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COVARIANT ELECTRODYNAMICS IN A MEDIUM

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THE UNIVERSITY OF OKLAHOMA

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COVARIANT ELECTRODYNAMICS IN A MEDIUM

A DISSERTATION

SUBMITTED TO THE GRADUATE FACULTY

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NAK HYUNG HONG

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ABSTRACT

The energy-momentum and angular momentum emission rates for an arbitrarily moving charge (whose speed is less than that of light in the medium) in a uniform transparent medium are calculated in manifestly covariant form. The calculations are executed for three types of stress tensors: Minkowski, Abraham and Marx. Among other things it is found that the energy-momentum emission rates for the latter two tensors are equal and differ from that of the former. Further, the angular momentum emission rates for all three tensors are found to be equal. Only for the Marx tensor is this rate independent of the orientation of the associated asymptotic space-like surface.

Then, the calculations are extended to a dyon of electric charge e and magnetic charge m. It is shown that all three tensors take the same form as those of the electric charge case except that e^2 is replaced by $e^2+(\varepsilon/\mu)m^2$. Thus, it may be concluded, as far as radiation is concerned, that a magnetic monopole behaves like a charged particle of effective charge $(\varepsilon/\mu)^{\frac{1}{2}}m$.

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COVARIANT ELECTRODYNAMICS IN A MEDIUM

CHAPTER I

INTRODUCTION

Background

The treatment of electrodynamics of systems where a medium is present has given rise to a long-lasting discussion in connection with the construction of the electromagnetic stress tensor. Some authors took a microscopic line of approach in constructing the electromagnetic stress tensor. De Groot and Suttorp developed a macroscopic electromagnetic stress tensor from the microscopic point of view, henceforth designated by a subscript G:[1]

$$T_G^{\mu\nu}=(1/4\pi)\left[F^{\mu\gamma}H_\gamma{}^\nu+\tfrac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}+(1/c^2)(F^{\mu\gamma}M_{\gamma\varepsilon}V^\varepsilon-\Delta^{\mu\gamma}M_{\gamma\varepsilon}F^{\varepsilon\beta}V_\beta)V^\nu\right] \tag{1.1}$$

where the antisymmetric field tensor $F^{\mu\nu}$ is composed of the electric field $\underline{E}=(F^{41},F^{42},F^{43})$ and the magnetic field $\underline{B}=(F^{23},F^{31},F^{12})$; the antisymmetric polarization tensor $M^{\mu\nu}$ is defined by the electric polarization $\underline{P}=(M^{14},M^{24},M^{34})$ and the magnetic polarization $\underline{M}=(M^{23},M^{31},M^{12})$; $H^{\mu\nu}\equiv F^{\mu\nu}-$

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[1] S. R. de Groot and L. G. Suttorp, Foundations of Electrodynamics (Amsterdam: North-Holland Publishing Co., 1972), Ch. V.

$4\pi M^{\mu\nu}$; $\Delta^{\mu\nu}=g^{\mu\nu}+(1/c^2)V^\mu V^\nu$; V^α is the 4-velocity of the medium; and the metric is given by $g^{\mu\nu}=\mathrm{diag}(1,1,1,-1)$.

Another example of the microscopic approach to the construction of the macroscopic stress tensor is that of Peierls, designated by P:[2]

$$S_P^{ik}=(1/4\pi)[\{(u/c)^2+\tau(1-u^2/c^2)^2\}E^iE^k-H^iH^k$$
$$+\tfrac{1}{2}g^{ik}\{1-\sigma(1-u^2/c^2)^2\}E^2+\tfrac{1}{2}g^{ik}H^2] \quad (i,k=1,2,3) \quad (1.2)$$

where c and u are speeds of light in vacuum and in the medium, respectively, and $\tau=\sigma=1/5$.

Even though the microscopic approach is advantageous from a fundamental point of view, the macroscopic stress tensor obtained in this way consists of terms in which the material and field parts are co-related in such a complicated way that there is no unique way of sorting the field part out of the total stress tensor. For example, the total stress tensor presented by de Groot and Suttorp is composed of many terms, among which eq. (1.1) is the only term that depends, in its explicit form, exclusively on the macroscopic fields, polarization and velocities.

The macroscopic approach, though less fundamental than the microscopic method, has the advantage over the microscopic method of being closely related to observation.[3]

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[2] R. Peierls, F. R. S., "The Momentum of Light in a Refracting Medium," <u>Proc. R. Soc. Lond., A37</u> (1976), pp. 475-491.

[3] For a fuller discussion of this matter, see, for example, I. Brevik, "Electromagnetic Energy-Momentum Tensor with Material Media," <u>Danske Vidensk. Selsk., 27</u>, no. 11 (1970); <u>37</u>, no. 13 (1970), and I. Brevik, <u>Experiments in Phenomenological Electrodynamics and the Electromagnetic Energy-Momentum Tensor</u> (Norway: University of Trondheim, 1978).

Some of the electromagnetic stress tensors obtained by macroscopic methods are:

Minkowski: $T_M^{\mu\nu}=(1/4\pi)(F^{\mu\alpha}H_\alpha^{\ \nu}+\frac{1}{4}g^{\mu\nu}F^{\alpha\beta}H_{\alpha\beta})$ (1.3)

Abraham: $T_A^{\mu\nu}=T_M^{\mu\nu}+(1/4\pi\mu)(1/\alpha u)^2(F^{\mu\alpha}F_\alpha^{\ \beta}V_\beta V^\nu - F^{\alpha\beta}F_\alpha^{\ \gamma}V_\beta V_\gamma V^\mu V^\nu/c^2)$ (1.4)

Marx: $T_S^{\mu\nu}=(u/c)^2[T_M^{\mu\nu}-(1/4\pi\mu)(1/\alpha u)^2\{V^\mu F^{\nu\alpha}F_\alpha^{\ \beta}V_\beta + \frac{1}{2}V^\mu V^\nu(F^{\alpha\beta}F_\alpha^{\ \gamma}V_\beta V_\gamma/\alpha^2u^2+\frac{1}{2}F^{\alpha\beta}F_{\alpha\beta})\}]$ (1.5)

Einstein-Laub:

$$T_E^{\mu\nu}=(1/4\pi)\begin{pmatrix} E^iD^k+H^iB^k-\frac{1}{2}g^{ik}(E^2+H^2), & (1/c)\underline{E}\mathrm{x}\underline{H} \\ (1/c)\underline{E}\mathrm{x}\underline{H}, & \text{undefined} \end{pmatrix} \tag{1.6}$$

where $\alpha^2=(1-u^2/c^2)^{-1}$, and μ is magnetic permeability of the medium.

The purpose of the present work is to apply some of these tensors--Minkowski, Abraham and Marx-- to calculate the asymptotic energy-momentum and angular momentum emission rates for an arbitrarily moving charged particle, whose speed is less than that of light in the medium, in a uniform transparent medium. Then the calculation will be extended to a dyon. We will calculate the field tensor and the electromagnetic stress tensor for a dyon of electric charge e and magnetic charge m. Then it will be shown that the dyon behaves like a particle of effective charge $e(1+\varepsilon m^2/\mu e^2)^{\frac{1}{2}}$.

It is, however, not intended to discuss the merits of one tensor over another. We are interested in developing a macroscopic formulation of electrodynamics in the presence of a medium that parallels the modern formulation of electrodynamics for the vacuum case.[4]

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[4]See, for example, F. Rohrlich, Classical Charged Particles (Reading, Mass.: Addison-Wesley Publishing Co., Inc., 1965).

Towards this end we will make extensive use of a paper found elsewhere.[5] In this paper a covariant formulation of electrodynamics in a transparent, uniform, boundaryless medium with constant permeabilities, ε and μ in the medium rest frame, is developed. As part of this formulation the relevant Maxwell's equations were found as well as an explicit expression for the field tensor--due to an arbitrarily moving point charge--expressed in terms of the charge's retarded kinematic properties.[6]

As the present discussion will draw heavily on certain portions of Cohn's paper, we first briefly summarize relevant results in that paper. Maxwell's equations take the form

$$\tilde{F}^{\mu\nu}{}_{,\nu}=F^{\mu\nu}{}_{,\nu}-(1/\alpha u)^2\{V_\alpha(V^\nu F^{\mu\alpha}-V^\mu F^{\nu\alpha})\}_{,\nu}=(4\pi\mu/c)J^\mu \tag{1.7}$$

where $u^2=c^2/\varepsilon\mu$; in the medium rest frame, henceforth denoted by a naught subscript, $J^\mu_{(0)}=(\underline{J},c\rho)_{(0)}$, and

$$F^{*\mu\nu}{}_{,\nu}=0 \tag{1.8}$$

where $F^{*\mu\nu}$ is defined as

$$F^{*\mu\nu}\equiv\tfrac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta} \tag{1.9}$$

and

$$F^{*}_{\mu\nu}\equiv g_{\mu\alpha}\,g_{\nu\beta}F^{*\alpha\beta} \tag{1.10}$$

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[5]J. Cohn, "Covariant Electrodynamics in a Medium, I," Annals of Physics, 114 (1978), pp. 467-478.

[6]Covariant formulation of Maxwell's equations similar to this are found in many standard works of electrodynamics. See, for example, A. Sommerfeld, Electrodynamics, Lectures on Theoretical Physics, Vol. III (New York: Academic Press, Inc., 1952), Pt. IV, or W. Pauli, Theory of Relativity (Oxford, London, New York, Paris: Pergamon Press, 1958), pp. 99-104. In their treatment, Maxwell's equations take the form $H^{\mu\nu}{}_{,\nu}=(4\pi/c)J^\mu$, and $F^{*\mu\nu}{}_{,\nu}=0$. And when arriving at this result, they assumed that $H^{\mu\nu}$ follows the Lorentz transformation.

Thus, $F^{*\mu\nu}$, in the medium rest frame, is

$$F^{*\mu\nu}_{(0)}=\begin{pmatrix} 0, & E_3, & -E_2, & B_1 \\ -E_3, & 0\ , & E_1, & B_2 \\ E_2, & -E_1, & 0\ , & B_3 \\ -B_1, & -B_2, & -B_3, & 0 \end{pmatrix}_{(0)} \tag{1.11}$$

The ε and μ entering here are the rest frame values.

In the case of a point charge moving arbitrarily in the medium, these equations were solved in a manner very like that in the vacuum case to give

$$\begin{aligned} F^{\mu\nu}=(2e\mu/cu)\Big[(1/\tilde{\rho}^2)\{&\tilde{\tilde{v}}^{[\nu}\tilde{v}^{\mu]}-\lambda\tilde{v}^{\alpha}V_{\alpha}\tilde{\tilde{v}}^{[\nu}V^{\mu]}+\tilde{v}^{\alpha}v_{\alpha}\tilde{\tilde{v}}^{[\nu}\Lambda^{\mu]} \\ &-\lambda v_{\beta}V^{\beta}\tilde{v}^{\alpha}V_{\alpha}\tilde{\tilde{v}}^{[\nu}\Lambda^{\mu]}\}+(1/\tilde{\rho})\{u\tilde{\tilde{a}}^{[\mu}\Lambda^{\nu]}-(\tilde{R}^{\alpha}\tilde{a}_{\alpha}/\tilde{\rho})\tilde{\tilde{v}}^{[\nu}\Lambda^{\mu]}\}\Big]_{ret} \end{aligned} \tag{1.12}$$

where: $G^{[\mu}H^{\nu]}\equiv\frac{1}{2}(G^{\mu}H^{\nu}-G^{\nu}H^{\mu})$; $\lambda\equiv(u-c)/c^3$; v^{μ} is the 4-velocity of the particle; $\tilde{v}^{\mu}_{(0)}=(\gamma\underline{v},\gamma u)_{(0)}=(v^{\mu}-\lambda v^{\alpha}V_{\alpha}V^{\mu})_{(0)}$; $\tilde{\tilde{v}}^{\mu}_{(0)}=(\gamma\underline{v},\gamma c/\varepsilon\mu)_{(0)}=(v^{\mu}-(\beta/\varepsilon\mu c^2)v^{\alpha}V_{\alpha}V^{\mu})_{(0)}$; $\tilde{\tilde{a}}^{\mu}=d\tilde{\tilde{v}}^{\mu}/d\tau$ and $\tilde{a}^{\mu}=d\tilde{v}^{\mu}/d\tau$, where $d\tau$ denotes the proper time interval, and $c^2d\tau^2=c^2dt^2-ds^2$; $\tilde{R}^{\mu}_{(0)}=(\underline{x}-\underline{z},|\underline{x}-\underline{z}|)_{(0)}$, where $\underline{x}$ and $\underline{z}$ denote the field and particle locations, respectively; $\tilde{\rho}$ is defined by the relation, $\tilde{R}^{\alpha}\tilde{v}_{\alpha}=-\tilde{\rho}u$; $\beta=1-\varepsilon\mu$; $\gamma^2=(1-v^2/c^2)^{-1}$; and

$$\Lambda_{\alpha}=(\tilde{R}_{\alpha}-\lambda\tilde{R}_{\sigma}V^{\sigma}V_{\alpha})/(\tilde{\rho}u) \tag{1.13}$$

Before being able to carry out the calculations we are ultimately interested in, we must first briefly mention certain basic relations that will be used over and over again. Towards this end we devote the next section.

Preliminaries

As in the vacuum case, we define $R^{\mu}_{(0)}=(\underline{x}-\underline{z},c\Delta t)_{(0)}$, where Δt

is the time required for the wave in the medium to go from $\underline{z}$ to $\underline{x}$. Then, we may write $R^{\mu}_{(0)}=\rho(U^{\mu}+V^{\mu}/u)_{(0)}$, where $U^{\mu}_{(0)}=(\hat{\underline{r}},0)_{(0)}$ is a unit position vector in the medium rest frame, and ρ is just $|\underline{x}-\underline{z}|_{(0)}$. Thus,

$$R^{\mu}=\rho(U^{\mu}+V^{\mu}/u) \tag{1.14}$$

Recalling the definition of $\tilde{R}^{\mu}$, we then have obviously,

$$\tilde{R}^{\mu}=\rho(U^{\mu}+V^{\mu}/c) \tag{1.15}$$

Further, from the relation, $\tilde{R}^{\alpha}V_{\alpha}\equiv\tilde{R}.V=-c\rho$, we have that

$$\Lambda^{\mu}=(\rho/u\tilde{\rho})(U^{\mu}+uV^{\mu}/c^{2}) \tag{1.16}$$

By direct calculation, we now have the relation,

$$V.v=-\gamma c^{2} \tag{1.17}$$

and thus

$$\tilde{\tilde{v}}^{\mu}=v^{\mu}-\gamma V^{\mu}/\alpha^{2} \tag{1.18}$$

Similarly,

$$\tilde{v}^{\mu}=v^{\mu}+\lambda\gamma c^{2}V^{\mu} \tag{1.19}$$

and

$$V.\tilde{v}=-\gamma uc \tag{1.20}$$

Again, from the general form for a^{μ}, we have, by calculating in the medium rest frame,

$$V^{\alpha}a_{\alpha}\equiv V.a=-\gamma^{4}\underline{v}.\underline{a} \tag{1.21}$$

where $\underline{v}$ and $\underline{a}$ refer to medium rest frame values.

Similarly, we have the additional relations,

$$\tilde{a}^{\mu}=a^{\mu}-\lambda V.aV^{\mu}, \qquad \tilde{\tilde{a}}^{\mu}=a^{\mu}+(1/\alpha c)^{2}V.aV^{\mu} \tag{1.22}$$

and

$$U.\tilde{\tilde{v}}=U.\tilde{v}=U.v, \qquad U.\tilde{\tilde{a}}=U.\tilde{a}=U.a \tag{1.23}$$

Further, we have

$$\tilde{\tilde{v}}.\tilde{\tilde{v}}=\gamma^2u^2(1/\alpha^2-1/\gamma'^2),\quad \tilde{\tilde{v}}.\tilde{\tilde{a}}=-(\gamma/\alpha^2)(2-1/\alpha^2)V.a \tag{1.24}$$

where $\gamma'^2=(1-v^2/u^2)^{-1}$.

Direct calculation now gives

$$\tilde{R}.\tilde{a}=\rho(U.a+uV.a/c^2)=\tilde{\rho}u\Lambda.a \tag{1.25}$$

as well as

$$\Lambda.U=\rho/u\tilde{\rho},\qquad \Lambda.V=-\rho/\tilde{\rho},\qquad \Lambda.\Lambda=\rho^2/\alpha^2u^2\tilde{\rho}^2 \tag{1.26}$$

and also

$$\begin{aligned} &\Lambda.a=(\rho/u\tilde{\rho})(U.a+uV.a/c^2),\ \Lambda.\tilde{a}=(\rho/u\tilde{\rho})(U.a+u^2V.a/c^3)\\ &\Lambda.\tilde{\tilde{a}}=(\rho/u\tilde{\rho})(U.a+u^3V.a/c^4),\ \Lambda.v=(\rho/u\tilde{\rho})(U.v-\gamma u)\\ &\Lambda.\tilde{v}=(\rho/u\tilde{\rho})(U.v-\gamma u^2/c),\ \Lambda.\tilde{\tilde{v}}=-1+\gamma\rho/\alpha^2\tilde{\rho} \end{aligned} \tag{1.27}$$

We finally mention that we shall choose our co-ordinate system—in the medium rest frame—so that $\underline{v}$ is along the 3-axis; $\underline{a}$ and $\underline{v}$ define the 1-3 plane; the angle between $\underline{v}$ and $\underline{r}$ is called θ, and the angle between $\underline{a}$ and $\underline{v}$ is called ξ.

Thus,

$$\begin{aligned} &\underline{v}\equiv v_{(0)}=(0,0,v);\ \underline{a}\equiv\underline{a}_{(0)}=(a.\sin\xi,\ 0,\ a.\cos\xi)\\ &U^{\mu}_{(0)}=(\sin\theta.\cos\varphi,\sin\theta.\sin\varphi,\cos\theta,\ 0)\\ &a^{\mu}_{(0)}=(\gamma^2a.\sin\xi,\ 0,\ -V.a/v,\ -V.a/c) \end{aligned} \tag{1.28}$$

With these preliminaries out of way, we are ready to begin the calculations proper.

CHAPTER II

EMISSION BY A CHARGE

Calculation of Stress Tensors

There seems to be several reasonable candidates for the electromagnetic stress tensor in a medium: In this work we shall consider the Minkowski, Abraham and Marx tensors,[7] and shall calculate asymptotic emissions for each. In this section we shall find the expressions for these tensors explicitly in terms of the charge's retarded kinematic properties.

Minkowski. Before we begin the calculation of any of these tensors we first note that one can substantially simplify the expression for the field tensor as given by eq. (1.12). This follows from the observation that—by direct calculation—

$$\tilde{\tilde{v}}^{[\nu}\tilde{v}^{\mu]}-\lambda V.\tilde{v}\tilde{\tilde{v}}^{[\nu}V^{\mu]}=0 \qquad (2.1)$$

and also

$$\tilde{v}.v-\lambda V.vV.\tilde{v}=\gamma^2c^2(1/\alpha^2-1/\gamma^2). \qquad (2.2)$$

Thus, we have the simpler expression

$$F^{\mu\nu}=(2e\mu/cu)\{(1/\tilde{\rho}^2)\gamma^2c^2(1/\alpha^2-1/\gamma^2)\tilde{\tilde{v}}^{[\nu}\Lambda^{\mu]}+(u/\hat{\rho})(\tilde{\tilde{a}}^{[\mu}\Lambda^{\nu]}-\Lambda.a\tilde{\tilde{v}}^{[\nu}\Lambda^{\mu]})\} \qquad (2.3)$$

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[7] I. Brevik, op. cit., discusses the merits of the various tensors.

Now, we take the Minkowski tensor as

$$T_M^{\mu\nu}=(1/4\pi\mu)(F^{\mu\alpha}\tilde{E}_\alpha{}^{\nu}+\tfrac{1}{4}g^{\mu\nu}F^{\alpha\beta}\tilde{F}_{\alpha\beta}) \tag{2.4}$$

In arriving at the final form for the expression of this tensor we shall present the calculations in some detail. We will then feel justified in presenting the similar calculations for the Abraham and Marx tensors much more briefly.

We begin by expressing $\tilde{F}^{\mu\nu}$ in terms of $F^{\mu\nu}$ according to the equation

$$\tilde{F}^{\mu\nu}=F^{\mu\nu}+(\beta/c^2)\,V_\alpha(V^\nu F^{\mu\alpha}-V^\mu F^{\nu\alpha}) \tag{2.5}$$

Then we have

$$T_M^{\mu\nu}=(1/4\pi\mu)\Big[F^{\mu\alpha}E_\alpha{}^{\nu}+(1/\alpha u)^2(F^{\mu\alpha}F^{\nu\beta}V_\alpha V_\beta-F^{\mu\beta}E_\beta{}^{\alpha}V_\alpha V^\nu)$$
$$+\tfrac{1}{4}g^{\mu\nu}(F^{\alpha\beta}E_{\alpha\beta}-(\sqrt{2}/\alpha u)^2F^{\alpha\beta}E_\alpha{}^{\gamma}V_\beta V_\gamma)\Big] \tag{2.6}$$

We now substitute the expression for $F^{\mu\nu}$ given by eq. (2.3) into the above, giving a somewhat lengthy expression. We exhibit a few of the "pieces" explicitly. For example, we find that

$$F^{\alpha\beta}E_{\alpha\beta}=(e\mu/cu)^2\Big[(\gamma^4c^4/\tilde{\rho}^4)(1/\alpha^2-1/\gamma^2)^2(-2+4\gamma\rho/\alpha^2\tilde{\rho}-2\gamma^2\rho^2/\alpha^2\gamma'^2\tilde{\rho}^2)$$
$$+(4\gamma^2c^2u/\tilde{\rho}^3)(1/\alpha^2-1/\gamma^2)\{(-\gamma\rho/\alpha^2\tilde{\rho}+\gamma^2\rho^2/\alpha^2\gamma'^2\tilde{\rho}^2)\Lambda.a$$
$$+V.a(\rho/\alpha^2c^2\tilde{\rho}+\gamma\rho^2/\alpha^4u^2\tilde{\rho}^2)\}+(u^2/\tilde{\rho}^2)\{-2(\gamma\rho/\alpha\gamma'\tilde{\rho})^2(\Lambda.a)^2$$
$$-4\gamma\rho^2/\alpha^4u^2\tilde{\rho}^2(V.a\Lambda.a)-(2\rho^2/\alpha^4c^4\tilde{\rho}^2)(V.a)^2$$
$$+(2\rho^2/\alpha^2u^2\tilde{\rho}^2)(\hat{\tilde{a}}.\hat{\tilde{a}})\}\Big] \tag{2.7}$$

and also that

$$F^{\mu\alpha}V_\alpha=(e\mu/cu)\Big[(\gamma c/\tilde{\rho})^2(1/\alpha^2-1/\gamma^2)(-\gamma u^2\Lambda^\mu+\rho\tilde{\tilde{v}}^\mu/\tilde{\rho})$$
$$+(u/\tilde{\rho})\{(-\rho\tilde{\tilde{a}}^\mu/\tilde{\rho}-V.\tilde{\tilde{a}}\Lambda^\mu)-\Lambda.a(-\gamma u^2\Lambda^\mu+\rho\tilde{\tilde{v}}^\mu/\tilde{\rho})\}\Big] \tag{2.8}$$

We also find that

$$F^{\alpha\beta}E_{\alpha\beta}-(2/\alpha^2u^2)F^{\alpha\beta}E_\alpha{}^{\gamma}V_\beta V_\gamma=-2(e\mu/cu)^2(\gamma c/\tilde{\rho})^4(1/\alpha^2-1/\gamma^2)^2 \tag{2.9}$$

From these expressions and others we then finally obtain the form

$$T_M^{\mu\nu}=(1/4\pi\mu)(e\mu/cu)^2\Big[(\gamma c/\tilde{\rho})^4(1/\alpha^2-1/\gamma^2)^2A^{\mu\nu}$$
$$+\gamma^2c^2u/\tilde{\rho}^3(1/\alpha^2-1/\gamma^2)B^{\mu\nu}+(u/\tilde{\rho})^2C^{\mu\nu}\Big] \qquad (2.10)$$

where

$$A^{\mu\nu}=(\gamma u/\gamma')^2\Lambda^\mu\Lambda^\nu-(\Lambda^\mu\tilde{\tilde{v}}^\nu+\Lambda^\nu\tilde{\tilde{v}}^\mu)-(1/\alpha u)^2(\rho/\hat{\rho})\tilde{\tilde{v}}^\mu V^\nu$$
$$-(\gamma/\alpha^2)\big(1-(\gamma/\gamma'^2)(\rho/\tilde{\rho})\big)\Lambda^\mu V^\nu-\tfrac{1}{2}g^{\mu\nu} \qquad (2.11)$$

$$B^{\mu\nu}=-2\gamma(V.a/\alpha^2+\gamma(u/\gamma')^2\Lambda.a)\Lambda^\mu\Lambda^\nu+(\Lambda^\mu\tilde{\tilde{a}}^\nu+\Lambda^\nu\tilde{\tilde{a}}^\mu)+\Lambda.a(\Lambda^\mu\tilde{\tilde{v}}^\nu+\Lambda^\nu\tilde{\tilde{v}}^\mu)$$
$$+(1/\alpha u)^2(\rho/\tilde{\rho})\tilde{\tilde{a}}^\mu V^\nu+(1/\alpha u)^2(\rho/\hat{\rho})\Lambda.a\tilde{\tilde{v}}^\mu V^\nu$$
$$-(1/\alpha u)^2\{\Lambda.a(-\gamma u^2+2\gamma^2u^2\rho/\gamma'^2\tilde{\rho})+V.a(u^2/c^2+2\gamma\rho/\alpha^2\tilde{\rho})\}\Lambda^\mu\Lambda^\nu \qquad (2.12)$$

$$C^{\mu\nu}=\{-\tilde{\tilde{a}}.\tilde{\tilde{a}}+(\gamma u/\gamma')^2(\Lambda.a)^2+2\gamma V.a\Lambda.a/\alpha^2+(uV.a/\alpha c^2)^2\}\Lambda^\mu(\Lambda^\nu+R^\nu/u\tilde{\rho}) \qquad (2.13)$$

The above is the form for the Minkowski tensor expressed in terms of the particle's retarded kinematical properties.

Abraham. We take Abraham's tensor to be

$$T_A^{\mu\nu}=T_A^{\nu\mu}=T_M^{\mu\nu}+(1/4\pi\mu\alpha^2u^2)(F^{\mu\alpha}F_\alpha{}^\beta V_\beta V^\nu-\tfrac{1}{c^2}F^{\alpha\beta}F_\alpha{}^\gamma V_\beta V_\gamma V^\mu V^\nu) \qquad (2.14)$$

And by a similar calculation to that for the Minkowski tensor, we find that

$$T_A^{\mu\nu}=(1/4\pi\mu)(e\mu/cu)^2\Big[(\gamma c/\tilde{\rho})^4(1/\alpha^2-1/\gamma^2)^2D^{\mu\nu}+(u/\tilde{\rho})^2G^{\mu\nu}$$
$$+\gamma^2c^2u/\tilde{\rho}^3(1/\alpha^2-1/\gamma^2)E^{\mu\nu}\Big] \qquad (2.15)$$

where

$$D^{\mu\nu}=(\gamma u/\gamma')^2\Lambda^\mu\Lambda^\nu-(\Lambda^\mu\tilde{\tilde{v}}^\nu+\Lambda^\nu\tilde{\tilde{v}}^\mu)-V^\mu V^\nu(2\gamma\rho/\alpha^2c^2\tilde{\rho}-\gamma^2\rho^2/\alpha^2c^2\gamma'^2\tilde{\rho}^2)$$
$$-\tfrac{1}{2}g^{\mu\nu} \qquad (2.16)$$

$$G^{\mu\nu}=(\Lambda^\mu\Lambda^\nu+\rho^2V^\mu V^\nu/\alpha^2u^2c^2\tilde{\rho}^2)(-\tilde{\tilde{a}}.\tilde{\tilde{a}}+(\gamma u\Lambda.a/\gamma')^2$$
$$+2\gamma V.a\Lambda.a/\alpha^2+(uV.a/\alpha c^2)^2) \qquad (2.17)$$

$$E^{\mu\nu}=-2\gamma(V.a/\alpha^2+\gamma u^2\Lambda.a/\gamma'^2)\Lambda^\mu\Lambda^\nu+(\Lambda^\mu\tilde{\tilde{a}}^\nu+\Lambda^\nu\tilde{\tilde{a}}^\mu)+\Lambda.a(\Lambda^\mu\tilde{\tilde{v}}^\nu+\Lambda^\nu\tilde{\tilde{v}}^\mu)$$
$$-V^\mu V^\nu[\Lambda.a(-2\gamma\rho/\alpha^2c^2\tilde{\rho}+2\gamma^2\rho^2/\alpha^2c^2\gamma'^2\tilde{\rho}^2)+V.a(\tfrac{2\gamma\rho^2}{\alpha^4c^2u^2\tilde{\rho}^2}+\tfrac{2\rho}{\alpha^2c^4\tilde{\rho}}) \qquad (2.18)$$

<u>Marx</u>. For Marx's tensor, we have

$$T_S^{\mu\nu}=T_S^{\nu\mu}=(u/c)^2\Big[T_M^{\mu\nu}-(1/4\pi\mu\alpha^2u^2)\Big\{V^\mu F^{\nu\alpha}E_\alpha{}^\beta V_\beta$$
$$+\tfrac{1}{2}V^\mu V^\nu\big(F^{\alpha\beta}E_\alpha{}^\gamma V_\beta V_\gamma/\alpha^2u^2+\tfrac{1}{2}F^{\alpha\beta}F_{\alpha\beta}\big)\Big\}\Big] \qquad (2.19)$$

Again, by direct calculation, we find that

$$T_S^{\mu\nu}=(e^2\mu/4\pi c^4)\Big\{(\gamma c/\tilde{\rho})^4(1/\alpha^2-1/\gamma^2)^2H^{\mu\nu}+(u/\tilde{\rho})^2L^{\mu\nu}$$
$$+(\gamma^2c^2u/\tilde{\rho}^3)(1/\alpha^2-1/\gamma^2)K^{\mu\nu}\Big\} \qquad (2.20)$$

where

$$H^{\mu\nu}=(\gamma u/\gamma')^2\Lambda^\mu\Lambda^\nu-(\Lambda^\mu\tilde{\tilde{v}}^\nu+\Lambda^\nu\tilde{\tilde{v}}^\mu)-(\rho/\alpha^2u^2\tilde{\rho})(\tilde{\tilde{v}}^\mu V^\nu+\tilde{\tilde{v}}^\nu V^\mu)$$
$$-(\gamma/\alpha^2)(1-\gamma/\gamma'^2)(\rho/\tilde{\rho})(\Lambda^\mu V^\nu+\Lambda^\nu V^\mu)$$
$$+(V^\mu V^\nu/2\alpha^2u^2)(1-4\gamma\rho/\alpha^2\tilde{\rho}+2\gamma^2\rho^2/\alpha^2\gamma'^2\tilde{\rho}^2)-\tfrac{1}{2}g^{\mu\nu} \qquad (2.21)$$

$$K^{\mu\nu}=-2\gamma\Lambda^\mu\Lambda^\nu\,(V.a/\alpha^2+(\gamma u^2/\gamma'^2)\Lambda.a)+(\Lambda^\mu\tilde{\tilde{a}}^\nu+\Lambda^\nu\tilde{\tilde{a}}^\mu)+\Lambda.a(\Lambda^\mu\tilde{\tilde{v}}^\nu+\Lambda^\nu\tilde{\tilde{v}}^\mu)$$
$$+(\rho/\alpha^2u^2\tilde{\rho})(\tilde{\tilde{a}}^\mu V^\nu+\tilde{\tilde{a}}^\nu V^\mu)+(\rho/\alpha^2u^2\tilde{\rho})\Lambda.a(\tilde{\tilde{v}}^\mu V^\nu+\tilde{\tilde{v}}^\nu V^\mu)$$
$$-(1/\alpha u)^2(\Lambda^\mu V^\nu+\Lambda^\nu V^\mu)\{-\gamma u^2(1-2\gamma\rho/\gamma'^2\tilde{\rho})\Lambda.a+V.a(1-1/\alpha^2+2\gamma\rho/\alpha^2\tilde{\rho})\}$$
$$-(2\rho/\alpha^4u^2\tilde{\rho})V^\mu V^\nu\{\gamma\Lambda.a(-1+\gamma\rho/\gamma'^2\tilde{\rho})+(V.a/u^2)(1-1/\alpha^2+\gamma\rho/\alpha^2\tilde{\rho})\} \qquad (2.22)$$

$$L^{\mu\nu}=(R^\mu R^\nu/u^2\tilde{\rho}^2)\Big\{-\tilde{\tilde{a}}.\tilde{\tilde{a}}+(\gamma u\Lambda.a/\gamma')^2+2\gamma V.a\Lambda.a/\alpha^2$$
$$+(uV.a/\alpha c^2)^2\Big\} \qquad (2.23)$$

For future reference we briefly comment here on the charge-free 4-divergences of these three tensors.

We first note that, from the definition of $\tilde{F}^{\mu\nu}$ (eq. (2.5)), we have that

$$F_{\alpha\beta}\tilde{F}^{\alpha\beta}{}_{,\nu}=\tilde{F}^{\alpha\beta}F_{\alpha\beta,\nu} \qquad (2.24)$$

and therefore that

$$T_M^{\mu\nu}{}_{,\nu}\equiv 0 \qquad \text{where } J^\sigma=0 \qquad (2.25)$$

We note here, however, that $T_M^{\mu\nu}{}_{,\mu} \neq 0$, due to the lack of symmetry of the tensor.

Using the relation, eq. (2.24), one similarly shows that

$$T_A^{\mu\nu}{}_{,\nu} = (1/4\pi\mu\varkappa^2 u^2)\left(F^{\mu\alpha}{}_{,\nu} E_\alpha{}^\beta V_\beta V^\nu + F^{\mu\alpha} E_\alpha{}^\beta{}_{,\nu} V_\beta V^\nu - (2/c^2) F^{\alpha\beta}{}_{,\nu} E_\alpha{}^\gamma V_\beta V_\gamma V^\mu V^\nu\right) \qquad (2.26)$$

which is not identically zero.

And finally, for the Marx tensor, we obtain, in the charge-free case, the relation

$$T_S^{\mu\nu}{}_{,\nu} \equiv 0 \qquad (2.27)$$

Energy-Momentum Emission

We are now ready to calculate the energy-momentum asymptotically emitted by an essentially arbitrarily moving charge in a medium. The only constraint on the charge's motion is that, at all times, $v < u$.

Before making this calculation for the three tensors, we pause to rather carefully discuss the involved concepts.

In the vacuum case we define the asymptotic energy-momentum emission as[8]

$$\Delta P^\mu = \lim_{\rho \to \infty} (1/c) \int_{(\Delta\sigma)} T^{\mu\nu} d\sigma_\nu \qquad (2.28)$$

where, as shown in figure 1, $(\Delta\sigma)$ is the space-like annulus trapped between the relevant light cones, at proper time τ and $\tau + \Delta\tau$. In this case we can replace the integration over $(\Delta\sigma)$ by one over a time-like "ribbon" surface (also indicated in figure 1) which is constructed to

[8] See, for example, F. Rohrlich, op. cit.

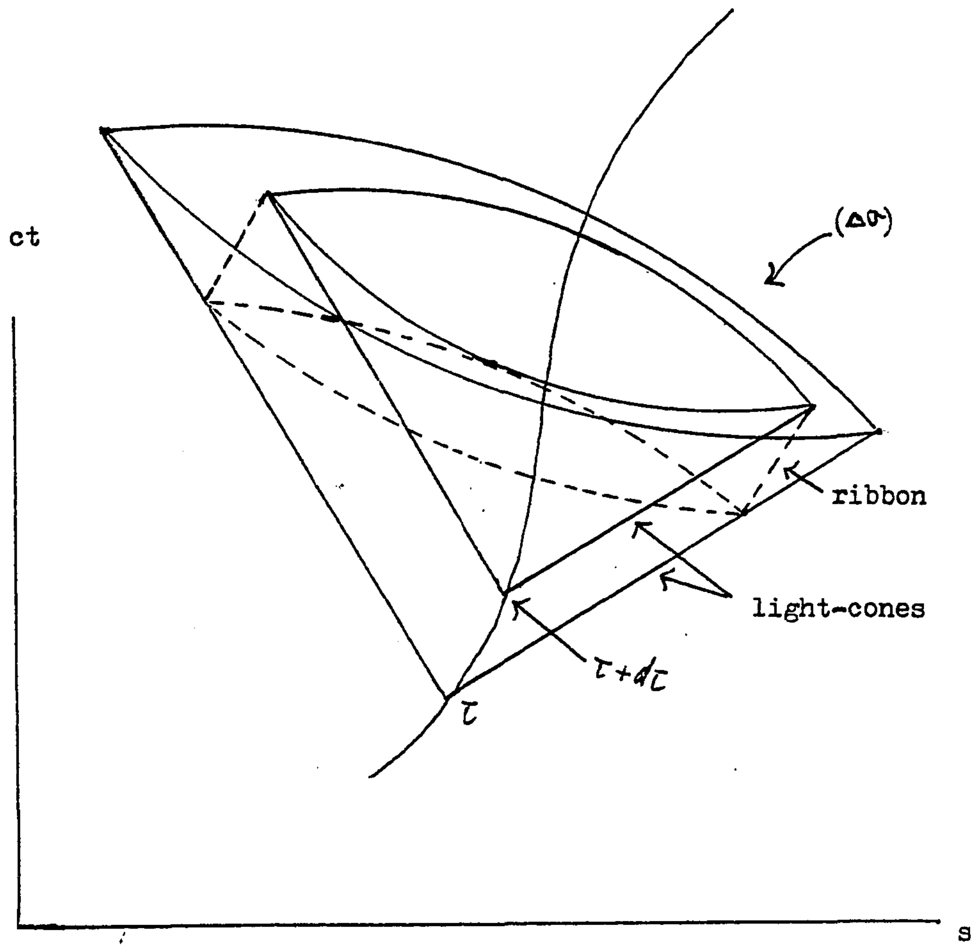

Fig. 1. Vacuum asymptotic surfaces involved in energy-momentum and angular momentum emission

be parallel to the charge's retarded 4-velocity v^μ. This replacement is possible since $T^{\mu\nu}{}_{,\nu}=0$ in charge free space here and because the contribution coming from integration over the relevant light-cone portion vanishes as $\rho\to\infty$. One thus ends up with the familiar expression

$$dP^\mu/d\tau=-\lim_{\rho\to\infty}\int T^{\mu\nu}u_\nu\rho^2 d\Omega \tag{2.29}$$

for the energy-momentum emission rate. And finally, the above statements also imply in this case that the energy-momentum emission rate is independent of the orientation of $(\Delta\sigma)$.

In the case where a medium is present we must be careful as some of the above statements may no longer hold. We begin by defining (see figure 2) in general, the asymptotic energy-momentum emitted as

$$\Delta P^\mu=\lim_{\rho\to\infty}(1/c)\int_{(\Delta\sigma)}T^{\mu\nu}d\sigma_\nu \tag{2.30}$$

where $(\Delta\sigma)$ is the space-like annulus trapped between the relevant "u-cones"[9], at times τ and $\tau+\Delta\tau$.

Now, in case $T^{\mu\nu}{}_{,\nu}\equiv 0$ (as with $T_M^{\mu\nu}$) we can again integrate over surfaces other than $(\Delta\sigma)$. So we again introduce (see figure 2) a time -like ribbon surface which, however, is to be parallel to V^μ (rather than v^μ as in the vacuum case) for ease of computation. The expression for ΔP^μ then becomes

$$\Delta P^\mu=\lim_{\rho\to\infty}(1/c)\left\{-\int T^{\mu\nu}d\sigma_\nu^{(t)}+\int T^{\mu\nu}d\sigma_\nu^{(u)}\right\} \tag{2.31}$$

where $d\sigma_\nu^{(t)}$ and $d\sigma_\nu^{(u)}$ denote, respectively, the vector area elements for portions along the ribbon and along the u-cone, and where $d\sigma_\nu^{(u)}$ is

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[9]We mean here, the analogue of the light cone; i.e. the wave surface as it propagates through the medium. The equation of the u-cone is given in Appendix II.

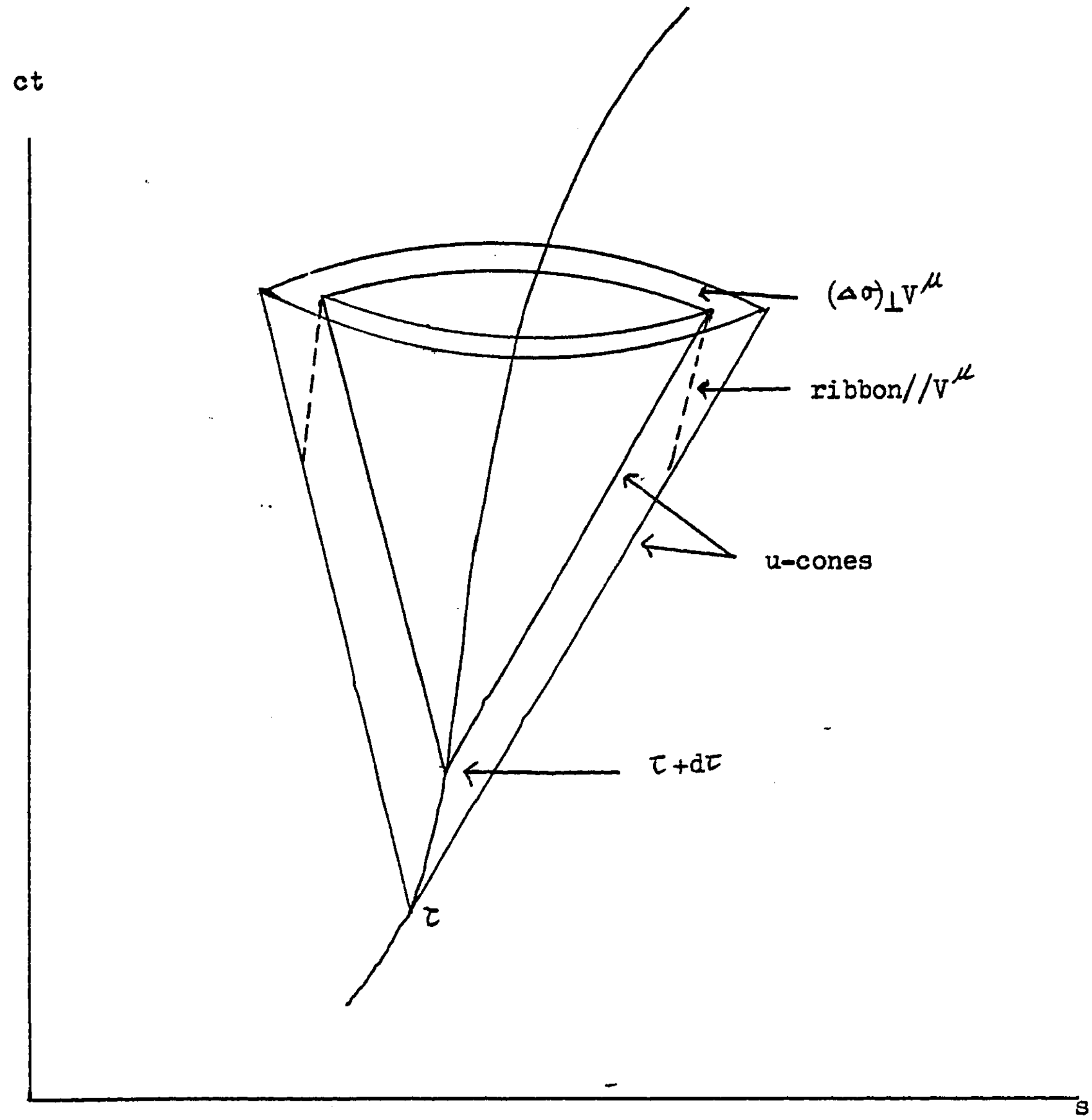

Fig. 2. Medium asymptotic surfaces involved in energy-momentum and angular momentum emission.

future pointing and $d\sigma_\nu^{(t)}$ points away from the retarded charge. In obtaining the signs in this equation due account must be paid to using the appropriate "indicators" in Gauss' law.[10]

In this case (i.e. where $T^{\mu\nu}{}_{,\nu}\equiv 0$) there is still a problem if the contribution coming from the u-cone does not vanish as $\rho\to\infty$. For then, the expression for ΔP^μ will depend on the orientation of $(\Delta\sigma)$, a rather unwelcome prospect. However, since a medium is present we will, in this case, choose $(\Delta\sigma)$ to be the preferred annulus orthogonal to V^μ. If, on the other hand, the u-cone portion vanishes, then we may choose any space-like orientation for $(\Delta\sigma)$. We shall find that, for both $T_M^{\mu\nu}$ and $T_S^{\mu\nu}$, the u-cone contributions vanish, so that for these tensors, the orientation of $(\Delta\sigma)$ is immaterial.

Again, in case that $T^{\mu\nu}{}_{,\nu}\neq 0$, as with $T_A^{\mu\nu}$, there is no point in introducing other surfaces and so we shall integrate directly over $(\Delta\sigma)$. We will learn, however, that even in this case the result is the same as if we had introduced other surfaces. Finally, in this case we also see that ΔP^μ will depend on the orientation of $(\Delta\sigma)$, which we then choose to be the preferred direction orthogonal to V^μ.

It is shown in Appendices I and II that the expression for the surface elements are, respectively,

$$d\sigma_\nu^{(t)}=U_\nu\rho\tilde{\rho}\, d\Omega c d\tau \tag{2.32}$$

and

$$d\sigma_\nu^{(u)}=u\hat{\rho}^2\lambda_\nu\, d\Omega c d\tau \tag{2.33}$$

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[10] See, for example, J. L. Synge, Relativity, The General Theory (Amsterdam: North-Holland Pub. Co., 1960), p. 46.

where $d\Omega = d\Omega_{(0)}$ is the invariant solid angle in 3-space.

Therefore, for a stress tensor $T^{\mu\nu}$ such that $T^{\mu\nu}{}_{,\nu} \equiv 0$, in charge free space, we have

$$dP^{\mu}/d\tau = \lim_{\tilde{\rho}\to\infty} (1/c)\{-\int T^{\mu\nu}U_{\nu}\, c\rho\tilde{\rho}\, d\Omega + \int T^{\mu\nu}u\tilde{\rho}^{2}\Lambda_{\nu} c\, d\Omega\} \qquad (2.34)$$

We are now ready to calculate the energy-momentum emission rate from the three tensors.

Minkowski. As can be seen from the above equation, we only need be concerned with terms of order $\tilde{\rho}^{-2}$ in $T_M^{\mu\nu}$. We denote such terms as $T^{\mu\nu}_{(-2)}$.

From eq. (2.10) we then have

$$T^{\mu\nu}_{M(-2)} = (\mu e^2/4\pi c^2)(1/\tilde{\rho}^2)(\Lambda^{\mu}R^{\nu}/u\tilde{\rho})\{-\tilde{\tilde{a}}.\tilde{\tilde{a}} + (\gamma u\Lambda.a/\gamma')^2$$
$$+(2\gamma/\alpha^2)V.a\Lambda.a + (uV.a/\alpha c^2)^2\} \qquad (2.35)$$

We immediately note that $T^{\mu\nu}_{M(-2)}\Lambda_{\nu} = 0$, since $R^{\mu}\Lambda_{\mu} = 0$, indicating that the u-cone contribution to the radiation rate vanishes, so that the associated emission rate is independent of the orientation of $(\Delta\sigma)$ in this case. Therefore, for the Minkowski case we only have to consider contributions from the ribbon surface. Concerning this expression we can write

$$T^{\mu\nu}_{M(-2)}U_{\nu} = (\mu e^2/4\pi c^2u^2)f(\rho^2/\tilde{\rho}^4)(U^{\mu} + uV^{\mu}/c^2) \qquad (2.36)$$

where we have used the fact that $R.U = \rho$, and f denotes the quantity in parenthesis in eq. (2.35)

Thus, we can write the emission rate in the form

$$dP^{\mu}_M/d\tau = -(\mu e^2/4\pi c^2u^2)(B^{\mu} + uAV^{\mu}/c^2) \qquad (2.37)$$

where

$$B^{\mu} \equiv \int (\rho/\tilde{\rho})^3 fU^{\mu} d\Omega \qquad (2.38)$$

and

$$A \equiv \int (\rho/\tilde{\rho})^3 d\Omega \qquad (2.39)$$

The evaluation of these quantities is very lengthy and we just present the results here. The relevant integrals used in the calculation are tabulated in Appendix III.

We find

$$A=(4\pi/3)(\gamma'^4/\gamma^3)\left[(-3\tilde{\tilde{a}}.\tilde{\tilde{a}}+a.a)+D(V.a/v)^2\right] \qquad (2.40)$$

where

$$D \equiv -2\gamma'^2/\alpha^2\gamma^2+6/\alpha^2-3/\alpha^4-4/\alpha^2\gamma^2+3/\alpha^4\gamma^2 \qquad (2.41)$$

and

$$B^\mu=(4\pi/3)(\gamma'^4/\gamma^3 u)\left[2V.aa^\mu+(-3\tilde{\tilde{a}}.\tilde{\tilde{a}}+a.a)(v^\mu/\gamma-V^\mu)\right.$$
$$\left.+(V.a/v)^2\{v^\mu(2+D)/\gamma-V^\mu(2/\gamma^2+D)\}\right] \qquad (2.42)$$

Thus, we finally have

$$dP^\mu_M/d\tau=-(\mu e^2\gamma'^4/3\gamma^3c^2u^3)\left[(-3\tilde{\tilde{a}}.\tilde{\tilde{a}}+a.a)(v^\mu/\gamma-V^\mu/\alpha^2)+2V.aa^\mu\right.$$
$$\left.+(V.a/v)^2\{(2+D)v^\mu/\gamma-V^\mu(2/\gamma^2+D/\alpha^2)\}\right] \qquad (2.43)$$

as the expression for the asymptotic energy-momentum emission rate in the Minkowski case, where terms on the right hand side take their retarded values.

It is interesting to compare this result with that of the customary vacuum case. To achieve the vacuum situation we must set $\varepsilon=1$, $\mu=1$, $\alpha=\infty$ and $\gamma'=\gamma$. Further, as the ribbon surfaces used in the two cases are defined with respect to the retarded charge and with respect to the medium, repectively, we must require that $v^\mu=V^\mu$. This, of course, implies that the retarded velocity of the charge relative to the medium is zero, so that $V.a=0$ in this case. We then obtain the relation

$$(dP^{\mu}/d\tau)_{vac.}=(2e^2/3c^5)a^2v^{\mu} \qquad (2.44)$$

which is the customary Larmor expression.[11]

Finally, it is informative to note that the expression, eq. (2.43), for the radiation rate does not imply that the energy emission rate, dE/dt, is an invariant, as it is in the vacuum case. We can see this as follows: To make the case simple, assume that, in the medium rest frame, the charge's 3-velocity and acceleration are orthogonal. Then $V.a=-\gamma^4\underline{v}.\underline{a}=0$. Now, setting $\mu=4$, eq. (2.43) yields

$$\bar{\gamma}dE/dt=-(\mu e^2/3c^2u^3)\left[(-3\tilde{\tilde{a}}.\tilde{\tilde{a}}+a.a)(\bar{\gamma}c/\gamma-\bar{\bar{\gamma}}c/\alpha^2)(\gamma'^4/\gamma^3)\right] \qquad (2.45)$$

where $\bar{\gamma}=(1-\bar{v}^2/c^2)^{-\frac{1}{2}}$ refers to the velocity $\bar{v}$ of the charge relative to the arbitrary frame under consideration, and involves the velocity of the medium in this frame. We note that $\bar{\gamma}\neq\bar{\bar{\gamma}}$, as the charge is moving relative to the medium in this case. Thus, dividing through by $\bar{\gamma}$ leaves a term, namely, $\bar{\bar{\gamma}}/\bar{\gamma}$, on the right hand side, which is frame dependent. So, dE/dt is not invariant. However, we do see that this rate is invariant if v is zero.

Abraham. In calculating $dP_A^{\mu}/d\tau$ we are again only interested in $T_{A(-2)}^{\mu\nu}$. From eq. (2.15) we have that

$$T_{A(-2)}^{\mu\nu}=(\mu e^2/4\pi c^2)(1/\tilde{\rho}^2)(\Lambda^{\mu}\Lambda^{\nu}+\rho^2V^{\mu}V^{\nu}/\alpha^2u^2c^2\tilde{\rho}^2) \qquad (2.46)$$
$$\times(-\tilde{\tilde{a}}.\tilde{\tilde{a}}+(\gamma u\Lambda.a/\gamma')^2+2\gamma V.a\Lambda.a/\alpha^2+(uV.a/\alpha c^2)^2)$$

Now, as discussed before, $T_A^{\mu\nu},_{\nu}\neq0$ in charge free space, so we shall integrate over $(\Delta\sigma)$ itself in calculating $dP_A^{\mu}/d\tau$. In appendix IV the expression for the 4-surface element for $(\Delta\sigma)$ is shown to be

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[11] F. Rohrlich, op. cit., p. 111.

$$d\sigma^{\mu}=\rho\tilde{\rho}(u/c)d\Omega d\tau V^{\mu} \tag{2.47}$$

where we select the preferred orientation of $(\Delta\sigma)$ to be orthogonal to V^{μ}.

Now,

$$dP^{\mu}_{A}/d\tau=\lim_{\rho\to\infty}(u/c^{2})\int\rho\tilde{\rho}\,T^{\mu\nu}_{A}V_{\nu}\,d\Omega \tag{2.48}$$

From eq. (2.46), we have that

$$(u/c^{2})T^{\mu\nu}_{A(-2)}V_{\nu}=-(\mu e^{2}/4\pi c^{4})(\rho/\tilde{\rho}^{4})fR^{\mu} \tag{2.49}$$

where f denotes the second expression in parenthesis on the right hand side in eq. (2.46), which is the same as the f introduced earlier.

As will be shown in the next subsection, the above relation is enough to guarantee that

$$dP^{\mu}_{A}/d\tau=dP^{\mu}_{S}/d\tau \tag{2.50}$$

So we now proceed to the consideration of the Marx emission rate.

We pause to point out, however, that in the above relation, $(\Delta\sigma)$ is chosen orthogonal to V^{μ} for the Abraham case, but can have any (space-like) orientation in the Marx case.

Marx. In calculating $dP^{\mu}_{S}/d\tau$, we again only need terms in $T^{\mu\nu}_{S}$ of order $\tilde{\rho}^{-2}$.

From eq. (2.20) we have that

$$T^{\mu\nu}_{S(-2)}=f(R^{\mu}R^{\nu}/u^{2}\tilde{\rho}^{2})(\mu e^{2}u^{2}/4\pi c^{4})(1/\tilde{\rho}^{2}) \tag{2.51}$$

where f has been defined earlier.

Since $T^{\mu\nu}_{S},_{\nu}\equiv0$ in charge free space, we have that

$$dP^{\mu}_{S}/d\tau=\lim_{\rho\to\infty}(1/c)\left\{\int-T^{\mu\nu}_{S}U_{\nu}\rho\tilde{\rho}\,c\,d\Omega+\int T^{\mu\nu}_{S}u\hat{\rho}^{2}c\Lambda_{\nu}\,d\Omega\right\} \tag{2.52}$$

From eq. (2.51) we find that $T^{\mu\nu}_{S(-2)}\Lambda_\nu=0$, which, with the fact that $T^{\mu\nu}_S{}_{,\nu}\equiv 0$ (no charge present), implies that any orientation of $(\Delta\sigma)$ is satisfactory here.

Now, from eq. (2.51) we have that

$$-T^{\mu\nu}_{S(-2)}U_\nu=-(\mu e^2/4\pi c^4)f(\rho/\hat{\rho}^4)R^\mu \tag{2.53}$$

and comparing this with eq. (2.49), we see that we have eq. (2.50).

Proceeding with the calculation then of $dP^\mu_S/d\tau$, we have

$$dP^\mu_S/d\tau=-(\mu e^2/4\pi c^4)(B^\mu+AV^\mu/u) \tag{2.54}$$

where A and B^μ here are the same as introduced in eqs. (2.38) and (2.39), and given in eqs. (2.40) and (2.42).

We then obtain

$$dP^\mu_A/d\tau=dP^\mu_S/d\tau=-(\mu e^2/3uc^4)(\gamma'^4/\gamma^3)\Big[(-3\tilde{\tilde{a}}.\tilde{\tilde{a}}+a.a)v^\mu/\gamma+2V.aa^\mu$$
$$+(V.a/v)^2\big\{(2+D)v^\mu/\gamma-2V^\mu/\gamma^2\big\}\Big] \tag{2.55}$$

as the emission rate for the Abraham and Marx cases.

Again, we see that in the vacuum case these expressions reduce to the customary Larmor rate given by eq. (2.44).

Further, we also notice that the emission rate above does not imply that dE/dt is an invariant in general. This is because the γ's involved in v^μ and V^μ are different in general, and no cancellation between them can occur. On the other hand, for the special case where $V.a=-\gamma^4\underline{v}.\underline{a}$ (say, if $\underline{v}\perp\underline{a}$, or $\underline{v}=0$) then we do see that no such cancellation is necessary and dE/dt is then invariant.

Angular Momentum Emission

In this section we calculate the asymptotic angular momentum emitted by an arbitrarily moving charge (with $v < u$) in the three cases.

We first briefly recall the definition of this quantity in the vacuum case, which is

$$\Delta J^{\mu\nu} = \lim_{\rho\to\infty} (1/c) \int_{(\Delta\sigma)} J^{\alpha\mu\nu} d\sigma_\alpha \qquad (2.56)$$

where

$$J^{\alpha\mu\nu} = T^{\alpha\nu} x^\mu - T^{\alpha\mu} x^\nu \qquad (2.57)$$

and $(\Delta\sigma)$ again denotes (see figure 1) the space-like annulus trapped between the light cones emitted at τ and $\tau+\Delta\tau$.

In the vacuum case, $J^{\alpha\mu\nu}{}_{,\alpha} \equiv 0$ (no charge present) as a consequence of the symmetry of $T^{\mu\nu}$ and the fact that it is divergenceless. As one can also show that contributions to $\Delta J^{\mu\nu}$ coming from the flux through the light-cone vanishes as $\rho\to\infty$, one also has that $\Delta J^{\mu\nu}$ is independent of the orientation of $(\Delta\sigma)$.[12] In this case one can also replace the integration over $(\Delta\sigma)$ by one over a suitable ribbon surface[13] yielding

$$dJ^{\mu\nu}/d\tau = -\lim_{\rho\to\infty} (1/c) \int J^{\alpha\mu\nu} c\rho^2 u_\alpha \, d\Omega \qquad (2.58)$$

In the medium case things are more complicated and seem somewhat less elegant. Recalling the properties of the three tensors we realize that the angular momentum emission rate in the Minkowski and Abraham cases will depend on the orientation of $(\Delta\sigma)$; in the former case because $T_M^{\mu\nu}$ is not symmetric, and in the latter case because $T_A^{\mu\nu}$ is not divergenceless. Only $T_S^{\mu\nu}$, since it is both symmetric and divergence-

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[12] J. Cohn and H. Wiebe, "Asymptotic Radiation from Spinning Charged Particles," J. of Math. Phys., 17 (1976), pp. 1496-1500.

[13] Ibid.

less might yield a divergenceless angular momentum density, which would therefore yield an emission rate independent of the orientation of $(\Delta\sigma)$. And, upon examining the quantity, $T^{\mu\nu}_{S(-3)}\Lambda_\nu$, we do find that it is zero, so that the u-cone contribution in this case can be neglected, so that in this case the emission is independent of the orientation of $(\Delta\sigma)$. Thus, in the former two cases we choose the orientation of $(\Delta\sigma)$ to be preferred direction orthogonal to V^μ, and in the Marx case the orientation of $(\Delta\sigma)$ is arbitrary. For convenience, however, even in this case, we also choose $(\Delta\sigma)$ orthogonal to V^μ.

We then take for the emission rate, in all three cases, the expression

$$dJ^{\mu\nu}/d\tau=\lim_{\rho\to\infty}(1/c)\int(T^{\alpha\nu}x^\mu-T^{\alpha\mu}x^\nu)\rho\tilde{\rho}(u/c)V_\alpha d\Omega \tag{2.59}$$

where $(\Delta\sigma)$ is orthogonal to V^μ, and we have used eq. (2.47).

We also choose our origin (in space-time) to be at the retarded charge event, so that $x^\mu=R^\mu$.

In the limit as $\rho\to\infty$, we shall have to pay attention to contributions coming from $T^{\mu\nu}_{(-3)}$ and $T^{\mu\nu}_{(-2)}$. So, we can write

$$dJ^{\mu\nu}/d\tau=dJ^{\mu\nu}_{(-2)}/d\tau+dJ^{\mu\nu}_{(-3)}/d\tau \tag{2.60}$$

where

$$dJ^{\mu\nu}_{(-2)}/d\tau=(u/c^2)\int\left[T^{\alpha\nu}_{(-2)}R^\mu-T^{\alpha\mu}_{(-2)}R^\nu\right]V_\alpha\rho\tilde{\rho}\,d\Omega \tag{2.61}$$

and

$$dJ^{\mu\nu}_{(-3)}/d\tau=(u/c^2)\int\left[T^{\alpha\nu}_{(-3)}R^\mu-T^{\alpha\mu}_{(-3)}R^\nu\right]V_\alpha\rho\tilde{\rho}\,d\Omega \tag{2.62}$$

We now consider $dJ^{\mu\nu}_{(-2)}/d\tau$.

Considering the expressions, eqs. (2.35), (2.46) and (2.49) for the three tensors, to order $\tilde{\rho}^{-2}$, we have that

$$T^{\alpha\nu}_{M(-2)}V_\alpha = T^{\alpha\nu}_{S(-2)}V_\alpha = T^{\alpha\nu}_{A(-2)}V_\alpha = -(\mu e^2/4\pi c^2)(f/u)(\rho/\tilde{\rho}^4)R^\nu \quad (2.63)$$

We see that these expressions have a form which is proportional to R^ν. Therefore, we have that, for all three tensors

$$dJ^{\mu\nu}_{(-2)}/d\tau = 0 \quad (2.64)$$

Now,

$$dJ^{\mu\nu}/d\tau = dJ^{\mu\nu}_{(-3)}/d\tau = (u/c^2)\int (T^{\alpha\nu}_{(-3)}R^\mu - T^{\alpha\mu}_{(-3)}R^\nu)V_\alpha \rho\tilde{\rho}\, d\Omega \quad (2.65)$$

and from eqs. (2.10), (2.15) and (2.19), we find the relevant expressions to be inserted in the right hand side above.

By direct calculation it can be shown that, $T^{\mu\nu}_{M(-3)}$, $T^{\mu\nu}_{A(-3)}$ and $T^{\mu\nu}_{S(-3)}$ all have this form

$$T^{\alpha\nu}_{(-3)}V_\alpha = K\{-\rho\tilde{\tilde{a}}^\nu/\tilde{\rho} - \rho\Lambda.a\tilde{\tilde{v}}^\nu/\tilde{\rho} + (\rho V.a/\alpha^2c^2\tilde{\rho} - \gamma\rho\Lambda.a/\alpha^2\tilde{\rho})V^\nu + CR^\nu\} \quad (2.66)$$

where K is the same in all three cases, and C is a scalar quantity that is different for the three tensors. Of course, because of the form of eq. (2.65) the term involving the factor C makes no contribution to the emission rate.

Thus, we obtain the relation

$$\begin{aligned} dJ^{\mu\nu}/d\tau \equiv dJ^{\mu\nu}_M/d\tau = dJ^{\mu\nu}_A/d\tau = dJ^{\mu\nu}_S/d\tau \\ = (\mu e^2/4\pi c^2)\gamma^2(1/\alpha^2 - 1/\gamma^2)\int 2(\rho/\tilde{\rho}^2)d\Omega\big[-(\rho/\tilde{\rho})R^{[\mu}\tilde{a}^{\nu]} \\ -(\rho/\tilde{\rho})\Lambda.aR^{[\mu}\tilde{v}^{\nu]} + (\rho/\tilde{\rho})(V.a/\alpha^2c^2 - \gamma\Lambda.a/\alpha^2)R^{[\mu}V^{\nu]}\big] \end{aligned} \quad (2.67)$$

where we use the notation that

$$A^{[\mu}B^{\nu]} = \tfrac{1}{2}(A^\mu B^\nu - A^\nu B^\mu)$$

Once again using the integrals in Appendix III, we then finally obtain the relation

$$dJ^{\mu\nu}/d\tau = -(4\mu e^2\gamma'^4/3uc^2\alpha^2)(1/\alpha^2 - 1/\gamma^2)v^{[\mu}a^{\nu]} \qquad (2.68)$$

as the asymptotic angular momentum emission rate for the Minkowski, Abraham and Marx cases.

We note that the vacuum case where $u=c$, $\alpha\to\infty$, $\gamma'=\gamma$, $\mu=1$, $\varepsilon=1$, this becomes

$$(dJ^{\mu\nu}/d\tau)_{vac.} = (2e^2/3c^3)(v^{\mu}a^{\nu} - v^{\nu}a^{\mu}) \qquad (2.69)$$

which is the customary vacuum result.[14]

Summary and Discussion

In this section we briefly summarize the results of the chapter.

We have calculated expressions for the asymptotic energy-momentum and angular momentum emissions from an arbitrarily moving charge in a medium, where $v < u$. These calculations were made for the Minkowski, Abraham and Marx stress tensors. In the case of energy-momentum emission, we find that the emission is independent of the orientation of $(\Delta\sigma)$ in the Minkowski and Marx cases, and dependent on the orientation in the Abraham case. In the latter case we took $(\Delta\sigma)$ to be orthogonal to V^{μ}. Also, the emission rates as given by the Abraham and Marx tensors were found to be the same. Furthermore, in the Minkowski case we found that the energy emission rate, dE/dt, is not an invariant as it is in the vacuum case, unless the retarded charge is momentarily at rest in the

[14] *Ibid*.

medium. In the Marx and Abraham cases, the energy emission rate is an invariant, provided that the particle's retarded 3-velocity and acceleration (relative to the medium rest frame) are orthogonal. The angular momentum emission rates were then calculated in the three cases, and found to be the same. Only in the Marx case was it found that $dJ^{\mu\nu}/d\tau$ was independent of the (space-like) orientation of $(\Delta\sigma)$.

CHAPTER III

EMISSION BY A DYON

Field of a Dyon

In this chapter we intend to examine the energy-momentum and angular momentum emission rates for a dyon of electric charge e and magnetic charge m in a uniform transparent medium. For this purpose we assume the validity of the following Maxwell's equations in the medium rest frame:

$$\nabla \times \underline{B} = (4\pi\mu/c)\underline{J} + (\varepsilon\mu/c)\dot{\underline{E}} \tag{3.1}$$

$$\nabla \times \underline{E} = -(1/c)\dot{\underline{B}} - (4\pi/c)\underline{S} \tag{3.2}$$

$$\nabla \cdot \underline{B} = 4\pi\sigma \tag{3.3}$$

$$\nabla \cdot \underline{E} = (4\pi/\varepsilon)\rho \tag{3.4}$$

where σ is the magnetic charge density and $\underline{S}$ is the magnetic current density, respectively, in the medium rest frame. Then, using the definition of $\tilde{F}^{\mu\nu}$ in eq. (2.5) and $F^{*\mu\nu}$ in eq. (1.9), we can write the generalized Maxwell's equations, eqs. (3.1)-(3.4), in the covariant form

$$\tilde{F}^{\mu\nu}_{(0)},_{\nu} = (4\pi\mu/c)J^{\mu}_{(0)} \tag{3.5}$$

$$F^{*\mu\nu}_{(0)},_{\nu} = -(4\pi/c)S^{\mu}_{(0)} \tag{3.6}$$

where $S^{\mu}_{(0)}=(\underline{S},\ c\sigma)_{(0)}$. Therefore, in any inertial frame, Maxwell's equations are

$$\tilde{F}^{\mu\nu}{}_{,\nu}=(4\pi\mu/c)J^{\mu} \tag{3.7}$$

$$F^{*\mu\nu}{}_{,\nu}=-(4\pi/c)S^{\mu} \tag{3.8}$$

We now wish to express $F^{\mu\nu}$ in terms of four potentials $\phi^{\mu}=(\underline{\phi},\ U)$ and $\psi^{\mu}=(\underline{\psi},\ V)$ which satisfy the following equations in the medium rest frame:

$$\Box'\underline{\phi}=-(4\pi\mu/c)\underline{J} \tag{3.9}$$

$$\Box'U=-(4\pi/\varepsilon)\rho \tag{3.10}$$

$$\Box'\underline{\psi}=-(4\pi/c)\underline{S} \tag{3.11}$$

$$\Box'V=-(4\pi/\varepsilon\mu)\sigma \tag{3.12}$$

where $\Box'\equiv\nabla^2-(\varepsilon\mu/c^2)(\partial^2/\partial t^2)$.

Towards this end we will, firstly, express $\underline{E}$ and $\underline{B}$ in terms of the potentials. By taking the curl of eq. (3.1), we obtain

$$\nabla x\nabla x\underline{B}=\nabla(\nabla.\underline{B})-\nabla^2\underline{B}=(4\pi\mu/c)\nabla x\underline{J}+(\varepsilon\mu/c)\nabla x\dot{\underline{E}} \tag{3.13}$$

Substitution of eq. (3.2) for $\nabla x\underline{E}$ and eq. (3.3) for $\nabla.\underline{B}$ yields

$$\Box'\underline{B}=4\pi\nabla\sigma-(4\pi\mu/c)\nabla x\underline{J}+(4\pi\varepsilon\mu/c^2)\dot{\underline{S}} \tag{3.14}$$

This becomes, after using eqs. (3.9)-(3.12),

$$\Box'\left[\underline{B}-\{-\varepsilon\mu\nabla V+\nabla x\underline{\phi}-(\varepsilon\mu/c)\dot{\underline{\psi}}\}\right]=0 \tag{3.15}$$

The argument of $\Box'$ is a certain vector $\underline{A}$ which satisfies

$$\Box'\underline{A}=0 \tag{3.16}$$

Then

$$\underline{B}=-\varepsilon\mu\nabla V+\nabla x\underline{\phi}-(\varepsilon\mu/c)\dot{\underline{\psi}}+\underline{A} \tag{3.17}$$

Similarly, by taking the curl of eq. (3.2) and using the above method, we obtain the expression for $\underline{E}$ in terms of the potentials:

$$\underline{E}=-\nabla U-\nabla\times\underline{\psi}-(1/c)\dot{\underline{\phi}}+\underline{C} \tag{3.18}$$

where $\Box^{\prime}\underline{C}=0$.

Since, from eqs. (3.17) and (3.18), $\underline{B}$ and $\underline{E}$ are left unchanged by the transformations

$$\begin{aligned}
\underline{\phi} &\longrightarrow \underline{\phi}+\nabla\Lambda\\
\underline{\psi} &\longrightarrow \underline{\psi}+\nabla\Gamma\\
U &\longrightarrow U-(1/c)\dot{\Lambda}\\
V &\longrightarrow V-(1/c)\dot{\Gamma}
\end{aligned} \tag{3.19}$$

where Λ and Γ are arbitrary, we have the freedom of choosing the potentials such that[15]

$$\nabla\cdot\underline{\phi}+(\varepsilon\mu/c)\dot{U}=0 \tag{3.20}$$

and

$$\nabla\cdot\underline{\psi}+(\varepsilon\mu/c)\dot{V}=0 \tag{3.21}$$

The gauges, eqs. (3.20) and (3.21), can be written in covariant form:

$$\partial_\alpha\tilde{\phi}^{\alpha}_{(0)}=0,\ \text{and}\ \ \partial_\alpha\tilde{\psi}^{\alpha}_{(0)}=0 \tag{3.22}$$

where

$$\tilde{\Phi}^{\mu}_{(0)}=(\underline{\phi},\varepsilon\mu U)_{(0)}=(\phi^{\mu}-(1/\alpha u)^2V_\alpha\phi^{\alpha}V^{\mu})_{(0)} \tag{3.23}$$

$$\tilde{\psi}^{\mu}_{(0)}=(\underline{\psi},\varepsilon\mu V)_{(0)}=(\psi^{\mu}-(1/\alpha u)^2V_\alpha\psi^{\alpha}V^{\mu})_{(0)} \tag{3.24}$$

Thus, in any inertial frame, the gauges for the potentials are

$$\partial_\alpha\tilde{\phi}^{\alpha}=0,\ \text{and}\ \ \partial_\alpha\tilde{\psi}^{\alpha}=0 \tag{3.25}$$

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[15] J. D. Jackson, Classical Electrodynamics (New York, London, Sydney: John Wiley & Sons, Inc., 1967), pp. 180-181.

Now we proceed to show that $\underline{A}$ and $\underline{C}$ can be absorbed into the gauge transformations, eq. (3.19).

From eq. (3.17), we have

$$\nabla\times\underline{A}=\nabla\times\underline{B}-\nabla(\nabla\cdot\underline{\phi})+\nabla^2\underline{\phi}+(\varepsilon\mu/c)\nabla\times\dot{\underline{\psi}} \tag{3.26}$$

Using eq. (3.1), and substituting $\Box'+(\varepsilon\mu/c^2)(\partial^2/\partial t^2)$ for ∇^2, we obtain

$$\nabla\times\underline{A}=(\varepsilon\mu/c^2)(\partial^2\underline{\phi}/\partial t^2)-\nabla(\nabla\cdot\underline{\phi})+(\varepsilon\mu/c)(\partial/\partial t)(\nabla\times\underline{\psi}+\underline{E}) \tag{3.27}$$

This equation finally becomes, utilizing eq. (3.18),

$$\nabla\times\underline{A}=-\nabla(\nabla\cdot\underline{\phi}+(\varepsilon\mu/c)\dot{U})+(\varepsilon\mu/c)(\partial\underline{C}/\partial t)$$
$$=(\varepsilon\mu/c)(\partial\underline{C}/\partial t) \tag{3.28}$$

And

$$\nabla\cdot\underline{A}=\nabla\cdot\underline{B}+\varepsilon\mu\nabla^2V+(\varepsilon\mu/c)\nabla\cdot\dot{\underline{\psi}}$$
$$=(\varepsilon\mu/c)(\partial/\partial t)(\nabla\cdot\underline{\psi}+(\varepsilon\mu/c)\dot{V})=0 \tag{3.29}$$

Thus, we can write

$$\underline{A}=\nabla\times\underline{\phi}' \tag{3.30}$$

Similarly, the curl and div of $\underline{C}$ are found to be

$$\nabla\times\underline{C}=-(1/c)(\partial\underline{A}/\partial t) \tag{3.31}$$

$$\nabla\cdot\underline{C}=0 \tag{3.32}$$

Thus, $\underline{C}$ can be written as

$$\underline{C}=-\nabla\times\underline{\psi}' \tag{3.33}$$

Substituting eq. (3.30) into eq. (3.31), and eq. (3.33) into eq. (3.28), we find that

$$\nabla\times\underline{A}=-(\varepsilon\mu/c)\nabla\times\dot{\underline{\psi}}' \tag{3.34}$$

and

$$\nabla\times\underline{C}=-(1/c)\nabla\times\dot{\underline{\phi}}' \tag{3.35}$$

Therefore, $\underline{A}$ and $\underline{C}$ are, in general, of the form

$$\underline{A}=-(\varepsilon\mu/c)\dot{\underline{\psi}}'-\varepsilon\mu\nabla V' \tag{3.36}$$

and

$$\underline{C}=-(1/c)\dot{\underline{\phi}}'-\nabla U' \qquad (3.37)$$

Now we observe that $\underline{A}$ and $\underline{C}$ are left unchanged by the transformations:

$$\begin{aligned} \underline{\phi}' &\longrightarrow \underline{\phi}'+\nabla\Lambda' \\ \underline{\psi}' &\longrightarrow \underline{\psi}'+\nabla\Gamma' \\ U' &\longrightarrow U'-\dot{\Lambda}'/c \\ V' &\longrightarrow V'-\dot{\Gamma}'/c \end{aligned} \qquad (3.38)$$

Again we have the freedom of choosing these potentials to satisfy

$$\nabla\cdot\underline{\phi}'+(\varepsilon\mu/c)\dot{U}'=0 \qquad (3.39)$$

and

$$\nabla\cdot\underline{\psi}'+(\varepsilon\mu/c)\dot{V}'=0 \qquad (3.40)$$

By taking the curl of eqs. (3.30) and (3.36), and using eqs. (3.33) and (3.37), we find that

$$\Box'\underline{\phi}'=\nabla(\nabla\cdot\underline{\phi}'+(\varepsilon\mu/c)\dot{U}')=0 \qquad (3.41)$$

Similarly, by taking the curl of eqs. (3.33) and (3.37), and using eqs. (3.30) and (3.36), we obtain

$$\Box'\underline{\psi}'=\nabla(\nabla\cdot\underline{\psi}'+(\varepsilon\mu/c)\dot{V}')=0 \qquad (3.42)$$

By taking the div instead of the curl, we obtain

$$\Box'V'=0 \qquad (3.43)$$

$$\Box'U'=0 \qquad (3.44)$$

Now we substitute $\underline{A}$ and $\underline{C}$ into eqs. (3.17) and (3.18), respectively, to obtain

$$\underline{B}=-\varepsilon\mu\nabla(V+\tfrac{1}{2}V')+\nabla x(\underline{\phi}+\tfrac{1}{2}\underline{\phi}')-(\varepsilon\mu/c)(\dot{\underline{\psi}}+\tfrac{1}{2}\dot{\underline{\psi}}') \qquad (3.45)$$

$$\underline{E}=-\nabla(U+\tfrac{1}{2}U')-\nabla x(\underline{\psi}+\tfrac{1}{2}\underline{\psi}')-(1/c)(\dot{\underline{\phi}}+\tfrac{1}{2}\dot{\underline{\phi}}') \qquad (3.46)$$

since $\underline{A}=\nabla x\underline{\phi}'=-(\varepsilon\mu/c)\dot{\underline{\psi}}'-\varepsilon\mu\nabla V'=\tfrac{1}{2}(\nabla x\underline{\phi}'-(\varepsilon\mu/c)\dot{\underline{\psi}}'-\varepsilon\mu\nabla V')$ and

$\underline{C}=-\nabla x\underline{\psi}'=-(1/c)\dot{\underline{\phi}}'-\nabla U'=-\tfrac{1}{2}(\nabla x\underline{\psi}'+(1/c)\dot{\underline{\phi}}'+\nabla U')$.

Furthermore, we observe that the "new" potentials ("old" potential+$\frac{1}{2}$ of "primed" potential) obey all the equations governing the "old" potentials, that is, eqs. (3.9)-(3.12) and (3.20)-(3.25). Thus, by choosing the "old" potentials to denote the "new" potentials, we restore these equations and

$$\underline{B}=-\varepsilon\mu\nabla V+\nabla\times\underline{\phi}-(\varepsilon\mu/c)\dot{\underline{\psi}} \tag{3.47}$$

$$\underline{E}=-\nabla U-\nabla\times\underline{\psi}-(1/c)\dot{\underline{\phi}} \tag{3.48}$$

We now wish to express the field tensor in the covariant form. From eqs. (3.47) and (3.48), we obtain

$$F^{14}=-E_1=(\partial U/\partial x)+(\partial\psi_3/\partial y-\partial\psi_2/\partial z)+(1/c)(\partial\phi_1/\partial t)$$

$$=(\phi^{4,1}-\phi^{1,4})+\varepsilon^{14\alpha\beta}\psi_{\beta,\alpha} \tag{3.49}$$

$$F^{12}=B_3=-\varepsilon\mu(\partial V/\partial z)+(\partial\phi_2/\partial x-\partial\phi_1/\partial y)-(\varepsilon\mu/c)(\partial\psi_3/\partial t)$$

$$=(\phi^{2,1}-\phi^{1,2})+\varepsilon\mu\,\varepsilon^{12\alpha\beta}\psi_{\beta,\alpha} \tag{3.50}$$

which may be generalized to give

$$F^{ij}=\phi^{j,i}-\phi^{i,j}+\varepsilon\mu\,\varepsilon^{ij\alpha\beta}\psi_{\beta,\alpha} \qquad (i,j=1,2,3) \tag{3.51}$$

$$F^{i4}=\phi^{4,i}-\phi^{i,4}+\varepsilon^{i4\alpha\beta}\psi_{\beta,\alpha} \tag{3.52}$$

Combining these two equations together, we finally have

$$F^{\mu\nu}=(\phi^{\nu,\mu}-\phi^{\mu,\nu})$$
$$+\{\varepsilon\mu\,\varepsilon^{\mu\nu\alpha\beta}+(1/\alpha u)^2V_\gamma(V^\nu\varepsilon^{\mu\gamma\alpha\beta}-V^\mu\varepsilon^{\nu\gamma\alpha\beta})\}\psi_{\beta,\alpha} \tag{3.53}$$

Inversely, it can be easily shown that the expression for $F^{\mu\nu}$ in eq. (3.53) with the gauges given by eq. (3.25) satisfies Maxwell's equations, eqs. (3.7) and (3.8).

For brevity of notation let us introduce new quantities:

$$f^{\mu\nu}\equiv\phi^{\nu,\mu}-\phi^{\mu,\nu} \tag{3.54}$$

and

$$b^{*\mu\nu} \equiv \psi^{\nu,\mu} - \psi^{\mu,\nu} \tag{3.55}$$

Then, $F^{\mu\nu}$ can, in general, be written as

$$F^{\mu\nu} = f^{\mu\nu} - b^{\mu\nu}_{\xi\mu} - (1/\alpha u)^2 V_\gamma (V^\nu b^{\mu\gamma} - V^\mu b^{\nu\gamma}) \tag{3.56}$$

where

$$b^{*\mu\nu} = \tfrac{1}{2}\varepsilon^{\mu\nu\alpha\beta} b_{\alpha\beta}\text{, and } b^*_{\mu\nu} = g_{\mu\alpha} g_{\nu\beta} b^{*\alpha\beta}.$$

The potentials ϕ^μ and ψ^μ for a dyon are the solutions to eqs. (3.9)-(3.12).

They are[16]

$$\phi^\mu = (e\mu/c)\tilde{\tilde{v}}^\mu/\tilde{\rho}\,]_{ret.} \tag{3.57}$$

$$\psi^\mu = (m/c)\tilde{\tilde{v}}^\mu/\tilde{\rho}\,]_{ret.} \tag{3.58}$$

where e and m are the electric and magnetic charge of the dyon.

We, thus, find that

$$f^{\mu\nu} = (2e\mu/cu)\{(\gamma c/\tilde{\rho})^2(1/\alpha^2 - 1/\gamma^2)\tilde{\tilde{v}}^{[\nu}\Lambda^{\mu]} + (u/\tilde{\rho})(\tilde{\tilde{a}}^{[\mu}\Lambda^{\nu]} - \Lambda.a\tilde{\tilde{v}}^{[\nu}\Lambda^{\mu]})\} \tag{3.59}$$

And this is the same as that of an electrically charged particle, eq. (2.3). Also,

$$b^{\mu\nu} = -(m/\mu e)f^{*\mu\nu} \tag{3.60}$$

The field tensor for a dyon, then, becomes

$$F^{\mu\nu} = f^{\mu\nu} + (m\varepsilon/e)f^{*\mu\nu} - (1/\alpha c)^2(m\varepsilon/e)y^{\mu\nu} \tag{3.61}$$

where

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[16] J. Cohn, <u>op. cit.</u>

$$y^{\mu\nu} \equiv V_\gamma (-f^{*\mu\gamma} V^\nu + f^{*\nu\gamma} V^\mu) \tag{3.62}$$

Electromagnetic Stress Tensor

In this section we will calculate various stress tensors--Minkowski, Abraham amd Marx--for the field produced by a dyon. As before the dyon is moving arbitrarily in the medium, and its speed is less than that of light in the medium.

Minkowski. The Minkowski tensor given by eq. (2.6) can be expressed as, using $f^{\mu\nu}$, $b^{\mu\nu}$ and $y^{\mu\nu}$,

$$T_M^{\mu\nu} = (1/4\pi\mu)\Big[(ff)^{\mu\nu} + (m\varepsilon/e)^2\{(f^*f^*)^{\mu\nu} - (1/\alpha c)^2(f^*y+yf^*)^{\mu\nu} + (1/\alpha c)^4(yy)^{\mu\nu}\} + (m\varepsilon/e)\{(ff^*+f^*f)^{\mu\nu} - (1/\alpha c)^2(fy+yf)^{\mu\nu}\}\Big] \tag{3.63}$$

where

$$(ab)^{\mu\nu} \equiv a^{\mu\alpha} b_\alpha{}^\nu + (1/\alpha u)^2 (a^{\mu\alpha} b^{\nu\beta} V_\alpha V_\beta - a^{\mu\beta} b_\beta{}^\alpha V_\alpha V^\nu) + \tfrac{1}{4} g^{\mu\nu}(a^{\alpha\beta} b_{\alpha\beta} + (2/\alpha^2 u^2) a^{\alpha\beta} b_\beta{}^\gamma V_\alpha V_\gamma) \tag{3.64}$$

$$(a+b)^{\mu\nu} \equiv (a)^{\mu\nu} + (b)^{\mu\nu}$$

and

$$((a+b)(c+d))^{\mu\nu} \equiv (ac+ad+bc+bd)^{\mu\nu} \tag{3.65}$$

The first term in eq. (3.63), which is the contribution of electric charge to the tensor, is already evaluated in terms of the particle's retarded kinematic properties and is given by eq. (2.10). We now wish to calculate the second term which is the contribution of magnetic charge.

From the definition eq. (3.64),

$$(f^*f^*)^{\mu\nu} = f^{*\mu\alpha} f^*_\alpha{}^\nu + (1/\alpha u)^2 (f^{*\mu\alpha} f^{*\nu\beta} V_\alpha V_\beta - f^{*\mu\beta} f^*_\beta{}^\alpha V_\alpha V^\nu) + \tfrac{1}{4} g^{\mu\nu}(f^{*\alpha\beta} f^*_{\alpha\beta} + (2/\alpha^2 u^2) f^{*\alpha\beta} f^*_\beta{}^\gamma V_\alpha V_\gamma) \tag{3.66}$$

From the definition of $y^{\mu\nu}$, eq. (3.62), $(yy)^{\mu\nu}$ is

$$(yy)^{\mu\nu}=y^{\mu\alpha}y_{\alpha}{}^{\nu}+(1/\alpha u)^2(y^{\mu\alpha}y^{\nu\beta}V_\alpha V_\beta-y^{\mu\beta}y_\beta{}^{\alpha}V_\alpha V^\nu)$$
$$+\tfrac{1}{4}g^{\mu\nu}(y^{\alpha\beta}y_{\alpha\beta}+(2/\alpha^2u^2)y^{\alpha\beta}y_\beta{}^{\gamma}V_\alpha V_\gamma) \qquad (3.67)$$

We will use the following identities, which can be easily shown by using the definition of $y^{\mu\nu}$, to express the above equation in terms of $f*^{\mu\nu}$:

$$y^{\mu\alpha}y_\alpha{}^{\nu}=V^\mu V^\nu f*^{\alpha\beta}f*_\beta{}^{\gamma}V_\alpha V_\gamma+c^2f*^{\mu\alpha}f*^{\nu\beta}V_\alpha V_\beta$$
$$y^{\mu\alpha}V_\alpha=c^2f*^{\mu\alpha}V_\alpha$$
$$y^{\mu\alpha}y^{\nu\beta}V_\alpha V_\beta=c^4f*^{\mu\alpha}f*^{\nu\beta}V_\alpha V_\beta$$
$$y^{\mu\alpha}y_\alpha{}^{\beta}V_\beta=-c^2V^\mu f*^{\alpha\beta}f*_\beta{}^{\gamma}V_\alpha V_\gamma \qquad (3.68)$$
$$y^{\alpha\beta}y_\beta{}^{\gamma}V_\alpha V_\gamma=c^4f*^{\alpha\beta}f*_\beta{}^{\gamma}V_\alpha V_\gamma$$

and

$$y^{\alpha\beta}y_{\alpha\beta}=2c^2f*^{\alpha\beta}f*_\beta{}^{\gamma}V_\alpha V_\gamma$$

Then, eq. (3.67) becomes

$$(yy)^{\mu\nu}=(c^4/u^2)f*^{\mu\alpha}f*^{\nu\beta}V_\alpha V_\beta+(c^2/u^2)V^\mu V^\nu f*^{\alpha\beta}f*_\beta{}^{\gamma}V_\alpha V_\gamma$$
$$+\tfrac{1}{2}g^{\mu\nu}(c^4/u^2)f*^{\alpha\beta}f*_\beta{}^{\gamma}V_\alpha V_\gamma \qquad (3.69)$$

And,

$$(f*y+yf*)^{\mu\nu}=f*^{\mu\alpha}y_\alpha{}^{\nu}+y^{\mu\alpha}f*_\alpha{}^{\nu}$$
$$+(1/\alpha u)^2(f*^{\mu\alpha}y^{\nu\beta}V_\alpha V_\beta+y^{\mu\alpha}f*^{\nu\beta}V_\alpha V_\beta$$
$$-f*^{\mu\beta}y_\beta{}^{\alpha}V_\alpha V^\nu-y^{\mu\beta}f*_\beta{}^{\alpha}V_\alpha V^\nu)$$
$$+\tfrac{1}{4}g^{\mu\nu}\{2f*^{\alpha\beta}y_{\alpha\beta}+(2/\alpha^2u^2)(f*^{\alpha\beta}y_\beta{}^{\gamma}V_\alpha V_\gamma+y^{\alpha\beta}f*_\beta{}^{\gamma}V_\alpha V_\gamma)\}$$
$$=(2c^2/u^2)f*^{\mu\alpha}f*^{\nu\beta}V_\alpha V_\beta$$
$$-V^\mu f*^{\nu\alpha}f*_\alpha{}^{\beta}V_\beta-(c/u)^2V^\nu f*^{\mu\alpha}f*_\alpha{}^{\beta}V_\beta$$

$$+(1/\alpha u)^2 V^\mu V^\nu f^{*\alpha\beta} f^{*}{}_\beta{}^\gamma V_\alpha V_\gamma + (c/u)^2 g^{\mu\nu} f^{*\alpha\beta} f^{*}{}_\beta{}^\gamma V_\alpha V_\gamma \tag{3.70}$$

where the following relations are used:

$$f^{*\mu\alpha} y_\alpha{}^\nu = f^{*\mu\alpha} f^{*\nu\beta} V_\alpha V_\beta - f^{*\mu\alpha} f^{*}{}_\alpha{}^\gamma V_\gamma V^\nu;\; y^{\mu\alpha} f^{*}{}_\alpha{}^\nu = f^{*\mu\alpha} f^{*\nu\beta} V_\alpha V_\beta$$
$$- V^\mu f^{*\nu\alpha} f^{*}{}_\alpha{}^\gamma V_\gamma;\; f^{*\alpha\beta} y_{\alpha\beta} = 2 f^{*\alpha\beta} f^{*}{}_\beta{}^\gamma V_\alpha V_\gamma;\; y^{\mu\alpha} f^{*\nu\beta} V_\alpha V_\beta = f^{*\mu\alpha} y^{\nu\beta} V_\alpha V_\beta$$
$$= c^2 f^{*\mu\alpha} f^{*\nu\beta} V_\alpha V_\beta;\; y^{\mu\alpha} f^{*}{}_\alpha{}^\nu V_\nu = -V^\mu f^{*\alpha\beta} f^{*}{}_\beta{}^\gamma V_\alpha V_\gamma;\; f^{*\mu\alpha} y_\alpha{}^\gamma V_\gamma =$$
$$c^2 f^{*\mu\alpha} f^{*}{}_\alpha{}^\gamma V_\gamma;\; \text{and } y^{\alpha\beta} f^{*}{}_\beta{}^\gamma V_\alpha V_\gamma = c^2 f^{*\alpha\beta} f^{*}{}_\beta{}^\gamma V_\alpha V_\gamma. \tag{3.71}$$

By combining eqs. (3.66), (3.69) and (3.70), we obtain

$$(f^*f^*)^{\mu\nu} - (1/\alpha c)^2 (f^*y + yf^*)^{\mu\nu} + (1/\alpha c)^4 (yy)^{\mu\nu} = f^{*\mu\alpha} f^{*}{}_\alpha{}^\nu$$
$$- (1/\alpha c)^2 (f^{*\mu\alpha} f^{*\nu\beta} V_\alpha V_\beta - V^\mu f^{*\nu\alpha} f^{*}{}_\alpha{}^\gamma V_\gamma)$$
$$+ \tfrac{1}{4} g^{\mu\nu} (f^{*\alpha\beta} f^{*}{}_{\alpha\beta} - (2/\alpha^2 c^2) f^{*\alpha\beta} f^{*}{}_\beta{}^\gamma V_\alpha V_\gamma) \tag{3.72}$$

To express eq. (3.72) in terms of $f^{\mu\nu}$, we will use the identities:

$$f^{*\mu\alpha} f^{*}{}_\alpha{}^\nu = f^{\mu\alpha} f_\alpha{}^\nu + \tfrac{1}{2} g^{\mu\nu} f^{\alpha\beta} f_{\alpha\beta};\; f^{*\alpha\beta} f^{*}{}_{\alpha\beta} = -f^{\alpha\beta} f_{\alpha\beta};$$
$$f^{*\mu\alpha} f^{*\nu\beta} V_\alpha V_\beta = c^2 f^{\mu\alpha} f_\alpha{}^\nu + V^\mu f^{\nu\alpha} f_\alpha{}^\beta V_\beta + V^\nu f^{\mu\alpha} f_\alpha{}^\beta V_\beta + \tfrac{1}{2} V^\mu V^\nu f^{\alpha\beta} f_{\alpha\beta}$$
$$- f^{\mu\alpha} f^{\nu\beta} V_\alpha V_\beta + \tfrac{1}{2} g^{\mu\nu} (c^2 f^{\alpha\beta} f_{\alpha\beta} - 2 f^{\alpha\beta} f_\beta{}^\gamma V_\alpha V_\gamma) \tag{3.73}$$

Then, finally, eq. (3.72) becomes

$$(f^*f^*)^{\mu\nu} - (1/\alpha c)^2 (f^*y + yf^*)^{\mu\nu} + (1/\alpha c)^4 (yy)^{\mu\nu} = (u/c)^2 (ff)^{\mu\nu} \tag{3.74}$$

Next, we will consider the last term which represents the interference between electric and magnetic charges. Straightforward calculation shows that

$$(ff^* + f^*f)^{\mu\nu} = 2 f^{(\mu\alpha} f^{*}{}_\alpha{}^{\nu)} + (1/\alpha u)^2 (2 f^{(\mu\alpha} f^{*\nu)\beta} V_\alpha V_\beta - f^{\mu\beta} f^{*}{}_\beta{}^\alpha V_\alpha V^\nu$$
$$- f^{*\mu\beta} f_\beta{}^\alpha V_\alpha V^\nu) + \tfrac{1}{4} g^{\mu\nu} (2 f^{\alpha\beta} f^{*}{}_{\alpha\beta} + (4/\alpha^2 u^2) f^{\alpha\beta} f^{*}{}_\beta{}^\gamma V_\alpha V_\gamma)$$
$$= (2/\alpha^2 u^2) f^{(\mu\alpha} f^{*\nu)\beta} V_\alpha V_\beta + \tfrac{1}{2} (1/\alpha u)^2 V^\mu V^\nu f^{\alpha\beta} f^{*}{}_{\alpha\beta}$$
$$+ \tfrac{1}{4} g^{\mu\nu} (c/\alpha u)^2 f^{\alpha\beta} f^{*}{}_{\alpha\beta} \tag{3.75}$$

where $a^{(\mu} b^{\nu)} = \tfrac{1}{2}(a^\mu b^\nu + a^\nu b^\mu)$, and

$$(fy+yf)^{\mu\nu}=(f^{\mu\alpha}y_{\alpha}{}^{\nu}+y^{\mu\alpha}f_{\alpha}{}^{\nu})+(1/\alpha u)^2(f^{\mu\alpha}y^{\nu\beta}V_{\alpha}V_{\beta}+y^{\mu\alpha}f^{\nu\beta}V_{\alpha}V_{\beta}$$
$$-f^{\mu\beta}y_{\beta}{}^{\alpha}V_{\alpha}V^{\nu}-y^{\mu\beta}f_{\beta}{}^{\alpha}V_{\alpha}V^{\nu})$$
$$+\tfrac{1}{4}g^{\mu\nu}(2f^{\alpha\beta}y_{\alpha\beta}+(2/\alpha u)^2f^{\alpha\beta}y_{\beta}{}^{\gamma}V_{\alpha}V_{\gamma})$$
$$=2(c/u)^2f^{(\mu\alpha}f^{*\nu)\beta}+\tfrac{1}{2}(c/u)^2V^{\mu}V^{\nu}f^{\alpha\beta}f^{*}_{\alpha\beta}$$
$$+(c^4/u^2)g^{\mu\nu}f^{\alpha\beta}f^{*}_{\alpha\beta} \qquad (3.76)$$

Thus,combining eqs. (3.75) and (3.76), we have

$$(ff^{*}+f^{*}f)^{\mu\nu}-(1/\alpha c)^2(fy+yf)^{\mu\nu}=0 \qquad (3.77)$$

where the following equation is used:

$$f^{\mu\alpha}f^{*}{}_{\alpha}{}^{\nu}=-\tfrac{1}{4}g^{\mu\nu}f^{\alpha\beta}f^{*}_{\alpha\beta} \qquad (3.78)$$

Therefore, by substituting eqs. (3.74) and (3.77) into eq. (3.63), we obtain the Minkowski tensor for a dyon:

$$T_M^{\mu\nu}=(1/4\pi\mu)(1+(u/c)^2(m\varepsilon/e)^2)(ff)^{\mu\nu}$$
$$=(1+(\varepsilon/\mu)(m/e)^2)T_{M(e)}^{\mu\nu} \qquad (3.79)$$

where we substituted $(u/c)^2=1/\varepsilon\mu$, and the subscript (e) indicates that it is the case for electric charge without any magnetic charge and is given by eq. (2.10).

We now proceed to calculate the Abraham and Marx tensors.

Abraham. The Abraham tensor is given by eq. (2.14):

$$T_A^{\mu\nu}=T_M^{\mu\nu}+(1/\alpha u)^2(F^{\mu\alpha}E_{\alpha}{}^{\beta}V_{\beta}V^{\nu}-(1/c)^2F^{\alpha\beta}E_{\alpha}{}^{\gamma}V_{\beta}V_{\gamma}V^{\mu}V^{\nu})(1/4\pi\mu) \qquad (3.80)$$

Using eqs. (3.68) and (3.71), we find that

$$F^{\mu\alpha}E_{\alpha}{}^{\beta}V_{\beta}=f^{\mu\alpha}f_{\alpha}{}^{\beta}V_{\beta}+(m\varepsilon/e)^2\{f^{*\mu\alpha}f^{*}{}_{\alpha}{}^{\beta}-(1/\alpha c)^2(f^{*\mu\alpha}y_{\alpha}{}^{\beta}+y^{\mu\alpha}f^{*}{}_{\alpha}{}^{\beta})$$
$$+(1/\alpha c)^4y^{\mu\alpha}y_{\alpha}{}^{\beta}\}V_{\beta}+(m\varepsilon/e)\{(f^{\mu\alpha}f^{*}{}_{\alpha}{}^{\beta}+f^{*\mu\alpha}f_{\alpha}{}^{\beta})$$
$$-(1/\alpha c)^2(f^{\mu\alpha}y_{\alpha}{}^{\beta}+y^{\mu\alpha}f_{\alpha}{}^{\beta})\}V_{\beta}$$

$$=f^{\mu\alpha}f_{\alpha}^{\ \beta}V_{\beta}+(u/c)^2(m\varepsilon/e)^2(f^{*\mu\alpha}f^*{}_{\alpha}^{\ \beta}V_{\beta}$$
$$+(1/\alpha c)^2f^{*\alpha\beta}f^*{}_{\beta}^{\ \gamma}V_{\alpha}V_{\gamma}V^{\mu})-\tfrac{1}{2}(u/c)^2(m\varepsilon/e)V^{\mu}f^{\alpha\beta}f^*{}_{\alpha\beta} \qquad (3.81)$$

and

$$F^{\alpha\beta}F_{\beta}^{\ \gamma}V_{\alpha}V_{\gamma}=f^{\alpha\beta}f_{\beta}^{\ \gamma}V_{\alpha}V_{\gamma}+(u/c)^2(m\varepsilon/e)^2(f^{*\gamma\alpha}f^*{}_{\alpha}^{\ \beta}V_{\beta}V_{\gamma}$$
$$-(1/\alpha^2)f^{*\alpha\beta}f^*{}_{\beta}^{\ \gamma}V_{\alpha}V_{\gamma})+\tfrac{1}{2}(m\varepsilon/e)u^2f^{\alpha\beta}f^*{}_{\alpha\beta} \qquad (3.82)$$

Thus, using eq. (3.73),

$$F^{\mu\alpha}F_{\alpha}^{\ \beta}V_{\beta}V^{\nu}+(1/c)^2F^{\alpha\beta}F_{\beta}^{\ \gamma}V_{\alpha}V_{\gamma}V^{\mu}V^{\nu}=(f^{\mu\alpha}f_{\alpha}^{\ \beta}V_{\beta}V^{\nu}$$
$$+(1/c)^2f^{\alpha\beta}f_{\beta}^{\ \gamma}V_{\alpha}V_{\gamma}V^{\mu}V^{\nu})+(u/c)^2(m\varepsilon/e)^2(f^{*\mu\alpha}f^*{}_{\alpha}^{\ \beta}V_{\beta}V^{\nu}$$
$$+(1/c)^2f^{*\alpha\beta}f^*{}_{\beta}^{\ \gamma}V_{\beta}V_{\gamma}V^{\mu}V^{\nu})$$
$$=(1+(u/c)^2(m\varepsilon/e)^2)(f^{\mu\alpha}f_{\alpha}^{\ \beta}V_{\beta}V^{\nu}$$
$$+(1/c)^2V^{\mu}V^{\nu}f^{\alpha\beta}f_{\beta}^{\ \gamma}V_{\alpha}V_{\gamma}) \qquad (3.83)$$

Substituting eqs. (3.79) and (3.83), we obtain

$$T_A^{\mu\nu}=(1+(\varepsilon/\mu)(m/e)^2)T_{A(e)}^{\mu\nu} \qquad (3.84)$$

Marx. The Marx tensor is given by eq. (2.19):

$$T_S^{\mu\nu}=(u/c)^2T_M^{\mu\nu}-(1/\alpha c)^2(1/4\pi\mu)\{V^{\mu}F^{\nu\alpha}F_{\alpha}^{\ \beta}V_{\beta}$$
$$+\tfrac{1}{2}V^{\mu}V^{\nu}((1/\alpha u)^2F^{\alpha\beta}F_{\alpha}^{\ \gamma}V_{\beta}V_{\gamma}+\tfrac{1}{2}F^{\alpha\beta}F_{\alpha\beta})\} \qquad (3.85)$$

Using eqs. (3.81) and (3.82) and

$$F^{\alpha\beta}F_{\alpha\beta}=f^{\alpha\beta}f_{\alpha\beta}+(m\varepsilon/e)^2(f^{*\alpha\beta}f^*{}_{\alpha\beta}-(2/\alpha^2c^2)f^{*\alpha\beta}y_{\alpha\beta}+(1/\alpha c)^4y^{\alpha\beta}y_{\alpha\beta}$$
$$+(m\varepsilon/e)(2f^{\alpha\beta}f^*{}_{\alpha\beta}-(2/\alpha^2c^2)f^{\alpha\beta}y_{\alpha\beta})$$
$$=f^{\alpha\beta}f_{\alpha\beta}+(m\varepsilon/e)^2(f^{*\alpha\beta}f^*{}_{\alpha\beta}-(2/\alpha^2c^2)(2-1/\alpha^2)f^{*\alpha\beta}f^*{}_{\beta}^{\ \gamma}V_{\alpha}V_{\gamma})$$
$$+(m\varepsilon/e)(2-1/\alpha^2)f^{\alpha\beta}f^*{}_{\alpha\beta} \qquad (3.86)$$

we obtain that

$$V^{\mu}F^{\nu\alpha}F_{\alpha}^{\ \beta}V_{\beta}+\tfrac{1}{2}V^{\mu}V^{\nu}(-(1/\alpha u)^2F^{\alpha\beta}F_{\beta}^{\ \gamma}V_{\alpha}V_{\gamma}+\tfrac{1}{2}F^{\alpha\beta}F_{\alpha\beta})=(1+(u/c)^2(m\varepsilon/e)^2)\times$$
$$\times\{V^{\mu}f^{\nu\alpha}f_{\alpha}^{\ \beta}V_{\beta}+\tfrac{1}{2}V^{\mu}V^{\nu}(-(1/\alpha u)^2f^{\alpha\beta}f_{\beta}^{\ \gamma}V_{\alpha}V_{\gamma}+\tfrac{1}{2}f^{\alpha\beta}f_{\alpha\beta})\} \qquad (3.87)$$

Substituting this and eq. (3.79) into eq. (3.85), we have

$$T_S^{\mu\nu}=(1+(\varepsilon/\mu)(m/e)^2)T_{S(e)}^{\mu\nu} \tag{3.88}$$

As we have observed in this section, all three tensors for a dyon differ from those of the electrically charged particle without magnetic charge by a factor of $1+(\varepsilon/\mu)(m/e)^2$. For all three tensors, explicitly, we have

$$T_{dyon}^{\mu\nu}=(1+(\varepsilon/\mu)(m/e)^2)T_{(e)}^{\mu\nu} \tag{3.89}$$

For further comparison with the electrical case, we will briefly mention the 4-divergences of the three tensors.

<u>Calculation of 4-divergence</u>. Using eq. (2.24),

$$\begin{aligned} F^{\mu\alpha}{}_{,\nu}\tilde{E}_\alpha{}^{\nu} &= -\tfrac{1}{4}\varepsilon^{\mu\alpha\beta\gamma}F^*{}_{\beta\gamma}{}^{,\nu}\varepsilon_{\alpha\nu\xi\eta}\tilde{F}^{*\xi\eta} \\ &= \tfrac{1}{4}\delta^{\mu\beta\gamma}_{\nu\xi\eta}F^*{}_{\beta\gamma}{}^{,\nu}\tilde{F}^{*\xi\eta} \\ &= -\tfrac{1}{2}g^{\mu\nu}F^{\alpha\beta}\tilde{F}_{\alpha\beta,\nu}-(4\pi/c)\tilde{F}^{*\mu\alpha}S_\alpha \end{aligned} \tag{3.90}$$

and

$$\begin{aligned} F^{\mu\alpha}{}_{,\nu}E_\alpha{}^{\nu} &= -\tfrac{1}{2}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta,\nu}-(4\pi/c)F^{*\mu\alpha}S_\alpha \\ &= \tfrac{1}{2}g^{\mu\nu}F^{*\alpha\beta}F^*{}_{\alpha\beta,\nu}-(4\pi/c)F^{*\mu\alpha}S_\alpha \end{aligned} \tag{3.91}$$

we find that

$$T_M^{\mu\nu}{}_{,\nu}=(1/c)(F^{\mu\alpha}J_\alpha-(1/\mu)\tilde{F}^{*\mu\alpha}S_\alpha) \tag{3.92}$$

and

$$\begin{aligned} T_S^{\mu\nu}{}_{,\nu}=(u^2/c^3)\{&(F^{\mu\alpha}J_\alpha-(1/\mu)\tilde{F}^{*\mu\alpha}S_\alpha) \\ &-(1/\alpha u)^2(F^{\alpha\beta}J_\beta-(1/\mu)F^{*\alpha\beta}S_\beta)V_\alpha V^\mu\} \end{aligned} \tag{3.93}$$

Therefore,

$$T_M^{\mu\nu}{}_{,\nu}\equiv 0,\ \text{when } J^\mu=S^\mu=0 \tag{3.94}$$

and

$$T_S^{\mu\nu}{}_{,\nu} \equiv 0, \text{ when } J^{\mu} = S^{\mu} = 0 \tag{3.95}$$

But $T_A^{\mu\nu}{}_{,\nu}$ is not identically zero, when there is no charge present; this can be shown from the following equation:

$$\begin{aligned} T_A^{\mu\nu}{}_{,\nu} = T_M^{\mu\nu}{}_{,\nu} + (1/4\pi\mu)(1/\alpha u)^2 (F^{\mu\alpha}{}_{,\nu} E_{\alpha}{}^{\beta} V_{\beta} V^{\nu} \\ -(2/c^2) F^{\alpha\beta}{}_{,\nu} E_{\alpha}^{\gamma} V_{\beta} V_{\gamma} V^{\mu} V^{\nu}) \end{aligned} \tag{3.96}$$

<u>Discussion</u>

The electromagnetic stress tensors--Minkowski, Abraham and Marx--for a dyon of electric charge e and magnetic charge m, moving arbitrarily in a uniform medium with the speed less than that of light in the medium,are the same as those for electric charge of "effective" charge e*, where

$$e^* = e(1 + (\varepsilon/\mu)(m/e)^2)^{\frac{1}{2}} \tag{3.97}$$

Furthermore, the 4-divergences of the three tensors, in the charge free case, have the similar relation, i. e., the Minkowski and Marx tensors are divergenceless in both electric charge and dyon case, and the 4-divergence of the Abraham tensor is not identically zero in both cases.

Therefore, the energy-momentum and angular momentum emission rates for a dyon moving arbitrarily (whose speed is less than that of light in the medium) in a uniform transparent medium are the same as those for electric charge, except that e is replaced by e*. And all the arguments discussed in the previous chapter concerning energy-momentum and angular momentum emitted by a charged particle are applicable to a dyon. We will not repeat them here.

Despite their resemblances in many ways, the difference between an elctric charge and a dyon is conspicuous. Since the difference arises due to the existence of magnetic charge, let us consider the magnetic monopole case. By setting e=0, we find that "effective" charge of a magnetic monopole, m*, is

$$m^* = (\varepsilon/\mu)^{\frac{1}{2}} m \tag{3.98}$$

In a vacuum we cannot distinguish the magnetic monopole from the electron qualitatively, in the sense that the electromagnetic tensors for them are indistinguishable, except for their charges. But in a medium the magnetic monopole behaves like a particle of medium-dependent charge. Thus, the phenomena related to it and its motion would be significantly different from those of the electron. If we could arrange a layer of media and observe certain phenomena such as radiation or the motion of a particle moving through the layer, it would be possible to tell the difference between the electron and the magnetic monopole.

The theory developed in this paper does not provide us with the tool to analyze radiation of a charged particle moving through an inhomogeneous medium, but we still expect that the effective charge of the magnetic monopole is medium dependent.[17]

It should be remarked here that, even though the theory presented in this chapter reveals certain properties of magnetic charge, we cannot claim that the tensors considered represent the real situation

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[17] P. B. Price, E. K. Shirk, W. Z. Osborne and L. S. Pinsky, "Evidence for Detection of a Moving Magnetic Monopole," Phys. Rev. Lett. 35(1975), pp. 487-490, in analyzing the image on Cherenkov detector, used the formula, Intensity$\propto m^2(n^2-(u/v)^2)$. This agrees with eq. (3.98), if we set μ=1. See, also, D. R. Tompkins, Phys. Rev. 138B (1965), pp. 248-250.

correctly. Nevertheless, it is suggestive that all three tensors-- Minkowski, Abraham and Marx-- agree on at least one point: A dyon behaves like a charged particle of effective charge e* and there is no interference effect due to the simultaneous existence of the two types of charge.

It should also be remarked that, if the existence of the dyon is assumed, it is possible that ordinary matter may contain a certain amount of magnetic charge, and Maxwell's equations (3.1)-(3.4) may have to be changed accordingly.[18]

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[18] L. L. Vant-Hull, "Experimental Upper Limit on the Magnetic Monopole Moment of Electrons, Protons, and Neutrons, Utilizing a Superconducting Quantum Interferometer," Phys. Rev., 173 (1968), pp. 1412-1413, measured the magnetic charge on the neutron and the difference m(proton)-m(electron) to be less than 2×10^{-41} Wb, and the electron less than 8×10^{-39} Wb. And $\nabla\cdot\underline{B}$, in the absence of magnetic monopole, is less than 1.2×10^{-14} Wb/kg.

APPENDIX I

RIBBON SURFACE AREA ELEMENT

Here we calculate the expression for the ribbon surface area element, $d\sigma_\nu^{(t)}$. We refer to figure 2.

Now, the ribbon surface is defined relative to the medium rest frame so that, in any frame, we have

$$d\sigma_\nu^{(t)} = \rho^2 U_\nu \, dx^4{}_{(0)} \, d\Omega \tag{I.1}$$

where $d\Omega \equiv d\Omega_{(0)}$.

We may write this as

$$d\sigma_\nu^{(t)} = \rho^2 U_\nu (\partial x^4{}_{(0)}/\partial\tau) d\tau \, d\Omega \tag{I.2}$$

But, since[19]

$$\partial x^4{}_{(0)}/\partial\tau = -\Lambda^{-1}_{4(0)} = c\hat{\rho}/\rho \tag{I.3}$$

we have

$$d\sigma_\nu^{(t)} = U_\nu \, c\rho\tilde{\rho} \, d\Omega \, d\tau \tag{I.4}$$

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[19] J. Cohn, Annals of Physics, 114 (1978), p. 476.

APPENDIX II

THE SURFACE AREA ELEMENT ON THE u-CONE

Here we calculate the expression for the surface area element on the u-cone. We refer to figure 3.

Now, the u-cone is defined, in the medium rest frame, by the condition that

$$\underline{r}^2=u^2t^2=(u/c)^2(x^4)^2 \qquad \text{(II.1)}$$

For variations on the u-cone we then have, in the same frame

$$\underline{r}\cdot\delta\underline{r}=(u/c)^2x^4\delta x^4 \quad \text{and} \quad \delta R^{\mu}_{(0)}=(\delta\underline{r},\delta x^4) \qquad \text{(II.2)}$$

Defining the vector C^{μ} by the relation

$$C^{\mu}_{(0)}=(\underline{r},\ (u/c)^2x^4) \qquad \text{(II.3)}$$

we then have

$$\delta R^{\mu}C_{\mu}=0 \qquad \text{(II.4)}$$

So, C^{μ} is orthogonal to the u-cone.

We note that C^{μ} is space-like, with a time varying magnitude, i. e.,

$$C^{\mu}C_{\mu}=u^2t^2/\gamma^2 \qquad \text{(II.5)}$$

Define the unit normal to the u-cone as

$$\eta^{\mu}\equiv C^{\mu}/|C^{\alpha}| \ ; \quad \eta^{\mu}_{(0)}=(\gamma/r)(\underline{r},u^2t/c) \qquad \text{(II.6)}$$

From figure 3 we have the relation between the measures of the area elements on the u-cone and on the associated projection on the space hyper-plane as

$$\Delta\sigma_u=(1/|\eta^\alpha n_\alpha|)\Delta\sigma \tag{II.7}$$

Therefore,

$$d\sigma_u^\nu=\eta^\nu d\sigma_u=\eta^\nu/|\eta^\alpha n_\alpha|\, d\sigma=\rho^2 d\rho\, d\Omega\, \eta^\nu/\eta^4_{(0)} \tag{II.8}$$

Now,

$$\eta^4_{(0)}=(\gamma/c)u \tag{II.9}$$

as follows from eqs. (II.5) and (II.6).

Thus we have

$$d\sigma_u^\nu=(c/\gamma u)\rho^2 d\rho\, d\Omega\, \eta^\nu \tag{II.10}$$

Further, we note that, from eq. (II.6), we can write

$$\eta^\mu=(\gamma/r)[\tilde{R}^\mu-\rho(1-u/c)V^\mu/c]=(\gamma/\rho)u\tilde{\rho}\Lambda^\mu \tag{II.11}$$

where we have used eq. (1.16).

So, we have as the momentary expression for the u-cone area element,

$$d\sigma_u^\nu=c\rho\tilde{\rho}\Lambda^\nu d\rho\, d\Omega \tag{II.12}$$

Now, we have from eq. (II.1) that

$$\rho\, d\rho=x^4_{(0)}dx^4_{(0)}(u/c)^2 \tag{II.13}$$

so that

$$\rho\, d\rho=x^4_{(0)}(\partial x^4_{(0)}/\partial\tau)(u/c)^2 d\tau=x^4_{(0)}(\tilde{\rho}/\rho)(u/c)^2 d\tau \tag{II.14}$$

Utilizing the relation

$$x^4_{(0)}/\rho=c/u \tag{II.15}$$

that follows from eq. (II.1), and inserting these results into eq. (II.12), then finally gives for the u-cone area element,

$$d\sigma_u^\nu = c\hat{\rho}^2 u \Lambda^\nu d\Omega d\tau \qquad \text{(II.16)}$$

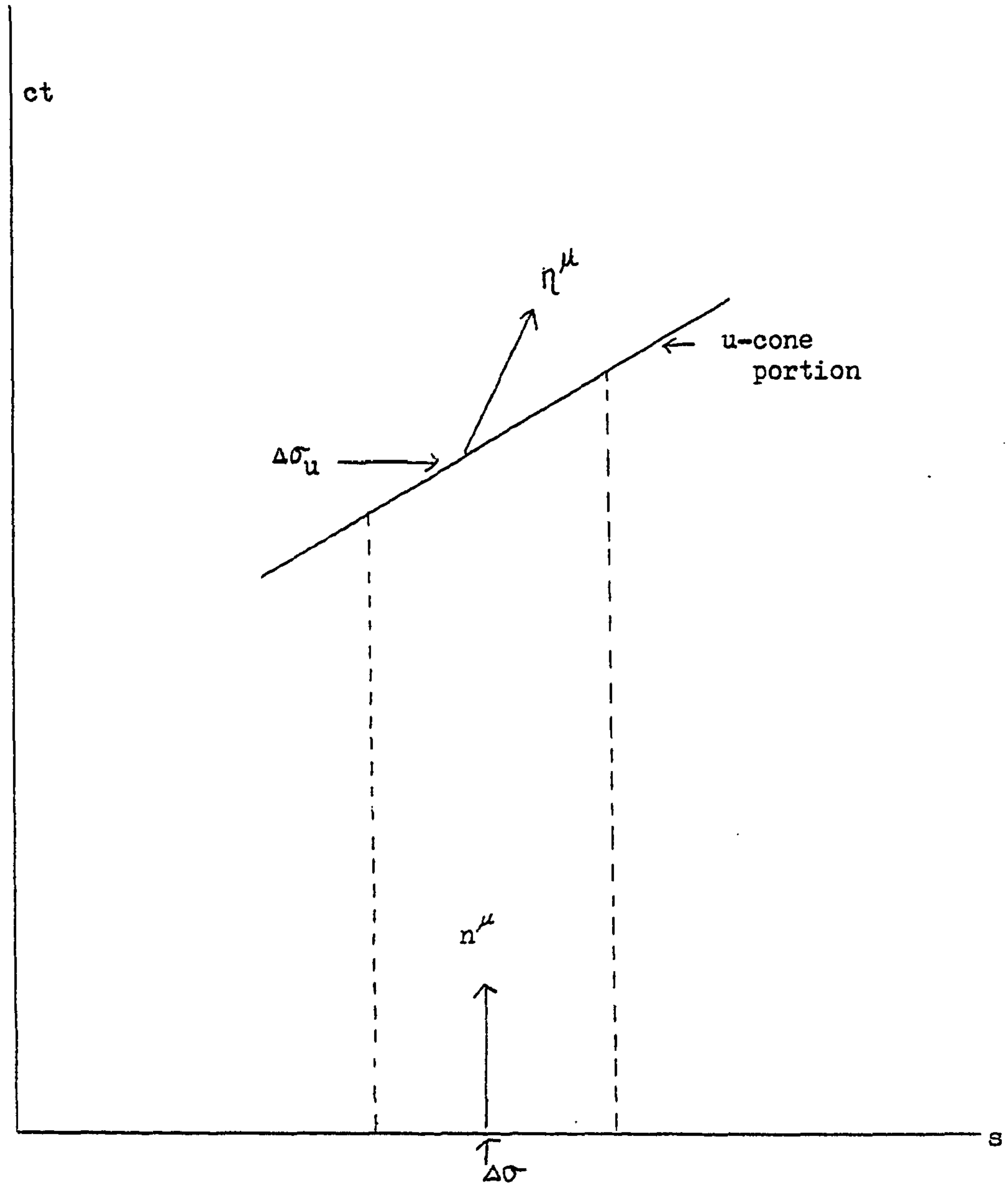

Fig. 3. Relevant surfaces for finding $d\sigma_u^{\nu}$.

APPENDIX III

RELEVANT INTEGRALS

Here we tabulate certain relevant integrals needed for evaluating the energy-momentum and angular momentum emission rates.

$$\int(\rho/\tilde{\rho})^3 U^\mu d\Omega=(4\pi\gamma'^4/u\gamma^4)(v^\mu-\gamma V^\mu) \tag{III.1}$$

$$\int(\rho/\tilde{\rho})^4 U.aU^\mu d\Omega=(4\pi\gamma'^4/3\gamma^4)[a^\mu-(4\gamma'^2V.a/\gamma u^2)v^\mu +(V.a/u^2)(4\gamma'^2+u^2/c^2)V^\mu] \tag{III.2}$$

$$\int(\rho/\tilde{\rho})^4 U^\mu d\Omega=(16\pi\gamma'^6/3u\gamma^5)(v^\mu-\gamma V^\mu) \tag{III.3}$$

$$\int(\rho/\tilde{\rho})^5 (U.a)^2 U^\mu d\Omega=(4\pi\gamma'^6/3u\gamma^6)[a.a+(V.a/v)^2\{3(2\gamma'^2-1) -1/\gamma^2\}](v^\mu-\gamma V^\mu) \tag{III.4}$$

$$\int(\rho/\tilde{\rho})^5 U.aU^\mu d\Omega=(4\pi\gamma'^6/3\gamma^5)[a^\mu+(6/\gamma u^2)V.av^\mu +(1/\gamma u^2)V.a(1-1/\alpha^2-6\gamma)V^\mu] \tag{III.5}$$

$$\int(\rho/\tilde{\rho})^5 U^\mu d\Omega=(4\pi\gamma'^6/3u\gamma^6)(6\gamma'^2-1)(v^\mu-\gamma V^\mu) \tag{III.6}$$

$$\int(\rho/\tilde{\rho})^3 d\Omega=4\pi\gamma'^4/\gamma^3 \tag{III.7}$$

$$\int(\rho/\tilde{\rho})^4 U.ad\Omega=-(16\pi/3)(v/u)(V.a/v)(\gamma'^6/\gamma^4) \tag{III.8}$$

$$\int(\rho/\tilde{\rho})^4 d\Omega=(4\pi\gamma'^4/3\gamma^4)(4\gamma'^2-1) \tag{III.9}$$

$$\int(\rho/\tilde{\rho})^5 (U.a)^2 d\Omega=(4\pi\gamma'^6/3\gamma^5)(V.a/v)^2(6\gamma'^2-5-1/\gamma^2) +(4\pi\gamma'^6/3\gamma^5)a.a \tag{III.10}$$

$$\int (\rho/\hat{\rho})^5 \mathrm{U.a}\,d\Omega = -(4\pi\gamma'^6/3\gamma^5)(v/u)(\mathrm{V.a}/v)(6\gamma'^2-1) \qquad \text{(III.11)}$$

$$\int (\rho/\tilde{\rho})^5 d\Omega = (4\pi\gamma'^6/\gamma^5)(2\gamma'^2-1) \qquad \text{(III.12)}$$

APPENDIX IV

THE SURFACE AREA ELEMENT ON $(\Delta\sigma)$

Here we calculate the expression for the surface area element on $(\Delta\sigma)$, where the orientation of $(\Delta\sigma)$ is chosen orthogonal to V^{μ}.

We refer to figure 4.

Now,

$$d\sigma^{\nu}=(V^{\nu}/c)\rho^{2}d\rho d\Omega \tag{IV.1}$$

In the medium rest frame, $\rho=(u/c)x^{4}_{(0)}$, so that $d\rho=(u/c)dx^{4}_{(0)}$. However, $dx^{4}_{(0)}=(\partial x^{4}_{(0)}/\partial\tau)d\tau=-\Lambda^{-1}_{4(0)}$, from eq. (I.3).

Thus,

$$d\sigma^{\nu}=-(V^{\nu}/c)\rho^{2}\Lambda^{-1}_{4(0)}(u/c)d\Omega d\tau \tag{IV.2}$$

However, $\Lambda^{4}_{(0)}=\rho/c\tilde{\rho}$, as a consequence of eq. (1.16), so that we finally have

$$d\sigma^{\nu}=(u/c)\rho\tilde{\rho}\, d\Omega d\tau\, V^{\nu} \tag{IV.3}$$

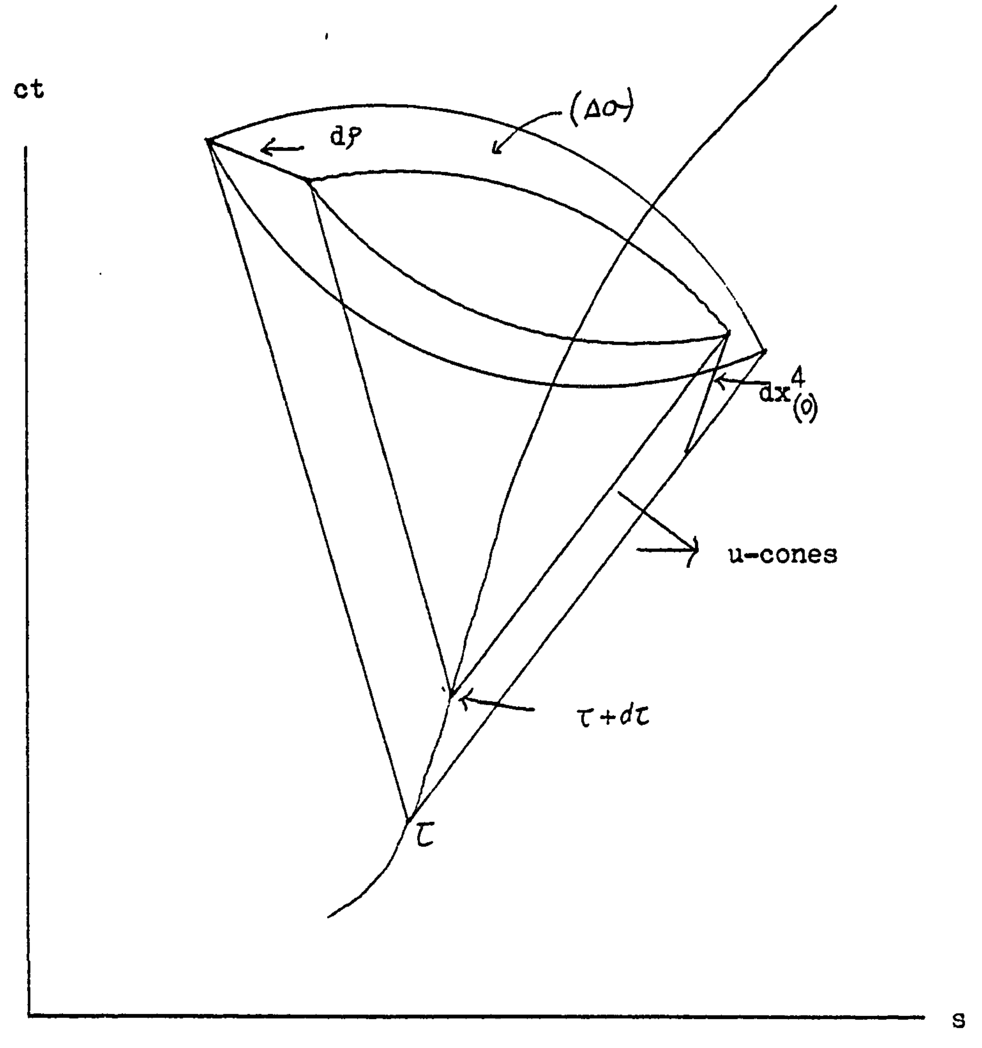

Fig. 4. Relevant construction for finding the area element $(\Delta\sigma)$.

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