THE UNIVERSITY OF OKLAHOMA

GRADUATE COLLEGE

SMALL DISTURBANCES IN DUSTY GASES

A DISSERTATION

SUBMITTED TO THE GRADUATE FACULTY

in partial fulfillment of the requirements for the

degree of

DOCTOR OF PHILOSOPHY

By

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Norman, Oklahoma

1979

SMALL DISTURBANCES IN DUSTY GASES

A DISSERTATION

APPROVED FOR THE SCHOOL OF AEROSPACE, MECHANICAL

AND NUCLEAR ENGINEERING

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ACKNOWLEDGMENTS

I am indebted to my advisor, Dr. Maurice L. Rasmussen, for suggesting the subject of this dissertation. Without his encouragement and help, this work would not have been completed. His warm personality and attitude towards human understanding have won my highest respect. Thanks are due to Dr. S. R. Gollahalli, Dr. M. C. Jischke, Dr. T. J. Love, and Dr. C. M. Sliepcevich for serving as members of my doctoral advisory committee. Their comments have stimulated me to rethink some of my ideas, and to put them on a more rigorous foundation.

To my wife Lee-Hwa in Taiwan, I would like to express my deepest gratitude for her personal sacrifices and understanding during this period of my advanced study in the United States. Her constant encouragement and patience were essential to this research, and I dedicate it to her as a gift of love.

The financial support of the Ministry of the Chinese Air Force is gratefully acknowledged.

ABSTRACT

The present study consists of two parts. The first part deals with small disturbances in inert dusty gases. The effects of viscosity, thermal conductivity, and particulates on the wave fronts are studied in detail by means of linearized theory. Laplace transforms are utilized to obtain asymptotic approximations for large times. The large-time solution shows a wave behavior with the wave fronts diffused out because of viscosity, thermal conductivity, and particulates. The interaction of nonlinearities and the above-mentioned dissipative effects is investigated. It is found that at the final stages the wave front is governed, to the lowest order, by Burger's equation.

The second part treats a model for transient flow in a porous particle. For simplicity, we assume that the gas inside the porous particle is identical to the outside so that no mass diffusion would occur. We also assume that the disturbances are so small that the linearized theory can be applied. Again, by means of Laplace transforms, the asymptotic solutions valid for small times and large times are obtained for the field properties. It is shown that the small-time solution is not valid physically since this region is governed by free-molecule flow. On the other hand, the large-time solution shows that there is a weak wave behavior outside the particle, and that the wave fronts diffuse out due to viscosity. In order to fit the problem concerning small distur-

bances in inert porous-particle dusty gases, the mass ejecting from the porous particles per unit time per unit volume, $\dot{\mu}$, is determined. With $\dot{\mu}$ known, the corresponding mass-, momentum-, and energy-source term appearing in the governing equations can be obtained. For more general models including vaporization and chemical reactions, the governing equations are derived in Appendix A.

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NOMENCLATURE

a	speed of sound
C	specific heat of particle material
C_p	specific heat of gas at constant pressure
e	specific internal energy of gas
e_p	specific internal energy of particles
h	specific enthalpy of gas
$\tilde{h}$	convective heat transfer coefficient between particles and gas
$\vec{f}_p$	force exerted by a single particle on gas
$\vec{F}_p$	force per unit volume exerted by particles on gas
$\dot{J}$	mass ejecting from a porous particle per unit time
k, k_g	thermal conductivity of gas
k_p	thermal conductivity of particle material
K	permeability of a porous particle
K^*	$\equiv KP_o/\mu_o\tilde{\nu}_o$, dimensionless parameter
m	mass of an individual particle
n	number density, i.e., number of particles per unit volume
N_u	$\equiv 2\bar{\sigma}\tilde{h}/k_o$, Nusselt number
p	gas pressure
P_r	gas Prandtl number
P_r^*	$\equiv C_{po}\tilde{\mu}_o/k_o$, effective gas Prandtl number

$\vec{q}$	gas heat flux vector
$\dot{Q}_p$	heat transfer rate per unit volume from particles to gas
$\vec{\bar{r}}$	dimensional position vector
$\vec{r}$	$\equiv a_o\vec{\bar{r}}/\tilde{\nu}_o$, dimensionless position vector
$\vec{\tilde{r}}$	$\equiv \varepsilon\, \vec{\bar{r}}/L = \varepsilon\vec{r}$, dimensionless position vector
R	gas constant
R_e	particle Reynolds number based on slip vector $\lvert\vec{V}_p-\vec{V}\rvert$
s	Laplace transform variable
t	dimensional time
T	gas temperature
T_p	particle temperature
u	gas velocity in $\bar{x}$ direction
u_p	particle velocity in $\bar{x}$ direction
u^*	gas velocity at the surface of the porous particle
$\vec{V}$	gas velocity vector
$\vec{V}_p$	particle velocity vector
x	nondimensional space coordinate in the direction of motion

Greek Letters

α	$\equiv C_p/C$, specific ratio
α_p	thermal diffusivity of the porous particle
α_p^*	$\equiv \alpha_p/\tilde{\nu}_o$, dimensionless parameter
$\bar{\delta}$	dimensional wave thickness
δ	dimensionless wave thickness
ε	$\equiv \tilde{\nu}_o/a_oL$, dimensionless perturbation parameter
$\overleftrightarrow{\varepsilon}$	rate of strain tensor

ζ a small dimensionless parameter determined from the boundary condition at the end wall, $T'(0,\tau)=2\zeta\sqrt{\tau/\pi}$

η $\equiv 1/\tau_v^*$

κ $\equiv \rho_p/\rho$, density ratio

λ second coefficient of viscosity of gas

μ first coefficient of viscosity of gas

$\tilde{\mu}$ $\equiv 2\mu + \lambda = 4\mu/3$

μ_s viscosity of particulate suspension

$\dot{\mu}$ mass rate ejecting from a unit volume of porous particles

ν kinematic viscosity of gas

$\tilde{\nu}$ $= \tilde{\mu}/\rho$

ν^* effective diffusion coefficient

ξ $\equiv 2\alpha/3P_{ro}\tau_v^*$

ρ gas density

ρ_p phase density of particles, i.e., the mass of particles per unit volume of gas-particle mixture particle radius

σ $\equiv a_o\bar{\sigma}/\tilde{\nu}_o$, dimensionless particle radius

τ_v velocity relaxation time of particles

τ_T temperature relaxation time of particles

τ_v^* $\equiv \tau_{vo}a^2/\tilde{\nu}_o$, dimensionless

τ^* $\equiv a_o t/\bar{\sigma}$, dimensionless wave time

τ $\equiv a_o^2\, t/\tilde{\nu}_o$, dimensionless time

τ_μ $\equiv m/4\pi\bar{\sigma}^2 a_o\rho_o$

τ_μ^* $\equiv a_o^2\, \tau_\mu/\tilde{\nu}_o$

$\tilde{\tau}$ $\equiv a_o t/L = \varepsilon\tau$, dimensionless time

$\overleftrightarrow{\tau}$ viscous stress tensor

ϕ	perturbation velocity potential of gas
ϕ_p	perturbation velocity potential of particle phase
Φ	$\equiv r\phi$
$\tilde{\Phi}$	volume fraction of particles

Superscripts

$(\ )'$	perturbation quantity, dimensionless
$(\tilde{\ })$	perturbation quantity, dimensionless

Subscripts

$(\ )_o$	variable evaluated at ambient condition
$(\ )_p$	particle variable

PART I

SMALL DISTURBANCES IN INERT DUSTY GASES

CHAPTER 1

INTRODUCTION

1.1 General Description of Dusty-Gas Flows

Interest in problems of mechanics of systems with more than one phase has developed rapidly in recent years. Situations which occur frequently are concerned with the motion of a gas containing small solid particles - the so-called dusty gas flow. Such situations occur, for example, in the problems of rocket nozzle flows, nuclear reactors, fuel sprays, air pollution, and lunar ash flows.

The dynamics of dusty gases is modified from the conventional gas dynamics by further taking into account the effects due to the presence of the particles, namely: (1) the collisions between individual solid particles; (2) the gas-particle interactions through viscous drag, heat transfer, as well as mass transfer by condensation, evaporation, or chemical reaction. Moreover, in view of the particles being unable to follow the rapid changes in the temperature and velocity of the gas, there may exhibit significant relaxation effects which make the flow problem of dusty gases greatly different from that of pure gases. The simplest flows of such a non-equilibrium nature are produced by shock waves that propagate through a particle suspension. The structure of the shock waves in such flows was first considered by Carrier (1958), who treated a stationary normal shock wave arising in a inviscid dusty gas and determined the dependence of the flow field on the parameter of interest. According to Carrier, the thickness of a gas dynamic shock is negligible in comparison with the momentum and thermal ranges of the particles so long as the solid particles are large with respect to molecular dimensions. Thus, the structure of a normal shock wave in a gas-particle mixture may be thought of as a conventional gas dynamic shock followed by a relaxation zone where particles and gas come to velocity and temperature equilibrium. A qualitative description of the structure of a normal shock wave in a gas-particle mixture is illustrated in Fig. 1. Upstream of the gas dynamic shock, the gas and dust are in equilibrium. Immediately downstream of the shock the velocity of the gas is smaller than that of the dust and the dust will then be decelerated. The dust will also accept heat from the gas since the gas temperature has been increased above that of ambient mixture by the shock. The flow configuration far downstream of the shock will be a steady one in which the gas and dust will achieve the same velocities and temperatures. Various aspects of the relaxation process have also been studied by other investigators, for instance, Soo (1961), Saffman (1962), Temkin and Dobbins (1966), and Schmitt-von Schubert (1969). These have relied for the most part on the simplifying assumptions needed to make the problem tractable.

The general conservation equations obtained on the basis of particle distribution function are discussed in detail by Marble (1963) in his paper, "Dynamics of a Gas Containing Small Solid Particles." In this paper he introduced many significant concepts and parameters which serve to give the physical insight into the behavior of dusty gases. Under the assumption of low-Reynolds-number flow of the particles relative to the gas, Marble proposed two important characteristic times τ_v and τ_T, corresponding to the velocity and temperature history of the spherical particles, which are defined by

$$\tau_v \equiv \frac{m}{6\pi\bar{\sigma}\ \mu} \tag{1.1.1}$$

$$\tau_T \equiv \frac{mC_p}{4\pi\bar{\sigma}\ k} = \frac{3}{2} P_r\ \tau_v \tag{1.1.2}$$

where m is the particle mass, $\bar{\sigma}$ the particle radius, μ the gas viscosity, C_p the specific heat of the gas at constant pressure, k the heat conductivity of the gas, and $P_r \equiv \frac{C_p \mu}{k}$ the gas Prandtl number. Physically, τ_v and τ_T are the relaxation times elapsed for a particle to adjust to changes in neighboring gas velocity and temperature. The introduction of these two parameters greatly clarifies the quantitative concept of the times for the relaxation processes occurring in the gas-particle flow. The structure of the flow field of a gas-particle mixture can be predicted by the value of τ_v (and hence τ_T): (1) if $\tau_v \to \infty$, then the flow is "frozen" in which no relaxation processes take place; (2) if $\tau_v \to 0$, then the flow is "equilibrium" for which relaxation can be infinitely fast; (3) if $0 < \tau_v < \infty$, one considers a "non-equilibrium" flow where finite relaxation processes may occur.

Marble's recent work (1970) in this field provides an extensive study of the flows of a dusty gas with examples of shock formation and linearized theory. More recently, Bhutani and Chandran (1977), using characteristic coordinate system, augmented Marble's study in linearized case to derive a general decay law for steady and nonsteady weak waves in an inviscid dusty gas.

As was made by previous investigators, our physical model of a dusty gas will be further simplifies by restrictive assumptions. All such simplifications, although they take us further away from physical reality, nevertheless allow us to obtain an insight into the nature of real dusty gases by providing us with physical models whose behavior can be analyzed.

1.2 Physical Models

It is well known that, in the linear acoustic theory for a uniform one-dimensional medium, a disturbance propagates at a constant velocity with constant wave form and the entropy of the fluid remains constant. In a more realistic case, this wave motion is modified by dissipative processes and by nonlinear effects due to convection. The term dissipative process is meant to include all processes which tend to increase the entropy of the fluid. Dissipative processes of importance are those due to viscosity, thermal conductivity, chemical reaction, mass diffusion, radiation, etc. The effects of these dissipative processes on wave propagation can be quite complex and quite different from each other. Both the form and speed of propagation of the waves may be modified by their influence.

The case in which viscosity and thermal conductivity are the only dissipative mechanisms was first considered by Lagerstrom, Cole, and Trilling (1949), and later by Lighthill (1956), in treating nonlinear one-dimensional piston problems. The mathematical analysis of this problem was further developed in detail by Moran and Shen (1966) by means of the method of matched asymptotic expansions and a composite solution uniformly valid for both small and large times was obtained. It is to be noted that immediately after the piston is set into motion, as pointed out by Lighthill, the velocity gradients are very steep and the viscous terms in Navier-Stokes equations are much more important then the non-linear convective terms. This implies that the linear theory is valid for some early time. As the wave propagates through the gas, viscosity and thermal conductivity cause the velocity gradients to ease, and the nonlinear effects become important. Eventually, for large time, the waveform-steepening effects of the nonlinear terms become comparable in strength to the waveform-easing effects of the viscous terms. The balance achieved between these opposing tendencies results in the formation of a shock wave which ultimately propagates steadily into the gas.

The other significant example concerning the dissipative effects on wave propagation was given by Rasmussen and Lake (1973) who, instead of treating one-dimensional piston problem, dealt with linearized wave propagation associated with weak spherical explosions. By making use of Laplace transform techniques, they obtained asymptotic solutions which are valid for small times and for large times. It was found that the small-time solution shows a diffusive character that is akin to the free-molecule flow theory, and that the large time solution shows a wave behavior with the wave front and contact discontinuity diffused out because of viscosity and heat conduction.

Much of the significant literature bearing upon real-gas effects on one-dimensional wave propagation induced by a piston, including various dissipative mechanisms with and without nonlinear effects due to convection, has been reviewed by Lick (1967).

All the work described above, however, refers to dissipative effects on wave propagation in pure gases. The purpose of this study is to extend the problem to account for the part played by the solid particles which are present in the gas flow. In the present study, we shall consider unsteady, one-dimensional, viscous flow of compressible dusty gases. The basic problem to be analyzed is that of a semi-infinite particulate suspension, initially in equilibrium, that is disturbed by an impulsive temperature increase at the boundary end wall. The end-wall disturbance affects the semi-infinite suspension by means of diffusive and wave mechanisms. The subsequent development of the flow field is to be determined. It is expected that the disturbances from the basic equilibrium flow are so small that, as a first order approximation, the governing equations can be linearized.

In order to formulate the problem in a reasonably simple manner and to bring out the essential features, the following assumptions regarding the motion of the gas-particle mixture are made:

(1) The gas is thermally perfect.

(2) The particles are spherical and are uniform in size.

(3) The particle diameter does not vary, i.e., any mass transfer between the particles and the gas is excluded.

(4) The temperature within each particle is uniform due to its small size and high conductivity relative to the gas.

(5) The gravity effect of the mixture is neglected so that there is no free convection.

(6) The particles do not interact. This assumption requires

that the particle number density is sufficiently small, so that the momentum exchange caused by collisions, and the force resulting from it, is small in comparison with the viscous resistance to particle motion.

(7) The volume occupied by the particles is negligible. In fact, the volume fraction of the particles $\tilde{\Phi}$ can be written as

$$\tilde{\Phi} = \kappa \frac{\rho}{\rho_s} \tag{1.2.1}$$

Thus, if $\kappa = 0\ (0.1)$, and $\rho/\rho_s = 0\ (10^{-3})$, then $\tilde{\Phi} = 0\ (10^{-4})$. The assumption is therefore satisfied if the density of the particle material ρ_s is high compared with the gas density ρ and if the density ratio κ is not extremely large.

(8) The random motion of the particles does not contribute to the pressure of the particle cloud. This assumption implies that the number of particles is negligible compared with the number of gas molecules in the same volume.

(9) The Reynolds number for the relative motion of the gas and particles is small enough that Stokes' drag law is applicable.

(10) Heat exchange between the particles and the gas occurs only through convection.

(11) The effect of heat radiation is neglected. It is known, however, that a cloud of particles is a good emitter and absorber of radiation compared with a pure gas.

Therefore, the hot particles upstream of the wave radiate heat to the cold particles downstream. This effect may increase the wave thickness, particularly as the size of the particle increases.

1.3 Methods of Attack

In the present analysis, a dusty gas with small κ and $\bar{\sigma}$ is considered. We start the formulation with the linearization of the governing equations based on the small perturbation model. The resultant linearized equations are further simplified by the assumption that the flow is irrotational due to geometric symmetry. Finally, a single seventh-order partial differential equation for the perturbation velocity potential is derived. Laplace transforms are utitlized to obtain asymptotic solutions valid for large times.

The large-time solution shows a wave behavior with the wave front diffused out due to: (1) the viscosity and thermal conductivity of the gas; (2) the size and number density of the particles. Further examination of the results indicates that the linearized theory breaks down at $\tau = O(\zeta^{-2})$, where τ is the dimensionless time and ζ is a small dimensionless parameter determined from the boundary condition at the end wall. The nonuniformity of the linear solution suggests that the problem at very large times should be attacked by singular perturbation methods. In this regard, the full equation with the nonlinear and transport dissipative terms is derived to the first order corrections. It is shown that at the final stages the wave front is governed by Burger's equation.

CHAPTER 2

FORMULATION OF THE PROBLEM

2.1 Gas-Particle Interaction

Before we consider the basic equations for the gas-particle flow, it is helpful to derive the terms in connection with the interaction between gas and particles. To begin with, denote the local gas velocity by $\vec{V}$ and the velocity of an individual particle by $\vec{V}_p$. Under the assumption that the particles obey Stoke's drag law, the force $\vec{f}_p$ exerted upon the gas by a single particle of radius $\bar{\sigma}$ is

$$\vec{f}_p = 6\,\pi\bar{\sigma}\mu\,(\vec{V}_p - \vec{V}) \tag{2.1.1}$$

Moreover, if n is the number of particles per unit volume, the total volumetric force exerted upon the gas by the particles is

$$\vec{F}_p = n\vec{f}_p = \frac{1}{\tau_v}\rho_p\,(\vec{V}_p - \vec{V}) \tag{2.1.2}$$

where

$$\rho_p \equiv n\,m \tag{2.1.3}$$

$$\tau_v \equiv \frac{m}{6\pi\bar{\sigma}\mu} = \frac{2}{9}\frac{\rho_s}{\rho}\frac{\bar{\sigma}^2}{\nu} \tag{2.1.4}$$

ρ_p being the local mass density of the particle phase, τ_v the velocity relaxation time of the particles, and ρ_s/ρ the ratio of the mass density of the particle material to the mass density of the gas. For metallic solids and gases at standard conditions, ρ_s/ρ is of the order of 10^3.

The definition of τ_v, Eq. (2.1.4), reveals that for a given gas τ_v is determined by the size of the individual particles; it is proportional to the square of the particle radius. Making the dust fine decreases τ_v, and making it coarse increases τ_v. Further, from the equation of motion for a single particle

$$m \frac{d\vec{V}_p}{dt} = - \vec{f}_p \qquad (2.1.5)$$

it can be shown that τ_v is the time required for a particle to decay its velocity relative to the gas, $\vec{V}_p - \vec{V}$, to e^{-1} of its original value.

The velocity relaxation time τ_v gives some indication of the gas-particle interaction process when compared with the characteristic time τ_c of the flow. When $\tau_v/\tau_c \gg 1$, the particle enters and leaves the region of interest before there is an opportunity to alter its state appreciably, and the particle motion depends largely on its initial conditions. On the other hand, when $\tau_v/\tau_c \ll 1$, the particle has time to adjust to the local gas motion before it has moved appreciably through the region, and hence the particle motion depends largely upon the local gas motion and is relatively independent of its previous history. For values of τ_v/τ_c that are neither very large nor very small, i.e., $\tau_v/\tau_c \approxeq 1$, the local particle motion is dependent upon its entire history.

To the same approximation of low Reynolds number flow, the rate of heat transfer per unit volume $\dot{Q}_p$ from the particles of temperature T_p to the gas at temperature T may be written as

$$\dot{Q}_p = n\ (4\pi\bar{\sigma}^2)\ \tilde{h}\ (T_p - T)$$

$$= \frac{1}{2}\ nmC_p\ (\frac{4\pi\bar{\sigma}k}{mC_p})\ (\frac{2\bar{\sigma}\tilde{h}}{k})\ (T_p - T)$$

$$= \frac{Nu}{2\tau_T}\ \rho_p\ C_p\ (T_p - T) \qquad (2.1.6)$$

where

$$\tau_T \equiv \frac{mC_p}{4\pi\bar{\sigma}k} \qquad (2.1.7)$$

$$N_u \equiv \frac{2\bar{\sigma}\tilde{h}}{k} \qquad (2.1.8)$$

τ_T is called the temperature relaxation time of the particles, and N_u is the Nusselt number based on the particle diameter. The Nusselt number is usually approximated by the formula for steady flow around a single sphere given by Knudsen and Katz (1958):

$$N_u = 2.0 + 0.60\ P_r^{1/3}\ R_e^{1/2} \qquad (2.1.9)$$

where P_r is the gas Prandtl number, and R_e is the particle Reynolds number. The simple formulation $N_u = 2$, which corresponds to pure steady-state heat conduction, has also been used (see, e.g., Soo 1961, Marble 1963, and Rudinger 1964). With $N_u = 2$, Eq. (2.1.6) becomes

$$\dot{Q}_p = \frac{1}{\tau_T}\ \rho_p\ C_p\ (T_p - T) \qquad (2.1.10)$$

The physical significance of τ_T is entirely similar to τ_v. Furthermore, τ_T can be written as

$$\tau_T = \frac{3}{2}\ P_r \tau_v \qquad (2.1.11)$$

Since the gas Prandtl number P_r is very nearly equal to 2/3, the velocity and temperature relaxation times are nearly equal. Consequently, the times for a particle to reach equilibrium velocity and temperature of the gas are approximately equal.

2.2 Basic Equations

Consider an ideal gas with local velocity $\vec{V}$, temperature T, and density ρ, containing a cloud of small, solid, spherical particles having a radius $\bar{\sigma}$. The particle cloud is also described by a set of continuum variables $\vec{V}_p$, T_p, and ρ_p = nm. Because the particles are noninteracting and too massive to have random motion, all particles in a local volume have the same velocity vector and temperature. Under the assumptions described in Section 1.2, the conservation equations for the two phases can be written separately as (see Marble 1963)

Mass

$$\frac{\partial \rho}{\partial t} + \bar{\nabla} \cdot \rho \vec{V} = 0 \tag{2.2.1}$$

$$\frac{\partial \rho_p}{\partial t} + \bar{\nabla} \cdot \rho_p \vec{V}_p = 0 \tag{2.2.2}$$

Momentum

$$\rho \frac{D\vec{V}}{Dt} = - \bar{\nabla} P + \bar{\nabla} \cdot \overleftrightarrow{\tau} + \vec{F}_p \tag{2.2.3}$$

$$\rho_p \frac{D_p \vec{V}_p}{Dt} = - \vec{F}_p \tag{2.2.4}$$

Energy

$$\rho \frac{De}{Dt} = - p \bar{\nabla} \cdot \vec{V} + \overleftrightarrow{\tau} : \overleftrightarrow{\varepsilon} - \bar{\nabla} \cdot \vec{q} + \dot{Q}_p + (\vec{V}_p - \vec{V}) \cdot \vec{F}_p \tag{2.2.5}$$

$$\rho_p \frac{D_p e_p}{Dt} = - \dot{Q}_p \qquad (2.2.6)$$

where

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \vec{V} \cdot \bar{\nabla} \qquad (2.2.7)$$

$$\frac{D_p}{Dt} \equiv \frac{\partial}{\partial t} + \vec{V}_p \cdot \bar{\nabla} \qquad (2.28)$$

$$\vec{F}_p = \frac{\rho_p}{\tau_v} (\vec{V}_p - \vec{V}) \qquad (2.2.9)$$

$$\tau_v = \frac{m}{6 \pi \bar{\sigma} \mu} \qquad (2.2.10)$$

$$\dot{Q}_p = \frac{1}{\tau_T} \rho_p C_p (T_p - T) \qquad (2.2.11)$$

$$\tau_T = \frac{m C_p}{4 \pi \bar{\sigma} k} \qquad (2.2.12)$$

$\overleftrightarrow{\tau}$, $\overleftrightarrow{\varepsilon}$, and $\vec{q}$ are the usual stress tensor, rate of strain tensor, and heat flux vector for the gas and given by

$$\overleftrightarrow{\tau} = 2 \mu \overleftrightarrow{\varepsilon} + \lambda (\bar{\nabla} \cdot \vec{V}) \overleftrightarrow{\Pi} \qquad (2.2.13)$$

$$\overleftrightarrow{\varepsilon} = \frac{1}{2} [\bar{\nabla}\vec{V} + (\bar{\nabla}\vec{V})^*] \qquad (2.2.14)$$

$$\vec{q} = -k \bar{\nabla} T \qquad (2.2.15)$$

where μ and λ are the first and second coefficients of viscosity of the gas, k the gas thermal conductivity, and $(\bar{\nabla}\vec{V})^*$ the transpose of $\bar{\nabla}\vec{V}$.

It is of interest to note that the limiting cases for $\tau_v \to 0$ and $\tau_\infty \to \infty$. With Eq. (2.2.9), the momentum equations (2.2.3) and (2.2.4) can be rewritten as

$$\rho \frac{D\vec{V}}{Dt} = - \bar{\nabla}P + \bar{\nabla} \cdot \overleftrightarrow{\tau} - \rho_p \frac{D_p\vec{V}_p}{Dt} \tag{2.2.16}$$

$$\rho_p \frac{D_p\vec{V}_p}{Dt} = - \frac{\rho_p}{\tau_v} (\vec{V}_p - \vec{V}) \tag{2.2.17}$$

When $\tau_v \to 0$, Eq. (2.2.17) gives

$$\vec{V}_p = \vec{V} \tag{2.2.18}$$

and thus Eq. (2.2.16) becomes

$$(1 + \kappa) \rho \frac{D\vec{V}}{Dt} = - \bar{\nabla}P + \bar{\nabla} \cdot \overleftrightarrow{\tau} \tag{2.2.19}$$

where

$$\kappa \equiv \frac{\rho_p}{\rho} \tag{2.2.20}$$

Eqs. (2.2.18) and (2.2.19) show that when $\tau_v \to 0$ (i.e. equilibrium flow) the particles move with the gas at each point, and that the flow for $\tau_v \to 0$ is identical with the flow of a pure gas with an effective density $\rho^* = (1 + \kappa)\rho$. The solution for the dusty gas flow at the Reynolds number R_e is then equivalent to the solution for a pure gas at the increased Reynolds number $R_e(1 + \kappa)$. In the other limiting case as $\tau_v \to \infty$, we have, from Eq. (2.2.17),

$$\frac{D_p\vec{V}_p}{Dt} = 0 \tag{2.2.21}$$

and hence Eq. (2.2.16) becomes

$$\rho \frac{D\vec{V}}{Dt} = - \bar{\nabla}P + \bar{\nabla} \cdot \overleftrightarrow{\tau} \tag{2.2.22}$$

Eqs. (2.2.21) and (2.2.22) reveal that, when $\tau_v \to \infty$ (i.e. frozen flow), the particles move with constant velocity, and the presence of the particles does not change the gas quantities.

It is also important to note that, owing to the lack of randomness in local particle motion and the rigidity of the small solid particles, there are no pressure and viscous stress terms appearing in the momentum equation for the particle phase , as shown in Eq. (2.2.4). And the usual relation for the first law of the gas, Eq. (2.2.5), is modified by (a) the heat addition $\dot{Q}_p$ from the particle phase, and (b) the dissipative work $(\vec{V}_p - \vec{V}) \cdot \vec{F}_p$ due to particles moving to the gas. Furthermore, it follows immediately from Eq. (2.2.4) that

$$\rho_p \frac{D_p}{Dt}\left(\frac{V_p^2}{2}\right) = - \vec{V}_p \cdot \vec{F}_p \qquad (2.2.23)$$

This means that the work resulting from gaseous drag on the particles affects only the kinetic energy of the particle cloud.

In the following analysis, it is more convenient to employ the energy equation for the gas in the enthalpy form:

$$\rho \frac{Dh}{Dt} = \frac{DP}{Dt} + \vec{\tau} : \bar{\nabla}\vec{V} - \bar{\nabla} \cdot \vec{q} + \dot{Q}_p + (\vec{V}_p - \vec{V}) \cdot \vec{F}_p \qquad (2.2.24)$$

With $dh = C_p dT$ and $de_p = CdT_p$, the governing equations can be recast as

Mass

$$\frac{\partial \rho}{\partial t} + \bar{\nabla} \cdot \rho\vec{V} = 0 \qquad (2.2.25)$$

$$\frac{\partial \rho_p}{\partial t} + \bar{\nabla} \cdot \rho_p\vec{V}_p = 0 \qquad (2.2.26)$$

Momentum

$$\rho \frac{D\vec{V}}{Dt} = - \bar{\nabla}P + \bar{\nabla} \cdot \overleftrightarrow{\tau} + \frac{\rho_p}{\tau_v}(\vec{V}_p - \vec{V}) \qquad (2.2.27)$$

$$\rho_p \frac{D_p \vec{V}_p}{Dt} = - \frac{\rho_p}{\tau_v} (\vec{V}_p - \vec{V}) \qquad (2.2.28)$$

Energy

$$\rho \, C_p \frac{DT}{Dt} = \frac{DP}{Dt} + \overleftrightarrow{\tau} \; : \bar{\nabla}\vec{V} + \bar{\nabla} \cdot (k \, \bar{\nabla} T)$$

$$+ \frac{2}{3} \frac{\rho_p C_p}{P_r \tau_v} (T_p - T) + \frac{\rho_p}{\tau_v} (\vec{V}_p - \vec{V})^2 \qquad (2.2.29)$$

$$\rho_p \, C \frac{D_p T_p}{Dt} = - \frac{2}{3} \frac{\rho_p C_p}{P_r \tau_v} (T_p - T) \qquad (2.2.30)$$

where C is the specific heat of the dust particles. The set of equations becomes complete with the addition of the equation of state for the perfect gas

$$P = \rho RT \qquad (2.2.31)$$

Therefore, there are eleven equations for eleven unknowns: ρ, ρ_p, P, $\vec{V}$, $\vec{V}_p$, T, and T_p.

2.3 Linearized Equations

Consider a uniform ambient state, denoted by the subscript nought, and the perturbations about this state. We introduce non-dimensional primed perturbation variables and write

$$
\begin{aligned}
P &= P_o\,(1 + p') \\
\rho &= \rho_o\,(1 + \rho') \\
\rho_p &= \rho_{po}(1 + \rho_p') \\
T &= T_o(1 + T') \\
T_p &= T_o\,(1 + T_p') \\
\vec{V} &= a_o\,\vec{V}' \\
\vec{V}_p &= a_o\,\vec{V}_p' \\
\mu &= \mu_o\,(1 + \mu_o') \\
k &= k_o\,(1 + k') \\
C_p &= C_{po}(1 + C_p') \\
\tau_v &= \tau_{vo}\,(1 + \tau_v')
\end{aligned}
\tag{2.3.1}
$$

where $|p'| << 1$, and so on for the other primed variables. a_o denotes the isentropic speed of sound in the ambient gas and is given by

$$a_o^2 = \gamma\, P_o/\rho_o \tag{2.3.2}$$

γ being the ratio of the specific heats of the gas. Further, we define the nondimensional quantities for time and position vector as

$$\tau = t/\tau_c \ , \ \tau_c \equiv \tilde{\nu}_o/a_o^2 \qquad (2.3.3)$$

$$\vec{r} = \vec{\tilde{r}}/L_c \ , \ L_c \equiv \tilde{\nu}_o/a_o \qquad (2.3.4)$$

where $\vec{\tilde{r}}$ is the dimensional position vector, and

$$\tilde{\nu}_o = \tilde{\mu}_o/\rho_o = (2\mu_o + \lambda_o)/\rho_o \qquad (2.3.5)$$

is the effective viscosity. If we evaluate the second coefficient of viscosity by Stokes approximation, $\lambda_o = -2\mu_o/3$, then we have

$$\tilde{\nu}_o = \frac{4}{3}\mu_o/\rho_o = \frac{4}{3}\nu_o \qquad (2.3.6)$$

The dimensionless del operator will be denoted by $\nabla \equiv \frac{\partial}{\partial \vec{r}}$.

The governing equations can now be linearized under the assumption that the primed quantities and their derivatives are small. In terms of the nondimensional variables, the linearized governing equations can be written as

Mass

$$\frac{\partial \rho'}{\partial \tau} + \nabla \cdot \vec{V}' = 0 \qquad (2.3.7)$$

$$\frac{\partial \rho'_p}{\partial \tau} + \nabla \cdot \vec{V}'_p = 0 \qquad (2.3.8)$$

Momentum

$$\frac{\partial \vec{V}'}{\partial \tau} = - \frac{\nabla P'}{\gamma} + \nabla(\nabla \cdot \vec{V}') - \frac{\nu_o}{\tilde{\nu}_o} \nabla \times (\nabla \times \vec{V}')$$

$$+ \frac{\kappa_o}{\tau_v^*}(\vec{V}'_p - \vec{V}') \tag{2.3.9}$$

$$\frac{\partial \vec{V}'_p}{\partial \tau} = - \frac{1}{\tau_v^*} (\vec{V}'_p - \vec{V}') \tag{2.3.10}$$

Energy

$$\frac{\partial T'}{\partial \tau} = \frac{\gamma-1}{\gamma} \frac{\partial p'}{\partial \tau} + \frac{1}{P_r^*} \nabla^2 T' + \frac{2}{3} \frac{\kappa_o}{P_{ro} \tau_v^*} (T'_p - T') \tag{2.3.11}$$

$$\frac{\partial T'_p}{\partial \tau} = - \frac{2}{3} \frac{\alpha}{P_{ro} \tau_v^*} (T'_p - T') \tag{2.3.12}$$

State

$$P' = \rho' + T' \tag{2.3.13}$$

where

$$P_{ro} \equiv \mu_o C_{po}/k_o$$

$$P_r^* \equiv \tilde{\mu}_o C_{po}/k_o = \frac{4}{3} P_{ro}$$

$$\tau_v^* \equiv \tau_{vo}/\tau_c \tag{2.3.14}$$

$$\alpha \equiv C_{po}/C$$

$$\kappa_o \equiv \rho_{po}/\rho_o$$

Eqs. (2.3.7) - (2.3.13) constitute a closed set for P', ρ', ρ'_p , $\vec{V}'$, $\vec{V}'_p$, T', and T'_p .

To obtain an idea of the numerical values of τ_{V_o}, τ_c, L_c, and n_o, consider air with $\mu_o = 1.912 \times 10^{-4}$ gm - cm^{-1} - sec^{-1}, $\rho_o = 1137.35 \times 10^{-6}$ gm - cm^{-3}, and $a_o = 35356.8$ cm - sec^{-1}. Then for spherical particles of radius $\bar{\sigma} = 1.00\,\mu$, $\rho_s = 2.0$ gm - cm^{-3} , and $\kappa_o = 0.01$, we have the values as shown in Table 1.

TABLE 1

NUMERICAL VALUES OF SOME PARAMETERS IN

THE AMBIENT GAS-PARTICLE MIXTURE

$\tau_{V_o} = \frac{2}{9} \frac{\rho_s}{\mu_o} \bar{\sigma}^2$	2.325×10^{-5} sec
$\tau_c = \frac{\tilde{\nu}_o}{a_o^2} = \frac{4}{3} \frac{\mu_o}{\rho_o a_o^2}$	1.000×10^{-10} sec
$L_c = \frac{\tilde{\nu}_o}{a_o} = \frac{4}{3} \frac{\mu_o}{\rho_o a_o}$	6.338×10^{-6} cm
$n_o = \frac{\rho_{p_o}}{m} = \frac{3\kappa_o \rho_o}{4\pi\bar{\sigma}^3 \rho_s}$	1.358×10^{6} cm^{-3}

where n_o is the number of particles in a unit volume of ambient mixture.

2.4 Reduced Equations

The linearized problem can be simplified further by restricting ourselves to flows that are irrotational by virture of symmetry; $\nabla \times \vec{V}' = 0$, $\nabla \times \vec{V}'_p = 0$ identically. For these irrotational flows, we write

$$\vec{V}' = \nabla \phi \tag{2.4.1}$$

$$\vec{V}'_p = \nabla \phi_p \tag{2.4.2}$$

where ϕ and ϕ_p are the gas and particle perturbation velocity potentials. The momentum equations (2.3.9) and (2.3.10) can now be integrated once and the time-function of integration can be set into zero without loss of generality. The governing linearized system of equations then becomes

Mass

$$\frac{\partial \rho'}{\partial \tau} + \nabla^2 \phi = 0 \tag{2.4.3}$$

$$\frac{\partial \rho'_p}{\partial \tau} + \nabla^2 \phi_p = 0 \tag{2.4.4}$$

Momentum

$$\phi_\tau = - \frac{P'}{\gamma} + \nabla^2 \phi + \frac{\kappa_o}{\tau_V^*} (\phi_p - \phi) \tag{2.4.5}$$

$$\phi_{p\tau} = - \frac{1}{\tau_V^*} (\phi_p - \phi) \tag{2.4.6}$$

Energy

$$\frac{\partial T'}{\partial \tau} = \frac{\gamma - 1}{\gamma} \frac{\partial P'}{\partial \tau} + \frac{1}{P_{ro}^*} \nabla^2 T' + \frac{2}{3} \frac{\kappa_o}{P_{ro} \tau_V^*} (T'_p - T') \tag{2.4.7}$$

$$\frac{\partial T'_p}{\partial \tau} = - \frac{2}{3} \frac{\alpha}{P_{ro} \tau_V^*} (T'_p - T') \tag{2.4.8}$$

State

$$P' = \rho' + T' \tag{2.4.9}$$

The equations (2.4.3) - (2.4.9) can be combined by elimination into a single seventh-order equation for ϕ:

$$\begin{aligned} &C_1 \nabla^4 \phi_{\tau\tau\tau} - \nabla^2 (C_2 \phi_{\tau\tau} - C_3 \nabla^2 \phi)_{\tau\tau} + C_4 \phi_{\tau\tau\tau\tau\tau} \\ &- \nabla^2 (C_5 \phi_{\tau\tau} - C_6 \nabla^2 \phi)_\tau + C_7 \phi_{\tau\tau\tau\tau} \\ &- \nabla^2 (C_8 \phi_{\tau\tau} - C_9 \nabla^2 \phi) + (C_{10} \phi_{\tau\tau} - C_{11} \nabla^2 \phi)_\tau = 0 \end{aligned} \tag{2.4.10}$$

where

$$C_1 = \frac{\gamma \tau_V^* \tau_T^*}{\alpha} \tag{2.4.11}$$

$$C_2 = \frac{\tau_V^* \tau_T^*}{\alpha} (\gamma + P_r^*) \tag{2.4.12}$$

$$C_3 = \frac{1}{\alpha} (\tau_V^* \tau_T^* + \gamma\tau_T^* + \alpha\gamma\tau_V^*) \tag{2.4.13}$$

$$C_4 = \frac{1}{\alpha} (\tau_V^* \tau_T^* P_r^*) \tag{2.4.14}$$

$$C_5 = \frac{1}{\alpha} \{\tau_V^* \tau_T^* P_r^* + \tau_T^*[P_r^* + \gamma(1+\kappa_0)] + \tau_V^*[(\gamma\kappa_0 + \alpha)P_r^* + \alpha\gamma]\} \tag{2.4.15}$$

$$C_6 = \frac{1}{\alpha}(\alpha\tau_V^* + \tau_T^* + \alpha\gamma) \tag{2.4.16}$$

$$C_7 = \frac{P_r^*}{\alpha} [\tau_T^* (1+\kappa_0) + \tau_V^* (\alpha + \gamma\kappa_0)] \tag{2.4.17}$$

$$C_8 = \frac{1}{\alpha} [\tau_T^* P_r^{'*} + \tau_v^* P_r^*(\alpha + \kappa_o) + P_r^*(\alpha + \gamma\kappa_o) + \alpha\gamma(1 + \kappa_o)] \tag{2.4.18}$$

$$C_9 = 1 \tag{2.4.19}$$

$$C_{10} = \frac{P_r^*}{\alpha} (1 + \kappa_o) (\alpha + \gamma\kappa_o) \tag{2.4.20}$$

$$C_{11} = \frac{P_r^*}{\alpha} (\alpha + \kappa_o) \tag{2.4.21}$$

Eq. (2.4.10) is the key equation in the present analysis, which is an extension of the linear wave equations for pure gases (i.e. $\kappa_o = 0$) studied by Lick (1967), Rasmussen and Lake (1973):

$$\gamma \nabla^4 \phi_\tau - \nabla^2 [(\gamma + P_r^*)\phi_{\tau\tau} - \nabla^2\phi]$$

$$+ P_r^* (\phi_{\tau\tau} - \nabla^2 \phi)_\tau = 0 \tag{2.4.22}$$

The last two terms in Eq. (2.4.10) are the lowest order terms and are related to the classical wave equation associated with inviscid acoustics. The higher-order terms reveal the combined effects due to the viscosity and thermal conductivity of the gas, and the presence of the particles.

2.5 Initial and Boundary Conditions

We now consider a semi-infinite particulate suspension, initially in equilibrium, that is disturbed at a given instant $\tau = 0$, by an impulsive temperature increase at the boundary end wall at $x = 0$, as shown in Fig. 2.

The initial conditions for the perturbation variables are

$$
\begin{aligned}
P'\ (x, 0) &= 0 \\
\rho'\ (x, 0) &= 0 \\
\rho'_p\ (x, 0) &= 0 \\
T'\ (x, 0) &= 0 \\
T'_p\ (x, 0) &= 0 \\
\phi\ (x, 0) &= 0 \\
\phi_p\ (x, 0) &= 0 \\
\phi_x\ (x, 0) &= 0 \\
\phi_{px}\ (x, 0) &= 0
\end{aligned}
\tag{2.5.1}
$$

The potential can be determined to be within an arbitrary constant, which is chosen such that ϕ and ϕ_p vanish at t = 0.

The boundary conditions are that the perturbations must vanish at infinity, and that the velocities $u' = \phi_x$ and $u_p = \phi_{px}$ must vanish at x = 0 for all times, i.e.,

$$u' = \phi_x\ (0, \tau) = 0 \tag{2.5.2}$$

$$u'_p = \phi_{px}\ (0, \tau) = 0 \tag{2.5.3}$$

The problem before us is to solve the equation

$$
\begin{aligned}
&C_1 \frac{\partial^4 \phi_{\tau\tau\tau}}{\partial x^4} - \frac{\partial^2}{\partial x^2} (C_2\ \phi_{\tau\tau} - C_3 \frac{\partial^2 \phi}{\partial x^2})_{\tau\tau} + C_4\ \phi_{\tau\tau\tau\tau\tau} \\
&- \frac{\partial^2}{\partial x^2} (C_5\ \phi_{\tau\tau} - C_6 \frac{\partial^2 \phi}{\partial x^2})_\tau + C_7\ \phi_{\tau\tau\tau\tau} \\
&- \frac{\partial^2}{\partial x^2} (C_8\ \phi_{\tau\tau} - C_9 \frac{\partial^2 \phi}{\partial x^2}) + (C_{10}\ \phi_{\tau\tau} - C_{11} \frac{\partial^2 \phi}{\partial x^2})_\tau = 0
\end{aligned}
\tag{2.5.4}
$$

subject to the above specified initial and boundary conditions.

CHAPTER 3

LINEAR SOLUTION

3.1 Solution by Laplace Transforms

The Laplace transform of a function f(x,t) with respect to t is defined as

$$L\,\{f(x,t)\} = \bar{f}(x,s) = \int_0^\infty e^{-st}\, f(x,t)\, dt \qquad (3.1.1)$$

where s is a transform variable. A solution can be obtained by transforming Eq. (2.5.4) by means of (3.1.1). Before this is done, however, it is necessary to evaluate the initial value of the time derivative of ϕ that are involved in the transform of Eq. (2.5.4). It can be shown, from the momentum equations (2.4.5) and (2.4.6), that ϕ_τ and higher time derivative of ϕ vanish at $\tau = 0$, that is,

$$\begin{aligned} \phi_\tau\,(x,0) &= 0 \quad , \quad \phi_{\tau\tau}\,(x,0) = 0 \\ \phi_{\tau\tau\tau}\,(x,0) &= 0 \quad , \quad \phi_{\tau\tau\tau\tau}\,(x,0) = 0 \quad \text{etc.} \end{aligned} \qquad (3.1.2)$$

With these conditions, the Laplace transform of Eq. (2.5.4) gives

$$\frac{d^4\bar{\phi}}{dx^4} + A_1\,\frac{d^2\bar{\phi}}{dx^2} + A_2\,\bar{\phi} = 0 \qquad (3.1.3)$$

Where $\bar{\phi}$ (x,s) is the Laplace transform of ϕ (x,τ), and

$$A_1 = \frac{-\,(C_2\, s^4 + C_5\, s^3 + C_8\, s^2 + C_{11}\, s)}{(C_1\, s^3 + C_3\, s^2 + C_6\, s + C_9)} \qquad (3.1.4)$$

$$A_2 = \frac{(C_4\ s^5 + C_7\ s^4 + C_{10}\ s^3)}{(C_1\ s^3 + C_3\ s^2 + C_6\ s + C_9)} \tag{3.1.5}$$

Eq. (3.1.3) is a fourth-order homogeneous ordinary differential equation. The general solution for $\bar{\phi}$ is

$$\bar{\phi}\ (x,\ s) = B_1\ e^{-\lambda_1 x} + B_2\ e^{-\lambda_2\ x} + B_3\ e^{\lambda_1 x} + B_4\ e^{\lambda_2\ x} \tag{3.1.6}$$

where B_1, B_2, B_3, and B_4 are constants of integration and

$$\lambda_{1,2} = \sqrt{\frac{-A_1 \pm \sqrt{A_1^2 - 4\ A_2}}{2}} \tag{3.1.7}$$

in which the plus and minus signs refer to λ_1 and λ_2, respectively. Since $\bar{\phi}$ vanishes at $x \to \infty$, B_3 and B_4 must be zero. Thus, we have

$$\bar{\phi}\ (x,\ s) = B_1\ e^{-\lambda_1\ x} + B_2\ e^{-\lambda_2\ x} \tag{3.1.8}$$

Further, by the aid of the boundary condition

$$\bar{\phi}_x\ (0,\ s) = 0$$

we find

$$B_2 = -\ \frac{\lambda_1}{\lambda_2} B_1 \tag{3.1.9}$$

Substitution of Eq. (3.1.9) into Eq. (3.1.8) results in

$$\bar{\phi}\ (x,\ s) = B_1\ (e^{-\lambda_1\ x} - \frac{\lambda_1}{\lambda_2}\ e^{-\lambda_2\ x}) \tag{3.1.10}$$

Hence, there remains only one constant for the solution.

The transformed flow variables can be found in terms of $\bar{\phi}$ by means of the transform of equations (2.4.3) - (2.4.8):

$$\bar{u}' \, (x, s) = \bar{\phi}_x \tag{3.1.11}$$

$$\bar{u}'_p \, (x, s) = \bar{\phi}_{px} = \frac{\bar{\phi}_x}{1 + s \, \tau_v^*} \tag{3.1.12}$$

$$\bar{T}' \, (x, s) = (\gamma + \frac{1}{s}) \, \bar{\phi}_{xx} - \gamma s \, (1 + \frac{\kappa_o}{1 + s \, \tau_v^*}) \bar{\phi} \tag{3.1.13}$$

$$\bar{T}'_p \, (x, s) = \frac{\xi}{s + \xi} \, \bar{T}' \tag{3.1.14}$$

$$\bar{\rho}'(x, s) = - \frac{1}{s} \, \bar{\phi}_{xx} \tag{3.1.15}$$

$$\bar{\rho}'_p \, (x, s) = - \frac{\bar{\phi}_{xx}}{s \, (1 + s \, \tau_v^*)} \tag{3.1.16}$$

$$\bar{P}' \, (x, s) = \gamma \bar{\phi}_{xx} - \gamma s \, (1 + \frac{\kappa_o}{1 + s \, \tau_v^*}) \, \bar{\phi} \tag{3.1.17}$$

where

$$\xi = \frac{2}{3} \frac{\alpha}{P_{ro} \, \tau_v^*} \tag{3.1.18}$$

$$\bar{\phi}_x = - B_1 \, \lambda_1 \, (e^{-\lambda_1 x} - e^{-\lambda_2 x}) \tag{3.1.19}$$

$$\bar{\phi}_{xx} = B_1 \lambda_1^2 \, (e^{-\lambda_1 x} - \frac{\lambda_2}{\lambda_1} e^{-\lambda_2 x}) \tag{3.1.20}$$

The constant B_1 can be determined by the boundary condition for the temperature at $x = 0$, which is given by

$$\bar{T}' \, (0, s) = \zeta / s^{3/2} \tag{3.1.21}$$

or

$$T'(0, \tau) = 2\zeta\sqrt{\frac{\tau}{\pi}} \tag{3.1.22}$$

Here, ζ is a small dimensionless parameter determined from Eq. (3.1.22) for some given value of τ such that $T'(0, \tau) << 1$. Substituting Eqs. (3.1.10) and (3.1.20) into Eq. (3.1.13) and evaluating the value at $x = 0$, we obtain

$$B_1 = \frac{\bar{T}'(0, s)}{(\lambda_1 - \lambda_2)\left[(\lambda + \frac{1}{s})\lambda_1 + \frac{\gamma s}{\lambda_2}\left(1 + \frac{\kappa_0}{1 + s\,\tau_v^*}\right)\right]} \tag{3.1.23}$$

With these results the solution for the transformed problem is now complete.

The above transformed expressions cannot be inverted exactly in closed forms. However, asymptotic expansions valid for large time, $\tau\to\infty$, can be obtained.

3.2 Solution for Large Time

The solution for large time, $\tau\to\infty$, can be found by expansion of the Laplace transform for small s. In the limit $s\to 0$, the appropriate expansions for λ_1 and λ_2, defined in Eq. (3.1.7), can be written as

$$\lambda_1 = a_1\sqrt{s}\,[1 + b_1 s + 0(s^2)] \tag{3.2.1}$$

$$\lambda_2 = a_2 s\,[1 + b_2 s + 0(s^2)] \tag{3.2.2}$$

where

$$a_1 = \sqrt{\frac{C_{11}}{C_9}} \tag{3.2.3}$$

$$b_1 = \frac{C_6\, C_{11}^2 - C_8\, C_9\, C_{11} + C_9^2\, C_{10}}{-2\, C_9\, C_{11}^2} \tag{3.2.4}$$

$$a_2 = \sqrt{\frac{C_{10}}{C_{11}}} \tag{3.2.5}$$

$$b_2 = \frac{C_9\, C_{10}^2 - C_8\, C_{10}\, C_{11} + C_7\, C_{11}^2}{2\, C_{10}\, C_{11}^2} \tag{3.2.6}$$

It is noted that, to the lowest order, λ_1 varies like $\sqrt{s}$ and hence is associated with diffusion; whereas λ_2 varies like s and hence leads to wave behavior. Furthermore, since a_1, b_1, a_2, and b_2 are functions of the coefficients of the lower-order terms of Eq. (2.5.4), it follows that, for large time, the lower-order terms of Eq. (2.5.4) dominate the motion of the suspension. The general features of the solution for large time can be deduced directly from Eq. (2.5.4). Initially the motion is diffusive in character as can be seen from the sixth and seventh terms. A wave which travels at the speed $(C_9/C_8)^{1/2}$ tends to form, as shown by the eighth and ninth terms. This wave decays and finally a wave propagating at the speed $(C_{11}/C_{10})^{1/2}$ is formed, as shown by the last two terms. The last wave then diffuses out due to the combined effects of viscosity, thermal conductivity, and particles.

Substitution for λ_1 and λ_2 from Eqs. (3.2.1) and (3.2.2) into Eq. (3.1.23) yields

$$B_1 = \frac{\bar{T}'(0,s)}{a_1^2}\,[1 + 0\,(\sqrt{s})]$$

or

$$B_1 = \frac{\zeta}{a_1^2 \, s^{3/2}} [1 + 0 (\sqrt{s})] \tag{3.2.7}$$

With the above asymptotic expansions for λ_1, λ_2, and B_1, we obtain, from Eqs. (3.1.10), (3.1.19), and (3.1.20), the following expressions for $\bar{\phi}$, $\bar{\phi}_x$, and $\bar{\phi}_{xx}$:

$$\bar{\phi}\ (x, s) \sim \frac{\zeta}{a_1^2} \left(\frac{e^{-\lambda_1 x}}{s^{3/2}} - \frac{a_1}{a_2} \frac{e^{-\lambda_2 x}}{s^2}\right) \tag{3.2.8}$$

$$\bar{\phi}_x\ (x, s) \sim - \frac{\zeta}{a_1} \left(\frac{e^{-\lambda_1 x}}{s} - \frac{e^{-\lambda_2 x}}{s}\right) \tag{3.2.8}$$

$$\bar{\phi}_{xx}\ (x, s) \sim \zeta \left(\frac{e^{-\lambda_1 x}}{s^{1/2}} - \frac{a_2}{a_1} e^{-\lambda_2 x}\right) \tag{3.2.10}$$

As a result, the Eqs. (3.1.11) - (3.1.17) can be rewritten asymptotically to the lowest order as

$$\frac{\bar{u}'}{\zeta} \sim - \frac{1}{a_1} \left(\frac{e^{-\lambda_1 x}}{s} - \frac{e^{-\lambda_2 x}}{s}\right) \tag{3.2.11}$$

$$\frac{\bar{u}_p'}{\zeta} \sim \frac{\bar{u}'}{\zeta} + \frac{1}{a_1} \left(\frac{e^{-\lambda_1 x}}{s + \eta} - \frac{e^{-\lambda_2 x}}{s + \eta}\right) \tag{3.2.12}$$

$$\frac{\bar{T}'}{\zeta} \sim \frac{e^{-\lambda_1 x}}{s^{3/2}} - D \frac{e^{-\lambda_2 x}}{s} \tag{3.2.13}$$

$$\frac{\bar{T}_p'}{\zeta} \sim \frac{\bar{T}'}{\zeta} - \frac{e^{-\lambda_1 x}}{(s + \xi) \sqrt{s}} + D \frac{e^{-\lambda_2 x}}{s + \xi} \tag{3.2.14}$$

$$\frac{\bar{\rho}'}{\zeta} \sim \frac{-e^{-\lambda_1 x}}{s^{3/2}} + \frac{a_2}{a_1} \frac{e^{-\lambda_2 x}}{s} \tag{3.2.15}$$

$$\frac{\bar{\rho}_p'}{\zeta} \sim \frac{\bar{\rho}'}{\zeta} + \frac{e^{-\lambda_1 x}}{(s + \eta)\sqrt{s}} - \frac{a_2}{a_1}\frac{e^{-\lambda_2 x}}{s + \eta} \tag{3.2.16}$$

$$\frac{\bar{p}'}{\zeta} \sim [\gamma - \frac{\gamma(1 + \kappa_0)}{a_1^2}] \frac{e^{-\lambda_1 x}}{\sqrt{s}} + \frac{\gamma(1 + \kappa_0)}{a_1 a_2} \frac{e^{-\lambda_2 x}}{s} \tag{3.2.17}$$

where

$$D = \frac{1}{a_1 a_2} [a_2^2 - \gamma(1 + \kappa_0)] \tag{3.2.18}$$

$$\eta = \frac{1}{\tau_v^*} \tag{3.2.19}$$

and again

$$\xi = \frac{2}{3} \frac{\alpha}{P_{ro} \tau_v^*}$$

$$\tau_v^* = \frac{\tau_{v_0}}{\tau_c}, \quad \tau_c = \frac{\tilde{\nu}_0}{a_0^2} \tag{3.2.20}$$

It can be easily seen from the above expressions that $\bar{u}_p'$, $\bar{T}_p'$, $\bar{\rho}_p'$ $= 0$ when $\tau_v^* \to \infty$, and that $\bar{u}_p' = \bar{u}'$, $\bar{T}_p' = \bar{T}'$, $\bar{\rho}_p' = \bar{\rho}'$ when $\tau_v^* = 0$. This is consistent with our discussion for the limiting cases when $\tau_v^* \to \infty$ (frozen flow) and $\tau_v^* = 0$ (equilibrium flow) in Sec. 2.2.

The inversion of the terms involving λ_1 can be accomplished by means of standard Laplace-transform tables. On the other hand, the terms involving λ_2 can be inverted by using an alternative asymptotic form for λ_2 (Rasmussen 1975):

$$\lambda_2 = 2\, a_2 b\, (\sqrt{s + b^2} - b) + 0\, (s^3) \tag{3.2.21}$$

where

$$b = \sqrt{\frac{-1}{4b_2}} \qquad b_2 < 0 \tag{3.2.22}$$

here a_2 and b being the parameters pertinent to the wave front. With these considerations in mind, the large-time approximations for the flow variables are obtained upon inversion of Eqs. (3.2.11) - (3.2.17).

3.2.1 Velocity Field for Large Time

The large time approximation for the velocity field is

$$\frac{u'}{\zeta} \sim -\frac{1}{a_1} [G_2\, (x, \tau; a_1) - H_1\, (x, \tau; a_2, b)] \tag{3.2.23}$$

$$\frac{u_p'}{\zeta} \sim \frac{u'}{\zeta} + \frac{1}{a_1} [G_4\, (x, \tau; a_1, \eta) - H_2\, (x, \tau; a_2, b, \eta)] \tag{3.2.24}$$

where

$$G_2\, (x, \tau; a_1) = L^{-1} \{\frac{e^{-\lambda_1 x}}{s}\} \quad , \quad \lambda_1 \sim a_1 \sqrt{s}$$

$$= \mathrm{erfc}\, (\frac{a_1 x}{2\sqrt{\tau}})$$

$$\sim \frac{1}{\sqrt{\pi}} e^{-z^2} [\frac{1}{z} - \frac{1}{2z^3} + \frac{3}{4z^5} - \ldots],$$

$$z = \frac{a_1 x}{2\sqrt{\tau}} \to \infty \tag{3.2.25}$$

$$H_1\, (x, \tau; a_2, b) = L^{-1} \{\frac{e^{-\lambda_2 x}}{s}\}, \lambda_2 \sim 2a_2 b\, (\sqrt{s + b^2} - b)$$

$$= \frac{1}{2} \{\mathrm{erfc} [\frac{b (a_2 x - \tau)}{\sqrt{\tau}}] + e^{4a_2 b^2 x} \mathrm{erfc} [\frac{b (a_2 x + \tau)}{\sqrt{\tau}}]\} \tag{3.2.26}$$

$$G_4 (x, \tau; a_1, \eta) = L^{-1} \{\frac{e^{-\lambda_1 x}}{s + \eta}\}$$

$$= \frac{1}{2} e^{-\eta\tau} [e^{-ia_1 x\sqrt{\eta}} \mathrm{erfc} (\frac{a_1 x}{2\sqrt{\tau}} - i \sqrt{\eta\tau})$$

$$+ e^{ia_1 x\sqrt{\eta}} \mathrm{erfc} (\frac{a_1 x}{2\sqrt{\tau}} + i \sqrt{\eta\tau})] , \eta = \frac{1}{\tau_v^*} \tag{3.2.27}$$

$$H_2 (x, \tau; a_2, b,\eta) = L^{-1} \{\frac{e^{-\lambda_2 x}}{s + \eta}\}$$

$$= \frac{1}{2} e^{2a_2 b^2 x - \eta\tau} \{e^{-2a_2 b^2 x \sqrt{1-(\eta/b^2)}} \mathrm{erfc} [\frac{a_2 bx}{\sqrt{\tau}} - b\sqrt{\tau} \sqrt{1-(\eta/b^2)}]$$

$$+ e^{2a_2 b^2 \sqrt{1-(\eta/b^2)}} \mathrm{erfc} [\frac{a_2 bx}{\sqrt{\tau}} + b\sqrt{\tau}\sqrt{1 - (\eta/b^2)}]\} \tag{3.2.28}$$

The features of the solution for the gas velocity distribution can be deduced from Eq. (3.2.23). Initially the motion is diffusive in nature as can be seen from the first term $G_2 (x, \tau; a_1)$. Far away from the wall $G_2 (x, \tau; a_1)$ dies out at least like $z^{-1}e^{-z^2}$, where $z = a_1 x/2\sqrt{\tau}$, and a wave traveling at the speed $(a_2)^{-1}$ is formed as shown by the second term $H_1 (x, \tau; a_2, b)$, which corresponds to a compression wave traveling to the right and an expansion wave to the left.

By combining Eqs. (3.2.5), (2.4.20), and (2.4.21), the wave speed can be written as

$$V_w' = \frac{1}{a_2} = \sqrt{\frac{\alpha + \kappa_0}{(1 + \kappa_0) (\alpha + \gamma\kappa_0)}} \tag{3.2.29}$$

or, for the dimensional speed,

$$V_w = a_o V_w' = a_o\sqrt{\frac{\alpha + \kappa_o}{(1 + \kappa_o)(\alpha + \gamma\kappa_o)}} \tag{3.2.30}$$

It can be readily seen from this equation that $V_w = a_o$ as $\kappa_o = 0$, i.e., in a pure gas the wave propagates at the isentropic speed of sound. It can also be seen from Eq. (3.2.30) that V_w decreases as κ_o increases. Note that $\kappa_o = \rho_{po}/\rho_o = n_o m/\rho_o$. Thus, the wave speed in a gas-particle mixture decreases when the number density n_o of the particles increases. Moreover, Eq. (3.2.30) shows that the wave speed is not affected by particle size; but, as will be seen later, the particle size does affect the wave thickness. The variation of the wave speed V_w' with κ_o is shown in Fig. 3.

To observe the behavior of the decay of the wave, let us define the thickness of the wave front as

$$\delta \equiv \frac{(\Delta H_1)_{max}}{\left(\frac{\partial H_1}{\partial x}\right)_{max}} \tag{3.2.31}$$

From this, we obtain

$$\delta = \frac{\sqrt{\pi\tau}}{2a_2 b} \tag{3.2.32}$$

or, in terms of the physical quantities,

$$\bar{\delta} = \frac{\sqrt{\pi \tilde{\nu}_o t}}{2a_2 b} \tag{3.2.33}$$

Thus, in the linearized theory the wave front spreads out like $\sqrt{\tilde{\nu}_o t}$. The effect of the thermal conductivity of the gas comes into play through Prandtl number $P_r^* = \frac{C_{p_o} \rho_o \tilde{\nu}_o}{k_o}$, and is then simply to diffuse the wave with a diffusivity $k_o/C_{po}\,\rho_o$ in the same manner as viscosity $\tilde{\nu}_o$. It is noted that

$$a_2 = f_n\,(\kappa_o, \alpha) \tag{3.2.34}$$

$$b = f_n\,(\kappa_o, \tau_v^*, P_r^*, \alpha) \tag{3.2.35}$$

and

$$\tau_v^* = \frac{\tau_{vo}}{\tau_c} = \frac{1}{\tau_c}\,(\frac{2}{9}\,\frac{\rho_s}{\mu_o})\,\bar{\sigma}^2 \tag{3.2.36}$$

where $\bar{\sigma}$ is the radius of the particles. Hence the effect of particle size on the wave thickness is reflected through the factor b. It can be shown that b decreases as $\bar{\sigma}$ increases. Therefore, from Eq. (3.2.33), <u>$\bar{\delta}$ increases when the particles become larger</u>. It can also be shown that a_2b decreases as κ_o $(= \frac{n_o m}{\rho_o})$ increases. Consequently, <u>$\bar{\delta}$ increases when the number density of the particles increases.</u>

The physical explanation of the diffusion of the wave front due to particle effect seems to be that the relative motion of the dust particles and the gas will dissipate energy because of the drag between the two phases. When the particles become larger, or the number density of the particles increases, more energy dissipation will be generated. And this causes the further decay of the wave.

It is interesting to note that the viscosity of a suspension of small particles is higher than that of a pure gas. Einstein (1905), in his studies on the theory of Brownian motions, found that the viscosity of a suspension of small, solid, spherical particles should be increased by a factor proportional to the total volume of the spheres suspended in a unit volume of the mixture. Einstein's formula can be written as

$$\mu_s = \mu_0 \, (1 + 2.5 \, \tilde{\Phi}_0) \tag{3.2.37}$$

or

$$\mu_s = \mu_0 \, (1 + 2.5 \, \kappa_0 \, \frac{\rho_0}{\rho_s}) \tag{3.2.38}$$

where μ_s is the viscosity of the suspension, μ_0 the viscosity of the ambient pure gas, and $\tilde{\Phi}_0$ the volume fraction of the particles in the ambient mixture. Eq. (3.2.38) reveals that the viscosity of the gas-particle mixture increases when the density ratio κ_0 increases. It follows that the energy associated with wave will decay faster in a suspension than in a pure gas. Einstein's theory greatly enhances the basic understanding of the present subject.

The effect of the particle size on the wave front is shown in Fig. 4 in which H_1 $(x, \tau; a_2, b)$ is plotted as a function of x for various values of particle radius $\bar{\sigma}$; while the effect of κ_0 on the wave front is illustrated in Fig. 5 where various values of τ have been examined.

The velocity distribution for the particle phase appears to be more complicated than that for the gas phase in that a certain relaxation time is needed for the particles to follow the gas. This is simply because a dust particle in the gas has a much larger inertia than the equivalent volume of gas and will not therefore participate as readily in the velocity fluctuations.

In order to have a physical insight into the field properties of the particle phase, it is necessary to further simplify the functions G_4 $(x, \tau; a_1, \eta)$ and H_2 $(x, \tau; a_2, b, \eta)$. Recall that

$\eta = 1/\tau_v^*$. In the usual case, τ_v^* is very large, i.e., η is very small (for example, $\tau_v^* = 2324.71$ and $\eta = 0.00043$ for $\bar{\sigma} = 0.1\ \mu$). The variation of τ_v^* with $\bar{\sigma}$ is shown in Table 2.

TABLE 2

VARIATION OF τ_v^* WITH $\bar{\sigma}$

$\bar{\sigma}$ (micron)	τ_v^*	$\bar{\sigma}$ (micron)	τ_v^*
0.1	2.32×10^3	0.6	83.56×10^3
0.2	9.30×10^3	0.7	113.88×10^3
0.3	20.92×10^3	0.8	148.74×10^3
0.4	37.18×10^3	0.9	188.24×10^3
0.5	58.10×10^3	1.0	232.47×10^3

This suggests that $G_4\ (x, \tau; a_1, \eta)$ and $H_2\ (x, \tau; a_2, b, \eta)$ may be expanded asymptotically in terms of η. After a series of tedious algebraic manipulations, we obtain

$$G_4\ (x, \tau; a_1, \eta) = G_2\ (x, \tau; a_1) + \eta\ G_4^*\ (x, \tau; a_1, \eta), \quad \eta << 1 \qquad (3.2.39)$$

$$H_2\ (x, \tau; a_2, b, \eta) = H_1\ (x, \tau; a_2, b) + \eta\ H_2^*\ (x, \tau; a_2, b, \eta), \quad \eta << 1 \qquad (3.2.40)$$

where

$$G_4^*\ (x, \tau; a_1, \eta) = G_8\ (x, \tau; a_1) + \eta\ G_9\ (x, \tau; a_1) + 0\ (\eta^2), \quad \eta << 1 \qquad (3.2.41)$$

$$H_2^* (x, \tau; a_2, b, \eta) = H_4 (x, \tau; a_2, b) + \eta\, H_5 (x, \tau; a_2, b) + 0 (\eta^2), \quad \eta << 1 \tag{3.2.42}$$

and where

$$G_8 (x, \tau; a_1) = a_1 x \sqrt{\frac{\tau}{\pi}}\, e^{-a_1^2 x^2/4\tau} - (\frac{1}{2} a_1^2 x^2 + \tau) \operatorname{erfc}(\frac{a_1 x}{2\sqrt{\tau}}) \tag{3.2.43}$$

$$G_9 (x, \tau; a_1) = (\frac{\tau^2}{2} + \frac{1}{2} a_1^2 x^2 \tau + \frac{a_1^4 x^4}{24}) \operatorname{erfc} (\frac{a_1 x}{2\sqrt{\tau}}) - \frac{\tau^2}{3\sqrt{\pi}} [5 (\frac{a_1 x}{2\sqrt{\tau}}) + 2 (\frac{a_1 x}{2\sqrt{\tau}})^3]\, e^{-a_1^2 x^2/4\tau} \tag{3.2.44}$$

$$H_4 (x, \tau; a_2, b) = \frac{1}{2} \{(a_2 x - \tau) \operatorname{erfc} [\frac{b (a_2 x - \tau)}{\sqrt{\tau}}] - (a_2 x + \tau)\, e^{4a_2 b^2 x} \operatorname{erfc} [\frac{b (a_2 x + \tau)}{\sqrt{\tau}}]\} \tag{3.2.45}$$

$$H_5 (x, \tau; a_2, b) = \frac{1}{2} \{ - \frac{a_2 x}{b} \sqrt{\frac{\tau}{\pi}}\, e^{-[\frac{b(a_2 x - \tau)}{\sqrt{\tau}}]^2} + [\frac{1}{2} (a_2 x - \tau)^2 + \frac{a_2 x}{4b^2}] \operatorname{erfc} [\frac{b (a_2 x - \tau)}{\sqrt{\tau}}] + [\frac{1}{2} (a_2 x + \tau)^2 - \frac{a_2 x}{4b^2}]\, e^{4a_2 b^2 x} \operatorname{erfc} [\frac{b(a_2 x + \tau)}{\sqrt{\tau}}] \tag{3.2.46}$$

Upon substituting Eqs. (3.2.39) and (3.2.40) into Eq. (3.2.24), we find that

$$\frac{u_p'}{\zeta} \sim \frac{\eta}{a_1} [G_4^* (x, \tau; a_1, \eta) - H_2^* (x, \tau; a_2, b, \eta)], \quad \eta << 1 \tag{3.2.47}$$

The function G_4^* describes the diffusive-type relaxation of the initial disturbance of the particles caused by the interaction with the gas through drag. As for the gas phase, the disturbance at $x = 0$ is balanced out by the wave term H_2^* $(x, \tau; a_2, b, \eta)$, which also corresponds to a compression wave traveling at the speed $(a_2)^{-1}$ to the right and an expansion wave at the same speed to the left. This therefore satisfies the no-slip boundary condition. In order to fit the zero-velocity boundary condition far away from the wall, there exists a maximum value near the wave front, as shown in Fig. 6 in which the velocities u'/ζ and u'_p/ζ are plotted as functions of x for various values of τ. When η is very small, the lower order terms G_8 $(x, \tau; a_1)$ and H_4 $(x, \tau; a_2, b)$ dominate the velocity distribution, as can be seen from Eqs. (3.2.41) and Eq. (3.2.42). Furthermore, it follows from Eq. (3.2.47) that $u_p' = 0$ when $\eta = 0$, and also from Eq. (3.2.24) that $u_p' = u'$ when $\eta \to \infty$. Figure 6 clearly shows the relaxation process experienced by the particles for a given finite value of η.

The resulting process associated with the initiation of the motion may be described as follows. The temperature jump at the boundary end wall causes a pressure wave that moves away from the wall through the ambient medium. The heat flow from the hot surface raises the adjacent layers of gas to a higher temperature and initiates an expansion of gas. The expansion of gas then causes a motion of the particles due to viscous drag.

3.2.2 Temperature Field for Large Time

The large time approximation for the temperature field is found to be

$$\frac{T'}{\zeta} \sim G_3\ (x,\ \tau;\ a_1) + D\ [G_2\ (x,\ \tau;\ a_1) - H_1\ (x,\ \tau;\ a_2,\ b)] \tag{3.2.48}$$

$$\frac{T_p'}{\zeta} \sim \frac{T'}{\zeta} - \{G_7\ (x,\ \tau;\ a_1,\ \xi) + D\ [G_6\ (x,\ \tau;\ a_1,\ \xi) - H_3(x,\tau;a_2,b,\xi)]\},$$

$$\xi \equiv \frac{2}{3}\ \frac{\alpha}{P_{ro}\ \tau_v^*} \tag{3.2.49}$$

where

$$G_3\ (x,\ \tau;\ a_1) = L^{-1}\ \{\frac{e^{-\lambda_1 x}}{s^{3/2}}\}$$

$$= 2\ \sqrt{\frac{\tau}{\pi}}\ e^{-a_1{}^2 x^2/4\tau} - a_1 x\, \mathrm{erfc}\, (\frac{a_1 x}{2\sqrt{\tau}}) \tag{3.2.50}$$

$$G_6\ (x,\ \tau;\ a_1,\ \xi) = L^{-1}\ \{\frac{e^{-\lambda_1 x}}{s + \xi}\}$$

$$= \frac{1}{2}\ e^{-\xi\tau} \{e^{-ia_1 x\sqrt{\xi}}\ \mathrm{erfc}\ (\frac{a_1 x}{2\sqrt{\tau}} - i\sqrt{\xi\tau})$$

$$+ e^{ia_1 x\sqrt{\xi}}\ \mathrm{erfc}\ (\frac{a_1 x}{2\sqrt{\tau}} + i\sqrt{\xi\tau}\ )\} \tag{3.2.51}$$

$$G_7\ (x,\ \tau;\ a_1,\ \xi) = L^{-1}\ \{\frac{e^{-\lambda_1 x}}{(s + \xi)\ \sqrt{s}}\}$$

$$= \frac{-ie^{-\xi\tau}}{2\sqrt{\xi}}\ [e^{-ia_1 x\sqrt{\xi}}\ \mathrm{erfc}\ (\frac{a_1 x}{2\sqrt{\tau}} - i\sqrt{\xi\tau}\ )$$

$$+ e^{ia_1 x\sqrt{\xi}} \operatorname{erfc} \left(\frac{a_1 x}{2\sqrt{\tau}} + i \sqrt{\xi\tau}\right)] \tag{3.2.52}$$

$$H_3 (x, \tau; a_2, b, \xi) = L^{-1} \{\frac{e^{-\lambda_2 x}}{s + \xi}\}$$

$$= \frac{1}{2} e^{2a_2 b^2 x - \xi\tau} \{e^{-2a_2 b^2 x\sqrt{1-(\xi/b^2)}} \operatorname{erfc}[\frac{a_2 bx}{\sqrt{\tau}} - b\sqrt{\tau} \sqrt{1-(\xi/b^2)}]$$

$$+ e^{2a_2 b^2 x\sqrt{1-(\xi/b^2)}} \operatorname{erfc}[\frac{a_2 bx}{\sqrt{\tau}} + b\sqrt{\tau} \sqrt{1 - (\xi/b^2)}]\} \tag{3.2.53}$$

In Eq. (3.2.48) the functions $G_3 (x, \tau; a_1)$ and $G_2 (x, \tau; a_1)$ are associated with diffusion and describe the relaxation of the initial temperature disturbance of the gas caused by the impulsive temperature increase at the end wall. At $x = 0$, $G_3 (x, \tau; a_1) = 2 \sqrt{\frac{\tau}{\pi}}$, and $G_2 (x, \tau; a_1)$ is balanced out by $H_1 (x, \tau; a_2, b)$. Therefore

$$T' (0, \tau) = 2\zeta \sqrt{\frac{\tau}{\pi}} \tag{3.2.54}$$

which is in agreement with the boundary condition, Eq. (3.1.22). Far away from the wall, G_3 and G_2 die out exponentially, and again a wave described by $H_1 (x, \tau; a_2, b)$ is formed.

Note from Eq. (3.2.54) that, for any given τ, $T' (0, \tau)$ can be made as small as desired by choosing ζ small enough. However, for a given ζ, $T' (0, \tau)$ becomes increasingly large as τ increases. The linear solution will therefore break down sooner or later for a given value of ζ. The range of validity for the linearized theory will be discussed in the later section.

The nature of the temperature distribution for the particle phase can be examined by further simplifying Eq. (3.2.49). Expanding the functions $G_6\ (x, \tau; a_1, \xi)$, $G_7\ (x, \tau; a_1, \xi)$, and $H_3\ (x, \tau; a_2, b, \xi)$ in terms of ξ, we have

$$G_6\ (x, \tau; a_1, \xi) = G_2\ (x, \tau; a_1) + \xi\ G_6^*\ (x, \tau; a_1, \xi),\ \xi \ll 1 \tag{3.2.55}$$

$$G_7\ (x, \tau; a_1, \xi) = G_3\ (x, \tau; a_1) + \xi\ G_7^*\ (x, \tau; a_1, \xi),\ \xi \ll 1 \tag{3.2.56}$$

$$H_3\ (x, \tau; a_2, b, \xi) = H_1\ (x, \tau; a_2, b) + \xi\ H_3^*\ (x, \tau; a_2, b, \xi),\quad \xi \ll 1 \tag{3.2.57}$$

where

$$G_6^*\ (x, \tau; a_1, \xi) = G_8\ (x, \tau; a_1) + \xi\ G_9\ (x, \tau; a_1) + 0\ (\xi^2),\quad \xi \ll 1 \tag{3.2.58}$$

$$G_7^*\ (x, \tau; a_1, \xi) = G_{10}\ (x, \tau; a_1) + \xi\ G_{11}\ (x, \tau; a_1) + 0\ (\xi^2),\quad \xi \ll 1 \tag{3.2.59}$$

$$H_3^*\ (x, \tau; a_2, b, \xi) = H_4\ (x, \tau; a_2, b) + \xi\ H_5\ (x, \tau; a_2, b) + 0(\xi^2),\quad \xi \ll 1 \tag{3.2.60}$$

and where

$$G_{10}\ (x, \tau; a_1) = (a_1\ x\ \tau + \frac{1}{6}\ a_1{}^3\ x^3)\ \mathrm{erfc}\ (\frac{a_1 x}{2\sqrt{\tau}}) - \frac{4}{3\sqrt{\pi}}\ [(\frac{a_1 x}{2\sqrt{\tau}})^2 + 1]\ \tau^{3/2}\ e^{-a_1{}^2 x^2/4\tau} \tag{3.2.61}$$

$$G_{11}\ (x,\tau; a_1) = -\ (\frac{1}{2}\ a_1 x \tau^2 + \frac{1}{6}\ a_1{}^3 x^3 \tau + \frac{1}{120}\ a_1{}^5 x^5)\mathrm{erfc}\ (\frac{a_1 x}{2\sqrt{\tau}})$$

$$- \frac{2\tau^{5/2}}{15\sqrt{\pi}} [4 + 9 \left(\frac{a_1 x}{2\sqrt{\tau}}\right)^2 + 2 \left(\frac{a_1 x}{2\sqrt{\tau}}\right)^4] e^{-a_1^2 x^2/4\tau} \qquad (3.2.62)$$

Substituting Eqs. (3.2.54) - (3.2.56) into Eq. (3.3.48) gives

$$\frac{T_p'}{\zeta} \sim - \xi\{G_7^* (x,\tau;a_1,\xi) + D[G_6^*(x,\tau;a_1,\xi) - H_3^*(x,\tau;a_2,b,\xi)]\},$$
$$\xi << 1 \qquad (3.2.63)$$

The functions G_7^* and G_6^* describe the diffusion of the initial temperature disturbance of the particles. At x = 0, G_6^* and H_3^* cancel each other, and G_7^* varies at least like $-\tau^{3/2}$ for small ξ. When x is large, the diffusion terms die out, and the motion of the particles is dominated by the wave term H_3^*. Finally, a wave propagating at the speed $(a_2)^{-1}$ is formed. The temperature relaxation process of the particles is shown in Fig. 7, where $\frac{T'}{\zeta}$ and $\frac{T_p'}{\zeta}$ are plotted as functions of x for various values of τ.

3.2.3 Density Field for Large Time

The corresponding large-time approximation for the density field is

$$\frac{\rho'}{\zeta} \sim - [G_3 (x, \tau; a_1) + DG_2 (x, \tau; a_1) - \frac{a_2}{a_1} H_1 (x, \tau; a_2, b)] \qquad (3.2.64)$$

$$\frac{\rho_p'}{\zeta} \sim \frac{\rho'}{\zeta} + [G_5 (x, \tau; a_1, \eta) + DG_4 (x, \tau; a_1, \eta)$$
$$- \frac{a_2}{a_1} H_2 (x, \tau; a_2, b, \eta)] \qquad (3.2.65)$$

where

$$G_5\ (x, \tau; a_1, \eta) = L^{-1}\ \{\frac{e^{-\lambda_1 x}}{(s + \eta)\ \sqrt{s}}\}$$

$$= \frac{i\ e^{-\eta\tau}}{2\sqrt{\eta}}\ [e^{-ia_1 x\sqrt{\eta}}\ \text{erfc}\ (\frac{a_1 x}{2\sqrt{\tau}} - i\ \sqrt{\eta\tau}\ )$$

$$- e^{ia_1 x\sqrt{\eta}}\ \text{erfc}\ (\frac{a_1 x}{2\sqrt{\tau}} + i\ \sqrt{\eta\tau})]$$

$$= G_3\ (x, \tau; a_1) + \eta\ G_5^*\ (x, \tau; a_1, \eta), \eta << 1 \tag{3.2.66}$$

where

$$G_5^*\ (x, \tau; a_1, \eta) = G_{10}\ (x, \tau; a_1) + \eta\ G_{11}\ (x, \tau; a_1) + 0\ (\eta^2), \quad \eta << 1 \tag{3.2.67}$$

And again

$$G_4\ (x, \tau; a_1, \eta) = G_2\ (x\ , \tau; a_1) + \eta\ G_4^*\ (x, \tau; a_1, \eta), \quad \eta << 1 \tag{3.2.39}$$

$$H_2\ (x, \tau; a_2, b, \eta) = H_1\ (x, \tau; a_2, b) + \eta\ H_2^*\ (x, \tau; a_2, b, \eta), \quad \eta << 1 \tag{3.2.40}$$

Substituting Eqs. (3.2.66), (3.2.39), and (3.2.40) into Eq. (3.2.65) gives

$$\frac{\rho_p'}{\zeta} \sim \eta\ [G_5^*\ (x, \tau; a_1, \eta) + DG_4^*\ (x, \tau; a_1, \eta) - \frac{a_2}{a_1} H_2^*\ (x, \tau; a_2, b, \eta)]\ , \quad \eta << 1 \tag{3.2.68}$$

The density distributions $\frac{\rho'}{\zeta}$ and $\frac{\rho_p'}{\zeta}$ for both phases are shown in

Fig. 8. Unlike the temperature distributions, the two terms involving G_2 $(x, \tau; a_1)$ and H_1 $(x, \tau; a_2, b)$ (or G_4^* $(x, \tau; a_1, \eta)$ and H_2^* $(x, \tau; a_2, b, \eta)$) do not cancel each other at $x = 0$. It is noted that near the wall the density disturbances are negative. This is due to the fact that the gas near the wall expands because of the sudden increase in temperature at the end wall. When x increases the mixture becomes denser and, like velocity distributions, in order to fit the zero-density boundary condition far away from the wall, there exist maximum values near the wave fronts.

3.2.4 Pressure Field for Large Time

The large time approximation for the pressure field is

$$\frac{p'}{\zeta} \sim \gamma \left[1 - \frac{(1 - \kappa_o)}{a_1^2}\right] G_1 \; (x, \tau; a_1)$$

$$+ \frac{\gamma \; (1 + \kappa_o)}{a_1 \; a_2} \; H_1 \; (x, \tau; a_2, b) \tag{3.2.69}$$

where

$$G_1 \; (x, \tau; a_1) = L^{-1}\left\{ \frac{e^{-\lambda_1 x}}{\sqrt{s}} \right\} = \frac{e^{-a_1^2 x^2/4\tau}}{\sqrt{\pi\tau}} \tag{3.2.70}$$

The function G_1 $(x, \tau; a_1)$ describes the diffusion of the initial pressure disturbance of the gas. The disturbance at $x = 0$ dies out like $\tau^{-1/2}$. Far away from the wall G_1 $(x, \tau; a_1)$ decays exponentially. In fact, the first term of Eq. (3.2.69) is very small. The pressure distribution is therefore dominated by the wave function H_1 $(x, \tau; a_2, b)$, as shown in Fig. 9 in which p'/ζ is plotted as a function of x for various value of τ. It is interesting to note that the effect of H_1

on the pressure distribution is more pronounced for a pure gas with $P_r^* = 1$ (i.e. $P_{ro} = 3/4$). For $\kappa_o = 0$, the parameters a_1 and a_2 become

$$a_1 = P_r^*$$

$$a_2 = 1$$

Thus, Eq. (3.2.69) can be written as

$$\frac{p'}{\zeta} \sim \gamma \left(1 - \frac{1}{P_r^*}\right) G_1 (x, \tau; a_1)$$

$$+ \frac{\gamma}{\sqrt{P_r^*}} H_1 (x, \tau; a_2, b) \tag{3.2.71}$$

It follows that the first term vanishes when $P_r^* = 1$. The pressure distribution for a pure gas with $P_r^* = 1$ is therefore totally dominated, to the lowest order, by the wave function $H_1 (x, \tau; a_2, b)$.

3.3 Limitation of the Linearized Theory

For large time, the linearized theory predicts a wave with a wave thickness increasing indefinitely with time like $\sqrt{\tau}$. It is expected, however, that the solution would be a steady-state wave front with a finite amplitude. The continual spreading of the wave front is an incorrect behavior that arises from the omission of the convective nonlinearities. On the other hand, as mentioned earlier, the boundary condition, Eq. (3.1.22),

$$T' (0, \tau) = 2 \zeta \sqrt{\frac{\tau}{\pi}}$$

shows that, for a given ζ, $T' (0, \tau)$ becomes increasingly large as τ increases. The linear solution is therefore not valid at $\tau \to \infty$, i.e., it is not uniformly valid. Eq. (3.1.22) leads to an estimate of the time at which the linear solution begins to break down. Under the assumption of linearized theory, the primed perturbation variables in Eq. (2.3.1) and their derivatives are all less than order of unity. This means that the linear solution becomes invalid when

$$T' \; (0, \; \tau) = 2 \; \zeta \sqrt{\frac{\tau}{\pi}} = 0 \; (1) \tag{3.3.1}$$

or

$$\tau \; = 0 \; (\zeta^{-2}) \tag{3.3.2}$$

CHAPTER 4

NONLINEAR INTERACTION

For large time, the wave front in linearized theory is dominated by dissipative transport effects. It is known, however, that, for pressure waves in gases, the dissipative and nonlinear effects tend to counteract each other. In order to delineate the balance between the dissipative linear terms and the steepening nonlinear terms, let us isolate the viscosity by means of the dimensionless parameter

$$\varepsilon \equiv \frac{\tilde{\nu}_o}{a_o L} \tag{4.1.1}$$

where L is some characteristic length pertinent to a given problem. The parameter ε is to be regarded as small so that the transport terms can be balanced with the nonlinear terms, which are also small. We then renormalize the variables appearing in Eqs. (2.3.3) and (2.3.4) and define non-dimensional time and distance as

$$\tilde{\tau} \equiv \frac{a_o t}{L} = \frac{a_o^2 t}{\tilde{\nu}_o} = \varepsilon \ \tau \tag{4.1.2}$$

$$\vec{\tilde{r}} \equiv \frac{\vec{\bar{r}}}{L} = \frac{\varepsilon \ a_o \ \vec{\bar{r}}}{\tilde{\nu}_o} = \varepsilon \ \vec{r} \tag{4.1.3}$$

We note that $\nabla = \varepsilon\tilde{\nabla}$, $\vec{V}' = \varepsilon\tilde{\nabla}\phi$, and $V_p' = \varepsilon\tilde{\nabla}\phi_p$. In the new variables, nth-order terms are proportional to ε^n, and we can rewrite Eq. (2.4.10) and display the lowest two orders as

$$(a_e^2 \tilde{\nabla}^2\phi - \phi_{\tilde{\tau}\tilde{\tau}})_{\tilde{\tau}} = \varepsilon[\frac{C_7}{C_{10}}\phi_{\tilde{\tau}\tilde{\tau}\tilde{\tau}\tilde{\tau}} - \tilde{\nabla}^2 (\frac{C_8}{C_{10}}\phi_{\tilde{\tau}\tilde{\tau}} - \frac{C_9}{C_{10}} \tilde{\nabla}^2\phi)]$$

$$+ 0 (\varepsilon^2) \tag{4.1.4}$$

where

$$a_e^2 = \frac{1}{a_2^2} = \frac{C_{11}}{C_{10}} = \frac{\alpha + \kappa_0}{(\alpha + \gamma\kappa_0)(1 + \kappa_0)} \tag{4.1.5}$$

Thus as the viscous transport effects go to zero, the classical wave operator prevails, and the first-order correction for small ε arises because of the next higher-order derivatives. The terms of order ε^2 in Eq. (4.1.4) correspond to the seventh-, sixth-, and fifth-order derivatives in Eq. (2.4.10).

Eq. (4.1.4) can be reduced further if we realize that to the lowest order $\tilde{\nabla}^2\phi = \phi_{\tilde{\tau}\tilde{\tau}}/a_e^2$. Replacing the linear space derivatives on the right-hand side with the equivalent time derivatives then yields, correct to order ε,

$$a_e^2 \tilde{\nabla}^2\phi - \phi_{\tilde{\tau}\tilde{\tau}} = \varepsilon (E_1 \phi_{\tilde{\tau}\tilde{\tau}\tilde{\tau}}) + 0 (\varepsilon^2) \tag{4.1.6}$$

where we have integrated once with respect to $\tilde{\tau}$ and set the function of integration to zero, and where

$$E_1 = \frac{C_7}{C_{10}} - \frac{C_8}{a_e^2 C_{10}} + \frac{C_9}{a_e^4 C_{10}} \tag{4.1.7}$$

Eq. (4.1.6) does not account for the nonlinearities in the problem. To obtain the lowest-order nonlinear correction, we return to the original equations, omit the transport terms for simplicity since they are already accounted for in Eq. (4.1.6), but retain the nonlinear convective terms. We replace the perturbation quantities in Eq. (2.3.1) by $P' = \varepsilon\tilde{P}$, and so on for the other primed variables, and utilize the normalizations Eqs. (4.1.2) and (4.1.3). The governing equations then become

Mass

$$\frac{\partial\tilde{\rho}}{\partial\tilde{\tau}} + \tilde{\nabla}\cdot\vec{\tilde{v}} + \varepsilon\,\tilde{\nabla}\cdot(\tilde{\rho}\,\vec{\tilde{v}}) + 0\,(\varepsilon^2) = 0 \qquad (4.1.8)$$

$$\frac{\partial\tilde{\rho}_p}{\partial\tilde{\tau}} + \tilde{\nabla}\cdot\vec{\tilde{v}}_p + \varepsilon\tilde{\nabla}\cdot(\tilde{\rho}_p\,\vec{\tilde{v}}_p) + 0\,(\varepsilon^2) = 0 \qquad (4.1.9)$$

Momentum

$$\frac{\partial\vec{\tilde{v}}}{\partial\tilde{\tau}} + \kappa_o\frac{\partial\vec{\tilde{V}}_p}{\partial\tilde{\tau}} + \frac{1}{\gamma}\tilde{\nabla}\tilde{P} + \varepsilon\,[\tilde{\rho}\,\frac{\partial\vec{\tilde{v}}}{\partial\tilde{\tau}} + \kappa_o\,\tilde{\rho}_p\,\frac{\partial\vec{\tilde{V}}_p}{\partial\tilde{\tau}} + \tilde{\nabla}\,(\frac{\tilde{v}^2}{2})$$

$$+ \kappa_o\,\tilde{\nabla}\,(\frac{\tilde{V}_p^{\,2}}{2})] + 0\,(\varepsilon^2) = 0 \qquad (4.1.10)$$

$$\vec{\tilde{V}}_p - \vec{\tilde{V}} + \varepsilon\,\tau_v^*\,\frac{\partial\vec{\tilde{V}}_p}{\partial\tilde{\tau}} + 0\,(\varepsilon^2) = 0 \qquad (4.1.11)$$

Energy

$$\frac{\partial\tilde{T}}{\partial\tilde{\tau}} + \frac{\kappa_o}{\alpha}\frac{\partial\tilde{T}_p}{\partial\tilde{\tau}} - \frac{\gamma-1}{\gamma}\frac{\partial\tilde{P}}{\partial\tilde{\tau}} + \varepsilon\{[\tilde{\rho} + (\frac{d\ln C_p}{d\ln T})_o\,\tilde{T}]\,\frac{\partial\tilde{T}}{\partial\tilde{\tau}}$$

$$+ [\tilde{\rho}_p + (\frac{d\ln C}{d\ln T_p})_o\,\tilde{T}_p]\,\frac{\kappa_o}{\alpha}\frac{\partial\tilde{T}}{\partial\tilde{\tau}}\} + 0\,(\varepsilon^2) = 0 \qquad (4.1.12)$$

$$\tilde{T}_p - \tilde{T} + \varepsilon\,\frac{3\,P_{ro}\,\tau_v^*}{2\alpha}\,\frac{\partial\tilde{T}_p}{\partial\tilde{\tau}} + 0\,(\varepsilon^2) = 0 \qquad (4.1.13)$$

State

$$\tilde{P} = \tilde{\rho} + \tilde{T} + \varepsilon\,(\tilde{\rho}\,\tilde{T}) + 0\,(\varepsilon^2) \tag{4.1.14}$$

where

$$\vec{V} = \tilde{\nabla}\phi$$

$$\vec{V}_p = \tilde{\nabla}\phi_p$$

These equations can be manipulated so that the following equation for the perturbation potential is obtained to the lowest two orders:

$$a_e^{\,2}\,\tilde{\nabla}^2\phi - \phi_{\tilde{\tau}\tilde{\tau}} = \varepsilon[E_2\,\phi_{\tilde{\tau}}\,\tilde{\nabla}^2\phi + \frac{\partial}{\partial\tilde{\tau}}\,(\tilde{\nabla}\phi)^2 + E_3\,\phi_{\tilde{\tau}\tilde{\tau}\tilde{\tau}}\,]$$
$$+\,0\,(\varepsilon^2) \tag{4.1.15}$$

where

$$E_2 = \frac{\alpha\,(\gamma - 1)}{\alpha + \gamma\,K_o} - \frac{G\,\alpha^3\,(\gamma - 1)^2}{(\alpha + \kappa_o)(\alpha + \gamma\kappa_o)^2} \tag{4.1.16}$$

$$E_3 = \frac{\kappa_o\,(1 - \gamma)}{(\alpha + \kappa_o)(\alpha + \gamma\kappa_o)}\,\tau_T^* - \frac{\kappa_o}{1 + \kappa_o}\,\tau_v^* \tag{4.1.17}$$

and where

$$G = (\frac{d\,\ln\,C_p}{d\,\ln\,T})_o + \frac{\kappa_o}{\alpha}\,(\frac{d\,\ln\,C}{d\,\ln\,T_p})_o \tag{4.1.18}$$

The first two terms of order ε in Eq. (4.1.15) are nonlinear; while the third term $E_3\,\phi_{\tilde{\tau}\tilde{\tau}\tilde{\tau}}$ is linear and is due to particle effect since it is the only dissipative source in the derivation of Eq. (4.1.15). This can also be easily observed from the fact that $E_3 = 0$ when $\kappa_o = 0$, i.e., for pure gas without particles the third term dissappears. Thus the diffusion term $E_3\,\phi_{\tilde{\tau}\tilde{\tau}\tilde{\tau}}$ is actually contained in

Eq. (4.1.6) where the particle effect was considered together with the other dissipative processes due to viscosity and thermal conductivity. It is also noted that, when C_p = constant and $\kappa_0 = 0$ (i.e., for thermally and colarically perfect gas without particles), $E_3 = 0$ and $E_2 = \gamma - 1$. Hence in this limit Eq. (4.1.15) reduces to the form obtained from classical inviscid potential theory.

By combining Eqs. (4.1.6) and (4.1.15), the full nonlinear equation can be written, to the first order correction, as

$$a_e^2 \tilde{\nabla}^2 \phi - \phi_{\tilde{\tau}\tilde{\tau}} = \varepsilon \left[E_1 \phi_{\tilde{\tau}\tilde{\tau}\tilde{\tau}} + E_2 \phi_{\tilde{\tau}} \tilde{\nabla}^2 \phi + \frac{\partial}{\partial \tilde{\tau}} (\tilde{\nabla}\phi)^2\right] + 0\,(\varepsilon^2) \tag{4.1.19}$$

Note that for pure gases ($\kappa_0 = 0$) with C_p constant Eq. (4.1.19) reads

$$a_e^2 \tilde{\nabla}^2 \phi - \phi_{\tilde{\tau}\tilde{\tau}} = \varepsilon \left[\frac{1 - \gamma - P_r^*}{P_r^*} \phi_{\tilde{\tau}\tilde{\tau}\tilde{\tau}} + (\gamma - 1)\, \phi_{\tilde{\tau}} \tilde{\nabla}^2 \phi + \frac{\partial}{\partial \tilde{\tau}} (\tilde{\nabla}\phi)^2\right] + 0\,(\varepsilon^2) \tag{4.1.20}$$

In Eq. (4.1.19) the second-degree nonlinear terms appear to order ε, as shown, and the third- and higher-degree nonlinearities appear to order ε^2. Therefore, the first-order corrections to inviscid acoustics are the sum of the first-order linear viscous correction and the first-order nonlinear inviscid correction. Our goal now is to examine Eq. (4.1.19) with the terms of ε^2 ignored.

The nature of the interaction of the dissipative and nonlinear mechanisms can be observed by considering a one-dimensional boundary-value problem. A series solution of Eq. (4.1.19) by a straight-forward expansion in powers of ε will lead to secular behavior such that the first-order correction becomes as large as the zeroth-order term when $\tilde{x} = 0\ (\varepsilon^{-1})$. We account for this behavior by the method of multiple scales and introduce new variables defined as

$$\xi^* = \tilde{x} - a_e\, \tilde{\tau}\ ,\ X = \varepsilon\, \tilde{x} \qquad (4.1.21)$$

We further expand ϕ in a series of the form

$$\phi\ \ (\tilde{x}, \tilde{\tau}; \varepsilon) = \phi_o\ (\xi^*, X; \varepsilon) + \varepsilon\, \phi_1\ (\xi^*, X; \varepsilon) + \ldots \qquad (4.1.22)$$

Applying Eqs. (4.1.21) and (4.1.22) to Eq. (4.1.19) leads to the folllowing equation for $\phi_o\ (\xi^*, X)$:

$$\frac{2}{E_2 + 2}\ \phi_{o\xi^* X} + \phi_{o\xi^*}\, \phi_{o\xi^*\xi^*} = - \frac{a_e^2\, E_1}{E_2 + 2}\, \phi_{o\xi^*\xi^*\xi^*} \qquad (4.1.23)$$

Note that $U_o \equiv \phi_{o\xi^*}$ is the lowest order velocity contribution, i.e.,

$$v' = \varepsilon\, U_o + 0\ (\varepsilon^2) \qquad (4.1.24)$$

Further, let

$$\eta^* \equiv \frac{E_2 + 2}{2}\, X = \frac{E_2 + 2}{2}\ \ \varepsilon\ \ \tilde{x} \qquad (4.1.25)$$

$$\nu^* \equiv\ -\ \frac{a_e^2\, E_1}{E_2 + 2} \qquad (4.1.26)$$

Then Eq. (4.1.23) becomes

$$U_{o\eta^*} + U_o\, U_{o\xi^*} = \nu^*\, U_{o\xi^*\xi^*} \qquad (4.1.27)$$

This is the standard form for Burgers' equation, orginally used by Burgers (1948) to study turbulent motion. It was first shown to be an approximate equation governing waves propagating in one direction in a compressible, viscous, heat-conducting fluid by Lagerstrom, Cole and

Trilling (1949). Lighthill (1956) later derived the same result more systematically. Burgers' equation can be solved exactly (Hope 1950, Cole 1951). A summary of the solutions of this equation is given by Benton and Platzman (1972).

Note that the nonlinear terms of order ε in Eq. (4.1.19) lead to the convective nonlinear term $U_o U_{o\xi^*}$ in Eq. (4.1.27), and the dissipative terms of ε in Eq. (4.1.19) lead to the diffusion term $U_{o\xi^*\xi^*}$ in Eq. (4.1.27). The effective diffusivity coefficient

$$\nu^* = - \frac{a_e^2 E_1}{E_2 + 2} > 0 \qquad (4.1.26)$$

is a combination of the dissipative and nonlinear effects, which can be seen from the parameters $E_1(<0)$ and E_2 in Eq. (4.1.19). Recall that

$$E_1 = \frac{C_7}{C_{10}} - \frac{C_8}{a_e^2 C_{10}} + \frac{C_9}{a_e^4 C_{10}} \qquad (4.1.7)$$

Thus, the origin of the dissipative part of ν^* can be traced back by identifying the factors C_7, C_8, C_9, and C_{10} in the original linearized equation (2.4.10). For pure gases ($\kappa_o = 0$, C_p = constant), Eq. (4.1.26) reduces to

$$\nu^* = \frac{\gamma - 1 + P_r^*}{P_r^* (\gamma + 1)} \qquad (4.1.28)$$

Eq. (4.1.28) is identical to the result obtained by Rasmussen (1977) when the term dealing with the binary-mixture diffusion mechanism is dropped, which is a partial check on the correctness of our present analysis.

Note also that ν^* is related to the parameter b in the linearized theory, Eq. (3.3.22), by the relation

$$\nu^{*} = \frac{a_e^2}{2\ (E_2 + 2)\ b^2} \tag{4.1.29}$$

Thus, the linearized theory yields the correct combination of the terms arising from viscous, thermal, and particle dissipation that contribute to the breadth of the wave front, but raised to the wrong power.

CHAPTER 5

CONCLUDING REMARKS

The problem of viscous, thermal conductivity, and particulates effects on wave propagation in dusty gases has been studied in detail by means of linearized theory. Laplace transforms were utilized to obtain asymptotoc approximations for large times. The large time solution shows a wave behavior with the wave front diffused out not only due to the viscosity and thermal conductivity of the gas, but also due to the size and number density of the particles. It was found that the wave front in linearized theory spreads out indefinitely like the square root of time, τ. This leads to an estimate of the time at which the linear solution breaks down. The failure of linearizations to yield a uniformly valid approximation to the solution suggests that the problem at large times, or far field, must be attacked by singular-perturbation methods.

It was shown that at the final stages the one-dimensional wave front is governed by Burgers' equation. The incorrect behavior pertaining to the associated linearized theory is delineated by the effective diffusion coefficient ν^* which is a combination of the lowest order dissipative and nonlinear effects.

The present problem can be extended to account for the other dissipative effects due to radiation, chemical reaction, mass diffusion, or electromagnetic fields. Also spherical and cylindrical disturbances might be studied fruitfully by this approach.

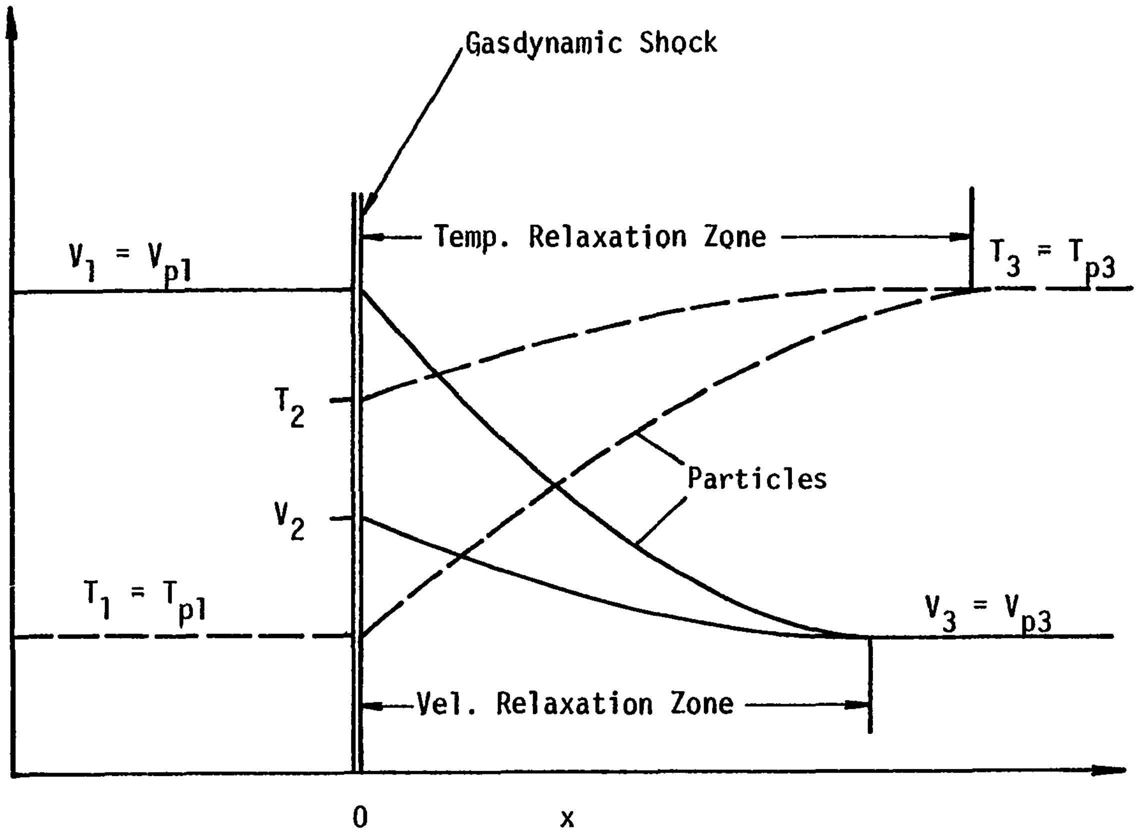

Fig. 1. Example of Relaxation Process Occurred in the Gas-Particle Flow: the Velocity and Temperature Distributions of the Gas and Particles for a Normal Shock (taken from Carrier 1958 and Marble 1963)

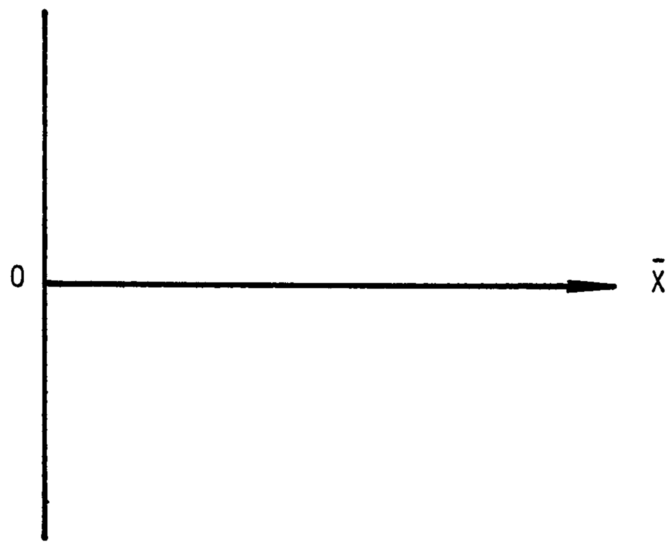

Fig. 2. Geometry of the Problem

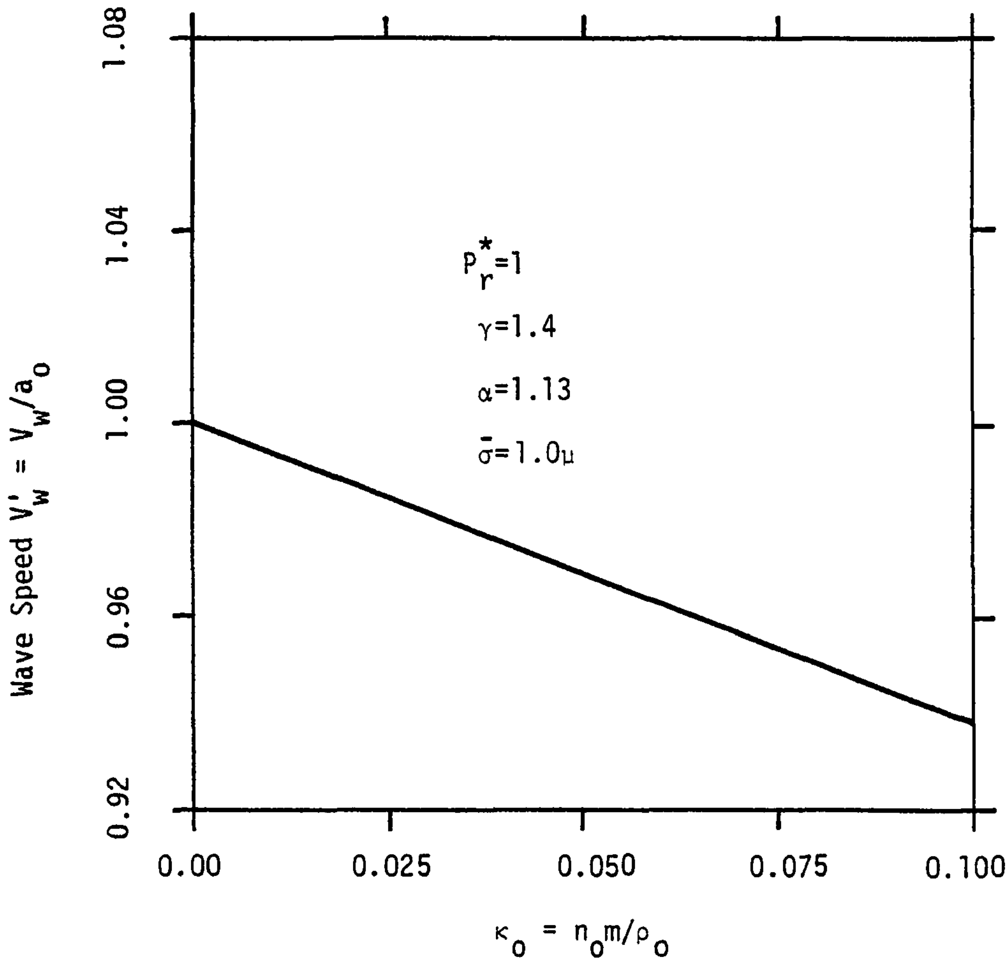

Fig. 3. Effect of the Number Density of Particles Upon Wave Speed

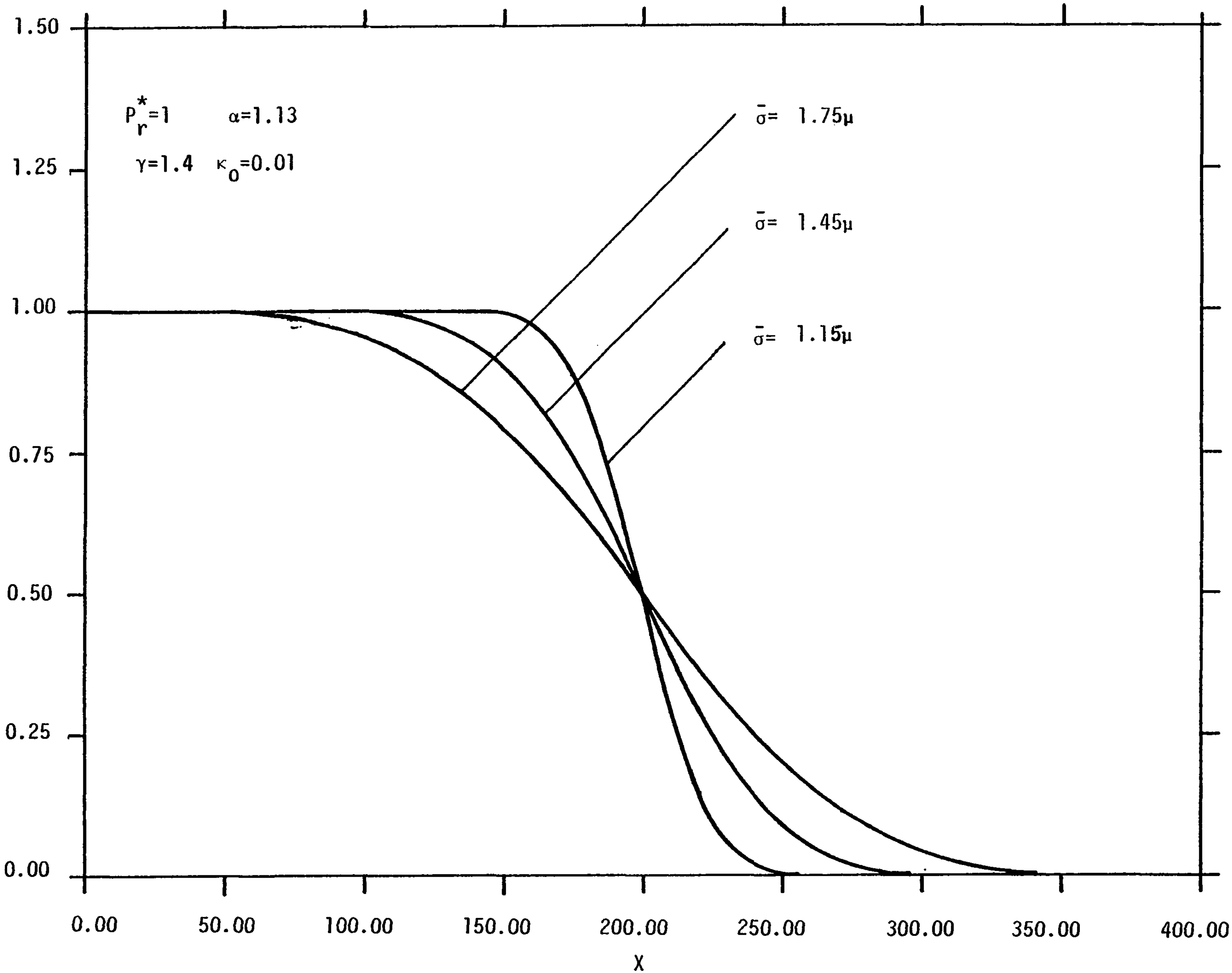

Fig. 4. Effect of Particle Radius Upon Large-Time Wave Fronts According to Linearized Theory

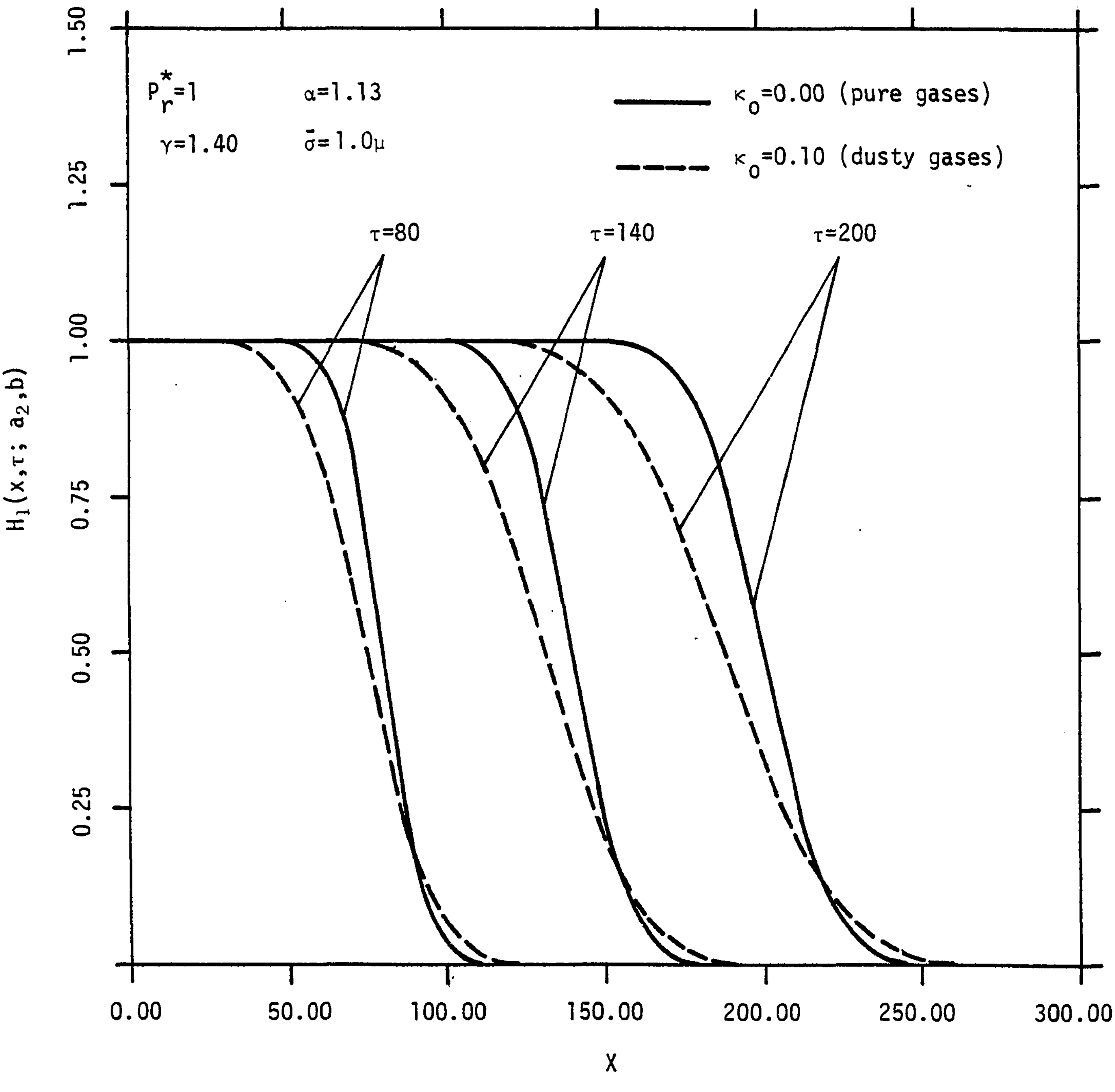

Fig. 5. Large-Time Behavior of Wave Fronts for Pure and Dusty Gases According to Linearized Theory

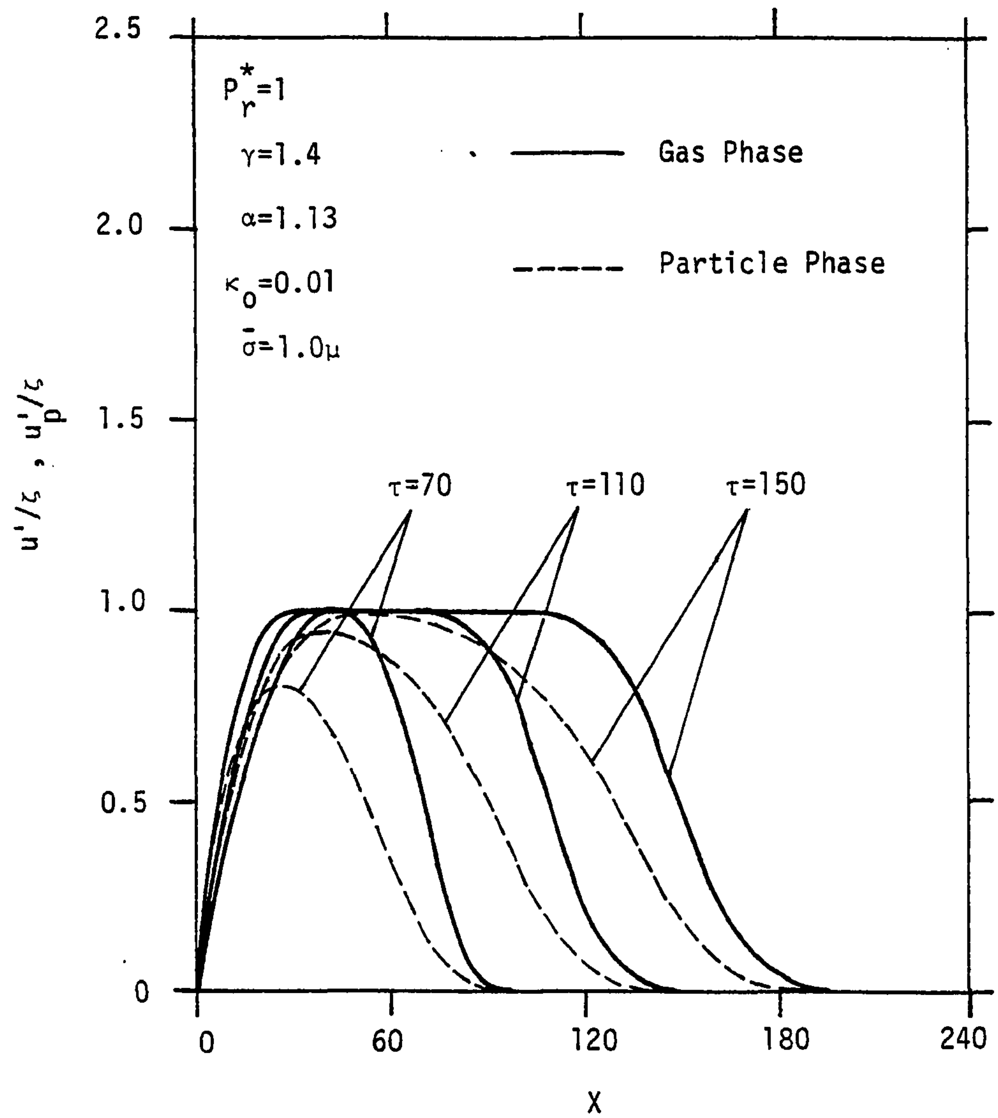

Fig. 6. Large-Time Behavior of Velocity for Gas and Particle Phases According to Linearized Theory

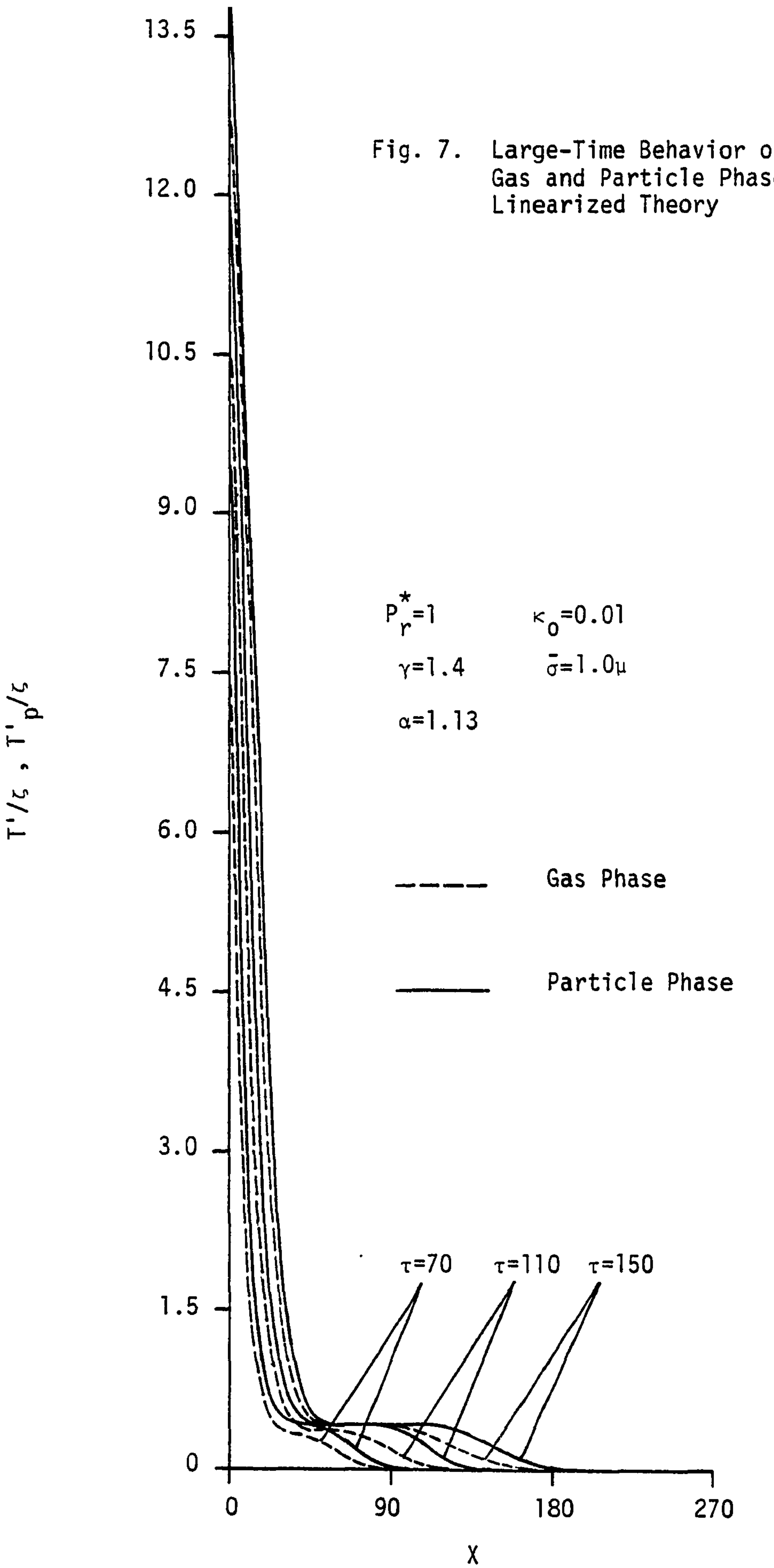

Fig. 7. Large-Time Behavior of Temperature for Gas and Particle Phases According to Linearized Theory

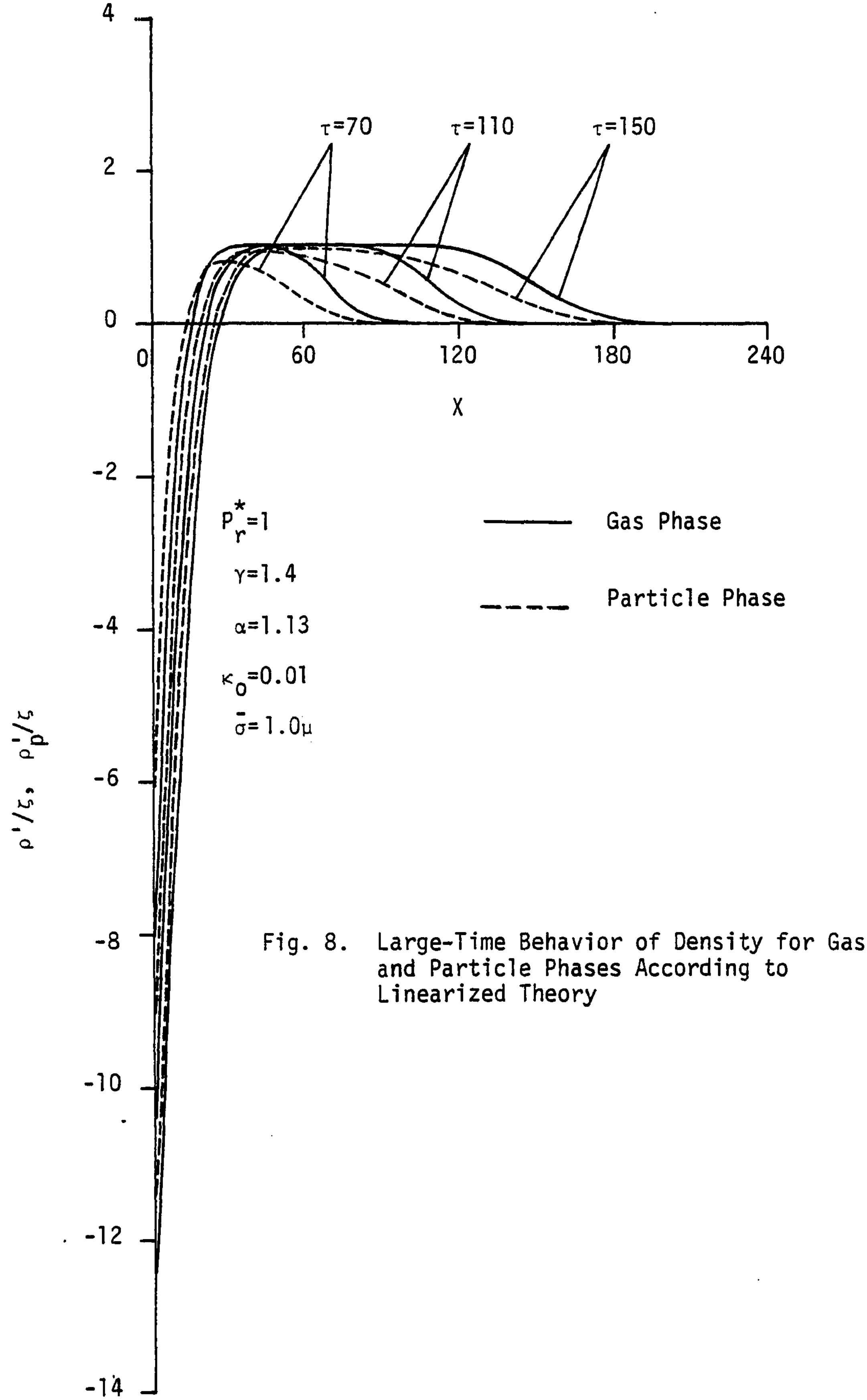

Fig. 8. Large-Time Behavior of Density for Gas and Particle Phases According to Linearized Theory

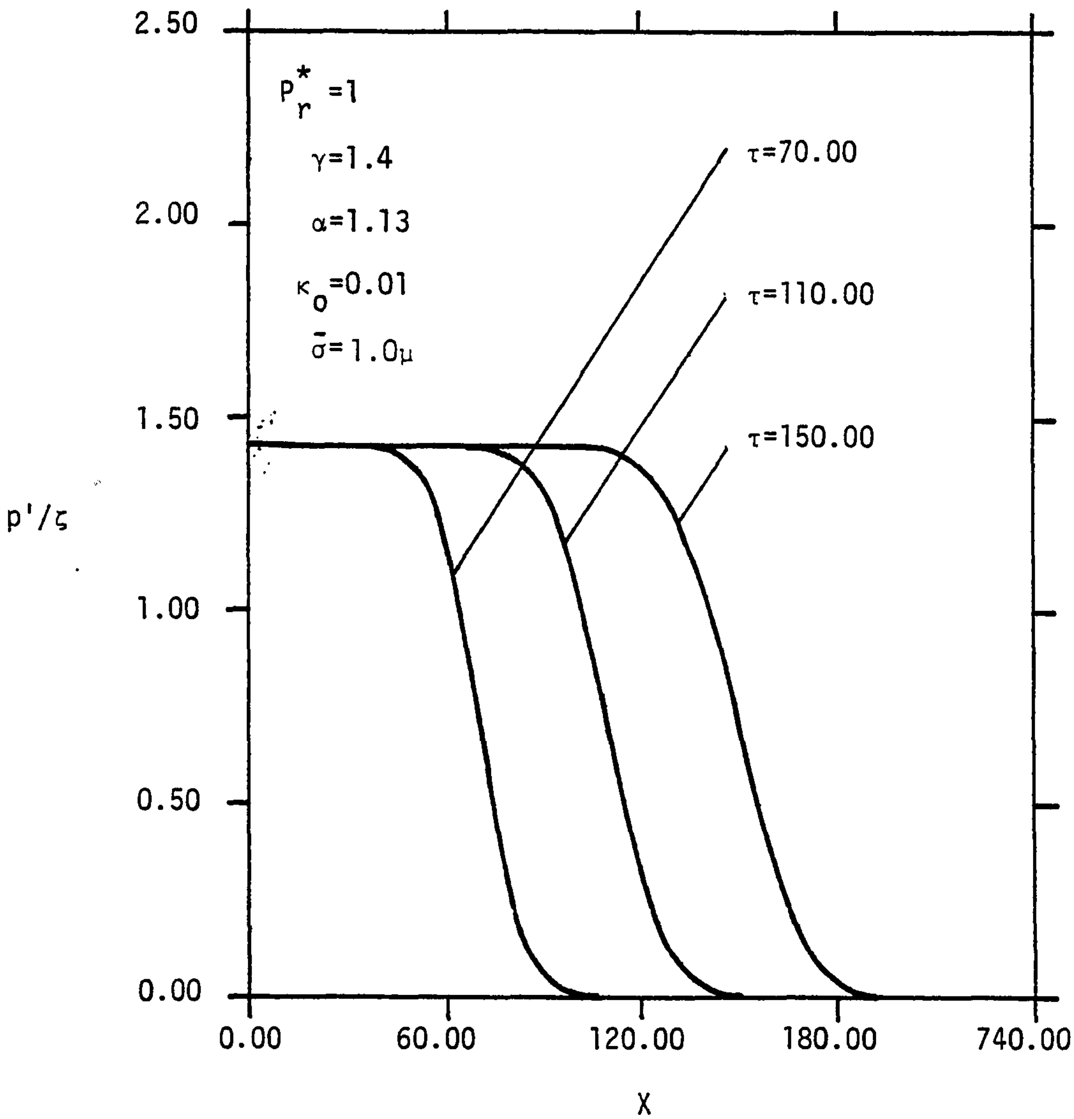

Fig. 9. Large-Time Behavior of Pressure for Gas Phase According to Linearized Theory

PART II

A MODEL FOR TRANSIENT FLOW IN A POROUS PARTICLE

CHAPTER I

INTRODUCTION

In Part I, the wave propagation in inert dusty gases has been studied. This problem may be extended to the case when the gas contains small porous particles. In this case, the field properties for the flow from a porous particle, especially the flow behavior on the surface of a porous particle, have to be determined. It is the purpose of Part II of the present study to consider this point. Before proceeding to the discussion of our physical model, a brief description of Darcy's law, which is essential to the flow through porous media, will be presented.

1.1 Darcy's Law

In 1856, as a result of experimental studies on the flow of water through unconsolidated sand filter beds, Henry Darcy formulated a law which bears his name. This law has been extended to describe, with some limitations, the motion of other liquids and gases in consolidated rocks and other porous media. Darcy's law states that the velocity of a homogeneous fluid in a porous medium is proportional to the pressure gradient, and inversely proportional to the fluid viscosity, that is,

$$v = - \frac{K}{\mu} \frac{\partial P}{\partial s} \quad , \qquad (1.1.1)$$

where v is the apparent velocity, μ the fluid viscosity, and $\frac{\partial P}{\partial s}$ the pressure gradient. The proportionality constant K is the permeability of the porous medium. It might be called fluid conductivity and is similar to the term thermal conductivity. The negative sign indicates that if the flow is taken as positive in the s-direction, then the pressure decreases in that direction so that the slope $\frac{\partial P}{\partial s}$ is negative.

The permeability K of a porous medium is one of its most useful properties. It is independent of the nature of the flüid and is determined solely by the structure of the porous medium. The capillary model (Parker, Boggs, and Blick, 1969) predicts that

$$K = \frac{\delta^2 \varepsilon}{32} \quad , \qquad (1.1.2)$$

where δ is the mean pore diameter, and ε is the effective porosity defined as the ratio of the interconnected void volume to the total volume. Thus, large mean pore diameters and large porosities are associated with large permeabilities. The value for K calculated from Eq. (1.1.2) is somewhat inaccurate, but it is useful for making estimates of K. For accurate values of the permeability one must resort to experiments. It can be seen from Eq. (1.1.2) that the permeability has the dimension of $(\text{length})^2$. The unit of permeability is the "darcy". A porous medium of one darcy permeability is one in which a fluid of one centipoise viscosity will move at a velocity of one centimeter per second under a pressure gradient of one atmosphere per centimeter.

The linear relationship between the pressure gradient and velocity described by Darcy's law, Eq. (1.1.1), is only valid in low-Reynolds-number flow; for in high-Reynolds-number flow, the pressure gradient increases faster than velocity. Therefore, in the high-Reynolds-number regime, the modification of Eq. (1.1.1) to account for the nonlinear effects is necessary. One suggestion for this would be to write the gradient $\frac{\partial P}{\partial s}$ as the sum of terms of several powers of v. This has been made by Forchheimer (1901) who set

$$\frac{\partial P}{\partial s} = av + bv^2 \tag{1.1.3}$$

and later by adding a third-order term:

$$\frac{\partial P}{\partial s} = av + bv^2 + cv^3 \tag{1.1.4}$$

Here, a, b, and c are constants. Another form was proposed by Missbach (1937), who set

$$\frac{\partial P}{\partial s} = av^n \tag{1.1.5}$$

with n undetermined between 1 and 2. Several experiments concerning the evaluation of the exponent n have been performed by many investigators. The exact value of n, however, seems to vary from case to case so that no universal correlation could be achieved.

The limiting range of Reynolds number below which Darcy's law is valid has been found up to the order of 10 (King 1940). Nevertheless, for the great majority of the practical cases, Darcy's law holds for $R_e \leq 1$ (see Muskat 1937, Sec. 2.2).

1.2 Physical Model

In this problem, we shall consider, at time t=0, a gas in equilibrium inside a spherical porous particle of radius $\bar{\sigma}$, as shown in Fig. 1, with temperature $T = T_0(1-\Delta)$ and density $\rho = \rho_0(1+\Delta)$, where $\Delta > 0$. Outside the particle the gas is in equilibrium with temperature and density denoted by $T = T_0$ and $\rho = \rho_0$. The pressures inside and outside the particle are the same, $P = P_0$. As time goes on, the temperature difference will induce a pressure gradient that, according to Darcy's law, causes the gas to eject from inside of the particle. The subsequent gaseous motion for $t > 0$ is to be determined. For simplicity we assume that the gases inside and outside the particle are of the same species so that no mass diffusion would occur. Further, since the particle is very small, it is reasonable to assume that the temperature of the gas inside the particle is identical to the temperature of the particle. In this case, the inner gas temperature can then be solved separately by the energy equation. Furthermore, we assume that the Reynolds number is low enough such that Darcy's law is applicable. Finally, we assume that the perturbation Δ is much less than unity so that the governing equations can be linearized.

The methods used here are essentially the same as those used in Part I. Asymptotic approximations valid for small times and large times are obtained. As pointed out by Rasmussen and Lake (1973), the Navier-Stokes equations are not valid when time becomes vanishingly small. However, the solution for small time is still of some mathematical interest. The large-time solution shows that there is a wave behavior outside the particle, and that the wave fronts diffuse out by virtue of viscosity. This wave behavior in low-Reynolds-number flow, however, is not so pronounced as that in high-Reynolds-number flow. The mass rate ejecting from the porous particles per unit volume, $\dot{\mu}$, will be formulated. With $\dot{\mu}$ known, the corresponding mass-, momentum-, and energy-source term which appear in the governing equations for the flow in porous-particle dusty gases can be found. These governing equations will be set up for future research. For more general cases including vaporization and chemical reactions, the governing equations will be derived in Appendix A.

CHAPTER 2

FORMULATION OF THE PROBLEM

2.1 Basic Equations

With the assumptions described in the previous section, the governing equations for the flow of gas are:

$\bar{r} < \bar{\sigma}$:

Mass $$\frac{\partial \rho}{\partial t} + \bar{\nabla} \cdot \rho \vec{V} = 0 \tag{2.1.1}$$

Momentum (Darcy's Law) $$\vec{V} = - \frac{K}{\mu} \bar{\nabla} P \tag{2.1.2}$$

Energy $$\frac{\partial T}{\partial t} = \alpha_p \bar{\nabla}^2 T \tag{2.1.3}$$

State $$P = \rho RT \tag{2.1.4}$$

$\bar{r} > \bar{\sigma}$:

Mass $$\frac{\partial \rho}{\partial t} + \bar{\nabla} \cdot \rho \vec{V} = 0 \tag{2.1.5}$$

Momentum $$\rho \frac{D\vec{V}}{dt} = - \bar{\nabla} p + \bar{\nabla} \cdot \overleftrightarrow{\tau} \tag{2.1.6}$$

Energy $$\rho \frac{Dh}{Dt} = \frac{Dp}{Dt} + \overleftrightarrow{\tau} : \bar{\nabla} \vec{V} - \bar{\nabla} \cdot \vec{q} \tag{2.1.7}$$

State $$p = \rho RT \tag{2.1.8}$$

where K and α_p are the permeability and thermal diffusivity of the porous particle. The other symbols are the same as those of Part I and are not re-defined here. It should be noted that the assumption that $T = T_p$ for the gas inside the porous particle has been used in Eq. (2.13), and that the gas is thermally perfect.

2.2 Linearized Equations

We now wish to linearize these basic equations about a uniform ambient state and write

$$
\begin{aligned}
T &= T_0(1 + T') \\
\rho &= \rho_0(1 + \rho') \\
P &= P_0(1 + P') \\
\vec{V} &= a_0\vec{V}' \\
\mu &= \mu_0(1 + \mu') \\
k &= k_0(1 + k') \\
C_p &= C_{po}(1 + C_p') \\
\alpha_p &= \alpha_{po}(1 + \alpha_p')
\end{aligned}
\tag{2.2.1}
$$

where, as before, a_0 defines the isentropic speed of sound at the ambient conditions:

$$a_0^2 = \gamma\, RT_0 \tag{2.2.2}$$

Further, we introduce the non-dimensional time, τ, and position vector, $\vec{r}$, defined by

$$\tau \equiv a_0^2 t/\tilde{\nu}_0 \tag{2.2.3}$$

$$\vec{r} \equiv a_0\vec{\bar{r}}/\tilde{\nu}_0 \tag{2.2.4}$$

where $\tilde{\nu}_0 = (2\mu_0 + \lambda_0)/\rho_0$ is the effective viscosity. In addition, since the present problem deals with spherical symmetry, we shall thus confine ourselves to irrotational flow, and to introduce a velocity potential such that

$$\vec{V}' = \nabla\phi \tag{2.2.5}$$

where $\nabla \equiv \frac{\partial}{\partial\vec{r}}$ (the barred space variables are dimensional). Based on the above definitions and the assumption that the perturbations are much less than unity, we obtain the following non-dimensional, linearized governing equations:

$r < \sigma$:

$$\frac{\partial\rho'}{\partial\tau} + \nabla^2\phi = 0 \tag{2.2.5}$$

$$\phi = - K^* p' \tag{2.2.6}$$

$$\frac{\partial T'}{\partial\tau} = \alpha_p^* \nabla^2 T' \tag{2.2.7}$$

$$P' = \rho' + T' \tag{2.2.8}$$

$r > \sigma$:

$$\frac{\partial\rho'}{\partial\tau} + \nabla^2\phi = 0 \tag{2.2.9}$$

$$P' = \gamma(\nabla^2\phi - \frac{\partial\phi}{\partial\tau}) \tag{2.2.10}$$

$$\frac{\partial T'}{\partial\tau} = \frac{\gamma-1}{\gamma}\frac{\partial P'}{\partial\tau} + \frac{1}{P_r^*}\nabla^2 T' \tag{2.2.11}$$

$$P' = \rho' + T' \tag{2.2.12}$$

where

$$K^* \equiv \frac{KP_o}{\mu_o\tilde{\nu}_o} \tag{2.2.13}$$

$$\alpha_p^* \equiv \alpha_p/\tilde{\nu}_o \tag{2.2.14}$$

$$P_r^* = \rho_o\tilde{\nu}_o\, C_{po}/k_o \tag{2.2.15}$$

Note that, in deriving equations (2.2.6) and (2.2.10), the momentum equations have been integrated once, and the time-functions have been set equal to zero.

It is convenient to find a single equation for ϕ. This can be done by the method of elimination. The equations for ϕ are thus found to be

$r < \sigma$:

$$\nabla^2\phi - \frac{1}{K^*}\frac{\partial\phi}{\partial\tau} = \frac{\partial T'}{\partial\tau} \tag{2.2.16}$$

$r > \sigma$:

$$\gamma\nabla^4\phi_\tau - (\gamma + P_r^*)\nabla^2\phi_{\tau\tau} + \nabla^4\phi + P_r^*(\phi_{\tau\tau} - \nabla^2\phi)_\tau = 0 \tag{2.2.17}$$

where, inside the particle, T' can be solved separately by using Eq. (2.2.7), and hence, in this region, ϕ satisfies a nonhomogeneous diffusion equation. Outside the particle the equation, Eq. (2.2.17), is fifth order and is identical to the form obtained by Rasmussen and Lake (1973). The lowest-order terms in Eq. (2.2.17) are related to the classical wave equation associated with inviscid acoustics.

For spherical symmetric flow, we have

$$u' = \frac{\partial\phi}{\partial r}$$

$$\nabla^2\phi = \frac{1}{r}(r\phi)_{rr} \tag{2.2.18}$$

$$\nabla^4\phi = \frac{1}{r}(r\phi)_{rrrr}$$

If we set $\Phi = r\phi$, the linearized governing equations in terms of Φ can then be written as

$r < \sigma$:

$$\Phi_{rr} - \frac{1}{K^*}\Phi_\tau = r\,\frac{\partial T'}{\partial\tau} \tag{2.2.19}$$

$r > \sigma$:

$$\gamma\Phi_{rrrr\tau} - (\gamma + P_r^*)\Phi_{rr\tau\tau} + \Phi_{rrrr} + P_r^*(\Phi_{\tau\tau} - \Phi_{rr})_\tau = 0 \tag{2.2.20}$$

which have the same form for one-dimensional flow.

2.3 Initial and Boundary Conditions

Here we consider a spherical porous particle of radius $\bar{\sigma}$ that, at time t=0, contains a gas with temperature perturbation $T' = -\Delta$, where $0 < \Delta << 1$. We assume that the pressure inside the porous particle is the same as outside and hence, $P' = 0$ everywhere at t=0. It follows from the equation of state, Eq. (2.2.8), that $\rho' = \Delta$ inside the porous particle. Outside the particle, of course, ambient uniform conditions prevail such that $\rho' = 0$. At t=0, the velocity is zero everywhere. Accordingly, the initial conditions can be summarized as

$r < \sigma$	$r > \sigma$
$T'(r,0) = -\Delta$	$T'(r,0) = 0$
$\rho'(r,0) = \Delta$	$\rho'(r,0) = 0$
$P'(r,0) = 0$	$P'(r,0) = 0$
$u'(r,0) = 0$	$u'(r,0) = 0$
$\phi(r,0) = 0$	$\phi(r,0) = 0$

where $\sigma \equiv a_0\bar{\sigma}/\tilde{\nu}_0$. The potential can be determined to within an arbitrary constant, which is chosen such that ϕ vanishes at t=0.

The boundary conditions are that all the perturbations must be zero at infinity and all quantitites must be finite at the origin, r=0. The velocity u' must vanish at the origin for all times.

The problem before us is to solve for the flow quantities for $t > 0$ by means of linearized governing equations subject to the above specified initial and boundary conditions. The method of Laplace transforms will be employed to obtain the asymptotic solutions for small times and for large times. The detailed analysis for the solution will be presented in the next chapter.

CHAPTER 3

SOLUTION OF THE PROBLEM

3.1 Solution by Laplace Transforms

In this section, we wish to solve Equations (2.2.19) and (2.2.20) by means of the Laplace transforms. Before doing so, it is necessary to evaluate the initial values of the time derivatives of Φ that are involved in the transform of Eqs. (2.2.19) and (2.2.20). Since $\phi(r,0) = 0$, $\Phi(r,0) = 0$. Further, all the space derivatives of Φ are zero when $t \leq 0$ for both $r < \sigma$ and $r > \sigma$ since there is no velocity disturbance for $t \leq 0$. At t=0, Eqs. (2.2.6) and (2.2.10) give

$$\phi_{\tau}(r,0) = 0 \quad , \quad 0 \leq r < \infty$$

and hence

$$\Phi_{\tau}(r,0) = 0 \quad , \quad 0 \leq r < \infty$$

The governing Equations (2.2.5)-(2.2.8) and (2.2.9)-(2.2.12) can be used to show that the higher time derivatives of Φ vanishes at t=0. With these conditions in mind, the Laplace transform of Eqs. (2.2.19) and (2.2.20) yields

$$\bar{\Phi}_{rr} - \frac{s}{K^*}\bar{\Phi} = s(r\bar{T}') + \Delta r \qquad r < \sigma \qquad (3.1.1)$$

$$(\gamma s + 1)\bar{\Phi}_{rrrr} - [(\gamma + P_r^*)s^2 + P_r^* s]\bar{\Phi}_{rr} + P_r^* s^3 \bar{\Phi} = 0 \qquad r > \sigma \qquad (3.1.2)$$

Note that the condition $T'(r,0) = -\Delta$ has been used in deriving Eq. (3.1.1). For clarity, in the following step the solution for the transformed variables in the two regions ($r < \sigma$ and $r > \sigma$) will be treated separately.

(1) <u>Solution for $r < \sigma$</u>

To solve for $\bar{\Phi}$ in Eq. (3.1.1), the temperature perturbation should be calculated first. This can be accomplished by making use of Eq. (2.2.7) which can be rewritten, for spherical symmetry, as

$$(rT')_\tau = \alpha_p^*(rT')_{rr} \tag{3.1.3}$$

Taking Laplace transform of Eq. (3.1.3), and noting that $T'(r,0) = -\Delta$, we have

$$(r\bar{T}')_{rr} - \frac{s}{\alpha_p^*}(r\bar{T}') = \frac{r\Delta}{\alpha_p^*} \tag{3.1.4}$$

This is an inhomogeneous, second-order, ordinary differential equation. Solving for $\bar{T}'$ gives

$$\bar{T}'(r<\sigma,s) = \frac{B'}{r}\cosh\left(r\sqrt{\frac{s}{\alpha_p^*}}\right) + \frac{B}{r}\sinh\left(r\sqrt{\frac{s}{\alpha_p^*}}\right) - \frac{\Delta}{s} \tag{3.1.5}$$

where B' and B are constants. Further, since $\bar{T}'$ is finite at the origin, $r=0$, B' should be zero. Thus, Eq. (3.1.5) becomes

$$\bar{T}'(r<\sigma,s) = \frac{B}{r}\sinh\left(r\sqrt{\frac{s}{\alpha_p^*}}\right) - \frac{\Delta}{s} \tag{3.1.6}$$

Substituting this equation into Eq. (3.1.1), we obtain

$$\bar{\Phi}_{rr} - \frac{s}{K^*}\bar{\Phi} = s\,B\sinh\left(r\sqrt{\frac{s}{\alpha_p^*}}\right) \tag{3.1.7}$$

Again, Eq. (3.1.7) is an inhomogeneous, second-order, ordinary differential equation. Solving for $\bar{\Phi}$ and using the condition that Φ is finite at $r=0$, we have

$$\bar{\Phi}(r<\sigma,s) = A\sinh\left(r\sqrt{\frac{s}{K^*}}\right) + B\,\frac{K^*\alpha_p^*}{K^*-\alpha_p^*}\sinh\left(r\sqrt{\frac{s}{\alpha_p^*}}\right) \tag{3.1.8}$$

or

$$\bar{\phi}(r<\sigma,s) = \frac{A}{r}\sinh\left(r\sqrt{\frac{s}{K^*}}\right) + \frac{B}{r}\frac{K^*\alpha_p^*}{K^*-\alpha_p^*}\sinh\left(r\sqrt{\frac{s}{\alpha_p^*}}\right) \tag{3.1.9}$$

It follows that

$$\bar{u}'(r<\sigma,s) = \bar{\phi}_r(r<\sigma,s)$$

$$= A\left\{\frac{1}{r}\sqrt{\frac{s}{K^*}}\cosh\left(r\sqrt{\frac{s}{K^*}}\right) - \frac{1}{r^2}\sinh\left(r\sqrt{\frac{s}{K^*}}\right)\right\}$$

$$+ B\frac{K^*\alpha_p^*}{K^*-\alpha_p^*}\left\{\frac{1}{r}\sqrt{\frac{s}{\alpha_p^*}}\cosh\left(r\sqrt{\frac{s}{\alpha_p^*}}\right) - \frac{1}{r^2}\sinh\left(r\sqrt{\frac{s}{\alpha_p^*}}\right)\right\} \tag{3.1.10}$$

On the other hand, taking Laplace transform of Eqs. (2.2.5) and (2.2.6) yields

$$\bar{\rho}'(r<\sigma,s) = -\frac{1}{rs}\bar{\Phi}_{rr}+\frac{\Delta}{s} \tag{3.1.11}$$

$$\bar{P}'(r<\sigma,s) = -\frac{\bar{\Phi}}{K^*r} \tag{3.1.12}$$

Further, substitution of $\bar{\Phi}$ from Eq. (3.1.8) into Eqs. (3.1.11) and (3.1.12) gives

$$\bar{\rho}'(r<\sigma,s) = -\frac{A}{rK^*}\sinh\left(r\sqrt{\frac{s}{K^*}}\right) - \frac{BK^*}{r(K^*-\alpha_p^*)}\sinh\left(r\sqrt{\frac{s}{\alpha_p^*}}\right) + \frac{\Delta}{s} \tag{3.1.13}$$

$$\bar{P}'(r<\sigma,s) = -\frac{A}{rK^*}\sinh\left(r\sqrt{\frac{s}{K^*}}\right) - \frac{B\alpha_p^*}{r(K^*-\alpha_p^*)}\sinh\left(r\sqrt{\frac{s}{\alpha_p^*}}\right) \tag{3.1.14}$$

The constants A and B appearing in Eqs. (3.1.6), (3.1.10), (3.1.13), and (3.1.14) can be determined from the boundary conditions at $r=\sigma$.

(2) <u>Solution for $r > \sigma$</u>

Solving for $\bar{\Phi}$ from Eq. (3.1.2) and taking into account the condition that $\bar{\Phi}$ vanishes as $r \to \infty$, we find for $r > \sigma$ that

$$\bar{\Phi}(r>\sigma,s) = C\ e^{-\lambda_1 r} + D\ e^{-\lambda_2 r} \tag{3.1.15}$$

where

$$\lambda_{1,2} = \left\{ \frac{(\gamma + P_r^*)s^2 + P_r^* s \pm s\sqrt{[(\gamma - P_r^*)s + P_r^*]^2 + 4(P_r^* - 1)P_r^* s}}{2(1+\gamma s)} \right\}^{\frac{1}{2}} \tag{3.1.16}$$

and C,D are the constants to be determined from the boundary conditions at $r=\sigma$.

For simplicity, we assume $P_r^* = 1$ and obtain

$$\begin{aligned} \lambda_1 &= \sqrt{s} \\ \lambda_2 &= s/(1+\gamma s)^{\frac{1}{2}} \end{aligned} \tag{3.1.17}$$

The assumption $P_r^* = 1$ was also used by Rasmussen and Lake (1973). Choosing $P_r^* = 1$ is not a severe limitation on the analysis and greatly simplifies the expressions for λ_1 and λ_2 so that the inversion of the transforms can be facilitated. Recall that the Prandtl number was defined as

$$P_r^* \equiv \frac{\rho_o \tilde{\nu}_o C_{po}}{k_o} = \frac{\tilde{\nu}_o}{\nu_o} \frac{\rho_o \nu_o C_{po}}{k_o} = \frac{2\mu_o + \lambda_o}{\mu_o} \frac{\mu_o C_{po}}{k_o}$$

If we evaluate the second coefficient of viscosity by the Stokes approximation, $\lambda_o = -2\mu_o/3$, then we have $P_r^* = 4\mu_o C_{po}/3\ k_o$. Thus, setting $P_r^* = 1$ is equivalent to setting the actual Prandtl number equal to 3/4, which is quite in accord with actual gases such as air. Therefore, the assumption $P_r^* = 1$ will be utilized from Eq. (3.1.17) on through the remainder of this analysis.

The transform of velocity can be found from $u' = \phi_r$ as

$$\bar{u}'(r>\sigma,s) = \frac{1}{r^2}(r\bar{\Phi}_r - \bar{\Phi}) \tag{3.1.18}$$

Moreover, by means of the transform of Eqs. (2.2.9) and (2.2.10), we find that

$$\bar{\rho}'(r>\sigma,s) = -\frac{\bar{\Phi}_{rr}}{rs} \tag{3.1.19}$$

$$\bar{P}'(r>\sigma,s) = \frac{\gamma}{r}(-s\bar{\Phi} + \bar{\Phi}_{rr}) \tag{3.1.20}$$

Finally, substituting for $\bar{\Phi}$ from Eq. (3.1.15) into Eqs. (3.1.18), (3.1.19), and (3.1.20), we obtain

$$\bar{u}'(r>\sigma,s) = -C\frac{(1+r\lambda_1)}{r^2} e^{-\lambda_1 r} - D\frac{(1+r\lambda_2)}{r^2} e^{-\lambda_2 r} \tag{3.1.21}$$

$$\bar{\rho}'(r>\sigma,s) = -C\frac{\lambda_1^2}{rs} e^{-\lambda_1 r} - D\frac{\lambda_2^2}{rs} e^{-\lambda_2 r} \tag{3.1.22}$$

$$\bar{P}'(r>\sigma,s) = C\frac{\gamma(\lambda_1^2 - s)}{r} e^{-\lambda_1 r} + D\frac{\gamma}{r}(\lambda_2^2 - s) e^{-\lambda_2 r} \tag{3.1.23}$$

The transform of the temperature can be found from the equation of state, $\bar{T}' = \bar{P}' - \bar{\rho}'$. We find that

$$\bar{T}'(r>\sigma,s) = \frac{C}{rs}[(1+\gamma s)\lambda_1^2 - \gamma s^2] e^{-\lambda_1 r}$$

$$+ \frac{D}{rs}[(1+\gamma s)\lambda_2^2 - \gamma s^2] e^{-\lambda_2 r} \tag{3.1.24}$$

Note that, since $\lambda_1 = \sqrt{s}$, $\lambda_1^2 - s = 0$. Thus, Eq. (3.1.23) becomes

$$\bar{P}'(r>\sigma,s) = D\frac{\gamma}{r}(\lambda_2^2 - s) e^{-\lambda_2 r} \tag{3.1.25}$$

There are four constants, A,B,C, and D in the above transformed expressions that remain to be determined. The four conditions needed to determine these constants are that $\bar{u}'$, $\bar{P}'$, $\bar{T}'$, and heat flux should be continuous at $r=\sigma$ when $t > 0$, that is,

$$\{\bar{P}'\}_{r=\sigma^-} - \{\bar{P}'\}_{r=\sigma^+} = 0 \tag{3.1.26}$$

$$\{\bar{u}'\}_{r=\sigma^-} - \{\bar{u}'\}_{r=\sigma^+} = 0 \tag{3.1.27}$$

$$\{\frac{\partial \bar{T}'}{\partial r}\}_{r=\sigma^-} - \frac{k_{go}}{k_{po}} \{\frac{\partial \bar{T}'}{\partial r}\}_{r=\sigma^+} = 0 \tag{3.1.28}$$

$$\{\bar{T}'\}_{r=\sigma^-} - \{\bar{T}'\}_{r=\sigma^+} = 0 \tag{3.1.29}$$

where k_{po} and k_{go} are the thermal conductivity of the porous particle and the gas. The above continuity conditions result in the following equations which, in matrix form, can be written as

$$\begin{bmatrix} \xi_{11} & \xi_{12} & \xi_{13} & \xi_{14} \\ \xi_{21} & \xi_{22} & \xi_{23} & \xi_{24} \\ \xi_{31} & \xi_{32} & \xi_{33} & \xi_{34} \\ \xi_{41} & \xi_{42} & \xi_{43} & \xi_{44} \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \Delta/s \end{bmatrix} \tag{3.1.30}$$

where

$$\begin{aligned} \xi_{11} &= \frac{1}{\sigma K^*} \sinh \left(\sigma \sqrt{\frac{s}{K^*}}\right) \\ \xi_{12} &= \frac{\alpha_p^*}{\sigma(K^* - \alpha_p^*)} \sinh \left(\sigma \sqrt{\frac{s}{\alpha_p^*}}\right) \\ \xi_{13} &= 0 \\ \xi_{14} &= \frac{\Upsilon}{\sigma} (\lambda_2^2 - s)\ e^{-\lambda_2 \sigma} \end{aligned} \tag{3.1.31}$$

$$\xi_{21} = \frac{1}{\sigma}\sqrt{\frac{s}{K^*}} \cosh \left(\sigma\sqrt{\frac{s}{K^*}}\right) - \frac{1}{\sigma^2} \sinh \left(\sigma\sqrt{\frac{s}{K^*}}\right)$$

$$\xi_{22} = \frac{K^* \alpha_p^*}{\sigma(K^* - \alpha_p^*)}\left[\sqrt{\frac{s}{\alpha_p^*}} \cosh \left(\sigma\sqrt{\frac{s}{\alpha_p^*}}\right) - \frac{1}{\sigma} \sinh \left(\sigma\sqrt{\frac{s}{\alpha_p^*}}\right)\right]$$

$$\xi_{23} = \frac{(1 + \sigma\lambda_1)}{\sigma^2} e^{-\lambda_1\sigma} \tag{3.1.32}$$

$$\xi_{24} = \frac{(1 + \sigma\lambda_2)}{\sigma^2} e^{-\lambda_2\sigma}$$

$$\xi_{31} = 0$$

$$\xi_{32} = \frac{1}{\sigma}\sqrt{\frac{s}{\alpha_p^*}} \cosh \left(\sigma\sqrt{\frac{s}{\alpha_p^*}}\right) - \frac{1}{\sigma^2} \sinh \left(\sigma\sqrt{\frac{s}{\alpha_p^*}}\right)$$

$$\xi_{33} = \frac{(1 + \sigma\lambda_1)}{\sigma^2 s} \frac{k_{go}}{k_{po}} [(1 + \gamma s)\lambda_1^2 - \gamma s^2] \tag{3.1.33}$$

$$\xi_{34} = \frac{(1 + \sigma\lambda_2)}{\sigma^2 s} \frac{k_{go}}{k_{po}} [(1 + \gamma s)\lambda_2^2 - \gamma s^2]$$

$$\xi_{41} = 0$$

$$\xi_{42} = \frac{1}{\sigma} \sinh \left(\sigma\sqrt{\frac{s}{\alpha_p^*}}\right)$$

$$\xi_{43} = \frac{-1}{\sigma s} [(1 + \gamma s)\lambda_1^2 - \gamma s^2] e^{-\lambda_1\sigma} \tag{3.1.34}$$

$$\xi_{44} = \frac{-1}{\sigma s} [(1 + \gamma s)\lambda_2^2 - \gamma s^2] e^{-\lambda_2\sigma}$$

Solving for A, B, C, and D gives

$$A = -\frac{\Delta}{s\Omega}\left[A_1 \sinh\left(\sigma\sqrt{\frac{s}{\alpha_p^*}}\right) + A_2 \cosh\left(\sigma\sqrt{\frac{s}{\alpha_p^*}}\right)\right] e^{-(\lambda_1+\lambda_2)\sigma} \quad (3.1.35)$$

$$B = \frac{\Delta}{s\Omega}\left[B_1 \sinh\left(\sigma\sqrt{\frac{s}{K^*}}\right) + B_2 \cosh\left(\sigma\sqrt{\frac{s}{K^*}}\right)\right] e^{-(\lambda_1+\lambda_2)\sigma} \quad (3.1.36)$$

$$\begin{aligned} C = -\frac{\Delta}{s\Omega}\Big[& C_1 \sinh\left(\sigma\sqrt{\frac{s}{K^*}}\right) \sinh\left(\sigma\sqrt{\frac{s}{\alpha_p^*}}\right) \\ & + C_2 \sinh\left(\sigma\sqrt{\frac{s}{K^*}}\right) \cosh\left(\sigma\sqrt{\frac{s}{\alpha_p^*}}\right) \\ & + C_3 \cosh\left(\sigma\sqrt{\frac{s}{K^*}}\right) \sinh\left(\sigma\sqrt{\frac{s}{\alpha_p^*}}\right) \\ & + C_4 \cosh\left(\sigma\sqrt{\frac{s}{K^*}}\right) \cosh\left(\sigma\sqrt{\frac{s}{\alpha_p^*}}\right)\Big] e^{-\lambda_2\sigma} \end{aligned} \quad (3.1.37)$$

$$\begin{aligned} D = \frac{\Delta}{s\Omega}\Big[& D_1 \sinh\left(\sigma\sqrt{\frac{s}{K^*}}\right) \sinh\left(\sigma\sqrt{\frac{s}{\alpha_p^*}}\right) \\ & + D_2 \sinh\left(\sigma\sqrt{\frac{s}{K^*}}\right) \cosh\left(\sigma\sqrt{\frac{s}{\alpha_p^*}}\right) \\ & + D_3 \cosh\left(\sigma\sqrt{\frac{s}{K^*}}\right) \sinh\left(\sigma\sqrt{\frac{s}{\alpha_p^*}}\right)\Big] e^{-\lambda_1\sigma} \end{aligned} \quad (3.1.38)$$

where

$$\begin{aligned} A_1 = & \frac{\alpha_p^*}{\sigma^5(K^*-\alpha_p^*)} \frac{k_{go}}{k_{po}} \frac{1}{s} (1+\sigma\lambda_1)(1+\sigma\lambda_2)[(1+\gamma s)(\lambda_2^2-\lambda_1^2)] \\ & - \frac{\gamma K^*\alpha_p^*}{\sigma^5(K^*-\alpha_p^*)} \frac{k_{go}}{k_{po}} \frac{1}{s} (1+\sigma\lambda_1)(\lambda_2^2-s)[(1+\gamma s)\lambda_1^2-\gamma s^2] \\ & + \frac{\gamma}{\sigma^5}(1+\sigma\lambda_1)(\lambda_2^2-s) \end{aligned} \quad (3.1.39)$$

$$A_2 = \frac{\gamma K^* \sqrt{\alpha_p^*}}{\sigma^4 (K^* - \alpha_p^*)} \frac{k_{go}}{k_{po}} \frac{1}{\sqrt{s}} (1 + \sigma\lambda_1)(\lambda_2^2 - s)[(1 + \gamma s)\lambda_1^2 - \gamma s^2]$$

$$- \frac{\gamma}{\sigma^4 \sqrt{\alpha_p^*}} \sqrt{s}\,(1 + \sigma\lambda_1)(\lambda_2^2 - s) \qquad (3.1.40)$$

$$B_1 = \frac{1}{\sigma^5 K^*} \frac{k_{go}}{k_{po}} \frac{1}{s} (1 + \sigma\lambda_1)(1 + \sigma\lambda_2)[(1 + \gamma s)(\lambda_2^2 - \lambda_1^2)]$$

$$- \frac{\gamma}{\sigma^5} \frac{k_{go}}{k_{po}} \frac{1}{s} (1 + \sigma\lambda_1)(\lambda_2^2 - s)[(1 + \gamma s)\lambda_1^2 - \gamma s^2] \qquad (3.1.41)$$

$$B_2 = \frac{\gamma}{\sigma^4} \frac{k_{go}}{k_{po}} \frac{1}{\sqrt{s}} (1 + \sigma\lambda_1)(\lambda_2^2 - s)[(1 + \gamma s)\lambda_1^2 - \gamma s^2] \qquad (3.1.42)$$

$$C_1 = \frac{1}{\sigma^5 K^*} (1 + \sigma\lambda_2) + \frac{\gamma}{\sigma^5} (\lambda_2^2 - s) \qquad (3.1.43)$$

$$C_2 = \frac{\sqrt{\alpha_p^*}}{\sigma^4 (K^* - \alpha_p^*)} \frac{k_{go}}{k_{po}} \frac{1}{\sqrt{s}} (1 + \sigma\lambda_2)[(1 + \gamma s)\lambda_2^2 - \gamma s^2]$$

$$- \frac{\sqrt{s}}{\sigma^4 K^* \sqrt{\alpha_p^*}} (1 + \sigma\lambda_2)$$

$$- \frac{\gamma\sqrt{s}}{\sigma^4 \sqrt{\alpha_p^*}} (\lambda_2^2 - s) \qquad (3.1.44)$$

$$C_3 = - \frac{\alpha_p^*}{\sigma^4 \sqrt{K^*}(K^* - \alpha_p^*)} \frac{k_{go}}{k_{po}} \frac{1}{\sqrt{s}} (1 + \sigma\lambda_2)[(1 + \gamma s)\lambda_2^2 - \gamma s^2]$$

$$- \frac{\gamma}{\sigma^4 \sqrt{K^*}} \sqrt{s}\,(\lambda_2^2 - s) \qquad (3.1.45)$$

$$C_4 = \frac{\gamma}{\sigma^5 \sqrt{K^* \alpha_p^*}} s\,(\lambda_2^2 - s) \qquad (3.1.46)$$

$$D_1 = \frac{1}{\sigma^5 K^*} (1 + \sigma\lambda_1) \qquad (3.1.47)$$

$$D_2 = \frac{\sqrt{\alpha_p^*}}{\sigma^4(K^* - \alpha_p^*)} \frac{k_{go}}{k_{po}} \frac{1}{\sqrt{s}} (1 + \sigma\lambda_1)[(1 + \gamma s)\lambda_1^2 - \gamma s^2]$$

$$- \frac{\sqrt{s}}{\sigma^4 K^* \sqrt{\alpha_p^*}} (1 + \sigma\lambda_1) \tag{3.1.48}$$

$$D_3 = \frac{-\alpha_p^*}{\sigma^4 \sqrt{K^*}(K^* - \alpha_p^*)} \frac{k_{go}}{k_{po}} \frac{1}{\sqrt{s}} (1 + \sigma\lambda_1)[(1 + \gamma s)\lambda_1^2 - \gamma s^2] \tag{3.1.49}$$

and

$$\Omega = \begin{vmatrix} \xi_{11} & \xi_{12} & \xi_{13} & \xi_{14} \\ \xi_{21} & \xi_{22} & \xi_{23} & \xi_{24} \\ \xi_{31} & \xi_{32} & \xi_{33} & \xi_{34} \\ \xi_{41} & \xi_{42} & \xi_{43} & \xi_{44} \end{vmatrix} \tag{3.1.50}$$

Expanding Ω gives

$$\Omega = [\Omega_1 \sinh\left(\sigma\sqrt{\frac{s}{K^*}}\right) \sinh\left(\sigma\sqrt{\frac{s}{\alpha_p^*}}\right)$$

$$+ \Omega_2 \sinh\left(\sigma\sqrt{\frac{s}{K^*}}\right) \cosh\left(\sigma\sqrt{\frac{s}{\alpha_p^*}}\right)$$

$$+ \Omega_3 \cosh\left(\sigma\sqrt{\frac{s}{K^*}}\right) \sinh\left(\sigma\sqrt{\frac{s}{\alpha_p^*}}\right)$$

$$+ \Omega_4 \cosh\left(\sigma\sqrt{\frac{s}{K^*}}\right) \cosh\left(\sigma\sqrt{\frac{s}{\alpha_p^*}}\right)] e^{-(\lambda_1 + \lambda_2)\sigma} \tag{3.1.51}$$

where

$$
\begin{aligned}
\Omega_1 = {} & \frac{1}{\sigma^6 K^* s} (1+\sigma\lambda_2)[(1+\gamma s)\lambda_1^2 - \gamma s^2] \\
& - \frac{1}{\sigma^6 K^* s} (1+\sigma\lambda_1)[(1+\gamma s)\lambda_2^2 - \gamma s^2] \\
& + \frac{1}{\sigma^6 K^*} \frac{k_{go}}{k_{po}} \frac{1}{s} (1+\sigma\lambda_1)(1+\sigma\lambda_2)[(1+\gamma s)(\lambda_2^2 - \lambda_1^2)] \\
& + \frac{\gamma}{\sigma^6} \frac{1}{s} (\lambda_2^2 - s)[(1+\gamma s)\lambda_1^2 - \gamma s^2] \\
& - \frac{\gamma}{\sigma^6} \frac{k_{go}}{k_{po}} \frac{1}{s} (1+\sigma\lambda_1)(\lambda_2^2 - s)[(1+\gamma s)\lambda_1^2 - \gamma s^2]
\end{aligned}
\tag{3.1.52}
$$

$$
\begin{aligned}
\Omega_2 = {} & \frac{\sqrt{\alpha_p^*}}{\sigma^4 (K^* - \alpha_p^*)} \frac{k_{go}}{k_{po}} \frac{1}{s^{3/2}} (\lambda_2 - \lambda_1)[(1+\gamma s)\lambda_1^2 - \gamma s^2][(1+\gamma s)\lambda_2^2 - \gamma s^2] \\
& + \frac{1}{\sigma^5 K^* \sqrt{\alpha_p^*}} \frac{(1+\sigma\lambda_1)}{\sqrt{s}} [(1+\gamma s)\lambda_2^2 - \gamma s^2] \\
& - \frac{1}{\sigma^5 K^* \sqrt{\alpha_p^*}} \frac{(1+\sigma\lambda_2)}{\sqrt{s}} [(1+\gamma s)\lambda_1^2 - \gamma s^2] \\
& - \frac{\gamma}{\sigma^5 \sqrt{\alpha_p^*}} \frac{(\lambda_2^2 - s)}{\sqrt{s}} [(1+\gamma s)\lambda_1^2 - \gamma s^2]
\end{aligned}
\tag{3.1.53}
$$

$$
\begin{aligned}
\Omega_3 = {} & \frac{\alpha_p^*}{\sigma^4 \sqrt{K^*}(K^* - \alpha_p^*)} \frac{k_{go}}{k_{po}} \frac{1}{s^{3/2}} (\lambda_1 - \lambda_2)[(1+\gamma s)\lambda_1^2 - \gamma s^2][(1+\gamma s)\lambda_2^2 - \gamma s^2] \\
& - \frac{\gamma}{\sigma^5 \sqrt{K^*}} \frac{1}{\sqrt{s}} (\lambda_2^2 - s)[(1+\gamma s)\lambda_1^2 - \gamma s^2] \\
& + \frac{\gamma}{\sigma^5 \sqrt{K^*}} \frac{k_{go}}{k_{po}} \frac{1}{\sqrt{s}} (1+\sigma\lambda_1)(\lambda_2^2 - s)[(1+\gamma s)\lambda_1^2 - \gamma s^2]
\end{aligned}
\tag{3.1.54}
$$

$$\Omega_4 = \frac{\gamma}{\sigma^4\sqrt{K^*\alpha_p^*}}(\lambda_2^2 - s)[(1+\gamma s)\lambda_1^2 - \gamma s^2] \tag{3.1.55}$$

Tne above transformed expressions with these coefficients can not be inverted in exact closed form. However, approximate expressions valied for small time, $\tau \to 0$, and for large time, $\tau \to \infty$, can be obtained.

3.2 Solution for Small Time

The solution for small time τ can be found by expansion of the Laplace transform for large s. For large s, the λ-functions take the form

$$\begin{aligned} \lambda_1 &= \sqrt{s} \\ \lambda_2 &= \frac{s}{\sqrt{1+\gamma s}} = \sqrt{\frac{s}{\gamma}} - \frac{1}{2\gamma^{3/2}}\frac{1}{\sqrt{s}} + O\left(\frac{1}{s^{3/2}}\right) \end{aligned} \tag{3.2.1}$$

and thus, to the lowest order, are the same order of magnitude. Substituting Eq. (3.2.1) into Eqs. (3.1.39)-(3.1.49) and Eqs. (3.1.52)-(3.1.55) yields

$$\begin{aligned} A_1 &= s^2[a_{11} + a_{12}\frac{1}{\sqrt{s}} + O(\frac{1}{s})] \\ A_2 &= s^2[a_{21} + a_{22}\frac{1}{\sqrt{s}} + O(\frac{1}{s})] \end{aligned} \tag{3.2.2}$$

$$\begin{aligned} B_1 &= s^2[b_{11} + b_{12}\frac{1}{\sqrt{s}} + O(\frac{1}{s})] \\ B_2 &= s^2[b_{21} + b_{22}\frac{1}{\sqrt{s}} + O(\frac{1}{s})] \end{aligned} \tag{3.2.3}$$

$$
\begin{aligned}
C_1 &= s[c_{11} + c_{12}\frac{1}{\sqrt{s}} + 0(\frac{1}{s})] \\
C_2 &= s^2[c_{21} + c_{22}\frac{1}{\sqrt{s}} + 0(\frac{1}{s})] \\
C_3 &= s^2[c_{31} + c_{32}\frac{1}{\sqrt{s}} + 0(\frac{1}{s})] \\
C_4 &= s^2[c_{41} + c_{42}\frac{1}{\sqrt{s}} + 0(\frac{1}{s})]
\end{aligned}
\tag{3.2.4}
$$

$$
\begin{aligned}
D_1 &= \sqrt{s}(d_{11} + d_{12}\frac{1}{\sqrt{s}}) \\
D_2 &= s(d_{21} + d_{22}\frac{1}{\sqrt{s}}) \\
D_3 &= s(d_{31} + d_{32}\frac{1}{\sqrt{s}})
\end{aligned}
\tag{3.2.5}
$$

$$
\begin{aligned}
\Omega_1 &= s^2[\omega_{11} + \omega_{12}\frac{1}{\sqrt{s}} + 0(\frac{1}{s})] \\
\Omega_2 &= s^2[\omega_{21} + \omega_{22}\frac{1}{\sqrt{s}} + 0(\frac{1}{s})] \\
\Omega_3 &= s^2[\omega_{31} + \omega_{32}\frac{1}{\sqrt{s}} + 0(\frac{1}{s})] \\
\Omega_4 &= s^2[\omega_{41} + \omega_{42}\frac{1}{s} + 0(\frac{1}{s^2})]
\end{aligned}
\tag{3.2.6}
$$

where

$$a_{11} = - \frac{(\gamma - 1)\alpha_p^* \, k_{go}}{\sqrt{\gamma}(K^* - \alpha_p^*)k_{po}} \tag{3.2.7}$$

$$a_{12} = [K^* - (\frac{1}{\sqrt{\gamma}} + 1)] \frac{(\gamma - 1)\alpha_p^* \, k_{go}}{\sigma^4(K^* - \alpha_p^*)k_{po}} - \frac{(\gamma - 1)}{\sigma^4}$$

$$a_{21} = - (\gamma - 1)[\frac{K^*\sqrt{\alpha_p^*} \, k_{go}}{\sigma^3(K^* - \alpha_p^*)k_{po}} - \frac{\gamma}{\sigma^4\sqrt{\alpha_p^*}}] \tag{3.2.8}$$

$$a_{22} = \sigma a_{21}$$

$$b_{11} = - \frac{(\gamma - 1)^{k_{go}}}{\sqrt{\gamma}\sigma^3 K^* k_{po}} \tag{3.2.9}$$

$$b_{12} = [1 + \frac{1}{K^*}(1 + \frac{1}{\sqrt{\gamma}})] \frac{(\gamma - 1) \, k_{go}}{\sigma^4 \, k_{po}}$$

$$b_{21} = - \frac{(\gamma - 1) \, k_{go}}{\sigma^3\sqrt{K^*} \, k_{po}} \tag{3.2.10}$$

$$b_{22} = b_{21}/\sigma$$

$$c_{11} = - (\gamma - 1)/\sigma^5 \tag{3.2.11}$$

$$c_{12} = \frac{1}{\sqrt{\gamma} \, \sigma^4 K^*}$$

$$c_{21} = - \frac{(\gamma - 1)\sqrt{\alpha_p^*} \, k_{go}}{\sqrt{\gamma} \, \sigma^3(K^* - \alpha_p^*)k_{po}} \tag{3.2.12}$$

$$c_{22} = \frac{-(\gamma - 1)\sqrt{\alpha_p^*} \, k_{go}}{\sigma^4(K^* - \alpha_p^*) \, k_{po}} + \frac{(\gamma - 1)}{\sigma^4\sqrt{\alpha_p^*}}$$

$$c_{31} = \frac{(\gamma - 1)\alpha_p^* k_{go}}{\sqrt{\gamma}\ \sigma^3\sqrt{K^*}(K^* - \alpha_p^*)k_{po}}$$

$$c_{32} = \frac{(\gamma - 1)\alpha_p^* k_{go}}{\sigma^4\sqrt{K^*}(K^* - \alpha_p^*)k_{po}} + \frac{(\gamma - 1)}{\sigma^4\sqrt{K^*}} \tag{3.2.13}$$

$$c_{41} = -\frac{(\gamma - 1)}{\sigma^5\sqrt{K^*\alpha_p^*}}$$

$$c_{42} = -\frac{1}{\gamma\sigma^5\sqrt{K^*\alpha_p^*}} \tag{3.2.14}$$

$$d_{11} = \frac{1}{\sigma^4 K^*}$$

$$d_{12} = d_{11}/\sigma \tag{3.2.15}$$

$$d_{21} = \frac{\sqrt{\alpha_p^*}\ k_{go}}{\sigma^3(K^* - \alpha_p^*)k_{po}} - \frac{1}{\sigma^3 K^*\sqrt{\alpha_p^*}}$$

$$d_{22} = d_{21}/\sigma \tag{3.2.16}$$

$$d_{31} = -\frac{\alpha_p^*\ k_{go}}{\sigma^3\sqrt{K^*}(K^* - \alpha_p^*)k_{po}}$$

$$d_{32} = d_{31}/\sigma \tag{3.2.17}$$

$$\omega_{11} = -\frac{(\gamma - 1)k_{go}}{\sqrt{\gamma}\ \sigma^4 K^* k_{po}}$$

$$\omega_{12} = (\gamma - 1)\left[\frac{k_{go}}{\sigma^5 k_{po}} - \left(1 + \frac{1}{\sqrt{\gamma}}\right)\frac{k_{go}}{\sigma^5 K^* k_{po}} + \frac{1}{\sigma^5 K^*}\right] \tag{3.2.18}$$

$$\omega_{21} = (\gamma - 1)[(1 - \frac{1}{\sqrt{\gamma}}) \frac{\sqrt{\alpha_p^*}\, k_{go}}{\sigma^4(K^* - \alpha_p^*)k_{po}} - \frac{1}{\sigma^4 K^* \sqrt{\alpha_p^*}}]$$

$$\omega_{22} = \frac{(\gamma - 1)}{\sigma^5\sqrt{\alpha_p^*}} (1 - \frac{1}{K^*}) \tag{3.2.19}$$

$$\omega_{31} = \frac{(\gamma - 1)}{\sigma^4\sqrt{K^*}} \frac{k_{go}}{k_{po}} [(\frac{1}{\sqrt{\gamma}} - 1) \frac{\alpha_p^*}{(K^* - \alpha_p^*)} - 1]$$

$$\omega_{32} = \frac{(\gamma - 1)}{\sigma^5\sqrt{K^*}} (1 - \frac{k_{go}}{k_{po}}) \tag{3.2.20}$$

$$\omega_{41} = \frac{1 - \gamma}{\sigma^4\sqrt{K^*\alpha_p^*}}$$

$$\omega_{42} = - \frac{1}{\gamma\sigma^4\sqrt{K^*\alpha_p^*}} \tag{3.2.21}$$

With the above expressions, the constants A, B, C, and D, given by Eqs. (3.1.35)-(3.1.38), can now be written for large s as

$$A = - \frac{a_{11} + a_{21}}{\omega^*} \frac{\Delta}{s \sinh (\sigma\sqrt{\frac{s}{K^*}})} [1 + a_1^* \frac{1}{\sqrt{s}} + O(\frac{1}{s})] \tag{3.2.22}$$

$$B = \frac{b_{11} + b_{21}}{\omega^*} \frac{\Delta}{s \sinh (\sigma\sqrt{\frac{s}{\alpha_p^*}})} [1 + b_1^* \frac{1}{\sqrt{s}} + O(\frac{1}{s})] \tag{3.2.23}$$

$$C = - \frac{C_{21} + C_{31} + C_{41}}{\omega^*} \frac{\Delta e^{\lambda_1\sigma}}{s} [1 + c_1^* \frac{1}{\sqrt{s}} + O(\frac{1}{s})] \tag{3.2.24}$$

$$D = \frac{d_{21} + d_{31}}{\omega^*} \frac{\Delta e^{\lambda_2\sigma}}{s} [1 + d_1^* \frac{1}{\sqrt{s}} + O(\frac{1}{s})] \tag{3.2.25}$$

where

$$\omega^* = \omega_{11} + \omega_{21} + \omega_{31} + \omega_{41} \tag{3.2.26}$$

$$a_1^* = \frac{a_{12} + a_{22}}{a_{11} + a_{21}} - \omega^{**} \tag{3.2.27}$$

$$b_1^* = \frac{b_{12} + b_{22}}{b_{11} + b_{21}} - \omega^{**} \tag{3.2.28}$$

$$c_1^* = \frac{c_{22} + c_{32}}{c_{21} + c_{31} + c_{41}} - \omega^{**} \tag{3.2.29}$$

$$d_1^* = \frac{d_{11} + d_{22} + d_{32}}{d_{21} + d_{31}} - \omega^{**} \tag{3.2.30}$$

in which ω^{**} is defined as

$$\omega^{**} = \frac{\omega_{12} + \omega_{22} + \omega_{32}}{\omega^*} \tag{3.2.31}$$

When the above approximations for large s are taken into account, inversions for velocity, temperature, and density can easily be found by means of standard tables of Laplace transforms. Therefore, for small time, we obtain

$$\begin{aligned}
\frac{u'(r<\sigma,\tau)}{\Delta} \sim &- \frac{a_{11} + a_{21}}{r^2\ \omega^*} \{[\frac{r}{\sqrt{K^*}} f_o^+(r,\tau;K^*) - f_1^-(r,\tau;K^*)] \\
&+ a_1^*[\frac{r}{\sqrt{K^*}} f_1^+(r,\tau;K^*) - f_2^-(r,\tau;K^*)]\} \\
&+ \frac{K^*\ \alpha_p^*}{K^* - \alpha_p^*} \frac{b_{11} + b_{21}}{r^2\ \omega^*} \{[\frac{r}{\sqrt{\alpha_p^*}} f_o^+(r,\tau;\alpha_p^*) - f_1^-(r,\tau;\alpha_p^*)] \\
&+ b_1^*[\frac{r}{\sqrt{\alpha_p^*}} f_1^+(r,\tau;\alpha_p^*) - f_2^-(r,\tau;\alpha_p^*)]\}
\end{aligned} \tag{3.2.32}$$

$$\frac{u'(r>\sigma,\tau)}{\Delta} \sim \frac{c_{21} + c_{31} + c_{41}}{r^2\ \omega^*} [r\ g_o(r,\tau) + (1 + c_1^* r) g_1(r,\tau) + c_1^* g_2(r,\tau)]$$

$$- \frac{d_{21} + d_{31}}{r^2\ \omega^*} [\frac{r}{\sqrt{\gamma}} h_1(r,\tau)] \tag{3.2.33}$$

$$\frac{T'(r<\sigma,\tau)}{\Delta} \sim \frac{b_{11} + b_{21}}{r\ \omega^*} [f_1^-(r,\tau;\alpha_p^*) + b_1^* f_2^-(r,\tau;\alpha_p^*)] - 1 \tag{3.2.34}$$

$$\frac{T'(r>\sigma,\tau)}{\Delta} \sim - \frac{c_{21} + c_{31} + c_{41}}{r\ \omega^*} [g_1(r,\tau) + c_1^* g_2(r,\tau)]$$

$$- \frac{(\gamma - 1)(d_{21} + d_{31})}{r\ \omega^*} [h_o(r,\tau) + d_1^* h_1(r,\tau)] \tag{3.2.35}$$

$$\frac{\rho'(r<\sigma,\tau)}{\Delta} \sim \frac{a_{11} + a_{21}}{r\ K^*\ \omega^*} [f_1^-(r,\tau;K^*) + a_1^* f_2^-(r,\tau;K^*)]$$

$$- \frac{K^*}{(K^* - \alpha_p^*)} \frac{b_{11} + b_{21}}{r\ \omega^*} [f_1^-(r,\tau;\alpha_p^*) + b_1^* f_2^-(r,\tau;\alpha_p^*)] + 1 \tag{3.2.36}$$

$$\frac{\rho'(r>\sigma,\tau)}{\Delta} \sim \frac{c_{21} + c_{31} + c_{41}}{r\ \omega^*} [g_1(r,\tau) + c_1^* g_2(r,\tau)]$$

$$- \frac{d_{21} + d_{31}}{r\ \omega^* \gamma} h_2(r,\tau) \tag{3.2.37}$$

where

$$f_o^+(r,\tau;K^*) = L^{-1}\{\frac{e^{-(\sigma-r)\sqrt{\frac{s}{K^*}}}}{\sqrt{s}} + \frac{e^{-(\sigma+r)\sqrt{\frac{s}{K^*}}}}{\sqrt{s}}\}$$

$$= \frac{1}{\sqrt{\pi\tau}} [e^{-\frac{(\sigma-r)^2}{4K^*\tau}} + e^{-\frac{(\sigma+r)^2}{4K^*\tau}}] \quad , \quad r < \sigma \tag{3.2.38}$$

$$f_1^{\pm}(r,\tau;K^*) = L^{-1}\{\frac{e^{-(\sigma-r)\sqrt{\frac{s}{K^*}}}}{s} \pm \frac{e^{-(\sigma+r)\sqrt{\frac{s}{K^*}}}}{s}\}$$

$$= \text{erfc}\ \{\frac{\sigma-r}{2\sqrt{K^*\tau}}\} \pm \text{erfc}\ \{\frac{\sigma+r}{2\sqrt{K^*\tau}}\} \quad , \quad r < \sigma \tag{3.2.39}$$

$$f_2^{-}(r,\tau;K^*) = L^{-1}\{\frac{e^{-(\sigma-r)\sqrt{\frac{s}{K^*}}}}{s^{3/2}} - \frac{e^{-(\sigma+r)\sqrt{\frac{s}{K^*}}}}{s^{3/2}}\}$$

$$= 2\sqrt{\frac{\tau}{\pi}}\ [e^{-\frac{(\sigma-\tau)^2}{4K^*\tau}} - e^{-\frac{(\sigma+\tau)^2}{4K^*\tau}}]$$

$$- [\frac{(\sigma-\tau)}{\sqrt{K^*}}\ \text{erfc}\ \{\frac{\sigma-r}{2\sqrt{K^*\tau}}\} - \frac{(\sigma+\tau)}{\sqrt{K^*}}\ \text{erfc}\ \{\frac{\sigma+r}{2\sqrt{K^*\tau}}\}], r < \sigma \tag{3.2.40}$$

$$g_0(r,\tau) = L^{-1}\{\frac{e^{-(r-\sigma)\sqrt{s}}}{\sqrt{s}}\}$$

$$= \frac{e^{-\frac{(r-\sigma)^2}{4\tau}}}{\sqrt{\pi\tau}} \quad , \quad r > \sigma \tag{3.2.41}$$

$$g_1(r,\tau) = L^{-1}\{\frac{e^{-(r-\sigma)\sqrt{s}}}{\sqrt{s}}\} = \text{erfc}\ \{\frac{r-\sigma}{2\sqrt{\tau}}\} \quad , \quad r > \sigma \tag{3.2.42}$$

$$g_2(r,\tau) = L^{-1}\{\frac{e^{-(r-\sigma)\sqrt{s}}}{s^{3/2}}\}$$

$$= 2\sqrt{\frac{\tau}{\pi}}\ e^{-\frac{(r-\sigma)^2}{4\tau}} - (r-\sigma)\ \text{erfc}\ \{\frac{r-\sigma}{2\sqrt{\tau}}\} \quad , \quad r > \sigma \tag{3.2.43}$$

$$h_0(r,\tau) = L^{-1}\{\frac{e^{-(r-\sigma)\sqrt{\frac{s}{\gamma}}}}{s}\} = \text{erfc}\ \{\frac{r-\sigma}{2\sqrt{\gamma\tau}}\} \quad , \quad r > \sigma \tag{3.2.44}$$

$$h_1(r,\tau) = L^{-1}\{\frac{e^{-(r-\sigma)\sqrt{\frac{s}{\gamma}}}}{s^{3/2}}\}$$

$$= 2\sqrt{\frac{\tau}{\pi}}\, e^{-\frac{(r-\sigma)^2}{4\gamma\tau}} - \frac{r-\sigma}{\sqrt{\gamma}} \operatorname{erfc}\{\frac{r-\sigma}{2\sqrt{\gamma\tau}}\} \quad , \qquad r > \sigma \qquad (3.2.45)$$

The functions $f_0^+(r,\tau;\alpha_p^*)$, $f_1^\pm(r,\tau;\alpha_p^*)$, and $f_2^-(r,\tau;\alpha_p^*)$ can be obtained simply by substituting α_p^* for K^* into the corresponding functions $f_0^+(r,\tau;K^*)$, $f_1^\pm(r,\tau;K^*)$, and $f_2^-(r,\tau;K^*)$. Furthermore, the pressure P' can be found from ρ' and T' by means of the equation of state.

It is useful to evaluate the non-dimensional time, $\tau = a_0^2 t/\tilde{\nu}_0$ in terms of Reynolds number, $\sigma \equiv \bar{\sigma} a_0/\tilde{\nu}_0$, based on the radius of the porous particle and the speed of sound. We write

$$\tau = \frac{a_0^2\, t}{\tilde{\nu}_0} = \frac{a_0\, \bar{\sigma}}{\tilde{\nu}_0}\, \frac{a_0\, t}{\bar{\sigma}} = \sigma\tau^* \qquad (3.2.46)$$

Here $\tau^* \equiv a_0 t/\bar{\sigma}$ is a non-dimensional "wave time", being unity when the isentropic acoustic wave has traveled the distance $\bar{\sigma}$. Thus, for a fixed physical time, the non-dimensional time τ is very small when the Reynolds number, $\bar{\sigma} a_0/\tilde{\nu}_0$, is very small. On the other hand, for a fixed Reynolds number, the time τ is very small for small values of the wave time τ^*, that is, for $t \ll \bar{\sigma}/a_0$.

It was mentioned by Rasmussen and Lake (1973) that the Navier-Stokes equations are not valid for the above small time regime. This fact can be seen from the following argument. For a gas to satisfy the continuum postulate, the molecular mean free path λ must be small compared with a significant characteristic length L pertinent to the flow field. By definition, the ratio λ/L is termed the Knudsen number, and is denoted by K_n, i.e.,

$$K_n \equiv \lambda/L$$

For small times, the characteristic length is the distance a molecule travels in the time t, approximately $L = c_o t$. For other times $L = \bar{\sigma}$. Since $\lambda \sim \tilde{\nu}_o/a_o$, we have

$$K_n \sim \tilde{\nu}_o/a_o^2 t = 1/\tau \qquad \text{(small times)} \qquad (3.2.47)$$

$$\sim \tilde{\nu}_o/a_o\bar{\sigma} = 1/\sigma \qquad \text{(other times)} \qquad (3.2.48)$$

Since the Navier-Stokes equations are valid only in the continuum limit of small Knudsen number, they are strictly valid only when τ and σ are large. Hence, the small time solution given above is not valid physically owing to the fact that the basic Navier-Stokes equations are not valid when τ is small.

The above solution, however, is of some theoretical interest. Examination of the above solution shows that the time t appears in conjunction with the space dimension in the form $(\bar{r}-\bar{\sigma})/\sqrt{t}$ which is the usual form found in heat-conduction problem, which are diffusive in behavior. The correct description of this problem is determined by free molecule flow theory (Bienkowski, 1964) wherein it is found that the corresponding space and time relation is $(\bar{r}-\bar{\sigma})/t$. Thus, although both solutions show a diffusive character, the functional dependence on time is not correct for the small-time Navier-Stokes solution.

3.3 Solution for Large Time

The solution for large time τ can be obtained by expansion of the Laplace transform for small s. The λ_1 function retains its exact form, $\lambda_1 = \sqrt{s}$, and λ_2 can be written as

$$\lambda_2 = s(1+\gamma s)^{-\frac{1}{2}}$$

$$= s - \frac{1}{2}\gamma s^2 + \frac{3}{8}\gamma^2 s^3 + 0(s^4) \quad , \qquad s \to 0 \qquad (3.3.1)$$

or

$$\lambda_2 = 2b[\sqrt{s+b^2} - b] + 0(s^3) \quad , \qquad s \to 0 \qquad (3.3.2)$$

where $b \equiv (2\gamma)^{-\frac{1}{2}}$. Substituting $\lambda_1 = \sqrt{s}$ and Eq. (3.3.1) into Eqs. (3.1.39)-(3.1.55), we have

$$\begin{aligned} A_1 &= A_{10} + A_{11}\sqrt{s} + A_{12}s + A_{13}s^{3/2} + A_{14}s^2 + A_{15}s^{5/2} + A_{16}s^3 + 0(s^{7/2}) \\ A_2 &= A_{23}s^{3/2} + A_{24}s^2 + A_{25}s^{5/2} + A_{26}s^3 + 0(s^{7/2}) \end{aligned} \qquad (3.3.3)$$

$$\begin{aligned} B_1 &= B_{10} + B_{11}\sqrt{s} + B_{12}s + B_{13}s^{3/2} + B_{14}s^2 + B_{15}s^{5/2} + B_{16}s^3 + 0(s^{7/2}) \\ B_2 &= B_{23}s^{3/2} + B_{24}s^2 + B_{25}s^{5/2} + B_{26}s^3 + 0(s^{7/2}) \end{aligned} \qquad (3.3.4)$$

$$\begin{aligned} C_1 &= C_{10} + C_{12}s + C_{14}s^2 + C_{16}s^3 + 0(s^4) \\ C_2 &= C_{21}\sqrt{s} + C_{23}s^{3/2} + C_{25}s^{5/2} + 0(s^{7/2}) \\ C_3 &= C_{33}s^{3/2} + C_{35}s^{5/2} + 0(s^{7/2}) \\ C_4 &= C_{44}s^2 + C_{46}s^3 + 0(s^{7/2}) \end{aligned} \qquad (3.3.5)$$

$$\begin{aligned} D_1 &= D_{10} + D_{11}\sqrt{s} \\ D_2 &= D_{21}\sqrt{s} + D_{22}s \\ D_3 &= D_{31}\sqrt{s} + D_{32}s \end{aligned} \qquad (3.3.6)$$

$$\Omega_1 = \Omega_{10} + \Omega_{11}\sqrt{s} + \Omega_{12}s + \Omega_{13}s^{3/2} + \Omega_{14}s^2 + \Omega_{15}s^{5/2} + \Omega_{16}s^3 + O(s^{7/2})$$

$$\Omega_2 = \Omega_{21}\sqrt{s} + \Omega_{23}s^{3/2} + \Omega_{24}s^2 + \Omega_{25}s^{5/2} + O(s^{7/2})$$

$$\Omega_3 = \Omega_{33}s^{3/2} + \Omega_{34}s^2 + \Omega_{35}s^{5/2} + \Omega_{36}s^3 + O(s^{7/2}) \tag{3.3.7}$$

$$\Omega_4 = \Omega_{44}s^2 + \Omega_{46}s^3 + O(s^4)$$

where

$$A_{10} = \frac{-\alpha_p^*}{\sigma^5(K^* - \alpha_p^*)} \frac{k_{go}}{k_{po}}$$

$$A_{11} = \sigma A_{10}$$

$$A_{12} = -[\sigma + (\gamma - 1)] \frac{\alpha_p^*}{\sigma^5(K^* - \alpha_p^*)} \frac{k_{go}}{k_{po}} + \frac{\gamma K^* \alpha_p^*}{\sigma^5(K^* - \alpha_p^*)} \frac{k_{go}}{k_{po}} - \frac{\gamma}{\sigma^5}$$

$$A_{13} = \sigma A_{12} \tag{3.3.8}$$

$$A_{14} = (1 - \frac{\gamma}{2}) \frac{\alpha_p^*}{\sigma^4(K^* - \alpha_p^*)} \frac{k_{go}}{k_{po}} \frac{\gamma K^* \alpha_p^*}{\sigma^5(K^* - \alpha_p^*)} \frac{k_{go}}{k_{po}} + \frac{\gamma}{\sigma^5}$$

$$A_{15} = \sigma A_{14}$$

$$A_{16} = \frac{\gamma}{8} [\gamma - 4] \frac{\alpha_p^*}{\sigma^4(K^* - \alpha_p^*)} \frac{k_{go}}{k_{po}} + \frac{\gamma^2 K^* \alpha_p^*}{\sigma^5(K^* - \alpha_p^*)} \frac{k_{go}}{k_{po}} - \frac{\gamma^2}{\sigma^5}$$

$$A_{23} = \frac{\gamma}{\sigma^4\sqrt{\alpha_p^*}} - \frac{\gamma K^*\sqrt{\alpha_p^*}}{\sigma^4(K^* - \alpha_p^*)} \quad \frac{k_{go}}{k_{po}}$$

$$A_{24} = \sigma A_{23} \tag{3.3.9}$$

$$A_{25} = \frac{\gamma K^*\sqrt{\alpha_p^*}}{\sigma^4(K^* - \alpha_p^*)} \frac{k_{go}}{k_{po}} - \frac{\gamma}{\sigma^4\sqrt{\alpha_p^*}}$$

$$A_{26} = \sigma A_{25}$$

$$B_{10} = -\frac{1}{\sigma^5 K^*} \frac{k_{go}}{k_{po}}$$

$$B_{11} = \sigma B_{10}$$

$$B_{12} = \frac{\gamma}{\sigma^5} \frac{k_{go}}{k_{po}} - [\sigma + (\gamma - 1)] \frac{1}{\sigma^5 K^*} \frac{k_{go}}{k_{po}}$$

$$B_{13} = \sigma B_{12} \tag{3.3.10}$$

$$B_{14} = (1 - \frac{\gamma}{2}) \frac{1}{\sigma^4 K^*} \frac{k_{go}}{k_{po}} - \frac{\gamma}{\sigma^5} \frac{k_{go}}{k_{po}}$$

$$B_{15} = \sigma B_{14}$$

$$B_{16} = \frac{\gamma}{2} (\frac{\gamma}{4} - 1) \frac{1}{\sigma^4 K^*} \frac{k_{go}}{k_{po}} + \frac{\gamma^2}{\sigma^5} \frac{k_{go}}{k_{po}}$$

$$B_{23} = -\frac{\gamma}{\sigma^4\sqrt{K^*}}\frac{k_{go}}{k_{po}}$$

$$B_{24} = \sigma B_{23}$$

$$B_{25} = -B_{23} \tag{3.3.11}$$

$$B_{26} = -\sigma B_{23}$$

$$C_{10} = \frac{1}{\sigma^5 K^*}$$

$$C_{12} = \frac{1}{\sigma^4 K^*} - \frac{\gamma}{\sigma^5}$$

$$C_{14} = \frac{\gamma}{\sigma^5} - \frac{\gamma}{2\sigma^4 K^*} \tag{3.3.12}$$

$$C_{16} = \frac{3\gamma^2}{8\sigma^4 K^*} - \frac{\gamma^2}{\sigma^5}$$

$$C_{21} = -\frac{1}{\sigma^4 K^*\sqrt{\alpha_p^*}}$$

$$C_{23} = \frac{\gamma}{\sigma^4\sqrt{\alpha_p^*}} - \frac{(\gamma-1)\sqrt{\alpha_p^*}}{\sigma^4(K^*-\alpha_p^*)}\frac{k_{go}}{k_{po}} - \frac{1}{\sigma^3 K^*\sqrt{\alpha_p^*}} \tag{3.3.13}$$

$$C_{25} = \frac{\gamma}{2\sigma^3 K^*\sqrt{\alpha_p^*}} - \frac{(\gamma-1)\sqrt{\alpha_p^*}}{\sigma^3(K^*-\alpha_p^*)}\frac{k_{go}}{k_{po}} - \frac{\gamma}{\sigma^4\sqrt{\alpha_p^*}}$$

$$C_{33} = \frac{(\gamma - 1)\alpha_p^* \, k_{go}}{\sigma^4\sqrt{K^*}(K^* - \alpha_p^*)k_{po}} + \frac{\gamma}{\sigma^4\sqrt{K^*}}$$

$$C_{35} = \frac{(\gamma - 1)\alpha_p^*}{\sigma^3\sqrt{K^*}(K^* - \alpha_p^*)} \frac{k_{go}}{k_{po}} - \frac{\gamma}{\sigma^4\sqrt{K^*}} \tag{3.3.14}$$

$$C_{44} = \frac{-\gamma}{\sigma^3\sqrt{K^*\alpha_p^*}}$$

$$C_{46} = - C_{44} \tag{3.3.15}$$

$$D_{10} = \frac{1}{\sigma^5 K^*}$$

$$D_{11} = \sigma D_{10} \tag{3.3.16}$$

$$D_{21} = \frac{\sqrt{\alpha_p^*}}{\sigma^4(K^* - \alpha_p^*)} \frac{k_{go}}{k_{po}} - \frac{1}{\sigma^4 K^* \sqrt{\alpha_p^*}}$$

$$D_{22} = \sigma D_{21} \tag{3.3.17}$$

$$D_{31} = \frac{- \alpha_p^*}{\sigma^4\sqrt{K^*}(K^* - \alpha_p^*)} \frac{k_{go}}{k_{po}}$$

$$D_{32} = \sigma D_{31} \tag{3.3.18}$$

$$\Omega_{10} = \frac{1}{\sigma^6 K^*}\left(1 - \frac{k_{go}}{k_{po}}\right)$$

$$\Omega_{11} = -\frac{1}{\sigma^5 K^*}\frac{k_{go}}{k_{po}}$$

$$\Omega_{12} = [\sigma + (\gamma - 1)]\frac{1}{\sigma^6 K^*} - [\sigma + (\gamma - 1)]\frac{1}{\sigma^6 K^*}\frac{k_{go}}{k_{po}} - \frac{\gamma}{\sigma^6}\left(1 + \frac{k_{go}}{k_{po}}\right)$$

$$\Omega_{13} = \frac{(\gamma - 1)}{\sigma^5 K^*} - [\sigma + (\gamma - 1)]\frac{1}{\sigma^5 K^*}\frac{k_{go}}{k_{po}} + \frac{\gamma}{\sigma^5}\frac{k_{go}}{k_{po}} \qquad (3.3.19)$$

$$\Omega_{14} = \left(1 - \frac{\gamma}{2}\right)\frac{1}{\sigma^5 K^*}\frac{k_{go}}{k_{po}} - \frac{\gamma}{2\sigma^5 K^*} - \frac{\gamma}{\sigma^6}\frac{k_{go}}{k_{po}} + \frac{\gamma}{\sigma^6}$$

$$\Omega_{15} = \left(1 - \frac{\gamma}{2}\right)\frac{1}{\sigma^4 K^*}\frac{k_{go}}{k_{po}} - \frac{\gamma}{\sigma^5}\frac{k_{go}}{k_{po}}$$

$$\Omega_{16} = \frac{3\gamma^2}{8\sigma^5 K^*} - \frac{\gamma}{8}(4 - \gamma)\frac{1}{\sigma^5 K^*}\frac{k_{go}}{k_{po}} - \frac{\gamma^2}{\sigma^6} + \frac{\gamma^2}{\sigma^6}\frac{k_{go}}{k_{po}}$$

$$\Omega_{21} = -\frac{1}{\sigma^5 K^* \sqrt{\alpha_p^*}}$$

$$\Omega_{23} = \frac{\gamma}{\sigma^5\sqrt{\sigma_p^*}} - [\sigma + (\gamma - 1)]\frac{1}{\sigma^5 K^* \sqrt{\alpha_p^*}}$$

$$(3.3.20)$$

$$\Omega_{24} = (\gamma - 1)\frac{\sqrt{\alpha_p^*}}{\sigma^4(K^* - \alpha_p^*)}\frac{k_{go}}{k_{po}} - \frac{(\gamma - 1)}{\sigma^4 K^* \sqrt{\alpha_p^*}}$$

$$\Omega_{25} = \frac{\gamma}{2\sigma^4 K^* \sqrt{\alpha_p^*}} - (\gamma - 1)\frac{\sqrt{\alpha_p^*}}{\sigma^4(K^* - \alpha_p^*)}\frac{k_{go}}{k_{po}} - \frac{\gamma}{\sigma^5\sqrt{\alpha_p^*}}$$

$$\Omega_{33} = (1 - \frac{k_{go}}{k_{po}}) \frac{\gamma}{\sigma^5\sqrt{K^*}}$$

$$\Omega_{34} = - \frac{(\gamma - 1)\alpha_p^*}{\sigma^4\sqrt{K^*}(K^* - \alpha_p^*)} \frac{k_{go}}{k_{po}} - \frac{\gamma}{\sigma^4\sqrt{K^*}} \frac{k_{go}}{k_{po}}$$

$$\Omega_{35} = - (1 - \frac{k_{go}}{k_{po}}) \frac{\gamma}{\sigma^5\sqrt{K^*}} + \frac{(\gamma - 1)\alpha_p^*}{\sigma^4\sqrt{K^*}(K^* - \alpha_p^*)} \frac{k_{go}}{k_{po}} \tag{3.3.21}$$

$$\Omega_{36} = \frac{\gamma}{\sigma^4\sqrt{K^*}} \frac{k_{go}}{k_{po}}$$

$$\Omega_{44} = - \frac{\gamma}{\sigma^4\sqrt{K^*\alpha_p^*}}$$

$$\Omega_{46} = - \Omega_{44} \tag{3.3.22}$$

Substituting Eqs. (3.3.3)-(3.3.7) into Eqs. (3.1.35)-(3.1.38), we obtain

$$A = - \frac{\Delta}{\Omega_o^* \; s \sinh (\sigma\sqrt{\frac{s}{K^*}})} [A_0^* + A_1^*\sqrt{s} + A_2^* s + A_3^* s^{3/2} + A_4^* s^2 + A_5^* s^{5/2} + O(s^3)] \tag{3.3.23}$$

$$B = \frac{\Delta}{\Omega_o^* \; s \sinh (\sigma\sqrt{\frac{s}{\alpha_p^*}})} [B_0^* + B_1^*\sqrt{s} + B_2^* s + B_3^* s^{3/2} + B_4^* s^2 + B_5^* s^{5/2} + O(s^3)] \tag{3.3.24}$$

$$C = - \frac{\Delta \; e^{\lambda_1 \sigma}}{\Omega_o^* \; s} [C_2^* s + C_3^* s^{3/2} + C_4^* s^2 + C_5^* s^{5/2} + O(s^3)] \tag{3.3.25}$$

$$D = \frac{\Delta \; e^{\lambda_2 \sigma}}{\Omega_o^* \; s} \; [D_2^* s + D_4^* s^2 + D_5^* s^{5/2} + O(s^3)] \tag{3.3.26}$$

where

$$\Omega_o^* = -\frac{1}{\sigma^6 K^*}\frac{k_{go}}{k_{po}} \tag{3.3.27}$$

$$\begin{aligned}
A_0^* &= A_0' \\
A_1^* &= A_1' + A_0'\Omega_1' = 0 \\
A_2^* &= A_2' + A_1'\Omega_1' + A_0'\Omega_2' \\
A_3^* &= A_3' + A_2'\Omega_1' + A_1'\Omega_2' + A_0'\Omega_3' \\
A_4^* &= A_4' + A_3'\Omega_1' + A_2'\Omega_2' + A_1'\Omega_3' + A_0'\Omega_4' \\
A_5^* &= A_5' + A_4'\Omega_1' + A_3'\Omega_2' + A_2'\Omega_3' + A_1'\Omega_4' + A_0'\Omega_5'
\end{aligned} \tag{3.3.28}$$

$$\begin{aligned}
B_0^* &= B_0' \\
B_1^* &= B_1' + B_0'\Omega_1' = 0 \\
B_2^* &= B_2' + B_1\Omega_1 + B_0'\Omega_2' \\
B_3^* &= B_3' + B_2'\Omega_1' + B_1'\Omega_2' + B_0'\Omega_3' \\
B_4^* &= B_4' + B_3'\Omega_1' + B_2'\Omega_2' + B_1'\Omega_3' + B_0'\Omega_4' \\
B_5^* &= B_5' + B_4'\Omega_1' + B_3'\Omega_2' + B_2'\Omega_3' + B_1'\Omega_4' + B_0'\Omega_5'
\end{aligned} \tag{3.3.29}$$

$$\begin{aligned}
C_2^* &= C_2' \\
C_3^* &= C_2'\Omega_1' \\
C_4^* &= C_4' + C_2'\Omega_2' \\
C_5^* &= C_4'\Omega_1' + C_2'\Omega_3'
\end{aligned} \tag{3.3.30}$$

$$\begin{aligned}
D_2^* &= D_2' \\
D_4^* &= D_4' + D_3'\Omega_1' + D_2'\Omega_2' \\
D_5^* &= D_3'\Omega_2' + D_2'\Omega_3'
\end{aligned} \tag{3.3.31}$$

in which

$$\begin{aligned}
A_0' &= A_{10} \\
A_1' &= \sigma A_0' \\
A_2' &= A_{12} + \frac{\sqrt{\alpha_p^*}}{\sigma} A_{23} \\
A_3' &= \sigma A_2' \\
A_4' &= A_{14} + \frac{\sigma}{3\sqrt{\alpha_p^*}} A_{23} + \frac{\sqrt{\alpha_p^*}}{\sigma} A_{25} \\
A_5' &= \sigma A_4'
\end{aligned} \tag{3.3.32}$$

$$B_0' = B_{10}$$
$$B_1' = \sigma B_0'$$
$$B_2' = B_{12} + \frac{\sqrt{K^*}}{\sigma} B_{23} \tag{3.3.33}$$
$$B_3' = \sigma B_2'$$
$$B_4' = B_{14} + \frac{\sigma}{3\sqrt{K^*}} B_{23} + \frac{\sqrt{K^*}}{\sigma} B_{25}$$
$$B_5' = \sigma B_4'$$

$$C_2' = C_{12} + \frac{\sigma}{3\sqrt{\alpha_p^*}} C_{21} + \frac{\sqrt{\alpha_p^*}}{\sigma} C_{23} + \frac{\sqrt{K^*}}{\sigma} C_{33} + \frac{\sqrt{K^*\alpha_p^*}}{\sigma^2} C_{44}$$

$$C_4' = C_{14} - \frac{\sigma^3}{45\alpha_p^{*5/2}} C_{21} + \frac{\sigma}{3\sqrt{\alpha_p^*}} C_{23} + \frac{\sqrt{\alpha_p^*}}{\sigma} C_{25} + \frac{\sigma}{3\sqrt{K^*}} C_{33} + \frac{\sqrt{K^*}}{\sigma} C_{35} \tag{3.3.34}$$
$$+ \frac{1}{3}\left(\sqrt{\frac{K^*}{\alpha_p^*}} + \sqrt{\frac{\alpha_p^*}{K^*}}\right) C_{44} + \frac{\sqrt{K^*\alpha_p^*}}{\sigma} C_{46}$$

$$D_2' = \frac{\sigma}{3\sqrt{\alpha_p^*}} D_{21} + \frac{\sigma}{3\sqrt{K^*}} D_{31}$$

$$D_3' = \sigma D_2' \tag{3.3.35}$$

$$D_4' = \frac{-\sigma^3}{45\alpha_p^{*3/2}} D_{21} - \frac{\sigma^3}{45K^{*3/2}} D_{31}$$

$$\Omega_1' = -\eta_1$$
$$\Omega_2' = \eta_1^2 - \eta_2$$
$$\Omega_3' = -\eta_3 + 2\eta_1\eta_2 - \eta_1^3 \tag{3.3.36}$$
$$\Omega_4' = -\eta_4 + (\eta_2^2 + 2\eta_1\eta_3) - 3\eta_1^2\eta_2 + \eta_1^4$$
$$\Omega_5' = -\eta_5 + (2\eta_1\eta_4 + 2\eta_2\eta_3) - (3\eta_1^2\eta_3 + 3\eta_1\eta_2^2) + 4\eta_1^3\eta_2 - \eta_1^5$$

In Eq. (3.3.36) η_i, i=1,2,...,5 are defined by

$$\eta_i \equiv \frac{\Omega_i^*}{\Omega_o^*} \quad , \quad i=1,2,\ldots,5 \qquad (3.3.37)$$

Here

$$\Omega_1^* = \Omega_{11}$$

$$\Omega_2^* = \Omega_{12} + \frac{\sigma}{3\sqrt{\alpha_p^*}}\Omega_{21} + \frac{\sqrt{\alpha_p^*}}{\sigma}\Omega_{23} + \frac{\sqrt{K^*}}{\sigma}\Omega_{33} + \frac{\sqrt{K^*\alpha_p^*}}{\sigma^2}\Omega_{44}$$

$$\Omega_3^* = \Omega_{13} + \frac{\sqrt{\alpha_p^*}}{\sigma}\Omega_{24} + \frac{\sqrt{K^*}}{\sigma}\Omega_{34} \qquad (3.3.38)$$

$$\Omega_4^* = \Omega_{14} - \frac{\sigma^3}{45\alpha_p^{*3/2}}\Omega_{21} + \frac{\sigma}{3\sqrt{\alpha_p^*}}\Omega_{23} + \frac{\sqrt{\alpha_p^*}}{\sigma}\Omega_{25} + \frac{\sigma}{3\sqrt{K^*}}\Omega_{33} + \frac{\sqrt{K^*}}{\sigma}\Omega_{35}$$

$$+ \frac{1}{3}\left(\sqrt{\frac{K^*}{\alpha_p^*}} + \sqrt{\frac{\alpha_p^*}{K^*}}\right)\Omega_{44} + \frac{\sqrt{K^*\alpha_p^*}}{\sigma^2}\Omega_{46}$$

$$\Omega_5^* = \Omega_{15} + \frac{\sigma}{3\sqrt{\alpha_p^*}}\Omega_{24} + \frac{\sigma}{3\sqrt{K^*}}\Omega_{34} + \frac{\sqrt{K^*}}{\sigma}\Omega_{36}$$

Further, if we expand the terms $\sinh\left(\sigma\sqrt{\frac{s}{K^*}}\right)$ and $\sinh\left(\sigma\sqrt{\frac{s}{\alpha_p^*}}\right)$ in Eqs. (3.3.23) and (3.3.24) as

$$\sinh(z\sqrt{s}) = \frac{1}{2}e^{z\sqrt{s}}(1 - e^{-2z\sqrt{s}})$$

$$= z\sqrt{s}\, e^{z\sqrt{s}}\,[1 - \frac{1}{2!}(2z\sqrt{s}) + \frac{1}{3!}(2z\sqrt{s})^2 - \frac{1}{4!}(2z\sqrt{s})^3$$

$$+ \frac{1}{5!}(2z\sqrt{s})^4 - \frac{1}{6!}(2z\sqrt{s})^5 + O(s^3)] \quad ,$$

$$z=\sigma/\sqrt{K^*} \text{ or } \sigma/\sqrt{\alpha_p^*} \qquad (3.3.39)$$

then A, B become

$$A = -\frac{\Delta\sqrt{K^*}}{\sigma\Omega_o^*}\frac{e^{-\sigma\sqrt{\frac{s}{K^*}}}}{s^{3/2}}[A_0^{**} + A_1^{**}\sqrt{s} + A_2^{**}s + A_3^{**}s^{3/2} + A_4^{**}s^2 + A_5^{**}s^{5/2} + O(s^3)] \tag{3.3.40}$$

$$B = \frac{\Delta\sqrt{\alpha_p^*}}{\sigma\Omega_o^*}\frac{e^{-\sigma\sqrt{\frac{s}{\alpha_p^*}}}}{s^{3/2}}[B_0^{**} + B_1^{**}\sqrt{s} + B_2^{**}s + B_3^{**}s^{3/2} + B_4^{**}s^2 + B_5^{**}s^{5/2} + O(s^3)] \tag{3.3.41}$$

where

$$\begin{aligned}
A_0^{**} &= A_0^* \\
A_1^{**} &= A_0^*\eta_1^*(s,K^*) \\
A_2^{**} &= A_2^* + A_0^*\eta_2^*(s,K^*) \\
A_3^{**} &= A_3^* + A_2^*\eta_1^*(s,K^*) + A_0^*\eta_3^*(s,K^*) \\
A_4^{**} &= A_4^* + A_3^*\eta_1^*(s,K^*) + A_2^*\eta_2^*(s,K^*) + A_0^*\eta_4^*(s,K^*) \\
A_5^{**} &= A_5^* + A_4^*\eta_1^*(s,K^*) + A_3^*\eta_2^*(s,K^*) + A_2^*\eta_3^*(s,K^*) + A_0^*\eta_5^*(s,K^*)
\end{aligned} \tag{3.3.42}$$

and

$$\begin{aligned}
B_0^{**} &= B_0^* \\
B_1^{**} &= B_0^*\eta_1^*(s;\alpha_p^*) \\
B_2^{**} &= B_2^* + B_0^*\eta_2^*(s;\alpha_p^*) \\
B_3^{**} &= B_3^* + B_2^*\eta_1^*(s;\alpha_p^*) + B_0^*\eta_3^*(s;\alpha_p^*) \\
B_4^{**} &= B_4^* + B_3^*\eta_1^*(s;\alpha_p^*) + B_2^*\eta_2^*(s;\alpha_p^*) + B_0^*\eta_4^*(s;\alpha_p^*) \\
B_5^{**} &= B_5^* + B_4^*\eta_1^*(s;\alpha_p^*) + B_3^*\eta_2^*(s;\alpha_p^*) + B_2^*\eta_3^*(s;\alpha_p^*) + B_0^*\eta_5^*(s;\alpha_p^*)
\end{aligned} \tag{3.3.43}$$

In Eqs. (3.3.42) and (3.3.43), the functions $\eta_i^*(s;K^*$ or $\alpha_p^*)$, i=1,2,...,5, are given by

$$
\begin{aligned}
\eta_1^* &= -\xi_1 \\
\eta_2^* &= -\xi_2 + \xi_1^2 \\
\eta_3^* &= -\xi_3 + 2\xi_1\xi_2 - \xi_1^3 \\
\eta_4^* &= -\xi_4 + \xi_2^2 + 2\xi_1\xi_3 - 3\xi_1^2\xi_2 + \xi_1^4 \\
\eta_5^* &= -\xi_5 + 2\xi_1\xi_4 + 2\xi_2\xi_3 - 3\xi_1^2\xi_3 - 3\xi_1\xi_2^2 + 4\xi_1^3\xi_2 - \xi_1^5
\end{aligned}
\tag{3.3.44}
$$

in which

$$
\begin{aligned}
\xi_1 &= -\frac{1}{2!}(2z\sqrt{s}) \\
\xi_2 &= \frac{1}{3!}(2z\sqrt{s})^2 \\
\xi_3 &= -\frac{1}{4!}(2z\sqrt{s})^3 \qquad (z=\sigma/\sqrt{K^*} \text{ or } \sigma/\sqrt{\alpha_p^*}) \\
\xi_4 &= \frac{1}{5!}(2z\sqrt{s})^4 \\
\xi_5 &= -\frac{1}{6!}(2z\sqrt{s})^5
\end{aligned}
\tag{3.3.45}
$$

With the above expressions for A, B, C, and D, Eqs. (3.3.25), (3.3.26), (3.3.40), and (3.3.41), the large-time approximations for the temperature, density, and velocity are found to be

$$
\frac{T'(r<\sigma,\tau)}{\Delta} \sim \frac{\sqrt{\alpha_p^*}}{2r\sigma\,\Omega_o^*} \sum_{n=0}^{5} B_n^{**} F_n^-(r,\tau;\alpha_p^*) - 1 \tag{3.3.46}
$$

$$
\frac{T'(r>\sigma,\tau)}{\Delta} \sim -\frac{1}{r\Omega_o^*} \sum_{n=1}^{4} C_{n+1}^* G_n(r,\tau) - \frac{(\gamma-1)}{r\Omega_o^*} D_2^* H_2(r,\tau) \tag{3.3.47}
$$

$$
\frac{\rho'(r<\sigma,\tau)}{\Delta} \sim \frac{1}{2r\sigma\Omega_o^*\sqrt{K^*}} \sum_{n=0}^{5} A_n^{**} F_n^-(r,\tau;K^*) - \frac{K^*\sqrt{\alpha_p^*}}{2r\sigma\Omega_o^*(K^*-\alpha_p^*)} \sum_{n=0}^{5} B_n^{**} F_n^-(r,\tau;\alpha_p^*) + 1 \tag{3.3.48}
$$

$$\frac{\rho'(r>\sigma,\tau)}{\Delta} \sim \frac{1}{r\Omega_o^*}\sum_{n=1}^{4} C_{n+1}^* G_n(r,\tau) - \frac{1}{r\Omega_o^*} D_2^* H_2(r,\tau) \tag{3.3.49}$$

$$\frac{u'(r<\sigma,\tau)}{\Delta} \sim \frac{-1}{2r\sigma\Omega_o^*}\sum_{n=0}^{4} A_n^{**} F_{n+1}^+(r,\tau;K^*) + \frac{K^*\alpha_p^*}{2r\sigma\Omega_o^*(K^*-\alpha_p^*)}\sum_{n=0}^{4} B_n F_{n+1}^+(r,\tau;\alpha_p^*)$$

$$+ \frac{\sqrt{K^*}}{2r^2\sigma\Omega_o^*}\sum_{n=0}^{5} A_n^{**} F_n^-(r,\tau;K^*) - \frac{K^*\alpha_p^{*3/2}}{2r^2\sigma\Omega_o^*(K^*-\alpha_p^*)}\sum_{n=0}^{5} B_n^{**} F_n^-(r,\tau;\alpha_p^*) \tag{3.3.50}$$

$$\frac{u'(r>\sigma,\tau)}{\Delta} \sim \frac{1}{r^2\Omega_o^*}\left[\sum_{n=1}^{4} C_{n+1}^* G_n(r,\tau) + r\sum_{n=2}^{4} C_n^* G_n(r,\tau)\right]$$

$$- \frac{1}{r^2\Omega_o^*}\left[D_2^* H_1(r,\tau) + (D_4^* + rD_2^*)H_2(r,\tau) + D_5^* H_3(r,\tau)\right] \tag{3.3.51}$$

where the functions F's and G's are associated with diffusion and the functions H's with wave behavior:

$$F_o^-(r<\sigma,\tau;K^*) = 2\sqrt{\frac{\tau}{\pi}}\left[e^{-\frac{(\sigma-r)^2}{4K^*\tau}} - e^{-\frac{(\sigma+r)^2}{4K^*\tau}}\right]$$

$$- \left[\frac{\sigma-r}{\sqrt{K^*}}\,\mathrm{erfc}\left\{\frac{\sigma-r}{2\sqrt{K^*\tau}}\right\} - \frac{\sigma+r}{\sqrt{K^*}}\,\mathrm{erfc}\left\{\frac{\sigma+r}{2\sqrt{K^*\tau}}\right\}\right] \tag{3.3.52}$$

$$F_1^\pm(r<\sigma,\tau;K^*) = \mathrm{erfc}\left\{\frac{\sigma-r}{2\sqrt{K^*\tau}}\right\} \pm \mathrm{erfc}\left\{\frac{\sigma+r}{2\sqrt{K^*\tau}}\right\} \tag{3.3.53}$$

$$F_2^\pm(r<\sigma,\tau;K^*) = \frac{1}{\sqrt{\pi\tau}}\left[e^{-\frac{(\sigma-r)^2}{4K^*\tau}} \pm e^{-\frac{(\sigma+r)^2}{4K^*\tau}}\right] \tag{3.3.54}$$

$$F_3^\pm(r<\sigma,\tau;K^*) = \frac{1}{2\sqrt{\pi K^*\tau^3}}\left[(\sigma-r)\,e^{-\frac{(\sigma-r)^2}{4K^*\tau}} \pm (\sigma+r)\,e^{-\frac{(\sigma+r)^2}{4K^*\tau}}\right] \tag{3.3.55}$$

$$F_4^{\pm}(r<\sigma,\tau;K^*) = \frac{1}{4\sqrt{\pi\tau^5}}\{[\frac{(\sigma-r)^2}{K^*} - 2\tau]\, e^{-\frac{(\sigma-r)^2}{4K^*\tau}} \pm [\frac{(\sigma+r)^2}{K^*} - 2\tau]\, e^{-\frac{(\sigma+r)^2}{4K^*\tau}}\} \tag{3.3.56}$$

$$F_5^{\pm}(r<\sigma,\tau;K^*) = \frac{\sigma - r}{4\sqrt{\pi K^*\tau^5}}\,[\frac{(\sigma-r)^2}{2K^*\tau} - 3]\, e^{-\frac{(\sigma-r)^2}{4K^*\tau}}$$

$$\pm \frac{\sigma + r}{4\sqrt{\pi K^*\tau}}\,[\frac{(\sigma+r)^2}{2K^*\tau} - 3]\, e^{-\frac{(\sigma+r)^2}{4K^*\tau}} \tag{3.3.57}$$

$$G_1(r>\sigma,\tau) = \frac{r-\sigma}{2\sqrt{\pi\tau^3}}\, e^{-\frac{(r-\sigma)^2}{4\tau}} \tag{3.3.58}$$

$$G_2(r>\sigma,\tau) = \frac{1}{4\sqrt{\pi\tau^5}}\,[(r-\sigma)^2 - 2\tau]\, e^{-\frac{(r-\sigma)^2}{4\tau}} \tag{3.3.59}$$

$$G_3(r>\sigma,\tau) = \frac{(r-\sigma)}{4\sqrt{\pi\tau^5}}\,[\frac{(r-\sigma)^2}{2\tau} - 3]\, e^{-\frac{(r-\sigma)^2}{4\tau}} \tag{3.3.60}$$

$$G_4(r>\sigma,\tau) = \frac{1}{4\sqrt{\pi\tau^5}}\,[\frac{(r-\sigma)^4}{4\tau^2} - \frac{3(r-\sigma)}{2\tau} + 3]\, e^{-\frac{(r-\sigma)^2}{4\tau}} \tag{3.3.61}$$

$$H_1(r>\sigma,\tau) = L^{-1}\{e^{-(r-\sigma)\lambda_2}\} = \frac{b(r-\sigma)}{\sqrt{\pi\tau^3}}\, e^{-\frac{b^2[r-\sigma)-\tau]^2}{\tau}} \tag{3.3.62}$$

$$H_2(r>\sigma,\tau) = L^{-1}\{s\, e^{-(r-\sigma)\lambda_2}\}$$

$$= \{\frac{b^2}{\tau}[(r-\sigma)-\tau]^2 + 2b^2[(r-\sigma)-\tau] - \frac{3}{2}\}\,\frac{b(r-\sigma)}{\sqrt{\pi\tau^5}}\, e^{-\frac{b^2[(r-\sigma)-\tau]^2}{\tau}} \tag{3.3.63}$$

$$H_3(r>\sigma,\tau) = L^{-1}\{s^{3/2}\, e^{-(r-\sigma)\lambda_2}\} \tag{3.3.64}$$

The functions $F_0^-(r,\tau;\alpha_p^*)$ and $F_i^{\pm}(r,\tau;\alpha_p^*)$, i=1,2,3,4,5, can be obtained by substituting α_p^* for K^* into the above corresponding functions. Note that the λ_2-function appearing in Eqs. (3.3.62), (3.3.63), and (3.3.64) is given by Eq. (3.3.2):

$$\lambda_2=2b[\sqrt{s + b^2} - b] + O(s^3) \quad , \; b=(2\gamma)^{-\frac{1}{2}}$$

The function H_3, Eq. (3.3.64) cannot be integrated analytically by the author at this time. However, its value on the surface of the particle, $r=\sigma$, can be found. For $r=\sigma$, H_3 becomes

$$H_3(\sigma,\tau) = L^{-1}\{s^{3/2}\} = \frac{3}{4\sqrt{\pi\tau^5}} \tag{3.3.65}$$

The solution for large time τ shows that, when $r>\sigma$, there is a compression wave traveling away from the particle, which arises from the functions H's. However, since these terms are very small compared with the diffusion terms, there is no dramatic wave behavior in this problem. It follows that the diffusion terms play a dominant role throughout all the flow field. Recall that

$$\tau = \sigma \; \tau^* \tag{3.3.66}$$

where

$$\sigma \equiv a_0\bar{\sigma}/\tilde{\nu}_0$$
$$\tau^* \equiv a_0 t/\bar{\sigma}$$

It follows from Eq. (3.2.48) that, in order for the Knudsen number K_n to be small, σ must be large. Thus, with large σ, τ will be large when τ^* is chosen as of order unity.

The large time behavior of the solution for the temperature perturbation as a function of $\bar{r}/\bar{\sigma}$ is shown in Figs. 2, 3, 4 for $\gamma = 1.4$, $\alpha_p^* = 0.0074$, $K^* = 1.3473$, $k_{go}/k_{po} = 0.1$, $\sigma = 80$, 90, 100, and various wave time τ^*. These curves indicate that, for a fixed wave time τ^*, the temperature perturbation decreases when $\sigma = a_o\bar{\sigma}/\tilde{\nu}_o$ decreases. This means that, for a gas with constant viscosity, the smaller the particle, the faster the rate for the temperature to achieve equilibrium. On the other hand, if the size of the particle is fixed, then the temperature will approach equilibrium at a faster rate for the gas with higher viscosity. It is noted that, on the surface of the particle, the temperature can be written as

$$\frac{T'(\sigma,\tau)}{\Delta} \sim \frac{1}{\tau\Omega_o^*}\left(\frac{C_3^*}{2\sqrt{\pi\tau^3}} - \frac{3C_5^*}{4\sqrt{\pi\tau^5}}\right) \tag{3.3.67}$$

and at the origin

$$\frac{T'(0,\tau)}{\Delta} \sim \frac{1}{\sigma\Omega_o^*}\sum_{n=0}^{5} B_n^{**}E_n(\tau;\alpha_p^*) - 1 \tag{3.3.68}$$

where

$$E_o(\tau;\alpha_p^*) = \text{erfc}\left\{\frac{\sigma}{2\sqrt{\alpha_p^*\tau}}\right\}$$

$$= 1 - \frac{2}{\sqrt{\pi}}\left\{\frac{\sigma}{2\sqrt{\alpha_p^*\tau}} - \frac{1}{3}\left(\frac{\sigma}{2\sqrt{\alpha_p^*\tau}}\right) + \dots\right\} \tag{3.3.69}$$

$$E_1(\tau;\alpha_p^*) = \frac{1}{\sqrt{\pi\tau}}\, e^{-\frac{\sigma^2}{4\alpha_\rho^*\tau}} \tag{3.3.70}$$

$$E_2(\tau;\alpha_p^*) = \frac{\sigma}{2\sqrt{\pi\alpha_p^*\tau^3}} e^{-\frac{\sigma^2}{4\alpha_p^*\tau}} \tag{3.3.71}$$

$$E_3(\tau;\alpha_p^*) = \frac{1}{4\sqrt{\pi\tau^5}} (\frac{\sigma^2}{\alpha_p^*} - 2\tau) e^{-\frac{\sigma^2}{4\alpha_p^*\tau}} \tag{3.3.72}$$

$$E_4(\tau;\alpha_p^*) = \frac{\sigma}{4\sqrt{\pi\tau^5}} (\frac{\sigma^2}{2\alpha_p^*\tau} - 3) e^{-\frac{\sigma^2}{4\alpha_p^*\tau}} \tag{3.3.73}$$

$$E_5(\tau;\alpha_p^*) = \frac{1}{4\sqrt{\pi\tau^5}} (\frac{\sigma^4}{4\alpha_p^{*2}\tau^2} - \frac{3\sigma^2}{2\alpha_p^*\tau} + 3) e^{-\frac{\sigma^2}{4\alpha_p^*\tau}} \tag{3.3.74}$$

Thus, on the surface of the particle, the temperature varies, to the lowest order, like $\tau^{-3/2}$ and like $\tau^{-1/2}$ at the origin. Far away from the particle, the temperature perturbation approaches zero. Furthermore, it should be emphasized here that at $r=\sigma$ the slope of the temperature curve is not continuous. This can be readily seen from the boundary condition, Eq. (3.1.28):

$$[\frac{\partial T'}{\partial r}]_{r=\sigma^-} = \frac{k_{go}}{k_{po}} [\frac{\partial T'}{\partial r}]_{r=\sigma^+}$$

Since $k_{go}/k_{po} < 1$, it is apparent that

$$[\frac{\partial T'}{\partial r}]_{r=\sigma^-} < [\frac{\partial T'}{\partial r}]_{r=\sigma^+} \tag{3.3.75}$$

Therefore, the temperature gradient for r approaches σ from inside is less than that from outside. As mentioned previously, the wave behavior is so weak that there are no wave fronts appearing in these figures. However, the effect of viscosity on the wave fronts can still be detected from the wave term $H = e^{-b^2[(r-\sigma)-\tau]^2/\tau}$. In Fig. 5, H is plotted as a function of $X = (r-\sigma)/\tau-1$ for $\tau^* = 1.0$ and $\sigma = 10, 15, 20$ It is found that the wave fronts become more diffuse as the Reynolds number σ becomes smaller, i.e., as the viscous effects become more pronounced.

The corresponding curves for the density-perturbation profiles are shown in Figs. 6, 7, 8. The variation of density with Reynolds number σ is similar to that of temperature. Note that

$$\frac{\rho'(\sigma,\tau)}{\Delta} \sim -\frac{1}{\sigma\Omega_0^*}\left[\frac{C_3^*}{2\sqrt{\pi\tau^3}} - \frac{3C_5^*}{4\sqrt{\pi\tau^5}}\right] \tag{3.3.76}$$

$$\frac{\rho'(0,\tau)}{\Delta} \sim \frac{1}{\sigma\Omega_0^* K^*}\sum_{n=0}^{5} A_n^{**}E_n(\tau;K^*) - \frac{K^*}{\sigma\Omega_0^*(K^* - \alpha_p^*)}\sum_{n=0}^{5} B_n^{**}E_n(\tau;\alpha_p^*) + 1 \tag{3.3.77}$$

where $E_i(\tau;K^*)$, i=1,2,3,4,5, can be found by substituting K^* for α_p^* into the $E_i(r,\tau;\alpha_p^*)$ functions, Eqs. (3.3.69)-(3.3.74). Thus, analogous to the temperature, the density perturbation varies at least like $\tau^{-3/2}$ at $r=\sigma$, and like $\tau^{-1/2}$ at r=0.

In order to have a physical insight for the pressure perturbation, let us return to Eq. (3.1.23):

$$\bar{P}'(r>\sigma,s) = C\,\frac{\gamma}{r}\,(\lambda_1^2 - s)\,e^{-\lambda_1 r} + D\,\frac{\gamma}{r}\,(\lambda_2^2 - s)\,e^{-\lambda_2 r} \tag{3.1.23}$$

It is noted that, for $\lambda_1 = \sqrt{s}$, the first term in the right hand side of Eq. (3.1.23) disappears, and the pressure becomes

$$\bar{P}'(r>\sigma,s) = D\,\frac{\gamma}{r}\,(\lambda_2^2 - s)\,e^{-\lambda_2 r} \tag{3.1.25}$$

Thus, under the assumption that $P_r^* = 1$, the pressure outside the particle is dominated by the wave term, which is due to λ_2. The asymptotic solution for the pressure perturbation is found to be

$$\frac{P'(r<\sigma,\tau)}{\Delta} \sim \frac{1}{2r\sigma\Omega_o^*\sqrt{K^*}} \sum_{n=0}^{5} A_n^{**} F_n^-(r,\tau;K^*) - \frac{\alpha_p^{*3/2}}{2r\sigma\Omega_o^*(K^* - \alpha_p^*)} \sum_{n=0}^{5} B_n^{**} F_n^-(r,\tau;\alpha_p^*) \tag{3.3.78}$$

$$\frac{P'(r>\sigma,\tau)}{\Delta} \sim \frac{\gamma}{r\Omega_o^*}\left[- D_2^* H_2(r,\tau) + (D_2^* - D_4^*)H_4(r,\tau) - D_5^* H_5(r,\tau)\right] \tag{3.3.79}$$

where

$$H_4(r>\sigma,\tau) = L^{-1}\{s^2\, e^{-(r-\sigma)\lambda_2}\}$$

$$= \left\{\frac{b^2[(r-\sigma)-\tau]^2}{\tau} + 2b^2[(r-\sigma)-\tau]\right\}\left\{\frac{b^2[(r-\sigma)-\tau]^2}{\tau} + 2b^2[(r-\sigma)-\tau] - \frac{3}{2}\right\}$$

$$\frac{b(r-\sigma)}{\sqrt{\pi\tau^7}}\, e^{-\frac{b^2[(r-\sigma)-\tau]^2}{\tau}} - \left\{\frac{b^2[(r-\sigma)-\tau]^2}{\tau} + 2b^2[(r-\sigma)-\tau] + 2b^2\tau\right\}$$

$$\frac{b(r-\sigma)}{\sqrt{\pi\tau^7}}\, e^{-\frac{b^2[(r-\sigma)-\tau]^2}{\tau}} - \left\{\frac{b^2[(r-\sigma)-\tau]^2}{\tau} + 2b^2[(r-\sigma)-\tau] - \frac{3}{2}\right\}$$

$$\frac{5b(r-\sigma)}{2\sqrt{\pi\tau^7}}\, e^{-\frac{b^2[(r-\sigma)-\tau]^2}{\tau}} \tag{3.3.80}$$

$$H_5(r>\sigma,\tau) = L^{-1}\{s^{5/2}\, e^{-(r-\sigma)\lambda_2}\} \tag{3.3.81}$$

At $r=\sigma$, Eq. (3.3.79) gives

$$\frac{P'(\sigma,\tau)}{\Delta} \sim -\frac{\gamma D_5^*}{\sigma\Omega_o^*} L^{-1}\{s^{5/2}\} \sim \tau^{-7/2} \tag{3.3.82}$$

Thus, on the surface of the porous particle, the pressure varies at least like $\tau^{-7/2}$, which is very small when τ is large. On the other hand, at $r=0$, we have

$$\frac{P'(0,\tau)}{\Delta} \sim \frac{1}{\sigma\Omega_o^* K^*} \sum_{n=0}^{5} A_n^{**} E_n(\tau;K^*) - \frac{\alpha_p^*}{\sigma\Omega_o^*(K^* - \alpha_p^*)} \sum_{n=0}^{5} B_n^{**} E_n(\tau;\alpha_p^*) \tag{3.3.83}$$

To the lowest order, $P'(0,\tau)/\Delta$ takes the form

$$\frac{P'(0,\tau)}{\Delta} \sim \frac{A_o^{**}}{\sigma\Omega_o^* K^*} E_o(\tau;K^*) - \frac{\alpha_p^* B_o^{**}}{\sigma\Omega_o^*(K^* - \alpha_p^*)} E_o(\tau;\alpha_p^*) \tag{3.3.84}$$

Since

$$A_o^{**} = \frac{\sigma\Omega_o^* K^* \alpha_p^*}{K^* - \alpha_p^*} \tag{3.3.85}$$

$$B_o^{**} = \sigma\Omega_o^*$$

$$E_o(\tau;K^*) = \text{erfc}\left\{\frac{\sigma}{2\sqrt{K^*\tau}}\right\} \sim 1 - \frac{\sigma}{\sqrt{\pi K^*\tau}} + \ldots \tag{3.3.87}$$

$$E_o(\tau;\alpha_p^*) = \text{erfc}\left\{\frac{\sigma}{2\sqrt{\alpha_p^*\tau}}\right\} \sim 1 - \frac{\sigma}{\sqrt{\pi\sigma_p^*\tau}} + \ldots \tag{3.3.88}$$

it immediately follows that

$$\frac{P'(0,\tau)}{\Delta} \sim \frac{\sigma\sqrt{\alpha_p^*}}{\sqrt{\pi K^*}(K^* - \alpha_p^*)} \tau^{-1/2} \tag{3.3.89}$$

Thus, at the origin of the particle, the pressure varies at least like $\tau^{-\frac{1}{2}}$, which is much larger than the pressure at $r=\sigma$. According to Darcy's law, it is this pressure gradient that causes the gas to eject from the porous particle.

The velocity profiles are shown in Figs. 9, 10, and 11. Because of the difficulty encountered in the integration of the term H_3, Eq. (3.3.64), only the velocity inside the porous particle is illustrated here. It can be seen from these figures that the velocity vanishes at the origin and reaches a maximum value at the surface of the particle. From Eq. (3.3.51), we have

$$\frac{u'(\sigma,\tau)}{\Delta} \sim \frac{1}{\sigma^2\Omega_0^*} [(C_3^* + \sigma C_2^*)\ L^{-1}\{\sqrt{s}\} + (C_5^* + \sigma C_4^* - D_5^*)\ L^{-1}\{s^{3/2}\}] \qquad (3.3.90)$$

Further, since

$$C_3^* + \sigma C_2^* = 0$$

Eq. (3.3.90) becomes

$$\frac{u'(\sigma,\tau)}{\Delta} \sim (C_5^* + \sigma\ C_4^* - D_5^*)\ \frac{3}{4\sigma^2\Omega_0^*\sqrt{\pi\tau^5}} \qquad (3.3.91)$$

that is, at $r=\sigma$, the velocity varies at least like $\tau^{-5/2}$.

3.4 Mass Efflux from Porous Particles

In order to fit the problem of small disturbances in porous-particle dusty gases, it is necessary to derive the term involving the mass ejecting from a unit volume of small porous particles per unit time. If we denote $\dot{J}$ as the mass ejecting from a porous particle per unit time, then we have

$$\dot{J} = 4\pi\,\bar{\sigma}^2\,\rho^* u^* \tag{3.4.1}$$

where ρ^* and u^* are the density and velocity of the gas at $\bar{r} = \bar{\sigma}$. Further, let n be the number of particles per unit volume and m be the mass of each individual particle, and note that $u^* = a_o u'(\sigma,\tau)$. Then the mass ejecting from a unit volume of small porous particles per unit time, $\dot{\mu}$, can be written as

$$\dot{\mu} = n\dot{J} = \frac{4\pi\bar{\sigma}^2}{m}\, a_o \rho_p \rho^*\, u'(\sigma,\tau) \tag{3.4.2}$$

where ρ_p = mn is the mass of the porous particles per unit volume. Now, recall from Eqs. (3.3.91) and (3.3.67) that, to the lowest order,

$$\frac{u'(\sigma,\tau)}{\Delta} \sim \frac{3}{4\sigma^2\Omega_o^*\sqrt{\pi\tau^5}}\,(C_5^* + \sigma C_4^* - D_5^*)$$

and

$$\frac{T'(\sigma,\tau)}{\Delta} \sim \frac{C_3^*}{2\sigma\Omega_o^*\sqrt{\pi\tau^3}}$$

Thus, we have

$$u'(\sigma,\tau) \sim C^*\,\frac{T'(\sigma,\tau)}{\tau} \tag{3.4.3}$$

where

$$C^* \equiv \frac{3(C_5^* + \sigma C_4^* - D_5^*)}{2\sigma C_3^*} \tag{3.4.4}$$

Substituting Eq. (3.4.3) into Eq. (3.4.2) yields

$$\dot{\mu} = \frac{4\pi\bar{\sigma}^2 a_o \rho_p \rho^* C^*}{m\tau}\, T'(\sigma,\tau) \tag{3.4.5}$$

Recall that $T'(\sigma,\tau)$ can be written as

$$T'(\sigma,\tau) = \frac{T(\bar{\sigma},t) - T_o}{T_o} \tag{3.4.6}$$

At this moment, if we treat $T(\bar{\sigma},t)$ as the temperature of the particles, T_p, and T_0 as the temperature of the gas surrounding the particles, T, then Eq. (3.4.6) can be rewritten as

$$T' = \frac{T_p - T}{T} \tag{3.4.7}$$

Substituting Eq. (3.4.7) into Eq. (3.4.5), we obtain

$$\dot{\mu} = \frac{C^* \rho^*}{\tau_\mu \tau T} (T_p - T) \tag{3.4.8}$$

where τ_μ has the dimension of time and is defined as

$$\tau_\mu \equiv \frac{m}{4\pi\bar{\sigma}^2 a_0 \rho_0} \tag{3.4.9}$$

Note that

$$\tau = a_0^2 t/\tilde{\nu}_0$$

Thus, Eq. (3.4.8) becomes

$$\dot{\mu} = \frac{C^* \rho^*}{\tau_\mu^* t T} \Delta T \tag{3.4.10}$$

where $\tau_\mu^* \equiv a_0^2\, \tau_\mu/\tilde{\nu}_0$ and $\Delta T = T_p - T$. Therefore, $\dot{\mu}$ is proportional to ΔT and varies like t^{-1}.

The governing equations for the flow in the inert porous-particle dusty gases are:

<u>Mass</u>

$$\frac{\partial \rho}{\partial t} + \bar{\nabla}\cdot(\rho \vec{V}) = \dot{\mu} \tag{3.4.11}$$

$$\frac{\partial \rho_p}{\partial t} + \bar{\nabla}\cdot(\rho_p \vec{V}_p) = -\dot{\mu} \tag{3.4.12}$$

Momentum

$$\rho \frac{D\vec{V}}{Dt} = \vec{F}_p + \rho\vec{g} - \bar{\nabla}P + \bar{\nabla}\cdot\overleftrightarrow{\tau} + \dot{\mu}(\vec{V}_p - \vec{V}) \tag{3.4.13}$$

$$\rho_p \frac{D_p\vec{V}_p}{Dt} = - \vec{F}_p + \rho_p\vec{g} \tag{3.4.14}$$

Energy

$$\rho \frac{De}{Dt} = - P\bar{\nabla}\cdot\vec{V} + \overleftrightarrow{\tau}:\nabla\vec{V} - \bar{\nabla}\cdot\vec{q} + \dot{Q}_p + (\vec{V}_p - \vec{V})\cdot\vec{F}_p$$

$$+ \dot{\mu}[h(T_p) - e + \frac{1}{2}(\vec{V}_p - \vec{V})^2] \tag{3.4.15}$$

$$\rho_p \frac{D_p e_p}{Dt} = - \dot{Q}_p - \dot{\mu}[h(T_p) - e_p] \tag{3.4.16}$$

where the terms involving $\dot{\mu}$ are the source terms due to the ejection of mass from the porous particles, and $\vec{F}_p$ and $\dot{Q}_p$ are given by Eqs. (2.2.9) and (2.2.11) in Part I:

$$\vec{F}_p = \frac{\rho_p}{\tau_V}(\vec{V}_p - \vec{V}) \tag{3.4.17}$$

$$\dot{Q}_p = \frac{\rho_p C_p}{\tau_T}(T_p - T) \tag{3.4.18}$$

The set of equations, (3.4.11)-(3.4.16), is a special case of the more general set of equations for the flow in reacting, vaporizing, dusty gases. These general equations are derived in Appendix A.

CHAPTER 4

CONCLUDING REMARKS

Viscous effects on the linearized wave propagation associated with the low-Reynolds-number flow from a porous particle has been investigated by means of the method developed in Part I. The asymptotic solution for small time is not valid physically since this region is governed by free-molecule flow. The large-time solution shows that the wave behavior in this problem is not so dramatic as that in high-Reynolds-number flow. The reason for this seems that the viscous effect in low-Reynolds-number flow is stronger than in high-Reynolds-number flow. As a result, the wave fronts in the flow at low Reynolds numbers dissipate so fast that the diffusion behavior plays a dominant role throughout all the flow field.

For the problem that involves small disturbances in inert porous-particle dusty gases, the mass-, momentum-, and energy-source term due to the ejection of mass from the porous particles should be taken into account. In this regard, the mass ejecting from a unit volume of small porous particles per unit time, $\dot{\mu}$, has been derived. For future research, the governing equations for the flow in inert porous-particle dusty gases have been built up.

Perhaps a more interesting development of this problem could be obtained by dealing with vaporizable particles and reacting gases. In this case, the basic equations would have to be modified to account for mass diffusion and chemical reactions.

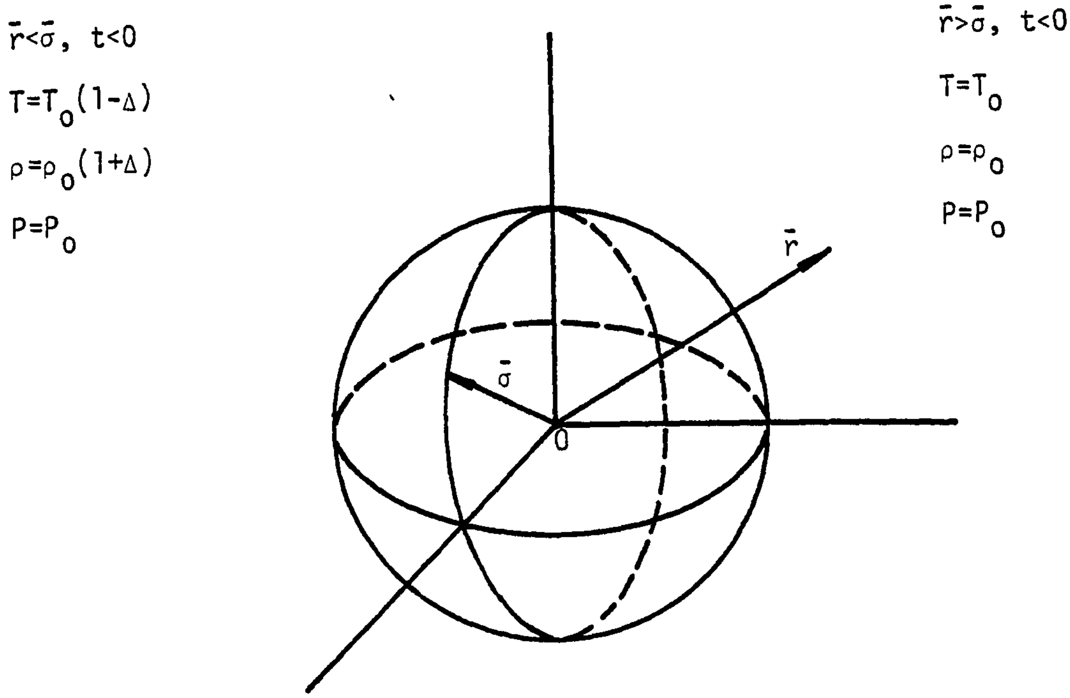

Fig. 1. Initial Conditions and Spherical Configuration

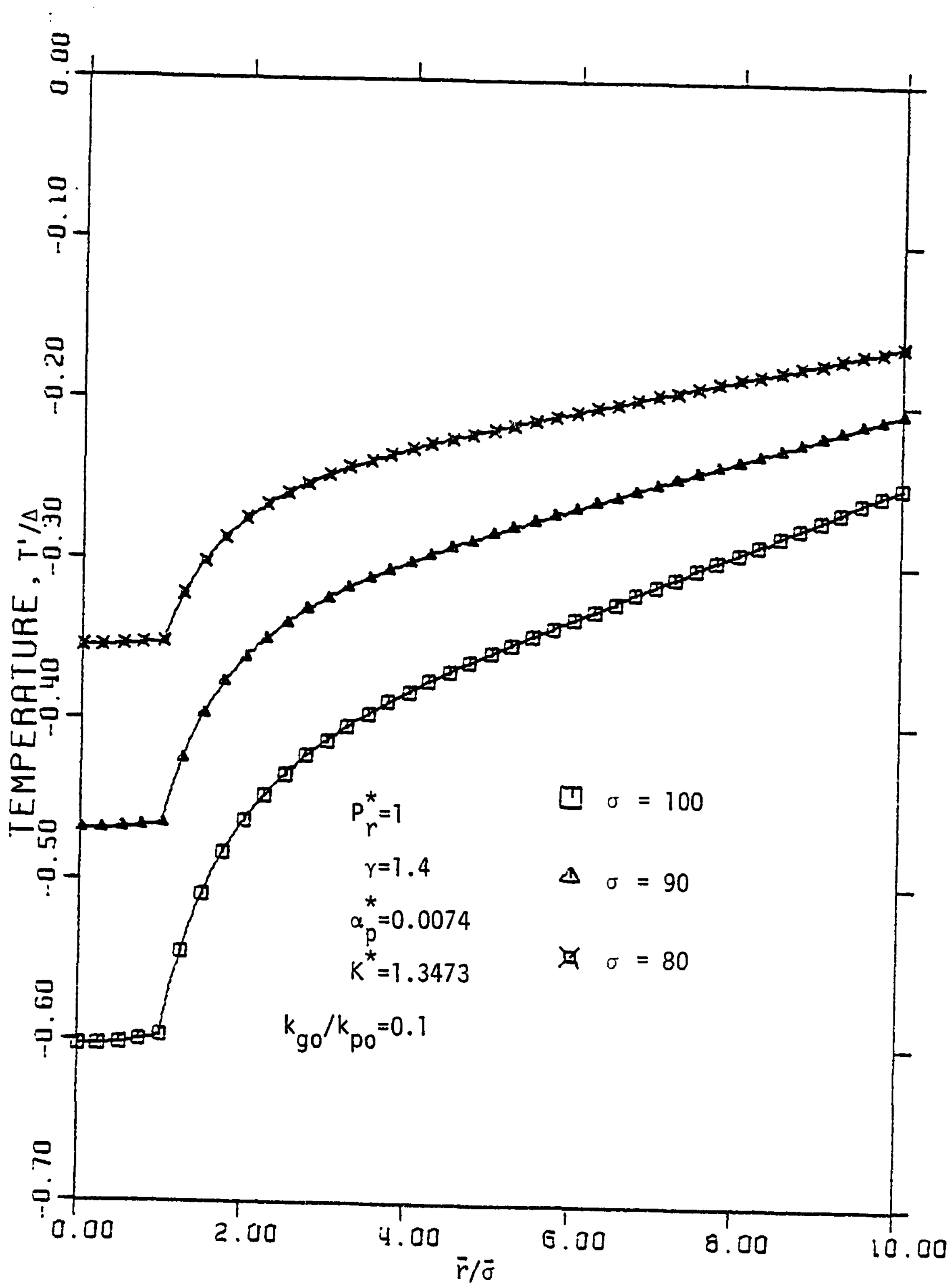

Fig. 2. LARGE-TIME BEHAVIOR OF TEMPERATURE FOR $\tau^* = 1.0$

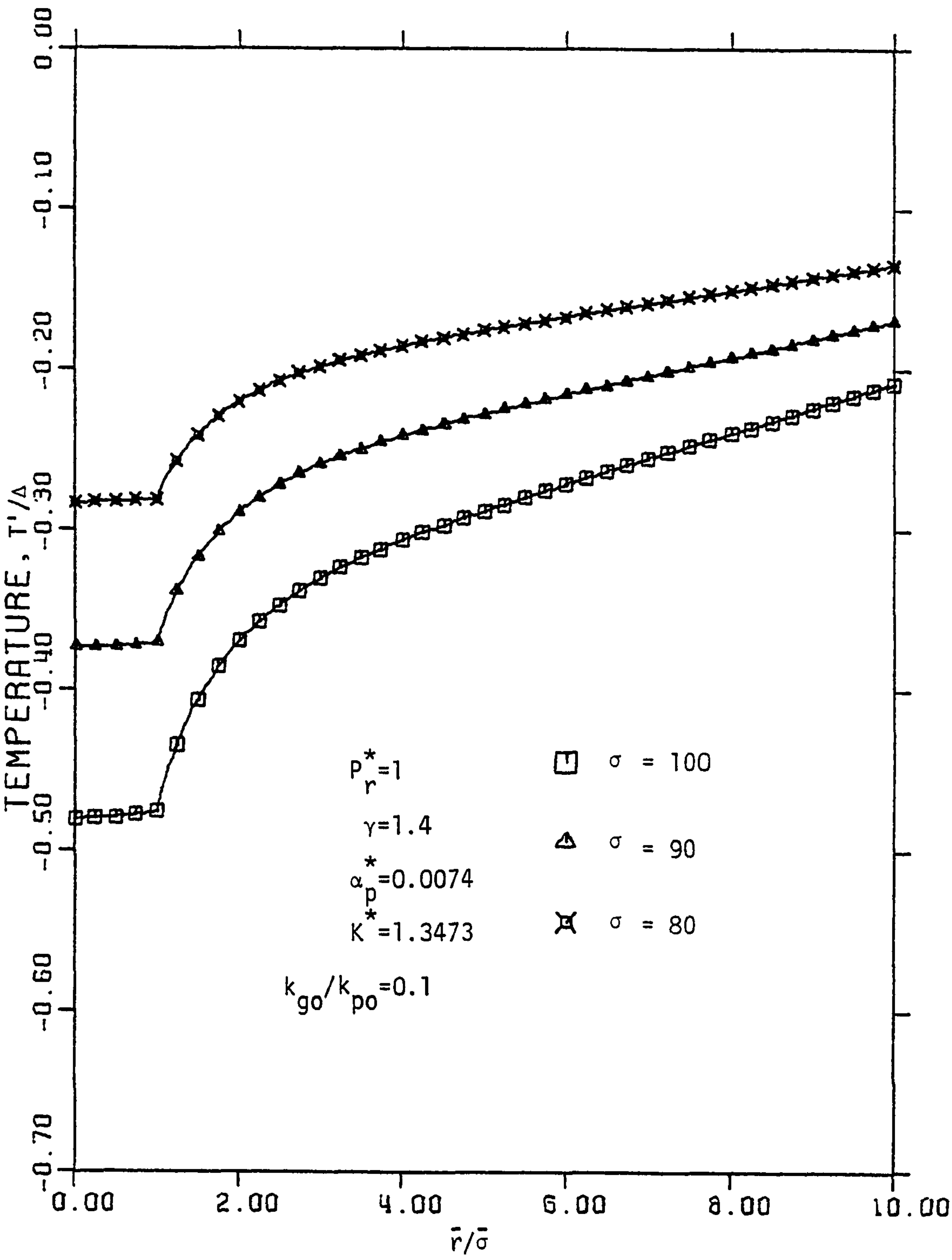

Fig. 3. LARGE-TIME BEHAVIOR OF TEMPERATURE FOR $\tau^* = 1.1$

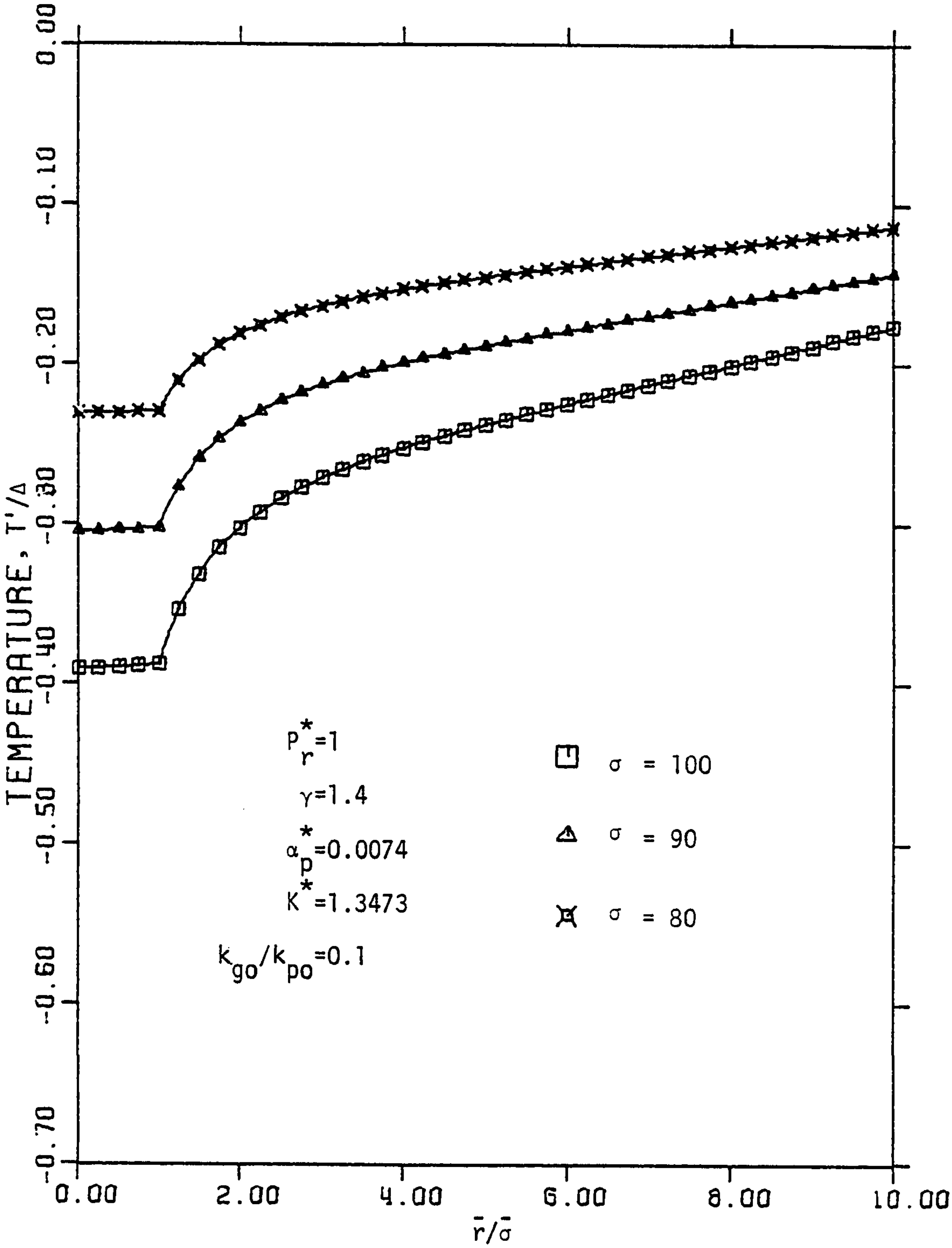

Fig. 4. LARGE-TIME BEHAVIOR OF TEMPERATURE FOR $\tau^* = 1.2$

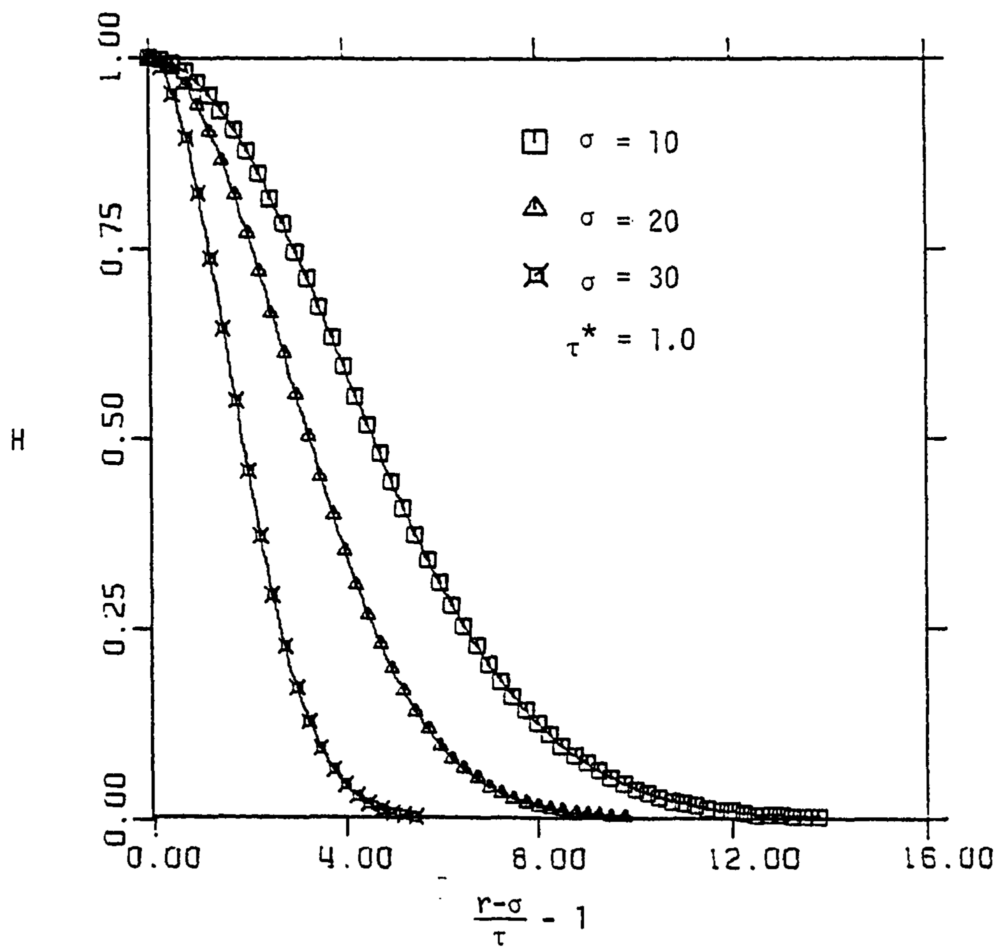

Fig. 5. LARGE-TIME WAVE BEHAVIOR

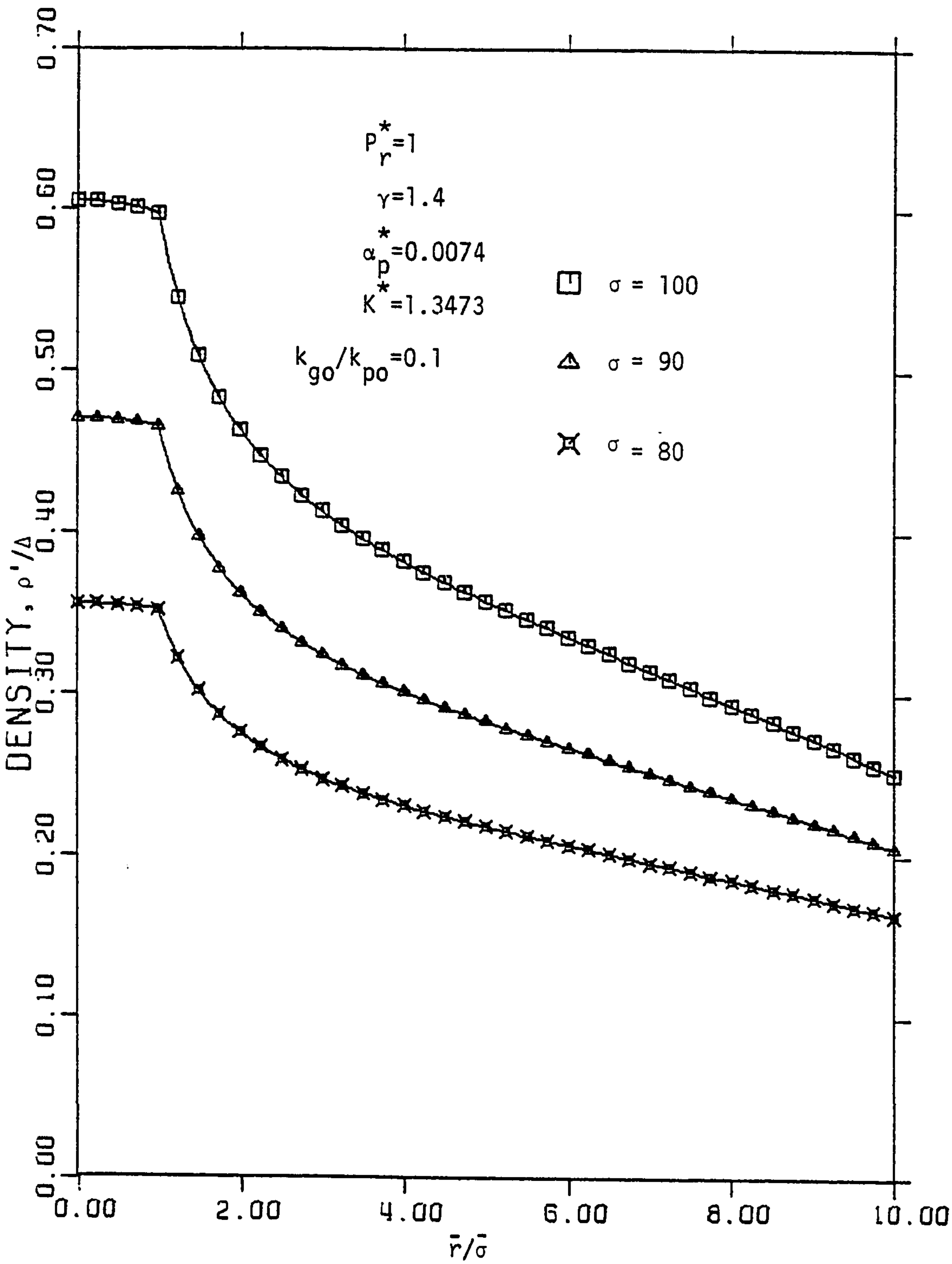

Fig. 6. LARGE-TIME BEHAVIOR OF DENSITY FOR $\tau^* = 1.0$

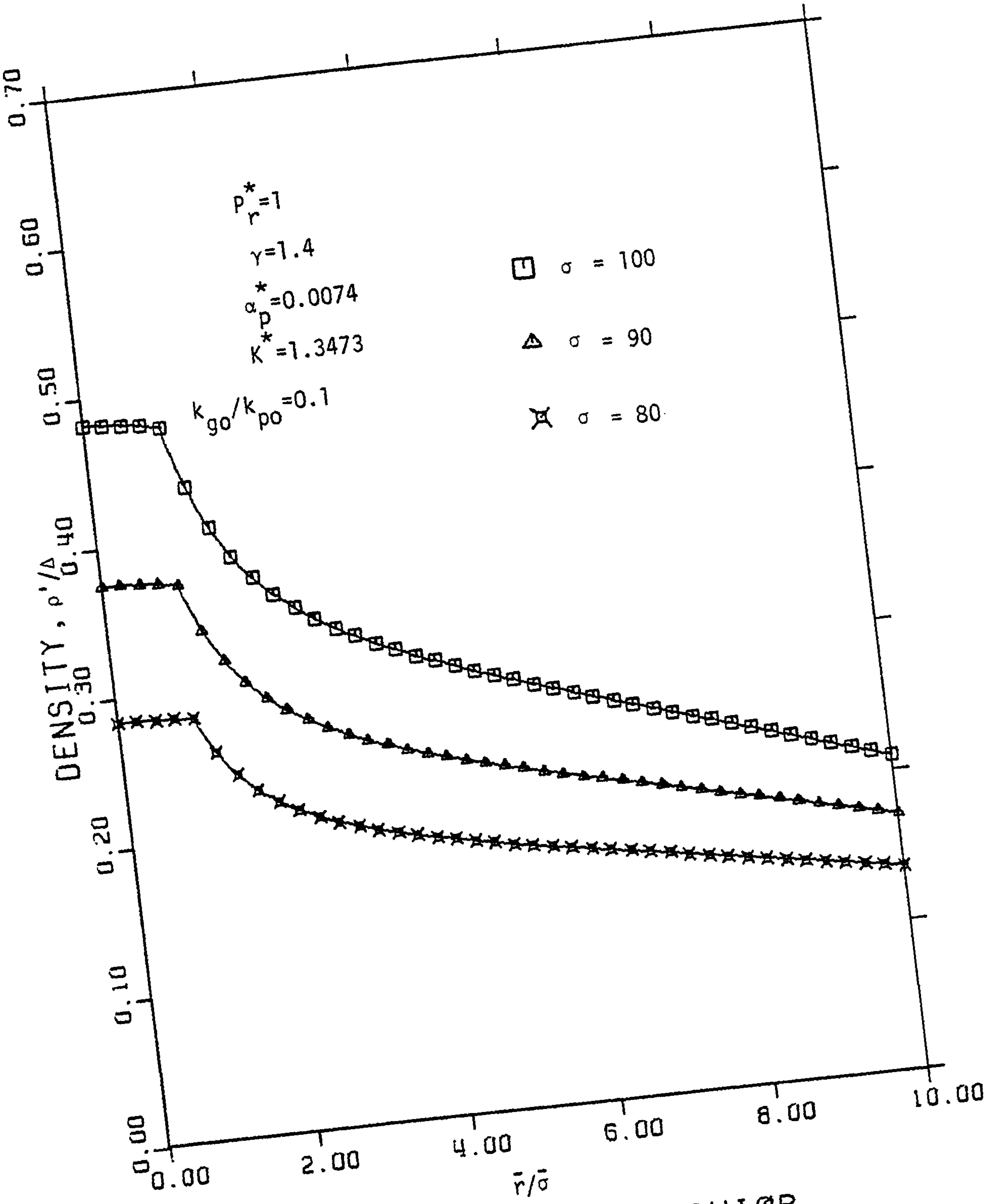

Fig. 7. LARGE-TIME BEHAVIOR OF DENSITY FOR $\tau^* = 1.1$

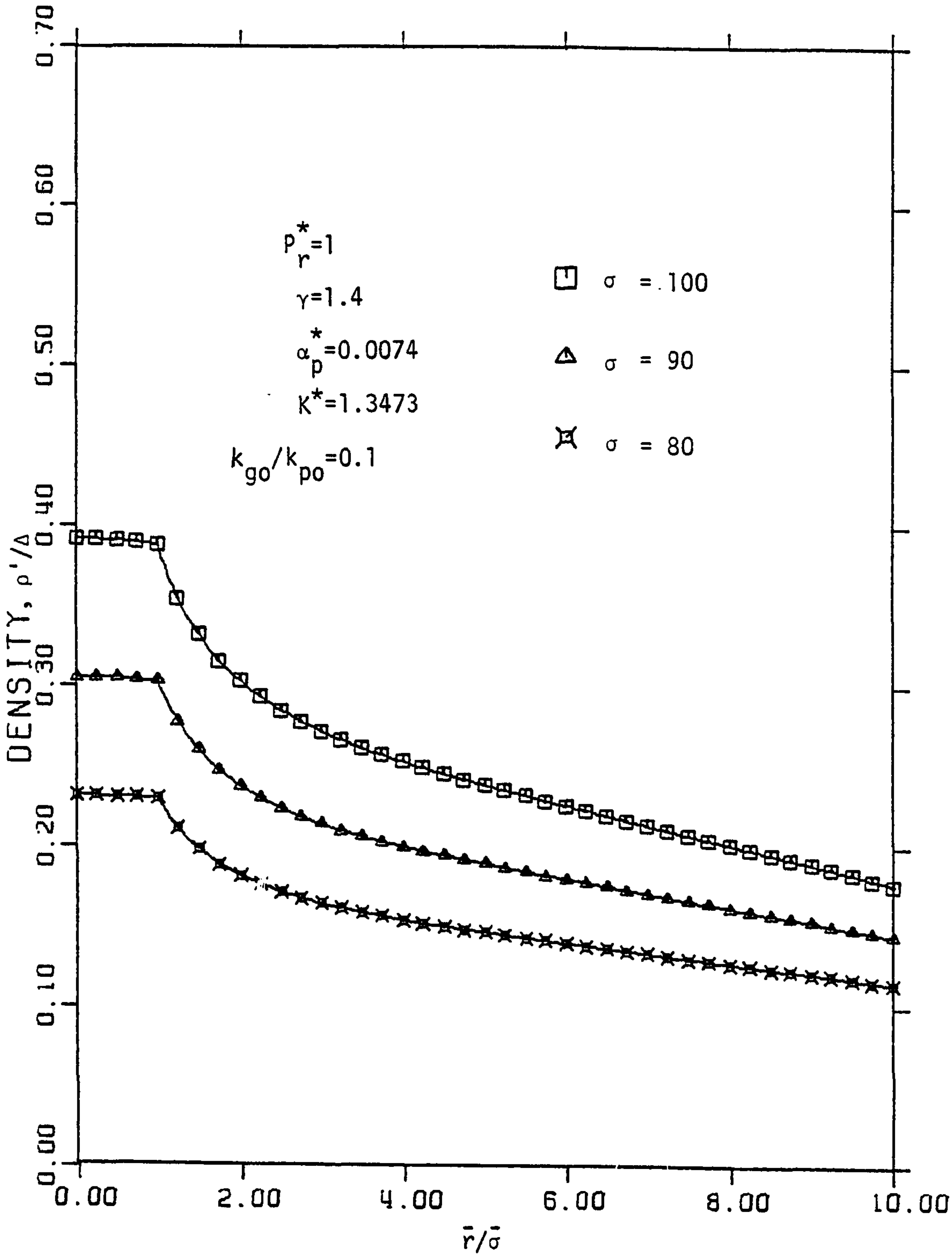

Fig. 8. LARGE-TIME BEHAVIOR OF DENSITY FOR $\tau^* = 1.2$

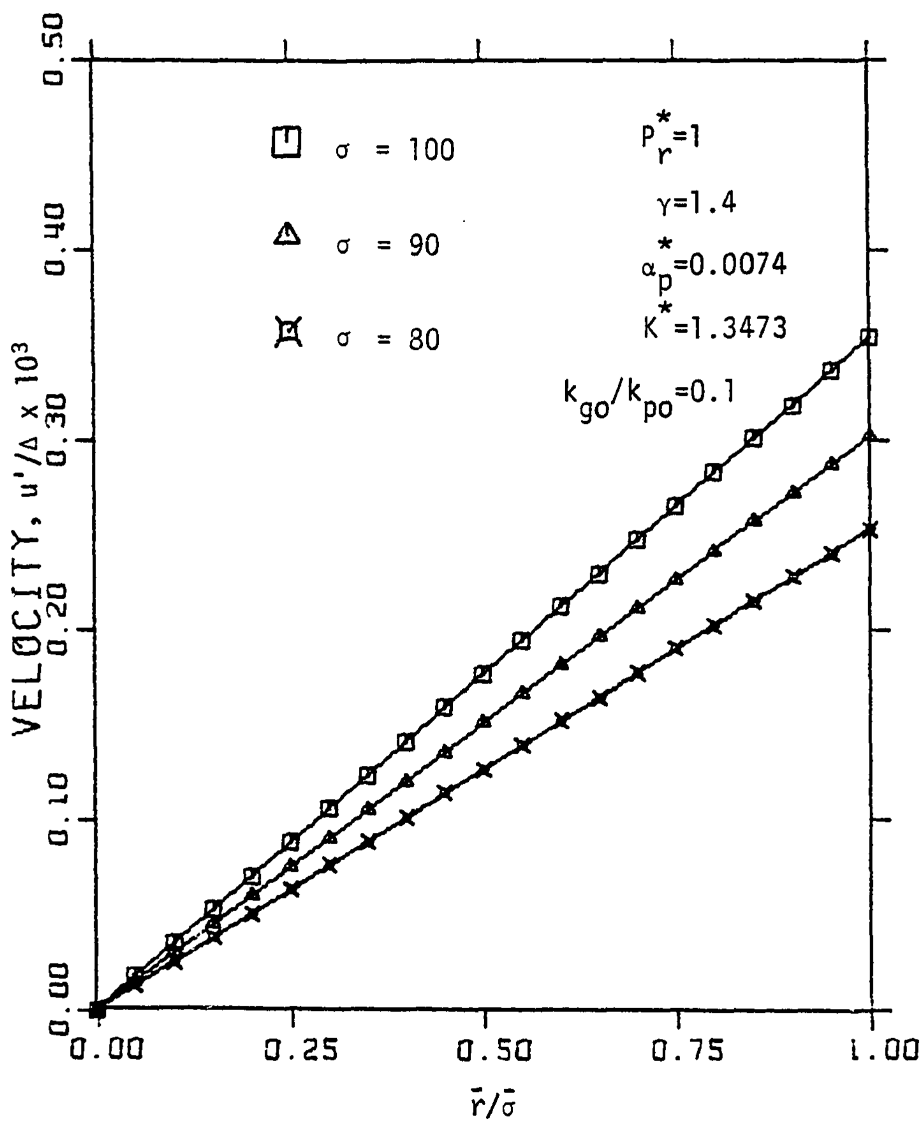

Fig. 9. LARGE-TIME BEHAVIOR OF VELOCITY FOR τ^* = 1.0

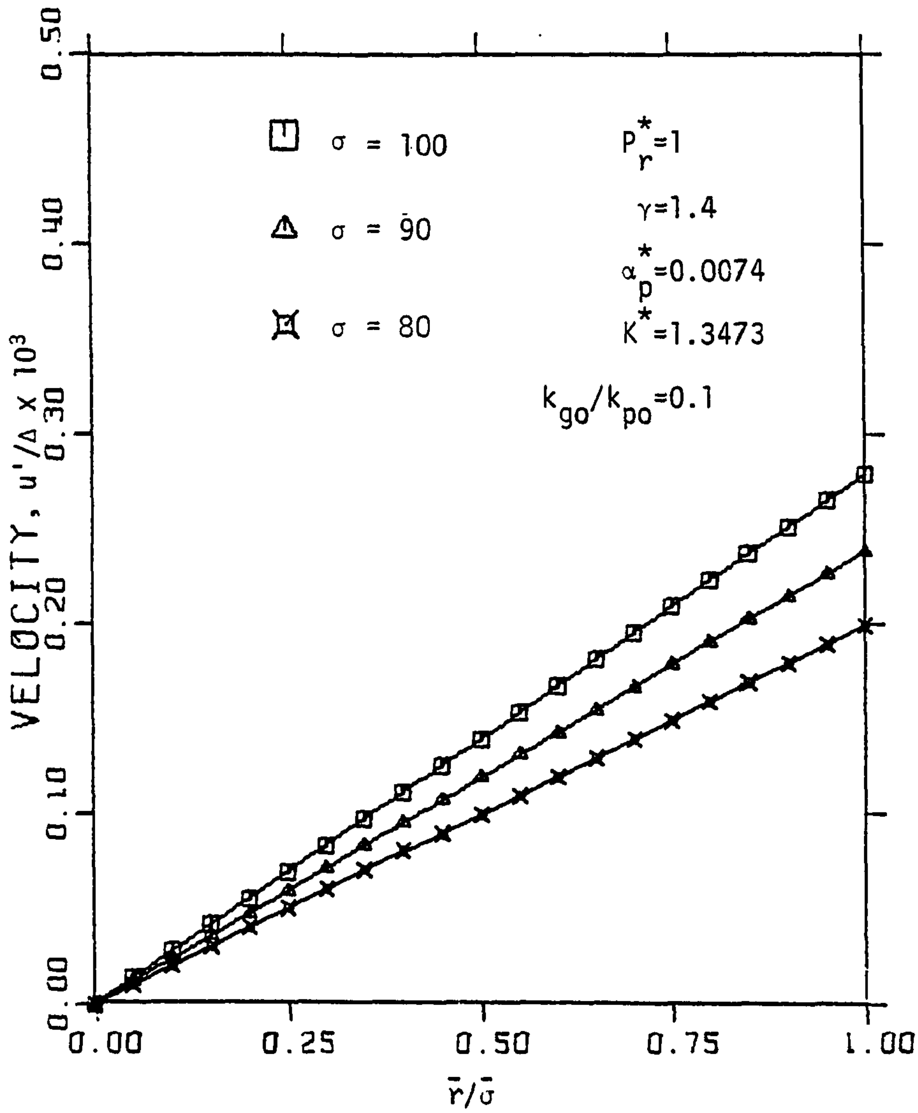

Fig. 10. LARGE-TIME BEHAVIOR OF VELOCITY FOR $\tau^* = 1.1$

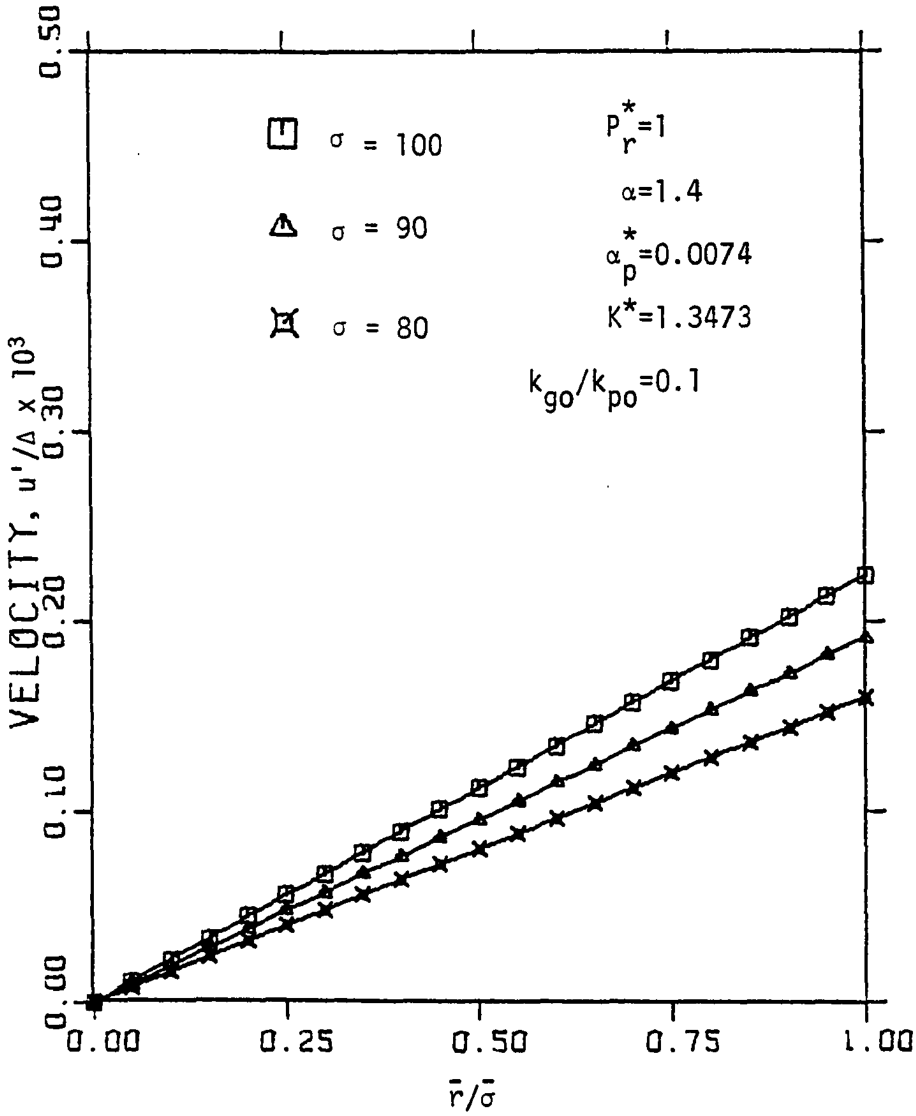

Fig. 11. LARGE-TIME BEHAVIOR OF VELOCITY FOR τ^* = 1.2

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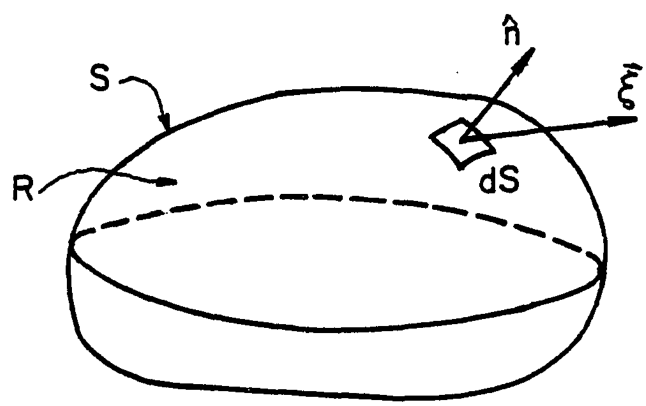
APPENDIX A

MECHANICS OF GAS-PARTICLE MIXTURES*

Time Derivative Following a Region

A Basic tool for understanding and manipulating mixtures of components that travel at different average velocities is that of Liebnitz' Rule. Imagine a closed surface S each point of which travels with a velocity $\vec{\xi}$. Let the closed surface S surround a region R. The time rate of change of the volume integral of some quantity $Q(\vec{r},t)$ is

$$\frac{d}{dt}\iiint_R Q(\vec{r},t)\, d\tau \equiv \iiint_R \frac{\partial Q}{\partial t}\, d\tau + \oint\!\!\oint_S Q\vec{\xi}\cdot\hat{n}\,dS \tag{1}$$



* This work follows the notes written by Dr. Maurice L. Rasmussen.

The nature of this time derivative thus depends upon the velocity of the surface, $\vec{\xi}$. The derivative is thus said to be taken following the region with velocity $\vec{\xi}$.

In fluid mechanics a particular velocity of interest is the fluid mass average velocity, denoted by $\vec{V}$. When $\vec{\xi}$ is set equal to $\vec{V}$ in Eq. (1), the region is said to be a fluid region, and the time derivative operator d/dt is given the special symbol D/Dt. Equation (1) then becomes

$$\frac{D}{Dt}\iiint_R Q(\vec{r},t)d\tau \equiv \iiint_R \frac{\partial Q}{\partial t}\, d\tau + \oiint_S Q\vec{V}\cdot\hat{n}dS \tag{2}$$

A fluid region is of particular interest because, since the boundary travels with the mass average velocity, no fluid mass crosses the boundary. Thus, the fluid mass inside a fluid region will be constant unless fluid mass is created inside the volume of the region. The derivative D/Dt is referred to as the fluid material derivative since it denotes following the fluid material.

The difference between the general time derivative d/dt and the fluid material derivative D/Dt can be found from the difference of Eqs. (1) and (2):

$$\frac{d}{dt}\iiint_R Q(\vec{r},t)d\tau \equiv \frac{D}{Dt}\iiint_R Q(\vec{r},t)d\tau + \oiint_S Q(\vec{\xi}-\vec{V})\cdot\hat{n}dS \tag{3}$$

The two time-derivative operations are not the same because the two surfaces involved travel at different velocities, the relative velocity being $\vec{\xi}$-V. Note at the given instant that the derivatives are evaluated by Eq. (3), the surfaces S are coincident and the regions R are the same, but at a later time they will be different since the boundary surfaces are moving relative to one another.

There are other velocities also of interest. Let $\vec{V}_\alpha$ denote the velocity of species α, and let $\vec{V}_p$ denote the average velocity of the particulate. Then we have

$$\frac{D_\alpha}{Dt} \iiint_R Q d\tau \equiv \iiint_R \frac{\partial Q}{\partial t} d\tau + \oint\!\!\oint_S Q\vec{V}_\alpha \cdot \hat{n} dS \tag{4}$$

$$\frac{D_p}{Dt} \iiint_R Q d\tau \equiv \iiint_R \frac{\partial Q}{\partial t} d\tau + \oint\!\!\oint_S Q\vec{V}_p \cdot \hat{n} dS \tag{5}$$

Infinitesimal Regions

Suppose the velocities $\vec{V}$, $\vec{V}_\alpha$, and $\vec{V}_p$ are continuous and have continuous first derivatives. By means of the divergence theorem we can convert the surface integrals in (2), (4), and (5) to volume integrals. Equation (2) becomes

$$\frac{D}{Dt} \iiint_R Q d\tau \equiv \iiint_R \left[\frac{\partial Q}{\partial t} + \mathrm{div}(\vec{V}Q)\right] d\tau \tag{6}$$

There are corresponding expressions for (4) and (5). For infinitesimal regions, that is, $R \to \delta\tau \to 0$, we can write (6) as

$$\frac{D}{Dt}(Q\delta\tau) \equiv \delta\tau\left[\frac{\partial Q}{\partial t} + \mathrm{div}(\vec{V}Q)\right] \tag{7}$$

where $\delta\tau$ is an infinitesimal volume element. Equations (4) and (5) yield the corresponding relations

$$\frac{D_\alpha}{Dt}(Q\delta\tau) \equiv \delta\tau \left[\frac{\partial Q}{\partial t} + \mathrm{div}(\vec{V}_\alpha Q)\right] \tag{8}$$

$$\frac{D_p}{Dt}(Q\delta\tau) \equiv \delta\tau[\frac{\partial Q}{\partial t} + \mathrm{div}(\vec{V}_p Q)] \qquad (9)$$

For Q = 1, we end up with the results

$$\frac{D}{Dt}(\delta\tau) \equiv \delta\tau \ \mathrm{div}\ \vec{V} \qquad (10)$$

$$\frac{D_\alpha}{Dt}(\delta\tau) \equiv \delta\tau \ \mathrm{div}\ \vec{V}_\alpha \qquad (11)$$

$$\frac{D_p}{Dt}(\delta\tau) \equiv \delta\tau \ \mathrm{div}\ \vec{V}_p \qquad (12)$$

The divergences of the velocities are thus seen to describe rates of volumetric strain (dilatation) following the volume element with fluid, species, or particulate velocities.

By means of vector operations we can also show that

$$\frac{DQ}{Dt} \equiv [\frac{\partial}{\partial t} + \vec{V}\cdot\mathrm{grad}]\ Q \qquad (13)$$

$$\frac{D_\alpha Q}{Dt} \quad [\frac{\partial}{\partial t} + \vec{V}_\alpha\cdot\mathrm{grad}]\ Q \qquad (14)$$

$$\frac{D_p Q}{Dt} \quad [\frac{\partial}{\partial t} + \vec{V}_p\cdot\mathrm{grad}]\ Q \qquad (15)$$

Equations of Change for Mass

We consider first the equation of change for the mass of species α. We consider the mass of species α in a region bounded by a surface that travels with the velocity $\vec{V}_\alpha$. Thus, no mass of species α crosses this surface. Changes of mass of species α come about because of source terms. There are two source terms that we shall consider here. One is that due to chemical reactions; we denote $\dot{\omega}_\alpha$ as the mass of species α per unit time per unit volume that is created by chemical reactions. We also suppose that the solid particles can vaporize and liberate gases into the gaseous mixture. We thus denote $\dot{\mu}_\alpha$ as the mass creation of species α per unit time per unit volume that arises because of vaporization of the particles. We thus write

$$\frac{D_\alpha}{Dt} \iiint_R \rho_\alpha \, d\tau = \iiint_R (\dot{\mu}_\alpha + \dot{\omega}_\alpha) d\tau \tag{16}$$

For infinitesimal regions, this becomes, by means of (8) with $Q \equiv \rho_\alpha$,

$$\frac{1}{\delta\tau} \frac{D_\alpha}{Dt} (\rho_\alpha \delta\tau) \equiv \frac{\partial\rho_\alpha}{\alpha t} + \mathrm{div}(\rho_\alpha \vec{V}_\alpha) = \dot{\mu}_\alpha + \dot{\omega}_\alpha \tag{17}$$

This equation holds for each species α. If we sum Eqs. (17) over all species α, then we obtain

$$\frac{\partial}{\partial t} (\Sigma \rho_\alpha) + \mathrm{div}(\Sigma \rho_\alpha \vec{V}_\alpha) = \Sigma \dot{\mu}_\alpha + \Sigma \dot{\omega}_\alpha \tag{18}$$

We define the total fluid density, ρ, and the fluid mass average velocity, $\vec{V}$, by

$$\rho \equiv \Sigma \rho_\alpha \quad , \quad \rho\vec{V} \equiv \Sigma \rho_\alpha \vec{V}_\alpha \tag{19}$$

Further we realize, since the mass of one species α that is created by chemical reactions comes at the expense of other species that are annihilated by the chemical reactions, that

$$\Sigma \dot{\omega}_\alpha = 0 \tag{20}$$

Hence, Eq. (18) becomes

$$\frac{\partial\rho}{\partial t} + \mathrm{div}\, \rho\vec{V} = \Sigma \dot{\mu}_\alpha \tag{21}$$

This is the equation of change for the fluid mixture as a whole. The term on the right-hand side reflects the fact that the mass of the fluid mixture increases by virtue of the vaporization of the particulate matter. In view of (7), one can write (21) alternatively as

$$\frac{D}{Dt}(\rho\delta\tau) = \delta\tau\, \Sigma\, \dot{\mu}_\alpha \tag{22}$$

THis equation reflects again that the mass of a fluid element is not conserved, but changes by virtue of vaporization of the particulate matter.

From the above results, we can see immediately that the change in particulate mass in a volume $\delta\tau$ as we follow it along with the velocity $\vec{V}_p$ is given by

$$\frac{D_p}{Dt}(\rho_p\delta\tau) = -\,\delta\tau\, \Sigma\, \dot{\mu}_\alpha \tag{23}$$

or

$$\frac{\partial\rho_p}{\partial t} + \mathrm{div}(\rho_p\vec{V}_p) = -\,\Sigma\, \dot{\mu}_\alpha \tag{24}$$

If we add Eqs. (21) and (24) together, we obtain

$$\frac{\partial}{\partial t}(\rho + \rho_p) + \mathrm{div}(\rho\vec{V} + \rho_p\vec{V}_p) = 0 \tag{25}$$

Let us define the total mass density, ρ_T, and the total mass average velocity, $\vec{V}_T$, by

$$\rho_T \equiv \rho + \rho_p \quad , \quad \rho_T V_T = \rho\vec{V} + \rho_p\vec{V}_p \tag{26}$$

Then, Eq. (25) can be written as

$$\frac{\partial\rho_T}{\partial t} + \mathrm{div}(\rho_T\vec{V}_T) = 0 \tag{27a}$$

or

$$\frac{D_T}{Dt}(\rho_T \delta\tau) = 0 \qquad (27b)$$

These last two equations assert that the total mass of the fluid and particulate taken together is conserved.

It is convenient to write the equation of change for each species mass in terms of the fluid velocity and not the species velocity. To do that, we note from (3) and the divergence theorem that we can write

$$\frac{D_\alpha}{Dt}(\rho_\alpha \delta\tau) \equiv \frac{D}{Dt}(\rho_\alpha \delta\tau) + \delta\tau \, \mathrm{div}\{\rho_\alpha(\vec{V}_\alpha - \vec{V})\} \qquad (28)$$

Thus, Eq. (17) can also be rewritten as

$$\frac{D}{Dt}(\rho_\alpha \delta\tau) = \delta\tau[- \mathrm{div}\{\rho_\alpha(\vec{V}_\alpha - \vec{V})\} + \dot{\mu}_\alpha + \dot{\omega}_\alpha] \qquad (29a)$$

or

$$\frac{\partial \rho_\alpha}{\partial t} + \mathrm{div}(\rho_\alpha \vec{V}) = - \mathrm{div}\{\rho_\alpha(\vec{V}_\alpha - \vec{V}) \quad + \dot{\mu}_\alpha + \dot{\omega}_\alpha \qquad (29b)$$

The relative velocity $\vec{V}_\alpha - \vec{V}$ is called the mass diffusion velocity for species α, and the combination

$$\vec{J}_\alpha \equiv \rho_\alpha(\vec{V}_\alpha - \vec{V}) \qquad (30)$$

is referred to as the mass diffusion flux vector for species α. From the definition (19), we have

$$\Sigma \, \vec{J}_\alpha = 0 \qquad (31)$$

The mass flux of a species that is diffusing in one direction is compensated for by the mass fluxes of the other species that are diffusing in the opposite direction. Summing Eqs. (29b) over all species leads to

Eq. (21), which, of course, should be the case.

The mass fraction for species α is defined as

$$c_\alpha \equiv \frac{\rho_\alpha}{\rho} \tag{32}$$

Since $\rho_\alpha \delta\tau = c_\alpha \rho \delta\tau$, we can write Eq. (29a) as

$$\rho\delta\tau \frac{Dc_\alpha}{Dt} + c_\alpha \frac{D}{Dt}(\rho\delta\tau) = \delta\tau[- \operatorname{div} \vec{j}_\alpha + \dot{\mu}_\alpha + \dot{\omega}_\alpha] \tag{33}$$

Use of Eq. (22) now leads to the following equation of change for the mass fraction of species α:

$$\rho \frac{Dc_\alpha}{Dt} = - \operatorname{div} \vec{j}_\alpha + \dot{\omega}_\alpha + \dot{\mu}_\alpha - c_\alpha \Sigma \dot{\mu}_\alpha \tag{34}$$

Since $\Sigma c_\alpha = 1$, both sides of the summation of Eqs. (34) for all species vanish identically.

Equations of Change for Momentum

Because our basic regions of interest are regions of varying mass, it is necessary to utilize Newton's second law of motion in a form appropriate for varying mass. For a region of fixed mass, $\vec{F} = m\vec{a}$ holds. For a continuum we write

$$\frac{d_{FM}}{dt} \iiint_R \rho d\tau = 0 \tag{35}$$

$$\frac{d_{FM}}{dt} \iiint_R \rho\vec{V}\, d\tau = \vec{F} \tag{36}$$

For a region whose bounding surface moves with the arbitrary surface velocity $\vec{\xi}$, we utilize Eq. (3) and rewrite (35) and (36) as

$$\frac{d}{dt}\iiint\limits_R \rho d\tau = \oiint\limits_S \rho(\vec{\xi} - \vec{V})\cdot\hat{n}dS \equiv \dot{m} \tag{37}$$

$$\begin{aligned}\frac{d}{dt}\iiint\limits_R \rho\vec{V}d\tau &= \vec{F} + \oiint\limits_S \rho\vec{V}(\vec{\xi} - \vec{V})\cdot\hat{n}dS \\ &= \vec{F} + \dot{m}V_s\end{aligned} \tag{38}$$

where

$$\vec{V}_s \equiv \frac{\oiint\limits_S \rho\vec{V}(\vec{\xi} - \vec{V})\cdot\hat{n}dS}{\oiint\limits_S \rho(\vec{\xi} - \vec{V})\cdot\hat{n}dS} \tag{39}$$

The term $\dot{m}\vec{V}_s$ is actually a momentum flux, but for a particulate suspension imbedded in a gaseous medium, this flux emanates from the surrounding surface of each vaporizing particle and thus manifests itself as an effective momentum-source term. The rate at which mass is added is $\dot{m}$, and the effective velocity at which this mass is added is given by $\vec{V}_s$, which is actually the average material velocity at the surface of the boundary S.

We can now write the momentum equation for the fluid mixture utilizing Eq. (38). The forces $\vec{F}$ acting on the fluid arise from the surface pressure, p, the surface viscous stresses, $\overleftrightarrow{\tau}$, the body force per unit volume due to gravity, $\rho\vec{g}$, and the effective body force exerted on the fluid by the particulate per unit volume, $\vec{F}_p$. Thus, we have

$$\begin{aligned}\frac{D}{Dt}\iiint\limits_R \rho\vec{V}\, d\tau = &- \oiint\limits_S p\hat{n}\, dS + \oiint\limits_S \hat{n}\cdot\overleftrightarrow{\tau}\, dS \\ &+ \iiint\limits_R [\rho\vec{g} + \vec{F}_p + \vec{V}_p \,\Sigma\, \dot{\mu}_\alpha]\, d\tau\end{aligned} \tag{40}$$

The term $\vec{V}_p \Sigma \dot{\mu}_\alpha$ is the effective momentum-source term, per unit volume, associated with $\dot{m}\vec{V}_s$ in Eq. (38). Using the divergence theorem and writing Eq. (40) for infinitesimal regions, we obtain

$$\frac{D}{Dt}(\rho\vec{V}\delta\tau) = \delta\tau[-\nabla p + \text{div}\overleftrightarrow{\tau} + \rho\vec{g} + \vec{F}_p + \vec{V}_p \Sigma \dot{\mu}_\alpha] \tag{41}$$

Expanding the left-hand side and utilizing Eq. (22) yields the further result

$$\rho\frac{D\vec{V}}{Dt} = -\nabla p + \text{div}\overleftrightarrow{\tau} + \rho\vec{g} + \vec{F}_p + (\vec{V}_p - \vec{V})\Sigma \dot{\mu}_\alpha \tag{42}$$

This is the familiar form of the momentum equation that allows for the interaction of a vaporizing particulate suspension. The interaction force $\vec{F}_p$ as well as the vaporizing rates $\dot{\mu}_\alpha$ are yet to be specified.

A similar equation to (41) exists for the particulate suspension, but the pressure and viscous stress terms are omitted. The reason for this is that the particulate particles have no random motion but undergo only a smooth streaming motion. They are sparsely distributed in the fluid and thus undergo no collisions with themselves, and hence exert no pressure or viscous stresses. Thus, for the particulate we have

$$\frac{D_p}{Dt}(\rho_p\vec{V}_p\delta\tau) = \delta\tau[\rho_p\vec{g} - \vec{F}_p - \vec{V}_p \Sigma \dot{\mu}_\alpha] \tag{43}$$

In this equation the signs on $\vec{F}_p$ and $\vec{V}_p \Sigma \dot{\mu}_\alpha$ are reversed from that of Eq. (41). Expanding the left-hand side and utilizing Eq. (23) yields

$$\rho_p\frac{D_p\vec{V}_p}{Dt} = \rho_p\vec{g} - \vec{F}_p \tag{44}$$

For this equation the vaporizing effect does not appear since there is no force induced by a momentum change relative to a particle where the vapor is emitted spherically symmetrically, as is assumed here.

Equations of Change for Kinetic Energy

The equation of change for kinetic energy are found by taking the scalar products of Eqs. (42) and (44) with their respective velocities. We obtain

$$\rho \frac{D}{Dt}\left(\frac{V^2}{2}\right) = -\vec{V}\cdot\nabla p + \vec{V}\cdot\mathrm{div}\,\overleftrightarrow{\tau} + \rho\vec{V}\cdot\vec{g} + \vec{V}\cdot\vec{F}_p + \vec{V}\cdot(\vec{V}_p - \vec{V})\,\Sigma\,\dot{\mu}_\alpha \tag{45}$$

$$\rho_p \frac{D_p}{Dt}\left(\frac{V_p^2}{2}\right) = \rho_p\,\vec{V}_p\cdot\vec{g} - \vec{V}_p\cdot\vec{F}_p \tag{46}$$

These equations are to be used in later developments.

Equations of Change for Energy

The equation of change for the First Law of Thermodynamics that applies for a region of variable mass, the boundary of which is moving at the arbitrary velocity $\vec{\xi}$, is

$$\frac{d}{dt}\iiint_R \rho\left(e+\frac{V^2}{2}\right)d\tau = \dot{Q} + \dot{W} + \oiint_S \rho\left(e+\frac{V^2}{2}\right)(\vec{\xi}-\vec{V})\cdot\hat{n}dS \tag{47}$$

Here e is the internal energy per unit mass, $\dot{Q}$ is the rate of heat added to the system from the surroundings, and $\dot{W}$ is the rate of work done <u>on</u> the system by the surroundings. Equation (47) for the energy corresponds to Eq. (38) for the momentum. To apply Eq. (47) to our fluid region, we note that the boundary surface is made up of two parts, an outer encompassing surface that moves with the fluid velocity itself, and internal subsurfaces that surround the particulate particles. The last term in Eq. (47) corresponds to the energy released by vaporization of the particulate. If we treat this as a volumetric source term, then Eq. (47) can be written as

$$\frac{D}{Dt} \iiint_R \rho\left(e + \frac{V^2}{2}\right) d\tau = \dot{Q} + \dot{W} + \iiint_R \Sigma \, \dot{\mu}_\alpha \left\{e_\alpha(T_p) + \frac{V_p^2}{2}\right\} d\tau \tag{48}$$

since the particulate vapor is released into the fluid at the particulate temperature, and the kinetic energy per unit mass of the particulate vapor is that of the particulate particles.

The rate of work is made up of a part arising from the pressure exerted on the vaporizing particulate, which we separate out as

$$\begin{aligned} \dot{W} &= \dot{W}' + \oiint_S p(\vec{\xi} - \vec{V}) \cdot \hat{n} dS \\ &= \dot{W}' + \iiint_R \left[\Sigma \, \dot{\mu}_\alpha \frac{p_\alpha}{\rho_\alpha}\right] d\tau \end{aligned} \tag{49}$$

In terms of the enthalpy $h \equiv e + (p/\rho)$, we write (48) as

$$\frac{D}{Dt} \iiint_R \rho\left(e + \frac{V^2}{2}\right) d\tau = \dot{Q} + \dot{W}' + \iiint_R \Sigma \, \dot{\mu}_\alpha \left\{h_\alpha(T_p) + \frac{V_p^2}{2}\right\} d\tau \tag{50}$$

where $\dot{W}'$ is the rate of work done on the fluid region that is not associated with the vaporizing of the particulate. The rate of work $\dot{W}'$ is made up of the usual rate of work done by the fluid pressure and viscous stresses on the surrounding bounding surface of the region, the work done by the gravity forces, and the work done by the forces exerted by the particulate, $\vec{V}_p \cdot \vec{F}_p$, per unit volume. The rate of heat added to the fluid region comes from the usual heat-flux vector at the surrounding surface plus a heat, $\dot{Q}_p$, per unit volume coming from the particulate. For infinitesimal regions, we thus write Eq. (50) as

$$\frac{D}{Dt}\left[\rho\delta\tau\left(e+\frac{V^2}{2}\right)\right] = \delta\tau\Big[-\operatorname{div}\vec{q} + \dot{Q}_p - \operatorname{div}(p\vec{V}) + \operatorname{div}(\overleftrightarrow{\tau}\cdot\vec{V}) + \rho\vec{V}\cdot\vec{g} + \vec{V}_p\cdot\vec{F}_p + \Sigma\,\dot{\mu}_\alpha\{h_\alpha(T_p) + \frac{V_p^2}{2}\}\Big] \tag{51}$$

Expanding the left-hand side and utilizing Eq. (22) then yields

$$\rho\frac{D}{Dt}\left(e+\frac{V^2}{2}\right) = -\operatorname{div}\vec{q} + \dot{Q}_p - \operatorname{div}(p\vec{V}) + \operatorname{div}(\overleftrightarrow{\tau}\cdot\vec{V}) + \rho\,\vec{V}\cdot\vec{g} + \vec{V}_p\cdot\vec{F}_p + \Sigma\,\dot{\mu}_\alpha\{h_\alpha(T_p) - e + \frac{V_p^2}{2} - \frac{V^2}{2}\} \tag{52}$$

Subtraction of the kinetic energy by means of (45) then gives the thermodynamic form

$$\rho\frac{De}{Dt} = -\operatorname{div}\vec{q} + \dot{Q}_p - p\operatorname{div}\vec{V} + \overleftrightarrow{\tau}:\nabla\vec{V} + (\vec{V}_p-\vec{V})\cdot\vec{F}_p + \Sigma\,\dot{\mu}_\alpha\{h_\alpha(T_p) - e + \frac{1}{2}(\vec{V}_p-\vec{V})^2\} \tag{53}$$

where

$$\overleftrightarrow{\tau}:\nabla\vec{V} \equiv \operatorname{div}(\overleftrightarrow{\tau}\cdot\vec{V}) - \vec{V}\cdot\operatorname{div}\overleftrightarrow{\tau} \tag{54}$$

and $\overleftrightarrow{\tau}$ is assumed to be symmetric.

The equation of energy for the particulate is analogous to (51) except that the signs on $\dot{Q}_p$, $\vec{F}_p$, and $\dot{\mu}_\alpha$ are reversed, and the fluid dynamic terms $\vec{q}$, p, and $\overleftrightarrow{\tau}$ do not appear. We have

$$\frac{D_p}{Dt}\left[\rho_p \delta\tau\left(e_p + \frac{V_p^2}{2}\right)\right] = \delta\tau\left[- \dot{Q}_p + \rho_p \vec{V}_p \cdot \vec{g} - \vec{V}_p \cdot \vec{F}_p - \Sigma\, \dot{\mu}_\alpha \left\{h_\alpha(T_p) + \frac{V_p^2}{2}\right\}\right] \tag{55}$$

Expanding the left-hand side and utilizing (23) then gives

$$\rho_p \frac{D_p}{Dt}\left(e_p + \frac{V_p^2}{2}\right) = - \dot{Q}_p + \rho_p \vec{V}_p \cdot \vec{g} - \vec{V}_p \cdot \vec{F}_p - \Sigma\, \dot{\mu}_\alpha \{h_\alpha(T_p) - e_p\} \tag{56}$$

Subtracting the kinetic energy by means of (46) then gives

$$\rho_p \frac{D_p e_p}{DT} = - \dot{Q}_p - \Sigma\, \dot{\mu}_\alpha \{h_\alpha(T_p) - e_p\} \tag{57}$$

The last term in this equation arises because of vaporization of the particulate.

It is useful to write the energy equation (53) in terms of the enthalpy $h \equiv e + (p/\rho)$. We note that

$$\rho \frac{D}{Dt}\left(\frac{p}{\rho}\right) \equiv \frac{Dp}{Dt} + p \operatorname{div} \vec{V} - \frac{p}{\rho} \Sigma\, \dot{\mu}_\alpha \tag{58}$$

by utilization of (22) and (10). Adding (58) to (53) then yields

$$\rho \frac{Dh}{Dt} = \frac{Dp}{Dt} + \overleftrightarrow{\tau} : \nabla\vec{V} - \operatorname{div} \vec{q} + \dot{Q}_p + (\vec{V}_p - \vec{V})\cdot\vec{F}_p + \Sigma\, \dot{\mu}_\alpha \left\{h_\alpha(T_p) - h + \frac{1}{2}(\vec{V}_p - \vec{V})^2\right\} \tag{59}$$