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COLLECTIONS OF COVERS WHICH IMPLY COMPACTNESS

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I. INTRODUCTION

The metrization problem is one of the most interesting and fruitful problems in general topology. It has been the motivation for the study of many classes of topological spaces which generalize the concept of a metric space. Some of the more important spaces of this type are Moore spaces, quasi-developable spaces, wΔ-spaces, and first countable spaces.

Many of these classes of generalized metric spaces can be characterized by the existence of a countable collection of open covers for the space which satisfy some condition. A natural question is whether there are conditions which could be placed on these defining collections of covers which would imply familiar topological properties. This paper will supply some conditions which characterize compactness and related properties.

The question of what happens if uncountable collections of covers are used instead of countable collections will be examined in a subsequent section. Conditions will be given for these uncountable collections of covers which imply certain sets have limit points. This case seems to be more complex than the corresponding case for countable collections, and, as might be expected, the results depend on the type of set theory assumed.

II. DEFINITIONS

2.1 If M is a set, x is a point, and G is a collection of sets, then star (M,G) denoted st(M,G) is $\bigcup\{V \in G : M \cap V \neq \phi\}$; $st(x,G) = st(\{x\},G)$. A sequence $G = G_0, G_1, G_2, \ldots$ of open covers of a topological space S is called a <u>development</u> for S if and only if for each $x \in S$ and open set U containing x there is a non-negative integer n such that $st(x,G_n) \subseteq U$. A space which admits a development is said to be a <u>developable</u> space, while a regular T_1 developable space is called a <u>Moore space</u>.

2.2 A development $G = G_0, G_1, G_2, \ldots$ for a topological space S is said to be <u>monotonic</u> if and only if $G_{n+1} \subseteq G_n$ for each non-negative integer n.

2.3 A monotonic development $G = G_0, G_1, G_2, \ldots$ for a Moore space S is said to have the <u>3-link property</u> if and only if for each pair p and q of points of S there is a non-negative integer n such that if U and V are mutually exclusive members of G_n containing p and q respectively then no member of G_n has nonempty intersection with both U and V.

2.4 A nonempty subset M of a topological space S is said to be <u>discrete</u> if and only if for each point x of M there is an open set U such that $U \cap M = \{x\}$. Thus, a discrete subset of a space S need not be closed in S. A collection of sets is discrete if the closures of the sets are mutually exclusive and the union of any subcollection of these closures is closed.

2.5 An ordinal number is the set of all ordinals which precede it. Throughout this paper ω and ω_o will denote the set of all finite ordinals and κ will denote an infinite cardinal number.

2.6 A topological space S is said to have <u>property D(κ)</u> if and only if for each closed discrete subset M of S with cardinality at most κ there is a collection H of mutually exclusive open sets such that (1) H covers M and each member of H contains only one point of M, and (2) if N is a set covered by H such that each member of H contains only one point of N then N has no limit point. A space which has property D(ω) is said to have <u>property D</u> (see [Mo; page 69]). A space S is said to have <u>property</u> wD (see J.E. Vaughan [Va]) if and only if for each countably infinite, closed, discrete subset M of S there exist an infinite subset M' of M and a disjoint open cover H of M' satisfying the condition of property D with M replaced by M'.

2.7 If P is a property of topological spaces, then a topological space S is said to be P-closed if and only if S is a closed subspace of every space having property P which contains it. In this paper the class of Moore-closed spaces will be of interest. Most of the results about Moore-closed spaces used here can be found in the papers of J.W. Green [Gr 1] and [Gr 2] .

2.8 Borges [Bo] defines a topological space S to be a $w\Delta$-space if and only if there is a sequence $G_0, G_1, G_2, \ldots$ of open covers of S such that if $F_0, F_1, F_2, \ldots$ is a decreasing sequence of nonempty, closed subsets of S and there is a point x of S such that $F_n \subseteq st(x, G_n)$ for each non-negative integer n, then $\cap\{F_n : n=0,1,2,\ldots\} \neq \phi$.

2.9 According to Hodel [Ho 1], a space S is said to have a G_δ^*-diagonal if and only if there is a sequence $G_0, G_1, G_2, \ldots$ of open covers of S such that if x and y are distinct points of S, there is a non-negative integer n such that $y \notin \overline{st(x, G_n)}$.

III. MOORE SPACES

In [Gr 2; Theorem 1.6] J.W. Green obtained the following result:

Theorem. A Moore space not having infinitely many isolated points is compact if and only if there is a monotonic development $G = G_0, G_1, G_2, \ldots$ such that (*) if $H = H_0, H_1, H_2, \ldots$ is a monotonic development, there is an increasing sequence $n_0, n_1, n_2, \ldots$ of non-negative integers such that for each i, G_{n_i} refines H_i. Furthermore, a monotonic development G for a compact Moore space satisfies (*) if and only if G has the three-link property.

In this section several types of developments which imply results similar to (*) in Green's theorem will be investigated.

3.1 Definition. If κ is an infinite cardinal, a cardinality κ collection $G = \{G_\alpha : \alpha \varepsilon \kappa\}$ of open covers of a topological space S is said to satisfy condition $A(\kappa)$ if and only if for each closed, discrete subset M of S and open set U containing M there is an $\alpha \varepsilon \kappa$ such that $st(M, G_\alpha) \subseteq U$.

3.2 Definition. If κ is an infinite cardinal, a cardinality κ collection $G = \{G_\alpha : \alpha \varepsilon \kappa\}$ of open covers of a topological space S is said to satisfy <u>condition B(κ)</u> if and only if for each closed, discrete subset M of S there is an $\alpha\epsilon\kappa$ such that if x and y are distinct points of M, then $st(x,G_\alpha) \cap st(y,G_\alpha) = \phi$.

3.3 <u>Definition</u>. A space S is said to satisfy condition A(κ) or B(κ) if it has a collection of covers satisfying that condition.

3.4 <u>Remark</u>. A collection of covers for a T_1 space which satisfies condition A(ω) is a development. A developable space satisfying A(ω) or B(ω) has a development which satisfies that condition. A development which satisfies condition A(ω) is said to be of <u>type A</u> and a development which satisfies condition B(ω) is said to be of <u>type B</u>. Spaces having developments of type A and type B as well as some modifications of those conditions will be examined in this section.

3.5 <u>Definition</u>. A development $G = G_0, G_1, G_2, \ldots$ for a space S is said to be of <u>type wA</u> if and only if for each countably infinite, closed, discrete subset D of S and open set U including D, there exist an infinite subset D' of D and a non-negative integer n such that $st(D', G_n) \subseteq U$.

3.6 <u>Definition</u>. A development $G = G_0, G_1, G_2, \ldots$ for a space S is said to be of <u>type C</u> if and only if for each disjoint pair of countable, closed, discrete subsets D and E of S, there exists a non-negative integer n such that $st(D,G_n) \cap st(E,G_n) = \phi$.

3.7 Definition. A development $G = G_0, G_1, G_2, \ldots$ for a space S is said to be of type wC if and only if for each pair of countably infinite, closed, discrete subsets D and E of S there exist an infinite subset D' of D and a non-negative integer n such that $st(D',G_n) \cap st(E,G_n) = \phi$.

3.8 Definition. A development $G = G_0, G_1, G_2, \ldots$ for a space S is said to be of type E if and only if for each countably infinite, closed, discrete subset D of S there is an infinite subset D' of D and a non-negative integer n such that if x and y are distinct points of D', then $st(st(x,G_n),G_n) \cap st(st(y,G_n),G_n) = \phi$.

3.9 Remark. If S is a compact Moore space, the only closed, discrete subsets of S are finite. Hence it follows that every development for a compact Moore space is of each of the types defined in 3.4 through 3.8.

3.10 Theorem. If a Moore space S is the disjoint union of sets K and M where K is compact and each point of M is an isolated point of S, then S is a metrizable space having a development which is of each type defined in 3.4 through 3.8.

Proof. S is paracompact and R.H. Bing [Bi] proved that paracompact Moore spaces are metrizable. There exist a metric d for S and for each non-negative integer n, a finite cover H_n of K by d-open balls each of which is centered at some point of K and has radius at most $1/2^n$. If $G_n = H_n \cup \{\{x\}: x \varepsilon M\}$ for each non-negative integer n, then $G_0, G_1, G_2, \ldots$ is a development for S. It is readily verified that this development is of each of the types defined in 3.4 through 3.8.

3.11 Lemma. A space which has a development of one of the types defined in 3.4 through 3.8 has a monotonic development of that type.

Proof. If $G_0, G_1, G_2, \ldots$ is a development of one of the types defined in 3.4 through 3.8 and $H = H_0, H_1, H_2, \ldots$ is a development such that for each non-negative integer n, H_n refines G_n, then H is a development of the respective type. Moreover, for each development $G = G_0, G_1, G_2, \ldots$ there is a monotonic development $H = H_0, H_1, H_2, \ldots$ with H_n a refinement of G_n for each non-negative integer n.

3.12 Lemma. A Moore space S having a development of type A satisfies property D.

Proof. Suppose, on the contrary, that S has a development of type A but fails to have property D. Thus, there is a closed, discrete, countable set $M = \{x_0, x_1, x_2, \ldots\}$ such that if $\{U_0, U_1, U_2, \ldots\}$ is a disjoint collection of open sets covering M each member of which contains exactly one point of M, then there is a sequence $z_0, z_1, z_2, \ldots$ such that for each n, z_n belongs to U_n and the sequence $z_0, z_1, z_2, \ldots$ has a limit point.

Let $\{U_0, U_1, U_2, \ldots\}$ be a pairwise disjoint open cover of M such that $x_n \varepsilon U_n$. Let $G = G_0, G_1, G_2, \ldots$ be a monotonic development for S of type A. There is a least positive integer n_0 such that $st(M, G_{n_0}) \subseteq \bigcup_{n \geq 0} U_n$. Let $z_0^0, z_1^0, z_2^0, \ldots$ be a sequence of points having a limit point y_0 such that z_n^0 belongs to the set $st(x_n, G_{n_0}) \cap U_n$. There exist an open set V_0 containing y_0 whose closure misses M and a least positive integer n_1 greater than n_0 and such that $st(M, G_{n_1}) \subseteq S - \overline{V}_0$. Let $z_0^1, z_1^1, z_2^1, \ldots$ be a sequence of points having a limit point y_1 such that z_n^1 belongs to the set $st(x_n, G_{n_1}) \cap U_n$. Let V_1 be an open set containing y_1 whose closure misses M. This process may be continued.

If $Y = \{y_0, y_1, y_2, \ldots\}$ is a closed subset of S, there is an open set W including Y whose closure misses M.

Consequently, there is a positive integer k such that $st(M,G_k) \subseteq S - \overline{W}$. But, the sequence $z_0^k, z_1^k, z_2^k, \ldots$ is included in $st(M,G_k)$ and has a limit point y_k contained in W, which is impossible.

If $Y = \{y_0, y_1, y_2, \ldots\}$ has a limit point y, then let $M' = M-\{y\}$. Let W be an open set containing y whose closure misses M'. There is a non-negative integer k such that $st(M',G_k) \subseteq S - \overline{W}$, and there is a positive integer i greater than k such that y_i is contained in W. The sequence $z_0^i, z_1^i, z_2^i, \ldots$ except for at most one point is contained in $st(M',G_k)$ and has a limit point y_i belonging to W, which is impossible.

3.13 <u>Lemma</u>. A Moore space S having a development of type B satisfies property D.

Proof. Suppose the contrary and M is as in the proof of lemma 3.12. There is a monotonic development $G = G_0, G_1, G_2, \ldots$ for S of type B, and a least non-negative integer n_0 such that if p and q are distinct points of M then $st(p,G_{n_0}) \cap st(q,G_{n_0}) = \phi$. Moreover, there is a sequence $z_0^0, z_1^0, z_2^0, \ldots$ with $z_i^0 \varepsilon st(x_i, G_{n_0})$ for each i, and having a limit point y_0. Let $M_0 = M \cup \{y_0\}$. There is a least positive integer n_1 greater than n_0 such that if p and q are distinct points of M_0 then $st(p,G_{n_1}) \cap st(q,G_{n_1}) = \phi$. Moreover, there is a sequence $z_0^1, z_1^1, z_2^1, \ldots$ with $z_i^1 \in st(x_i, G_{n_1})$, and having a limit point y_1. Let $M_1 = M_0 \cup \{y_1\}$. This process may be continued.

Now $Y = \{y_0, y_1, y_2, \ldots\}$ is a discrete sequence of points. If Y is closed let $N = M \cup Y$. If Y is not closed, let y be a limit point of Y and $N = M \cup \{y\}$. In either case, N is a countable, closed, discrete subset of S for which there does not exist a non-negative integer k such that if p and q are distinct points of N, then $st(p,G_k) \cap st(q,G_k) = \phi$. This is a contradiction.

3.14 <u>Lemma</u>. A Moore space S having a development of type C satisfies property D.

Proof. The proof of this lemma is very similar to the proofs of Lemmas 3.12 and 3.13.

3.15 <u>Lemma</u>. If S is a Moore space containing no infinite, discrete, closed and open subset and having a development of type wA, then S is Moore-closed.

Proof. Suppose S is not Moore-closed. By Theorem 5 of [Re] there is an infinite discrete open collection U. There exist an infinite set $X = \{x_0, x_1, x_2, \ldots\}$ of points and a subcollection $\{U_0, U_1, U_2, \ldots\}$ of U such that x_n is a limit point of U_n for each non-negative integer n. Let $G_0, G_1, G_2, \ldots$ be a monotonic development for S of <u>type wA</u>. There is a set $Y = \{y_0, y_1, y_2, \ldots\}$ such that for each non-negative integer n $y_n \varepsilon\ st(x_n, G_n) \cap U_n - \{x_n\}$. It follows that Y is a closed subset of S, and there exist a positive integer k and an infinite subset X' of X such that $st(X', G_k) \subseteq S - Y$. This is a contradiction.

3.16 <u>Lemma</u>. If S is a Moore space containing no infinite, discrete, closed, and open subset, and having a development of type wC, then S is Moore-closed.

Proof. This Lemma may be established by a modification of the proof of Lemma 3.15.

3.17 <u>Theorem</u>. If S is a Moore space containing no infinite, discrete, open and closed subset, the following statements are equivalent:

(1) S is a compact metric space.

(2) S has a development of type A.

(3) S has a development of type B.

(4) S has a development of type C.

(5) S has a development of type wA and satisfies property wD.

(6) S has a development of type wC and satisfies property wD.

(7) S is Moore-closed and has a development of type E.

Proof: As indicated in 3.7, (1) implies (2) - (7). Suppose (2) holds but S is not compact. Suppose $G = G_0, G_1, G_2, \ldots$ is a monotonic development of type A for S. Now a non-compact Moore space is not countably compact. It follows that there is a countably infinite subset $M = \{x_0, x_1, x_2, \ldots\}$ of limit points of S such that M has no limit point. By lemma 3.12 S has property D and thus there is a disjoint collection $U = \{U_0, U_1, U_2, \ldots\}$ such that $x_n \varepsilon U_n$ and if T is any set such that each point of T belongs to some member of the collection U and no member of U contains more than one point of T, then T has no limit point. For each non-negative integer n, let y_n be a point of $st(x_n, G_n) \cap U_n - \{x_n\}$. Then, $Y = \{y_0, y_1, y_2, \ldots\}$ is a closed subset of S. Moreover, there is a non-negative integer k such that $st(M, G_k) \subseteq S - Y$. This is impossible since y_k is a point of $st(M, G_k)$. Hence (2) implies (1).

The arguments that (3) implies (1) and (4) implies (1) are analogous.

If S has a development of type wA or wC, then S is Moore-closed. A Moore-closed space with property wD is compact by the proof of Theorem 2.10 of [Gr 1].

Suppose S is Moore-closed and $G = G_0, G_1, G_2, \ldots$ is a monotonic development for S of type E. Moreover, suppose S fails to be compact. There is a countably infinite subset M of S which has no limit point. Also, there exist an infinite subset M' of M and a non-negative integer k such that if x and y are distinct points of M' then $st(st(x,G_k),G_k) \cap st(st(y,G_k),G_k) = \phi$. By Theorem 7 of [Re] there is a finite set $\{z_0, z_1, z_2, \ldots, z_n\}$ such that for each point x of S, there are intersecting members V and W of G_k such that x belongs to V and W contains at least one of the points $z_0, z_1, z_2, \ldots, z_n$. But then, for some non-negative integer i, z_i belongs to the set $st(st(x,G_k),G_k)$ for infinitely many points x of M'. This is a contradiction. Thus, (7) implies (1).

3.18 <u>Theorem</u>. For a Moore space S, the following statements are equivalent:

(1) S is a Moore space which is the union of a compact set and a discrete set.

(2) S is a metric space and the set of all limit points of S is compact.

(3) S has a development of type A.

(4) S has a development of type B.

(5) S has a development of type C.

Proof. That (1) implies (2) is Theorem 3.10. The remaining implications can be shown by a slight modification of the proof of Theorem 3.17.

3.19 Remark. The previous results may be used to prove the following familiar theorem which gives metric space analogues of developments of types A, B, and C.

Theorem. A metric space X having at most finitely many isolated points is compact if and only if there is a metric d for X which satisfies at least one of the properties:

(1) If M is a countable, closed, discrete subset of X and U is an open set containing M, there is a positive number ε such that the d-open neighborhood of M of radius ε is a subset of U.

(2) If M is a countable, closed, discrete subset of X, there is a positive number ε such that if x and y are distinct points of M, then $d(x,y) \geq \varepsilon$.

(3) If M and N are disjoint, countable, closed, discrete subsets of X, there is a positive number ε such that $d(M,N) \geq \varepsilon$.

3.19 An interesting generalization of development, due to E.E. Grace [Be], is the concept of a quasi-development. A sequence $G = G_1, G_2, G_3, \ldots$ of collections of open subsets of a topological space S is called a <u>quasi-development</u> for S provided for each point p of S and open set U containing p there is a positive integer n such that some member of G_n contains p and $st(p,G_n) \subseteq U$.

3.20 <u>Definition</u>. A quasi-development $G = G_1, G_2, G_3, \ldots$ for a space S is said to be of <u>type B</u> if and only if for every countable, closed, discrete subset D of S, there exists a positive integer n such that G_n covers D, and if x and y are distinct points of D then $st(x,G_n) \cap st(y,G_n) = \phi$.

3.21 <u>Definition</u>. A quasi-development $G = G_1, G_2, G_3, \ldots$ for a space S is said to be of <u>type C</u> if and only if for every disjoint pair of countable, closed, discrete subsets D and E of S, there exists a positive integer n such that G_n covers both D and E and $st(D,G_n) \cap st(E,G_n) = \phi$.

3.22 <u>Theorem</u>. If S is a regular T_1, quasi-developable space containing no infinite, discrete, closed and open subset, the following statements are equivalent:

(1) S is compact metric.

(2) S has a quasi-development of type B.

(3) S has a quasi-development of type C.

Proof. That (1) implies (2) and (3) follows from Theorem 3.17.

Modifying the arguments, only slightly, it can be shown first that (3) or (2) implies that S satisfies property D, and then that S is countably compact. H. H. Wicke and J. M. Worrell showed in [Wi and Wo; Theorem 2.10] that countably compact quasi-developable spaces are compact metric.

IV. FIRST COUNTABLE SPACES

Heath in [He] and Hodel in [Ho 2] gave elegant characterizations of various classes of generalized metric spaces. Suppose X is a topological space, and g is a function from $\omega \times X$ to the topology on X such that for each point x of X and each non-negative integer n, x belongs to g(n,x). They found that placing conditions on g yields characterizations of familiar classes of topological spaces. For example, consider these conditions:

(I) If x belongs to $g(n,x_n)$ and y_n belongs to $g(n,x_n)$ for each n, then x is a cluster point of the sequence $y_0, y_1, y_2, \ldots$.

(II) If x belongs to $g(n,x_n)$ and y_n belongs to $g(n,x_n)$ for each n, then the sequence $y_0,y_1,y_2,\ldots$ has a cluster point.

(III) If x_n belongs to $g(n,x)$ for each n, then x is a cluster point of the sequence $x_0,x_1,x_2,\ldots$.

Heath [He] showed that (I) characterizes developable spaces, while Hodel [Ho 2] showed that (II) characterizes $w\Delta$-spaces and (III) characterizes first countable spaces. This approach suggests the definitions which follow.

Throughout this section g will be a function from $\omega \times X$ into the topology on X such that for each point x of X and non-negative integer n, x belongs to $g(n,x)$. For each non-negative integer n, $G_n = \{g(n,x):x\varepsilon X\}$ is an open cover of X.

4.1 Definition. The function g is said to satisfy condition B if and only if the collection $G = \{G_0,G_1,G_2,\ldots\}$ satisfies condition $B(\omega)$.

4.2 Definition. The function g is said to satisfy condition A' if and only if for each countable, closed, discrete subset D of X and open set U containing D there is a non-negative integer k such that if $x \varepsilon D$ then $g(k,x) \subseteq U$.

4.3 <u>Definition</u>. The function g is said to satisfy <u>condition B'</u> if and only if for each countable, closed, discrete subset D of X there is a non-negative integer k such that if x and y are distinct points of D then $g(k,x) \cap g(k,y) = \phi$.

4.4 <u>Definition</u>. The function g is said to satisfy <u>condition C</u> if and only if for each disjoint pair D and E of countable, closed, discrete subsets of X there is a non-negative integer k such that $st(D,G_k) \cap st(E,G_k) = \phi$.

4.5 <u>Definition</u>. The function g is said to satisfy <u>condition C'</u> if and only if for each disjoint pair D and E of countable, closed, discrete subsets of X there is a non-negative integer k such that if x belongs to D and y belongs to E, then $g(k,x) \cap g(k,y) = \phi$.

4.6 <u>Remark</u>. If there is a function which satisfies one of the conditions in definitions 4.1 through 4.5, then there is a function g satisfying that condition such that for each point x of S and each non-negative integer n, $g(n+1,x) \subseteq g(n,x)$. A function having this additional property will be called a <u>monotonic function</u>.

4.7 <u>Theorem</u>. A regular T_1 space X having a function g which satisfies condition B is a Moore space having a development of type B.

Proof. Suppose g is a function which satisfies condition B. It will be shown that g satisfies (II) and, hence, X is a wΔ-space. Suppose $x_0, x_1, x_2, \ldots$ and $y_0, y_1, y_2, \ldots$ are sequences and x is a point of X such that both x and y_n belong to $g(n, x_n)$ for each non-negative integer n. If $Y = \{y_0, y_1, y_2, \ldots\}$ has no cluster point, then $Y' = Y \cup \{x\}$ is a countable, closed discrete subset of X. There is a positive integer k such that if p and q are distinct points of Y', then $st(p, G_k) \cap st(q, G_k) = \phi$. But this is impossible since x and y_k both belong to $g(k, x_k)$. Thus, Y has a cluster point, and X is a wΔ-space.

If x and y are distinct points of X, there is a non-negative integer k such that $st(x, G_k) \cap st(y, G_k) = \phi$. Consequently X has a G_δ*-diagonal. Hodel in [Ho 1; Theorem 2.5] showed that every wΔ- space with a G_δ*-diagonal is developable. Every development $H = H_0, H_1, H_2, \ldots$ for X such that for each non-negative integer n, H_n is a refinement of G_n is a development of type B.

4.8 Theorem. A regular T_1 space X having a function g which satisfies condition C is a Moore space having a development of type C.

Proof. An argument identical to the one given for theorem 4.7 also proves this result.

4.9 Lemma. A regular T_1 space having a function g which satisfies one of conditions A', B', or C', satisfies property D.

Proof. Suppose g is a monotonic function satisfying one of the conditions A', B', or C'. An argument essentially the same as the corresponding argument in Lemma 3.12, Lemma 3.13, or 3.14 will prove the respective result.

4.10 Theorem. For a regular T_1 space X containing no infinite, discrete, open and closed subset, the following statements are equivalent.

(1) X is a first countable, countably compact space.

(2) There is a function g for X which satisfies condition A'.

(3) There is a function g for X which satisfies condition B'.

(4) There is a function g for X which satisfies condition C'.

Proof. Suppose X is a first countable, countably compact space. There is a function g such that for each point x of X, $\{g(n,x) : n \in \omega\}$ is a base for the neighborhoods of x and, moreover, $g(n+1,x) \subseteq g(n,x)$. This function g will satisfy each of the conditions A', B', and C', since the only discrete subsets of X are finite.

To show that each of the statements (2), (3), or (4) implies that X is countably compact all that is needed is to repeat the corresponding argument from 3.17. Each point of X is a G_δ assuming one of the statements (2), (3), or (4). A countably compact space in which each point is a G_δ is first countable [Wi;17F.7].

4.11 Remark. As a result of Theorem 4.10 none of conditions A', B', and C' imply developability. The space of countable ordinals with the usual order topology is a first countable, countably compact space and, hence, has a function g satisfying each of conditions A', B', and C' but is not developable and not compact.

4.12 Theorem. For a first countable, regular T_1 space X the following statements are equivalent:

(1) The set of all limit points of X is countably compact.

(2) There is a function g for X which satisfies condition A'.

(3) There is a function g for X which satisfies condition B'.

(4) There is a function g for X which satisfies condition C'.

Proof. There is a function g from $\omega \times X$ to the topology of X satisfying (1) $g(n,x) = \{x\}$ for each isolated point x of X and non-negative integer n, (2) for each limit point y of X, $\{g(n,y): n \in \omega\}$ is a local base at y and, moreover, for each non-negative integer n, $g(n+1,y) \subseteq g(n,y)$. This function g satisfies condition A', B', and C'.

The implications (2) implies (1), (3) implies (1), and (4) implies (1) are established by applying the respective parts of the proof of 3.17 to a monotonic function of the respective type.

V. SET THEORY

This section will provide a summary of the results from set theory which will be referred to in the remainder of this paper. The notation used will be essentially that found in Jech [Je]. The usual axioms for set theory, the Zermelo-Fraenkel axioms including the axiom of choice, will be denoted by ZFC.

5.1 The continuum hypothesis, denoted CH, is the assertion that 2^{ω_0} is ω_1. The notation c is used for 2^{ω_0}.

The work of Gödel [Go] and Cohen [Co] showed that CH is independent of ZFC.

5.2 A topological space X is said to satisfy the <u>countable chain condition</u> if and only if there does not exist an uncountable pairwise disjoint collection of open subsets of X. <u>Martin's Axiom</u> [Ma & So], denoted MA, is the assertion that no compact, countable chain condition, Hausdorff space X is the union of fewer than c nowhere dense sets. Note that CH implies MA. It has been shown [Ma & So] that MA together with the negation of CH is consistent with ZFC.

5.3 If α is a limit ordinal, the <u>cofinality</u> of α, denoted $cf(\alpha)$, is the least ordinal β such that there is a function f from β into α such that $\sup\{f(x):x\varepsilon\beta\}=\alpha$.

5.4 The set ${}^{\omega}\omega$ of all functions from ω to ω has two natural partial orders. If f and g are functions from ω to ω then $f<g$ if and only if for each non-negative integer n, $f(n)<g(n)$, and $f<^*g$ if and only if there is a positive integer m such that for each positive integer n greater than m, $f(n)<g(n)$.

5.5 A subset S of ${}^{\omega}\omega$ is called a <u>scale</u> if and only if for each $f\varepsilon{}^{\omega}\omega$ there is a $g\varepsilon S$ such that $f<^*g$. A scale S

which is well ordered by $<^*$ and order isomorphic to the ordinal α is called an α-scale.

5.6 A subset $\mathcal{S}$ of ${}^{\omega}\omega$ is called a <u>dominating family</u> if and only if for each $f \in {}^{\omega}\omega$ there is a $g \in \mathcal{S}$ such that $f < g$.

5.7 <u>Remark</u>. If there is a scale with cardinality κ, there is a dominating family of cardinality κ.

5.8 In [Hc 1], Stephen Hechler showed (1) for each cardinal κ such that $\omega_1 \leq cf(\kappa) \leq \kappa \leq c$, it is consistent with ZFC that there exist a scale whose cardinality is κ, (2) the existence of a scale with cardinality ω_1 implies the existence of an ω_1-scale, and (3) MA implies all scales have cardinality c.

Hausdorff [Ha] showed that CH implies the existence of an ω_1-scale. A consequence of one of Hechler's theorems [He 1; Theorem 8.1] is that for each cardinal κ of uncountable cofinality, it is consistent with ZFC that there is a scale with cardinality ω_1 and $c = \kappa$. In particular, c may be assumed to have cofinality greater than ω_1.

VI. UNCOUNTABLE COLLECTIONS OF COVERS

The results of previous sections showed that for regular T_1 spaces, which contain no infinite, discrete, closed and open subset, the statement S is a compact metric space is equivalent to each of the following statements: (I) S satisfies $A(\omega)$, (II) S satisfies $B(\omega)$.

In the arguments which showed that each of statements (I) and (II) imply compactness, it is shown that every infinite subset of S has a limit point. For an infinite cardinal κ, a topological space having the property that each of its subsets of cardinality κ has a limit point will be called <u>κ-compact</u>. In T_1 spaces, ω_o-compactness is equivalent to countable compactness. In this section cardinality κ collections of open covers will be used to explore κ-compact spaces.

6.1 <u>Theorem</u>. If κ is an infinite cardinal, and S is a regular T_1 space having property $D(\kappa)$, satisfying either $A(\kappa)$ or $B(\kappa)$ and containing no infinite, discrete, open and closed subset of cardinality κ, then S is κ-compact.

Proof. Suppose the theorem is false. Then there exists a closed, discrete subset $M = \{x_\alpha:\alpha\varepsilon\kappa\}$ of limit points of S. There is a disjoint collection of open sets $U = \{U_\alpha:\alpha\varepsilon\kappa\}$ which satisfies the definition of property $D(\kappa)$ for M and, moreover, $x_\alpha \varepsilon U_\alpha$.

There is a cardinality κ collection $\{G_\alpha:\alpha\varepsilon\kappa\}$ which satisfies condition $A(\kappa)$. For each $\alpha\varepsilon\kappa$, let

$y_\alpha \varepsilon U_\alpha \cap st(x_\alpha, G_\alpha) - \{x_\alpha\}$. Then, the set $Y = \{y_\alpha : \alpha\varepsilon\kappa\}$ is a closed, discrete subset of S. There does not exist an $\alpha\varepsilon\kappa$ such that $st(M, G_\alpha) \subseteq S - Y$. This is a contradiction. If condition $A(\kappa)$ is replaced by condition $B(\kappa)$ and $M' = M \cup Y$, a similar contradiction is reached.

6.2 <u>Remark</u>. Whether the converse to Theorem 6.1 is true, even for metric spaces, depends on the type of set theory assumed. The case for ω_1-compactness is of particular interest, since in metric spaces ω_1-compact, Lindelöf, separable, and second countable are equivalent and important.

6.3 <u>Theorem</u>. CH implies that if S is a metric space which has no uncountable, closed and open subset, the following are equivalent:

(1) S is ω_1-compact.

(2) S has a cardinality ω_1 collection of open covers which satisfy condition $A(\omega_1)$.

(3) S has a cardinality ω_1 collection of open covers which satisfy condition $B(\omega_1)$.

Proof. That each of statements (2) and (3) imply statement (1) follows from Theorem 6.1.

If S is an ω_1-compact metric space, then there is a countable base B for the topology on S. There is a collection G consisting of all subsets of B which cover S. The cardinality of G is ω_1 and the collection G will satisfy either (2) or (3).

6.4 Example. If $\kappa < cf(c)$, there is a subspace S of the real line such that if $G = \{G_\alpha : \alpha \epsilon \kappa\}$ is a collection of open covers of S, there is a closed, discrete subset D of S, such that if $\beta \epsilon \kappa$, there is a member of G_β containing more than one point of D.

There is a subset S of the set R of real numbers such that both S and R-S have cardinality c and, moreover, both S and R-S intersect every uncountable, closed subset of R [Br]. Suppose there is a collection $G = \{G_\alpha : \alpha \epsilon \kappa\}$ of open covers of S contrary to the claim. For each point t in the complement of S, there is a sequence $t_0, t_1, t_2, \ldots$ of points of S which converges to t. The set of terms of this sequence is discrete and closed in the subspace topology of S. For each $\alpha \epsilon \kappa$, let T_α be the set of all points t belonging to R-S such that no member of G contains more than one point of the sequence $t_0, t_1, t_2, \ldots$. Since $cf(c) > \kappa$ and $\bigcup_{\alpha \epsilon \kappa} T_\alpha = R - S$ for some $\alpha \epsilon \kappa$, T_α has cardinality c. The closure in R of T_α contains a point p of S. This point p belongs to some member V of G_α. There is a set U open in R such that $V = U\cap S$. Moreover, $U\cap T_\alpha \neq \phi$. If $t\varepsilon U\cap T$, U contains a tail of the sequence $t_0, t_1, t_2, \ldots$ associated with t, but then so does V. This is a contradiction.

6.5 Remark. In particular, the Example 6.7 shows that if $\omega_1 < cf(c)$, there is no ω_1-cardinality collection of open covers of S which satisfies either condition $A(\omega_1)$ or condition $B(\omega_1)$ even though the example is ω_1-compact! Since a metric space has property $D(\kappa)$ for all κ, the converse of Theorem 6.1 fails if $\omega_1 < cf(c)$.

6.6 In what follows the space Y will denote the set to which a point x belongs if and only if x is a non-negative integer or for non-negative integers n and k, $x = n - \frac{1}{k+2}$. The topology on Y is the subspace topology that Y inherits as a closed subset of the set of real numbers with the usual topology.

6.7 Lemma. If S is a metric space and the set of all limit points of S is not compact, then S includes a closed subspace which is homeomorphic to the space Y.

Proof. The set M of all limit points of S is not compact and thus not countably compact. Hence, M has a countably infinite subset $D = \{x_0, x_1, x_2, \ldots\}$ with no limit point. Every metric space satisfies property D, and, consequently, there is a disjoint collection $U = \{U_0, U_1, U_2, \ldots\}$ with $x_n \varepsilon U_n$ for all n, and which satisfies the condition in property D. For each n, there is a sequence $z_0{}^n, z_1{}^n, z_2{}^n, \ldots$ of distinct points of U_n which converges to x_n. The set X, to which a point belongs if and only if it is a point of D or it is $z_n{}^k$ for some non-negative integers n and k, is a closed subset of S homeomorphic to Y.

6.8 <u>Lemma</u>. If κ is an infinite cardinal, there is a cardinality κ collection of open covers of Y which satisfies condition $A(\kappa)$ or $B(\kappa)$ if and only if there is a dominating family of cardinality κ.

Proof. If $G = \{G_\alpha : \alpha \varepsilon \kappa\}$ is a collection of open covers of Y, define a function f_α for each $\alpha\varepsilon\kappa$ as follows. For each non-negative integer n, let $f_\alpha(n) = \inf\{i : i$ is a positive integer and $n - \frac{1}{i+2} \varepsilon \operatorname{st}(n, G_\alpha)\}$. The set $\{f_\alpha : \alpha\varepsilon\kappa\}$ forms a dominating family provided the collection of covers $\{G_\alpha : \alpha\varepsilon\kappa\}$ satisfies condition $A(\kappa)$ or condition $B(\kappa)$. For if $g \varepsilon {}^\omega\omega$, then $M = \{n - \frac{1}{i+2} : i \leq g(n) + 1$, and $n \varepsilon \omega\}$ is a closed subset of

Y. If $\{G_\alpha:\alpha\epsilon\kappa\}$ satisfies condition $A(\kappa)$, there is an $\alpha\epsilon\kappa$ such that for each non-negative integer n, $st(n,G_\alpha) \subseteq Y - M$. It then follows that $f_\alpha(n)>g(n)$ for each n. On the other hand, if $G = \{G_\alpha:\alpha\epsilon\kappa\}$ satisfies condition $B(\kappa)$ then $D = M\cup\omega$ is a closed discrete set, and there is an $\alpha\epsilon\kappa$ such that if x and y are distinct points of D, then $st(x,G_\alpha)\cap st(y,G_\alpha) = \phi$. It again follows that $f_\alpha(n)>g(n)$ for each n.

For the converse, if n and k are non-negative integers and $f \in {}^\omega\omega$, let $U(n,k) = \{n\}\bigcup\{n-\frac{1}{i+2} : i \geq k\}$ and $G_f = \{\{n - \frac{1}{k+2}\} : n,k \in \omega\} \bigcup \{U(n,f(n)):n \in \omega\}$. If S is a dominating family of cardinality κ, then $\{G_f:f\epsilon S\}$ is a cardinality κ collection of open covers of Y which satisfies both condition $A(\kappa)$ and condition $B(\kappa)$.

6.9 <u>Theorem</u>. If κ is an infinite cardinal and S is a σ-compact metric space whose set of limit points is not compact, the following statements are equivalent:

(1) There is a dominating family of cardinality κ.

(2) There is a cardinality κ collection of open covers of S satisfying condition $A(\kappa)$.

(3) There is a cardinality κ collection of open covers of S satisfying condition $B(\kappa)$.

Proof. Suppose there is a dominating family $\mathcal{S}$ with cardinality κ. There is an increasing sequence $F_0, F_1, F_2, \ldots$ of compact sets whose union is S. For each pair of non-negative integers n and k, let G_n^k be a cover of F_n by open balls, centered at a point x of F_n, with radius less than $1/2^{k+1}$. For each $g \varepsilon \mathcal{S}$ define $G_g = \bigcup\{G_n^{g(n)} : n = 0,1,2,\ldots\}$. Corresponding to each closed discrete subset D of S, there is a function $f \varepsilon {}^{\omega}\omega$ such that if x and y are distinct points of D, at least one of which belongs to F_n, then $d(x,y) > 1/2^{f(n)}$. There is a $g \varepsilon \mathcal{S}$ such that $g > f$. If x and y are distinct points of D, $st(x,G_g) \bigcap st(y,G_g) = \phi$. This shows (1) implies (3). A similar argument shows the same collection also satisfies condition $A(\kappa)$ and hence yields (1) implies (2).

The remaining implications are obtained by noting that by 6.7 S includes a closed subspace M which is homeomorphic to the space Y. The existence of a cardinality κ collection of open covers satisfying either condition $A(\kappa)$ or condition $B(\kappa)$ is hereditary on closed subsets. Lemma 6.8 gives the desired result.

6.10 Remark. The next two results give interesting applications of Theorem 6.9 to the set R of real numbers. The set of rational numbers is denoted by Q, and the set of all discrete subsets of Q which are closed in R is denoted by F. The collection F has cardinality c.

6.11 Theorem. If there is a dominating family of cardinality ω_1, and $\omega_1 < cf(c)$, then every subcollection H of F with cardinality c has a subcollection H' with cardinality c and such that $\bigcup H'$ has no irrational limit point.

Proof. It follows with the aid of Theorem 6.9 there is a collection $G = \{G_\alpha : \alpha \in \omega_1\}$ of open covers of R which has the following properties:

(1) For each closed, discrete subset D of R there is a $\beta \in \omega_1$ such that if x and y are distinct points of D, then $st(x, G_\beta) \cap st(y, G_\beta) = \phi$.

(2) For each $\alpha \in \omega_1$ the cardinality of G_α is ω_0.

(3) For each $\alpha \in \omega_1$ the cover G_α is locally finite.

For $\alpha \in \omega$, let $G_\alpha = \{V_n : n = 0,1,2,\ldots\}$. For each non-negative integer n, $x_0^n, x_1^n, x_2^n, \ldots$ are the points of the set $Q \cap V_n$. There is a dominating family $\mathcal{S}$ with cardinality ω_1. For $f \in \mathcal{S}$ define $D_f^\alpha = \bigcup_{n \in \omega} \{x_t^n : t \leq f(n)\}$. The set $C = \{D_f^\alpha : \alpha \in \omega_1 \text{ and } f \in \mathcal{S}\}$ has cardinality at most ω_1, being the union of ω_1 collections each with cardinality ω_1. Each D_f^α is a closed discrete subset of R.

Suppose D belongs to F. There is an $\alpha\varepsilon\omega_1$ such that whenever x and y are distinct points of D, $st(x,G_\alpha)\cap st(y,G_\alpha)=\phi$. Define a function r from ω to ω as follows: If $V_n \cap D = \phi$ then $r(n) = 1$. If $V_n \cap D \neq \phi$ then $V_n \cap D = \{x_k^n\}$ for some non-negative integer k. In this case define $r(n) = k$. There is a member f of S such that $r<f$. Then $D \subseteq D_f^{\alpha}$. Thus every member of F is contained in some member of C. If H is a subcollection of F having cardinality c, there is an $\alpha\varepsilon\omega_1$ and an $f\varepsilon S$ with the property that D_f^{α} includes each member of some cardinality c subcollection H' of H. Then $\bigcup H' \subseteq D_f^{\alpha}$ which is a closed discrete set.

6.12 Theorem. Assuming CH, there is a subcollection H of F having cardinality c and such that if H' is any subcollection of H with cardinality c, then $\bigcup H'$ has an irrational limit point.

Proof. For each non-negative integer n, let $x_0^n, x_1^n, x_2^n, \ldots$ denote the rational numbers in $(n,n+1)$. CH implies there is an ω_1-scale S. For each $f\varepsilon S$ let $D_f = \{w_t^n : t \leq f(n)$ and $n = 0,1,2,\ldots\}$. Let $H = \{D_f : f\varepsilon S\}$. For each subset H' of H having cardinality c, $S' = \{f\varepsilon S : D_f \varepsilon H'\}$ is cofinal in S. Suppose for some subset H' of H having cardinality c, $\bigcup H'$ has no irrational limit point. For each non-negative integer n, there is a non-negative integer k_n such that if $i \geq k_n$ then $x_i^n \notin \bigcup H'$. There is a function r from ω to ω such that

$r(n) = k_n$ for each non-negative integer n. Moreover, there exists a function $f \varepsilon S'$ such that $r <^* f$, and a non-negative integer n such that $r(n) < f(n)$. Thus $x_{f(n)}{}^n \varepsilon \bigcup H'$ which is a contradiction.

6.13 Remark. Whether Theorem 6.9 remains true if σ-compactness is replaced by a weaker condition as part of its hypothesis is unknown to the author. Some of the results which follow seem relevant to this question.

In Theorem 6.14 P is the set of irrational real numbers with its usual subspace topology.

6.14 Theorem. The existence of a dominating family of cardinality κ implies the existence of a cardinality κ collection $G = \{G_\alpha : \alpha \varepsilon \kappa\}$ of open covers of P having the property that if D is an infinite, closed, discrete subset of P, in the subspace topology of P, and U is an open set containing D, then there exist an $\alpha \varepsilon \kappa$ and an infinite subset D' of D such that $st(D', G_\alpha) \subseteq U$.

Proof. There is a collection $H = \{H_\alpha : \alpha\varepsilon\kappa\}$ of open covers of the set R of real numbers which satisfy $A(\kappa)$ for R. For each point q of the set Q of rational numbers there is a collection $H^q = \{H_\alpha{}^q : \alpha\varepsilon U\}$ of open covers of the set $R - \{q\}$ which satisfies condition $A(\kappa)$ for the space $R - \{q\}$.

If $H' = H\bigcup(\bigcup\{H^q : q\varepsilon Q\})$, the cardinality of H' is κ. For $F'\varepsilon H'$, let $F = \{V\cap P : V\varepsilon F'\}$, and $G = \{F : F'\varepsilon H'\}$.

If D is an infinite, closed, discrete subset of P which is also closed in R and U is a set open in P which contains D, there exist a set U' open in R and an $\alpha\varepsilon\kappa$ such that $U'\cap P = U$ and $st(D,H_\alpha) \subseteq U'$. There is an $F\varepsilon G$ corresponding to H_α and $st(D,F) \subseteq U$.

If D is an infinite, closed, discrete subset of P having a limit point q in Q, there is an infinite sequence $x_0, x_1, x_2, \ldots$ of points of D which converges to q. The set X of all terms of this sequence is a closed discrete subset of $R - \{q\}$. If U is an open set in P containing X, then there exist a set U' open in R and an $\alpha\varepsilon\kappa$ such that $U'\cap P = U$ and $st(D,H_\alpha{}^q) \subseteq U'$. There is an $F\varepsilon G$ corresponding to $H_\alpha{}^q$ and $st(D,F) \subseteq U$.

6.15 <u>Definition</u>. A monotonic development $G_0, G_1, G_2,$ ... for a Moore space is said to be <u>semicomplete</u> or <u>Rudin complete</u> [Ru] if and only if whenever $U_0, U_1, U_2, \ldots$ is a sequence of sets such that for each non-negative integer n, $U_n \varepsilon G_n$ and $\overline{U_{n+1}} \subseteq U_n$, then $\bigcap U_n \neq \emptyset$. A Moore space having a semicomplete monotonic development is said to be semicomplete.

6.16 <u>Definition</u>. A development $G = G_0, G_1, G_2, \ldots$ for a space S is said to be of <u>type wB</u> if and only if for each countably infinite, closed, discrete subset D of S there exist an infinite subset D' of D and a non-negative integer n such that if x and y are distinct points of D' then

$st(x, G_n) \cap st(y, G_n) = \emptyset$.

6.17 <u>Theorem</u>. Every monotonic development for a Moore space of type wB is semicomplete.

Proof. Suppose that $G_0, G_1, G_2, \ldots$ is a monotonic development for the Moore space of type wB which is not semicomplete. Then there is a sequence $U_0, U_1, U_2, \ldots$ of open sets with the property that $U_n \varepsilon G_n$ and $\overline{U_{n+1}} \subseteq U_n$ for all n, such that $\cap U_n = \emptyset$.

Let $x_0 \varepsilon U_0$. There is a least integer n_1 such that $x_0 \not\varepsilon \overline{U_{n_1}}$.

Let $x_1 \varepsilon U_{n_1}$. This process may be continued. The set

$X = \{x_0, x_1, x_2, \ldots\}$ is a closed, discrete subset of S for which

the condition in type wB is not satisfied.

6.18 <u>Theorem.</u> A metric space is complete if and only if it has a development of type wB.

Proof. Suppose there is a development of type wB for the metric space X. By Theorem 6.17 X is semicomplete. M.E. Rudin [Ru; Theorem 7] proved that for metric spaces semicompleteness is equivalent to complete metrizability.

Conversely, suppose X has a complete metric d. Let $B(x,\varepsilon) = \{y \varepsilon X : d(x,y) < \varepsilon\}$.

For each non-negative integer n let

$G_n = \{B(x, 1/2^{n+1}) : n \varepsilon N\}$. Let M be a countably infinite, closed, discrete subset of X. For each non-negative integer n, $A_n = \{x \varepsilon M : st(st(x,G_n),G_n) \cap M \text{ is finite}\}$. Suppose A_n is finite for each n. There is a point $x_0 \varepsilon M-A_0$. Then there is a point $x_1 \varepsilon st(st(x_0,G_0),G_0) \cap M-A_1-\{x_0\}$. There is a point $x_2 \varepsilon st(st(x_1,G_1),G_1) \cap M-A_2-\{x_0,x_1\}$. This process may be continued. The sequence $x_0,x_1,x_2,\ldots$ of distinct points of X is Cauchy and hence converges to a point y. Thus y is a limit point of M which is impossible. Let k be a non-negative integer such that A_k is infinite. Let $\{a_0,a_1,a_2,\ldots\}$ be the points of A_k. There is a least positive integer n_1 such that $a_{n_1} \not\varepsilon st(st(a_0,G_k),G_k)$. There is a least positive integer n_2 such that $a_{n_2} \not\varepsilon st(st(\{a_0,a_{n_1}\},G_k),G_k)$. This process may be continued. The set $A = \{a_0,a_{n_1},a_{n_2},\ldots\}$ has the property that if a and b are distinct members of A, then $st(a,G_k) \cap st(b,G_k) = \emptyset$.

6.19 <u>Remark</u>. The author does not at this time know any interesting characterizations of the class of Moore spaces having developments of type wB.

The proof of Theorem 6.18 may be used to prove the following familiar result about metric spaces.

Theorem. If X is a metric space having a metric d and M is an infinite subset of X with the property that any Cauchy sequence of points of M is eventually constant, then there is an infinite subset M' of M and a positive number ε such that if x and y are distinct points of M', then $d(x,y) \geq \varepsilon$.

6.20. Theorem. If there is a dominating family with cardinality ω_1 and S is an ω_1-compact metric space which is the union of ω_1, compact sets, there is a collection G of each type below.

(1) $G = \{G_\alpha : \alpha \varepsilon \omega_1\}$ is a collection of sets of open subsets of S having the property that if D is a closed, discrete subset of S, there is an $\alpha \varepsilon \omega_1$ such that G_α covers D and if x and y are distinct points of D, then $st(x,G_\alpha) \cap st(y,G_\alpha) = \phi$.

(2) $G = \{G_\alpha : \alpha \varepsilon \omega_1\}$ is a collection of sets of open subsets of S having the property that if D is a closed, discrete subset of S and U is an open set including D, there is an $\alpha \varepsilon \omega_1$ such that G_α covers D and $st(D,G_\alpha) \subseteq U$.

Proof. There is a collection $\{F_\alpha : \alpha\varepsilon\omega_1\}$ of compact subsets of S whose union is S. For each $\alpha\varepsilon\omega_1$ the collection $\{F_\beta : \beta\varepsilon\alpha\}$ is a countable collection of compact sets. Use the construction in the proof of Theorem 6.9 to show the existence of a cardinality ω_1 collection of open covers of $\bigcup\{F_\beta : \beta\varepsilon\alpha\}$ satisfying the condition (1) and (2) for any closed, discrete subset which is contained in $\bigcup\{F_\beta : \beta\varepsilon\alpha\}$.

If G is the union of all the collections for each $\alpha\varepsilon\omega_1$, then G is the union of ω_1 collections each having ω_1 members and hence G has cardinality ω_1. The collection G has the desired properties since if D is a closed discrete subset of S, then for some $\alpha\varepsilon\omega_1$, $D \subseteq \bigcup\{F_\beta : \beta\varepsilon\alpha\}$.

6.21 Remark. If there is a dominating family of cardinality ω_1, then the irrationals--indeed, every metric space which is the continuous image of the irrationals--is ω_1-compact and is the union of a cardinality ω_1 collection of compact sets (see [Hc 2]). If CH is false, then the space of Example 6.4 is not the union of a cardinality ω_1 collection of compact sets.

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