.\mathcal{Z}(f) = \{t.x : x \in V \text{ and } f(x) = 0\}$.

Definition 5.5.35. Let $f \in R(V)^*$ be an irreducible function where V is a subvariety of the B-variety X. Let $D_1 = f^{-1}(0)$ and let $D_2 = f^{-1}(\infty)$. Then $(f) = D_1 - D_2$ where $D_i \subseteq X$ is a codimension one subvariety of X. Let $\mathcal{D}_j \subseteq \mathbb{C}^* \times X$ be a subscheme over $\mathbb{C}^*$ with fibers $t.D_j$. Then $t.(f) = t.D_1 - t.D_2$. We define $\lim_{t \longrightarrow 0} t.(f) = (\overline{\mathcal{D}_1})_0 - (\overline{\mathcal{D}_2})_0$.

Definition 5.5.36. The group of T-invariant k-cycles on X, denoted $Z_k^T(X)$, is the free abelian group on the T-invariant k-dimensional subvarieties of X; to a subvariety V of X corresponds $[V_T]$ in $Z_k^T(X)$. The T-invariant k-cycles rationally equivalent to zero form a subgroup $R_k^T(X)$ of $Z_k^T(X)$. The group of T-invariant k-cycles modulo T-invariant rational equivalence on X is the factor group $A_k^T(X) = Z_k^T(X)/R_k^T(X)$. Define $Z_*^T(X)$ (resp.$A_*^T(X)$) to be the direct sum of $Z_k^T(X)$ (resp.$A_k^T(X)$) for $k =$

$0, ..., \dim X$. A T-invariant cycle (resp. T-invariant cycle class) on X is an element of $Z_*^T(X)$(resp. $A_*^T(X)$).

Notation. Let $\mathcal{G}$ denote an irreducible variety of dimension $m > 0$. The notation "$a \in \mathcal{G}$" will be used to denote a regular, closed point of $\mathcal{G}$(Appendix B.1 [14]). By abuse of notation we will write t in place of $Spec(k(t))$, where $k(t)$ is the residue field of the local ring of $\mathcal{G}$ at the point, and we denote by

$$t : \{t\} \longrightarrow \mathcal{G}$$

the canonical inclusion of $Spec(k(t))$ in $\mathcal{G}$. The assumption that the point is regular means that t is a regular embedding of codimension m. Script letters will be used to denote schemes over $\mathcal{G}$, with corresponding Latin letters, subscripted by t, denoting the fiber over $t \in \mathcal{G}$. Given the morphism of schemes $p : \mathcal{Y} \longrightarrow \mathcal{G}$ then $Y_t = p^{-1}(t)$; Y_t is regarded as an algebraic scheme over the ground field $k(t)$.

Fact 5.5.37. Let $\mathcal{G}$ be a smooth variety, $t \in \mathcal{G}$ be a regular closed point of $\mathcal{G}$. Any $(k+m)-$cycle α on $\mathcal{Y}$, or more generally any rational equivalence class $\alpha \in A_{k+m}(\mathcal{Y})$, determines a family of k-cycle classes $\alpha_t \in A_k(Y_t)$, for all $t \in \mathcal{G}$, by the formula

$$\alpha_t = t^!(\alpha)$$

where $t^! : A_{k+m}(\mathcal{Y}) \longrightarrow A_k(Y_t)$ is the refined Gysin homomorphism defined from the fiber square

$$\begin{array}{ccc} Y_t & \longrightarrow & \mathcal{Y} \\ \downarrow & & \downarrow \\ \{t\} & \longrightarrow & \mathcal{G} \end{array}$$

by construction of section 6.2 in [17].

Fact 5.5.38. If a $(k+1)-$cycle β is rationally equivalent to zero on $\mathcal{Y}$ then $t^!\beta$ is rationally equivalent to zero in Y_t. This follows easily because $t^!$ is a group homomorphism.

Let $\overline{\beta}$ denote the cycle class of β then $\overline{\beta} = \overline{0} \in A_{k+1}(\mathcal{Y})$. So $t^!(\overline{\beta}) = t^!(\overline{0}) = \overline{0} \in A_k(Y_t)$. But $t^!(\overline{\beta}) := \overline{t^!(\beta)} = \overline{0} \in A_k(Y_t)$. Therefore $t^!(\beta) \in R_k(Y_t)$. Hence $t^!(\beta)$ is rationally equivalent to zero in Y_t.

Fact 5.5.39. If $\alpha = [\mathcal{W}]$ where $\mathcal{W}$ is a subvariety of $\mathcal{Y}$ of pure dimension $k+m$, then $\alpha_t = [\mathcal{W}]_t = \{s(W_t, \mathcal{W})\}_k$ where $W_t = \mathcal{W} \cap Y_t$, and $s(W_t, \mathcal{W})$ is the Segre class of W_t in $\mathcal{W}$. (This follows from Proposition 6.1 (a) in [17] and the fact that the normal bundle to t in $\mathcal{G}$ is trivial). In particular, if $\mathcal{W} \subset Y_t$, then $[\mathcal{W}]_t = 0$. If $\mathcal{G}$ is a curve and α is a $(k+1)-$cycle on $\mathcal{Y}$, then α_t is well-defined as a k-cycle on Y_t. For if $\alpha = \sum n_i[\mathcal{W}_i]$, with $\mathcal{W}_i$ is a variety then we define $\alpha_t = \sum\limits_{\mathcal{W}_i \not\subseteq \mathcal{Y}_t} n_i[(W_i)_t]$ where $(W_i)_t = \mathcal{W}_i \cap Y_t$.

Our goal in the next two pages is to verify that $0^![(F)] = [\lim\limits_{t \longrightarrow 0} t.(f)]$. Let $f \in R(V)^*$ where V is a subvariety of the B-variety X. Let $\mathcal{V} \subset \mathbb{C}^* \times X$ be a family with fibers $t.V$. We define the rational function $F^*(t,x)$ on the total space of the family $\mathcal{V}$ by $F^*(t,x) = (t.f)(x)$ (t is not fixed). Let $F \in R(\overline{\mathcal{V}})^*$ such that $F(t,x) \mid_{\mathcal{V}} = F^*(t,x)$. First note that

$$[(F)] - [(F \mid_{\mathcal{V}})] = \sum_{\eta_{V_i} \in \overline{\mathcal{V}}} m_i[V_i] - \sum_{\eta_{V_i} \in \mathcal{V}} m_i[V_i] = \sum_{\eta_{V_i} \in \overline{\mathcal{V}} - \mathcal{V}} m_i[V_i] = \sum_{\eta_{V_i} \in (\overline{\mathcal{V}})_0} m_i[V_i] \quad (83)$$

where $\eta_{V_i} \in V_i$ is the generic point of the codimension one subvariety $V_i \subseteq \overline{\mathcal{V}}$ and $m_i = \mathrm{ord}_{V_i}(F)$ (definition 5.5.25). It follows

$$[(F)] - [(F \mid_{\mathcal{V}})] = \sum_{V_i \subseteq (\overline{\mathcal{V}})_0} m_i[V_i] \quad (84)$$

Note that $\overline{\mathcal{V}}$ is a variety because $\mathcal{V}$ is a variety and the closure of a variety is a variety. The fact that $\mathcal{V}$ is a variety can be checked easily. Consider the isomorphism $P_1 \times \varphi : \mathbb{C}^* \times X \longrightarrow \mathbb{C}^* \times X$ where P_1 is the projection to the first factor, i.e $\varphi(t,x) = (t, t^{-1}x)$ (the inverse map is defined by $(t,y) \longmapsto (t, ty)$). Then

$(P_1 \times \varphi) \mid_{\mathcal{V}}: \mathcal{V} \longrightarrow \mathbb{C}^* \times V$ is an isomorphism. But $\mathbb{C}^* \times V$ is a variety so $\mathcal{V}$ is a variety.Suppose that $\overline{V_i} \not\subseteq (\overline{\mathcal{V}})_0$. CLAIM: $\mathrm{ord}_{V_i}(F) = \mathrm{ord}_{\overline{V_i}}(F)$.

PROOF. Since $F \in R(\overline{\mathcal{V}})^* = K(\mathcal{O}_{V_i,\overline{\mathcal{V}}})$, where $K(\mathcal{O}_{V_i,\overline{\mathcal{V}}})$ is the quotient field of the domain $\mathcal{O}_{V_i,\overline{\mathcal{V}}}$. Then $F = g/h$ where g, h $\in \mathcal{O}_{V_i,\overline{\mathcal{V}}}$. So $\mathrm{ord}_{\overline{V_i}}(F) = \mathrm{ord}_{\overline{V_i}}(g) - \mathrm{ord}_{\overline{V_i}}(h) = \ell(\mathcal{O}_{\overline{V_i},\overline{\mathcal{V}}}/(g)) - \ell(\mathcal{O}_{\overline{V_i},\overline{\mathcal{V}}}/(h))$ (definition 5.5.26). But $\mathcal{O}_{\overline{V_i},\overline{\mathcal{V}}} = \mathcal{O}_{V_i,\mathcal{V}}$: this follows easily because $K(\mathcal{O}_{\overline{V_i},\overline{\mathcal{V}}}) = R(\overline{\mathcal{V}})^* = R(\mathcal{V})^* = K(\mathcal{O}_{V_i,\mathcal{V}})$. So $K(\mathcal{O}_{\overline{V_i},\overline{\mathcal{V}}}) = K(\mathcal{O}_{V_i,\mathcal{V}})$. Now since $\mathcal{O}_{\overline{V_i},\overline{\mathcal{V}}}$, $\mathcal{O}_{V_i,\mathcal{V}}$ are domains with the same quotient field. It follows $\mathcal{O}_{\overline{V_i},\overline{\mathcal{V}}} = \mathcal{O}_{V_i,\mathcal{V}}$. Therefore $\ell(\mathcal{O}_{\overline{V_i},\overline{\mathcal{V}}}/(g)) - \ell(\mathcal{O}_{\overline{V_i},\overline{\mathcal{V}}}/(h)) = \ell(\mathcal{O}_{V_i,\mathcal{V}}/(g)) - \ell(\mathcal{O}_{V_i,\mathcal{V}}/(h)) = \mathrm{ord}_{V_i}(g) - \mathrm{ord}_{V_i}(h) = \mathrm{ord}_{V_i}(g/h) = \mathrm{ord}_{V_i}(F)$. So $\mathrm{ord}_{\overline{V_i}}(F) = \mathrm{ord}_{V_i}(F)$.

Let η_{V_i} be the generic point of V_i. Then

$$\begin{aligned}
[(F)] &= \sum_{\eta_{V_i} \in \mathcal{V}} m_i[\eta_{V_i}] + \sum_{\eta_{V_i} \in (\overline{\mathcal{V}})_0} m_i[\eta_{V_i}] \\
&= \sum_{\eta_{V_i} \in \mathcal{V}} \mathrm{ord}_{V_i}(F)[\eta_{V_i}] + \sum_{\eta_{V_i} \in (\overline{\mathcal{V}})_0} m_i[\eta_{V_i}] \\
&= \sum_{\eta_{V_i} \in \mathcal{V}} \mathrm{ord}_{\overline{V_i}}(F)[\eta_{V_i}] + \sum_{\eta_{V_i} \in (\overline{\mathcal{V}})_0} m_i[\eta_{V_i}]
\end{aligned}$$

But $\eta_{V_i} = \eta_{\overline{V_i}}$. It follows

$$\begin{aligned}
[(F)] &= \sum_{\eta_{V_i} \in \mathcal{V}} \mathrm{ord}_{\overline{V_i}}(F)[\eta_{\overline{V_i}}] + \sum_{\eta_{V_i} \in (\overline{\mathcal{V}})_0} m_i[\eta_{V_i}] \\
&= \sum_{V_i \not\subseteq (\overline{\mathcal{V}})_0} m_i[\overline{V_i}] + \sum_{V_i \subseteq (\overline{\mathcal{V}})_0} m_i[V_i]
\end{aligned}$$

So $[(F \mid_{\mathcal{V}})] = \sum_{V_i \not\subseteq (\overline{\mathcal{V}})_0} m_i[\overline{V_i} \cap \mathcal{V}]$ (note that $\overline{V_i} \cap \mathcal{V}$ is empty if $V_i \subseteq (\overline{\mathcal{V}})_0$). So $[(F \mid_{\mathcal{V}})] = \sum m_i[V_i]$. But $[(F \mid_{\mathcal{V}})] = [\mathcal{D}_1] - [\mathcal{D}_2]$. Therefore

$$[(F)] = [\overline{\mathcal{D}_1}] - [\overline{\mathcal{D}_2}] + \sum_{V_i \subseteq (\overline{\mathcal{V}})_0} m_i[V_i] \tag{85}$$

Thus $0^![(F)] = 0^![\overline{\mathcal{D}_1}] - 0^![\overline{\mathcal{D}_2}] + 0^!(\sum_{V_i \subseteq (\overline{\mathcal{V}})_0} m_i[V_i]) = [(\overline{\mathcal{D}_1})_0] - [(\overline{\mathcal{D}_2})_0] = [(\overline{\mathcal{D}_1})_0 - (\overline{\mathcal{D}_2})_0]$. But $[(\overline{\mathcal{D}_1})_0 - (\overline{\mathcal{D}_2})_0] = [\lim_{t \longrightarrow 0} t.D_1 - \lim_{t \longrightarrow 0} t.D_2] = [\lim_{t \longrightarrow 0}(t.D_1 - t.D_2)] = [\lim_{t \longrightarrow 0} t.(f)]$. So $0^![(F)] = [\lim_{t \longrightarrow 0} t.(f)]$. But the cycle $[(F)]$ is rationally equivalent to zero on the total space of the family $\overline{\mathcal{V}}$, being a cycle of a rational function, implies (by fact 5.5.38) that $0^![(F)]$ is rationally equivalent to zero in the fiber $(\overline{\mathcal{V}})_0$. So $[\lim_{t \longrightarrow 0} t.(f)]$ is rationally equivallent to zero in $(\overline{\mathcal{V}})_0 = \lim_{t \longrightarrow 0} t.V$. Therefore $\lim_{t \longrightarrow 0} t.(f)$ is linearly equivalent to zero in $\lim_{t \longrightarrow 0} t.V$.

Geometrically we observed from examples that the graph of $\lim_{t \longrightarrow 0} t.f$ consists of several components and is not necessairly a graph of a function. Consider the function F on the total space of the family $\overline{\mathcal{V}}$ then the graph of the divisor (F) in $\overline{\mathcal{V}}$ has components that live in $(\overline{\mathcal{V}})_0$. When we apply the map $0^!$ to the cycle $[(F)]$ we kill those components in $(\overline{\mathcal{V}})_0$. So if $\gamma = [(F)] = \sum_{D_i \subseteq (\overline{\mathcal{V}})_0} n_i[C_i] + \sum_{E_i \nsubseteq (\overline{\mathcal{V}})_0} m_i[E_i]$ then $0^!\gamma = 0^![(F)] = \sum_{E_i \nsubseteq (\overline{\mathcal{V}})_0} m_i 0^![E_i]$.

Definition 5.5.40. Let X be an n-dimensional B-variety and let *Div* X be the free abelian group generated by prime divisors. We define the cycle $[\sum n_i D_i]$ by setting $[\sum n_i D_i] = \sum n_i[D_i]$ where $[D_i]$ is the cycle that corresponds to the divisor D_i.

Lemma 5.5.41. Let X be a B-variety, and let Z be a k-dimensional subvariety of X such that $Z \sim 0$ in V. Let $\psi : \mathbb{C}^* \longrightarrow T$ be a 1-parameter subgroup of T such that $\lim_{t \longrightarrow 0} \psi(t).Z$ and $\lim_{t \longrightarrow 0} \psi(t).V$ are T-invariant. Then $\lim_{t \longrightarrow 0} \psi(t).Z$ is a subset of $\lim_{t \longrightarrow 0} \psi(t).V$.

Proof. Since $Z \subset V$ then $(Z, V) \in \mathcal{C}$ (fact 5.5.23). Define the map $f : \mathbb{C}^* \longrightarrow \mathrm{Hilb}^p X$ (resp.$g : \mathbb{C}^* \longrightarrow \mathrm{Hilb}^q X$) by $f(t) = t.Z$ (rep.$g(t) = t.V$) where $\mathrm{Hilb}^p X$ (resp.$\mathrm{Hilb}^q X$) is the component of the Hilbert scheme $\mathrm{Hilb}\, X$ containing Z (resp.containing V). Since $\mathrm{Hilb}\, X$ is projective there exists a unique lifting $f' : \mathbb{C} \longrightarrow \mathrm{Hilb}^p X$ (resp.$g' : \mathbb{C} \longrightarrow$

$\text{Hilb}^q X$) such that $\lim_{t\longrightarrow 0}\psi(t).Z = f'(0)$ (resp. $\lim_{t\longrightarrow 0}\psi(t).V = g'(0)$). Define the limit of the family $\{(t.Z, t.V)\}_{t\in\mathbb{C}^*}$ by $\lim_{t\longrightarrow 0}(t.Z, t.V) = (f'(0), g'(0))$. Then $\{(t.Z, t.V)\}_{t\in\mathbb{C}^*}$ is a family in $\mathcal{C}$ because $t.Z \in \text{Hilb}^p X$, $t.V \in \text{Hilb}^q X$, and $t.Z \subset t.V$ (since $Z \subset V$). Also $\{(t.Z, t.V)\}_{t\in\mathbb{C}^*}$ converges to $(f'(0), g'(0))$. So $(f'(0), g'(0)) \in \overline{\mathcal{C}} = \mathcal{C}$. Therefore $f'(0) \subseteq g'(0)$).

Lemma 5.5.42. Let X be a B-variety, and let Z be a k-dimensional subvariety of X such that $Z \sim 0$ in V. Let $\psi : \mathbb{C}^* \longrightarrow T$ be a 1-parameter subgroup of T such that $\lim_{t\longrightarrow 0}\psi(t).Z$ and $\lim_{t\longrightarrow 0}\psi(t).V$ are T-invariant. Then $\lim_{t\longrightarrow 0}\psi(t).Z \sim 0$ in $\lim_{t\longrightarrow 0}\psi(t).V$.

Proof. For simplicity we replace $\psi(t)$ by t. Let $Z' = \lim_{t\longrightarrow 0} t.Z$ and let $V' = \lim_{t\longrightarrow 0} t.V$. By lemma 5.5.41 $Z' \subset V'$. We will show that $Z' \sim 0 \subset V'$. Since $Z \sim 0 \subset V$ then $Z = (f)$ where $f : V \longrightarrow \mathbb{P}^1$ is a rational function on the $(k+1)$-dimensional variety V. So $[Z] = [(f)]$. Let $\mathcal{V} \subseteq \mathbb{C}^* \times X$ be a family with fibers $\mathcal{V}_t = t.V$. Let $(f(x)) = D_1 - D_2$ where $D_1 = f^{-1}(0)$, $D_2 = f^{-1}(\infty)$. Fix $t \in \mathbb{C}^*$ then $(t.f(x)) = t.D_1 - t.D_2$. Let $\mathcal{D}_i \subseteq \mathbb{C}^* \times X$ be the subscheme with fibers $t.D_i$. Let $F^*(t,x) = (t.f)(x) \in R(V)^*$ (here t is not fixed). Note that F^* is a nonzero rational function on the total space of the family $\mathcal{V}$. It follows $(F^*) = \mathcal{D}_1 - \mathcal{D}_2$. Let $F \in R(\overline{\mathcal{V}})^*$ such that $F\mid_{\mathcal{V}} = F^*$. As explained in fact 5.5.39 we have

$$[(F)] = [\overline{\mathcal{D}_1}] - [\overline{\mathcal{D}_2}] + \sum_{V_i \subseteq (\overline{\mathcal{V}})_0} m_i[V_i] \tag{86}$$

Thus $0^![(F)] = 0^![\overline{\mathcal{D}_1}] - 0^![\overline{\mathcal{D}_2}] + 0^!(\sum_{V_i \subseteq (\overline{\mathcal{V}})_0} m_i[V_i]) = [(\overline{\mathcal{D}_1})_0] - [(\overline{\mathcal{D}_2})_0] = [(\overline{\mathcal{D}_1})_0 - (\overline{\mathcal{D}_2})_0]$. But $[(\overline{\mathcal{D}_1})_0 - (\overline{\mathcal{D}_2})_0] = [\lim_{t\longrightarrow 0} t.D_1 - \lim_{t\longrightarrow 0} t.D_2] = [\lim_{t\longrightarrow 0}(t.D_1 - t.D_2)] = [\lim_{t\longrightarrow 0} t.(f)]$. So $0^![(F)] = [\lim_{t\longrightarrow 0} t.(f)]$. But the cycle $[(F)]$ is rationally equivalent to zero on the total space of the family $\overline{\mathcal{V}}$, being a cycle of a rational function, implies (by fact 5.5.38) that $0^![(F)]$ is rationally equivalent to zero in the fiber $(\overline{\mathcal{V}})_0$. So $[\lim_{t\longrightarrow 0} t.(f)]$ is rationally equivalent to zero in $(\overline{\mathcal{V}})_0 = \lim_{t\longrightarrow 0} t.V$. Therefore $\lim_{t\longrightarrow 0} t.(f)$ is rationally equivallent to zero in $\lim_{t\longrightarrow 0} t.V$. But $Z' = lim_{t\longrightarrow 0} t.Z = \lim_{t\longrightarrow 0} t.(f)$ and $V' = \lim_{t\longrightarrow 0} t.V$. It follows Z' is linearly equivalent to zero in V'. Thus lemma 5.5.42 is proven.

Remark 5.5.43. Let V be a subvariety of a B-variety X. It can be checked easily that if V is T-invariant, i.e $t.V = V$ then $\lim\limits_{t \longrightarrow 0} \psi(t).V = V$. Let $\mathcal{V} \subseteq \mathbb{C}^* \times X$ be the subscheme with fibers $t.V$. Consider the fiber square

$$\begin{array}{ccc} \mathbb{C}^* \times V & \longrightarrow & V \\ \big\downarrow{\scriptstyle Id \times i} & & \big\downarrow{\scriptstyle i} \\ \mathbb{C}^* \times X & \xrightarrow{\varphi} & X \end{array} \tag{87}$$

Since the closed imbedding $\mathbb{C}^* \times V \hookrightarrow \mathbb{C}^* \times X$ is $Id \times i$ it follows $\mathcal{V} = \mathbb{C}^* \times V$. So the closure $\overline{\mathcal{V}}$ of $\mathcal{V}$ in $\mathbb{C} \times V$ is equal to $\mathbb{C} \times V$ which is equal to the closure of $\mathcal{V}$ in $\mathbb{C} \times X$ because $\mathbb{C} \times V$ is a closed subset in $\mathbb{C} \times X$ containing $\mathbb{C}^* \times V$. Therefore $(\overline{\mathcal{V}})_0 = \{0\} \times V = V$. But $\lim\limits_{t \longrightarrow 0} t.V = (\overline{\mathcal{V}})_0$. Therefore $\lim\limits_{t \longrightarrow 0} t.V = V$.
In the following theorem we gave a necessary and sufficient condition for two T-invariant subvarieties D_1, $D_2 \subset X$ of dimension k to be T-invariantly rationally equivalent to zero. This condition is expressed using the weights of the characters $\chi_i(t) = t_i$ where $t \in T$, and and T-invariant subvarieties $Z \subset X$ of dimension $k+1$. Using this theorem we can calculate the dimension of the $\mathbb{C}-$module $R^k_T(X)$ and determine the BETTI NUMBERS of X by calculating the dimension of $A^k_T(X) = Z^k_T(X)/R^k_T(X)$.

Theorem 5.5.44. Let X be a B-variety, and Let D_1, D_2 be T-invariant subvarieties of dimension k. Let λ_i be the weight of the character $\chi_i(t) = t_i$ where $t \in T$. Then $D_1 \overset{\text{rat}}{\sim} D_2$ iff $[D_{1T} - D_{2T}] \in$ Span $\{\lambda_i.[Z_{jT}]$: where $Z_j \subset X$ are the T-invariant subvarieties of dimension $k+1\}$.

Proof. ($\Longrightarrow$) Suppose $D_1 \overset{\text{rat}}{\sim} D_2$ where D_1, D_2 are T-invariant subvariety of dimension k. Then $D_1 - D_2 = \sum_{i=1}^{m} E_i$ such that $E_i \sim 0$ in C_i where $\dim E_i =$ k, and $\dim C_i = k+1$. Let Hilb_{E_i} (resp. Hilb_{C_i}) be the component of the Hilbert scheme $\text{Hilb}(X)$ containing E_i (resp. C_i). By theorem 5.5.22 there exists a T-representation

$W_i = \oplus_{j=1}^{m_i} W_{\chi_{ij}}$ (resp. $N_i = \oplus_{k=1}^{r_i} W_{\rho_{ik}}$) such that Hilb_{E_i} (resp. Hilb_{C_i}) is embedded T-equivariantly in $\mathbb{P}(W_i)$ (resp. $\mathbb{P}(N_i)$), $i = 1, ..., m$. Consider the set of characters $S = \{\chi_{ij} : i = 1, ..., m,\ j = 1, ..., m_i\} \cup \{\rho_{ik} : i = 1, ..., m,\ k = 1, ..., r_i\}$. Choose a one parameter subgroup $\psi : \mathbb{C}^* \longrightarrow T$ such that $\mu - \nu$ does not live in $H_\psi = \{\chi :< \psi, \chi >= 0\}$ for each $\mu,\ \nu \in S$. Then by lemma 5.5.13 $\lim_{t \longrightarrow 0} \psi(t).\overline{x}$ is T-invariant $\forall \overline{x} \in \mathbb{P}(\oplus_{i=1}^{m} W_i)$ and $\forall \overline{x} \in \mathbb{P}(\oplus_{i=1}^{m} N_i)$. But $E_i \in \mathrm{Hilb}_{E_i} \subset \mathbb{P}(W_i)$ implies $\lim_{t \longrightarrow 0} \psi(t).E_i$ is T-invariant $\forall i = 1, ..., m$. Also $C_i \in \mathrm{Hilb}_{C_i} \subset \mathbb{P}(N_i)$ implies $\lim_{t \longrightarrow 0} \psi(t).C_i$ is T-invariant $\forall i = 1, ..., m$. Let $E_i' = \lim_{t \longrightarrow 0} \psi(t).E_i$, and let $C_i' = \lim_{t \longrightarrow 0} \psi(t).C_i$ then E_i', and C_i' are T-invariant and by lemma 5.5.41 $E_i' \subset C_i'$ because $E_i \subset C_i$. By lemma 5.5.42 it follows that E_i' is linearly equivalent to zero on C_i'. Consider $\psi(t).(D_1 - D_2) = \psi(t).\sum_{i=1}^{m} E_i = \sum_{i=1}^{m} \psi(t).E_i$. It follows that $\lim_{t \longrightarrow 0} \psi(t).(D_1 - D_2) = \sum_{i=1}^{m} \lim_{t \longrightarrow 0} \psi(t).E_i$. Let $\mathcal{E}_i \subseteq \mathbb{C}^* \times X$ be the subscheme with fibers $t.E_i$. Then $\lim_{t \longrightarrow 0} \psi(t).(D_1 - D_2) = \sum_{i=1}^{m} (\overline{\mathcal{E}_i})_0$. So $\lim_{t \longrightarrow 0} \psi(t).D_1 - \lim_{t \longrightarrow 0} t.D_2 = \sum_{i=1}^{m} (\overline{\mathcal{E}_i})_0$. But D_i is T-invariant, i.e $t.D_i = D_i$ implies $\lim_{t \longrightarrow 0} \psi(t).D_i = D_i$ (remark 5.5.43). It follows that $D_1 - D_2 = \sum_{i=1}^{m} (\overline{\mathcal{E}_i})_0$. Now since the Hilbert definition of the limit and the closure definition of the limit agree (proposition 5.5.12), it follows that $(\overline{\mathcal{E}_i})_0$ is T-invariant. Let $E_i' = (\overline{\mathcal{E}_i})_0$ then $D_1 - D_2 = \sum_{i=1}^{m} E_i'$ such that $E_i' \subset C_i'$ where E_i', C_i' are T-invariant and $E_i' \sim 0 \subset C_i'$. It follows from part (1) of theorem 5.4.1 that the equivariant cycle $[E_{iT}'] \in$ Span $\{\lambda_k.[C_{iT}'] : k = 1, ..., r\}$. Thus $[D_{1T} - D_{2T}] = \sum_{i=1}^{m} [E_{iT}'] \in$ Span $\{\lambda_k.[C_{iT}'] : i = 1, ..., m,\ k = 1, ..., r$ where C_i' are T-invariant subvarieties of dimension $k+1\} \subset$ Span $\{\lambda_k.[Z_{jT}] : k = 1, ..., r$ where Z_j are T-invariant subvarieties of dimension $k+1\}$.

($\Longleftarrow$) Suppose that $[D_{1T} - D_{2T}] \in$ Span $\{\lambda_i.[Z_{jT}]$: where $Z_j \subset X$ is a T-invariant subvariety of dimension $k+1\}$. Then $[D_{1T} - D_{2T}] = \sum_{q=1}^{r} \sum_{l=1}^{n} a_{ql} \lambda_q.[Z_{lT}]$ where $a_{ql} \in \mathbb{Z}$. Recall the map $i_X^* : H_T^*(X) \longrightarrow H^*(X)$ where $i_X^*([W_T]) = [W]$. It follows that $[D_1 - D_2] = \sum_{q=1}^{r} \sum_{l=1}^{n} a_{ql} i_X^*(\lambda_q.[Z_{lT}])$. So $[D_1 - D_2] = \sum_{q=1}^{r} \sum_{l=1}^{n} a_{ql} i_X^*(\lambda_q).i_X^*([Z_{lT}])$.

But $i_X^*(\lambda_q) = 0$ implies $[D_1 - D_2] = 0$. Thus $D_1 \overset{\text{rat}}{\sim} D_2$.

Definition 5.5.45. A scheme X has a cellular decomposition if there is a filtration $X = X_n \supset X_{n-1} \supset ... \supset X_0 \supset X_{-1} = \phi$ by closed subschemes with each $X_i - X_{i-1}$ a disjoint union of schemes U_{ij} isomorphic to affine spaces $\mathbb{A}^{n_{ij}}$. The U_{ij}'s are called the cells of the decomposition.

Proposition 5.5.46. Let X an n-dimensional B-variety then $A_*(X)$ is generated by the closure of T-invariant classes. If X has a cellular decomposition then the k-th Chow group $A_k(X)$ is generated by the classes of the closures of the k-dimensional cells. See [12].

Remark 5.5.47. Let X be a B-variety and let $t \in T$ where T is the torus acting on X. Define $\mu : X \longrightarrow X$ by $\mu(x) = t.x$. Then μ is a bijection. Let $V \subset X$ be a T-invariant subset. It follows that $\mu(X - V) = \mu(X) - \mu(V) = X - V$ $(\mu(V) = V$ because V is T-invariant). So $X - V$ is T-invariant.

Definition 5.5.48. Let X be a B-variety. A k-cycle α is T-equivariantly rationally equivalent to zero, written $\alpha \overset{\text{T}}{\sim} 0$, if there exists a finite number of T-invariant $(k+1)-$dimensional subvarieties W_i of X, and $f_i \in R(W_i)^*$, such that $t.[(f_i)] = [(f_i)]$ with $\alpha = \sum[(f_i)]$.

Remark 5.5.49. Let q $\in \mathbb{P}^1$. Form the fiber square

$$\begin{array}{ccc} X \times \{q\} & \hookrightarrow & X \times \mathbb{P}^1 \\ \downarrow & & \downarrow \\ \{q\} & \hookrightarrow & \mathbb{P}^1 \end{array} \tag{88}$$

Let $V \subset X \times \mathbb{P}^1$ be a $(k+1)-$dimensional subvariety such that the projection to the second factor induces a surjective morphism f from V to $\mathbb{P}^1$. Form the fiber square

$$\begin{array}{ccc} V \times_{\mathbb{P}^1} \{q\} & \hookrightarrow & V \\ \downarrow & & \downarrow f \\ \{q\} & \hookrightarrow & \mathbb{P}^1 \end{array} \tag{89}$$

Let P be the projection from $X \times \mathbb{P}^1$ to X. Note that the scheme-theoretic fiber $f^{-1}(q) = V \times_{\mathbb{P}^1} \{q\}$ is a subscheme of $X \times \{q\}$, which P maps isomorphically onto a subscheme of X; we denote this subscheme by $V(q)$. Note in particular that $P_*[f^{-1}(q)] = [V(q)]$ in $Z_k(X)$.

Definition 5.5.50. (Hirschowitz) Let Y be an algebraic variety acted on by an algebraic group G. Let U and V be two invariant cycles on Y under G and let R be a cycle of $Y \times \mathbb{P}^1$ which gives the rational equivalence between U,V, i.e (for example) $U = R(0)$, $V = R(\infty)$. We say that the rational equivalence R is equivariant if there exists an algebraic action of G on $\mathbb{P}^1$ which fixes 0, ∞, such that the cycle R is invariant under the corresponding action of G on $Y \times \mathbb{P}^1$. And we say that the rational equivalence R is invariant if it is the trivial action of G on $\mathbb{P}^1$ which makes the rational equivalence equivariant ([26]).

Fact 5.5.51. Definition 5.5.48 and Definition 5.5.50 are equivalent if $G = T$ is a torus.

To see that we will prove the following theorem.

Theorem 5.5.52. Let X be a B-variety. A k-cycle α is T-equivariantly rationally equivalent to zero, written $\alpha \overset{\mathrm{T}}{\sim} 0$, if and only if there exists a finite number of $(k+1)-$dimensional equivariant subvarieties V_1, ..., V_s of $X \times \mathbb{P}^1$, i.e there exists an action on $\mathbb{P}^1$ which fixes the two points x_0, x_∞ in $\mathbb{P}^1$ such that the cycle $[V_i]$ is invariant under the corresponding action of T on $X \times \mathbb{P}^1$, with $\alpha = \sum_{i=1}^{s}[V_i(0)] - [V_i(\infty)]$ in $Z_k(X)$.

Proof. ($\Longleftarrow$): Suppose that there exists a finite number of $(k+1)-$dimensional equivariant subvarieties V_1, ..., V_s of $X \times \mathbb{P}^1$ such that $\alpha = \sum_{i=1}^{s}[V_i(0)] - [V_i(\infty)]$ in $Z_k(X)$. Let $f_i : V_i \longrightarrow \mathbb{P}^1$ be the morphism induced by the projection to the second factor, i.e $f_i = \pi_2 \mid_{V_i}$ where $\pi_2 : X \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ is the projection morphism. Let P be the projection from $X \times \mathbb{P}^1$ to X. It follows that $\alpha = \sum_{i=1}^{s}[V_i(0)] - [V_i(\infty)] = \sum_{i=1}^{s} P_*[(f_i)]$. But $P_*[(f_i)] = [(N(f_i))]$ (Proposition 1.4 in [17]) where $N(f_i) \in R(P(V_i))$ is the norm of f_i, i.e the determinant of the $R(P(V_i))$-linear endomorphism given by multiplication by f_i. We check that V_i is equivariant implies $P(V_i)$ is T-invariant. Note that T is the group acting on X by $\rho : T \longrightarrow Aut(X)$ where $Aut(X)$ is the group of automorphisms on X, and on $\mathbb{P}^1$ by $\sigma : T \longrightarrow Aut(\mathbb{P}^1)$ ($Aut(\mathbb{P}^1)$ is the group of automorphisms on $\mathbb{P}^1$). Now if $x \in X$, $y \in \mathbb{P}^1$ then we have the actions: $t.x = \rho(t)x$, $t.y = \sigma(t)y$, and $t.(x, y) = (\rho(t)x, \sigma(t)y)$. Let $x \in P(V_i) \subset X$. We check that $t.x \in P(V_i)$ where $t \in T$. Let $x = P(z)$ where $z \in V_i$. Then $t.x = t.P(z) = P(t.z)$ because P is an equivariant map ($P(t.(a, b)) = P(\rho(t)a, \sigma(t)b) = \rho(t)a = t.a = t.P(a, b)$). But $t.z \in V_i$ because V_i is equivariant. It follows that $P(t.z) \in P(V_i)$. So $t.x \in P(V_i)$.

It remains to check that $[(N(f_i))]$ is T-invariant. Note that $[(N(f_i))] = P_*[(f_i)] = [V_i(0)] - [V_i(\infty)]$ and $V_i(0) = P(V_i \times_{\mathbb{P}^1} \{0\})$ where $V_i \times_{\mathbb{P}^1} \{0\} = f_i^{-1}(0) = (\pi_2 \mid_{V_i})^{-1}(0) \subset X \times \{0\} \subset X \times \mathbb{P}^1$. We check that $V_i \times_{\mathbb{P}^1} \{0\}$ is $T-$invariant. Let $s \in T$. Then $s.(V_i \times_{\mathbb{P}^1} \{0\}) := s.V_i \times_{\mathbb{P}^1} s.\{0\}$. But V_i is equivariant, i.e V_i is $T-$invariant and $s.\{0\} = \{\rho(t).0\} = \{0\}$. It follows that $s.V_i \times_{\mathbb{P}^1} s.\{0\} = V_i \times_{\mathbb{P}^1} \{0\}$. So $V_i \times_{\mathbb{P}^1} \{0\}$ is $T-$invariant. Therefore $P(V_i \times_{\mathbb{P}^1} \{0\}) = V_i(0)$ is T-invariant. Hence $[V_i(0)]$ is a T-invariant cycle. By a similar argument $[V_i(\infty)]$ is T-invariant. So $[(N(f_i))] = [V_i(0)] - [V_i(\infty)]$ is a T-invariant cycle.

($\Longrightarrow$): Let $\alpha = [(r)]$, $r \in R(W)^*$ where W is a T-invariant $(k+1)-$dimensional subvariety of X. Let U be an open subset of W such that r is defined on U. Note that $t.r = \chi(t)r$ where $\chi(t)$ is a character. Define an action on $\mathbb{P}^1$ by $t.y = \chi^{-1}(t)y$ where $y \in \mathbb{P}^1$. Let T be the group acting on X by $\rho : T \longrightarrow Aut(X)$ where $Aut(X)$

is the group of automorphisms on X, and on $\mathbb{P}^1$ by $\chi : T \longrightarrow Aut(\mathbb{P}^1)$ ($Aut(\mathbb{P}^1)$ is the group of automorphisms on $\mathbb{P}^1$). Now if $x \in X$, $y \in \mathbb{P}^1$ then we have the actions: $t.x = \rho(t)x$, $t.y = \chi^{-1}(t)y$, and $t.(x, y) = (\rho(t)x, \chi^{-1}(t)y)$. Let $\tilde{U} = \cup_{t \in T} t.U$. Then $\tilde{U}$ is an open subset of W. Let V be the closure of the graph of $r \mid_{\tilde{U}}$ in $X \times \mathbb{P}^1$ (which is equal to the closure of the graph of $r \mid_{\tilde{U}}$ in $W \times \mathbb{P}^1$ because $W \times \mathbb{P}^1$ is a closed subset of $X \times \mathbb{P}^1$). Let $Id : \tilde{U} \longrightarrow \tilde{U}$ be the identity morphism. Then $V = \overline{(Id \times r \mid_{\tilde{U}})(\tilde{U})}$. Let P be the projection from $X \times \mathbb{P}^1$ to X. Then P maps V birationally and properly onto W. Note that P is proper being a composition of proper maps. Also P is birational because P is an isomorphism of $(Id \times r \mid_{\tilde{U}})(\tilde{U})$ with $\tilde{U}$. We check that $V = \overline{(Id \times r \mid_{\tilde{U}})(\tilde{U})}$ is $T-$invariant. It is enough to check that $(Id \times r \mid_{\tilde{U}})(\tilde{U})$ is $T-$invariant. So let $(x, r(x)) \in (Id \times r \mid_{\tilde{U}})(\tilde{U})$ and let $t \in T$. Then $t.(x, r(x)) = (t.x, t.r(x)) = (\rho(t)x, \chi^{-1}(t)r(x))$. But $t.r(x) = \chi(t)r(x)$ implies $t^{-1}.r(x) = \chi^{-1}(t)r(x)$. So $r(t.x) = \chi^{-1}(t)r(x)$. It follows that $(t.x, \chi^{-1}(t)r(x)) = (t.x, r(t.x)) \in (Id \times r \mid_{\tilde{U}})(\tilde{U})$ because $x \in \tilde{U}$ and $\tilde{U}$ is $T-$invariant. Therefore $(Id \times r \mid_{\tilde{U}})(\tilde{U})$ is $T-$invariant. So $\overline{(Id \times r \mid_{\tilde{U}})(\tilde{U})}$ is $T-$invariant, i.e $\overline{(Id \times r \mid_{\tilde{U}})(\tilde{U})}$ is equivariant.

Note that $N(r) = r$, where $N(r)$ is the norm of r, because $R(W) \simeq R(V)$. Then $\alpha = [(r)] = [(N(r))]$. Let f be the induced rational map from V to $\mathbb{P}^1$ (r induces f). But $[(N(r))] = P_*[(r)]($ proposition 1.4 in [17]). So $\alpha = P_*[(r)] = P_*[(f)] = [V(0)] - [V(\infty)]$.

Theorem 5.5.53. Let X be an n-dimensional B-variety then $A_T^k(X) \simeq A^k(X)$.

For a proof of this theorem see [26]. We will provide a new proof for this theorem.

Proof. Let $[C]$ be the cycle class corresponding to C where $C \subset X$ is a T-invariant subvariety of codimension k in X. We define the group homomorphism $\phi : A_T^k(X) \longrightarrow A^k(X)$ by $\phi([C]) = [C]$. The surjectivity of ϕ follows from Proposition 5.5.46. To show that ϕ is injective, it is enough to show that $Z_k^T(X) \cap R_k(X) = R_k^T(X)$.

It is clear that $Z^k_T(X) \cap R^k(X) \supset R^k_T(X)$. We will show that the inclusion

$$Z^k_T(X) \cap R^k(X) \subset R^k_T(X)$$

holds. Let $[Z] \in Z^k_T(X) \cap R^k(X)$ then $[Z]$ is a T-invariant $(n-k)-cycle$ and $[Z] \sim 0$ on X. So there exists a finite number of $(n-k+1)-$dimensional subvarieties V_i of X, and $f_i \in R(V_i)^*$, such that $[Z] = \sum_{i=0}^{n}[(f_i)]$. Let $Z_i = (f_i)$ and let Hilb_{V_i} (resp. Hilb_{Z_i}) be the component of $\text{Hilb}(X)$ containing V_i (resp. Z_i) then there exists a T-representation $W_i = \oplus_{j=1}^{m_i} W_{\chi_{ij}}$ (resp. $N_i = \oplus_{j=1}^{r_i} W_{\rho_{ij}}$) such that Hilb_{V_i} (resp. Hilb_{Z_i}) is embedded T-equivariantly in $\mathbb{P}(W_i)$ (resp. $\mathbb{P}(N_i)$), $i = 0, ..., n$. Consider the set of characters

$$S = \{\chi_{ij} : i = 0, ..., n, j = 1, ..., m_i\} \cup \{\rho_{ik} : i = 0, ..., n, k = 1, ..., r_i\}.$$

Choose a 1-parameter subgroup $\psi : \mathbb{C}^* \longrightarrow T$ such that $\chi_{ik} - \chi_{jl} \notin H_\psi$ and $\rho_{ik} - \rho_{jl} \notin H_\psi$. Then by lemma 5.5.13 $\lim_{t \longrightarrow 0} \psi(t).\overline{x}$ is T-invariant $\forall \overline{x} \in \mathbb{P}(W_i)$ and $\forall \overline{x} \in \mathbb{P}(N_i)$. But $V_i \in \text{Hilb}_{V_i} \subset \mathbb{P}(W_i)$. So $V'_i = \lim_{t \longrightarrow 0} \psi(t).V_i$ is T-invariant $\forall i = 0, ..., n$. Also $Z_i \in \text{Hilb}_{Z_i} \subset \mathbb{P}(N_i)$. So $Z'_i = \lim_{t \longrightarrow 0} \psi(t).Z_i$ is T-invariant $\forall i = 0, ..., n$.
For simplicity we will use t to denote for $\psi(t)$. Let $\mathcal{V}_i \subset \mathbb{C}^* \times X$ (resp.$\xi \subset \mathbb{C} \times X$) be the subscheme with fibers $t.V_i$ (resp.$t.Z$). Since $f_i \in R(V_i)^*$ then $(f_i) = D_{i1} - D_{i2}$ where $D_{i1} = f_i^{-1}(0)$, $D_{i2} = f_i^{-1}(\infty)$. Let $\mathcal{D}_{ij} \subset \mathbb{C}^* \times X$ be the subscheme with fibers $t.D_{ij}$ where $j = 1, 2$, $i = 0, 1, ..., n$. Let $F_i^*(t, x) = (t.f_i)(x)$ (t is not fixed). Note that $F_i^*(t, x)$ is a non-zero rational function on the total space of the family $\mathcal{V}_i$. It follows $(F_i^*) = \mathcal{D}_{i1} - \mathcal{D}_{i2}$. Let $F \in R(\overline{\mathcal{V}_i})^*$ such that $F_i \mid \mathcal{V}_i = F_i^*$. Then as explained in fact 5.5.39 (see (85)) we have

$$[(F_i)] = [\overline{\mathcal{D}_{i1}}] - [\overline{\mathcal{D}_{i2}}] + \sum_{V_{ij} \subseteq (\overline{\mathcal{V}})_0} m_{ij}[V_{ij}] \tag{90}$$

Thus $0^![(F_i)] = 0^![\overline{\mathcal{D}_{i1}}] - 0^![\overline{\mathcal{D}_{i2}}] + 0^!(\sum_{V_{ij}\subseteq(\overline{\mathcal{V}_i})_0} m_i[V_i]) = [(\overline{\mathcal{D}_{i1}})_0] - [(\overline{\mathcal{D}_{i2}})_0] = [(\overline{\mathcal{D}_{i1}})_0 - (\overline{\mathcal{D}_{i2}})_0]$. But $[(\overline{\mathcal{D}_{i1}})_0 - (\overline{\mathcal{D}_{i2}})_0] = [\lim_{t\longrightarrow 0} t.D_{i1} - \lim_{t\longrightarrow 0} t.D_{i2}] = [\lim_{t\longrightarrow 0}(t.D_{i1} - t.D_{i2}] = [\lim_{t\longrightarrow 0} t.(f_i)]$. But the cycle $[(F_i)]$ is rationally equivalent to zero on the total space of the family $\overline{\mathcal{V}_i}$, being a cycle of a rational function on $\overline{\mathcal{V}_i}$ implies (by fact 5.5.38) that $0^![(F_i)]$ is rationally equivalent to zero in the fiber $(\overline{\mathcal{V}^i})_0 = \lim_{t\longrightarrow 0} t.V_i = V_i'$. But $Z_i' = \lim_{t\longrightarrow 0} t.(f_i)$ is T-invariant (resp. V_i' is T-invariant). So $[\lim_{t\longrightarrow 0} t.(f_i)]$ is a T-invariant cycle. So $0^![(F_i)]$ is a T-invariant cycle. Therefore $0^![(F_i)]$ is a T-invariant cycle which is rationally equivalent to zero in the fiber $(\overline{\mathcal{V}^i})_0$ and $(\overline{\mathcal{V}^i})_0$ is T-invariant ($(\overline{\mathcal{V}^i})_0 = \lim_{t\longrightarrow 0} t.V_i = V_i'$ and V_i' is T-invariant). Thus $0^![(F_i)]$ is T-invariantly rationally equivalent to zero. Now $t.[Z] = \sum_{i=0}^n[((t.f_i)(x))]$ (here t is not fixed) implies $[\xi] = \sum_{i=0}^n[(F_i(t,x))]$ where $F_i \in R(\overline{\mathcal{V}_i})^*$. Applying $0^!$ we get $0^![\xi] = \sum_{i=0}^n 0^![(F_i(t,x))]$. But $[(F_i)]$ is T-invariantly rationally equivalent to zero. So $\sum_{i=0}^n[(F_i)]$ is T-invariantly rationally equivalent to zero. So $0^![\xi]$ is T-invariantly rationally equivalent to zero. But $0^![\xi] = [(\overline{\xi^*})_0] = [\lim_{t\longrightarrow 0} t.Z]$. Also $[Z]$ is a T-invariant cycle, i.e $t.[Z] = [Z]$, implies $t.Z = Z$. Now by remark 5.5.43 it follows $\lim_{t\longrightarrow 0} t.Z = Z$. Therefore $0^![\xi] = [Z]$. Thus $[Z]$ is T-invariantly rationally equivalent to zero.

6 Application

Let $Hilb^2\mathbb{P}^2$ be the Hilbert scheme parameterizing finite subschemes of length 2 in the projective plane (see [12]). Consider the $\mathbb{C}^*$-action on $\mathbb{P}^2$ given by

$$t.(x_0, x_1, x_2) = (t^{-a}x_0, t^{-b}x_1, t^{-c}x_2) \tag{91}$$

where a, b, c are integers. We will use $\mathcal{H}$ to denote $Hilb^2\mathbb{P}^2$. Let $\mathcal{H}^T = \{(x^2, y), (x, y^2), (x^2, z), (x, z^2), (y^2, z), (y, z^2), (yz, x), (xz, y), (xy, z)\}$ be the fixed point locus of the $\mathbb{C}^*$-action. Let $P_1 = (y^2, z)$, $P_2 = (x, z^2)$, $P_3 = (x, y^2)$, $P_4 = (x^2, y)$, $P_5 = (x^2, z)$, $P_6 = (y, z^2)$, $P_7 = (xy, z)$, $P_8 = (yz, x)$, $P_9 = (xz, y)$. Let $A_k(\mathcal{H})$ be the k-th Chow group of the Hilbert scheme $Hilb^2\mathbb{P}^2$. We will use our method namely, theorem 5.5.44

to determine the rational equivalences for the Chow ring $A_k(\mathcal{H})$. This new method will work to give new results for more complicated B-varieties. We will leave this for future work.

Consider the above $\mathbb{C}^*$-action on $\mathbb{P}^2$ such that $a \neq b$, $a \neq c$, and $b \neq c$. Let $E = \{e_i\}_{i=1}^3$ be the fixed point locus of $\mathbb{C}^*$-action on $\mathbb{P}^2$ where the i-th coordinate of e_i is nonzero, all other coordinates being zero. Let the line $L_{e_0} = (x = 0)$, $L_{e_1} = (y = 0)$, and $L_{e_2} = (z = 0)$. Then $A_*(\mathcal{H})$ is generated by the following T-invariant cycles (see Proposition 5.5.46)

(1) $A_0(\mathcal{H})$ is generated by the nine classes $[P_{iT}]$ where $[P_{iT}]$ denote the class of the closure of the cell P_i below:

$P_1 = \{\{e_0, e_0\} \in \mathcal{H} : \{e_0, e_0\}$ is contained in the line $L_{e_2}\}$

$P_2 = \{\{e_1, e_1\} \in \mathcal{H} : \{e_1, e_1\}$ is contained in the line $L_{e_0}\}$

$P_3 = \{\{e_2, e_2\} \in \mathcal{H} : \{e_2, e_2\}$ is contained in the line $L_{e_0}\}$

$P_4 = \{\{e_2, e_2\} \in \mathcal{H} : \{e_2, e_2\}$ is contained in the line $L_{e_1}\}$

$P_5 = \{\{e_1, e_1\} \in \mathcal{H} : \{e_1, e_1\}$ is contained in the line $L_{e_2}\}$

$P_6 = \{\{e_0, e_0\} \in \mathcal{H} : \{e_0, e_0\}$ is contained in the line $L_{e_1}\}$

$P_7 = \{\{e_0, e_1\} \in \mathcal{H} : e_0, e_1 \in E\}$

$P_8 = \{\{e_1, e_2\} \in \mathcal{H} : e_1, e_2 \in E\}$

$P_9 = \{\{e_0, e_2\} \in \mathcal{H} : e_0, e_2 \in E\}$

(2) $A_1(\mathcal{H})$ is generated by the nine classes $[l_{iT}]$, $[m_{iT}]$ where $[l_{iT}]$ (resp.$[m_{iT}]$) denote the class of the closure of the cell l_i (resp.m_i) below:

$l_1 = \{\{e_0, s\} \in \mathcal{H} : e_0 \in E,\ s \in L_{e_2}\}$

$l_2 = \{\{e_0, s\} \in \mathcal{H} : e_0 \in E,\ s \in L_{e_1}\}$

$l_3 = \{\{e_1, s\} \in \mathcal{H} : e_1 \in E,\ s \in L_{e_2}\}$

$l_4 = \{\{e_1, s\} \in \mathcal{H} : e_1 \in E,\ s \in L_{e_0}\}$

$l_5 = \{\{e_2, s\} \in \mathcal{H} : e_2 \in E,\ s \in L_{e_1}\}$

$l_6 = \{\{e_2, s\} \in \mathcal{H} : e_2 \in E,\ s \in L_{e_0}\}$

$m_1 = \{\{e_0, e_0\} \in \mathcal{H} : e_0 \in E$ and there exists a line of $\mathbb{P}^2$ containing the double point

$\{e_0, e_0\}\}$

$m_2 = \{\{e_1, e_1\} \in \mathcal{H} : e_1 \in E$ and there exists a line of $\mathbb{P}^2$ containing the double point $\{e_1, e_1\}\}$

$m_3 = \{\{e_2, e_2\} \in \mathcal{H} : e_2 \in E$ and there exists a line of $\mathbb{P}^2$ containing the double point $\{e_2, e_2\}\}$

(3) $A_2(\mathcal{H})$ is generated by the nine classes $[p_{iT}]$, $[q_{iT}]$, and $[r_{iT}]$ where $[p_{iT}]$ (resp.$[q_{iT}]$, $[r_{iT}]$) denote the class of the closure of the cell p_i (resp.q_i, r_i) below:

$p_1 = \{\{w, s\} \in \mathcal{H} : w, s \in L_{e_2}\}$

$p_2 = \{\{w, s\} \in \mathcal{H} : w, s \in L_{e_0}\}$

$p_3 = \{\{w, s\} \in \mathcal{H} : w, s \in L_{e_1}\}$

$q_1 = \{\{e_0, w\} \in \mathcal{H} : e_0 \in E,\ w \in \mathbb{P}^2\}$

$q_2 = \{\{e_1, w\} \in \mathcal{H} : e_1 \in E,\ w \in \mathbb{P}^2\}$

$q_3 = \{\{e_2, w\} \in \mathcal{H} : e_2 \in E,\ w \in \mathbb{P}^2\}$

$r_1 = \{\{v, v\} \in \mathcal{H} : \{v, v\} \in L_{e_2}$ and there exists a line of $\mathbb{P}^2$ containing the double point $\{v, v\}\}$

$r_2 = \{\{v, v\} \in \mathcal{H} : \{v, v\} \in L_{e_0}$ and there exists a line of $\mathbb{P}^2$ containing the double point $\{v, v\}\}$

$r_3 = \{\{v, v\} \in \mathcal{H} : \{v, v\} \in L_{e_1}$ and there exists a line of $\mathbb{P}^2$ containing the double point $\{v, v\}\}$

(4) $A_3(\mathcal{H})$ is generated by the six classes $[t_{iT}]$, $[u_{iT}]$ where $[t_{iT}]$ (resp.$[u_{iT}]$) denote the class of the closure of the cell t_i (resp.u_i) below:

$t_1 = \{\{s, v\} \in \mathcal{H} : s \in L_{e_2},\ v \in \mathbb{P}^2\}$

$t_2 = \{\{s, v\} \in \mathcal{H} : s \in L_{e_1},\ v \in \mathbb{P}^2\}$

$t_3 = \{\{s, v\} \in \mathcal{H} : s \in L_{e_0},\ v \in \mathbb{P}^2\}$

$u_1 = \{\{s, v\} \in \mathcal{H}$: there exists a line of $\mathbb{P}^2$ containing $e_0, s, v\}$

$u_2 = \{\{s, v\} \in \mathcal{H}$: there exists a line of $\mathbb{P}^2$ containing $e_1, s, v\}$

$u_3 = \{\{s, v\} \in \mathcal{H}$: there exists a line of $\mathbb{P}^2$ containing $e_2, s, v\}$

(5) $A_4(\mathcal{H})$ is generated by one class which is the class of the closure of the cell

$\{\{p,q\} : p,q \in \mathbb{P}^2\}$.

Case(I): We will determine the rational equivalences in the Chow ring $A_0(\mathcal{H})$ using theorem 5.5.44. Now according to theorem 5.5.44

$$\sum_{i=1}^{9} a_i[P_{iT}] \sim 0 \text{ iff } \sum_{i=1}^{9} a_i[P_{iT}] \in Span\{\lambda.[l_{iT}]\}_{i=1}^{6} \oplus Span\{\lambda.[m_{iT}]\}_{i=1}^{3} \tag{92}$$

So

$$\sum_{i=1}^{9} a_i[P_{iT}] \sim 0 \text{ iff } \sum_{i=1}^{9} a_i[P_{iT}] = \sum_{i=1}^{6} f_i(\lambda).[l_{iT}] + \sum_{i=1}^{3} g_i(\lambda).[m_{iT}] \tag{93}$$

Recall the map $i_j^* : A_T^*(\mathcal{H}) \longrightarrow A_T^*(P_j)$. In order to determine the rational equivalences in the Chow ring $A_0(\mathcal{H})$, we need to solve the following linear system which we get by applying the map i_j^* to the previous equation above:

System I

$$a_1 i_1^*[p_{1T}] = f_1(\lambda) i_1^*[l_{1T}] + g_1(\lambda) i_1^*[m_{1T}] \tag{94}$$

$$a_2 i_2^*[p_{2T}] = f_4(\lambda) i_2^*[l_{4T}] + g_2(\lambda) i_2^*[m_{2T}] \tag{95}$$

$$a_3 i_3^*[p_{3T}] = f_6(\lambda) i_3^*[l_{6T}] + g_3(\lambda) i_3^*[m_{3T}] \tag{96}$$

$$a_4 i_4^*[p_{4T}] = f_5(\lambda) i_4^*[l_{5T}] + g_3(\lambda) i_4^*[m_{3T}] \tag{97}$$

$$a_5 i_5^*[p_{5T}] = f_3(\lambda) i_5^*[l_{3T}] + g_2(\lambda) i_5^*[m_{2T}] \tag{98}$$

$$a_6 i_6^*[p_{6T}] = f_2(\lambda) i_6^*[l_{2T}] + g_2(\lambda) i_6^*[m_{2T}] \tag{99}$$

$$a_7 i_7^*[p_{7T}] = f_3(\lambda) i_7^*[l_{3T}] + f_1(\lambda) i_7^*[l_{1T}] \tag{100}$$

$$a_8 i_8^*[p_{8T}] = f_4(\lambda) i_8^*[l_{4T}] + f_6(\lambda) i_8^*[l_{6T}] \tag{101}$$

$$a_9 i_9^*[p_{9T}] = f_2(\lambda) i_9^*[l_{2T}] + f_5(\lambda) i_9^*[l_{5T}] \tag{102}$$

We evaluate $i_j^*[P_{jT}]$, $i_j^*[l_{kT}]$, and $i_j^*[m_{kT}]$ as follows:

(a) To evaluate $i_j^*[P_{jT}]$ we calculate the weights of the normal bundle $\mathcal{N}_{P_j|\mathcal{H}} = T_{P_j}\mathcal{H}$. Then $i_j^*[P_{jT}]$ is the product of those weights.

1) $i_1^*[P_{1T}] = (2a-2b)(a-b)(a-c)(b-c)$

2) $i_2^*[P_{2T}] = (b-a)(2b-2c)(b-c)(c-a)$

3) $i_3^*[P_{3T}] = (c-a)(b-a)(2c-2b)(c-b)$

4) $i_4^*[P_{4T}] = (2c-2a)(c-a)(c-b)(a-b)$

5) $i_5^*[P_{5T}] = (b-a)(2b-2a)(b-c)(a-c)$

6) $i_6^*[P_{6T}] = (2a-2c)(a-b)(a-c)(c-b)$

7) $i_7^*[P_{7T}] = (a-b)(a-c)(b-a)(b-c)$

8) $i_8^*[P_{8T}] = (b-a)(c-b)(b-c)(c-a)$

9) $i_9^*[P_{9T}] = (a-b)(a-c)(c-a)(c-b)$

(b) To evaluate $i_j^*[l_{kT}]$ we calculate the weights of the normal bundle of l_k in $\mathcal{H}$ at P_j namely, the weights of $(\mathcal{N}_{l_k|\mathcal{H}})_{P_j}$ (see example 4.0.23). Then $i_j^*[l_{kT}]$ is the product of those weights. Similarly $i_j^*[m_{kT}]$ is the product of the weights of $(\mathcal{N}_{m_k|\mathcal{H}})_{P_j}$.

(c) As in (1) and (2) above $i_j^*[p_{kT}]$ = product of the weights of $(\mathcal{N}_{p_k|\mathcal{H}})_{P_j}$. Similarly $i_j^*[q_{kT}]$ = product of the weights of $(\mathcal{N}_{q_k|\mathcal{H}})_{P_j}$, and $i_j^*[r_{kT}]$ = product of the weights of $(\mathcal{N}_{r_k|\mathcal{H}})_{P_j}$.

(d) As explained above $i_j^*[t_{kT}]$ = product of the weights of $(\mathcal{N}_{t_k|\mathcal{H}})_{P_j}$, and $i_j^*[u_{kT}]$ =

product of the weights of $(\mathcal{N}_{u_k|\mathcal{H}})_{P_j}$.

In table 3(on page 89), we calculated all the deformations between the fixed points of $\mathcal{H}$. Note that ab denote the weight $a-b$, ba denote the weight $b-a$, $2ba$ denote the weight $2(b-a)$,... etc.

In table 4(on page 89) the first entry $i_1^* : (2a-2b)(a-c)(b-c)$ denotes $i_1^*[l_{1T}] = (2a-2b)(a-c)(b-c)$,...etc. Therefore using table 3, 4, and the calculations above we substitute for $i_j^*[P_{kT}]$, $i_j^*[l_{kT}]$, and $i_j^*[m_{kT}]$ in the previous linear system above to get the following linear system of nine equations:

System I (weights)

$$a_1(2a-2b)(a-b)(a-c)(b-c)\lambda^4 = f_1(\lambda)(2a-2b)(a-c)(b-c)+g_1(\lambda)(2a-2b)(a-b)(a-c) \tag{103}$$

$$a_2(b-a)(2b-2c)(b-c)(c-a)\lambda^4 = f_4(\lambda)(b-a)(2b-2c)(c-a)+g_2(\lambda)(b-a)(2b-2c)(b-c) \tag{104}$$

$$a_3(c-a)(b-a)(2c-2b)(c-b)\lambda^4 = f_6(\lambda)(c-a)(b-a)(2c-2b)+g_3(\lambda)(c-a)(2c-2b)(c-b) \tag{105}$$

$$a_4(2c-2a)(c-a)(c-b)(a-b)\lambda^4 = f_5(\lambda)(2c-2a)(c-b)(a-b)+g_3(\lambda)(2c-2a)(c-a)(c-b) \tag{106}$$

$$a_5(b-a)(2b-2a)(b-c)(a-c)\lambda^4 = f_3(\lambda)(2b-2a)(b-c)(a-c)+g_2(\lambda)(b-a)(2b-2a)(b-c) \tag{107}$$

Fixed point	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9
P_1			ac		$2ab$	bc	ab		
P_2			$2bc$		ca	ba		bc	
P_3	ca	$2cb$		ba				cb	
P_4				ab	cb	$2ca$			ca
P_5	$2ba$	ac		bc			ba		
P_6	cb	ab		$2ac$					ac
P_7	ba				ab			ac	bc
P_8		cb	bc				ca		ba
P_9				ac		ca	cb	ab	

Table 3: deformations between the fixed points of $\mathcal{H}$

$cell/i_j^*$		
l_1	$i_1^* : (2a-2b)(a-c)(b-c)$	$i_7^* : (a-c)(a-b)(b-c)$
l_2	$i_6^* : (2a-2c)(a-b)(c-b)$	$i_9^* : (a-b)(a-c)(c-b)$
l_3	$i_5^* : (2b-2a)(b-c)(a-c)$	$i_7^* : (b-a)(a-c)(b-c)$
l_4	$i_2^* : (b-a)(2b-2c)(c-a)$	$i_8^* : (b-a)(b-c)(c-a)$
l_5	$i_4^* : (2c-2a)(c-b)(a-b)$	$i_9^* : (a-b)(c-a)(c-b)$
l_6	$i_3^* : (c-a)(b-a)(2c-2b)$	$i_8^* : (b-a)(c-b)(c-a)$
m_1	$i_1^* : (2a-2b)(a-b)(a-c)$	$i_6^* : (2a-2c)(a-b)(a-c)$
m_2	$i_2^* : (b-a)(2b-2c)(b-c)$	$i_5^* : (b-a)(2b-2a)(b-c)$
m_3	$i_3^* : (c-a)(2c-2b)(c-b)$	$i_4^* : (2c-2a)(c-a)(c-b)$

Table 4: the pull backs $i_j^*[l_{kT}]$, and $i_j^*[m_{kT}]$

$$a_6(2a-2c)(a-b)(a-c)(c-b)\lambda^4 = f_2(\lambda)(2a-2c)(a-b)(c-b)+g_1(\lambda)(2a-2c)(a-b)(a-c) \quad (108)$$

$$a_7(a-b)(a-c)(b-a)(b-c)\lambda^4 = f_3(\lambda)(a-c)(b-a)(b-c)+f_1(\lambda)(a-b)(a-c)(b-c) \quad (109)$$

$$a_8(b-a)(c-b)(b-c)(c-a)\lambda^4 = f_4(\lambda)(b-a)(b-c)(c-a)+f_6(\lambda)(b-a)(c-b)(c-a) \quad (110)$$

$$a_9(a-b)(a-c)(c-a)(c-b)\lambda^4 = f_2(\lambda)(a-b)(a-c)(c-b)+f_5(\lambda)(a-b)(c-a)(c-b) \quad (111)$$

Using Maple we solve the linear system to get the relation $\sum_{i=1}^{9} a_i = 0$.

Case(II): We will determine the rational equivalences in the Chow ring $A_1(\mathcal{H})$ using theorem 5.5.44. Now according to theorem 5.5.44

$$\sum_{i=1}^{6} a_i[l_{iT}]+\sum_{i=1}^{3} b_i[m_{iT}] \sim 0 \text{ iff } \sum_{i=1}^{6} a_i[l_{iT}]+\sum_{i=1}^{3} b_i[m_{iT}] \in Span\{\lambda.[p_{iT}], \lambda.[q_{iT}], \lambda[r_{iT}]\}_{i=1}^{3} \quad (112)$$

So $\sum_{i=1}^{6} a_i[l_{iT}] + \sum_{i=1}^{3} b_i[m_{iT}] \sim 0$ iff

$$\sum_{i=1}^{6} a_i[l_{iT}] + \sum_{i=1}^{3} b_i[m_{iT}] = \sum_{i=1}^{3} f_i(\lambda).[p_{iT}] + \sum_{i=1}^{3} g_i(\lambda).[q_{iT}] + \sum_{i=1}^{3} h_i(\lambda).[r_{iT}] \quad (113)$$

In order to determine the rational equivalences in the Chow ring $A_1(\mathcal{H})$, we need to solve the following linear system which we get by applying the map i_j^* to the previous equation above

System II

$$a_1 i_1^*[l_{1T}] + b_1 i_1^*[m_{1T}] = f_1(\lambda) i_1^*[p_{1T}] + g_1(\lambda) i_1^*[q_{1T}] + h_1(\lambda) i_1^*[r_{1T}] + h_3(\lambda) i_1^*[r_{3T}] \quad (114)$$

$$a_4 i_2^*[l_{4T}] + b_2 i_2^*[m_{2T}] = f_2(\lambda) i_2^*[p_{2T}] + g_2(\lambda) i_2^*[q_{2T}] + h_1(\lambda) i_2^*[r_{1T}] + h_2(\lambda) i_2^*[r_{2T}] \quad (115)$$

$$a_6 i_3^*[l_{6T}] + b_3 i_3^*[m_{3T}] = f_2(\lambda) i_3^*[p_{2T}] + g_3(\lambda) i_3^*[q_{3T}] + h_2(\lambda) i_3^*[r_{2T}] + h_3(\lambda) i_3^*[r_{3T}] \quad (116)$$

$$a_5 i_4^*[l_{5T}] + b_3 i_4^*[m_{3T}] = f_3(\lambda) i_4^*[p_{3T}] + g_3(\lambda) i_4^*[q_{3T}] + h_2(\lambda) i_4^*[r_{2T}] + h_3(\lambda) i_4^*[r_{3T}] \quad (117)$$

$$a_3 i_5^*[l_{3T}] + b_2 i_5^*[m_{2T}] = f_1(\lambda) i_5^*[p_{1T}] + g_2(\lambda) i_5^*[q_{2T}] + h_1(\lambda) i_5^*[r_{1T}] + h_2(\lambda) i_5^*[r_{2T}] \quad (118)$$

$$a_2 i_6^*[l_{2T}] + b_1 i_6^*[m_{1T}] = f_3(\lambda) i_6^*[p_{3T}] + g_1(\lambda) i_6^*[q_{1T}] + h_1(\lambda) i_6^*[r_{1T}] + h_3(\lambda) i_6^*[r_{3T}] \quad (119)$$

$$a_1 i_7^*[l_{1T}] + a_3 i_7^*[l_{3T}] = f_1(\lambda) i_7^*[p_{1T}] + g_1(\lambda) i_7^*[q_{1T}] + g_2(\lambda) i_7^*[q_{2T}] \quad (120)$$

$$a_4 i_8^*[l_{4T}] + a_6 i_8^*[l_{6T}] = f_2(\lambda) i_8^*[p_{2T}] + g_2(\lambda) i_8^*[q_{2T}] + g_3(\lambda) i_8^*[q_{3T}] \quad (121)$$

$$a_2 i_9^*[l_{2T}] + a_5 i_9^*[l_{5T}] = f_3(\lambda) i_9^*[p_{3T}] + g_1(\lambda) i_9^*[q_{1T}] + g_3(\lambda) i_9^*[q_{3T}] \quad (122)$$

We evaluate $i_j^*[l_{kT}]$, $i_j^*[m_{kT}]$, $i_j^*[p_{kT}]$, $i_j^*[q_{kT}]$, and $i_j^*[r_{kT}]$ using table 3, 4, and the calculations below :

$$i_1^*[p_{1T}] = (a-c)(b-c)$$

$$i_5^*[p_{1T}] = (b-c)(a-c)$$

$$i_7^*[p_{1T}] = (a-c)(b-c)$$

$$i_2^*[p_{2T}] = (b-a)(c-a)$$

$$i_3^*[p_{2T}] = (c-a)(b-a)$$

$$i_8^*[p_{2T}] = (b-a)(c-a)$$

$$i_4^*[p_{3T}] = (c-b)(a-b)$$

$$i_6^*[p_{3T}] = (a-b)(c-b)$$

$$i_9^*[p_{3T}] = (a-b)(c-b)$$

$$i_1^*[q_{1T}] = (2a-2b)(a-c)$$

$$i_6^*[q_{1T}] = (2a-2c)(a-b)$$

$$i_7^*[q_{1T}] = (a-b)(a-c)$$

$$i_9^*[q_{1T}] = (a-b)(a-c)$$

$$i_2^*[q_{2T}] = (b-a)(2b-2c)$$

$$i_5^*[q_{2T}] = (2b-2a)(b-c)$$

$$i_7^*[q_{2T}] = (b-a)(b-c)$$

$$i_8^*[q_{2T}] = (b-a)(b-c)$$

$$i_3^*[q_{3T}] = (c-a)(2c-2b)$$

$$i_4^*[q_{3T}] = (2c-2a)(c-b)$$

$$i_8^*[q_{3T}] = (c-b)(c-a)$$

$$i_9^*[q_{3T}] = (c-a)(c-b)$$

$$i_1^*[r_{1T}] = (2a-2b)(a-c)$$

$$i_2^*[r_{1T}] = (2b-2c)(b-c)$$

$$i_5^*[r_{1T}] = (2b-2a)(b-c)$$

$$i_6^*[r_{1T}] = (2a-2c)(a-c)$$

$$i_2^*[r_{2T}] = (b-a)(2b-2c)$$

$$i_3^*[r_{2T}] = (c-a)(2c-2b)$$

$$i_4^*[r_{2T}] = (2c-2a)(c-a)$$

$i_5^*[r_{2T}] = (b-a)(2b-2a)$

$i_1^*[r_{3T}] = (2a-2b)(a-b)$

$i_3^*[r_{3T}] = (2c-2b)(c-b)$

$i_4^*[r_{3T}] = (2c-2a)(c-b)$

$i_6^*[r_{3T}] = (a-b)(2a-2c)$

Using Maple we solve the linear system that we get after substituting the values of $i_j^*[l_{kT}]$, $i_j^*[m_{kT}]$, $i_j^*[p_{kT}]$, $i_j^*[q_{kT}]$, and $i_j^*[r_{kT}]$ to get the relation $\sum_{i=1}^3 a_i = 0$, $\sum_{i=1}^3 b_i = 0$.

Case(III): We will determine the rational equivalences in the Chow ring $A_2(\mathcal{H})$ using theorem 5.5.44. Now according to theorem 5.5.44
$\sum_{i=1}^3 a_i[p_{iT}]+\sum_{i=1}^3 b_i[q_{iT}]+\sum_{i=1}^3 c_i[r_{iT}] \sim 0$ iff $\sum_{i=1}^3 a_i[p_{iT}]+\sum_{i=1}^3 b_i[q_{iT}]+\sum_{i=1}^3 c_i[r_{iT}]$ belongs to Span $\{\lambda.[t_{iT}], \lambda.[u_{iT}]\}_{i=1}^3$. So
$\sum_{i=1}^3 a_i[p_{iT}]+\sum_{i=1}^3 b_i[q_{iT}]+\sum_{i=1}^3 c_i[r_{iT}] \sim 0$ iff $\sum_{i=1}^3 a_i[p_{iT}]+\sum_{i=1}^3 b_i[q_{iT}]+\sum_{i=1}^3 c_i[r_{iT}]$ is equal to $\sum_{i=1}^3 f_i(\lambda)[t_{iT}] + \sum_{i=1}^3 g_i(\lambda)[u_{iT}]$

In order to determine the rational equivalences in the Chow ring $A_2(\mathcal{H})$, we need to solve the following linear system which we get by applying the map i_j^* to the previous equation above

System III

$$a_1 i_1^*[p_{1T}] + b_1 i_1^*[q_{1T}] + c_1 i_1^*[r_{1T}] + c_3 i_1^*[r_{3T}] = f_1 i_1^*[t_{1T}] + f_2 i_1^*[t_{2T}] + g_1 i_1^*[u_{1T}] + g_2 i_1^*[u_{2T}] \quad (123)$$

$$a_2 i_2^*[p_{2T}] + b_2 i_2^*[q_{2T}] + c_1 i_2^*[r_{1T}] + c_2 i_2^*[r_{2T}] = f_1 i_2^*[t_{1T}] + f_3 i_2^*[t_{3T}] + g_2 i_2^*[u_{2T}] + g_3 i_2^*[u_{3T}] \quad (124)$$

$$a_2 i_3^*[p_{2T}] + b_3 i_3^*[q_{3T}] + c_2 i_3^*[r_{2T}] + c_3 i_3^*[r_{3T}] = f_2 i_3^*[t_{2T}] + f_3 i_3^*[t_{3T}] + g_2 i_3^*[u_{2T}] + g_3 i_3^*[u_{3T}] \tag{125}$$

$$a_3 i_4^*[p_{3T}] + b_3 i_4^*[q_{3T}] + c_2 i_4^*[r_{2T}] + c_3 i_4^*[r_{3T}] = f_2 i_4^*[t_{2T}] + f_3 i_4^*[t_{3T}] + g_1 i_4^*[u_{1T}] + g_3 i_4^*[u_{3T}] \tag{126}$$

$$a_1 i_5^*[p_{1T}] + b_2 i_5^*[q_{2T}] + c_1 i_5^*[r_{1T}] + c_2 i_5^*[r_{2T}] = f_1 i_5^*[t_{1T}] + f_3 i_5^*[t_{3T}] + g_1 i_5^*[u_{1T}] + g_2 i_5^*[u_{2T}] \tag{127}$$

$$a_3 i_6^*[p_{3T}] + b_1 i_6^*[q_{1T}] + c_1 i_6^*[r_{1T}] + c_3 i_6^*[r_{3T}] = f_1 i_6^*[t_{1T}] + f_2 i_6^*[t_{2T}] + g_1 i_6^*[u_{1T}] + g_3 i_6^*[u_{3T}] \tag{128}$$

$$a_1 i_7^*[p_{1T}] + b_1 i_7^*[q_{1T}] + b_2 i_7^*[q_{2T}] = f_1 i_7^*[t_{1T}] + f_2 i_7^*[t_{2T}] + f_3 i_7^*[t_{3T}] + g_1 i_7^*[u_{1T}] + g_2 i_7^*[u_{2T}] \tag{129}$$

$$a_2 i_8^*[p_{2T}] + b_2 i_8^*[q_{2T}] + b_3 i_8^*[q_{3T}] = f_1 i_8^*[t_{1T}] + f_2 i_8^*[t_{2T}] + f_3 i_8^*[t_{3T}] + g_2 i_8^*[u_{2T}] + g_3 i_8^*[u_{3T}] \tag{130}$$

$$a_3 i_9^*[p_{3T}] + b_1 i_9^*[q_{1T}] + b_3 i_9^*[q_{3T}] = f_1 i_9^*[t_{1T}] + f_2 i_9^*[t_{2T}] + f_3 i_9^*[t_{3T}] + g_1 i_9^*[u_{1T}] + g_3 i_9^*[u_{3T}] \tag{131}$$

we evaluated $i_j^*[p_{kT}]$,$i_j^*[q_{kT}]$, and $i_j^*[r_{kT}]$ above. Now we calculate $i_j^*[t_{kT}]$, $i_j^*[u_{kT}]$:

1) $i_1^*[t_{1T}] = (2a - 2c)$, $i_2^*[t_{1T}] = (2b - 2c)$, $i_5^*[t_{1T}] = (2b - 2c)$, $i_6^*[t_{1T}] = (2a - 2c)$, $i_7^*[t_{1T}] = (a + b - 2c)$, $i_8^*[t_{1T}] = (b - c)$, $i_9^*[t_{1T}] = (a - c)$

2) $i_1^*[t_{2T}] = (2a - 2b)$, $i_3^*[t_{2T}] = (2c - 2b)$, $i_4^*[t_{2T}] = (2c - 2b)$, $i_6^*[t_{2T}] = (2a - 2b)$,

$i_7^*[t_{2T}] = (a-b)$, $i_7^*[t_{2T}] = (c-b)$, $i_9^*[t_{2T}] = (a+c-2b)$

3) $i_2^*[t_{3T}] = (2b-2a)$, $i_3^*[t_{3T}] = (2c-2a)$, $i_4^*[t_{3T}] = (2c-2a)$, $i_5^*[t_{3T}] = (2b-2a)$, $i_7^*[t_{3T}] = (b-a)$, $i_8^*[t_{3T}] = (b+c-2a)$, $i_9^*[t_{3T}] = (c-a)$

4) $i_1^*[u_{1T}] = (a-c)$, $i_4^*[u_{1T}] = (a-b)$, $i_5^*[u_{1T}] = (a-c)$, $i_6^*[u_{1T}] = (a-b)$, $i_7^*[u_{1T}] = (a-c)$, $i_9^*[u_{1T}] = (a-b)$

5) $i_1^*[u_{2T}] = (b-c)$, $i_2^*[u_{2T}] = (b-a)$, $i_3^*[u_{2T}] = (b-a)$, $i_5^*[u_{2T}] = (b-c)$, $i_7^*[u_{2T}] = (b-c)$, $i_8^*[u_{2T}] = (b-a)$

6) $i_2^*[u_{3T}] = (c-a)$, $i_3^*[u_{3T}] = (c-a)$, $i_4^*[u_{3T}] = (c-b)$, $i_6^*[u_{3T}] = (c-b)$, $i_8^*[u_{3T}] = (c-a)$, $i_9^*[u_{3T}] = (c-b)$

Using Maple we solve the linear system that we get after substituting the values of $i_j^*[p_{kT}]$, $i_j^*[q_{kT}]$, $i_j^*[r_{kT}]$, $i_j^*[t_{kT}]$, and $i_j^*[u_{kT}]$ to get the relations $\sum_{i=1}^3 a_i = 0$, $\sum_{i=1}^3 b_i = 0$, and $\sum_{i=1}^3 c_i = 0$.

Case(IV): We will determine the rational equivalences in the Chow ring $A_3(\mathcal{H})$ using theorem 5.5.44. Now according to theorem 5.5.44
$\sum_{i=1}^3 a_i[t_{iT}] + \sum_{i=1}^3 b_i[u_{iT}] \sim 0$ iff $\sum_{i=1}^3 a_i[t_{iT}] + \sum_{i=1}^3 b_i[u_{iT}] \in \oplus_{i=1}^9$ Span $\{\lambda.1\}$. So $\sum_{i=1}^3 a_i[t_{iT}] + \sum_{i=1}^3 b_i[u_{iT}] \sim 0$ iff $\sum_{i=1}^3 a_i[t_{iT}] + \sum_{i=1}^3 b_i[u_{iT}] = f(\lambda)$.

In order to determine the rational equivalences in the Chow ring $A_3(\mathcal{H})$, we need to solve the following linear system which we get by applying the map i_j^* to the previous equation above

System IV

$$a_1 i_1^*[t_{1T}] + a_2 i_1^*[t_{2T}] + b_1 i_1^*[u_{1T}] + b_2 i_1^*[u_{2T}] = f(\lambda) \tag{132}$$

$$a_1 i_2^*[t_{1T}] + a_3 i_2^*[t_{3T}] + b_2 i_2^*[u_{2T}] + b_3 i_2^*[u_{3T}] = f(\lambda) \tag{133}$$

$$a_2 i_3^*[t_{2T}] + a_3 i_3^*[t_{3T}] + b_2 i_3^*[u_{2T}] + b_3 i_3^*[u_{3T}] = f(\lambda) \tag{134}$$

$$a_2 i_4^*[t_{2T}] + a_3 i_4^*[t_{3T}] + b_1 i_4^*[u_{1T}] + b_3 i_4^*[u_{3T}] = f(\lambda) \tag{135}$$

$$a_1 i_5^*[t_{1T}] + a_3 i_5^*[t_{3T}] + b_1 i_5^*[u_{1T}] + b_2 i_5^*[u_{2T}] = f(\lambda) \tag{136}$$

$$a_1 i_6^*[t_{1T}] + a_2 i_6^*[t_{2T}] + b_1 i_6^*[u_{1T}] + b_3 i_6^*[u_{3T}] = f(\lambda) \tag{137}$$

$$a_1 i_7^*[t_{1T}] + a_2 i_7^*[t_{2T}] + a_3 i_7^*[t_{3T}] + b_1 i_7^*[u_{1T}] + b_2 i_7^*[u_{2T}] = f(\lambda) \tag{138}$$

$$a_1 i_8^*[t_{1T}] + a_2 i_8^*[t_{2T}] + a_3 i_8^*[t_{3T}] + b_2 i_8^*[u_{2T}] + b_3 i_8^*[u_{3T}] = f(\lambda) \tag{139}$$

$$a_1 i_9^*[t_{1T}] + a_2 i_9^*[t_{2T}] + a_3 i_9^*[t_{3T}] + b_1 i_9^*[u_{1T}] + b_3 i_9^*[u_{3T}] = f(\lambda) \tag{140}$$

We evaluated $i_j^*[t_{kT}]$, and $i_j^*[u_{kT}]$ above. Using Maple we solve the linear system that we get after substituting the values of $i_j^*[t_{kT}]$, and $i_j^*[u_{kT}]$. We get the relations $\sum_{i=1}^3 a_i = 0$, $\sum_{i=1}^3 b_i = 0$.

Remark 6.0.54. In case III we calculate $i_7^*[t_{1T}]$ as follows: first note that $P_7 = (xy, z)$ is a singular point because because P_7 lives in the in all the cells of dimension three and it has four tangent weights. Now to calculate $i_7^*[t_{1T}]$ we give local coordinates for $\mathcal{H}$. Let $\{(1, u, v)\}$, $\{(w, 1, s)\}$ with origin $(u, v, w, s) = (0, 0, 0, 0)$ corresponds to the point P_7. In these coordinates t_1 has the equation $vs = 0$. Now to calculate the weights corresponding to v and s, note that $t.(1, u, v) = (t^{-a}, t^{-b}u, t^{-c}v) = (1, t^{a-b}u, t^{a-c}v)$. So the weight corresponding to v is $a - c$. Similarly since $t.(w, 1, s) = (t^{b-a}w, , t^{b-c}s)$ then the weight corresponding to s is $b - c$. Therefore $i_7^*[t_{1T}] = a - c + b - c = a + b - 2c$. Similarly $i_8^*[t_{3T}] = b + c - 2a$, and $i_9^*[t_{2T}] = a + c - 2b$.

Calculations.

Recall that the Chow ring $A_2(\mathcal{H})$ is generated by the classes $[p_{iT}]$, $[q_{iT}]$, and $[r_{iT}]$, $i = 1, 2, 3$. In the following calculations we used localization to compute the following intersections: $[p_{iT}].[p_{jT}]$, $[q_{iT}].[q_{jT}]$, $[r_{iT}].[r_{jT}]$, $[p_{iT}].[q_{jT}]$, $[p_{iT}].[r_{jT}]$, and $[q_{iT}].[r_{jT}]$.

$$\begin{aligned}\int [p_{1T}].[q_{1T}] &= \frac{(2a-2c)(b-c)(2a-2b)(a-c)}{(2a-2b)(a-b)(a-c)(b-c)} \\ &+ \frac{(a-c)(b-c)(a-b)(a-c)}{(a-b)(a-c)(b-a)(b-c)} \\ &= \frac{(a-c)}{(a-b)} + \frac{(a-c)}{(b-a)} \\ &= 0\end{aligned}$$

$$\begin{aligned}\int [p_{1T}].[q_{2T}] &= \frac{(b-c)(a-c)(2b-2a)(b-c)}{(b-a)(2b-2a)(b-c)(a-c)} \\ &+ \frac{(a-c)(b-c)(b-a)(b-c)}{(a-b)(a-c)(b-a)(b-c)} \\ &= \frac{(b-c)}{(b-a)} + \frac{(b-c)}{(a-b)} \\ &= 0\end{aligned}$$

$$\int [p_{2T}].[q_{2T}] = \frac{(b-a)(c-a)(b-a)(2b-2c)}{(b-a)(2b-2c)(b-c)(c-a)}$$

$$+ \frac{(b-a)(c-a)(b-a)(b-c)}{(b-a)(c-b)(b-c)(c-a)}$$

$$= \frac{(b-a)}{(b-c)} + \frac{(b-a)}{(c-b)}$$

$$= 0$$

$$\int [p_{2T}].[q_{3T}] = \frac{(c-a)(b-a)(c-a)(2c-2b)}{(c-a)(b-a)(2c-2b)(c-b)}$$

$$+ \frac{(b-a)(c-a)(c-b)(c-a)}{(b-a)(c-b)(b-c)(c-a)}$$

$$= \frac{(c-a)}{(c-b)} + \frac{(c-a)}{(b-c)}$$

$$= 0$$

$$
\begin{aligned}
\int [p_{3T}].[q_{1T}] &= \frac{(a-b)(c-b)(2a-2c)(a-b)}{(2a-2c)(a-b)(a-c)(c-b)} \\
&+ \frac{(a-b)(c-b)(a-b)(a-c)}{(a-b)(a-c)(c-a)(c-b)} \\
&= \frac{(a-b)}{(a-c)} + \frac{(a-b)}{(c-a)} \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\int [p_{3T}].[q_{3T}] &= \frac{(c-b)(a-b)(2c-2a)(c-b)}{(2c-2a)(c-a)(c-b)(a-b)} \\
&+ \frac{(a-b)(c-b)(c-a)(c-b)}{(a-b)(a-c)(c-a)(c-b)} \\
&= \frac{(c-b)}{(c-a)} + \frac{(c-b)}{(a-c)} \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\int [p_{1T}].[r_{1T}] &= \frac{(a-c)(b-c)(2a-2b)(a-c)}{(2a-2b)(a-b)(a-c)(b-c)} \\
&+ \frac{(b-c)(a-c)(2b-2a)(b-c)}{(b-a)(2b-2a)(b-c)(a-c)} \\
&= \frac{2(a-c)}{(2a-2b)} + \frac{2(b-c)}{(2b-2a)} \\
&= \frac{(a-b)}{(a-b)} \\
&= 1
\end{aligned}
$$

$$
\begin{aligned}
\int [p_{1T}].[r_{2T}] &= \frac{(b-c)(a-c)(b-a)(2b-2a)}{(b-a)(2b-2a)(b-c)(a-c)} \\
&= 1
\end{aligned}
$$

$$
\begin{aligned}
\int [p_{1T}].[r_{3T}] &= \frac{(a-c)(b-c)(2a-2b)(a-b)}{(2a-2b)(a-b)(a-c)(b-c)} \\
&= 1
\end{aligned}
$$

$$\int [p_{2T}].[r_{1T}] = \frac{(b-a)(c-a)(2b-2c)(b-c)}{(b-a)(2b-2c)(b-c)(c-a)}$$

$$= 1$$

$$\int [p_{2T}].[r_{2T}] = \frac{(b-a)(c-a)(b-a)(2b-2c)}{(b-a)(2b-2c)(b-c)(c-a)}$$

$$+ \frac{(c-a)(b-a)(c-a)(2c-2b)}{(c-a)(b-a)(2c-2b)(c-b)}$$

$$= \frac{2(b-a)}{(2b-2c)} + \frac{2(c-a)}{(2c-2b)}$$

$$= \frac{2(b-c)}{(2b-2c)}$$

$$= 1$$

$$\int [p_{2T}].[r_{3T}] = \frac{(c-a)(b-a)(2c-2b)(c-b)}{(c-a)(b-a)(2c-2b)(c-b)}$$

$$= 1$$

$$\int [p_{3T}].[r_{1T}] = \frac{(a-b)(c-b)(2a-2c)(a-c)}{(2a-2c)(a-b)(a-c)(c-b)}$$

$$= 1$$

$$\int [p_{3T}].[r_{2T}] = \frac{(c-b)(a-b)(2c-2a)(c-a)}{(2c-2a)(c-a)(c-b)(a-b)}$$

$$= 1$$

$$\int [p_{3T}].[r_{3T}] = \frac{(c-b)(a-b)(2c-2a)(c-b)}{(2c-2a)(c-a)(c-b)(a-b)}$$

$$+ \frac{(a-b)(c-b)(a-b)(2a-2c)}{(2a-2c)(a-b)(a-c)(c-b)}$$

$$= \frac{2(c-b)}{(2c-2a)} + \frac{2(a-b)}{(2a-2c)}$$

$$= \frac{2(c-a)}{(2c-2a)}$$

$$= 1$$

$$\int [q_{1T}].[r_{1T}] = \frac{(2a-2b)(a-c)(2a-2b)(a-c)}{(2a-2b)(a-b)(a-c)(b-c)}$$

$$+ \frac{(2a-2c)(a-b)(2a-2c)(a-c)}{(2a-2c)(a-b)(a-c)(c-b)}$$

$$= \frac{2(a-c)}{(b-c)} + \frac{(2a-2c)}{(c-b)}$$

$$= 0$$

$$\int [q_{1T}].[r_{3T}] = \frac{(2a-2b)(a-c)(2a-2b)(a-b)}{(2a-2b)(a-b)(a-c)(b-c)}$$

$$+ \frac{(2a-2c)(a-b)(a-b)(2a-2c)}{(2a-2c)(a-b)(a-c)(c-b)}$$

$$= \frac{(2a-2b)}{(b-c)} + \frac{2(a-b)}{(c-b)}$$

$$= 0$$

$$\int [q_{2T}].[r_{1T}] = \frac{(b-a)(2b-2c)(2b-2c)(b-c)}{(b-a)(2b-2c)(b-c)(c-a)}$$

$$+ \frac{(2b-2a)(b-c)(2b-2a)(b-c)}{(b-a)(2b-2a)(b-c)(a-c)}$$

$$= \frac{(2b-2c)}{(c-a)} + \frac{2(b-c)}{(a-c)}$$

$$= 0$$

$$\int [q_{2T}].[r_{2T}] = \frac{(b-a)(2b-2c)(b-a)(2b-2c)}{(b-a)(2b-2c)(b-c)(c-a)}$$

$$+ \frac{(2b-2a)(b-c)(b-a)(2b-2a)}{(b-a)(2b-2a)(b-c)(a-c)}$$

$$= \frac{2(b-a)}{(c-a)} + \frac{(2b-2a)}{(a-c)}$$

$$= 0$$

$$\int [q_{3T}].[r_{2T}] = \frac{(c-a)(2c-2b)(c-a)(2c-2b)}{(c-a)(b-a)(2c-2b)(c-b)}$$

$$+ \frac{(2c-2a)(c-b)(2c-2a)(c-a)}{(2c-2a)(c-a)(c-b)(a-b)}$$

$$= \frac{2(c-a)}{(b-a)} + \frac{(2c-2a)}{(a-b)}$$

$$= 0$$

$$\int [q_{3T}].[r_{3T}] = \frac{(c-a)(2c-2b)(2c-2b)(c-b)}{(c-a)(b-a)(2c-2b)(c-b)}$$

$$+ \frac{(2c-2a)(c-b)(2c-2a)(c-b)}{(2c-2a)(c-a)(c-b)(a-b)}$$

$$= \frac{(2c-2b)}{(b-a)} + \frac{2(c-b)}{(a-b)}$$

$$= 0$$

$$\int [p_{1T}]^2 = \frac{(a-c)^2(b-c)^2}{(2a-2b)(a-b)(a-c)(b-c)}$$

$$+ \frac{(b-c)^2(a-c)^2}{(b-a)(2b-2a)(b-c)(a-c)}$$

$$+ \frac{(a-c)^2(b-c)^2}{(a-b)(a-c)(b-a)(b-c)}$$

$$= \frac{(a-c)(b-c)+(b-c)(a-c)-(2a-2c)(b-c)}{(2a-2b)(a-b)} = 0$$

$$\int [p_{2T}]^2 = \frac{(b-a)^2(c-a)^2}{(b-a)(2b-2c)(b-c)(c-a)}$$

$$+ \frac{(c-a)^2(b-a)^2}{(c-a)(b-a)(2c-2b)(c-b)}$$

$$+ \frac{(b-a)^2(c-a)^2}{(b-a)(c-b)(b-c)(c-a)}$$

$$= \frac{(b-a)(c-a)+(c-a)(b-a)-(2b-2a)(c-a)}{(2b-2c)(b-c)} = 0$$

$$
\begin{aligned}
\int [p_{3T}]^2 &= \frac{(c-b)^2(a-b)^2}{(2c-2a)(c-a)(c-b)(a-b)} \\
&+ \frac{(a-b)^2(c-b)^2}{(2a-2c)(a-b)(a-c)(c-b)} \\
&+ \frac{(a-b)^2(c-b)^2}{(a-b)(a-c)(c-a)(c-b)} \\
&= \frac{(c-b)(a-b)+(a-b)(c-b)-(2a-2b)(c-b)}{(2c-2a)(c-a)} \\
&= 0
\end{aligned}
$$

$$\int [q_{1T}]^2 = \frac{(2a-2b)^2(a-c)^2}{(2a-2b)(a-b)(a-c)(b-c)}$$

$$+ \frac{(2a-2c)^2(a-b)^2}{(2a-2c)(a-b)(a-c)(c-b)}$$

$$+ \frac{(a-b)^2(a-c)^2}{(a-b)(a-c)(b-a)(b-c)}$$

$$+ \frac{(a-b)^2(a-c)^2}{(a-b)(a-c)(c-a)(c-b)}$$

$$= \frac{(2a-2b)(a-c)(a-c)-(2a-2c)(a-b)(a-b)-(a-b)(a-c)(a-c)}{(a-b)(b-c)(a-c)}$$

$$+ \frac{(a-b)(a-c)(a-b)}{(a-b)(b-c)(a-c)}$$

$$= \frac{(a-c)-(a-b)}{(b-c)}$$

$$= 1$$

$$
\begin{aligned}
\int [q_{2T}]^2 &= \frac{(b-a)^2(2b-2c)^2}{(b-a)(2b-2c)(b-c)(c-a)} \\
&+ \frac{(2b-2a)^2(b-c)^2}{(b-a)(2b-2a)(b-c)(a-c)} \\
&+ \frac{(b-a)^2(b-c)^2}{(a-b)(a-c)(b-a)(b-c)} \\
&+ \frac{(b-a)^2(b-c)^2}{(b-a)(c-b)(b-c)(c-a)} \\
&= \frac{(2a-2b)(a-c)(a-c)-(2a-2c)(a-b)(a-b)-(a-b)(a-c)(a-c)}{(a-b)(b-c)(a-c)} \\
&+ \frac{(a-b)(a-c)(a-b)}{(a-b)(b-c)(a-c)} \\
&= \frac{(a-c)-(a-b)}{(b-c)} \\
&= 1
\end{aligned}
$$

$$\begin{aligned}
\int [q_{3T}]^2 &= \frac{(c-a)^2(2c-2b)^2}{(c-a)(b-a)(2c-2b)(c-b)} \\
&+ \frac{(2c-2a)^2(c-b)^2}{(2c-2a)(c-a)(c-b)(a-b)} \\
&+ \frac{(c-b)^2(c-a)^2}{(b-a)(c-b)(b-c)(c-a)} \\
&+ \frac{(c-a)^2(c-b)^2}{(a-b)(a-c)(c-a)(c-b)} \quad . \\
&= \frac{(2c-2a)}{(b-a)} + \frac{(2c-2b)}{(a-b)} + \frac{(a-c)}{(b-a)} + \frac{(b-c)}{(a-b)} \\
&= \frac{(2c-2a)-(2c-2b)+(a-c)-(b-c)}{(b-a)} \\
&= \frac{(2b-2a)+(a-b)}{(b-a)} \\
&= 1
\end{aligned}$$

$$\int [r_{1T}]^2 = \frac{(2a-2b)^2(a-c)^2}{(2a-2b)(a-b)(a-c)(b-c)}$$

$$+ \frac{(2b-2c)^2(b-c)^2}{(b-a)(2b-2c)(b-c)(c-a)}$$

$$+ \frac{(2b-2a)^2(b-c)^2}{(b-a)(2b-2a)(b-c)(a-c)}$$

$$+ \frac{(2a-2c)^2(a-c)^2}{(2a-2c)(a-b)(a-c)(c-b)}$$

$$= \frac{2(a-c)}{(b-c)} + \frac{(2b-2c)(b-c)}{(b-a)(c-a)} + \frac{2(b-c)}{(a-c)} + \frac{(2a-2c)(a-c)}{(a-b)(c-b)}$$

$$= \frac{(2a-2c)(a-c)(c-b) + (2b-2c)(b-c)(a-c)}{(a-b)(a-c)(b-c)}$$

$$= \frac{-2(a-c) + 2(b-c)}{(a-b)}$$

$$= -2$$

$$\begin{aligned}
\int [r_{2T}]^2 &= \frac{(b-a)^2(2b-2c)^2}{(b-a)(2b-2c)(b-c)(c-a)} \\
&+ \frac{(c-a)^2(2c-2b)^2}{(c-a)(b-a)(2c-2b)(c-b)} \\
&+ \frac{(2c-2a)^2(c-a)^2}{(2c-2a)(c-a)(c-b)(a-b)} \\
&+ \frac{(b-a)^2(2b-2a)^2}{(b-a)(2b-2a)(b-c)(a-c)} \\
&= \frac{(2b-2a)}{(c-a)} + \frac{(2c-2a)}{(b-a)} + \frac{(2c-2a)^2}{(c-b)(a-b)} + \frac{(2b-2a)^2}{(b-c)(a-c)} \\
&= \frac{2(b-a)(a-b)(c-a) - 2(c-a)(a-c)(b-a)}{(b-c)} \\
&= \frac{-2(b-a) + 2(c-a)}{(b-c)} \\
&= -2
\end{aligned}$$

$$
\begin{aligned}
\int [r_{3T}]^2 &= \frac{(2a-2b)^2(a-b)^2}{(2a-2b)(a-b)(a-c)(b-c)} \\
&+ \frac{(2c-2b)^2(c-b)^2}{(c-a)(b-a)(2c-2b)(c-b)} \\
&+ \frac{(2c-2a)^2(c-b)^2}{(2c-2a)(c-a)(c-b)(a-b)} \\
&+ \frac{(a-b)^2(2a-2c)^2}{(2a-2c)(a-b)(a-c)(c-b)} \\
&= \frac{2(a-b)^2}{(a-c)(b-c)} + \frac{2(c-b)^2}{(c-a)(b-a)} + \frac{2(c-b)}{(a-b)} + \frac{2(a-b)}{(c-b)} \\
&= \frac{(2a-2b)^2(c-b) + 2(c-b)(b-c)(a-b)}{(a-b)(a-c)(b-c)} \\
&= \frac{2(b-a)+(c-b)}{(a-c)} \\
&= -2
\end{aligned}
$$

$$\int [r_{1T}].[r_{2T}] = \frac{(2b-2c)(b-c)(b-a)(2b-2c)}{(b-a)(2b-2c)(b-c)(c-a)}$$

$$+ \frac{(2b-2a)(b-c)(b-a)(2b-2a)}{(b-a)(2b-2a)(b-c)(a-c)}$$

$$= \frac{2(b-c)}{(c-a)} + \frac{2(b-a)}{(a-c)}$$

$$= -2$$

$$\int [r_{1T}].[r_{3T}] = \frac{(2a-2b)(a-c)(2a-2b)(a-b)}{(2a-2b)(a-b)(a-c)(b-c)}$$

$$+ \frac{(2a-2c)(a-c)(a-b)(2a-2c)}{(2a-2c)(a-b)(a-c)(c-b)}$$

$$= \frac{2(a-b)}{(b-c)} + \frac{2(a-c)}{(c-b)}$$

$$= \frac{2(a-b)+2(c-a)}{(b-c)}$$

$$= \frac{2(c-b)}{(b-c)}$$

$$= -2$$

$$\int [r_{2T}].[r_{3T}] = \frac{(c-a)(2c-2b)(2c-2b)(c-b)}{(c-a)(b-a)(2c-2b)(c-b)}$$

$$+ \frac{(2c-2a)(c-a)(2c-2a)(c-b)}{(2c-2a)(c-a)(c-b)(a-b)}$$

$$= \frac{2(c-b)}{(b-a)} + \frac{2(c-a)}{(a-b)}$$

$$= \frac{2(a-b)}{(b-a)}$$

$$= -2$$

$$\int [q_{1T}].[q_{2T}] = \frac{(a-b)(a-c)(b-a)(b-c)}{(a-b)(a-c)(b-a)(b-c)}$$

$$= 1$$

$$\int [q_{1T}].[q_{3T}] = \frac{(a-b)(a-c)(c-a)(c-b)}{(a-b)(a-c)(c-a)(c-b)}$$

$$= 1$$

$$\int [q_{2T}].[q_{3T}] = \frac{(b-a)(b-c)(c-b)(c-a)}{(b-a)(c-b)(b-c)(c-a)}$$

$$= 1$$

The following integrals are equal to zero since the cells are disjoint: $\int[p_{1T}].[p_{2T}]$, $\int[p_{1T}].[p_{3T}]$, $\int[p_{2T}].[p_{3T}]$, $\int[p_{1T}].[q_{3T}]$, $\int[p_{2T}].[q_{1T}]$, $\int[p_{3T}].[q_{2T}]$, $\int[q_{1T}].[r_{2T}]$, $\int[q_{2T}].[r_{3T}]$, and $\int[q_{3T}].[r_{1T}]$.

Remark 6.0.55. We will calculate the following intersections using geometry and check the intersection multiplicity using local coordinates:
(I) Recall $[p_{iT}]$, $[q_{iT}]$, $[r_{iT}] \in A_2(\mathcal{H})$ generate the Chow ring $A_2(\mathcal{H})$ where $[p_{iT}]$ (resp.$[q_{iT}]$, $[r_{iT}]$) denote the class of the closure of the cell p_i (resp.p_i, q_i). Note that the class $[p_{iT}] = [p_{jT}]$ (resp.$[r_{iT}] = [r_{jT}]$, $[q_{iT}] = [q_{jT}]$) because the cycles p_{iT}, p_{jT} (resp.r_{iT}, r_{jT}, and q_{iT}, q_{jT}) are rationally equivalent. So we let $[p_T]$ (resp. $[q_T]$, $[r_T]$) denote $[p_{iT}]$ (resp.$[q_{iT}]$, $[r_{iT}]$). First we calculate $[q_{1T}].[q_{2T}]$ using geometry. Recall $q_1 = \{\{e_0, w\} \in \mathcal{H} : e_0 \in E,\ w \in \mathbb{P}^2\}$, $q_2 = \{\{e_1, w\} \in \mathcal{H} : e_1 \in E,\ w \in \mathbb{P}^2\}$. Clearly the closures of the cells q_1, q_2 intersect at the point $P_7 = (xy, z) \in \mathcal{H}$ where $\mathcal{H} = \mathrm{Hilb}^2\,\mathbb{P}^2$. Now we check that the intersection multiplicity is equal to one. Let $\{(1, a, b), (c, 1, d)\}$ with origin $(a, b, c, d) = (0, 0, 0, 0)$ corresponds to the point $P_7 \in \mathcal{H}$.

Consider the ideal $(x-c, z-d) \cap (y-a, z-b)$ then in the local coordinates a, b, c, d the cell q_1 has the equations $a = b = 0$, and the cell q_2 has the equations $c = d = 0$. So $I(q_1) = I(Z(a,b)) = \sqrt{(a,b)} = (a,b)$. Similarly $I(q_2) = (c,d)$. Now since the equations of $I(q_1)$ and $I(q_2)$ are linear in a,b,c,d it follows that the intersection multiplicity is equal to one. So $[q_{1T}].[q_{2T}] = 1$. Therefore $[q_T]^2 = 1$. Second to calculate $[p_{1T}].[r_{2T}]$ using geometry. Recall $p_1 = \{\{w,s\} \in \mathcal{H} : w, s \in L_{e_2}\}$, $r_2 = \{\{v,v\} \in \mathcal{H} : \{v,v\} \in L_{e_0}$ and there exists a line of $\mathbb{P}^2$ containing the double point $\{v,v\}\}$. Clearly the closures of the cells p_1, r_2 intersect at the point $P_5 = (x^2, z) \in \mathcal{H}$ where $\mathcal{H} = \mathrm{Hilb}^2\,\mathbb{P}^2$. Now we check that the intersection multiplicity is equal to one. Consider the ideal $(x^2 + ax + b, z + cx + d)$. Then in the local coordinates a,b,c,d the cell r_2 has the equations $a^2 - 4b = b = 0$, and the cell p_1 has equations $c = d = 0$. Clearly the cell p_1 has equations $c = d = 0$ because when $c = d = 0$ we get the ideal $(x^2 + ax + b, z)$ which gives the cell p_1. So $p_1 = Z(c,d)$. To check that the cell r_2 has equations $a^2 - 4b = b = 0$, first note that the quadratic equation $x^2 + ax + b = 0$ has to be a complete square, i.e the discriminant $a^2 - 4b = 0$ is equal to zero. second to get the ideal $(x^2, z + cx + d)$ which gives the cell r_2 we must set $b = 0$. So $r_2 = Z(a^2 - 4b,$ b). But $I(r_2) = I(Z(a^2 - 4b, b)) = \sqrt{(a^2 - 4b, b)} = (a,b)$ ($\sqrt{(a^2 - 4b, b)}$ is the radical of the ideal $(a^2 - 4b, b)$). Similarly $I(p_1) = I(Z(c,d)) = \sqrt{(c,d)} = (c,d)$. Now since the equations of $I(p_1)$, $I(r_2)$ are linear in a,b,c,d it follows that the intersection multiplicity is equal to one. So $[p_{1T}].[r_{2T}] = 1$. Therefore $[p_T].[r_T] = 1$. Recall $q_3 = \{\{e_2, w\} \in \mathcal{H} : e_2 \in E,\ w \in \mathbb{P}^2\}$, $r_1 = \{\{v,v\} \in \mathcal{H} : \{v,v\} \in L_{e_2}$ and there exists a line of $\mathbb{P}^2$ containing the double point $\{v,v\}\}$, $p_1 = \{\{w,s\} \in \mathcal{H} : w, s \in L_{e_2}\}$, $p_2 = \{\{w,s\} \in \mathcal{H} : w, s \in L_{e_0}\}$. Using geometry it is clear that the cells q_3 and r_1 (resp. p_1 and p_2, p_1 and q_3) are disjoint. It follows $[q_3].[r_1] = [p_1].[p_2] = [p_1].[q_3] = 0$. Therefore $[q].[r] = [p]^2 = [p].[q] = 0$.

(II) Recall $[l_{iT}]$, $[m_{iT}] \in A_1(\mathcal{H})$ generates the Chow ring $A_1(\mathcal{H})$ and $[t_{iT}]$, $[u_{iT}] \in A_3(\mathcal{H})$ generates the Chow ring $A_3(\mathcal{H})$. Note that the class $[l_{iT}] = [l_{jT}]$ and $[m_{iT}] =$

$[m_{jT}]$ (resp.$[t_{iT}] = [t_{jT}]$, $[u_{iT}] = [u_{jT}]$) because the cycles l_{iT}, l_{jT} (resp.t_{iT} and t_{jT}, u_{iT} and u_{jT}) are rationally equivalent. So we let $[l_T]$ and $[m_T]$ (resp. $[t_T]$, $[u_T]$) denote $[l_{iT}]$ and $[m_{iT}]$ (resp.$[t_{iT}]$, $[u_{iT}]$). Recall $l_1 = \{\{e_0, s\} \in \mathcal{H} : e_0 \in E,\ s \in L_{e_2}\}$, $m_1 = \{\{e_0, e_0\} \in \mathcal{H} : e_0 \in E$ and there exists a line of $\mathbb{P}^2$ containing the double point $\{e_0, e_0\}\}$, $t_1 = \{\{s, v\} \in \mathcal{H} : s \in L_{e_2},\ v \in \mathbb{P}^2\}$, $u_2 = \{\{s, v\} \in \mathcal{H}$: there exists a line of $\mathbb{P}^2$ containing $e_1, s, v\}$. We calculate intersections using geometry. Clearly the closures of the cells l_1 and u_3 (resp.m_1 and t_1) are disjoint. So $[l_{1T}].[u_{3T}] = 0$ (resp.$[m_{1T}].[t_{1T}] = 0$). So $[l_T].[u_T] = 0$ (resp.$[m_T].[t_T] = 0$). Next we calculate $[m_{1T}].[u_{2T}]$ using geometry. Clearly the closures of the cells m_1, u_2 intersect at the point $P_1 = (y^2, z) \in \mathcal{H}$ where $\mathcal{H} = \mathrm{Hilb}^2\,\mathbb{P}^2$. Consider the ideal $(y^2+ay+b, z+cy+d)$. Then as explained in (I) above, the cell m_1 has equations $a^2 - 4b = b = d = 0$, and the cell u_2 has equations $c = 0$. But $I(m_1) = I(Z(a^2 - 4b, b, d)) = \sqrt{(a^2 - 4b, b, d)} = (a, b, d)$. Similarly $I(u_2) = I(Z(c)) = \sqrt{(c)} = (c)$. Now since the equations of $I(m_1)$, $I(u_2)$ are linear in a,b,c,d it follows that the intersection multiplicity is equal to one. Therefore $[m_T].[u_T] = 1$.

(III) we calculate $[l_{1T}].[t_{3T}]$ using geometry. Clearly the closures of the cells l_1, t_3 intersect at the point $P_7 = (z, xy) \in \mathcal{H}$. Now we check that the intersection multiplicity is equal to one. Let $\{(1, a, b), (c, 1, d)\}$ corresponds to the point $P_7 \in \mathcal{H}$. Consider the ideal $(z - c, x - d) \cap (z - a, y - b)$. Then as explained in (I) above the cell l_1 has equations $a = b = c = 0$ and the cell t_3 has equations $d = 0$. So $I(l_1) = (a, b, c)$, $I(t_3) = (d)$. Now since the equations of $I(l_1)$, $I(t_3)$ are linear in a,b,c,d it follows that the intersection multiplicity is equal to one. So $[l_{1T}].[t_{3T}] = 1$. Therefore $[l_T].[t_T] = 1$.

(IV) Let $[\alpha_{iT}]$denote the class of the closure of the cell α_i below:

$\alpha_1 = \{\{p, q\} \in \mathcal{H} : p \in L_{e_0},\ q \in L_{e_1}\}$

$\alpha_2 = \{\{p, q\} \in \mathcal{H} : p \in L_{e_0},\ q \in L_{e_2}\}$

$\alpha_3 = \{\{p, q\} \in \mathcal{H} : p \in L_{e_1},\ q \in L_{e_2}\}$

Consider table 5 below which gives the weights of the normal bundle of α_k in $\mathcal{H}$ at P_j, namely the weights of $(\mathcal{N}_{\alpha_k|\mathcal{H}})_{P_j}$.

Fixed pt	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9
α_1			$ca.2cb$	$cb.2ca$			$ba.ab$	$ba.cb$	$ab.ca$
α_2		$2bc.ba$			$2ba.bc$		$ba.ac$	$ca.bc$	$ac.ca$
α_3	$ac.2ab$					$ab.2ac$	$ab.bc$	$cb.bc$	$cb.ac$

Table 5: the pull backs $i_j^*[\alpha_1]$, $i_j^*[\alpha_2]$, and $i_j^*[\alpha_3]$

Note that the intersection of the closures of the cells α_i, α_j contains a line. So we can not calculate $[\alpha_{iT}].[\alpha_{jT}]$ by geometry. So we will use table 5 to calculate $[\alpha_{iT}].[\alpha_{jT}]$ by localization:

$$\int [\alpha_{1T}].[\alpha_{2T}] = \frac{(b-a)(a-b)(b-a)(a-c)}{(b-a)(a-b)(a-c)(b-c)}$$

$$+ \frac{(b-a)(c-b)(c-a)(b-c)}{(c-b)(b-c)(c-a)(b-a)}$$

$$+ \frac{(a-b)(c-a)(a-c)(c-a)}{(a-c)(c-a)(c-b)(a-b)}$$

$$= \frac{(b-a)}{(b-c)} + 1 + \frac{(c-a)}{(c-b)} = 2$$

$$\int [\alpha_{1T}].[\alpha_{3T}] = \frac{(b-a)(a-b)(a-b)(b-c)}{(b-a)(a-b)(a-c)(b-c)}$$

$$+ \frac{(b-a)(c-b)(c-b)(b-c)}{(c-b)(b-c)(c-a)(b-a)}$$

$$+ \frac{(a-b)(c-a)(c-b)(a-c)}{(a-c)(c-a)(c-b)(a-b)}$$

$$= 1 + \frac{(a-b)}{(a-c)} + \frac{(c-b)}{(c-a)} = 2$$

$$\int [\alpha_{2T}].[\alpha_{3T}] = \frac{(b-a)(a-c)(a-b)(b-c)}{(b-a)(a-b)(a-c)(b-c)}$$

$$+ \frac{(c-a)(b-c)(c-b)(b-c)}{(c-b)(b-c)(c-a)(b-a)}$$

$$+ \frac{(a-c)(c-a)(c-b)(a-c)}{(a-c)(c-a)(c-b)(a-b)}$$

$$= \frac{(b-c)}{(b-a)} + \frac{(a-c)}{(a-b)} + 1 = 2$$

(V) we calculate $[p_{1T}].[\alpha_{1T}]$ using geometry. Clearly the closures of the cells p_1, α_1 intersect at the point $P_7 = (z, xy) \in \mathcal{H}$. Now we check that the intersection multiplicity is equal to one. Let $\{(1, a, b), (c, 1, d)\}$ corresponds to the point $P_7 \in \mathcal{H}$. Consider the ideal $(z - c, x - d) \cap (z - a, y - b)$. Then as explained in (I) above the cell p_1 has equations $a = c = 0$ and the cell α_1 has equations $d = b = 0$. So $I(p_1) = (a, c)$, $I(\alpha_1) = (d, b)$. Now since the equations of $I(p_1)$, $I(\alpha_1)$ are linear in a,b,c,d it follows that the intersection multiplicity is equal to one. So $[p_{1T}].[\alpha_{1T}] = 1$. Therefore $[p_T].[\alpha_T] = 1$.

(VI) Consider table 6 below which gives the weights of the normal bundle of p_k, q_k, and r_k in $\mathcal{H}$ at P_j.

	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9
p_1	$ac.bc$				$bc.ac$		$ac.bc$		
p_2		$ba.ca$	$ca.ba$					$ba.ca$	
p_3				$cb.ab$		$ab.cb$			$ab.cb$
q_1	$2ab.ac$					$2ac.ab$	$ab.ac$		$ab.ac$
q_2		$ba.2bc$			$2ba.bc$		$ba.bc$	$ba.bc$	
q_3			$ca.2cb$	$2ca.cb$				$cb.ca$	$ca.cb$
r_1	$2ab.ac$	$2bc.bc$			$2ba.bc$	$2ac.ac$			
r_2		$ba.2bc$	$ca.2cb$	$2ca.ca$	$ba.2ba$				
r_3	$2ab.ab$		$2cb.cb$	$2ca.cb$		$ab.2ac$			

Table 6: the product of the weights of the normal bundle of p_k, q_k, and r_k in $\mathcal{H}$ at P_j.

Note that the intersection of the closures of the cells q_i, α_j contains a line. So we can not calculate $[q_{iT}].[\alpha_{jT}]$ using geometry. We will calculate $[q_{iT}].[\alpha_{jT}]$ by localization:

$$\int [q_{1T}].[\alpha_{1T}] = \frac{(a-b)(a-c)(b-a)(a-b)}{(b-a)(a-b)(a-c)(b-c)}$$

$$+ \frac{(a-b)(a-c)(a-b)(c-a)}{(a-c)(c-a)(c-b)(a-b)}$$

$$= \frac{(a-b)}{(b-c)} + \frac{(a-b)}{(c-b)} = 0$$

Similarly $[q_{iT}].[\alpha_{jT}] = 0$.

(VII) we calculate $[r_{1T}].[\alpha_{1T}]$ using geometry. Clearly the closures of the cells r_1, α_1 are disjoint. So $[r_{1T}].[\alpha_{1T}] = 0$. Therefore $[r_T].[\alpha_T] = 0$

Note that $r^2 = -2$ by localization. Also note that r_i, r_j are not transverse so we can not calculate r^2 using geometry. On the other hand the intersections of α with each of p, q, and r can be found by geometry. Now using theorem 5.5.44

$\sum_{i=1}^{3} a_i[p_{iT}] + \sum_{i=1}^{3} b_i[q_{iT}] + \sum_{i=1}^{3} c_i[r_{iT}] + \sum_{i=1}^{3} e_i[\alpha_{iT}] \sim 0$ iff $\sum_{i=1}^{3} a_i[p_{iT}] + \sum_{i=1}^{3} b_i[q_{iT}] + \sum_{i=1}^{3} c_i[\alpha_{iT}] \in \text{Span } \{\lambda.[t_{iT}], \lambda.[u_{iT}]\}_{i=1}^{3}$. So

$\sum_{i=1}^{3} a_i[p_{iT}] + \sum_{i=1}^{3} b_i[q_{iT}] + \sum_{i=1}^{3} c_i[r_{iT}] + \sum_{i=1}^{3} e_i[\alpha_{iT}] \sim 0$ iff $\sum_{i=1}^{3} a_i[p_{iT}] + \sum_{i=1}^{3} b_i[q_{iT}] + \sum_{i=1}^{3} c_i[r_{iT}] + \sum_{i=1}^{3} e_i[\alpha_{iT}] = \sum_{i=1}^{3} f_i'(\lambda)[t_{iT}] + \sum_{i=1}^{3} g_i'(\lambda)[u_{iT}]$

Now we apply the map i_j^* to the previous equation above to get the following system of linear equations

System V

$$a_1 i_1^*[p_{1T}] + b_1 i_1^*[q_{1T}] + \sum_{k\in\{1,3\}} c_k i_k^*[r_{kT}] + e_3 i_1^*[\alpha_{3T}] = \sum_{k\in\{1,2\}} f_k' i_k^*[t_{kT}] + \sum_{k\in\{1,2\}} g_k' i_k^*[u_{kT}] \tag{141}$$

$$a_2 i_2^*[p_{2T}] + b_2 i_2^*[q_{2T}] + \sum_{k\in\{1,2\}} c_k i_k^*[r_{kT}] + e_2 i_2^*[\alpha_{2T}] = \sum_{k\in\{1,3\}} f_k' i_k^*[t_{kT}] + \sum_{k\in\{2,3\}} g_k' i_k^*[u_{kT}] \tag{142}$$

$$a_2 i_3^*[p_{2T}] + b_3 i_3^*[q_{3T}] + \sum_{k\in\{2,3\}} c_k i_k^*[r_{kT}] + e_1 i_3^*[\alpha_{1T}] = \sum_{k\in\{2,3\}} f_k' i_k^*[t_{kT}] + \sum_{k\in\{2,3\}} g_k' i_k^*[u_{kT}] \tag{143}$$

$$a_3 i_4^*[p_{3T}] + b_3 i_4^*[q_{3T}] + \sum_{k\in\{2,3\}} c_k i_k^*[r_{kT}] + e_1 i_4^*[\alpha_{1T}] = \sum_{k\in\{2,3\}} f_k' i_k^*[t_{kT}] + \sum_{k\in\{1,3\}} g_k' i_k^*[u_{kT}] \tag{144}$$

$$a_1 i_5^*[p_{1T}] + b_2 i_5^*[q_{2T}] + \sum_{k\in\{1,2\}} c_k i_k^*[r_{kT}] + e_2 i_5^*[\alpha_{2T}] = \sum_{k\in\{1,3\}} f_k' i_k^*[t_{kT}] + \sum_{k\in\{1,2\}} g_k' i_k^*[u_{kT}] \tag{145}$$

$$a_3 i_6^*[p_{3T}] + b_1 i_6^*[q_{1T}] + \sum_{k\in\{1,3\}} c_k i_k^*[r_{kT}] + e_3 i_6^*[\alpha_{3T}] = \sum_{k\in\{1,2\}} f_k' i_k^*[t_{kT}] + \sum_{k\in\{1,3\}} g_k' i_k^*[u_{kT}] \tag{146}$$

$$a_1 i_7^*[p_{1T}] + b_1 i_7^*[q_{1T}] + b_2 i_7^*[q_{2T}] + \sum_{i=1}^{3} e_i i_7^*[\alpha_{iT}] = \sum_{k\in\{1,2,3\}} f_k' i_k^*[t_{kT}] + \sum_{k\in\{1,2\}} g_k' i_k^*[u_{kT}] \tag{147}$$

$$a_2 i_8^*[p_{2T}] + b_2 i_8^*[q_{2T}] + b_3 i_8^*[q_{3T}] + \sum_{i=1}^{3} e_i i_8^*[\alpha_{iT}] = \sum_{k\in\{1,2,3\}} f_k' i_k^*[t_{kT}] + \sum_{k\in\{2,3\}} g_k' i_k^*[u_{kT}] \tag{148}$$

$$a_3 i_9^*[p_{3T}]+b_1 i_9^*[q_{1T}]+b_3 i_9^*[q_{3T}]+\sum_{i=1}^{3} e_i i_9^*[\alpha_{iT}] = \sum_{k\in\{1,2,3\}} f_k' i_k^*[t_{kT}]+ \sum_{k\in\{1,3\}} g_k' i_k^*[u_{kT}] \quad (149)$$

First note that we evaluated $i_j^*[p_{kT}]$ in case(III) above. Using table 5 we evaluate $i_j^*[\alpha_{kT}]$. Second we use maple to solve the linear system that we get after substituting for the values of $i_j^*[p_{kT}]$, $i_j^*[q_{kT}]$, $i_j^*[r_{kT}]$, $i_j^*[\alpha_{kT}]$, $i_j^*[t_{kT}]$, and $i_j^*[u_{kT}]$. The result is the following relations:

$$\sum_{i=1}^{3} a_i + 2\sum_{i=1}^{3} e_i = 0 \quad (150)$$

$$\sum_{i=1}^{3} b_i = 0 \quad (151)$$

$$\sum_{i=1}^{3} c_i + \sum_{i=1}^{3} e_i = 0 \quad (152)$$

Subtracting (152) from (150) we get the relation

$$\sum_{i=1}^{3} a_i - \sum_{i=1}^{3} c_i + \sum_{i=1}^{3} e_i = 0 \quad (153)$$

Similarly (150)$-2\times$(152), (150)$\pm$(151), and (152)$\pm$(151) give the following relations:

$$\sum_{i=1}^{3} a_i - 2\sum_{i=1}^{3} c_i = 0 \quad (154)$$

$$\sum_{i=1}^{3} a_i \pm \sum_{i=1}^{3} b_i + 2\sum_{i=1}^{3} e_i = 0 \quad (155)$$

$$\pm\sum_{i=1}^{3} b_i + \sum_{i=1}^{3} c_i + \sum_{i=1}^{3} e_i = 0 \quad (156)$$

Let

$$a_1p_1 + a_2p_2 + a_3p_3 + c_1r_1 + e_1\alpha_1 = 0 \quad (157)$$

in the Chow group $A_2(\mathcal{H})$. Using the equations above we have the following relations between a_1, a_2, a_3, c_1, and e_1:

$$a_1 + a_2 + a_3 + 2e_1 = 0 \tag{158}$$

$$c_1 + e_1 = 0 \tag{159}$$

Take $a_1 = 1$, $a_2 = -1$, $a_3 = 0$, $c_1 = 0$, and $e_1 = 0$. Clearly these values satisfy the relations above. Now substituting these values in (157) we get $p_1 = p_2$. Similarly take $a_1 = 1$, $a_2 = 0$, $a_3 = -1$, $c_1 = 0$, and $e_1 = 0$ to get $p_1 = p_3$. Therefore $p_1 = p_2 = p_3$.
To check that $q_1 = q_2 = q_3$, let

$$b_1 q_1 + b_2 q_2 + b_3 q_3 = 0 \tag{160}$$

in the Chow group $A_2(\mathcal{H})$. Using the equations above we have the relation $\sum_{i=1}^{3} b_i$. Take $b_1 = 1$, $b_2 = -1$, and $b_3 = 0$. Clearly these values satisfy the previous relation. Now substituting these values in (160) we get $q_1 = q_2$. Similarly take $b_1 = 1$, $b_2 = 0$, and $b_3 = -1$ to get $q_1 = q_3$. Therefore $q_1 = q_2 = q_3$.
To check that $r_1 = r_2 = r_3$, let

$$a_1 p_1 + c_1 r_1 + c_2 r_2 + c_3 r_3 + e_1 \alpha_1 = 0 \tag{161}$$

in the Chow group $A_2(\mathcal{H})$. Using the equations above we have the relations:

$$a_1 + 2e_1 = 0 \tag{162}$$

$$c_1 + c_2 + c_3 + e_1 \alpha_1 = 0 \tag{163}$$

between a_1, c_1, c_2, c_3, and e_1. Take $a_1 = 0$, $c_1 = 1$, $c_2 = -1$, $c_3 = 0$, and $e_1 = 0$. Clearly these values satisfy the previous relations. Now substituting these values in (161) we get $r_1 = r_2$. Similarly take $a_1 = 0$, $c_1 = 1$, $c_2 = 0$, $c_3 = -1$, and $e_1 = 0$ to get $r_1 = r_3$. Therefore $r_1 = r_2 = r_3$.

Let

$$a_1 p_1 + b_1 q_1 + c_1 r_1 + e_1 \alpha_1 = 0 \tag{164}$$

in the Chow group $A_2(\mathcal{H})$. Using the equations above we have the following relations between a_1, b_1, c_1, and e_1:

$$a_1 + 2e_1 = 0 \tag{165}$$

$$b_1 = 0 \tag{166}$$

$$c_1 + e_1 = 0 \tag{167}$$

$$a_1 - c_1 + e_1 = 0 \tag{168}$$

$$a_1 - 2c_1 = 0 \tag{169}$$

$$a_1 \pm b_1 + 2e_1 = 0 \tag{170}$$

$$\pm b_1 + c_1 + e_1 = 0 \tag{171}$$

Take $a_1 = 2$, $b_1 = 0$, $c_1 = 1$, and $e_1 = -1$. Clearly these values satisfy the relations above. Now substituting these values in (157) we get $2p_1 + r_1 - \alpha_1 = 0$. Therefore $2p + r - \alpha = 0$ since the $p_i's$ (resp.$r_i's$, $\alpha_i's$) are rationally equivalent. So $r = -2p + \alpha$. It follows $r^2 = -2p.r + \alpha.r$. But $p.r = 1$, $\alpha.r = 0$. So $r^2 = -2$.

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VITA

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