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THIRD-ORDER DIFFERENTIAL INEQUALITIES AND SINGULAR PERTURBATIONS

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THIRD-ORDER DIFFERENTIAL INEQUALITIES AND SINGULAR PERTURBATIONS

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τὸν καλὸν ἀγῶνα ἠγώνισμαι.

II Timothy, IV.7.

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INTRODUCTION

Recently, Howes [Ho2] has employed differential inequalities to establish the exist.nce, and to study the asymptotic behavior of certain classes of second-order nonlinear boundary value problems having a small parameter multiplying the highest derivative. In this paper, we study third-order problems of the form

(0.1) $$\varepsilon y''' = f(t,y,y',y'',\varepsilon) , \quad 0 < t < 1$$

(0.2) $$y(0) = A , \ y'(0) = A' , \ y(1) = B \quad \text{or}$$

(0.2′) $$y(0) = A , \ y'(0) = A' , \ y'(1) = B' ,$$

where ε is a small positive parameter and f is, in general, a nonlinear continuously differentiable function. Our goal is to obtain sufficient conditions for which solutions to the above problems exist, and deduce the behavior, as $\varepsilon \to 0^+$, of such solutions. The existence of appropriate approximate solutions will be assumed, and growth conditions with respect to y'' and y' will be necessary. To set the stage as to approximations and type of behavior which is studied, consider the example

$$\varepsilon y''' = y'' , \quad 0 < t < 1$$

$$y(0,\varepsilon) = 0 , \ y'(0,\varepsilon) = 0 , \ y(1,\varepsilon) = 1 ,$$

having solution $y(t,\varepsilon) = (t + \varepsilon - \varepsilon\exp(t/\varepsilon))(1 + \varepsilon - \varepsilon\exp(1/\varepsilon))^{-1}$. Note that $y(t,\varepsilon) \to 0$ as $\varepsilon \to 0^+$, $t \neq 1$. $\lim_{\varepsilon\to 0^+} \lim_{t\to 1} y(t,\varepsilon) = 1 \neq 0 = \lim_{t\to 1} \lim_{\varepsilon\to 0^+} y(t,\varepsilon)$, a phenomenon often termed, there is a "boundary layer" at $t = 1$. Formally setting $\varepsilon = 0$, in the above equation, to obtain $u'' = 0$, and requesting $u'(0) = 0$ and $u(0) = 0$, we have $u \equiv 0$ as an approximate solution (reduced solution). We will basically follow this pattern, i.e., to obtain an approximate or reduced solution and use such to complete the aim.

We will make use of the concept of lower and upper solutions to (0.1), the reduced or approximate solution being the basis of constructing such, and then apply various existence and comparison theorems. The first chapter is a presentation of existence results used throughout our treatment of singular perturbations, with note of the various forms we allow f to display. Chapters Two and Three are, respectively, treatments of the problems (0.1), (0.2′) and (0.1), (0.2), where comparison is made to reduced solutions ($\varepsilon = 0$). Chapter Four is comparison with several types of approximate solutions, where questions of admissibility of the order of approximation, and verification that an approximation is valid are answered.

Throughout this paper, o and O will be the standard Landau symbols of order of magnitude, and terms such as $o(1)$ and $O(\varepsilon)$ are meant to hold uniformly in all other variables. Partial derivatives will be denoted by subscripts, e.g., $f_y = \partial f/\partial y$. C^i is the class of i times continuously differentiable functions.

THIRD-ORDER DIFFERENTIAL INEQUALITIES

AND SINGULAR PERTURBATIONS

CHAPTER I

THIRD-ORDER DIFFERENTIAL INEQUALITIES

The technique, used in this treatment of singular perturbation problems, requires existence and comparison theorems presented in this chapter. We consider the third-order boundary value problems

(1.1) $\quad x''' = f(t,x,x',x'') \ , \quad a < t < b$

(1.2) $\quad x(a) = A \ , \ x'(a) = A' \ , \ x(b) = B$ or

(1.2′) $\quad x(a) = A \ , \ x'(a) = A' \ , \ x'(b) = B' \ ;$

where the function f is assumed to be continuous on $[a,b] \times R^3$; A , A' , B and B' being real constants.

<u>Definition</u>: Let $\alpha,\beta \in C^3[a,b]$, such that $\alpha(t) \le \beta(t)$, $t \in [a,b]$. $\alpha = \alpha(t)$ is called a <u>lower solution of equation</u> (1.1), with respect to β on $[a,b]$, if $\alpha'''(t) \ge f(t,x,\alpha'(t),\alpha''(t))$; $\beta = \beta(t)$ is an <u>upper solution of equation</u> (1.1), with respect to α on $[a,b]$, if $\beta'''(t) \le f(t,x,\beta'(t),\beta''(t))$; where x is such that $\alpha(t) \le x \le \beta(t)$.

The following is Theorem 3.1, found in Kelley [Ke], stated in terms of the third-order problem (1.1), (1.2′).

<u>Theorem 1.1</u>. Assume:

(1) $\alpha,\beta \in C^3[a,b]$ are lower and upper solutions, respectively and with respect to one another, of (1.1); such that $\alpha'(t) \le \beta'(t)$, $a \le t \le b$, and $\alpha(a) \le A \le \beta(a)$, $\alpha'(a) \le A' \le \beta'(a)$, $\alpha'(b) \le B' \le \beta'(b)$.

(2) Solutions $x(t)$ of (1.1) have the property: If $x'(t)$ is bounded on $[a,b]$, then $x''(t)$ is bounded on $[a,b]$.

Then (1.1), (1.2′) has a solution $x(t)$ such that

$$\alpha(t) \le x(t) \le \beta(t) \quad \text{and} \quad \alpha'(t) \le x'(t) \le \beta'(t) \ , \quad a < t < b \ .$$

A theorem of similar nature, regarding the problem (1.1), (1.2) begins here, using devices of a nature similar to those used in second-order problems, [BL] and [Ja]. The following is a standard consequence from the Schauder fixed-point theorem.

<u>Lemma 1.2</u>. If f is bounded on $[a,b] \times R^3$, then (1.1), (1.2) has a solution.

<u>Definition</u>: Let $\alpha,\beta \in C^2[a,b]$ with $\alpha(t) \le \beta(t)$, $a < t < b$. Choose positive constants $N,c > 0$ such that $c > \max\{|\alpha'(t)|,|\beta'(t)| : a \le t \le b\}$ and $N > \max\{|\alpha''(t)|,|\beta''(t)| : a \le t \le b\}$. We define for f continuous on $[a,b] \times R^3$

$$f^*(t,x,x',x'') = \begin{cases} f(t,x,x',N) & \text{if} \quad x'' > N \\ f(t,x,x',x'') & \text{if} \quad |x''| \le N \\ f(t,x,x',-N) & \text{if} \quad x'' < -N \end{cases}$$

$$F^*(t,x,x',x'') = \begin{cases} f^*(t,x,c,x'') & \text{if } x' > c \\ f^*(t,x,x',x'') & \text{if } |x'| \le c \\ f^*(t,x,-c,x'') & \text{if } x' < -c \end{cases}$$

$$F(t,x,x',x'') = \begin{cases} F^*(t,\beta(t),x',x'') & \text{if } x > \beta(t) \\ F^*(t,x,x',x'') & \text{if } \alpha(t) \le x \le \beta(t) \\ F^*(t,\alpha(t),x',x'') & \text{if } x < \alpha(t) \end{cases} .$$

The function F will be called <u>the modification of</u> f <u>associated with</u> $(\alpha(t),\beta(t),c,N)$.

It follows, from the continuity of f , that F is continuous on $[a,b] \times R^3$, and also that F is bounded thereon. Consider, now, the boundary value problem (1.1), (1.2).

<u>Theorem 1.3</u>. (1) Let $\alpha,\beta \in C^3[a,b]$ be, respectively, lower and upper solutions of (1.1) with respect to one another for $a \le t \le b$; such that $\alpha(a) \le A \le \beta(a)$, $\alpha'(a) \le A' \le \beta'(a)$, $\alpha(b) \le B \le \beta(b)$.

(2) f is increasing with respect to x' .

(3) F is the modification of f associated with $(\alpha(t),\beta(t),c,N)$.

Then the boundary value problem

(1.3) $$x''' = F(t,x,x',x'') , \quad a < t < b$$

(1.4) $$x(a) = A , \ x'(a) = A' , \ x(b) = B$$

has a solution $x = x(t) \in C^3[a,b]$ such that $\alpha(t) \le x(t) \le \beta(t)$.

<u>Proof</u>: F being continuous and bounded on $[a,b] \times R^3$, Lemma 1.2 declares the boundary value problem (1.3), (1.4) has a solution $x(t)$. Thus we need only establish the concluding assertion. We show the case $x(t) \le \beta(t)$. $\alpha(t) \le x(t)$ is proved in an analogous fashion.

Assume there exist a $\tau \in [a,b]$ and an $\eta > 0$ such that $x(\tau) = \beta(\tau)$ with $\alpha(t) \le x(t) \le \beta(t)$ if $a \le t \le \tau$, and $x(t) > \beta(t)$

if $\tau < t < \tau + \eta$. Since $x(b) = B \le \beta(b)$, there exists a maximal interval $[t_1,t_2] \subset [a,b]$ containing τ , upon which $x' - \beta' \ge 0$, with strict inequality at places therein. Thus we may conclude the existence of $t_0 \in (t_1,t_2)$ such that $x' - \beta'$ achieves a positive maximum thereat. And we have $x''(t_0) = \beta''(t_0)$ and $x'''(t_0) \le \beta'''(t_0)$. However,

i) if $t_0 \ge \tau$, $x(t_0) \ge \beta(t_0)$ and so

$$x'''(t_0)-\beta'''(t_0) \ge F(t_0,x(t_0),x'(t_0),x''(t_0))-f(t_0,\beta(t_0),\beta'(t_0),\beta''(t_0))$$
$$= F^*(t_0,\beta(t_0),x'(t_0),\beta''(t_0))-F^*(t_0,\beta(t_0),\beta'(t_0),\beta''(t_0)) > 0$$

as F^* inherits the monotonicity of f with respect to x' .

ii) if $t_0 < \tau$, $\alpha(t_0) \le x(t_0) \le \beta(t_0)$, $x'''(t_0) - \beta'''(t_0)$
$$\ge F^*(t_0,x(t_0),x'(t_0),\beta''(t_0)) - F^*(t_0,x(t_0),\beta'(t_0),\beta''(t_0)) > 0 .$$

Therefore the assumption that $x' - \beta'$ achieves a positive maximum at t_0 is invalid, and so the existence of τ and η is denied. We conclude then that $x(t) \le \beta(t)$.

<u>Lemma 1.4</u>. Assume:

(1) $f(t,x,x',x'')$ is continuous on $[a,b] \times R^3$, and of class C^1 with respect to x' and x'' .

(2) $f_{x''}(t,x,x',x'') \ge 0$ and $f_{x'}(t,x,x',0) \ge 0$.

(3) $x = x(t)$ is a solution of (1.1), (1.2) such that $\alpha(t) \le x(t) \le \beta(t)$, $a \le t \le b$; where $\alpha,\beta \in C^3[a,b]$ are lower and upper solutions of (1.1) in the sense of assumption (1) in Theorem 1.3.

Then there exists a positive number M , dependent only on α , β and f , such that $|x''(a)| \le M$.

<u>Proof</u>: $f(t,x,x',x'')$

$$= f(t,x,0,0) + f_{x'}(t,x,\theta_1 x',0)x' + f_{x''}(t,x,x',\theta_2 x'')x'' ,$$

$0 < \theta_i < 1$, $i = 1,2$. We consider the initial value problem

(1.5) $$x''' = f(t,x,x',x'')$$

(1.6) $$x(a) = A \ , \ x'(a) = A' \ , \ x''(a) = M \gg 0 \ .$$

The question to be resolved is then: under supposition (3), how large is M in (1.6) allowed to become.

f being continuous, condition (3) implies $f(t,x,0,0)$ is bounded for $a \le t \le b$, $\alpha(t) \le x(t) \le \beta(t)$. And as long as $x''(t) \ge 0$, then $f_{x'}(t,x,\theta_1 x',0)x'$ has a lower bound. Thus for $a \le t \le b$ and $\alpha(t) \le x(t) \le \beta(t)$, as long as $x'' \ge 0$, there is a number $Q > 0$ such that $x''' > -6Q$. Employing Taylor's formula, we have $x(t) = A + A'(t - a) + M(t - a)^2/2 + x'''(\xi)(t - a)^3/6$, $a < \xi < t$. Thus, as long as $x'' \ge 0$, $x(t) > A + A'(t - a) + M(t - a)^2/2 - Q(t - a)^3$. If M is chosen such that $M > (N/3 + 2Q)(b - a) + \alpha''(a)$, $|\alpha'''(t)| \le N$, $a \le t \le b$, $x(t) \ge \alpha(t)$. But if M were so large that $A + A'(\tau - a) + M(\tau - a)^2/2 - Q(\tau - a)^3 > \beta(\tau)$ we contradict $x(t) \le \beta(t)$. Therefore we must have $x''(a) < M$ for some positive M .

Turning to the initial value problem (1.5) with

(1.7) $$x(a) = A \ , \ x'(a) = A' \ , \ x''(a) = -M \ll 0 \ ;$$

as long as $x'' \le 0$, $f_{x'}(t,x,\theta_1 x',0)x'$ has an upper bound. Thus when we have $a \le t \le b$ and $\alpha(t) \le x(t) \le \beta(t)$, as long as $x'' \le 0$, $x'''(t) < 6Q$, for some $Q > 0$. And so, $x(t) < A + A'(t - a) - M(t - a)^2/a + Q(t - a)^3 \le \beta(t)$ if $-M < \beta''(a) - (N/3 + 2Q)(b - a)$, where $|\beta'''(t)| \le N$, $a \le t \le b$. But if $-M$ is chosen so small that $A + A'(\tau - a) - M(\tau - a)^2/2 + Q(\tau - a)^3 < \alpha(\tau)$ we contradict $\alpha(t) \le x(t)$. Therefore, $x''(a)$ is bounded below also, thus the conclusion of this lemma.

The existence of an a priori bound for $x''(a)$, given α and β, leads to speculation that other such confinements may be deduced; hopefully with respect to x' and x'' across the entire interval $[a,b]$. An affirmative response to this situation, coupled with previous results, explains the particular structure of the modified function we defined.

Lemma 1.5. If f satisfies the conditions of Lemma 1.4, and $x = x(t)$ is a solution of (1.1), (1.2) such that $\alpha(t) \le x(t) \le \beta(t)$ as above; then there exist positive values c and N such that $|x'(t)| < c$ and $|x''(t)| < N$ for $a \le t \le b$.

Proof: Suppose there exists a sequence of solutions $\{x_n(t)\}$ of (1.1), (1.2); where $\alpha(t) \le x_n(t) \le \beta(t)$, $a \le t \le b$, such that $|x_n''(t_n)| > n$ for some $t_n \in (a,b]$. We thus obtain a sequence of initial values $z_n(a) = (A,A',x_n''(a))$ for (1.1), $z' = g(t,z) = (x',x'',f(t,x,x',x''))$. By Lemma 1.4 the sequence $\{z_n(a)\}$ is bounded, and so has a convergent subsequence $z_{n(k)}(a) \to (A,A',x_0''(a))$. Whence, by a standard convergence theorem ([Ha], p.14), there exists a solution $z_0 = z_0(t)$ of (1.1) (and subsequently of (1.1), (1.2)) such that $z_0(a) = (A,A',x_0''(a))$ and $z_{n(k)}(t) = (x_{n(k)}(t),x_{n(k)}'(t),x_{n(k)}''(t)) \to z_0(t)$ uniformly as $k \to \infty$, $a \le t \le b$. Thus the boundedness of $x_0''(t)$ implies that of $z_{n(k)}(t)$, contradicting the choice of $\{x_n\}$. Therefore the existence of the bound N is established. c is a natural consequence of N.

Theorem 1.6. Suppose solutions of initial value problems for (1.1) extend to $[a,b]$ or become unbounded. Also assume:

(1) f is continuous on $[a,b] \times R^3$; and of class C^1 with respect to x' and x'', such that $f_{x''} \ge 0$ and $f_{x'} > 0$.

(2) $\alpha,\beta \in C^3[a,b]$ are, respectively, lower and upper solutions of (1.1) satisfying: $\alpha(t) \leq \beta(t)$, $a < t < b$; $\alpha(a) \leq A \leq \beta(a)$, $\alpha'(a) \leq A' \leq \beta'(a)$, $\alpha(b) \leq B \leq \beta(b)$.

Then the boundary value problem (1.1), (1.2) has a solution $x = x(t) \in C^3[a,b]$, with $\alpha(t) \leq x(t) \leq \beta(t)$.

Proof: Let c and N be positive constants, which satisfy the conclusion of Lemma 1.5; such that $c > \max\{|\alpha'(t)|,|\beta'(t)| : a \leq t \leq b\}$ and $N > \max\{|\alpha''(t)|,|\beta''(t)| : a \leq t \leq b\}$. Thus, if F is the modification of f with respect to (α,β,c,N) ; there exists a solution $x = x(t)$ to $x''' = F(t,x,x',x'')$ satisfying boundary conditions (1.2) such that $\alpha(t) \leq x(t) \leq \beta(t)$ (Theorem 1.3). But then, $|x'| < c$ and $|x''| < N$; and we thus find $F(t,x(t),x'(t),x''(t)) = f(t,x(t),x'(t),x''(t))$. Therefore, x is a solution of (1.1), (1.2).

Note that the above theorem also applies to the problem where $x''' = f(t,x,x')$, as we may note $f_{x''} \equiv 0$. In the particular case when $x''' = f(t,x,x'')$, modification of proof to effect conclusion is that we choose α and β such that $\alpha(t) < \beta(t)$. Also we demand that the inequalities expressed in the definitions of lower and upper solutions be strict. We have:

Theorem 1.7. Suppose solutions of initial value problems for $x''' = f(t,x,x'')$ extend to $[a,b]$ or become unbounded. Also assume:

(1) f is continuous on $[a,b] \times R^2$ and of class C^1 with respect to x'' , such that $f_{x''} \geq 0$.

(2) $\alpha,\beta \in C^3[a,b]$ are, respectively, strict lower and upper solutions of $x''' = f(t,x,x'')$ satisfying: $\alpha(t) < \beta(t)$, $a < t < b$, with $\alpha(a) \leq A \leq \beta(a)$, $\alpha'(a) \leq A' \leq \beta'(a)$, and $\alpha(b) \leq B \leq \beta(b)$.

Then the boundary value problem

$$x''' = f(t,x,x'') , \quad a < t < b \tag{1.8}$$

$$x(a) = A , \ x'(a) = A' , \ x(b) = B \tag{1.9}$$

has a solution $x = x(t)$ such that $\alpha(t) \le x(t) \le \beta(t)$.

<u>Proof</u>: We form a modification F of f as in the definition, using N as in the conclusion of Lemma 1.5, here ignoring F^* . By Lemma 1.2, the boundary value problem $x''' = F(t,x,x'')$, (1.9) has a solution $x = x(t)$. If we show then, that $\alpha(t) \le x(t) \le \beta(t)$, Lemma 1.5 will establish that $F(t,x,x'') = f(t,x,x'')$ and we are done. To this end, proceed as in the proof of Theorem 1.3, here we show $\alpha(t) \le x(t)$.

Assume that there exist $\tau \in [a,b]$ and $\eta > 0$ such that $x(\tau) = \alpha(\tau)$ with $\alpha(t) \le x(t) \le \beta(t)$ if $a \le t \le \tau$, and $x(t) < \alpha(t)$ if $\tau < t < \tau + \eta$. Since $x(b) = B \ge \alpha(b)$, there exists a maximal interval $[t_1,t_2] \subset [a,b]$ containing τ , upon which $x' - \alpha' \le 0$, with strict inequality at places therein. Thus we conclude the existence of $t_0 \in (t_1,t_2)$ such that $x' - \alpha'$ achieves a negative minimum there. Also we have $x''(t_0) = \alpha''(t_0)$ and $x'''(t_0) \ge \alpha'''(t_0)$. But i) if $t_0 \ge \tau$, $x(t_0) \le \alpha(t_0)$ and so

$$\begin{aligned} x'''(t_0) - \alpha'''(t_0) &< F(t_0,x(t_0),x''(t_0)) - f(t_0,\alpha(t_0),\alpha''(t_0)) \\ &= f(t_0,\alpha(t_0),\alpha''(t_0)) - f(t_0,\alpha(t_0,\alpha''(t_0)) = 0 , \end{aligned}$$

ii) if $t_0 < \tau$, $\alpha(t_0) \le x(t_0) \le \beta(t_0)$ and so

$$\begin{aligned} x'''(t_0) - \alpha'''(t_0) &< F(t_0,x(t_0),x''(t_0)) - f(t_0,x(t_0),\alpha''(t_0)) \\ &= f(t_0,x(t_0),\alpha''(t_0)) - f(t_0,x(t_0),\alpha''(t_0)) = 0 . \end{aligned}$$

Thus the conclusion that $x' - \alpha'$ has a negative minimum at t_0 is negated, along with the existence of τ and η . We conclude then that $\alpha(t) \le x(t)$. $x(t) \le \beta(t)$ follows similarly.

Henceforth, we will restrict attention to third-order singularly perturbed problems

$$\varepsilon y''' = f(t,y,y',y'',\varepsilon) \tag{1.10}$$

where ε is a small positive parameter. Our method of attack will be to fix an ε, construct appropriate bounding lower and upper solutions $\alpha(t,\varepsilon)$ and $\beta(t,\varepsilon)$ and make use of Theorems 1.1, 1.6, and 1.7 to establish existence of solutions to (1.10) for given boundary values. We shall also be able to learn, then, how these solutions behave as a function of both t and ε.

CHAPTER II

SINGULAR PERTURBATIONS WITH BOUNDARY LAYERS AT THE DERIVATIVE LEVEL

We consider, herein, the existence and asymptotic behavior, as $\varepsilon \to 0^+$, of the solutions $y = y(t,\varepsilon)$ of the nonlinear boundary value problem

(2.1) $$\varepsilon y''' = f(t,y,y',y'',\varepsilon)\ ,\quad 0 < t < 1$$

(2.2) $$y(0,\varepsilon) = A\ ,\ y'(0,\varepsilon) = A'\ ,\ y'(1,\varepsilon) = B'\ ,$$

to the solution of the corresponding "reduced" initial value problem

(2.3) $$0 = f(t,u,u',u'',0)$$

(2.4) $$u(0) = A\ ,\ u'(0) = A'\ .$$

We will assume that (2.3), (2.4) possesses a solution $u = u(t)$ of class C^3 on $[0,1]$, and that $f_{y''}$ exists and is nonnegative. With additional restrictions on f, we are able to conclude sufficient conditions such that the "full" problem (2.1), (2.2) has a class C^3 solution. And we will be able to study the limiting behavior of $y(t,\varepsilon)$ and $y'(t,\varepsilon)$ for uniform approach, as $\varepsilon \to 0^+$, to $u(t)$ and $u'(t)$, respectively. Also of interest is the magnitude of the boundary layers which appear. The following theorem is now presented, as a prototype of existence theorems for this problem.

Theorem 2.1. Suppose solutions $y(t,\varepsilon)$ of (2.1) have the property: If $y'(t,\varepsilon)$ is bounded on $[0,1]$, then $y''(t,\varepsilon)$ is bounded on $[0,1]$. Also assume:

(1) There exists a solution $u = u(t) \in C^3[0,1]$ satisfying (2.3), (2.4).

(2) f is continuous in (t,y,y',y'',ε) and of class C^1 with respect to y, y', y'' for a region $R : 0 \le t \le 1$, $|y - u(t)| \le d_1$, $|y' - u'(t)| \le d_2$, $|y''| < \infty$, $0 < \varepsilon \ll 1$; where $|B' - u'(1)| \le d_2$.

(3) $f(t,u(t),u'(t),u''(t),\varepsilon) = O(\varepsilon)$, $0 \le t \le 1$.

(4) $f_{y''} \ge m > 0$ in R.

(5) $f_{y'}(t,y,y',u''(t),\varepsilon) = O(1)$ and $f_y(t,y,u'(t),u''(t),\varepsilon) = O(1)$ in R.

Then there exists an $\varepsilon_0 > 0$, and for each ε, $0 < \varepsilon \le \varepsilon_0$, there exists a solution $y = y(t,\varepsilon)$ of (2.1), (2.2) such that

$$y(t,\varepsilon) - u(t) = O(\varepsilon),$$

$$|y'(t,\varepsilon) - u'(t)| \le |u'(1) - B'|\exp[(t-1)c\varepsilon^{-1}] + O(\varepsilon),$$

where c is a positive constant independent of ε.

Proof: In view of Theorem 1.1, we show the existence of a solution to (2.1), (2.2) and deduce our estimates for y and y' by construction of appropriate lower and upper solutions of (2.1).

Let k and ℓ be such that $|f_{y'}(t,y,y',u'',\varepsilon)| \le k$ and $|f_y(t,y,u'(t),u''(t),\varepsilon)| \le \ell$. Condition (1) implies that there exists an $M > 0$ such that $|u'''(t)| \le M$, $0 \le t \le 1$; and condition (3) declares there is an $N > 0$ for which $|f(t,u(t),u'(t),u''(t),\varepsilon)| \le N\varepsilon$.

We define $P = |u'(1) - B'|$, $\lambda = (m + (m^2 - 8k\varepsilon)^{\frac{1}{2}})(2\varepsilon)^{-1}$, for ε sufficiently small, so as to have λ a real root of

$$-\varepsilon\lambda^2 + m\lambda - 2k = 0 .$$

$\mu = -(k + (k^2 + 4m(\ell+M+N+1))^{\frac{1}{2}})(2m)^{-1}$ is a root for

$$m\mu^2 + k\mu - (\ell+M+N+1) = 0 .$$

We construct lower and upper solutions of (2.1) as follows, for $t \in [0,1]$:

$$\alpha(t) = \alpha(t,\varepsilon) = u(t) - \lambda^{-1}P\exp[(t - 1)\lambda] - \varepsilon\exp(-\mu t) ,$$

$$\beta(t) = \beta(t,\varepsilon) = u(t) + \lambda^{-1}P\exp[(t - 1)\lambda] + \varepsilon\exp(-\mu t) .$$

It is of interest to note, here, that the estimates regarding the level $y - u$ will be determinable as $\lambda^{-1} = O(\varepsilon)$, and μ is fixed for all ε . We now demonstrate that α and β are indeed lower and upper solutions of (2.1) with respect to each other, and with the desirable properties listed in Theorem 1.1.

$$\alpha(t) = u(t) - \lambda^{-1}P\exp[(t - 1)\lambda] - \varepsilon\exp(-\mu t)$$

and so

$$\alpha'(t) = u'(t) - P\exp[(t - 1)\lambda] + \mu\varepsilon\exp(-\mu t) ,$$

$$\alpha''(t) = u''(t) - \lambda P\exp[(t - 1)\lambda] - \mu^2\varepsilon\exp(-\mu t) , \quad \text{and}$$

$$\varepsilon\alpha'''(t) = \varepsilon u'''(t) - \varepsilon\lambda^2 P\exp[(t - 1)\lambda] + \varepsilon\mu^3\varepsilon\exp(-\mu t) .$$

Similarly

$$\beta(t) = u(t) + \lambda^{-1}P\exp[(t - 1)\lambda] + \varepsilon\exp(-\mu t) ,$$

$$\beta'(t) = u'(t) + P\exp[(t - 1)\lambda] - \mu\varepsilon\exp(-\mu t) ,$$

$$\beta''(t) = u''(t) + \lambda P\exp[(t - 1)\lambda] + \mu^2\varepsilon\exp(-\mu t) ,$$

$$\varepsilon\beta'''(t) = \varepsilon u'''(t) + \varepsilon\lambda^2 P\exp[(t - 1)\lambda] - \varepsilon\mu^2\varepsilon\exp(-\mu t) .$$

Note that $\beta(t) - \alpha(t) = 2\lambda^{-1}P\exp[(t - 1)\lambda] + 2\varepsilon\exp(-\mu t) \geq 0$. Also, we have $\beta'(t) - \alpha'(t) = 2P\exp[(t - 1)\lambda] - 2\mu\varepsilon\exp(-\mu t) \geq 0$. At the endpoints: $\alpha(0) = A - \lambda^{-1}P\exp(-\lambda) - \varepsilon < A < A + \lambda^{-1}P\exp(-\lambda) + \varepsilon = \beta(0)$, $\alpha'(0) = A' - P\exp(-\lambda) + \mu\varepsilon \le A' \le A' + P\exp(-\lambda) - \mu\varepsilon = \beta'(0)$, and $\alpha'(1) = u'(1) - P + \mu\varepsilon\exp(-\mu) \le B' \le u'(1) + P - \mu\varepsilon\exp(-\mu) = \beta'(1)$.

Thus, if it is shown that the inequalities expressed in the definitions of lower and upper solutions are here valid, our conclusions will follow. Testing, by inserting α' , α'' , $\varepsilon\alpha'''$ and some z , $\alpha(t) \le z \le \beta(t)$ into equation (2.1) and "turning corners" via the mean value theorem, we have

$$\begin{aligned}
&\varepsilon\alpha'''(t) - f(t,z,\alpha'(t),\alpha''(t),\varepsilon) \\
&\quad = \varepsilon u'''(t) - \varepsilon\lambda^2 P\exp[(t-1)\lambda] + \varepsilon\mu^3\varepsilon\exp(-\mu t) - f_{y''}(t,z,\alpha'(t),\cdot,\varepsilon) \\
&\qquad \times \{-\lambda P\exp[(t-1)\lambda] - \mu^2\varepsilon\exp(-\mu t)\} - f_{y'}(t,z,\cdot\cdot,u''(t),\varepsilon) \\
&\qquad \times \{-P\exp[(t-1)\lambda] + \mu\varepsilon\exp(-\mu t)\} - f_y(t,\cdot\cdot\cdot,u'(t),u''(t),\varepsilon) \\
&\qquad \times \{\theta\lambda^{-1}P\exp[(t-1)\lambda] + \theta\varepsilon\exp(-\mu t)\} - f(t,u(t),u'(t),u''(t),\varepsilon) ,
\end{aligned}$$

where $|\theta| < 1$ and $\cdot$, $\cdot\cdot$, $\cdot\cdot\cdot$ denote, respectively, values somewhere between $\alpha''(t)$ and $u''(t)$, $\alpha'(t)$ and $u'(t)$, and z and $u(t)$. Continuing and comparing, we have

$$\begin{aligned}
&\varepsilon\alpha'''(t) - f(t,z,\alpha'(t),\alpha''(t),\varepsilon) \\
&\quad \ge -\varepsilon M - \varepsilon\lambda^2 P\exp[(t-1)\lambda] + \varepsilon\mu^3\varepsilon\exp(-\mu t) + m\lambda P\exp[(t-1)\lambda] + m\mu^2\varepsilon\exp(-\mu t) \\
&\qquad -kP\exp[(t-1)\lambda] + k\mu\varepsilon\exp(-\mu t) - \ell\lambda^{-1}P\exp[(t-1)\lambda] - \ell\varepsilon\exp(-\mu t) - \varepsilon N
\end{aligned}$$

(with M , N , m , k and ℓ as defined above)

$$= (-\varepsilon\lambda^2+m\lambda-k-\ell\lambda^{-1})P\exp[(t-1)\lambda] + (\varepsilon\mu^3+m\mu^2+k\mu-\ell)\varepsilon\exp(-\mu t) - \varepsilon(M+N) = * .$$

Since $\lambda^{-1} = O(\varepsilon)$, $\varepsilon\mu^3 + m\mu^2 + k\mu - \ell \ge m\mu^2 + k\mu - \ell - 1 > 0$, for sufficiently small values of ε , and $\exp(-\mu t)$ being an increasing function, we have

$$* \ge (-\varepsilon\lambda^2 + m\lambda - 2k)P\exp[(t-1)\lambda] + (m\mu^2 + k\mu - (\ell+M+N+1))\varepsilon = 0 \text{ , i.e.,}$$

$$\varepsilon\alpha'''(t) - f(t,z,\alpha'(t),\alpha''(t),\varepsilon) \ge 0 .$$

Via a symmetric argument we ascertain

$$f(t,z,\beta'(t),\beta''(t),\varepsilon) - \varepsilon\beta'''(t)$$

$$= -\varepsilon u'''(t) - \varepsilon\lambda^2 P\exp[(t-1)\lambda] + \varepsilon\mu^3\varepsilon\exp(-\mu t) + f_{y''}(t,z,\beta'(t),:,\varepsilon)$$

$$\times \{\lambda P\exp[(t-1)\lambda] + \mu^2\varepsilon\exp(-\mu t)\} + f_{y'}(t,z,::,u''(t),\varepsilon)$$

$$\times \{P\exp[(t-1)\lambda] - \mu\varepsilon\exp(-\mu t)\} + f_y(t,:::,u'(t),u''(t),\varepsilon)$$

$$\times \{\theta\lambda^{-1}P\exp[(t-1)\lambda] + \theta\varepsilon\exp(-\mu t)\} + f(t,u(t),u'(t),u''(t),\varepsilon)$$

$$\geq (-\varepsilon\lambda^2+m\lambda-k-\ell\lambda^{-1})P\exp[(t-1)\lambda] + (\varepsilon\mu^3+m\mu^2+k\mu-\ell)\varepsilon\exp(-\mu t) - \varepsilon(M+N) \geq 0 .$$

Therefore $\alpha(t,\varepsilon)$ and $\beta(t,\varepsilon)$ are indeed lower and upper solutions of (2.1), whence Theorem 1.1 asserts the existence of a solution $y = y(t,\varepsilon)$ of (2.1), (2.2) such that $\alpha(t,\varepsilon) \leq y(t,\varepsilon) \leq \beta(t,\varepsilon)$ and $\alpha'(t,\varepsilon) \leq y'(t,\varepsilon) \leq \beta'(t,\varepsilon)$.

It is well to remark here that the boundedness requirement of $f_{y'}$ in assumption (5) may be replaced with the assumption

(5′) there is a $k > 0$, such that $f_{y'}(t,y,y',u''(t),\varepsilon) \geq k$ in R .

Then we would let λ be as above, but change μ so that $\mu = (k - (k^2 + 4m(\ell + M + N + 1))^{\frac{1}{2}}(2m)^{-1}$, a negative root of $m\mu^2 - k\mu - (\ell + M + N + 1) = 0$.

<u>Corollary 2.2</u>. Making the same assumptions as in Theorem 2.1 (condition (5) may be replaced by (5′)), there exists $\delta = \delta(\varepsilon) = O(\varepsilon)$ such that for $0 \leq t \leq 1 - \delta$,

$$|y''(t,\varepsilon) - u''(t)| = O(1/\varepsilon)\exp[-m\varepsilon^{-1}(1 - \delta - t)] + O(\varepsilon) .$$

<u>Proof</u>: Define $z = z(t,\varepsilon) = y(t,\varepsilon) - u(t)$, where y is the solution obtained in Theorem 2.1. $\varepsilon z''' + \varepsilon u''' = f(t,z + u,z' + u',z'' + u'',\varepsilon) = f(t,u,u',u'',\varepsilon) + f_y z + f_{y'}z' + f_{y''}z''$, where the partial derivatives are evaluated at $[\tau] = (t,u + \theta z,u' + \theta z',u'' + \theta z'',\varepsilon)$, $0 < \theta < 1$.

Thus z satisfies the boundary value problem

$$(2.5) \qquad \varepsilon z''' - f_{y''} z'' = r \ , \quad 0 < t < 1$$

$$z(0,\varepsilon) = 0 \ , \ z'(0,\varepsilon) = 0 \ , \ z'(1,\varepsilon) = B' - u'(1) \ ,$$

where $r = f_{y'} z' + f_y z + O(\varepsilon)$, $r = O(z' + \varepsilon)$. Thus we know that

$$|r| \le c\{\exp[(t-1)\varepsilon^{-1}] + \varepsilon\} \ .$$

Set $Q(t) = \varepsilon^{-1} \int_t^{1-\delta} f_{y''}$. We multiply (2.5) by $\varepsilon^{-1}\exp[Q(t)]$ to obtain

$$(z''\exp[Q(t)])' = \varepsilon^{-1} r(t) \exp[Q(t)] \ .$$

Integrating between t and $1 - \delta$, observe:

$$z''(t) = z''(1-\delta)\exp[-Q(t)] - \varepsilon^{-1} \int_t^{1-\delta} r(s)\exp[Q(s) - Q(t)]ds \ .$$

Arguing in a manner analogous to one by Coddington and Levenson [CL], as $z' = y' - u' = O(\varepsilon)$ outside of the boundary layer at $t = 1$, the derivative z'' is increasing in magnitude as t approaches 1 , and so there is a $\delta = \delta(\varepsilon) = O(\varepsilon)$ at which $z''(1-\delta) = O(1/\varepsilon)$. This is the δ above. And so

$$|z''(t)| \le |z''(1-\delta)|\exp[\varepsilon^{-1} \int_t^{1-\delta} -m ds] + \varepsilon^{-1} \int_t^{1-\delta} |r(s)|\exp[-\varepsilon^{-1} m(s-t)]$$

$$= O(1/\varepsilon)\exp[-m\varepsilon^{-1}(1 - \delta - t)] + O(\varepsilon) \ .$$

An interesting phenomenon may occur when we consider the boundary value problem

$$(2.6) \qquad y''' = f(t,y,y',\varepsilon) \ , \quad 0 < t < 1$$

$$(2.7) \qquad y(0,\varepsilon) = A \ , \ y'(0,\varepsilon) = A' \ , \ y'(1,\varepsilon) = B' \ .$$

If we let $\varepsilon = 0$, a first-order equation results, and so hoping to attain two of the boundary conditions for the reduced problem will usually be impossible. Here we set our reduced problem to be

$$(2.8) \qquad 0 = f(t,u,u',0)\ , \quad 0 < t < 1$$

$$(2.9) \qquad u(0) = A\ .$$

In fact, we show an order $\varepsilon^{\frac{1}{2}}$ convergence for solutions of (2.6), (2.7) with respect to (2.8), (2.9), and boundary layers of considerably larger magnitude than those exhibited in Theorem 2.1.

Theorem 2.3. Suppose solutions $y(t,\varepsilon)$ of (2.6) have the property: if $y'(t,\varepsilon)$ is bounded on $[0,1]$, then $y''(t,\varepsilon)$ is bounded on $[0,1]$. Also assume:

(1) There exists a solution $u = u(t,\varepsilon) \in C^3[0,1]$ satisfying (2.8), (2.9).

(2) f is continuous in (t,y,y',ε) and of class C^1 with respect to y and y' for a region R : $0 \le t \le 1$, $|y - u(t)| \le d_1$, $|y' - u'(t)| \le d_2$, $0 < \varepsilon \ll 1$ ($d_1,d_2 > 0$); where $|A' - u'(0)|$, $|B' - u'(t)| \le d_2$.

(3) $f(t,u(t),u'(t),\varepsilon) = O(\varepsilon^{\frac{1}{2}})$, $0 < t < 1$.

(4) $f_{y'} \ge k > 0$ in R .

(5) $f_y(t,y,u'(t),\varepsilon) = O(1)$ in R .

Then there exists an $\varepsilon_0 > 0$, such that for each ε , $0 < \varepsilon \le \varepsilon_0$, there exists a solution $y = y(t,\varepsilon)$ of (2.6), (2.7) such that

$$y(t,\varepsilon) - u(t) = O(\varepsilon^{\frac{1}{2}}) \quad \text{and}$$

$$|y'(t,\varepsilon) - u'(t)| \le |u'(0) - A'|\exp[-c_1\varepsilon^{-\frac{1}{2}}t] + |u'(1) - B'|\exp[c_2(t-1)\varepsilon^{-\frac{1}{2}}] + O(\varepsilon^{\frac{1}{2}})\ ,$$

where c_1 and c_2 are positive constants independent of ε .

Proof: We form lower and upper solutions of (2.6). Let ℓ be such that $|f_y(t,y,u'(t),\varepsilon)| \le \ell$. Conditions (1) and (3) provide positive constants M and N , where $|u'''(t)| \le M$ and $|f(t,u(t),u'(t),\varepsilon)| \le N\varepsilon$,

$0 \le t \le 1$. $Q = |u'(0) - A'|$, $P = |u'(1) - B'|$. Then

$\alpha(t) = \alpha(t,\varepsilon) = u(t) - \gamma^{-1}Q[\exp(\gamma t)-1] - \lambda^{-1}P\exp[(t-1)\lambda] - \varepsilon^{\frac{1}{2}}\exp(-\mu t)$,

$\beta(t) = \beta(t,\varepsilon) = u(t) + \gamma^{-1}Q[\exp(\gamma t)-1] + \lambda^{-1}P\exp[(t-1)\lambda] + \varepsilon^{\frac{1}{2}}\exp(-\mu t)$,

where γ is the negative root of

$$-\varepsilon\gamma^2 + k = 0 ,$$

λ is the positive root of

$$-\varepsilon\lambda^2 + k/2 = 0 .$$

$\lambda^{-1} = O(\varepsilon^{\frac{1}{2}})$, and in particular $\gamma^{-1} = O(\varepsilon^{\frac{1}{2}})$, whence there is an $r \ge 0$ such that $|\ell\gamma^{-1}Q| \le r\varepsilon^{\frac{1}{2}}$. Define μ to be the root of

$$-k\mu - (\ell + M + N + r + 1) = 0 .$$

$\beta(t) - \alpha(t) = 2\gamma^{-1}Q[\exp(\gamma t)-1] + 2\lambda^{-1}\exp[(t-1)\lambda] + 2\varepsilon^{\frac{1}{2}}\exp(-\mu t)$ and $\beta'(t) - \alpha'(t) = 2Q\exp(\gamma t) + 2P\exp[(t - 1)\lambda] - 2\mu\varepsilon^{\frac{1}{2}}\exp(-\mu t)$. Also we have $\alpha(0) = A - \lambda^{-1}P\exp(-\lambda) - \varepsilon^{\frac{1}{2}} < A < A + \lambda^{-1}P\exp(-\lambda) + \varepsilon^{\frac{1}{2}} = \beta(0)$, $\alpha'(0) = u'(0) - Q - P\exp(-\lambda) + \mu\varepsilon^{\frac{1}{2}} \le A' \le u'(0) + Q + P\exp(-\lambda) - \mu\varepsilon^{\frac{1}{2}} = \beta'(0)$, and $\alpha'(1) = u'(1) - Q\exp(\gamma) - P + \mu\varepsilon^{\frac{1}{2}}\exp(-\mu) \le B' \le u'(1) + Q\exp(\gamma) + P - \mu\varepsilon^{\frac{1}{2}}\exp(-\mu) = \beta'(1)$. Thus, if we show that $\varepsilon\alpha''' \ge f(t,z,\alpha',\varepsilon)$ and $\varepsilon\beta''' \le f(t,z,\beta',\varepsilon)$, Theorem 1.1 will provide our conclusion. We show the case α explicitly, β follows by symmetry.

$$\begin{aligned}
&\varepsilon\alpha'''(t) - f(t,z,\alpha'(t),\varepsilon)\\
&\quad = \varepsilon u'''(t) - \varepsilon\gamma^2Q\exp(\gamma t) - \varepsilon\lambda^2P\exp[(t-1)\lambda] - \varepsilon\mu^3\varepsilon^{\frac{1}{2}}\exp(-\mu t) - f_{y'}(t,z,\cdot,\varepsilon)\\
&\qquad \times \{-Q\exp(\gamma t)-P\exp[(t-1)\lambda]+\mu\varepsilon^{\frac{1}{2}}\exp(-\mu t)\} - f_y(t,\cdot\cdot,u'(t),\varepsilon)\\
&\qquad \times \{\theta\gamma^{-1}Q[\exp(\gamma t)-1]+\theta\lambda^{-1}P\exp[(t-1)\lambda]+\theta\varepsilon^{\frac{1}{2}}\exp(-\mu t)\} - f(t,u(t),u'(t),\varepsilon) ,
\end{aligned}$$

where $|\theta| \le 1$, and $\cdot$, $\cdot\cdot$ denote values between $\alpha'(t)$ and $u'(t)$, and z and $u(t)$. Comparing, using assumptions (1), (3), (4) and (5);

$$\begin{aligned}
&\varepsilon\alpha'''(t) - f(t,z,\alpha'(t),\varepsilon)\\
&\quad \ge -\varepsilon M - \varepsilon\gamma^2Q\exp(\gamma t) - \varepsilon\lambda^2P\exp[(t-1)\lambda] - \varepsilon\mu^3\varepsilon^{\frac{1}{2}}\exp(-\mu t) + kQ\exp(\gamma t)
\end{aligned}$$

$$+ kP\exp[(t-1)\lambda] - k\mu\varepsilon^{\frac{1}{2}}\exp(-\mu t) - \ell\gamma^{-1}Q\exp(\gamma t)$$
$$- \ell\lambda^{-1}P\exp[(t-1)\lambda] - \ell\varepsilon^{\frac{1}{2}}\exp(-\mu t) + \ell\gamma^{-1}Q - \varepsilon N$$
$$\geq -(M+N+r)\varepsilon^{\frac{1}{2}} + (-\varepsilon\gamma^{2}+k-\ell\gamma^{-1})Q\exp(\gamma t) + (-\varepsilon\lambda^{2}+k-\ell\lambda^{-1})$$
$$\times P\exp[(t-1)\lambda] + (-\varepsilon\mu^{3}-k\mu-\ell)\varepsilon^{\frac{1}{2}}\exp(-\mu t) \quad .$$

If ε is sufficiently small, we use the definitions of γ, λ and μ to conclude that α is indeed a lower solution of (2.6).

To illustrate the behavior exhibited by solutions to equations discussed in the above theorem, consider the linear example

$$\varepsilon y''' = y' \ ; \ y(0) = 0 \ , \ y'(0) = -1 \ \text{ and } \ y'(1) = 1 \ .$$

The explicit solution is $y(t,\varepsilon) = \varepsilon^{\frac{1}{2}}[\cosh(t\varepsilon^{-\frac{1}{2}}) + \cosh((t-1)\varepsilon^{-\frac{1}{2}}) - \varepsilon^{\frac{1}{2}}\cosh(\varepsilon^{-\frac{1}{2}})]/\sinh(\varepsilon^{-\frac{1}{2}})$, which indeed converges $O(\varepsilon^{\frac{1}{2}})$ to the solution $u \equiv 0$ of the reduced problem. Note also that $y'(t,\varepsilon) = [\sinh(t\varepsilon^{-\frac{1}{2}}) + \sinh((t-1)\varepsilon^{-\frac{1}{2}})]/\sinh(\varepsilon^{-\frac{1}{2}})$ converges to $u' \equiv 0$ away from the two boundary layers, with the boundary layers being of the indicated magnitude.

The assumption that the solution, $u = u(t)$, of the reduced problem for the above theorems be of class $C^3[a,b]$ can be weakened to that of being $C^2[a,b]$ with $u''(t)$ piecewise continuous. To illustrate this on the interval $[-1,1]$ with $u'''(t)$ having a "jump" discontinuity at $t = 0$, we first tender the following lemma to provide C^3 estimates of u .

<u>Lemma 2.4</u>. Suppose $x \in C^2[-1,1] \cap C^3\{[-1,0) \cup (0,1]\}$ is such that $x'''(0^-)$ and $x'''(0^+)$ exist, are finite and not equal. Then for given small values of ε , there exist functions $x_\ell(t,\varepsilon), x_u(t,\varepsilon) \in C^3[-1,1]$ such that

i) $x_\ell^{(i)}(t,\varepsilon) \leq x^{(i)}(t) \leq x_u^{(i)}(t,\varepsilon)$, $-1 \leq t \leq 1$ and

ii) $x^{(i)} - x_\ell^{(i)} = O(\varepsilon)$ and $x^{(i)} - x_u^{(i)} = O(\varepsilon)$, $i = 0,1,2$.

<u>Proof:</u> Let $r^3 = x'''(0^+) - x'''(0^-)$. We define x_ℓ and x_u as follows:

$$x_\ell = x_\ell(t,\varepsilon) = \begin{cases} \left.\begin{cases} x(t) + \varepsilon^3\exp(rt/\varepsilon) - \varepsilon\gamma\exp(t+1) & t < 0 \\ x(t)+\varepsilon^3+\varepsilon^2 rt+\varepsilon r^2t^2/2-\varepsilon\gamma\exp(t+1) & t > 0 \end{cases}\right\} r > 0 \\ \left.\begin{cases} x(t) - \varepsilon^3\exp(-rt/\varepsilon) - \varepsilon\gamma\exp(t+1) & t < 0 \\ x(t)-\varepsilon^3+\varepsilon^2 rt-\varepsilon r^2t^2/2-\varepsilon\gamma\exp(t+1) & t > 0 \end{cases}\right\} r < 0 \end{cases}$$

$$x_u = x_u(t,\varepsilon) = \begin{cases} \left.\begin{cases} x(t)+\varepsilon^3+\varepsilon^2 rt+\varepsilon r^2t^2/2+\varepsilon\gamma\exp(t+1) & t < 0 \\ x(t) + \varepsilon^3\exp(-rt/\varepsilon) + \varepsilon\gamma\exp(t+1) & t > 0 \end{cases}\right\} r > 0 \\ \left.\begin{cases} x(t)-\varepsilon^3-\varepsilon^2 rt-\varepsilon r^2t^2/2+\varepsilon\gamma\exp(t+1) & t < 0 \\ x(t) - \varepsilon^3\exp(rt/\varepsilon) + \varepsilon\gamma\exp(t+1) & t > 0 \end{cases}\right\} r < 0 \end{cases}$$

where γ is chosen to be a large enough positive constant to effect the conditions i) and ii) for sufficiently small ε. Noting, and so defining for $t = 0$, that $x_\ell^{(i)}(0^-) = x_\ell^{(i)}(0^+)$ and $x_u^{(i)}(0^-) = x_u^{(i)}(0^+)$ for $i = 0,1,2,3$. Therefore we have $x_\ell, x_u \in C^3[-1,1]$.

We may now state the following theorems, which are versions of Theorems 2.1 and 2.3 for piecewise thrice differentiable reduced solutions. The boundary value problem is

(2.10) $$\varepsilon y''' = f(t,y,y',y'',\varepsilon)\ , \quad -1 < t < 1$$

(2.11) $$y(-1,\varepsilon) = A\ ,\ y'(-1,\varepsilon) = A'\ ,\ y'(1,\varepsilon) = B'$$

with corresponding reduced equation

(2.12) $$0 = f(t,u,u',u'',0)\ , \quad -1 < t < 1$$

(2.13) $$u(-1) = A\ ,\ u'(-1) = A' \quad \text{or}$$

(2.13′) $u(-1) = A$, when f does not depend on u''.

<u>Theorem 2.5.</u> Assume the hypotheses of Theorem 2.1 for the interval $-1 \le t \le 1$, but replace assumption (1) by

(1′) There exists a solution $u = u(t) \in C^2[-1,1] \cap C^3\{[-1,0) \cup (0,1]\}$

satisfying (2.12), (2.13) such that $u'''(0^-)$ and $u'''(0^+)$ exist and are finite.

Then there exists an $\varepsilon_0 > 0$, and for each ε, $0 < \varepsilon \le \varepsilon_0$, there exists a solution $y(t,\varepsilon)$ of (2.10), (2.11) such that

$$y(t,\varepsilon) - u(t) = O(\varepsilon) ,$$

$$|y'(t,\varepsilon) - u'(t)| \le |u'(1) - B'|\exp[(t-1)c\varepsilon^{-1}] + O(\varepsilon) ,$$

where c is a positive constant independent of ε.

Proof: Letting $u_\ell(t,\varepsilon)$ and $u_u(t,\varepsilon)$ be C^3 approximations of $u(t)$ as in Lemma 2.4, we construct lower and upper solutions of (2.10).

$$\alpha(t) = \alpha(t,\varepsilon) = u_\ell(t,\varepsilon) - \lambda^{-1}P\exp[(t-1)\lambda] - \varepsilon\exp[-\mu(t+1)]$$

$$\beta(t) = \beta(t,\varepsilon) = u_u(t,\varepsilon) + \lambda^{-1}P\exp[(t-1)\lambda] + \varepsilon\exp[-\mu(t+1)]$$

where λ and P are as formulated in the proof of Theorem 2.1, and μ is here taken as the negative root of

$$m\mu^2 - k\mu - (\ell+M+N+S+1) = 0 ,$$

S being a positive constant, with definition to follow. M is here taken to be an upper bound for $\{|u_\ell'''(t)|, |u_u'''(t)| : -1 \le t \le 1\}$. $\alpha \le \beta$ and $\alpha' \le \beta'$ as before, and the boundary inequalities are also proven directly. Thus, we need only verify that α and β are indeed lower and upper solutions.

We test the case where $r > 0$ ($r^3 = u'''(0^+) - u'''(0^-)$), for $\alpha(t,\varepsilon)$ with special note to the size and inequality directions of the derivatives of u_ℓ and u_u.

$$\begin{aligned}
&\varepsilon\alpha'''(t) - f(t,z,\alpha'(t),\alpha''(t),\varepsilon) \\
&\quad = \varepsilon u'''(t) + \varepsilon r^3\exp(rt/\varepsilon) - \varepsilon^2\gamma\exp(t+1) - \varepsilon\lambda^2P\exp[(t-1)\lambda] \\
&\qquad + \varepsilon\mu^3\varepsilon\exp[-\mu(t+1)] - f_{y''}(t,z,\alpha'(t),\cdot,\varepsilon) \\
&\qquad \times \{-\lambda P\exp[(t-1)\lambda] - \mu^2\varepsilon\exp[-\mu(t+1)] + O(\varepsilon)\} - f_{y'}(t,z,\cdot\cdot,u''(t),\varepsilon)
\end{aligned}$$

$$\times \{-P\exp[(t-1)\lambda] + \mu\varepsilon\exp[-\mu(t+1)] + O(\varepsilon)\}$$

$$- f_y(t,\cdots,u'(t),u''(t),\varepsilon)$$

$$\times \{\theta\lambda^{-1}P\exp[(t-1)\lambda] + \theta\varepsilon\exp[-\mu(t+1)] + O(\varepsilon)\}$$

$$- f(t,u(t),u'(t),u''(t),\varepsilon) = *$$

for $t \le 0$, $|\theta| \le 1$; $\cdot$, $\cdot\cdot$, $\cdots$ being appropriate intermediate values.

$$* \ge (-\varepsilon\lambda^2 + m\lambda - k - \ell\lambda^{-1})P\exp[(t-1)\lambda] - \varepsilon(M+N)$$
$$+ (\varepsilon\mu^3 + m\mu^2 + k\mu - \ell)\varepsilon\exp[-\mu(t+1)] + O(\varepsilon) ,$$

the last $O(\varepsilon)$ obtained from the modifications u_ℓ and u_u of u . Whence we have $S > 0$, and $\varepsilon\alpha'''(t) - f(t,z,\alpha'(t),\alpha''(t),\varepsilon) \ge$ $(-\varepsilon\lambda^2 + m\lambda - k - \ell\lambda^{-1})P\exp[(t-1)\lambda] + (\varepsilon\mu^3 + m\mu^2 + k\mu - \ell - M - N - S)\varepsilon \ge 0$. When $t \ge 0$, the difference in the above expansion is that the term $\varepsilon r^3\exp(rt/\varepsilon)$ would be absent. Similar calculations for $r < 0$, and the β's follow directly, as in the proof of Theorem 2.1.

Theorem 2.6. Assume the hypotheses of Theorem 2.3 for the interval $-1 \le t \le 1$, but replacing assumption (1) by

(1′) There exists a solution $u = u(t) \in C^2[-1,1] \cap C^3\{[-1,0) \cup (0,1]\}$ satisfying (2.12), (2.13′); such that $u'''(0^-)$ and $u'''(0^+)$ exist and are finite.

Then there exists an $\varepsilon_0 > 0$, and for each ε , $0 < \varepsilon \le \varepsilon_0$, there exists a solution $y(t,\varepsilon)$ of (2.10), (2.11) such that

$$y(t,\varepsilon) - u(t) = O(\varepsilon^{\frac{1}{2}}) \text{ and}$$

$$|y'(t,\varepsilon) - u'(t)| \le |u'(-1) - A'|\exp[-c_1(t+1)\varepsilon^{-\frac{1}{2}}]$$
$$+ |u'(1) - B'|\exp[c_2(t-1)\varepsilon^{-\frac{1}{2}}] + O(\varepsilon^{\frac{1}{2}}) ,$$

c_1 and c_2 being positive constants independent of ε .

Proof: Lower and upper solutions are formed as in the proof of Theorem

2.3, where we now take μ to satisfy

$$-k\mu - (\ell+M+N+\mathcal{r}+S+1) = 0 .$$

$\mathcal{r}$ and S arising in the expansions of $f(t,z,\alpha',\varepsilon)$ and $f(t,z,\beta',\varepsilon)$. $P = |u'(-1) - A'|$, $Q = |u'(1) - B'|$ and $u_\ell(t,\varepsilon^{\frac{1}{2}})$ and $u_u(t,\varepsilon^{\frac{1}{2}})$ being C^3 approximations of $u(t)$ as in Lemma 2.4, we have

$$\alpha(t) = \alpha(t,\varepsilon) = u_\ell(t,\varepsilon^{\frac{1}{2}}) - \gamma^{-1}Q\{\exp[\gamma(t+1)] - 1\} - \lambda^{-1}P\exp[(t-1)\lambda] - \varepsilon^{\frac{1}{2}}\exp[-\mu(t+1)] ,$$

$$\beta(t) = \beta(t,\varepsilon) = u_u(t,\varepsilon^{\frac{1}{2}}) + \gamma^{-1}Q\{\exp[\gamma(t+1)] - 1\} + \lambda^{-1}P\exp[(t-1)\lambda] + \varepsilon^{\frac{1}{2}}\exp[-\mu(t+1)] .$$

Verification follows by direct calculation.

The choice of boundary conditions for which the above reduced solutions are asked to satisfy is not arbitrary, but follows from The Cancellation Law as stated by O'Malley ([OM], p.49) for linear boundary value problems as to number and position. As a guide to phenomena, i.e., order of convergence and width of boundary layers, second-order equations of the form $\varepsilon x'' = f(t,x,x',\varepsilon)$ as treated by Howes in [Ho1] and [Ho2] display equivalent action at the $x(t,\varepsilon)$ level as $y'(t,\varepsilon)$ here show.

CHAPTER III

SINGULAR PERTURBATIONS WITH BOUNDARY LAYERS AT THE FUNCTION LEVEL

Again, we study the nonlinear differential equation

$$\varepsilon y''' = f(t,y,y',y'',\varepsilon)\ , \quad 0 < t < 1\ ; \tag{3.1}$$

but with the boundary values now being

$$y(0,\varepsilon) = A\ ,\ y'(0,\varepsilon) = A'\ ,\ y(1,\varepsilon) = B\ . \tag{3.2}$$

We study the existence and asymptotic behavior, as $\varepsilon \to 0^+$, of solutions $y = y(t,\varepsilon)$ of (3.1), (3.2) to the solution of the "reduced" initial value problem

$$0 = f(t,u,u',u'',0) \tag{3.3}$$

$$u(0) = A\ ,\ u'(0) = A'\ . \tag{3.4}$$

As in Chapter II, we will assume (3.3), (3.4) has a solution $u = u(t) \in C^3[0,1]$ (or perhaps merely C^2), $f_{y''}$ is nonnegative, and here especially that $f_{y'}$ is positively bounded away from 0 .

<u>Theorem 3.1</u>. Suppose initial value problems (3.1) extend to $[0,1]$ or become unbounded. Also assume:

(1) There exists a solution $u = u(t) \in C^3[0,1]$ satisfying (3.3), (3.4).

(2) f is continuous in (t,y,y',y'',ε) and of class C^1 with respect to y' and y'' with $f_{y''} \geq 0$ and $f_{y'} > 0$. And for a region $R : 0 \leq t \leq 1$, $|y - u(t)| \leq d$, $|y'| < \infty$, $|y''| < \infty$, $0 \leq \varepsilon \ll 1$, ($d > 0$), with $|B - u(1)| < d$.

(3) $f(t,u(t),u'(t),u''(t),\varepsilon) = O(\varepsilon)$, $0 \leq t \leq 1$.

(4) There exists a $k > 0$ such that $f_{y'}(t,y,y',u'',\varepsilon) \geq k$.

(5) $f_y(t,y,u'(t),u''(t),\varepsilon) = O(1)$.

Then there exists an $\varepsilon_0 > 0$, such that for all ε, $0 < \varepsilon \leq \varepsilon_0$; there exists a solution $y = y(t,\varepsilon)$ of (3.1), (3.2) such that

i) $|y(t,\varepsilon) - u(t)| \leq |B - u(1)|\exp[(t-1)c_1\varepsilon^{-1}] + O(\varepsilon)$ provided $f_{y''} \geq m > 0$ in R;

ii) $|y(t,\varepsilon) - u(t)| \leq |B - u(1)|\exp[(t-1)c_2\varepsilon^{-\frac{1}{2}}] + O(\varepsilon)$ otherwise.

<u>Proof</u>: Case i). $f_{y''} \geq m > 0$. Let λ, $-2m/\varepsilon < \lambda < -m/\varepsilon$ be the root of

$$\varepsilon\lambda^3 + m\lambda^2 - k\lambda - \ell = 0 ;$$

and μ be the largest negative root of

$$\varepsilon\mu^3 + m\mu^2 - k\mu - (\ell+M+N+1) = 0 ,$$

where $|f_y(t,y,u'(t),u''(t),\varepsilon)| \leq \ell$, $|u'''(t)| \leq M$ and $|f(t,u(t),u'(t),u''(t),\varepsilon)| \leq N\varepsilon$. $P = |u(1) - B|$, we form lower and upper solutions of (3.1).

$$\alpha(t) = \alpha(t,\varepsilon) = u(t) - P\exp[\lambda(1-t)] - \varepsilon\exp(-\mu t)$$

$$\beta(t) = \beta(t,\varepsilon) = u(t) + P\exp[\lambda(1-t)] + \varepsilon\exp(-\mu t) .$$

Observe that $\beta(t) - \alpha(t) = 2P\exp[\lambda(1-t)] + 2\varepsilon\exp(-\mu t)$ and that

$\alpha(0) = A - P\exp(\lambda) - \varepsilon \leq A \leq A + P\exp(\lambda) + \varepsilon = \beta(0)$,

$\alpha(1) - u(1) - |u(1)-B| - \exp(-\mu) \leq B \leq u(1) + |u(1)-B| + \exp(-\mu) = \beta(1)$, and $\alpha'(0) = A' + \lambda P\exp(\lambda) + \mu\varepsilon \le A' \le A' - \lambda P\exp(\lambda) - \mu\varepsilon = \beta'(0)$.

Using the mean value theorem, and turning corners, as in the previous chapter, α and β are shown to be lower and upper solutions of (3.1).

$$\begin{aligned}
&\varepsilon\alpha'''(t) - f(t,z,\alpha'(t),\alpha''(t),\varepsilon) \\
&\quad = \varepsilon u'''(t) + \varepsilon\lambda^3 P\exp[\lambda(1-t)] + \varepsilon\mu^3\varepsilon\exp(-\mu t) - f_{y''}(t,z,\alpha'(t),\cdot,\varepsilon) \\
&\qquad \times \{-\lambda^2 P\exp[\lambda(1-t)] - \mu^2\varepsilon\exp(-\mu t)\} - f_{y'}(t,z,\cdot\cdot,u''(t),\varepsilon) \\
&\qquad \times \{\lambda P\exp[\lambda(1-t)] + \mu\varepsilon\exp(-\mu t)\} - f_{y}(t,\cdot\cdot\cdot,u'(t),u''(t),\varepsilon) \\
&\qquad \times \{\theta P\exp[\lambda(1-t)] + \theta\varepsilon\exp(-\mu t)\} - f(t,u(t),u'(t),u''(t),\varepsilon) \quad ,
\end{aligned}$$

where $z = z(t)$ is such that $\alpha(t) \le z \le \beta(t)$, and so $|\theta| \le 1$; $\cdot$, $\cdot\cdot$, $\cdot\cdot\cdot$ denoting, respectively, values between $\alpha''(t)$ and $u''(t)$, $\alpha'(t)$ and $u'(t)$ and z and $u(t)$. Using our assumptions for comparative purposes, we find then

$$\begin{aligned}
&\varepsilon\alpha'''(t) - f(t,z,\alpha'(t),\alpha''(t),\varepsilon) \\
&\quad \ge -\varepsilon M + \varepsilon\lambda^3\exp[\lambda(1-t)] + \varepsilon\mu^3\varepsilon\exp(-\mu t) + m\lambda^2 P\exp[\lambda(1-t)] \\
&\qquad + m\mu^2\varepsilon\exp(-\mu t) - k\lambda P\exp[\lambda(1-t)] - k\mu\varepsilon\exp(-\mu t) \\
&\qquad - \ell\theta P\exp[\lambda(1-t)] - \ell\theta\varepsilon\exp(-\mu t) - \varepsilon N \\
&\quad \ge (\varepsilon\lambda^3 + m\lambda^2 - k\lambda - \ell)P\exp[\lambda(1-t)] - (M+N)\varepsilon \\
&\qquad + (\varepsilon\mu^3 + m\mu^2 - k\mu - \ell)\exp(-\mu t) \quad .
\end{aligned}$$

As $\mu \le 0$, $\exp(-\mu t)$ is nondecreasing and so we have, noting the definitions of λ and μ ,

$$\begin{aligned}
&\varepsilon\alpha'''(t) - f(t,z,\alpha'(t),\alpha''(t),\varepsilon) \\
&\qquad \ge (\varepsilon\mu^3+m\mu^2-k\mu-\ell-M-N) > (\varepsilon\mu^3+m\mu^2-k\mu-\ell-M-N-1) = 0 ,
\end{aligned}$$

that is to say α is a lower solution with respect to β . For β we deduce

$$f(t,z,\beta'(t),\beta''(t),\varepsilon) - \varepsilon\beta'''(t) \geq -\varepsilon(M+N) + (\varepsilon\lambda^3 + m\lambda^2 - k\lambda - \ell\theta)$$
$$\times\ P\exp[\lambda(1-t)] + (\varepsilon\mu^3 + m\mu^2 - k\mu - \ell\theta)\varepsilon\exp(-\mu t) > 0\ ,$$

β is an upper solution. Thus Theorem 1.6 allows us to declare a solution $y(t,\varepsilon)$, $\alpha(t) \leq y(t,\varepsilon) \leq \beta(t)$ exists.

Things of interest in the above are that as $\varepsilon \to 0^+$, $\mu \to [k - (k^2 + m(\ell + M + N + 1))^{\frac{1}{2}}](2m)^{-1}$; and that $\varepsilon\alpha''' > f(t,z,\alpha',\alpha'',\varepsilon)$ and $\varepsilon\beta''' < f(t,z,\beta',\beta'',\varepsilon)$. The latter observation is useful in dealing with the situation where f is independent of y' , as in Theorem 1.7. We return for the

Case ii). $f_{y''} \geq 0$. Let λ , $-(k/\varepsilon)^{\frac{1}{2}} < \lambda < -(k/4\varepsilon)^{\frac{1}{2}}$ be a negative root of

$$\varepsilon\lambda^3 - k\lambda - \ell = 0$$

and μ the largest negative root of

$$\varepsilon\mu^3 - k\mu - (\ell+M+N+1) = 0\ ,$$

($\mu \to (\ell + M + N + 1)/k$ as $\varepsilon \to 0^+$), ℓ , k , M , and N as above. Letting $P = |u(1) - B|$, we form α and β

$$\alpha(t) = \alpha(t,\varepsilon) = u(t) - P\exp[\lambda(1-t)] - \varepsilon\exp(-\mu t)$$
$$\beta(t) = \beta(t,\varepsilon) = u(t) + P\exp[\lambda(1-t)] + \varepsilon\exp(-\mu t)\ .$$

Again, $\beta(t) - \alpha(t) > 0$; and $\alpha(0) < A < \beta(0)$, $\alpha'(0) \leq A' \leq \beta'(0)$, and $\alpha(1) < B < \beta(1)$. We test β :

$$\begin{aligned}
&f(t,z,\beta'(t),\beta''(t),\varepsilon) - \varepsilon\beta'''(t) \\
&\quad = f_{y''}(t,z,\beta'(t),:,\varepsilon) \\
&\qquad \times \{\lambda^2 P\exp[\lambda(1-t)] + \mu^2\varepsilon\exp(-\mu t)\} + f_{y'}(t,z,::,u''(t),\varepsilon) \\
&\qquad \times \{-\lambda P\exp[\lambda(1-t)] - \mu\varepsilon\exp(-\mu t)\} + f_y(t,:::,u'(t),u''(t),\varepsilon) \\
&\qquad \times \{\theta P\exp[\lambda(1-t)] + \theta\varepsilon\exp(-\mu t)\} + f(t,u(t),u'(t),u''(t),\varepsilon) \\
&\qquad - \varepsilon u'''(t) + \varepsilon\lambda^3 P\exp[\lambda(1-t)] + \varepsilon\mu^3\varepsilon\exp(-\mu t)
\end{aligned}$$

$$\geq -k\lambda P\exp[\lambda(1-t)] - k\mu\varepsilon\exp(-\mu t) - \ell P\exp[\lambda(1-t)]$$
$$- \ell\varepsilon\exp(-\mu t) - \varepsilon N - \varepsilon M + \varepsilon\lambda^3\exp[\lambda(1-t)] + \varepsilon\mu^3\varepsilon\exp(-\mu t)$$
$$= (\varepsilon\lambda^3 - k\lambda - \ell)P\exp[\lambda(1-t)] + (\varepsilon\mu^3 - k\mu - \ell)\varepsilon\exp(-\mu t) - \varepsilon(M+N)$$
$$\geq (\varepsilon\mu^3 - k\mu - \ell - M - N)\varepsilon > 0 \qquad ,$$

due to the definitions of λ and μ. Therefore, β is an upper solution. A symmetric argument for α produces $\varepsilon\alpha'''(t) > f(t,z,\alpha'(t),\alpha''(t),\varepsilon)$ for z, $\alpha(t) \leq z \leq \beta(t)$. Theorem 1.6 may now be employed to give the indicated conclusion.

<u>Theorem 3.2.</u> Consider the differential equation

$$\text{(3.5)} \qquad \varepsilon y''' = f(t,y,y'',\varepsilon) \ , \quad 0 < t < 1 \ .$$

Assume that initial value problems (3.5) extend to $[0,1]$ or become unbounded. Also assume:

(1) There exists a solution $u = u(t) \in C^3[0,1]$ satisfying $0 = f(t,u,u'',0)$, $u(0) = A$, $u'(0) = A'$.

(2) f is continuous in (t,y,y'',ε) and of class C^1 regarding y and y'' in region $R: 0 \leq t \leq 1$, $|y - u(t)| \leq d$, $|y''| < \infty$, $0 \leq \varepsilon \ll 1$.

(3) $f(t,u(t),u''(t),\varepsilon) = O(\varepsilon)$, $0 \leq t \leq 1$.

(4) $f_{y''}(t,y,y'',\varepsilon) \geq m > 0$ in R, for some m.

(5) $f_y(t,y,u''(t),\varepsilon) = O(1)$ in R.

Then there is an $\varepsilon_0 > 0$, such that for all ε, $0 < \varepsilon \leq \varepsilon_0$, there exists a solution $y = y(t,\varepsilon)$ of (3.5), (3.2) such that

$$|y(t,\varepsilon) - u(t)| \leq |B - u(t)|\exp[(t-1)c\varepsilon^{-1}] + O(\varepsilon) \ .$$

<u>Proof:</u> We make use here of Theorem 1.7, and construct lower and upper solutions accordingly.

$$\alpha(t) = \alpha(t,\varepsilon) = u(t) - P\exp[(1-t)\lambda] - \varepsilon\exp(-\mu t)$$

$$\beta(t) = \beta(t,\varepsilon) = u(t) + P\exp[(1-t)\lambda] + \varepsilon\exp(-\mu t) \ ,$$

where we define λ and μ as roots of

$$\varepsilon\lambda^3 + m\mu^2 - \ell = 0 \ , \quad -m/\varepsilon \le \lambda \le -m/2\varepsilon$$

for small ε ,

$$\varepsilon\mu^3 + m\mu^2 - (\ell+M+N+1) = 0 \ ,$$

with $\mu \to (\ell + M + N + 1)^{\frac{1}{2}}m^{-\frac{1}{2}}$ as $\varepsilon \to 0^+$; ℓ , M , N being positive constants such that $|f_y(t,z,\dddot{u}'',\varepsilon)| \le \ell$, $|\dddot{u}'''(t)| \le M$, and $|f(t,\dddot{u},u'',\varepsilon)| \le N\varepsilon$.

$$\begin{aligned}
&\varepsilon\dddot{\alpha}'''(t) - f(t,z,\dddot{\alpha}''(t),\varepsilon) \\
&\quad = \varepsilon\dddot{u}'''(t) + \varepsilon\lambda^3 P\exp[(1-t)\lambda] + \varepsilon\mu^3\varepsilon\exp(-\mu t) - f_{y''}(t,z,*,\varepsilon) \\
&\qquad \times \{-\lambda^2 P\exp[(1-t)\lambda] - \mu^2\varepsilon\exp(-\mu t)\} - f_y(t,**,u''(t),\varepsilon) \\
&\qquad \times \{\theta P\exp[(1-t)\lambda] + \theta\varepsilon\exp(-\mu t)\} - f(t,u(t),u''(t),\varepsilon) \quad ,
\end{aligned}$$

with $\alpha(t) \le z \le \beta(t)$, $|\theta| \le 1$, $*$ and $**$ being some values between $\alpha''(t)$ and $u''(t)$, and z and $u(t)$, respectively. Comparing we find:

$$\begin{aligned}
&\varepsilon\dddot{\alpha}'''(t) - f(t,z,\dddot{\alpha}''(t),\varepsilon) \\
&\quad \ge -\varepsilon M + \varepsilon\lambda^3 P\exp[(1-t)\lambda] + \varepsilon\mu^3\varepsilon\exp(-\mu t) + m\lambda^2 P\exp[(1-t)\lambda] \\
&\qquad + m\mu^3\varepsilon\exp(-\mu t) - \ell P\exp[(1-t)\lambda] - \ell\varepsilon\exp(-\mu t) - \varepsilon N \\
&\quad = (\varepsilon\lambda^3 + m\lambda^2 - \ell)P\exp[(1-t)\lambda] - \varepsilon(M+N) + (\varepsilon\mu^3 + m\mu^2 - \ell)\varepsilon\exp(-\mu t) \\
&\quad > (\varepsilon\mu^3 + m\mu^2 - \ell - M - N - 1)\varepsilon = 0 \quad ,
\end{aligned}$$

i.e., α is a strict lower solution of (3.5) with respect to β . Also, one has $\varepsilon\dddot{\beta}'''(t) < f(t,z,\beta''(t),\varepsilon)$, whence Theorem 1.7 provides our conclusion.

Heretofore, u was requested to satisfy both of the boundary conditions at $t = 0$. But, as in the previous chapter, if f is independent of y'' , solutions behave quite differently. Convergence was much slower, away from the boundary layers, and the boundary layers, themselves, were magnified. These same phenomena may also be observed for the type of boundary conditions we deal herewith. The problem is

$$\varepsilon y''' = f(t,y,y',\varepsilon) , \quad 0 < t < 1 \tag{3.6}$$

$$y(0,\varepsilon) = A , \; y'(0,\varepsilon) = A' , \; y(1,\varepsilon) = B ; \tag{3.7}$$

where the reduced problem will be

$$0 = f(t,u,u',0) , \quad 0 < t < 1 \tag{3.8}$$

$$u(0) = A . \tag{3.9}$$

Theorem 3.3. Suppose initial value problems (3.6) extend to $[0,1]$ or become unbounded. Also assume:

(1) There exists a solution $u = u(t) \in C^3[0,1]$ satisfying (3.8) and (3.9).

(2) f is continuous in (t,y,y',ε) and of class C^1 with respect to y and y' in a region R : $0 \le t \le 1$, $|y - u(t)| \le d$, $|y'| < \infty$, $0 \le \varepsilon \ll 1$.

(3) $f(t,u(t),u'(t),\varepsilon) = O(\varepsilon)$, $0 \le t \le 1$.

(4) There exists a $k > 0$, such that $f_{y'} \ge k$ in R .

(5) $f_y(t,y,u'(t),\varepsilon) = O(1)$ in R .

Then there exists an $\varepsilon_0 > 0$, such that for each ε , $0 < \varepsilon \le \varepsilon_0$, there exists a solution $y = y(t,\varepsilon)$ of the boundary value problem (3.6), (3.7). Also,

$$|y(t,\varepsilon) - u(t)| \le |B - u(1)|\exp[(t-1)c\varepsilon^{-\frac{1}{2}}] + O(\varepsilon^{\frac{1}{2}}) , \quad 0 \le t \le 1 ,$$

for c , a positive constant independent of ε .

Proof: In order to employ Theorem 1.6, we construct functions $\alpha < \beta$ where $\alpha(0,\varepsilon) \le A \le \beta(0,\varepsilon)$, $\alpha'(0,\varepsilon) \le A' \le \beta'(0,\varepsilon)$ and $\alpha(1,\varepsilon) \le B \le \beta(1,\varepsilon)$, such that α and β are lower and upper solutions of (3.6).

We designate $P = |u(1) - B|$ and $Q = |u'(0) - A'|$, and define for $t \in [0,1]$,

$\alpha(t) = \alpha(t,\varepsilon) = u(t) - P\exp[(1-t)\lambda] + \gamma^{-1}Q\{\exp(-\gamma t)-1\} - \varepsilon^{\frac{1}{2}}\exp(-\mu t)$,

$\beta(t) = \beta(t,\varepsilon) = u(t) + P\exp[(1-t)\lambda] - \gamma^{-1}Q\{\exp(-\gamma t)-1\} + \varepsilon^{\frac{1}{2}}\exp(-\mu t)$;

where γ is the positive root, $\gamma = O(\varepsilon^{-\frac{1}{2}})$, for

$$-\varepsilon\gamma^2 + k/2 = 0 \ ,$$

$-(k/\varepsilon)^{\frac{1}{2}} < \lambda < -(k/4\varepsilon)^{\frac{1}{2}}$ satisfies

$$\varepsilon\lambda^3 - k\lambda - \ell = 0 \ .$$

If ℓ is the bound for f_y in R , for $-1 \le \theta \le 1$ we have then $|\ell\theta\gamma^{-1}Q| \le S\varepsilon^{\frac{1}{2}}$, and so we define μ as the largest negative root of

$$\varepsilon\mu^3 - k\mu - \ell - M - N - S = 0 \ ,$$

M and N obtained from conditions (1) and (3) in the usual way.

By construction, $\alpha \le \beta$, $\alpha(0) \le A \le \beta(0)$, $\alpha'(0) \le A' \le \beta'(0)$ and $\alpha(1) \le B \le \beta(1)$. We verify explicitly that $\varepsilon\beta'''(t) \le f(t,z,\beta'(t),\varepsilon)$ and $\varepsilon\alpha'''(t) \ge f(t,z,\alpha'(t),\varepsilon)$, where $\alpha(t) \le z \le \beta(t)$. For β , substituting and expanding

$$\begin{aligned} & f(t,z,\beta'(t),\varepsilon) - \varepsilon\beta'''(t) \\ &\quad = f_{y'}(t,z,*,\varepsilon) \times \{-\lambda P\exp[(1-t)\lambda] - \mu\varepsilon^{\frac{1}{2}}\exp(-\mu t) + Q\exp(-\gamma t)\} + f_y(t,**,u'(t),\varepsilon) \\ &\qquad \times \{\theta P\exp[(1-t)\lambda] + \theta\varepsilon^{\frac{1}{2}}\exp(-\mu t) - \theta\gamma^{-1}Q[\exp(-\gamma t)-1]\} + f(t,u(t),u'(t),\varepsilon) \\ &\qquad - \varepsilon u'''(t) + \varepsilon\lambda^3 P\exp[(1-t)\lambda] + \varepsilon\mu^3\varepsilon^{\frac{1}{2}}\exp(-\mu t) - \varepsilon\gamma^2 Q\exp(-\gamma t) \ , \end{aligned}$$

for some θ , $|\theta| \le 1$, and $*$ and $**$ standing in for appropriate intermediate values. Making use of the fact that $f_{y'} \ge k > 0$ and $|f_y| \le \ell$,

$$\begin{aligned} & f(t,z,\beta'(t),\varepsilon) - \varepsilon\beta'''(t) \\ &\quad \ge (\varepsilon\lambda^3 - k\lambda - \ell)P\exp[(1-t)\lambda] + (-\varepsilon\gamma^2 + k + \ell\theta\gamma^{-1}) \\ &\qquad \times Q\exp(-\gamma t) + (\varepsilon\mu^3 - k\mu - \ell)\varepsilon^{\frac{1}{2}}\exp(-\mu t) - \varepsilon M - \varepsilon N - \ell\theta\gamma^{-1}Q \ . \end{aligned}$$

For ε sufficiently small, we note that $|\ell\theta\gamma^{-1}| < k/2$, and since $\exp(-\mu t)$ is increasing, we then have

$$f(t,z,\beta'(t),\varepsilon) - \varepsilon\beta'''(t)$$
$$\geq (\varepsilon\lambda^3 - k\lambda - \ell)P\exp[(1-t)\lambda]$$
$$+ (-\varepsilon\gamma^2 + k/2)Q\exp(-\gamma t) + (\varepsilon\mu^3 - k\mu - \ell - M - N - S)\varepsilon^{\frac{1}{2}} = 0 .$$

For α , we obtain in the same manner,

$$\varepsilon\alpha'''(t) - f(t,z,\alpha'(t),\varepsilon)$$
$$\geq \varepsilon u'''(t) + \varepsilon\lambda^3 P\exp[(1-t)\lambda] + \varepsilon\mu^3\varepsilon^{\frac{1}{2}}\exp(-\mu t)$$
$$- \varepsilon\gamma^2 Q\exp(-\gamma t) - f_{y'}(t,z,:,\varepsilon)$$
$$\times \{\lambda P\exp[(1-t)\lambda] + \mu\varepsilon^{\frac{1}{2}}\exp(-\mu t) - Q\exp(-\gamma t)\} - f_y(t,::,u'(t),\varepsilon)$$
$$\times \{\theta P\exp[(1-t)\lambda] + \theta\varepsilon^{\frac{1}{2}}\exp(-\mu t) - \theta\gamma^{-1}Q[\exp(-\gamma t)-1]\} - f(t,u(t),u'(t),\varepsilon)$$
$$\geq (\varepsilon\lambda^3 - k\lambda - \ell)P\exp[(1-t)\lambda] + (-\varepsilon\gamma^2+k/2)Q\exp(-\gamma t) + (\varepsilon\mu^3-k\mu-\ell-M-N-S)\varepsilon^{\frac{1}{2}}$$
$$= 0 .$$

We conclude, by Theorem 1.6, that the boundary value problem (3.6), (3.7) has a solution, of the desired size.

Due to particular nature which f might possess, the assumption that the solution to the reduced equation be thrice continuously differentiable may not be attainable. If we but consider, as example, the piecewise defined linear equation

$$\varepsilon y''' = \begin{cases} y'' & -1 \leq t \leq 0 \\ y'' + t & 0 < t \leq 1 \end{cases}$$

whose $C^3[-1,1]$ solution is obtained by elementary means, for arbitrary boundary values, does indeed act like solutions already mentioned, in regard to the reduced solution. But the reduced solution must satisfy: $u'''(t) = 0$, $-1 \leq t \leq 0$ and $u'''(t) = -1$, $0 < t \leq 1$. Boundary value problems of this type may be handled using, as before, approximations which do not deviate excessively from the reduced solution at any of the derivative levels expressed in f . Recall, then, the C^3 approximations u_ℓ and u_u obtained in Lemma 2.4, for functions $u \in C^2[-1,1] \cap C^3\{[-1,0) \cup (0,1]\}$. We state the following theorems for the boundary value problem

$$\varepsilon y''' = f(t,y,y',y'',\varepsilon) , \quad -1 < t < 1 \tag{3.10}$$

$$y(-1,\varepsilon) = A , \; y'(-1,\varepsilon) = A' , \; y(1,\varepsilon) = B . \tag{3.11}$$

The reduced problem is

$$0 = f(t,u,u',u'',\varepsilon) , \quad -1 < t < 1 \tag{3.12}$$

$$u(-1) = A \quad (\text{and} \quad u'(-1) = A'), \tag{3.13}$$

of which solutions are required to be C^2 and piecewise C^3 .

<u>Theorem 3.4</u>. Make the same assumptions as in Theorem 3.1 for the interval $-1 \le t \le 1$; replacing assumption (1) by

(1′) There exists a solution $u = u(t) \in C^2[-1,1] \cap C^3\{[-1,0) \cup (0,1]\}$, such that $u'''(0^-)$ and $u'''(0^+)$ exist and are finite.

Then there exists an $\varepsilon_0 > 0$, such that for all ε , $0 < \varepsilon \le \varepsilon_0$, there is a solution $y = y(t,\varepsilon)$ of (3.10), (3.11) such that

i) $|y(t,\varepsilon) - u(t)| \le |B - u(1)|\exp[(t-1)c_1\varepsilon^{-1}] + O(\varepsilon)$ provided $f_{y''} \ge m > 0$, or elsewise

ii) $|y(t,\varepsilon) - u(t)| \le |B - u(1)|\exp[(t-1)c_2\varepsilon^{-\frac{1}{2}}] + O(\varepsilon)$, where c_1 and c_2 are positive constants independent of ε .

<u>Proof</u>: We but note that this is the analogue of Theorem 3.1 for the situation described in condition (1′) . We form lower and upper solutions via the same device as in Theorems 2.5 and 2.6, i.e., where $r^3 = u'''(0^+) - u'''(0^-)$ we replace $u(t)$ in the definitions of $\alpha(t,\varepsilon)$

$\beta(t,\varepsilon)$ by $u_\ell(t,\varepsilon)$ and $u_u(t,\varepsilon)$, respectively. Adjustment of μ follows accordingly.

Indeed, this usage of u_ℓ and u_u instead of u for situations where the assumption (1′) is more practicable, also yields the same conclusions as would apply where f is of the nature found in Theorems 3.2 and 3.3, for the interval $[-1,1]$. The construction procedure even allows one to extend such conclusions where the reduced solutions are C^2 and piecewise C^3, without loss, other than the aesthetics in writing out the lower and upper solutions in explicit form.

CHAPTER IV

APPROXIMATING SOLUTIONS

Perturbation theory presents many practical means of "solving" problems of the sort we deal with; asymptotic sequences, matched and composite expansions, averaging techniques, to name a few. A question raised in applying these to a given problem is often that which alludes to the validity of a returned form being a good approximation. We are presented

(4.1) $\varepsilon y''' = f(t,y,y',y'',\varepsilon)$, $0 < t < 1$

(4.2) $y(0,\varepsilon) = A$, $y'(0,\varepsilon) = A'$, $y(1,\varepsilon) = B$ or

(4.2′) $y(0,\varepsilon) = A$, $y'(0,\varepsilon) = A'$, $y'(1,\varepsilon) = B'$.

The determination of the existence and asymptotic behavior of solutions to (4.1), (4.2) (or (4.2′)) when an approximate solution exists, is the goal. To lessen any deviation from what has gone before, we first consider approximations in the sense that the initial value problem

(4.3) $\varepsilon u''' = f(t,u,u',u'',\varepsilon) + O(\eta)$, $0 < t < 1$

(4.4) $u(0,\varepsilon) = A + O(\eta)$, $u'(0,\varepsilon) = A' + O(\eta)$

is satisfied, where $\eta = \eta(\varepsilon) > 0$, $\eta = o(1)$. $u(t,\varepsilon)$ is called an approximate solution of (4.1), "nearly" satisfying the boundary conditions at the left-hand endpoint. Indeed, as in Theorems 2.1-2.3 and 3.1-3.3, C^3 solutions of reduced problems are examples of such approximate solutions, when viewed in light of the standard assumption that $f(t,u,u',u'',\varepsilon) = O(\varepsilon)$.

Theorem 4.1. Suppose solutions $y(t,\varepsilon)$ of (4.1) have the property: if $y'(t,\varepsilon)$ is bounded on $[0,1]$, then $y''(t,\varepsilon)$ is bounded on $[0,1]$. Also,

(1) There exists an approximate solution $u = u(t,\varepsilon) \in C^3[0,1]$ satisfying (4.3), (4.4) with $\eta > 0$, $\eta = o(1)$.

(2) f is continuous in (t,y,y',y'',ε) and of class C^1 with respect to y , y' , and y'' for a region R : $0 \le t \le 1$, $|y - u(t)| \le d_1$, $|y' - u'(t)| \le d_2$, $|y'' - u''(t)| \le d_3(1 + \varepsilon^{-1}\exp[m\varepsilon^{-1}(t-1)])$, $0 \le \varepsilon \le \varepsilon_1$ ($d_i, \varepsilon_1 > 0$).

(3) There is a constant $m > 0$ such that $f_{y''} \ge m$ in R .

(4) $f_{y'}(t,y,y',u''(t),\varepsilon) = O(1)$ and $f_y(t,y,u'(t),u''(t),\varepsilon) = O(1)$ in R .

Then there exists an $\varepsilon_0 > 0$, such that for each ε , $0 < \varepsilon \le \varepsilon_0$, there exists a solution $y = y(t,\varepsilon)$ of the exact problem (4.1), (4.2′). Also,

i) $y(t,\varepsilon) - u(t) = O(\eta)$,

ii) $|y'(t,\varepsilon) - u'(t)| \le |B' - u'(1)|\exp[(t-1)c\varepsilon^{-1}] + O(\eta)$,

where c is a positive constant independent of ε .

Proof: In view of our initial assumption, the proof consists of construction of appropriate lower and upper solutions for (4.1). Let

$$\alpha(t) = u(t,\varepsilon) - \lambda^{-1}P\exp[(t-1)\lambda] - \eta\gamma\exp(-\mu t)$$

$$\beta(t) = u(t,\varepsilon) + \lambda^{-1}P\exp[(t-1)\lambda] + \eta\gamma\exp(-\mu t) ;$$

where λ , $m/2\varepsilon < \lambda < m/\varepsilon$, is a real root of $-\varepsilon\lambda^2 + m\lambda - 2k = 0$; μ is the negative root for $m\mu^2 + k\mu - (\ell + \sigma_3 + 1) = 0$ and $\gamma = \max\{\sigma_1, \sigma_2/-\mu, 1\}$ being a positive constant; k and ℓ being the bounds described in assumption (4), $|u(0) - A| \le \sigma_1\eta$, $|u'(0) - A'| \le \sigma_2\eta$ and $|u''' - f(t,u,u',u'',\varepsilon)| \le \sigma_3\eta$, $P = |u'(1) - B'|$. Clearly $\alpha \le \beta$ and $\alpha' \le \beta'$; also,

$$\alpha(0) \le A+\sigma_1\eta-\lambda^{-1}P\exp(-\lambda)-\eta\gamma \le A \le A-\sigma_1\eta+\lambda^{-1}P\exp(-\lambda)+\eta\gamma \le \beta(0) ,$$

$$\alpha'(0) \le A'+\sigma_2\eta-P\exp(-\lambda)+\mu\eta\gamma \le A' \le A'-\sigma_2\eta+P\exp(-\lambda)-\mu\eta\gamma \le \beta'(0) ,$$

and

$$\alpha'(1) = u'(1)-P+\mu\eta\gamma\exp(-\mu t) \le B' \le u'(1)+P-\mu\eta\gamma\exp(-\mu) = \beta'(1) .$$

All that remains to be shown is that α is a lower solution of (4.1) and β an upper. We let z be such that $\alpha(t) \le z \le \beta(t)$, substitute and expand to receive

$$\begin{aligned}
&\varepsilon\alpha''' - f(t,z,\alpha',\alpha'',\varepsilon)\\
&\quad = \varepsilon u''' - \varepsilon\lambda^2 P\exp[(t-1)\lambda] + \varepsilon\mu^3\eta\gamma\exp(-\mu t)\\
&\qquad - f_{y''}(t,z,\alpha',\theta_2(\alpha'' - u'') + u'',\varepsilon)\\
&\qquad \times \{-\lambda P\exp[(t-1)\lambda] - \mu^2\eta\gamma\exp(-\mu t)\} - f_{y'}(t,z,\theta_1(\alpha' - u') + u',u'',\varepsilon)\\
&\qquad \times \{-P\exp[(t-1)\lambda] + \mu\eta\gamma\exp(-\mu t)\} - f_y(t,\theta_0(z - u) + z,u',u'',\varepsilon)\\
&\qquad \times \{\theta\lambda^{-1}P\exp[(t-1)\lambda] + \theta\eta\gamma\exp(-\mu t)\} - f(t,u,u',u'',\varepsilon) ,
\end{aligned}$$

$0 < \theta_i < 0$, $i = 0,1,2$, $|\theta| \le 1$. Continuing, using comparison,

$$\begin{aligned}
&\varepsilon\alpha''' - f(t,z,\alpha',\alpha'',\varepsilon)\\
&\quad \ge (-\varepsilon\lambda^2 + m\lambda - k - \ell\theta\lambda^{-1})P\exp[(t-1)\lambda]\\
&\qquad + (\varepsilon\mu^3 + m\mu^2 + k\mu - \ell\theta)\eta\gamma\exp(-\mu t) - \sigma_3\eta = * .
\end{aligned}$$

Since $\lambda^{-1} = O(\varepsilon)$, $\gamma \ge 1$ and $\exp(-\mu t) \ge 1$ we conclude

$$* \ge (-\varepsilon\lambda^2+m\lambda-2k)P\exp[(t-1)\lambda] + (m\mu^2+k\mu-\ell-\sigma_3-1) = 0 ,\quad \text{i.e.,}$$

α is a lower solution. In the situation β , we find

$$
\begin{aligned}
& f(t,z,\beta',\beta'',\varepsilon) - \varepsilon\beta''' \\
&\quad = f_{y''}(t,z,\beta',\cdot,\varepsilon) \\
&\qquad \times \{\lambda P\exp[(t-1)\lambda] + \mu^2\eta\gamma\exp(-\mu t)\} + f_{y'}(t,z,\cdot\cdot,u'',\varepsilon) \\
&\qquad \times \{P\exp[(t-1)\lambda] - \mu\eta\gamma\exp(-\mu t)\} + f_y(t,\cdot\cdot\cdot,u',u'',\varepsilon) \\
&\qquad \times \{\theta\lambda^{-1}P\exp[(t-1)\lambda] + \theta\eta\gamma\exp(-\mu t)\} + f(t,u,u',u'',\varepsilon) \\
&\qquad - \varepsilon u''' - \varepsilon\lambda^2 P\exp[(t-1)\lambda] + \varepsilon\mu^3\eta\gamma\exp(-\mu t) \\
&\quad \geq (-\varepsilon\lambda^2 + m\lambda - k - \ell\lambda^{-1})P\exp[(t-1)\lambda] \\
&\qquad + (\varepsilon\mu^3 + m\mu^2 + k\mu - \ell)\eta\gamma\exp(-\mu t) - \sigma_3\eta \geq 0 \quad .
\end{aligned}
$$

Thus, β is an upper solution. Therefore, we use Theorem 1.1 to establish the existence of a solution to (4.1), (4.2′) such that $\alpha(t) \leq y(t,\varepsilon) \leq \beta(t)$ and $\alpha'(t) \leq y'(t,\varepsilon) \leq \beta'(t)$.

<u>A nonlinear example</u>. We illustrate the power of the above theorem with a description of the velocity of a fluid at high Reynolds number flowing in a two-dimensional channel with porous walls. Proudman [Pr] developes the differential equation

$$\varepsilon v''' = vv'' - v'^2 - k \ , \quad 0 < s < 1$$

$$v(0) = A \ , \ v'(0) = 0 \ , \ v(1) = B \ , \ v'(1) = 0 \ ;$$

where the walls are located at $s = 0$ and $s = 1$. In the situation that $B > A > 0$ and $k = A^2[\cosh^{-1}(B/A)]^2$, we find that

$$u(s) = A\cosh[(s)\cosh^{-1}(B/A)]$$

is an approximate solution for the differential equation, which satisfies the first three boundary conditions. For ε sufficiently small, we construct functions α and β and use them to deduce that $u(s)$ is an order ε approximation to the boundary value problem which replaces $v(1)$ by $v'(1)$. We note here that $f_{y''} = v$, $f_{y'} = -2v'$, and $f_y = v''$, which for the regions described satisfy all hypotheses. Also we conclude that $u'(s)$ is an order ε approximation to v' away from the boundary layer.

The situation for the problem (4.1), (4.2), where the approximation (4.3), (4.4) exists, is taken care of in a similar vein; i.e., adjusting a previous theorem for our purposes.

Theorem 4.2. Suppose solutions to initial value problems (4.1) extend $[0,1]$ or become unbounded. Also assume:

(1) There exists an approximate solution $u = u(t)$ satisfying (4.3), (4.4) with $\eta > 0$, $\eta = o(1)$;

(2) f is continuous in (t,y,y',y'',ε) and of class C^1 with respect to y , y' , and y'' , with $f_{y''} \geq 0$ and $f_{y'} > 0$; and for a region R : $0 \leq t \leq 1$, $|y - u| \leq d_1$, $|y' - u'| \leq d_2$ $(1 + \varepsilon^{-1}\exp[-m\varepsilon^{-1}(1-t)])$, and $|y'' - u''| \leq d_3$ $(1 + \varepsilon^{-2}\exp[-m\varepsilon^{-1}(1-t)])$, $0 \leq \varepsilon \leq \varepsilon_1$ ($d_i, \varepsilon_1 > 0$);

(3) there exists constants $m,k > 0$ such that $f_{y''} \geq m$, $f_{y'}(t,y,y',u'',\varepsilon) \geq k$ in R ;

(4) $f_y(t,y,u',u'',\varepsilon) = O(1)$ in R .

Then there exists an $\varepsilon_0 > 0$, such that for each ε , $0 < \varepsilon \leq \varepsilon_0$, there exists a solution $y = y(t,\varepsilon)$ of the exact problem (4.1), (4.2) with

$$|y(t,\varepsilon) - u(t,\varepsilon)| \leq |B - u(1)|\exp[(t-1)c\varepsilon^{-1}] + O(\eta) ,$$

c being a positive constant independent of ε

Proof: We proceed in a manner as seen in the proof of Theorem 3.1, where $\lambda = O(\varepsilon^{-1})$ is the smallest root of $\varepsilon\lambda^3 + m\lambda^2 - k\lambda - \ell = 0$, μ the largest negative root of $\varepsilon\mu^3 + m\mu^2 - k\mu - \ell - \sigma_3 - 1 = 0$; with $|\varepsilon u''' - f(t,u,u',u'',\varepsilon)| \leq \sigma_3$. Letting $P = |u(1) - B|$ we define lower and upper solutions of (4.1):

$$\alpha(t) = u(t,\varepsilon) - P\exp[\lambda(1-t)] - \eta\gamma\exp(-\mu t) ,$$

$$\beta(t) = u(t,\varepsilon) + P\exp[\lambda(1-t)] + \eta\gamma\exp(-\mu t) ,$$

where $\gamma \geq \max\{1,\sigma_1,-\sigma_2/\mu\}$ is a positive constant, $|u(0) - A| \leq \sigma_1\eta$, $|u'(0) - A'| \leq \sigma_2\eta$. $\alpha \leq \beta$ and the definition of γ provides $\alpha(0) \leq A \leq \beta(0)$, $\alpha'(0) \leq A' \leq \beta'(0)$, also $\alpha(1) \leq B \leq \beta(1)$. Letting z be such that $\alpha(t) \leq z \leq \beta(t)$, we show that α is a lower solution (β being an upper solution will follow by symmetry). Substituting z , α' , α'' , and $\varepsilon\alpha'''$ into (4.1) and expanding in our usual manner:

$$\begin{aligned}
&\varepsilon\alpha''' - f(t,z,\alpha',\alpha'',\varepsilon) \\
&\quad = \varepsilon u''' + \varepsilon\lambda^3 P\exp[\lambda(1-t)] + \varepsilon\mu^3\eta\gamma\exp(-\mu t) - f_{y''}(t,z,\alpha',:,\varepsilon) \\
&\qquad \times \{-\lambda^2 P\exp[\lambda(1-t)] - \mu^2\eta\gamma\exp(-\mu t)\} - f_{y'}(t,z,::,u'',\varepsilon) \\
&\qquad \times \{\lambda P\exp[\lambda(1-t)] + \mu\eta\gamma\exp(-\mu t)\} - f(t,:::,u',u'',\varepsilon) \\
&\qquad \times \{\theta P\exp[\lambda(1-t)] + \theta\eta\gamma\exp(-\mu t)\} - f(t,u,u',u'',\varepsilon) \quad ,
\end{aligned}$$

$|\theta| \leq 1$. Making use now of assumptions (1), (3) and (4)

$$\begin{aligned}
&\varepsilon\alpha''' - f(t,z,\alpha',\alpha'',\varepsilon) \\
&\quad \geq \varepsilon\lambda^3 P\exp[\lambda(1-t)] + \varepsilon\mu^3\eta\gamma\exp(-\mu t) + m\lambda^2 P\exp[\lambda(1-t)] + m\mu^2\eta\gamma\exp(-\mu t) \\
&\qquad - k\lambda P\exp[\lambda(1-t)] - k\mu\eta\gamma\exp(-\mu t) - \ell P\exp[\lambda(1-t)] - \ell\eta\gamma\exp(-\mu t) - \sigma_3\eta \\
&\quad = (\varepsilon\lambda^3 + m\lambda^2 - k\lambda - \ell)P\exp[\lambda(1-t)] + (\varepsilon\mu^3 + m\mu^2 - k\mu - \ell) \\
&\qquad \times \eta\gamma\exp(-\mu t) - \sigma_3\eta \quad .
\end{aligned}$$

$\gamma \geq 1$, and $\exp(-\mu t)$ having minimum value 1 , the definitions of λ and μ provide that $\varepsilon\alpha''' - f(t,z,\alpha',\alpha'',\varepsilon) \geq 0$. Theorem 1.6 now establishes our conclusions.

The above theorems, as noted earlier, are in one sense restatements of Theorems 2.1 and 3.1. Note, also, that 2.3, 3.2 and 3.3 have similar analogues, proved by the same devices, mainly replacing M + N in the definition of μ by σ_3 and choosing a large enough γ to multiply the μ term so to effect the boundary inequalities.

If the boundary conditions desired are given in the forms

$y(0,\varepsilon) = A(\varepsilon)$, $y'(0,\varepsilon) = A'(\varepsilon)$, $y(1,\varepsilon) = B(\varepsilon)$ or

$y(0,\varepsilon) = A(\varepsilon)$, $y'(0,\varepsilon) = A'(\varepsilon)$, $y'(1,\varepsilon) = B'(\varepsilon)$,

where A , A' , B , B' are independent functions of ε , continuous for a region $0 \le \varepsilon \le \varepsilon_1$. We would have the reduced problem asked to satisfy $u(0) = A(0)$ and $u'(0) = A'(0)$. Continuity arguments establish the η in (4.4) and all results follow.

Another type of approximation which may be considered is $u = u(t,\varepsilon)$ which satisfies

(4.5) $$\varepsilon u''' = f(t,u,u',u'',\varepsilon) + O(\eta) + O\{\eta\varepsilon^{-1}\exp[rm(t-1)\varepsilon^{-1}]\} , \quad 0 < t < 1$$

(4.6) $$u(0,\varepsilon) = A + O(\eta) , \quad u'(0,\varepsilon) = A' + O(\eta) , \quad u(1,\varepsilon) = B + O(\eta) \quad \text{or}$$

(4.6′) $$u(0,\varepsilon) = A + O(\eta) , \quad u'(0,\varepsilon) = A' + O(\eta) , \quad u'(1,\varepsilon) = B' + O(\eta) ,$$

where r is a positive constant independent of ε . Indeed, such is the type of approximation quite often sought using various asymptotic methods. We are of good fortune to almost attain the boundary conditions and have the equation fair approximate away from the normal position of the boundary layers.

Theorem 4.3. Suppose solutions $y(t,\varepsilon)$ of (4.1) have the property: if $y'(t,\varepsilon)$ is bounded on $[0,1]$, then $y''(t,\varepsilon)$ is bounded on $[0,1]$. Also assume:

(1) For each $\varepsilon > 0$, small, there exists a function $u = u(t,\varepsilon)$

satisfying (4.5), (4.6′), with $\eta > 0$, $\eta = o(1)$ and $r > 1$;

(2) f is continuous in (t,y,y',y'',ε) and of class C^1 with respect to y , y' , y'' in a region R : $0 \le t \le 1$, $|y - u| \le d_1$, $|y' - u'| \le d_2$, and $|y'' - u''| \le d_3\{1 + \varepsilon^{-1}\eta\exp[m(t-1)\varepsilon^{-1}]\}$, $0 \le \varepsilon \le \varepsilon_1$; $d_i, \varepsilon_1 > 0$;

(3) there exists $m > 0$ such that $f_{y''} \ge m$ in R ;

(4) $f_{y'}(t,y,y',u'',\varepsilon) = O(1)$ and $f_y(t,y,u',u'',\varepsilon) = O(1)$ in R .

Then there exists $\varepsilon_0 > 0$, such that for ε , $0 < \varepsilon \le \varepsilon_0$, there exists a solution $y(t,\varepsilon)$ of (4.1), (4.2′) with

$$y(t,\varepsilon) - u(t,\varepsilon) = O(\eta) \quad \text{and}$$
$$y'(t,\varepsilon) - u'(t,\varepsilon) = O(\eta) .$$

<u>Proof</u>: We establish the existence and estimates of $y(t,\varepsilon)$ by constructing lower and upper solutions of (4.1) as follows,

$$\alpha(t) = u(t,\varepsilon) - (\varepsilon\eta/m)\Gamma\exp[m(t-1)(2\varepsilon)^{-1}] - \eta\gamma\exp(-\mu t)$$
$$\beta(t) = u(t,\varepsilon) + (\varepsilon\eta/m)\Gamma\exp[m(t-1)(2\varepsilon)^{-1}] + \eta\gamma\exp(-\mu t) ,$$

where Γ and γ are positive constants, whose values will be specified during the proof; and μ is the negative root of $m\mu^2 + k\mu - \ell - \sigma - 1 = 0$ ($|f_{y'}| \le k$, $|f_y| \le \ell$, and σ arising from the magnitude relation in (4.5)). We have $\alpha \le \beta$ and $\alpha' \le \beta'$; and we must choose γ so that $\alpha(0) \le A \le \beta(0)$, $\alpha'(0) \le A' \le \beta'(0)$, and $\alpha'(1) \le B' \le \beta'(1)$.

$$\alpha(0) = u(0,\varepsilon) = (\varepsilon\eta/m)\Gamma\exp[-m(2\varepsilon)^{-1}] - \eta\gamma$$
$$\le A + \sigma_1\eta - (\varepsilon\eta/m)\Gamma\exp[-m(2\varepsilon)^{-1}] - \eta\gamma \le A \quad \text{if} \quad \gamma \ge \sigma_1 ,$$
$$\alpha'(0) = u'(0,\varepsilon) - (\eta/2)\Gamma\exp[-m(2\varepsilon)^{-1}] + \eta\mu\gamma$$
$$\le A' + \sigma_2 - (\eta/2)\Gamma\exp[-m(2\varepsilon)^{-1}] + \eta\mu\gamma \le A' \quad \text{if} \quad \gamma \ge \sigma_2/-\mu ,$$

$$\alpha'(1) = u'(1,\varepsilon) - (\eta/2)\Gamma + \eta\mu\gamma\exp(-\mu)$$
$$\le B' + \sigma_3 - (\eta/2)\Gamma + \eta\mu\gamma\exp(-\mu) \le B' \quad \text{if} \quad \gamma \ge \sigma_3/(-\mu\exp(-\mu)) \ .$$

Similarly, the inequalities for β are determined, and we then choose $\gamma = \max\{\sigma_1, -\sigma_2/\mu, -\sigma_3/(\mu\exp(-\mu))\}$.

We now show that α is a lower solution of (4.1). Expanding f by cornering and substituting, we find

$$\begin{aligned}
&\varepsilon\alpha''' - f(t,z,\alpha',\alpha'',\varepsilon) \\
&\quad = \varepsilon u''' - (m^2\eta/8\varepsilon)\Gamma\exp[m(t-1)(2\varepsilon)^{-1}] \\
&\qquad + \varepsilon\mu^3\eta\gamma\exp(-\mu t) - f_{y''}(t,z,\alpha',\cdot,\varepsilon) \\
&\qquad \times \{-(m\eta/4\varepsilon)\Gamma\exp[m(t-1)(2\varepsilon)^{-1}] - \mu^2\eta\gamma\exp(-\mu t)\} - f_{y'}(t,z,\cdot\cdot,u'',\varepsilon) \\
&\qquad \times \{-(\eta/2)\Gamma\exp[m(t-1)(2\varepsilon)^{-1}] + \mu\eta\gamma\exp(-\mu t)\} - f(t,\cdot\cdot\cdot,u',u'',\varepsilon) \\
&\qquad \times \{\theta(\varepsilon\eta/m)\Gamma\exp[m(t-1)(2\varepsilon)^{-1}] + \theta\eta\gamma\exp(-\mu t)\} - f(t,u,u',u'',\varepsilon) \ ,
\end{aligned}$$

where $|\theta| \le 1$. Making use of our size restrictions for the partial derivatives of f and recalling σ we find

$$\begin{aligned}
&\varepsilon\alpha''' - f(t,z,\alpha',\alpha'',\varepsilon) \\
&\quad \ge -\sigma\eta - \sigma\eta\varepsilon^{-1}\exp[rm(t-1)\varepsilon^{-1}] - (m^2/8)\eta\varepsilon^{-1}\Gamma\exp[m(t-1)(2\varepsilon)^{-1}] \\
&\qquad + \varepsilon\mu^3\eta\gamma\exp(-\mu t) + (m^2/4)\eta\varepsilon^{-1}\Gamma\exp[m(t-1)(2\varepsilon)^{-1}] + m\mu^2\eta\gamma\exp(-\mu t) \\
&\qquad - (k/2)\eta\Gamma\exp[m(t-1)(2\varepsilon)^{-1}] + k\mu\eta\gamma\exp(-\mu t) \\
&\qquad - (\ell/m)\varepsilon\eta\Gamma\exp[m(t-1)(2\varepsilon)^{-1}] - \ell\eta\gamma\exp(-\mu t) \\
&\quad = [(m^2/8)\varepsilon^{-1} - (k/2) - (\ell/m)\varepsilon]\eta\Gamma\exp[m(t-1)(2\varepsilon)^{-1}] \\
&\qquad - \sigma\eta\varepsilon^{-1}\exp[rm(t-1)\varepsilon^{-1}] + (\varepsilon\mu^3 + m\mu^2 + k\mu - \ell)\eta\gamma\exp(-\mu t) - \sigma\eta \ .
\end{aligned}$$

If we have $\gamma \ge 1$ also, and $\exp(-\mu t)$ being monotone increasing, this reduces to having

$$\begin{aligned}
&\varepsilon\alpha''' - f(t,z,\alpha',\alpha'',\varepsilon) \\
&\quad \ge [(m^2/8) - (k\varepsilon/2) - (\ell\varepsilon^2/m)]\eta\varepsilon^{-1}\Gamma\exp[m(t-1)(2\varepsilon)^{-1}] \\
&\qquad - \sigma\eta\varepsilon^{-1}\exp[rm(t-1)\varepsilon^{-1}] \qquad .
\end{aligned}$$

As $r \ge 1$, $\exp[m(t-1)(2\varepsilon)^{-1}] \ge \exp[rm(t-1)\varepsilon^{-1}]$, $0 \le t \le 1$, hence, if Γ is chosen such that $\Gamma \ge 9\sigma m^{-2}$ we have $\varepsilon\alpha''' - f(t,z,\alpha',\alpha'',\varepsilon) \ge 0$, therefore α is a lower solution. By symmetry, β is an upper solution of (4.1). Theorem 1.1 is now applied, and we conclude that for each small ε, there is a solution $y(t,\varepsilon)$ of (4.1), (4.2′) with $\alpha(t) \le y(t,\varepsilon) \le \beta(t)$ and $\alpha'(t) \le y'(t,\varepsilon) \le \beta'(t)$, $0 \le t \le 1$.

The above problem, when $f_{y'} \ge k > 0$, a solution also exists; the only adjustment of proof is to define μ to satisfy $m\mu^2 - k\mu - \ell - \sigma - 1 = 0$.

The problem (4.1), (4.2), viewed in the light of an approximate solution satisfying (4.5), (4.6) offers a bit of a surprise in that the construction of bounding functions is parallel to those we used to determine the existence of a solution for (4.1), (4.2′).

Theorem 4.4. Suppose solutions to initial value problems for (4.1) extend to $[0,1]$ or become unbounded. Also assume:

(1) For each $\varepsilon > 0$, small, there exists a function $u = u(t,\varepsilon)$ satisfying (4.5), (4.6) with $\eta > 0$, $\eta = o(1)$ and $r > 1$;

(2) f is continuous in (t,y,y',y'',ε) and of class C^1 with respect to y, y' and y'' such that $f_{y''} \ge 0$ and $f_{y'} > 0$. And for a region R: $0 \le t \le 1$, $|u - y| \le d_1$, $|u' - y'| \le d_2$, $|u'' - y''| \le d_3\{1 + \varepsilon^{-1}\eta \times \exp[m(t-1)\varepsilon^{-1}]\}$, $0 \le \varepsilon \le \varepsilon_1$ ($d_i,\varepsilon_1 > 0$);

(3) there exists an $m > 0$ such that $f_{y''} \ge m > 0$, in R;

(4) there exists an $k > 0$ such that $f_{y'}(t,y,y',u'',\varepsilon) \ge k$ in R;

(5) $f_y(t,y,u',u'',\varepsilon) = O(1)$ in R.

Then there exists an $\varepsilon_0 > 0$, such that for all ε, $0 < \varepsilon \le \varepsilon_0$

there exists a solution $y(t,\varepsilon)$ of (4.1), (4.2) with

$$y(t,\varepsilon) - u(t,\varepsilon) = O(\eta) .$$

Proof: In order to make use of our basic existence theorems (here Theorem 1.6), we construct lower and upper solutions to (4.1).

$$\alpha(t) = u(t,\varepsilon) - (\varepsilon\eta/m)\Gamma\exp[m(t-1)(2\varepsilon)^{-1}] - \eta\gamma\exp(\mu t) ,$$

$$\beta(t) = u(t,\varepsilon) + (\varepsilon\eta/m)\Gamma\exp[m(t-1)(2\varepsilon)^{-1}] + \eta\gamma\exp(\mu t) ,$$

where Γ and γ are positive constants, to be declared later, and μ is the positive root of $m\mu^2 + k\mu - \ell - \sigma - 1 = 0$, ℓ the bound in assumption (5) and σ arising from the order relation in (4.5). We have $\alpha < \beta$; and so our first task is to set γ so that $\alpha(0) \le A \le \beta(0)$, $\alpha'(0) \le A' \le \beta'(0)$ and $\alpha(1) \le B \le \beta(1)$.

$$\beta(0) = u(0,\varepsilon) + (\varepsilon\eta/m)\Gamma\exp[-m(2\varepsilon)^{-1}] + \eta\gamma$$

$$\ge A - \sigma_1\eta + (\varepsilon\eta/m)\Gamma\exp[-m(2\varepsilon)^{-1}] + \eta\gamma \ge A \quad \text{if} \quad \gamma \ge \sigma_1 ,$$

$$\beta(1) = u(1,\varepsilon) + (\varepsilon\eta/m)\Gamma + \eta\gamma\exp(\mu)$$

$$\ge B - \sigma_2\eta + (\varepsilon\eta/m)\Gamma + \eta\gamma\exp(\mu) \ge B \quad \text{if} \quad \gamma \ge \sigma_2/\exp(\mu) , \text{ and}$$

$$\beta'(0) = u'(0,\varepsilon) + (\eta/2)\Gamma\exp[-m(2\varepsilon)^{-1}] + \eta\mu\gamma$$

$$\ge A' - \sigma_3\eta + (\eta/2)\Gamma\exp[-m(2\varepsilon)^{-1}] + \eta\mu\gamma \ge A' \quad \text{if} \quad \gamma \ge \sigma_3/\mu .$$

The same relations on γ apply in the case of α , and so if we let $\gamma = \max\{\sigma_1,\sigma_2/\exp(\mu),\sigma_3/\mu\}$ our boundary conditions are met for α and β . With z such that $\alpha(t) \le z \le \beta(t)$, we show that β is an upper solution. Substituting z , β' , β'' and $\varepsilon\beta'''$ into (4.1) and expanding we obtain:

$$f(t,x,\beta',\beta'',\varepsilon) - \varepsilon\beta'''$$

$$= f_{y''}(t,z,\beta',\cdot,\varepsilon)$$

$$\times \{(m/4)\varepsilon^{-1}\eta\Gamma\exp[m(t-1)(2\varepsilon)^{-1}] + \eta\mu^2\gamma\exp(\mu t)\} + f_{y'}(t,z,\cdot\cdot,u'',\varepsilon)$$

$$\times \{(\eta/2)\Gamma\exp[m(t-1)(2\varepsilon)^{-1}] + \eta\mu\gamma\exp(\mu t)\} + f_y(t,\cdot\cdot\cdot,u',u'',\varepsilon)$$

$$\times \{\theta(\varepsilon\eta/m)\Gamma\exp[m(t-1)(2\ \ )^{-1}] + \theta\eta\gamma\exp(\mu t)\} + f(t,u,u',u'',\varepsilon)$$

$$- \varepsilon u''' - (m^2\eta/8)\varepsilon^{-1}\Gamma\exp[m(t-1)(2\varepsilon)^{-1}] - \varepsilon\eta\mu^3\gamma\exp(\mu t) \quad ,$$

where $|\theta| \le 1$. Noting conditions (1), (3), (4) and (5) we ascertain

$$f(t,z,\beta',\beta'',\varepsilon) - \varepsilon\beta'''$$

$$\ge (m^2/4)\varepsilon^{-1}\eta\Gamma\exp[m(t-1)(2\varepsilon)^{-1}] + m\mu^2\eta\gamma\exp(\mu t)$$

$$+ (k/2)\eta\Gamma\exp[m(t-1)(2\varepsilon)^{-1}] + k\mu\eta\gamma\exp(\mu t)$$

$$- (\ell/m)\varepsilon\eta\Gamma\exp[m(t-1)(2\varepsilon)^{-1}] - \ell\eta\gamma\exp(\mu t)$$

$$- (m^2/8)\varepsilon^{-1}\eta\Gamma\exp[m(t-1)(2\varepsilon)^{-1}] - \varepsilon\mu^3\eta\gamma\exp(\mu t) - \sigma\eta - \sigma\eta\varepsilon^{-1}\exp[rm(t-1)\varepsilon^{-1}]$$

$$= [(m^2/8) + (k/2)\varepsilon - (\ell/m)\varepsilon^2]\varepsilon^{-1}\eta\Gamma\exp[m(t-1)(2\varepsilon)^{-1}]$$

$$+ (-\varepsilon\mu^3 + m\mu^2 + k\mu - \ell)\eta\gamma\exp(\mu t) - \sigma\eta - \sigma\eta\varepsilon^{-1}\exp[rm(t-1)\varepsilon^{-1}] \quad .$$

As $\mu > 0$ is fixed, and if also $\gamma \ge 1$, we find that $f(t,z,\beta',\beta'',\varepsilon) - \varepsilon\beta''' \ge (m^2/9)\varepsilon^{-1}\eta\Gamma\ \exp[m(t-1)(2\varepsilon)^{-1}] - \sigma\varepsilon^{-1}\eta\exp[rm(t-1)\varepsilon^{-1}] \ge 0$, since $r \ge 1$ and if $\Gamma \ge (9\sigma/m^2)$.

Symmetry establishes that α is a lower solution, and so Theorem 1.6 applies and we conclude that there exists a solution $y(t,\varepsilon)$ with $\alpha(t) \le y(t,\varepsilon) \le \beta(t)$.

In the above proof, we note that the definition of μ actually would force our inequalities to be sharp. Whence, if f were independent of y' , Theorem 1.7 could be used with note that $k = 0$.

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