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THE UNIVERSITY OF OKLAHOMA, PH.D., 1979

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THE UNIVERSITY OF OKLAHOMA

GRADUATE COLLEGE

GENERAL RELATIVISTIC PULSATION THEORY FOR CHARGED FLUIDS

A DISSERTATION

SUBMITTED TO THE GRADUATE FACULTY

in partial fulfillment of the requirements for the

degree of

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BY

IRVING GLAZER

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GENERAL RELATIVISTIC PULSATION THEORY FOR CHARGED FLUIDS

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ABSTRACT

General Relativistic Pulsation Theory as developed by Chandrasekhar is extended to include charged fluids with arbitrary charge distributions.

We then obtain an equilibrium solution for the charged homogeneous model, and use the theory developed earlier to investigate the effects of electric charge upon dynamical stability. We find that electric charge decreases the minimum radius at which dynamical stability is possible, and increases the frequency of the fundamental mode of the permitted pulsation modes.

Finally, we investigate the stability of Bonner's model for charged dust. His model is found to be neutrally stable.

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GENERAL RELATIVISTIC PULSATION THEORY FOR CHARGED FLUIDS

CHAPTER I.

INTRODUCTION

This study is concerned with an extension of general relativistic pulsation theory to charged fluids. The application of General Relativity to pulsation theory is due to Chandrasekhar (1), and is a recent development. This is because the gravitational fields of most stars are usually very weak, and consequently the use of Newtonian pulsation theory is well justified. However, this is not the case for stellar systems such as white dwarfs, neutron stars, and supermassive stars. For these, and astrophysical systems such as black holes, general relativistic theory is indispensable.

Most of the observed pulsating stars appear to be doing so predominantly in a radial, spherically symmetric mode. We shall, therefore, concentrate exclusively on this type of motion. Rotation and non-radial oscillations produce enormous complications in pulsation theory. Their roles are not yet clarified, and we will not consider them.

We shall also assume that the oscillations are adiabatic. Of course, this is not strictly true. Nevertheless, it is generally used and gives an excellent description of the gross dynamical properties of stars, in particular their periods. The reason for this is that the time necessary for departure from thermodynamic equilibrium is enormously greater than the pulsation period of a star, hence not much heat can leak out during a period.

It is worth pointing out that a star pulsating radially and adiabatically is a perfectly conservative system. Spherically symmetric oscillations do not permit losses through gravitational radiation, and since we are assuming a perfect fluid, there are no dissipative losses such as viscosity. Hence, a theory based upon spherically symmetric, adiabatic oscillations cannot provide information about the origin or cause of the oscillations.

We mentioned above that the development of a formal body of general relativistic pulsation theory is due to Chandrasekhar. Using a normal mode analysis, he developed the theory of adiabatic, infinitesimal, radial oscillations of a gas sphere and was able to establish a criterion for the onset of dynamical instability.

Chandrasekhar's method begins by assuming that the quantities $\mathcal{F}_n$ which characterize the dynamical state of the fluid differ from the equilibrium quantities $\mathcal{F}_{0n}$ only to first order. Similarly, the fluid displacements are taken as first order. Then the time dependent equations of motion are linear equations with the first order differences $\delta\mathcal{F}_n = \mathcal{F}_n - \mathcal{F}_{0n}$ as dependent variables. If the motions are isentropic and hence reversible, the time dependence can be factored out by assuming a functional dependence of the form $\delta\mathcal{F}(r, x^0) = \delta\mathcal{F}(r)\, e^{i\sigma x^0}$. The equations for $\delta\mathcal{F}_n(r)$ are then transformed into a Sturm-Liouville eigenvalue problem with σ^2 as the eigenvalue. It is well known that the eigenvalues form an infinite sequence, beginning with some lowest value which may be negative. If the lowest value vanishes, the condition is called neutral or labile equilibrium.

A system is considered to be dynamically stable with respect to small perturbations if it has no unstable modes of oscillation. Hence, the condition for dynamical stability is $\sigma^2 > 0$. What is the physical motivation for determining the dynamical stability for a star? It is simply that a body which is unstable with respect to small perturbations is incapable of existence in general.

In Newtonian theory, the condition for dynamical stability of a spherical star with a constant ratio of specific heats γ is $\gamma > 4/3$. The general relativistic analogue of this, as Chandrasekhar showed, is

$$\gamma - 4/3 > K \left[\frac{2GM}{c^2}\right]$$

where M is the star's mass, G is the constant of gravitation, c is the velocity of light, and K is a numerical factor depending upon the mass distribution of the star and generally lying between 0.5 and 1.5. For the uncharged, homogeneous model, i.e. a perfect fluid sphere of constant energy density, Chandrasekhar obtained as the condition for dynamical stability.

$$R > \frac{19}{42(\gamma - 4/3)}\left[\frac{2GM}{c^2}\right]$$

where R is the star's radius. The physical interpretation of this condition is that even with $\gamma > 4/3$, the star will become dynamically unstable if its radius does not satisfy the above inequality. It is quite remarkable that general relativistic effects which are too small to affect the static structure of a star can nevertheless have an important effect upon its dynamical stability.

Now we briefly discuss the motivation for this investigation and summarize the following chapters.

Stettner (2) appears to have been the first to use Chandrasekhar's work in order to investigate how electric charge influences dynamical stability. However, since Chandrasekhar's theory deals only with uncharged fluids, Stettner's choice of charge distribution was limited to being on the surface of the sphere. The results of his work indicate that the addition of a small amount of charge to a system might actually increase its dynamical stability. It seems desirable therefore, to extend Chandrasekhar's theory to charged fluids with arbitrary charge distributions in order to more fully investigate what General Relativity has to say about the effects of electric charge upon dynamical stability.

In the next chapter we extend Chandrasekhar's theory to include charged fluids. In Chapter III, the theory is applied to the charged homogeneous model, and the effects of charge upon dynamical stability are explicitly shown. In Chapter IV, we analyze Bonner's (3) model for charged dust and shown it to be neutrally stable.

PULSATION THEORY FOR CHARGED FLUIDS

Consider a perfect fluid sphere with matter energy density ϵ, and electrical energy density η. For the metric, we choose

$$ds^2 = e^{\nu}(dx^0)^2 - e^{\lambda}(dr)^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \quad (2\text{-}1)$$

The field equations are given by (4)

$$\frac{8\pi G}{c^4} T_0^{\;0} = -e^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda'}{r}\right) + \frac{1}{r^2} \quad (2\text{-}2)$$

$$\frac{8\pi G}{c^4} T_1^{\;1} = -e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) + \frac{1}{r^2} \quad (2\text{-}3)$$

$$\frac{8\pi G}{c^4} T_0^{\;1} = -e^{-\lambda}\frac{\dot{\lambda}}{r} \quad (2\text{-}4)$$

$$\frac{8\pi G}{c^4} T_2^{\;2} = \frac{8\pi G}{c^4} T_3^{\;3} = -e^{-\lambda}\left(\frac{\nu''}{2} + \frac{(\nu')^2}{4} + \frac{\nu' - \lambda'}{2r} - \frac{\nu'\lambda'}{4}\right)$$
$$+ e^{-\nu}\left(\frac{\ddot{\lambda}}{2} + \frac{(\dot{\lambda})^2}{4} - \frac{\dot{\nu}\dot{\lambda}}{4}\right) \quad (2\text{-}5)$$

where the primes indicate differentiation with respect to r, and the dots with respect to x^0.

The energy-momentum tensor for the fluid is given by

$$M_\mu^{\;\nu} = (\epsilon + P)u^\nu u_\mu - \delta_\mu^{\;\nu} P \quad (2\text{-}6)$$

where P is the pressure in the rest frame of the fluid, and u^ν is the four-velocity of the fluid.

For the metric we have chosen:

$$u^{\nu}u_{\nu} = 1, \qquad u^{\nu} = \frac{dx^{\nu}}{ds} \tag{2-7}$$

The electromagnetic energy tensor is given by

$$E_{\mu}{}^{\nu} = \frac{1}{4\pi}\left[F_{\mu\alpha}F^{\alpha\nu} + \frac{1}{4}\delta_{\mu}{}^{\nu}F_{\alpha\beta}F^{\alpha\beta}\right] \tag{2-8}$$

where $F_{\nu\mu}$ is the electromagnetic field tensor. Because of spherical symmetry, the only non-zero field component is F_{10}, where

$$F_{10} = -\frac{4\pi}{r^2}e^{\frac{\nu+\lambda}{2}}\int_0^r \rho r^2 e^{\lambda/2}dr \tag{2-9}$$

In (2-9), ρ is the proper charge density.

Putting F_{10} into (2-8), we get for $E_0{}^0$

$$\begin{aligned}
E_0{}^0 &= \frac{1}{4\pi}\left[F_{0\alpha}F^{\alpha 0} + \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\right] \\
&= \frac{1}{4\pi}\left[F_{01}F^{10} + \frac{1}{4}\left(F_{01}F^{01} + F_{10}F^{10}\right)\right] \\
&= \frac{1}{4\pi}\left[-F_{10}F^{10} + \frac{1}{4}\left(F_{10}F^{10} + F_{10}F^{10}\right)\right] \\
&= -\frac{F_{10}F^{10}}{8\pi} = -g^{00}g^{11}\frac{(F_{10})^2}{8\pi} \\
&= \frac{e^{-(\nu+\lambda)}}{8\pi}(F_{10})^2
\end{aligned} \tag{2-10}$$

where $g^{\mu\nu}$ are the contravariant components of the metric tensor.

For $E_1{}^1$, we obtain:

$$E_1{}^1 = \frac{1}{4\pi}\left[F_{1\alpha}F^{\alpha 1} + \frac{1}{4}\left(F_{01}F^{01} + F_{10}F^{10}\right)\right]$$

$$= \frac{1}{4\pi}\left[-F_{10}F^{10} + \frac{F_{10}F^{10}}{2}\right]$$

$$= \frac{F_{10}F^{10}}{8\pi} = E_0{}^0$$

Since η is the electrical energy density, we get from the definition of the electromagnetic energy tensor

$$E_0{}^0 = E_1{}^1 = \eta \tag{2-11}$$

Similarly:

$$E_2{}^2 = E_3{}^3 = -\eta \tag{2-12}$$

The other components of $E_\mu{}^\nu$ are zero.

A consequence of the field equations is the vanishing of the covariant divergence of the total energy-momentum tensor.

$$T_\mu{}^\nu{}_{;\nu} = \left(M_\mu{}^\nu + E_\mu{}^\nu\right)_{;\nu} = 0 \tag{2-13}$$

The covariant divergence of a tensor is given by:

$$T_{\mu\ ;\nu}^{\ \nu} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^{\nu}} \left(\sqrt{-g}\, T_{\mu}^{\ \nu} \right) - \frac{T^{\nu\alpha}}{2} \frac{\partial g_{\nu\alpha}}{\partial x^{\mu}} \tag{2-14}$$

where g is the determinant of the metric tensor, and

$$\sqrt{-g} = r^2 \sin\theta\, e^{\frac{\nu+\lambda}{2}} \tag{2-15}$$

Letting $\mu = 1$, we expand (2-13) as

$$T_{1;\nu}^{\ \nu} = T_{1;0}^{\ 0} + T_{1;1}^{\ 1} + T_{1;2}^{\ 2} + T_{1;3}^{\ 3} = 0 \tag{2-16}$$

We consider, in turn, each term on the r.h.s.

$$T_{1;0}^{\ 0} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^0} \left(\sqrt{-g}\, T_1^{\ 0} \right) - \frac{T^{0\alpha}}{2} \frac{\partial g_{0\alpha}}{\partial r}$$

$$= \frac{1}{\sqrt{-g}} \left[\sqrt{-g} \frac{\partial T_1^{\ 0}}{\partial x^0} + T_1^{\ 0} \sqrt{-g} \frac{\partial}{\partial x^0} \left(\frac{\nu+\lambda}{2} \right) \right] - \frac{T^{00}}{2} \frac{\partial g_{00}}{\partial r}$$

$$= \frac{\partial T_1^{\ 0}}{\partial x^0} + T_1^{\ 0} \frac{\partial}{\partial x^0} \left(\frac{\nu+\lambda}{2} \right) - \frac{e^{\nu}}{2} \nu' g^{00} T_0^{\ 0}$$

$$= \frac{\partial T_1^{\ 0}}{\partial x^0} + T_1^{\ 0} \frac{\partial}{\partial x^0} \left(\frac{\nu+\lambda}{2} \right) - \frac{\nu'}{2} T_0^{\ 0} \tag{2-17}$$

Similarly

$$T_{1;1}^{1} = \frac{\partial T_1^1}{\partial r} + T_1^1 \frac{\nu'}{2} + \frac{2}{r} T_1^1 \tag{2-18}$$

$$T_{1;2}^{2} = -\frac{T_2^2}{r} \tag{2-19}$$

$$T_{1;3}^{3} = -\frac{T_3^3}{r} \tag{2-20}$$

Substituting (2-17, 2-18, 2-19, 2-20) into (2-16) gives

$$\frac{\partial T_1^0}{\partial x^0} + \frac{\partial T_1^1}{\partial r} + \frac{T_1^0}{2}\frac{\partial}{\partial x^0}(\nu+\lambda) + \frac{\nu'}{2}\left(T_1^1 - T_0^0\right) + \frac{2T_1^1}{r} - \frac{T_2^2}{r} - \frac{T_3^3}{r} = 0 \tag{2-21}$$

The Equilibrium Equations

In the equilibrium state, none of the quantities depend on x^0, and of the four-velocities only u^0 is not zero. Designating the equilibrium state by a zero subscript, the total energy-momentum tensor is given by

$$T_0^0 = \epsilon_0 + \eta_0, \quad T_1^1 = \eta_0 - P_0, \quad T_2^2 = T_3^3 = -\eta_0 - P_0 \tag{2-22}$$

Equations (2-2, 2-3, 2-21) become

$$\frac{d}{dr}\left(r e^{-\lambda_0}\right) = 1 - \frac{8\pi G}{c^4}\left(\epsilon_0 + \tau_0\right) r^2 \tag{2-23}$$

$$\frac{e^{-\lambda_0}}{r}\,\frac{d\nu_0}{dr} = \frac{1}{r^2}\left(1 - e^{-\lambda_0}\right) + \frac{8\pi G}{c^4}\left(P_0 - \tau_0\right) \tag{2-24}$$

$$P_0' - \tau_0' = -\frac{\nu_0'}{2}\left(\epsilon_0 + P_0\right) + \frac{4\tau_0}{r} \tag{2-25}$$

Using (2-2) and (2-3), we form

$$\frac{8\pi G}{c^4}\left(T_0^{\,0} - T_1^{\,1}\right) = \frac{e^{-\lambda_0}}{r}\left(\lambda_0' + \nu_0'\right) \tag{2-26}$$

Substituting (2-22) in the above, we get

$$\frac{8\pi G}{c^4}\left(\epsilon_0 + P_0\right) = \frac{e^{-\lambda_0}}{r}\left(\lambda_0' + \nu_0'\right) \tag{2-27}$$

<u>The Perturbation Equations</u>

Now suppose that the system is perturbed so that infinitesimal radial oscillations result. Since spherical symmetry is preserved, we can write the metric as

$$ds^2 = e^{\nu(r,x^0)}\left(dx^0\right)^2 - e^{\lambda(r,x^0)}\,dr^2 \tag{2-28}$$

In obtaining the equations governing the perturbed state, we shall retain only linear terms in the motions. Let

$$\lambda(r,x^0) = \lambda_0(r) + \delta\lambda(r,x^0)\ ,\quad \nu(r,x^0) = \nu_0(r) + \delta\nu(r,x^0) \tag{2-29}$$

Substituting (2-29) into (2-28), we get

$$1 = e^{\nu_0+\delta\nu}\left(\frac{dx^0}{ds}\right)^2 - e^{\lambda_0+\delta\lambda}\left(\frac{dr}{ds}\right)^2 \tag{2-30}$$

The components of the four-velocity are given by

$$U^0 = \frac{dx^0}{ds}\ ,\quad U^1 = \frac{dr}{ds} = \frac{dr}{dx^0}\,\frac{dx^0}{ds} = v\,\frac{dx^0}{ds} \tag{2-31}$$

where

$$v = \frac{dr}{dx^0} \tag{2-32}$$

is the velocity in the radial direction in units of the velocity of light. Putting (2-31) into (2-30) gives

$$\frac{dx^0}{ds} = \left[e^{\nu_0+\delta\nu} - v^2\, e^{\lambda_0+\delta\lambda}\right]^{-1/2} \tag{2-33}$$

Since the second term under the radical is second order in the motion, we have

$$u^0 = \frac{dx^0}{ds} = e^{-\nu_0/2}\left(1-\frac{\delta\nu}{2}\right) \tag{2-34}$$

$$u_0 = g_{00}\, u^0 = e^{\nu_0+\delta\nu} u^0 = e^{\nu_0/2}\left(1+\frac{\delta\nu}{2}\right) \tag{2-35}$$

$$u^1 = v\frac{dx^0}{ds} = v\, e^{-\nu_0/2} \tag{2-36}$$

$$u_1 = g_{11}\, u^1 = -e^{\lambda_0+\delta\lambda} u^1 = -v\, e^{\lambda_0-\nu_0/2} \tag{2-37}$$

Next we determine the changes in the total energy-momentum tensor which result from the oscillations. Let

$$P(r,x^0) = P_0(r) + \delta P(r,x^0)\,, \quad \epsilon = \epsilon_0 + \delta\epsilon, \quad \eta = \eta_0 + \delta\eta \tag{2-38}$$

where $\delta P, \delta\epsilon, \delta\eta$ are the Eulerian changes in the pressure and energy densities as a result of the oscillations. Then the components of the total energy-momentum tensor become

$$T_0^{\ 0} = \epsilon_0 + \eta_0 + \delta\epsilon + \delta\eta \tag{2-39}$$

$$T_1^{\ 1} = \eta_0 - P_0 + \delta\eta - \delta P \tag{2-40}$$

$$T_0^{\ 1} = (\epsilon_0 + P_0)\, v \tag{2-41}$$

$$T_2^{\ 2} = T_3^{\ 3} = -\eta_0 - P_0 - \delta\eta - \delta P \tag{2-42}$$

It is convenient to first evaluate $\delta\eta$. Let

$$F_{10} = E_0, \quad \mathcal{F}_{10} = E_0 + \delta E, \quad \rho = \rho_0 + \delta\rho \tag{2-43}$$

Maxwell's source equations in covariant form are

$$\frac{\partial}{\partial x^{\nu}}\left(\sqrt{-g}\,\mathcal{F}^{\mu\nu}\right) = -\frac{4\pi}{c}\sqrt{-g}\,J^{\mu} \tag{2-44}$$

where the current density J^{μ} is given by

$$J^{\mu} = \rho c \frac{dx^{\mu}}{ds} \tag{2-45}$$

Since $\mathcal{F}^{10}$ is the only non-zero field component, (2-44) becomes

$$\frac{\partial}{\partial x^{0}}\left(\sqrt{-g}\,\mathcal{F}^{10}\right) = -4\pi\sqrt{-g}\,\rho u^{1} \tag{2-46}$$

Now:

$$\mathcal{F}^{10} = g^{00}g^{11}\mathcal{F}_{10} = -e^{-(\nu+\lambda)}\mathcal{F}_{10} = -e^{-(\nu_0+\lambda_0)}(1-\delta\nu-\delta\lambda)(E_0+\delta E)$$

or

$$\mathcal{F}^{10} = -e^{-(\nu_0+\lambda_0)}\left(E_0+\delta E - E_0\delta\nu - E_0\delta\lambda\right) \tag{2-47}$$

to first order.

Putting (2-47) into the left side of (2-46) gives

$$\frac{\partial}{\partial x^0}\left(\sqrt{-g}\,\mathfrak{J}^{10}\right) = -\frac{\partial}{\partial x^0}\left\{r^2\sin\theta\, e^{-\frac{\nu_0+\lambda_0}{2}}\left(1+\frac{\delta\nu}{2}+\frac{\delta\lambda}{2}\right)\left[e^{(\nu_0+\lambda_0)}\left(E_0+\delta E-E_0\delta\nu-E_0\delta\lambda\right)\right]\right\}$$

$$= -\frac{\partial}{\partial x^0}\left\{r^2\sin\theta\, e^{-\frac{\nu_0+\lambda_0}{2}}\left[E_0+\delta E-E_0\delta\nu-E_0\delta\lambda+E_0\frac{\delta\nu}{2}+E_0\frac{\delta\lambda}{2}\right]\right\}$$

Since $r^2\sin\theta\, e^{-\frac{\nu_0+\lambda_0}{2}}$ is not a function of x^0, we get

$$\frac{\partial}{\partial x^0}\left(\sqrt{-g}\,\mathfrak{J}^{10}\right) = -r^2\sin\theta\, e^{-\frac{\nu_0+\lambda_0}{2}}\frac{\partial}{\partial x^0}\left[\delta E-E_0\frac{\delta\nu}{2}-E_0\frac{\delta\lambda}{2}\right] \qquad (2\text{-}48)$$

The right side of (2-46) becomes

$$-4\pi\sqrt{-g}\,\rho u' = -4\pi r^2\sin\theta\, e^{\frac{\nu_0+\lambda_0}{2}}\left(1+\frac{\delta\nu}{2}+\frac{\delta\lambda}{2}\right)\rho_0 v\, e^{-\nu_0/2}$$

or

$$-4\pi\sqrt{-g}\,\rho u' = -4\pi r^2\sin\theta\, e^{\lambda_0/2}\rho_0\frac{\partial\xi}{\partial x^0} \qquad (2\text{-}49)$$

where

$$v = \frac{\partial\xi}{\partial x^0} \qquad (2\text{-}50)$$

Equation (2-50) defines the "Lagrangian displacement" $\xi(r, x^0)$. It is the displacement of a fluid element from r to $r+\xi$ as a result of the oscillation. Equating (2-48) to (2-49) gives

$$\delta E - E_0 \frac{\delta\nu}{2} - E_0 \frac{\delta\lambda}{2} = 4\pi\rho_0\, e^{(\lambda_0+\nu_0/2)}\xi$$

or

$$2\frac{\delta E}{E_0} - \delta\nu - \delta\lambda = \frac{8\pi}{E_0}\, e^{(\lambda_0+\nu_0/2)}\rho_0\,\xi \tag{2-51}$$

Referring to (2-10) the time dependent case for the electromagnetic energy tensor is

$$E_0{}^0 = \frac{e^{-(\nu+\lambda)}}{8\pi}(F_{10})^2 = \frac{e^{-(\nu_0+\lambda_0)}}{8\pi}E_0^2\left(1 + \frac{2\delta E}{E_0} - \delta\nu - \delta\lambda\right) \tag{2-52}$$

Comparing (2-51) with (2-52), we see that

$$E_0{}^0 = \frac{e^{-(-\nu_0+\lambda_0)}}{8\pi}E_0^2 + E_0\rho_0\, e^{-\nu_0/2}\xi \tag{2-53}$$

Therefore, the change in the components of the electromagnetic energy tensor due to the oscillations is

$$\delta\eta = E_0\rho_0\, e^{-\nu_0/2}\xi \tag{2-54}$$

Next we return to (2-2) and rewrite it as

$$\frac{\partial}{\partial r}\left(r e^{-\lambda}\right) = 1 - \frac{8\pi G}{c^4} r^2 T_0{}^0$$

or

$$\frac{\partial}{\partial r}\left[r e^{-\lambda_0}(1-\delta\lambda)\right] = 1 - \frac{8\pi G r^2}{c^4}\left[\epsilon_0 + \eta_0 + \delta\epsilon + \delta\eta\right]$$

cancelling out the static solution, we are left with

$$\frac{\partial}{\partial r}\left[r e^{-\lambda_0}\,\delta\lambda\right] = \frac{8\pi G}{c^4} r^2\left[\delta\epsilon + \delta\eta\right] \qquad (2\text{-}55)$$

We return to (2-3) and rewrite it as

$$\frac{e^{-\lambda}}{r}\frac{\partial\nu}{\partial r} = \frac{1}{r^2}\left(1 - e^{-\lambda}\right) - \frac{8\pi G}{c^4} T_1{}^1$$

or

$$\frac{e^{-\lambda_0}}{r}(1-\delta\lambda)\left[\frac{\partial\nu_0}{\partial r} + \frac{\partial\delta\nu}{\partial r}\right] = \frac{1}{r^2}\left[1 - e^{-\lambda_0}(1-\delta\lambda)\right] - \frac{8\pi G}{c^4}\left[\eta_0 - P_0 + \delta\eta - \delta P\right]$$

Collecting terms:

$$\frac{e^{-\lambda_0}}{r}\left[\frac{d\nu_0}{dr} + \frac{\partial\delta\nu}{\partial r} - \delta\lambda\frac{d\nu_0}{dr}\right] = \frac{1}{r^2}\left[1 - e^{-\lambda}(1-\delta\lambda)\right] - \frac{8\pi G}{c^4}\left[\eta_0 - P_0 + \delta\eta - \delta P\right]$$

Cancelling out the static solution, we have

$$\frac{e^{-\lambda_c}}{r}\left[\frac{\partial \delta\nu}{\partial r} - \delta\lambda \frac{d\nu_c}{dr}\right] = \frac{e^{-\lambda_c}}{r^2}\,\delta\lambda - \frac{8\pi G}{c^4}\left[\delta\eta - \delta P\right] \tag{2-56}$$

From (2-4) and (2-41), we get

$$\frac{e^{-\lambda_c}}{r}\,\frac{\partial \delta\lambda}{\partial x^0} = -\frac{8\pi G}{c^4}\left[\epsilon_c + P_0\right]\frac{\partial \xi}{\partial x^0} \tag{2-57}$$

We can integrate (2-57) directly to get

$$\frac{e^{-\lambda_c}}{r}\,\delta\lambda = -\frac{8\pi G}{c^4}\left[\epsilon_0 + P_c\right]\xi \tag{2-58}$$

The integration constant is zero because $\delta\lambda = 0$ when $\xi = 0$.

Comparing (2-58) with (2-27), we see that

$$\delta\lambda = -\left(\lambda_c' + \nu_c'\right)\xi \tag{2-59}$$

Combining (2-55) and (2-58) produces

$$\delta\epsilon = -\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2(\epsilon_0 + P_0)\xi\right] - \delta\eta \tag{2-60}$$

We can rewrite (2-60) as

$$\delta\epsilon = -\xi\frac{d\epsilon_0}{dr} - \xi\frac{dP_0}{dr} - \frac{(\epsilon_0+P_0)}{r^2}\frac{\partial}{\partial r}(r^2\xi) - \delta\eta \qquad (2\text{-}61)$$

Substituting for P_0' from (2-25) gives

$$\delta\epsilon = -\xi\frac{d\epsilon_0}{dr} - (\epsilon_0+P_0)\frac{e^{\nu_0/2}}{r^2}\frac{\partial}{\partial r}(r^2 e^{-\nu_0/2}\xi) - \xi\frac{d\eta_0}{dr} - \frac{4\eta_0}{r}\xi - \delta\eta$$

Next we substitute (2-58) into (2-56)

$$\frac{e^{-\lambda_0}}{r}\frac{\partial\delta\nu}{\partial r} = \frac{8\pi G}{c^4}\left[\delta P - \delta\eta - (\epsilon_0+P_0)\left(\frac{d\nu_0}{dr}+\frac{1}{r}\right)\xi\right] \qquad (2\text{-}62)$$

In (2-62) we substitute for $\frac{e^{-\lambda_0}}{r}$ obtained from (2-27). We get

$$(\epsilon_0+P_0)\frac{\partial\delta\nu}{\partial r} = \left[\delta P - \delta\eta - (\epsilon_0+P_0)\left(\frac{d\nu_0}{dr}+\frac{1}{r}\right)\xi\right]\left(\frac{d\nu_0}{dr}+\frac{d\lambda_0}{dr}\right) \qquad (2\text{-}63)$$

Now we evaluate (2-21) term by term.

$$\frac{\partial T_1^{\,0}}{\partial x^0} = -\frac{\partial}{\partial x^0}(\epsilon_0 + P_0)\, v\, e^{\lambda_0 - \nu_0} = -e^{\lambda_0 - \nu_0}(\epsilon_0 + P_0)\frac{\partial v}{\partial x^0} \tag{2-64}$$

$$\frac{\partial T_1^{\,1}}{\partial r} = \frac{\partial}{\partial r}(\eta_0 - P_0 + \delta\eta - \delta P) \tag{2-65}$$

$$\frac{T_1^{\,0}}{2}\frac{\partial(\nu+\lambda)}{\partial x^0} = -\frac{(\epsilon_0 + P_0)\, v\, e^{\lambda_0 - \nu_0}}{2}\left[\frac{\partial \nu_0}{\partial x^0} + \frac{\partial \delta\nu}{\partial x^0} + \frac{\partial \lambda_0}{\partial x^0} + \frac{\partial \delta\lambda}{\partial x^0}\right]$$

$$= -\frac{(\epsilon_0 + P_0)}{2}\, v\, e^{\lambda_0 - \nu_0}\left[\frac{\partial \delta\nu}{\partial x^0} + \frac{\partial \delta\lambda}{\partial x^0}\right] = 0 \tag{2-66}$$

to first order

$$\frac{(T_1^{\,1} - T_0^{\,0})}{2}\frac{\partial \nu}{\partial r} = \frac{1}{2}\left[\eta_0 - P_0 + \delta\eta - \delta P - \epsilon_0 - \eta_0 - \delta\epsilon - \delta\eta\right]\left(\frac{d\nu_0}{dr} + \frac{\partial \delta\nu}{\partial r}\right)$$

$$= -\frac{1}{2}(\epsilon_0 + P_0)\left[\frac{d\nu_0}{dr} + \frac{\partial \delta\nu}{\partial r}\right] - \frac{1}{2}\left[\delta P + \delta\epsilon\right]\frac{d\nu_0}{dr} \tag{2-67}$$

$$\frac{2}{r}\left[T_1^{\,1} - \frac{T_2^{\,2}}{2} - \frac{T_3^{\,3}}{2}\right] = \frac{2}{r}\left[\eta_0 - P_0 + \delta\eta - \delta P + \eta_0 + P_0 + \delta\eta + \delta P\right]$$

$$= \frac{4}{r}\left[\eta_0 + \delta\eta\right] \tag{2-68}$$

Then (2-21) becomes

$$e^{\lambda_0-\nu_0}(\epsilon_0+P_0)\frac{\partial v}{\partial x^0}+\frac{\partial}{\partial r}\left[P_0+\delta P-\eta_0-\delta\eta\right]+\left[\frac{\epsilon_0+P_0}{2}\right]\left(\frac{d\nu_0}{dr}+\frac{d\,\delta\nu}{dr}\right)$$

$$+\left[\frac{\delta\epsilon+\delta P}{2}\right]\frac{d\nu_0}{dr}-\frac{4}{r}\left[\eta_0+\delta\eta\right]=0 \qquad (2\text{-}69)$$

Referring to (2-25), (2-69) becomes

$$e^{\lambda_0-\nu_0}(\epsilon_0+P_0)\frac{\partial v}{\partial x^0}+\frac{\partial}{\partial r}(\delta P-\delta\eta)+\left[\frac{\epsilon_0+P_0}{2}\right]\frac{\partial\,\delta\nu}{\partial r}+\left[\frac{\delta\epsilon+\delta P}{2}\right]\frac{d\nu_0}{dr}$$

$$-\frac{4}{r}\,\delta\eta=0 \qquad (2\text{-}70)$$

We assume that the perturbations vary as $e^{i\sigma x^0}$ and substitute (2-63) into (2-70)

$$\sigma^2 e^{\lambda_0-\nu_0}(\epsilon_0+P_0)\xi=\frac{d}{dr}\delta P+\delta P\frac{d}{dr}\left(\frac{\lambda_0}{2}+\nu_0\right)+\frac{\delta\epsilon}{2}\frac{d\nu_0}{dr}$$

$$-\left[\frac{\epsilon_0+P_0}{2}\right]\left(\frac{d\nu_0}{dr}+\frac{1}{r}\right)\left(\frac{d\lambda_0}{dr}+\frac{d\nu_0}{dr}\right)\xi-\frac{d\,\delta\eta}{dr}-\frac{4}{r}\delta\eta-\frac{\delta\eta}{2}\left[\frac{d\lambda_0}{dr}+\frac{d\nu_0}{dr}\right] \qquad (2\text{-}71)$$

where it should be noted that $\delta P, \delta\epsilon, \delta\eta$, and ξ are functions of r and x^0 only.

In (2-71) we have explicit expressions for $\delta\eta$ and $\delta\epsilon$ given by (2-54) and (2-61) respectively. We need to obtain the change in pressure δP. We take note of Lupanov's (5) observation that for an adiabatic process:

$$\frac{\delta P}{\delta \epsilon} = \frac{\gamma P_0}{\epsilon_0 + P_0} \tag{2-72}$$

where γ is the ratio of specific heats.

Using (2-72) in (2-61), we get

$$\delta P = -\xi\left(\frac{\gamma P_0}{\epsilon_0 + P_0}\right)\frac{d\epsilon_0}{dr} - \left(\frac{\gamma P_0}{\epsilon_0 + P_0}\right)\left(\epsilon_0 + P_0\right)\left(\frac{e^{\lambda_0/2}}{r^2}\right)\frac{\partial}{\partial r}\left(r^2 e^{-\lambda_0/2}\xi\right)$$

$$-\xi\left(\frac{\gamma P_0}{\epsilon_0 + P_0}\right)\left(\frac{d\eta_0}{dr} + \frac{4\eta_0}{r}\right) - \left(\frac{\gamma P_0}{\epsilon_0 + P_0}\right)\delta\eta \tag{2-73}$$

which we rewrite as

$$\delta P = -\xi\frac{dP_0}{dr} - \frac{\gamma P_0}{r^2}e^{\lambda_0/2}\frac{\partial}{\partial r}\left(r^2 e^{-\lambda_0/2}\xi\right) - \frac{dP_0}{d\epsilon_0}\delta\eta$$

$$-\xi\frac{dP_0}{d\epsilon_0}\left(\frac{d\eta_0}{dr} + \frac{4\eta_0}{r}\right) \tag{2-74}$$

Putting (2-60) and (2-74) into (2-71) produces

$$
\begin{aligned}
\sigma^2 e^{\lambda_0-\nu_0}(\epsilon_0+P_0)\xi = &- \frac{d}{dr}\left[\xi\frac{dP_0}{dr}\right] - \frac{dP_0}{dr}\left[\frac{\lambda'}{2}+\nu'\right]\xi \\
&- \left(\frac{\epsilon_0+P_0}{2}\right)\left(\frac{d\nu_0}{dr}+\frac{1}{r}\right)\left(\lambda_0'+\nu_0'\right)\xi - \frac{\nu_0'}{2}\left\{\frac{d}{dr}\left(\epsilon_0+P_0\right)\xi\right. \\
&\left. + \frac{2}{r}\left(\epsilon_0+P_0\right)\xi\right\} - e^{-\frac{(\lambda_0+2\nu_0)}{2}}\frac{d}{dr}\left[e^{\frac{(\lambda_0+3\nu_0)}{2}}\frac{\gamma P_0}{r^2}\cdot\right. \\
&\left.\frac{d}{dr}\left(r^2 e^{-\nu_0/2}\xi\right)\right] - \frac{d}{dr}\left\{\xi\frac{dP_0}{d\epsilon_0}\left[\frac{dn_0}{dr}+\frac{4n_0}{r}\right]+\delta n\frac{dP_0}{d\epsilon_0}\right\} \\
&- \left[\frac{\lambda_0'}{2}+\nu_0'\right]\left\{\xi\frac{dP_0}{d\epsilon_0}\left[\frac{dn_0}{dr}+\frac{4n_0}{r}\right]+\delta n\frac{dP_0}{d\epsilon_0}\right\} \\
&- \frac{d}{dr}\delta n - \frac{4}{r}\delta n - \delta n\left[\frac{\lambda_0'+\nu_0'}{2}\right] - \frac{\nu_0'}{2}\delta n \qquad \text{(2-75)}
\end{aligned}
$$

Substituting for P_0' its value given by (2-25) in the first two terms of (2-75) gives

$$\sigma^2 e^{\lambda_0-\nu_0}(\epsilon_0+P_0)\xi = -\left\{\frac{d}{dr}\left[-\left(\frac{\epsilon_0+P_0}{2}\right)\xi\right]\frac{d\nu_0}{dr} - \xi\left(\frac{\epsilon_0+P_0}{2}\right)\frac{d^2\nu_0}{dr^2}\right\}$$

$$+\left[\frac{\lambda_0'}{2}+\nu_0'\right]\left(\frac{\nu_0'}{2}\left[\frac{\epsilon_0+P_0}{2}\right]\xi\right) - \left(\frac{\epsilon_0+P_0}{2}\right)\left[\frac{d\nu_0}{dr}+\frac{1}{r}\right]\left(\frac{d\lambda_0}{dr}+\frac{d\nu_0}{dr}\right)\xi$$

$$-\frac{\nu_0'}{2}\left\{\frac{d}{dr}\left[\left(\epsilon_0+P_0\right)\xi\right]+\frac{2}{r}\left(\epsilon_0+P_0\right)\xi\right\}$$

$$-e^{-\frac{(\lambda_0+2\nu_0)}{2}}\frac{d}{dr}\left[e^{\frac{(\lambda_0+3\nu_0)}{2}}\frac{\gamma P}{r^2}\frac{d}{dr}\left(r^2 e^{-\nu_0/2}\xi\right)\right]$$

$$-\frac{d}{dr}\left[\xi\left(\frac{4\eta_0}{r}+\eta_0'\right)\right]-\xi\left[\frac{\lambda_0'}{2}+\nu_0'\right]\left(\frac{4\eta_0}{r}+\eta_0'\right)$$

$$-\frac{d}{dr}\left\{\xi\frac{dP_0}{d\epsilon_0}\left[\frac{4\eta_0}{r}+\eta_0'\right]+\delta\eta\frac{dP_0}{dr}\right\}$$

$$-\left[\frac{\lambda_0'}{2}+\nu_0'\right]\left\{\xi\frac{dP_0}{d\epsilon_0}\left[\frac{4\eta_0}{r}+\eta_0'\right]+\delta\eta\frac{dP_0}{d\epsilon_0}\right\}$$

$$-\frac{d\,\delta\eta}{dr}-\frac{4}{r}\delta\eta-\delta\eta\left[\frac{\lambda_0'}{2}+\nu_0'\right] \tag{2-76}$$

Combining the first four terms on the r.h.s. of (2-76) gives

$$\left(\frac{\epsilon_0+P_0}{2}\right)\left[\frac{d^2\nu_0}{dr^2} - \frac{\lambda_0'\nu_0'}{2} - \frac{\lambda_0'}{r} - \frac{3\nu_0'}{r}\right]\xi \tag{2-77}$$

On the other hand, according to (2-5) under conditions of equilibrium, we get using (2-22)

$$\frac{16\pi G}{c^4}\left(P_0+\eta_0\right)e^{\lambda_0} = \frac{d^2\nu_0}{dr^2} - \frac{\lambda_0'\nu_0'}{2} + \frac{1}{2}\left(\frac{d\nu_0}{dr}\right)^2 + \frac{1}{r}\left(\nu_0'-\lambda_0'\right) \tag{2-78}$$

Using (2-78) in (2-77) produces

$$\left(\frac{\epsilon_0+P_0}{2}\right)\left[\frac{16\pi G}{c^4}\left(P_0+\eta_0\right)e^{\lambda_0} - \frac{(\nu_0')^2}{2} - \frac{4\nu_0'}{r}\right]\xi \tag{2-79}$$

or

$$\frac{8\pi G}{c^4}e^{\lambda_0}P_0(P_0+\epsilon_0)\xi - \nu_0'\left(\frac{P_0+\epsilon_0}{4}\right)\left[\frac{8}{r}+\nu_0'\right]\xi + \frac{8\pi G}{c^4}e^{\lambda_0}\eta_0(P_0+\epsilon_0)\xi \tag{2-80}$$

Using (2-25) in (2-80) we get

$$\frac{8\pi G}{c^4}e^{\lambda_0}P_0(P_0+\epsilon_0)\xi + \left[\left(\frac{P_0'-\eta_0'}{2}\right) - \frac{2\eta_0}{r}\right]\left(\frac{8}{r} - \frac{2(P_0'-\eta_0')}{P_0+\epsilon_0} + \frac{8\eta_0}{r(P_0+\epsilon_0)}\right)\xi$$

$$+\ \frac{8\pi G}{c^4}e^{\lambda_0}\eta_0\,(P_0+\epsilon_0)\xi \tag{2-81}$$

Expanding (2-81):

$$\frac{8\pi G}{c^4} e^{\lambda_0} P_0 (P_0+\epsilon_0)\xi + \frac{P_0'}{2}\left[\frac{8}{r} - \frac{2P_0'}{P_0+\epsilon_0}\right]\xi + \frac{P_0'}{2}\left[\frac{2\eta_0'}{P_0+\epsilon_0} + \frac{8\eta_0}{r(P_0+\epsilon_0)}\right]\xi$$

$$-\frac{\eta_0'}{2}\left[\frac{8}{r} - 2\left(\frac{P_0'-\eta_0'}{P_0+\epsilon_0}\right) + \frac{8\eta_0}{r(P_0+\epsilon_0)}\right]\xi - \frac{2\eta_0}{r}\left[\frac{8}{r} - 2\left(\frac{P_0'-\eta_0'}{P_0+\epsilon_0}\right) + \frac{8\eta_0}{r(P_0+\epsilon_0)}\right]\xi$$

$$+ \frac{8\pi G}{c^4} e^{\lambda_0} \eta_0 (P_0+\epsilon_0)\xi$$

Simplifying:

$$\frac{4}{r}P_0'\xi + \frac{8\pi G}{c^4} e^{\lambda_0} P_0 (P_0+\epsilon_0)\xi - \frac{(P_0')^2}{P_0+\epsilon_0}\xi$$

$$+\left[\frac{P_0'\eta_0'}{P_0+\epsilon_0} + \frac{4\eta_0 P_0'}{r(P_0+\epsilon_0)}\right]\xi - \eta_0'\left[\frac{4}{r} - \left(\frac{P_0'-\eta_0'}{P_0+\epsilon_0}\right) + \frac{4\eta_0}{r(P_0+\epsilon_0)}\right]\xi$$

$$-\left[\frac{16\eta_0}{r^2} - \frac{4\eta_0(P_0'-\eta_0')}{r(P_0+\epsilon_0)} + \frac{16\eta_0^2}{r^2(P_0+\epsilon_0)}\right]\xi$$

$$+\frac{8\pi G}{c^4} e^{\lambda_0} \eta_0 (P_0+\epsilon_0)\xi \qquad \text{(2-82)}$$

Remembering that (2-82) represents the first four terms on the r.h.s. of (2-76), we get

$$
\begin{aligned}
\sigma^2 e^{(\lambda_0-\nu_0)}(\epsilon_0+P_0)\xi &= \frac{4}{r}\frac{dP_0}{dr}\xi - e^{-\frac{(\lambda_0+2\nu_0)}{2}}\frac{d}{dr}\left[e^{\frac{(\lambda_0+3\nu_0)}{2}}\frac{\gamma P}{r^2}\frac{d}{dr}\left(r^2 e^{-\nu_0/2}\xi\right)\right] \\
&+ \frac{8\pi G}{c^4}e^{\lambda_0}P_0(P_0+\epsilon_0)\xi - \frac{(P_0')^2\xi}{\epsilon_0+P_0} - \frac{d}{dr}\left[\xi\left(\frac{4\eta_0}{r}+\eta_0'\right)\right] \\
&- \xi\left[\frac{\lambda_0'}{2}+\nu_0'\right]\left(\frac{4\eta_0}{r}+\eta_0'\right) - \frac{d}{dr}\left\{\xi\left[\eta_0'+\frac{4\eta_0}{r}\right]\frac{dP_0}{d\epsilon_0}+\frac{dP_0}{d\epsilon_0}\delta\eta\right\} \\
&- \left[\frac{\lambda_0'}{2}+\nu_0'\right]\left\{\xi\left[\eta_0'+\frac{4\eta_0}{r}\right]\frac{dP_0}{d\epsilon_0}+\frac{dP_0}{d\epsilon_0}\delta\eta\right\} - \frac{d}{dr}\delta\eta - \frac{4}{r}\delta\eta \\
&- \left[\frac{\lambda_0'}{2}+\nu_0'\right]\delta\eta + \left[\frac{P_0'\eta_0'}{P_0+\epsilon_0}+\frac{4\eta_0 P_0'}{r(P_0+\epsilon_0)}\right]\xi + \left[\eta_0'\frac{(P_0'-\eta_0')}{P_0+\epsilon_0}-\frac{4\eta_0'}{r}-\frac{4\eta_0\eta_0'}{r(P_0+\epsilon_0)}\right]\xi \\
&+ \left[\frac{4\eta_0}{r}\frac{(P_0'-\eta_0')}{(P_0+\epsilon_0)}-\frac{16\eta_0}{r^2}-\frac{16\eta_0^2}{r^2(P_0+\epsilon_0)}\right]\xi + \frac{8\pi G}{c^4}e^{\lambda_0}\eta_0(P_0+\epsilon_0)\xi
\end{aligned}
$$

(2-83)

Collecting terms this becomes

$$\sigma^2 e^{\lambda_0-\nu_0}(\epsilon_0+P_0)\xi = \frac{4P_0'\xi}{r} - e^{-\frac{(\lambda_0+2\nu_0)}{2}}\frac{d}{dr}\left[e^{\frac{(\lambda_0+3\nu_0)}{2}}\frac{\gamma P}{r^2}\frac{d}{dr}\left(r^2 e^{-\nu_0/2}\xi\right)\right]$$

$$+\frac{8\pi G}{c^4}e^{\lambda_0}P_0(P_0+\epsilon_0)\xi - \frac{(P_0')^2\xi}{P_0+\epsilon_0} - \frac{d}{dr}\left[\xi\left(\frac{4\eta_0}{r}+\eta_0'\right)\left(1+\frac{dP_0}{d\epsilon_0}\right)\right]$$

$$-\frac{d}{dr}\left[\delta\eta\,\frac{dP_0}{d\epsilon_0}\right] - \xi\left[\left(\frac{4\eta_0}{r}+\eta_0'\right)\left(1+\frac{dP_0}{d\epsilon_0}\right)\left(\frac{\lambda_0'}{2}+\nu_0'\right)\right]$$

$$-\delta\eta\left[\frac{dP_0}{d\epsilon_0}\left(\frac{\lambda_0'}{2}+\nu_0'\right)+\frac{4}{r}+\left(\frac{\lambda_0'}{2}+\nu_0'\right)\right] - \frac{d}{dr}\delta\eta + \left[\frac{P_0'\eta_0'}{P_0+\epsilon_0}+\frac{4\eta_0P_0'}{r(P_0+\epsilon_0)}\right]\xi$$

$$+\xi\left[\frac{\eta_0'(P_0'-\eta_0')}{P_0+\epsilon_0} - \frac{4\eta_0'}{r} - \frac{4\eta_0\eta_0'}{r(P_0+\epsilon_0)}\right] + \xi\left[\frac{4\eta_0(P_0'-\eta_0')}{r(P_0+\epsilon_0)} - \frac{16\eta_0}{r^2} - \frac{16\eta_0^2}{r^2(P_0+\epsilon_0)}\right]$$

$$+\frac{8\pi G}{c^4}e^{\lambda_0}\eta_0(P_0+\epsilon_0)\xi \qquad (2\text{-}84)$$

We continue to collect terms and use (2-72)

$$
\begin{aligned}
\sigma^2 e^{\lambda_0-\nu_0}(\epsilon_0+P_0)\xi &= \frac{4}{r}P_0'\xi - e^{-\frac{(\lambda_0+3\nu_0)}{2}}\frac{d}{dr}\left[e^{\frac{\lambda_0-3\nu_0}{2}}\frac{\gamma P_0}{r^2}\frac{d}{dr}\left(r^2 e^{-\nu_0/2}\xi\right)\right] \\
&- \frac{d}{dr}\left[\delta\eta\left(\frac{\gamma P_0}{P_0+\epsilon_0}+1\right)\right] - \frac{d}{dr}\left[\xi\left(\frac{4\eta_0}{r}+\eta_0'\right)\left(1+\frac{\gamma P_0}{P_0+\epsilon_0}\right)\right] \\
&- \delta\eta\left[\frac{\gamma P_0}{P_0+\epsilon_0}\left(\frac{\lambda_0'}{2}+\nu_0'\right)+\frac{4}{r}+\left(\frac{\lambda_0'}{2}+\nu_0'\right)\right] \\
&- \xi\left[\left(\frac{4\eta_0}{r}+\eta_0'\right)\left(1+\frac{\gamma P_0}{P_0+\epsilon_0}\right)\left(\frac{\lambda_0'}{2}+\nu_0'\right)\right] \\
&+ \xi\left[8\eta_0\frac{(P_0'-\eta_0')}{r(P_0+\epsilon_0)} - \frac{(-\eta_0')^2+(P_0')^2-2\eta_0'P_0'}{P_0+\epsilon_0} - \frac{16\eta_0^2}{r^2(P_0+\epsilon_0)}\right. \\
&\left. - \frac{16\eta_0}{r^2} - \frac{4\eta_0'}{r}\right] + \xi\left[\frac{8\pi G}{c^4}e^{\lambda_0}\left(P_0+\epsilon_0\right)\left(P_0+\eta_0\right)\right] \qquad \text{(2-85)}
\end{aligned}
$$

Substituting for $\delta\eta$ as derived in the appendix, and dropping the subscripts as no longer necessary, we obtain the pulsation equation

$$\sigma^2 e^{\lambda-\nu}(\epsilon+P)\xi = \frac{4}{r}P'\xi - e^{-\frac{(\lambda+2\nu)}{2}}\frac{d}{dr}\left[e^{\frac{(\lambda+3\nu)}{2}}\frac{\gamma P}{r^2}\frac{d}{dr}\left(r^2 e^{-\nu/2}\xi\right)\right]$$

$$+\xi\left[\frac{8\eta(P'-\eta')}{r(P+\epsilon)} - \frac{(\eta'-P')^2}{P+\epsilon} - \frac{16\eta^2}{r^2(P+\epsilon)}\right.$$

$$\left.+\frac{8\pi G}{c^4}e^{\lambda}(P+\epsilon)(P+\eta)\right] \qquad (2\text{-}86)$$

Solutions of the pulsation equation must satisfy the boundary conditions:

$$\xi = 0 \text{ at } r = 0, \quad \Delta P = 0 \text{ at } r = R = R_0 + \xi \qquad (2\text{-}87)$$

where ΔP is the Lagrangian variation in pressure.

Physically, the first boundary condition states that the center of the star remains fixed during an oscillation; the second insists that the pressure at the surface remains zero.

CHAPTER III

THE CHARGED HOMOGENEOUS MODEL

Introduction

The homogeneous model is a perfect fluid sphere of constant energy density. It is one of the earliest models to be studied in general relativity, and has been used as a model for elementary particles as well as for stars. Kyle and Martin (6) have commented that solutions of the field equations for extended charge distributions may be useful and perhaps necessary when considering questions of quantum field theory in a Riemannian manifold. Toward this end, the solution provided by them is that of a perfect fluid sphere with a constant charge density. The solution which we shall provide below is for a perfect fluid sphere with constant electrical energy density.

It may also be useful to use the charged homogeneous model in order to study the dynamical stability of charged stars. Shvartsman (7) has discussed exchange processes between stars and their surrounding medium, and has shown that they might acquire an electrical charge as a result of these processes. Although astrophysical systems are usually considered to be electrically neutral, Shvartsman's analysis suggests that this may not always be the case. It is, therefore, of interest to determine how the dynamical stability of a perfect fluid sphere is affected by the presence of electrical charge.

Of course, since gravitational forces are extremely weak compared to electromagnetic forces, highly charged stars are impossible. In fact, if a sphere of neutral hydrogen lost one electron in 10^{18}, it would be sufficient to produce gravitational instability. Saakyan (8) has discussed this point in greater detail.

Equilibrium Solution

Consider a perfect fluid sphere with matter energy density ϵ, and electrical energy density η, where ϵ and η are constants. Using the metric and energy-momentum tensors defined in the previous chapter, the components of the total energy-momentum tensor are given by

$$T_0^{\ 0} = \epsilon + \eta, \quad T_1^{\ 1} = \eta - P, \quad T_2^{\ 2} = T_3^{\ 3} = -\eta - P \tag{3-1}$$

We take η/ϵ and P/ϵ to be first order quantities, and will neglect second order quantities in the solution of the field equations.

Substituting (3-1) into (2-2) we rewrite it as

$$\frac{8\pi G(\epsilon+\eta)r^2}{c^4} = \frac{d}{dr}\left(r - re^{-\lambda}\right)$$

From which we immediately get

$$e^{-\lambda} = 1 - \frac{8\pi G(\epsilon+\eta)r^2}{3c^4} + \frac{A}{r} \tag{3-2}$$

where the constant of integration A represents a point mass at the origin. In order that the metric be regular at the origin, we set A = 0.

$$e^{-\lambda} = 1 - \frac{8\pi G(\epsilon+\eta)r^2}{3c^4} = 1 - \frac{r^2}{L^2} \tag{3-3}$$

$$\lambda' = \frac{16\pi G(\epsilon+\eta)r}{3c^4 - 8\pi G(\epsilon+\eta)r^2} = \frac{2r}{L^2(1-r^2/L^2)} \tag{3-4}$$

Substituting (3-1) into (2-21), and remembering that none of the quantities depend on x^0, we get

$$P' = -\frac{\nu'}{2}(\epsilon+P) + \frac{4\eta}{r} \tag{3-5}$$

(3-5) is rewritten as

$$\frac{P'}{\epsilon+P} = -\frac{\nu'}{2} + \frac{4\eta}{r(\epsilon+P)}$$

$$\frac{P'}{\epsilon(1+P/\epsilon)} = -\frac{\nu'}{2} + \frac{4\eta}{\epsilon(1+P/\epsilon)r}$$

$$\frac{P'}{\epsilon}(1+P/\epsilon)^{-1} = -\frac{\nu'}{2} + \frac{4\eta}{\epsilon r}(1+P/\epsilon)^{-1}$$

Expanding to first order:

$$\frac{P'}{\epsilon}\left(1-P/\epsilon\right) = -\frac{\nu'}{2} + \frac{4\eta}{\epsilon r}\left(1-P/\epsilon\right) \tag{3-6}$$

Solving for ν' we have

$$\nu' = -\frac{2P}{\epsilon}\left(1-P/\epsilon\right) + \frac{8\eta}{\epsilon r}\left(1-P/\epsilon\right) \tag{3-7}$$

From the previous chapter we reproduce (2-27) below

$$\frac{8\pi G}{c^4}(\epsilon+P) = \frac{e^{-\lambda}}{r}(\lambda'+\nu')$$

Substituting for ν' from (3-7) we get

$$\frac{8\pi G}{c^4}(\epsilon+P) = \frac{e^{-\lambda}}{r}\lambda' + \frac{e^{-\lambda}}{r}\left[\frac{8\eta}{\epsilon r}(1-P/\epsilon) - \frac{2P'}{\epsilon}(1-P/\epsilon)\right]$$

$$\frac{2P'(1-P/\epsilon)e^{-\lambda}}{\epsilon r} + \frac{8\pi G P}{c^4} = \frac{e^{-\lambda}\lambda'}{r} + \frac{8\eta(1-P/\epsilon)e^{-\lambda}}{\epsilon r^2} - \frac{8\pi G\epsilon}{c^4}$$

Since

$$\frac{8\pi G}{c^4} = \frac{3}{(\epsilon+\eta)L^2}, \text{ and } \qquad \lambda' = \frac{2re^{\lambda}}{L^2}$$

the above becomes:

$$P' + \frac{3\epsilon r P(1-P/\epsilon)^{-1}}{2(\epsilon+\eta)(1-r^2/L^2)L^2} = \frac{\epsilon r(1-P/\epsilon)^{-1}}{(1-r^2/L^2)L^2} + \frac{4\eta}{r} - \frac{3\epsilon^2 r(1-P/\epsilon)^{-1}}{2(\epsilon+\eta)(1-r^2/L^2)L^2}$$

Expanding to first order in P/ϵ, we get

$$P' + \frac{(3\epsilon r P + 3\epsilon^2 r)(1+P/\epsilon)}{2(\epsilon+\eta)(1-r^2/L^2)L^2} = \frac{\epsilon r(1+P/\epsilon)}{(1-r^2/L^2)L^2} + \frac{4\eta}{r}$$

$$P' + \frac{(3\epsilon^2 r P/\epsilon + 3\epsilon^2 r)(1+P/\epsilon)}{2(\epsilon+\eta)(1-r^2/L^2)L^2} = \frac{\epsilon r(1+P/\epsilon)}{(1-r^2/L^2)L^2} + \frac{4\eta}{r}$$

$$P' + \frac{3\epsilon^2 r(1+P/\epsilon)^2}{2(\epsilon+\eta)(1-r^2/L^2)L^2} = \frac{\epsilon r(1+P/\epsilon)}{(1-r^2/L^2)L^2} + \frac{4\eta}{r}$$

To first order in P/ϵ this becomes

$$P' + \frac{3\epsilon^2 r}{2(\epsilon+\eta)(1-r^2/L^2)L^2} + \frac{3\epsilon r P}{(\epsilon+\eta)(1-r^2/L^2)L^2} = \frac{\epsilon r(1+P/\epsilon)}{(1-r^2/L^2)L^2} + \frac{4\eta}{r}$$

Finally we get

$$P' + \left[\left(\frac{3\epsilon}{\epsilon+\eta} - 1\right)\left(\frac{r}{L^2(1-r^2/L^2)}\right)\right]P = \frac{4\eta}{r} + \left[\left(\epsilon - \frac{3\epsilon^2}{2(\epsilon+\eta)}\right)\left(\frac{r}{L^2(1-r^2/L^2)}\right)\right] \tag{3-8}$$

Before solving (3-8) we want to show that this equation is precisely the condition that insures satisfaction of the $T_2^{\ 2}$ component of the field equations.

Solving (2-27) for ν' gives

$$\nu' = \frac{8\pi G}{c^4}\left[\epsilon r e^{\lambda} + P r e^{\lambda}\right] - \lambda' = \frac{3}{(\epsilon+\pi)L^2}\left[\epsilon r e^{\lambda} + P r e^{\lambda}\right] - \frac{2re^{\lambda}}{L^2} \tag{3-9}$$

$$(\nu')^2 = \frac{9e^{2\lambda}}{(\epsilon+\pi)^2 L^4}\left[\epsilon^2 r^2 + 2\epsilon r^2 P\right] - \frac{12e^{2\lambda}}{(\epsilon+\pi)L^2}\left[\frac{r^2}{L^2}(\epsilon+P)\right] + \frac{4r^2e^{2\lambda}}{L^4} \tag{3-10}$$

where again $P^2 \ll \epsilon^2$.

$$(\nu')^2 = \frac{9e^{2\lambda}}{(\epsilon+\pi)L^2}\left[\frac{\epsilon^2 r^2}{L^2} + \frac{2\epsilon r^2 P}{L^2}\right] - \frac{12e^{2\lambda}}{(\epsilon+\pi)L^2}\left[\frac{\epsilon r^2}{L^2} + \frac{P r^2}{L^2}\right] + \frac{4r^2e^{2\lambda}}{L^4} \tag{3-11}$$

Then

$$\nu'' = \frac{3}{(\epsilon+\pi)L^2}\left[\epsilon r e^{\lambda}\left(\frac{2re^{\lambda}}{L^2}\right) + \epsilon e^{\lambda} + P r e^{\lambda}\left(\frac{2re^{\lambda}}{L^2}\right) + r e^{\lambda}P' + P e^{\lambda}\right] - \frac{2}{L^2}\left[r e^{\lambda}\left(\frac{2re^{\lambda}}{L^2}\right) + e^{\lambda}\right]$$

$$\nu'' = \frac{3}{(\epsilon+\pi)L^2}\left[\frac{2\epsilon r^2 e^{2\lambda}}{L^2} + \epsilon e^{\lambda} + \frac{2r^2 P e^{2\lambda}}{L^2} + rP'e^{\lambda} + Pe^{\lambda}\right] - \frac{4r^2e^{2\lambda}}{L^4} - \frac{2e^{\lambda}}{L^2} \tag{3-12}$$

Turning now to the $T_2^{\;2}$ component of the field equations given by (2-5), we have for the l.h.s.

$$\frac{8\pi G\, T_2^{\;2}}{c^4} = \frac{3}{L^2(\epsilon+\eta)}\left[P+\eta\right] \tag{3-13}$$

For the r.h.s.:

$$e^{-\lambda}\left\{\frac{\nu''}{2}+\frac{(\nu')^2}{4}+\frac{\nu'-\lambda'}{2r}-\frac{\nu'\lambda'}{4}\right\} \tag{3-14}$$

Using (3-9), (3-11), and (3-12), we substitute into (3-14)

$$e^{-\lambda}\left\{\frac{3}{(\epsilon+\eta)L^2}\left[\frac{\epsilon r^2 e^{-\lambda}}{L^2}+\frac{\epsilon e^{\lambda}}{2}+\frac{Pr^2e^{2\lambda}}{L^2}+\frac{rP'e^{\lambda}}{2}+\frac{Pe^{\lambda}}{2}\right]-\frac{2r^2e^{2\lambda}}{L^4}\right.$$

$$-\frac{e^{\lambda}}{L^2}+\frac{9e^{2\lambda}}{4(\epsilon+\eta)^2L^2}\left[\frac{\epsilon^2r^2}{L^2}+\frac{2\epsilon r^2P}{L^2}\right]-\frac{3e^{2\lambda}}{(\epsilon+\eta)L^2}\left[\frac{\epsilon r^2}{L^2}+\frac{Pr^2}{L^2}\right]+\frac{r^2e^{2\lambda}}{L^4}$$

$$\left.+\frac{3}{(\epsilon+\eta)L^2}\left[\frac{\epsilon e^{\lambda}}{2}+\frac{Pe^{\lambda}}{2}\right]-\frac{e^{\lambda}}{L^2}-\frac{e^{\lambda}}{L^2}-\frac{3}{(\epsilon+\eta)L^2}\left[\frac{\epsilon r^2e^{2\lambda}}{2L^2}+\frac{Pr^2e^{2\lambda}}{2L^2}\right]+\frac{r^2e^{2\lambda}}{L^4}\right\}$$

Simplifying:

$$\frac{3}{(\epsilon+\eta)L^2}\left[\frac{\epsilon r^2e^{\lambda}}{L^2}+\frac{\epsilon}{2}+\frac{Pr^2e^{\lambda}}{L^2}+\frac{rP'}{2}+\frac{P}{2}-\frac{\epsilon r^2e^{\lambda}}{L^2}-\frac{Pr^2e^{\lambda}}{L^2}+\frac{\epsilon}{2}+\frac{P}{2}\right.$$

$$\left.-\frac{\epsilon r^2e^{\lambda}}{2L^2}-\frac{Pr^2e^{\lambda}}{2L^2}\right]+\frac{9e^{\lambda}}{4(\epsilon+\eta)^2L^2}\left[\frac{\epsilon r^2}{L^2}+\frac{2\epsilon r^2P}{L^2}\right]-\frac{3}{L^2}$$

Collecting terms:

$$\frac{3}{(\epsilon+\eta)L^2}\left[\epsilon+P+\frac{rP'}{2}-\frac{\epsilon r^2 e^\lambda}{2L^2}-\frac{Pr^2e^\lambda}{2L^2}\right]+\frac{9e^\lambda}{4L^2(\epsilon+\eta)^2}\left[\frac{\epsilon^2r^2}{L^2}+\frac{2\epsilon Pr^2}{L^2}\right]-\frac{3}{L^2}$$

or

$$\frac{3}{(\epsilon+\eta)L^2}\left[P-\eta+\frac{rP'}{2}-\frac{\epsilon r^2e^\lambda}{2L^2}-\frac{Pr^2e^\lambda}{2L^2}\right]+\frac{9e^\lambda}{4(\epsilon+\eta)^2L^2}\left[\frac{\epsilon^2r^2}{L^2}+\frac{2\epsilon Pr^2}{L^2}\right]$$

This is rewritten as

$$\frac{3}{(\epsilon+\eta)L^2}\left[P+\eta\right]+\frac{3}{(\epsilon+\eta)L^2}\left[\frac{rP'}{2}-\frac{\epsilon r^2e^\lambda}{2L^2}-\frac{r^2Pe^\lambda}{2L^2}-2\eta\right]+\frac{9e^\lambda}{4(\epsilon+\eta)^2L^2}\left[\frac{\epsilon^2r^2}{L^2}+\frac{2\epsilon r^2P}{L^2}\right] \tag{3-15}$$

Comparing (3-15) with (3-13), we see that the condition for T_2^2 to be satisfied is

$$\frac{3}{(\epsilon+\eta)L^2}\left[\frac{rP'}{2}-\frac{\epsilon r^2e^\lambda}{2L^2}-\frac{Pr^2e^\lambda}{2L^2}-2\eta\right]+\frac{9e^\lambda}{4(\epsilon+\eta)^2L^2}\left[\frac{\epsilon^2r^2}{L^2}+\frac{2\epsilon r^2P}{L^2}\right]=0 \tag{3-16}$$

or

$$\frac{rP'}{2}-\frac{\epsilon r^2e^\lambda}{2L^2}-\frac{Pr^2e^\lambda}{2L^2}-2\eta+\frac{3}{4(\epsilon+\eta)}\left[\frac{\epsilon r^2}{L^2}+\frac{2\epsilon r^2P}{L^2}\right]=0$$

$$P'-\frac{\epsilon re^\lambda}{L^2}-\frac{Pre^\lambda}{L^2}-\frac{4\eta}{r}+\frac{3\epsilon}{2(\epsilon+\eta)}\left[\frac{\epsilon re^\lambda}{L^2}+\frac{2rPe^\lambda}{L^2}\right]=0$$

$$P'+\left[\frac{3\epsilon re^\lambda}{(\epsilon+\eta)L^2}-\frac{re^\lambda}{L^2}\right]P=\frac{4\eta}{r}+\frac{\epsilon re^\lambda}{L^2}-\frac{3\epsilon^2re^\lambda}{2(\epsilon+\eta)L^2}$$

This is rewritten as

$$P' + \left[\left(\frac{3\epsilon}{\epsilon+\eta} - 1\right)\left(\frac{r}{L^2(1-r^2/L^2)}\right)\right]P = \frac{4\gamma}{r} + \left[\left(\epsilon - \frac{3\epsilon^2}{2(\epsilon+\eta)}\right)\left(\frac{r}{L^2(1-r^2/L^2)}\right)\right] \tag{3-17}$$

which agrees with (3-8).

The solution of (3-17) is given by

$$P = K e^{-\left(\frac{3\epsilon}{(\epsilon+\eta)L^2} - \frac{1}{L^2}\right)\int \frac{r\,dr}{1-r^2/L^2}} + e^{-\left(\frac{3\epsilon}{(\epsilon+\eta)L^2} - \frac{1}{L^2}\right)\int \frac{r\,dr}{1-r^2/L^2}}$$

$$\cdot \int e^{\left(\frac{3\epsilon}{(\epsilon+\eta)L^2} - \frac{1}{L^2}\right)} \left[\frac{4\gamma}{r} + \left(\frac{\epsilon}{L^2} - \frac{3\epsilon^2}{2(\epsilon+\eta)L^2}\right)\left(\frac{r}{1-r^2/L^2}\right)\right] dr \tag{3-18}$$

where K is an integration constant.

Since: $$\int \frac{r\,dr}{1-r^2/L^2} = -\frac{L^2}{2} \ln\left(1 - r^2/L^2\right)$$

We have:

$$e^{\left(\frac{3\epsilon}{(\epsilon+\eta)L^2} - \frac{1}{L^2}\right)\int \frac{r\,dr}{1-r^2/L^2}} = e^{\left(\frac{3\epsilon}{(\epsilon+\eta)L^2} - \frac{1}{L^2}\right)\left(-\frac{L^2}{2}\ln\left(1-r^2/L^2\right)\right)}$$

$$= e^{-\left(\frac{3\epsilon L^2}{2(\epsilon+\eta)L^2} - \frac{L^2}{2L^2}\right)\ln\left(1-r^2/L^2\right)}$$

$$= e^{\ln\left(1-r^2/L^2\right)^{-\left(\frac{3\epsilon}{2(\epsilon+\eta)} - \frac{1}{2}\right)}}$$

$$= \left(1 - r^2/L^2\right)^{-\left(\frac{3\epsilon}{2(\epsilon+\eta)} - \frac{1}{2}\right)}$$

Then (3-18) becomes

$$P = K\left(1-r^2/L^2\right)^{\left(\frac{3\epsilon}{2(\epsilon+\eta)}-\frac{1}{2}\right)} + \left(1-r^2/L^2\right)^{\left(\frac{3\epsilon}{2(\epsilon+\eta)}-\frac{1}{2}\right)} \int \left(1-r^2/L^2\right)^{-\left(\frac{3\epsilon}{2(\epsilon+\eta)}-\frac{1}{2}\right)} \left[\frac{4\eta}{r} + \left(\frac{\epsilon}{L^2} - \frac{3\epsilon^2}{2L^2(\epsilon+\eta)}\right)\left(\frac{r}{1-r^2/L^2}\right)\right] dr \tag{3-19}$$

Since:

$$\left(1-r^2/L^2\right)^{-\left(\frac{3\epsilon}{2(\epsilon+\eta)}-\frac{1}{2}\right)} \simeq 1 + \left[\frac{3\epsilon}{2(\epsilon+\eta)} - \frac{1}{2}\right]\frac{r^2}{L^2}$$

The integral in (3-19) becomes

$$\int \left\{1 + \left[\frac{3\epsilon}{2(\epsilon+\eta)} - \frac{1}{2}\right]\frac{r^2}{L^2}\right\}\left\{\frac{4\eta}{r} + \left[\frac{\epsilon}{L^2} - \frac{3\epsilon^2}{2(\epsilon+\eta)L^2}\right]\left(\frac{r}{1-r^2/L^2}\right)\right\} dr \tag{3-20}$$

We begin to integrate (3-20)

$$\frac{\eta}{L^2}\left[\frac{6\epsilon}{\epsilon+\eta} - 2\right]\int r\,dr + \frac{1}{L^4}\left[\frac{9\epsilon^2}{4(\epsilon+\eta)} - \frac{9\epsilon^3}{4(\epsilon+\eta)^2} - \frac{\epsilon}{2}\right]\int \frac{r^3\,dr}{1-r^2/L^2} =$$

$$\left[\frac{6\epsilon}{\epsilon+\eta} - 2\right]\frac{\eta r^2}{2L^2} + \frac{1}{L^4}\left[\frac{9\epsilon^2}{4(\epsilon+\eta)} - \frac{9\epsilon^3}{4(\epsilon+\eta)^2} - \frac{\epsilon}{2}\right]\left[L^2\int \frac{r\,dr}{1-r^2/L^2} - L^2\int r\,dr\right] =$$

$$\left[\frac{3\epsilon}{\epsilon+\eta} - 1\right]\frac{\eta r^2}{L^2} - \left[\frac{9\epsilon^2}{8(\epsilon+\eta)} - \frac{9\epsilon^3}{8(\epsilon+\eta)^2} - \frac{\epsilon}{4}\right]\left[\frac{r^2}{L^2} + \ln\left(1-r^2/L^2\right)\right] \tag{3-21}$$

Combining (3-20) and (3-21) we have

$$4\eta \ln r + \left[\frac{9\epsilon^3}{8(\epsilon+\eta)^2} - \frac{3\epsilon^2}{8(\epsilon+\eta)} - \frac{\epsilon}{4}\right] \ln\left(1 - r^2/L^2\right)$$

$$+ \left[\frac{9\epsilon^3}{8(\epsilon+\eta)^2} - \frac{9\epsilon^2}{8(\epsilon+\eta)} + \frac{\epsilon}{4} + \frac{3\epsilon\eta}{\epsilon+\eta} - \eta\right]\frac{r^2}{L^2} \tag{3-22}$$

To the first order in η/ϵ we have

$$\left[\frac{9\epsilon^3}{8(\epsilon+\eta)^2} - \frac{3\epsilon^2}{8(\epsilon+\eta)} - \frac{\epsilon}{4}\right] = \epsilon\left[\frac{9}{8(1+2\eta/\epsilon)} - \frac{3}{8(1+\eta/\epsilon)} - \frac{1}{4}\right]$$

$$\left[\frac{9\epsilon^3}{8(\epsilon+\eta)^2} - \frac{9\epsilon^2}{8(\epsilon+\eta)} + \frac{\epsilon}{4} + \frac{3\epsilon\eta}{\epsilon+\eta} - \eta\right] = \epsilon\left[\frac{9}{8(1+2\eta/\epsilon)} - \frac{9}{8(1+\eta/\epsilon)} + \frac{1}{4} + \frac{2\eta}{\epsilon}\right]$$

Then (3-22) becomes

$$4\eta \ln r + \epsilon\left[\frac{9}{8}\left(1 - \frac{2\eta}{\epsilon}\right) - \frac{3}{8}\left(1 - \frac{\eta}{\epsilon}\right) - \frac{1}{4}\right] \ln\left(1 - r^2/L^2\right) \tag{3-23}$$

$$+ \epsilon\left[\frac{9}{8}\left(1 - \frac{2\eta}{8}\right) - \frac{9}{8}\left(1 - \frac{\eta}{\epsilon}\right) + \frac{1}{4} + \frac{2\eta}{\epsilon}\right]\frac{r^2}{L^2}$$

Turning to (3-19), the pressure becomes

$$P = \left(1 - r^2/L^2\right)^{\left(\frac{3\epsilon}{2(\epsilon+\eta)} - \frac{1}{2}\right)} \left[K + 4\eta \ln r + \epsilon\left(\frac{1}{2} - \frac{15\eta}{8\epsilon}\right) \ln\left(1 - r^2/L^2\right)\right.$$

$$\left. + \frac{\epsilon r^2}{L^2}\left(\frac{1}{4} + \frac{7\eta}{8\epsilon}\right)\right]$$

Since $\left(1-r^2/L^2\right)^{\left(\frac{3\epsilon}{2(\epsilon+\eta)}-\frac{1}{2}\right)} \simeq 1-r^2/L^2$

to first order.

$$P=\left(1-r^2/L^2\right)\left[K+4\eta\ln r+\epsilon\left(\frac{1}{2}-\frac{15\eta}{8\epsilon}\right)\ln\left(1-r^2/L^2\right)+\frac{\epsilon r^2}{L^2}\left(\frac{1}{4}+\frac{7\eta}{8\epsilon}\right)\right] \tag{3-24}$$

When r = R, P = o

$$K=-\left[4\eta\ln R+\epsilon\left(\frac{1}{2}-\frac{15\eta}{8\epsilon}\right)\ln\left(1-R^2/L^2\right)+\frac{\epsilon R^2}{L^2}\left(\frac{1}{4}+\frac{7\eta}{8\epsilon}\right)\right] \tag{3-25}$$

Finally, the pressure is given by

$$P=\left(1-r^2/L^2\right)\left[4\eta\ln\frac{r}{R}+\epsilon\left(\frac{1}{2}-\frac{15\eta}{8\epsilon}\right)\ln\frac{1-r^2/L^2}{1-R^2/L^2}+\epsilon\left(\frac{1}{4}+\frac{7\eta}{8\epsilon}\right)\left(\frac{r^2}{L^2}-\frac{R^2}{L^2}\right)\right] \tag{3-26}$$

A discussion of this equation and its non-relativistic limit is given in the appendix.

$$P'=\left(1-r^2/L^2\right)\left[\frac{4\eta}{r}-\frac{2\epsilon r}{L^2}\frac{\left(1/2-15\eta/8\epsilon\right)}{\left(1-r^2/L^2\right)}+\frac{2\epsilon r}{L^2}\left(\frac{1}{4}+\frac{7\eta}{8\epsilon}\right)\right] \tag{3-27}$$

$$-\frac{2r}{L^2}\left[4\eta\ln\frac{r}{R}+\epsilon\left(\frac{1}{2}-\frac{15\eta}{8\epsilon}\right)\ln\frac{1-r^2/L^2}{1-R^2/L^2}+\epsilon\left(\frac{1}{4}+\frac{7\eta}{8\epsilon}\right)\left(\frac{r^2}{L^2}-\frac{R^2}{L^2}\right)\right]$$

Next we obtain $\nu'(r)$. Putting (3-26) into (3-9) gives

$$\nu'(r) = \frac{8\pi G \epsilon r e^{\lambda}}{c^4} - \lambda' + \frac{8\pi G P r e^{\lambda}}{c^4}$$

$$= \frac{8\pi G\epsilon}{c^4}\left(\frac{r}{1-r^2/L^2}\right) - \frac{2r}{L^2(1-r^2/L^2)} + \frac{8\pi G}{c^4}\left(\frac{r}{1-r^2/L^2}\right)$$

$$\cdot\left\{(1-r^2/L^2)\left[4\eta \ln\frac{r}{R} + \epsilon\left(\frac{1}{2}-\frac{15\eta}{8\epsilon}\right)\ln\frac{1-r^2/L^2}{1-R^2/L^2}\right.\right.$$

$$\left.\left.+\,\epsilon\left(\frac{1}{4}+\frac{7\eta}{8\epsilon}\right)\left(\frac{r^2}{L^2}-\frac{R^2}{L^2}\right)\right]\right\}$$

Since: $$\frac{8\pi G}{c^4} = \frac{3}{L^2(\epsilon+\eta)}$$

$$\nu'(r) = \left[\frac{3\epsilon}{(\epsilon+\eta)L^2} - \frac{2}{L^2}\right]\left(\frac{r}{1-r^2/L^2}\right) + \frac{3r}{(\epsilon+\eta)L^2}\left[4\eta \ln\frac{r}{R}\right.$$

$$\left.+\,\epsilon\left(\frac{1}{2}-\frac{15\eta}{8\epsilon}\right)\ln\frac{1-r^2/L^2}{1-R^2/L^2} + \epsilon\left(\frac{1}{4}+\frac{7\eta}{8\epsilon}\right)\left(\frac{r^2}{L^2}-\frac{R^2}{L^2}\right)\right] \tag{3-28}$$

We evaluate the integral of each of the terms in (3-28)

$$\left[\frac{3\epsilon}{\epsilon+\eta}-2\right]\int\frac{r\,dr}{L^2(1-r^2/L^2)}=\left[1-\frac{3\epsilon}{2(\epsilon+\eta)}\right]\ln\left(1-r^2/L^2\right)$$

$$=\left[\frac{3\eta}{2\epsilon}-\frac{1}{2}\right]\ln\left(1-r^2/L^2\right) \tag{3-29}$$

$$\frac{12\eta}{(\epsilon+\eta)L^2}\int r\ln\frac{r}{R}\,dr=\frac{12\eta R^2}{(\epsilon+\eta)L^2}\int\frac{r}{R}\ln\frac{r}{R}\,d\left(\frac{r}{R}\right)$$

$$=\frac{12\eta R^2}{(\epsilon+\eta)L^2}\left[\frac{r^2}{L^2}\left(\frac{1}{2}\ln\frac{r}{R}-\frac{1}{4}\right)\right]=\frac{6\eta}{(\epsilon+\eta)}\left[\frac{r^2}{L^2}\ln\frac{r}{R}-\frac{r^2}{2L^2}\right] \tag{3-30}$$

$$\left(\frac{3\epsilon}{(\epsilon+\eta)L^2}\right)\left(\frac{1}{2}-\frac{15\eta}{8\epsilon}\right)\int r\ln\frac{1-r^2/L^2}{1-R^2/L^2}\,dr=\frac{3}{L^2}\left(\frac{1}{2}-\frac{19\eta}{8\epsilon}\right)\int r\ln\frac{1-r^2/L^2}{1-R^2/L^2}\,dr$$

$$=\frac{3}{L^2}\left(\frac{1}{2}-\frac{19\eta}{8\epsilon}\right)\left[\frac{r^2}{2}\ln\frac{1-r^2/L^2}{1-R^2/L^2}+\frac{1}{L^2}\int\frac{r^3\,dr}{1-r^2/L^2}\right]$$

$$=\frac{3}{L^2}\left(\frac{1}{2}-\frac{19\eta}{8\epsilon}\right)\left[\frac{r^2}{2}\ln\frac{1-r^2/L^2}{1-R^2/L^2}+\frac{1}{L^2}\left(L^2\int\frac{r\,dr}{1-r^2/L^2}-L^2\int r\,dr\right)\right]$$

$$=\frac{3}{L^2}\left(\frac{1}{2}-\frac{19\eta}{8\epsilon}\right)\left[\frac{r^2}{2}\ln\frac{1-r^2/L^2}{1-R^2/L^2}-\frac{r^2}{2}-\frac{L^2}{2}\ln\left(1-r^2/L^2\right)\right]$$

$$=\left(\frac{3}{4}-\frac{57\eta}{16\epsilon}\right)\left[\frac{r^2}{L^2}\ln\frac{1-r^2/L^2}{1-R^2/L^2}-\frac{r^2}{L^2}-\ln\left(1-r^2/L^2\right)\right] \tag{3-31}$$

where we have integrated by parts.

$$\left(\frac{3\epsilon}{\epsilon+\eta}\right)\left(\frac{1}{4}+\frac{7\eta}{8\epsilon}\right)\int\left(\frac{r^3}{L^4}-\frac{rR^2}{L^4}\right)dr = 3\left(1-\frac{\eta}{\epsilon}\right)\left(\frac{1}{4}+\frac{7\eta}{8\epsilon}\right)\left[\frac{r^4}{4L^4}-\frac{r^2R^2}{2L^4}\right]$$

$$= \left(\frac{3}{4}+\frac{15\eta}{8\epsilon}\right)\left[\frac{r^4}{4L^4}-\frac{r^2R^2}{2L^4}\right]$$

$$= \left(\frac{3}{16}+\frac{15\eta}{32\epsilon}\right)\left[\frac{r^4}{L^4}-\frac{2r^2R^2}{L^4}\right] \qquad (3\text{-}32)$$

Using (3-29) - (3-32) we form $v(r)$.

$$v(r) = \left(\frac{3\eta}{2\epsilon}-\frac{1}{2}\right)\ln\left(1-r^2/L^2\right)+\frac{6\eta}{\epsilon}\left(1-\frac{\eta}{\epsilon}\right)\left(\frac{r^2}{L^2}\ln\frac{r}{R}-\frac{r^2}{2L^2}\right)$$

$$+\left(\frac{3}{4}-\frac{57\eta}{16\epsilon}\right)\left[\frac{r^2}{L^2}\ln\frac{1-r^2/L^2}{1-R^2/L^2}-\frac{r^2}{L^2}-\ln\left(1-r^2/L^2\right)\right]$$

$$+\left(\frac{3}{16}+\frac{15\eta}{32\epsilon}\right)\left[\frac{r^4}{L^4}-\frac{2r^2R^2}{L^4}\right]+\text{const.} \qquad (3\text{-}33)$$

$$\nu(r) = \left(\frac{81\eta}{16\epsilon} - \frac{5}{4}\right) \ln\left(1 - r^2/L^2\right) + \left(\frac{3}{4} - \frac{57\eta}{16\epsilon}\right)\left[\frac{r^2}{L^2} \ln \frac{1 - r^2/L^2}{1 - R^2/L^2} - \frac{r^2}{L^2}\right]$$

$$+ \frac{6\eta}{\epsilon}\left(1 - \frac{\eta}{\epsilon}\right)\left(\frac{r^2}{L^2} \ln \frac{r}{R} - \frac{r^2}{2L^2}\right) + \left(\frac{3}{16} + \frac{15\eta}{32\epsilon}\right)\left[\frac{r^4}{L^4} - \frac{2r^2R^2}{L^4}\right] + \text{const.} \tag{3-34}$$

Since $\nu = -\lambda$ at $r = R$ we have

$$\ln\left(1 - R^2/L^2\right) = \left(\frac{81\eta}{16\epsilon} - \frac{5}{4}\right) \ln\left(1 - R^2/L^2\right) + \left(\frac{3}{4} - \frac{57\eta}{16\epsilon}\right)\left[-\frac{R^2}{L^2}\right]$$

$$- \frac{3\eta R^2}{\epsilon L^2}\left(1 - \frac{\eta}{\epsilon}\right) + \left(\frac{3}{16} + \frac{15\eta}{32\epsilon}\right)\left[-\frac{R^4}{L^4}\right] + \text{const}$$

$\therefore$

$$\nu(r) = \left(\frac{81\eta}{16\epsilon} - \frac{5}{4}\right) \ln \frac{1 - r^2/L^2}{1 - R^2/L^2} + \left(\frac{3}{4} - \frac{57\eta}{16\epsilon}\right)\left[\frac{r^2}{L^2} \ln \frac{1 - r^2/L^2}{1 - R^2/L^2}\right.$$

$$\left. + \frac{R^2}{L^2} - \frac{r^2}{L^2}\right] + \left[\frac{3\eta}{\epsilon}\left(\frac{R^2}{L^2} - \frac{r^2}{L^2}\right) + \frac{6\eta r^2}{\epsilon L^2} \ln \frac{r}{R}\right]\left(1 - \frac{\eta}{\epsilon}\right)$$

$$+ \left(\frac{3}{16} + \frac{15\eta}{32\epsilon}\right)\left[\frac{r^4}{L^4} + \frac{R^4}{L^4} - \frac{2r^2R^2}{L^4}\right] + \ln\left(1 - R^2/L^2\right) \tag{3-35}$$

In order to obtain e^{ν}, we write

$$e^{\nu} \simeq 1+\nu = 1-\frac{5}{4}\ln\frac{1-r^2/L^2}{1-R^2/L^2}+\frac{3}{4}\left[\frac{R^2}{L^2}-\frac{r^2}{L^2}\right]+\ln\left(1-r^2/L^2\right)$$

$$= 1+\frac{5r^2}{4L^2}-\frac{5R^2}{4L^2}+\frac{3}{4}\left[\frac{R^2}{L^2}-\frac{r^2}{L^2}\right]-\frac{R^2}{L^2}$$

$$= 1+\frac{r^2}{2L^2}-\frac{3R^2}{2L^2} \qquad (3\text{-}36)$$

to first order

where we have used

$$\ln\left(1-r^2/L^2\right) \simeq -\frac{r^2}{L^2}-\frac{r^4}{2L^4}$$

In order to determine the charge density that produces a constant electrical energy density, we refer to (2-9) and (2-10).

$$F^{10} = (8\pi\eta)^{1/2} e^{-\frac{\nu+\lambda}{2}} = \frac{4\pi}{r^2}e^{-\frac{\nu+\lambda}{2}}\int_0^r \rho r^2 e^{\lambda/2}dr$$

$$\therefore\ \eta^{1/2} = \frac{\sqrt{2\pi}}{r^2}\int_0^r \rho r^2 e^{\lambda/2}dr \qquad (3\text{-}37)$$

By inspection, we obtain

$$\rho = \left(\frac{2\eta}{\pi}\right)^{1/2} \frac{e^{-\lambda/2}}{r} = \left(\frac{2\eta}{\pi}\right)^{1/2} \left(\frac{1}{r^2} - \frac{1}{L^2}\right)^{1/2} \tag{3-38}$$

The singularity in charge density at $r=0$ is physically acceptable since the total charge is finite.

Stability Analysis

In the pulsation equation given by (2-86), we set

$$\eta' = 0 \, , \quad \frac{8\pi G}{c^4}(\epsilon + P)\, e^{\lambda} = \frac{\lambda' + \nu'}{r}$$

Then:

$$\sigma^2 e^{\lambda-\nu}(\epsilon+P)\xi = \frac{4P'}{r}\xi - e^{-\frac{\lambda+2\nu}{2}} \frac{d}{dr}\left[e^{\frac{\lambda+3\nu}{2}} \frac{\gamma P}{r^2} \frac{d}{dr}\left(r^2 e^{-\nu/2}\xi\right)\right]$$
$$+ \xi\left[\frac{8\eta P'}{r(P+\epsilon)} - \frac{(P')^2}{P+\epsilon} - \frac{16\eta^2}{r^2(P+\epsilon)} + \frac{(\lambda'+\nu')(P+\epsilon)}{r}\right] \tag{3-39}$$

Consider the second term on the right in (3-39)

$$e^{-\frac{\lambda+2\nu}{2}} \frac{d}{dr}\left[e^{\frac{\lambda+2\nu}{2}} \frac{\gamma P}{r^2} e^{\nu/2} \frac{d}{dr}\left(r^2 e^{-\nu/2}\xi\right)\right] =$$

$$e^{-\frac{\lambda+2\nu}{2}} \frac{d}{dr}\left[e^{\frac{\lambda+2\nu}{2}} \frac{\gamma P}{r^2} e^{\nu/2}\left(2r e^{-\nu/2}\xi - \frac{\nu' r^2}{2} e^{-\nu/2}\xi + r^2 e^{-\nu/2}\xi'\right)\right] =$$

$$e^{-\frac{\lambda+2\nu}{2}}\frac{d}{dr}\left[\gamma e^{\frac{\lambda+2\nu}{2}}\left(\frac{2P}{r}\xi - \frac{\nu' P\xi}{2} + P\xi'\right)\right]=$$

$$e^{-\frac{\lambda+2\nu}{2}}\left[\gamma e^{\frac{\lambda+2\nu}{2}}\left(\frac{2P}{r}\xi' + \frac{2P'}{r}\xi - \frac{2P}{r^2}\xi - \frac{P\nu'\xi'}{2} - \frac{\nu'P'\xi}{2}\right.\right.$$

$$\left. - \frac{\nu''P\xi}{2} + P\xi'' + P'\xi'\right) + \gamma e^{\frac{\lambda+2\nu}{2}}\left(\frac{\lambda'}{2}+\nu'\right)$$

$$\left.\cdot\left(\frac{2P}{r}\xi - \frac{\nu'P\xi}{2} + P\xi'\right)\right] =$$

$$\xi''\gamma P + \xi'\gamma\left[\frac{2P}{r} + P' + \frac{P\lambda'}{2} + P\nu' - \frac{P\nu'}{2}\right]$$

$$+\xi\gamma\left[\frac{2P'}{r} - \frac{2P}{r^2} - \frac{\nu'P'}{2} - \frac{P\nu''}{2} + \frac{P\lambda'}{r} + \frac{2P\nu'}{r} - \frac{P\nu'\lambda'}{4} - \frac{P(\nu')^2}{2}\right] =$$

Continued on the next page.

$$\xi''\gamma P + \xi'\gamma\left[P' + P\left(\frac{2}{r} + \frac{\lambda'}{2} + \frac{\nu'}{2}\right)\right] + \xi\left[\gamma P'\left(\frac{2}{r} - \frac{\nu'}{2}\right)\right.$$

$$\left. + \gamma P\left(\frac{\lambda'}{r} + \frac{2\nu'}{r} - \frac{2}{r^2} - \frac{\nu''}{2} - \frac{\nu'\lambda'}{4} - \frac{(\nu')^2}{2}\right)\right] \qquad (3\text{-}40)$$

Putting (3-40) into (3-39) gives

$$\sigma^2 e^{\lambda-\nu}(\epsilon+P)\xi = \frac{4}{r}P'\xi - \xi''\gamma P - \xi'\gamma\left[P' + P\left(\frac{2}{r} + \frac{\lambda'}{2} + \frac{\nu'}{2}\right)\right]$$

$$- \xi\left[\gamma P'\left(\frac{2}{r} - \frac{\nu'}{2}\right) + \gamma P\left(\frac{\lambda'}{r} + \frac{2\nu'}{r} - \frac{2}{r^2} - \frac{\nu''}{2} - \frac{\nu'\lambda'}{4} - \frac{(\nu')^2}{2}\right)\right.$$

$$\left. - \frac{8\chi P'}{r(P+\epsilon)} + \frac{(P')^2}{P+\epsilon} + \frac{16\chi^2}{r^2(P+\epsilon)} - \frac{(\lambda'+\nu')(\chi+P)}{r}\right] \qquad (3\text{-}41)$$

Multiplying by $r^2(\epsilon+P)$ and rearranging we get

$$\xi''\gamma r^2 P(\epsilon+P) + \xi'\gamma\left[r^2P'(\epsilon+P) + P(\epsilon+P)\left(2r + \frac{\lambda' r^2}{2} + \frac{\nu' r^2}{2}\right)\right]$$

$$+ \xi\left[\gamma P'(\epsilon+P)\left(2r - \frac{\nu' r^2}{2}\right) + \gamma P(\epsilon+P)\left(\lambda' r + 2\nu' r - \frac{\nu'' r^2}{2} - \frac{\nu'\lambda' r^2}{4}\right.\right.$$

$$\left. - \frac{(\nu')^2 r^2}{2} - 2\right) - 8\chi r P' + r^2(P')^2 + 16\chi^2 - (\lambda'+\nu')(\epsilon+P)(\chi+P)r$$

$$\left. - 4rP'(\epsilon+P) + \sigma^2 e^{\lambda-\nu} r^2(\epsilon+P)^2\right] = 0 \qquad (3\text{-}42)$$

This is the pulsation equation.

We make the following observations. In evaluating each of the terms of the pulsation equation (3-42), we shall find that they consist of first, second, and higher order terms. Since these terms are all to be substituted back into the pulsation equation, the higher order terms would there be dropped, and the pulsation equation would consist of first and second order terms. Anticipating this elimination of the higher order terms, we will drop them as they appear in the terms to be evaluated below. Hence, if we should drop third order terms in an expression consisting only of second and third order terms, the reader should keep in mind that we are evaluating the pulsation equation to first order only.

Now we evaluate (3-42) term by term

$$\gamma r^2 P(\epsilon + P) = \gamma \epsilon^2 r^2 \frac{P}{\epsilon}\left[1 + \frac{P}{\epsilon}\right]$$

$$= \gamma \epsilon^2 r^2 \left\{ (1 - r^2/L^2)\left[\frac{4\eta}{\epsilon} \ln\frac{r}{R} + \left(\frac{1}{2} - \frac{15\eta}{8\epsilon}\right) \ln\frac{1 - r^2/L^2}{1 - R^2/L^2} \right.\right.$$

$$\left. + \left(\frac{1}{4} + \frac{7\eta}{8\epsilon}\right)\left(\frac{r^2}{L^2} - \frac{R^2}{L^2}\right)\right]\left[1 + (1 - r^2/L^2)\left[\frac{4\eta}{\epsilon} \ln\frac{r}{R} + \left(\frac{1}{2} - \frac{15\eta}{8\epsilon}\right) \ln\frac{1 - r^2/L^2}{1 - R^2/L^2}\right.\right.$$

$$\left.\left.\left. + \left(\frac{1}{4} + \frac{7\eta}{8\epsilon}\right)\left(\frac{r^2}{L^2} - \frac{R^2}{L^2}\right)\right]\right]\right\}$$

$$= \gamma \epsilon^2 r^2 \left\{ (1 - r^2/L^2)\left[\frac{4\eta}{\epsilon} \ln\frac{r}{R} + \left(\frac{1}{2} - \frac{15\eta}{8\epsilon}\right) \ln\frac{1 - r^2/L^2}{1 - R^2/L^2} + \left(\frac{1}{4} + \frac{7\eta}{8\epsilon}\right)\left(\frac{r^2}{L^2} - \frac{R^2}{L^2}\right)\right] + \right.$$

Continued on the next page.

$$\left(1-r^2/L^2\right)^2\left[\frac{16\eta^2}{\epsilon^2}\ln^2\frac{r}{R}+\left(\frac{1}{2}-\frac{15\eta}{8\epsilon}\right)^2\ln^2\frac{1-r^2/L^2}{1-R^2/L^2}+\left(\frac{1}{4}+\frac{7\eta}{8\epsilon}\right)^2\left(\frac{r^4}{L^4}-\frac{2r^2R^2}{L^4}+\frac{R^4}{L^4}\right)\right.$$

$$+\frac{8\eta}{\epsilon}\left(\frac{1}{2}-\frac{15\eta}{8\epsilon}\right)\left(\ln\frac{1-r^2/L^2}{1-R^2/L^2}\right)\left(\ln\frac{r}{R}\right)+\frac{8\eta}{\epsilon}\left(\frac{1}{4}+\frac{7\eta}{8\epsilon}\right)\left(\frac{r^2}{L^2}-\frac{R^2}{L^2}\right)\ln\frac{r}{R}$$

$$\left.\left.+\,2\left(\frac{1}{2}-\frac{15\eta}{8\epsilon}\right)\left(\frac{1}{4}+\frac{7\eta}{8\epsilon}\right)\left(\frac{r^2}{L^2}-\frac{R^2}{L^2}\right)\ln\frac{1-r^2/L^2}{1-R^2/L^2}\right]\right\}$$

$$=\gamma\epsilon^2r^{-2}\left\{\frac{4\eta}{\epsilon}\ln\frac{r}{R}+\left(\frac{1}{2}-\frac{15\eta}{8\epsilon}\right)\ln\frac{1-r^2/L^2}{1-R^2/L^2}+\left(\frac{1}{4}+\frac{7\eta}{8\epsilon}\right)\left(\frac{r^2}{L^2}-\frac{R^2}{L^2}\right)\right.$$

$$-\frac{4\eta r^2}{\epsilon L^2}\ln\frac{r}{R}-\frac{r^2}{2L^2}\ln\frac{1-r^2/L^2}{1-R^2/L^2}-\frac{r^4}{4L^4}+\frac{r^2R^2}{4L^4}$$

$$+\left(1-\frac{2r^2}{L^2}\right)\left[\frac{16\eta^2}{\epsilon^2}\ln^2\frac{r}{R}+\frac{1}{4}\ln^2\frac{1-r^2/L^2}{1-R^2/L^2}+\frac{r^4}{16L^4}-\frac{r^2R^2}{8L^4}+\frac{R^4}{16L^4}\right.$$

$$+\frac{4\eta}{\epsilon}\left(\ln\frac{1-r^2/L^2}{1-R^2/L^2}\right)\ln\frac{r}{R}+\frac{2\eta r^2}{\epsilon L^2}\ln\frac{r}{R}-\frac{2\eta R^2}{\epsilon L^2}\ln\frac{r}{R}$$

$$\left.\left.+\frac{r^2}{4L^2}\ln\frac{1-r^2/L^2}{1-R^2/L^2}-\frac{R^2}{4L^2}\ln\frac{1-r^2/L^2}{1-R^2/L^2}\right]\right\}$$

$$=\gamma\epsilon^2r^2\left\{\frac{4\eta}{\epsilon}\ln\frac{r}{R}+\left(\frac{1}{2}-\frac{15\eta}{8\epsilon}\right)\ln\frac{1-r^2/L^2}{1-R^2/L^2}\right.$$

$$+\left(\frac{1}{4}+\frac{7\eta}{8\epsilon}\right)\left(\frac{r^2}{L^2}-\frac{R^2}{L^2}\right)-\frac{4\eta r^2}{\epsilon L^2}\ln\frac{r}{R}$$

Continued on the next page.

$$-\frac{r^2}{2L^2}\ln\frac{1-r^2/L^2}{1-R^2/L^2}-\frac{r^4}{4L^4}+\frac{r^2R^2}{4L^4}+\frac{16\eta^2}{\epsilon^2}\ln^2\frac{r}{R}+\frac{1}{4}\ln^2\frac{1-r^2/L^2}{1-R^2/L^2}$$

$$+\frac{r^4}{16L^4}-\frac{r^2R^2}{8L^4}+\frac{R^4}{16L^4}+\frac{4\eta}{\epsilon}\left(\ln\frac{1-r^2/L^2}{1-R^2/L^2}\right)\ln\frac{r}{R}+\frac{2\eta r^2}{\epsilon L^2}\ln\frac{r}{R}$$

$$-\frac{2\eta R^2}{\epsilon L^2}\ln\frac{r}{R}+\frac{r^2}{4L^2}\ln\frac{1-r^2/L^2}{1-R^2/L^2}-\frac{R^2}{4L^2}\ln\frac{1-r^2/L^2}{1-R^2/L^2}\Bigg\}$$

$$=\gamma\epsilon^2r^2\Bigg[\left(\frac{4\eta}{\epsilon}-\frac{2\eta r^2}{\epsilon L^2}-\frac{2\eta R^2}{\epsilon L^2}+\frac{16\eta^2}{\epsilon^2}\ln\frac{r}{R}\right)\ln\frac{r}{R}$$

$$+\left(\frac{1}{2}-\frac{15\eta}{8\epsilon}-\frac{r^2}{4L^2}-\frac{R^2}{4L^2}+\frac{4\eta}{\epsilon}\ln\frac{r}{R}+\frac{1}{4}\ln\frac{1-r^2/L^2}{1-R^2/L^2}\right)\ln\frac{1-r^2/L^2}{1-R^2/L^2}$$

$$+\frac{r^2}{4L^2}-\frac{R^2}{4L^2}+\frac{r^2R^2}{8L^4}-\frac{3r^4}{16L^4}+\frac{R^4}{16L^4}+\frac{7\eta r^2}{8\epsilon L^2}-\frac{7\eta R^2}{8\epsilon L^2}\Bigg] \qquad (3\text{-}43)$$

$$\gamma r^2P'(\epsilon+P)=\gamma\epsilon^2r^2\frac{P'}{\epsilon}\left(1+\frac{P}{\epsilon}\right)$$

$$=\gamma\epsilon^2r^2\Bigg\{\left(1-r^2/L^2\right)\Bigg[\frac{4\eta}{\epsilon r}-\frac{2r}{L^2}\frac{\left(1/2-15\eta/8\epsilon\right)}{\left(1-r^2/L^2\right)}$$

$$+\frac{2r}{L^2}\left(\frac{1}{4}+\frac{7\eta}{8\epsilon}\right)\Bigg]$$

Continued on the next page.

$$-\frac{2r}{L^2}\left[\frac{4\eta}{\epsilon}\ln\frac{r}{R}+\left(\frac{1}{2}-\frac{15\eta}{8\epsilon}\right)\ln\frac{1-r^2/L^2}{1-R^2/L^2}+\left(\frac{1}{4}+\frac{7\eta}{8\epsilon}\right)\left(\frac{r^2}{L^2}-\frac{R^2}{L^2}\right)\right]\Bigg\}\left(1+\frac{P}{\epsilon}\right)$$

$$=\gamma\epsilon^2 r\Bigg\{(1-r^2/L^2)\left[\frac{4\eta}{\epsilon}-\frac{r^2}{L^2}\left(1-\frac{15\eta}{4\epsilon}\right)\left(1+\frac{r^2}{L^2}\right)+\frac{r^2}{L^2}\left(\frac{1}{2}+\frac{7\eta}{4\epsilon}\right)\right]$$

$$-\frac{r^2}{L^2}\left[\frac{8\eta}{\epsilon}\ln\frac{r}{R}+\left(1-\frac{15\eta}{4\epsilon}\right)\ln\frac{1-r^2/L^2}{1-R^2/L^2}+\left(\frac{1}{2}+\frac{7\eta}{4\epsilon}\right)\left(\frac{r^2}{L^2}-\frac{R^2}{L^2}\right)\right]\Bigg\}\left(1+\frac{P}{\epsilon}\right)$$

$$=\gamma\epsilon^2 r\Bigg\{(1-r^2/L^2)\left[\frac{4\eta}{\epsilon}-\frac{r^2}{L^2}-\frac{r^4}{L^4}+\frac{15\eta r^2}{4\epsilon L^2}+\frac{r^2}{2L^2}+\frac{7\eta r^2}{4\epsilon L^2}\right]$$

$$-\frac{8\eta r^2}{\epsilon L^2}\ln\frac{r}{R}-\frac{r^2}{L^2}\ln\frac{1-r^2/L^2}{1-R^2/L^2}-\frac{r^4}{2L^4}+\frac{r^2R^2}{2L^4}\Bigg\}\left(1+\frac{P}{\epsilon}\right)$$

$$=\gamma\epsilon^2 r\Bigg\{(1-r^2/L^2)\left[\frac{4\eta}{\epsilon}-\frac{r^2}{2L^2}-\frac{r^4}{L^4}+\frac{11\eta r^2}{2\epsilon L^2}\right]-\frac{8\eta r^2}{\epsilon L^2}\ln\frac{r}{R}$$

$$-\frac{r^2}{L^2}\ln\frac{1-r^2/L^2}{1-R^2/L^2}-\frac{r^4}{2L^4}+\frac{r^2R^2}{2L^4}\Bigg\}\left(1+\frac{P}{\epsilon}\right)$$

$$=\gamma\epsilon^2 r\Bigg\{\frac{4\eta}{\epsilon}-\frac{r^2}{2L^2}-\frac{r^4}{L^4}+\frac{11\eta r^2}{2\epsilon L^2}-\frac{4\eta r^2}{\epsilon L^2}+\frac{r^4}{2L^4}-\frac{8\eta r^2}{\epsilon L^2}\ln\frac{r}{R}$$

$$-\frac{r^2}{L^2}\ln\frac{1-r^2/L^2}{1-R^2/L^2}-\frac{r^4}{2L^4}+\frac{r^2R^2}{2L^4}\Bigg\}\left(1+\frac{P}{\epsilon}\right)$$

Continued on the next page.

$$= \gamma\epsilon^2 r\left[\frac{4\eta}{\epsilon} - \frac{r^2}{2L^2} + \frac{r^2R^2}{2L^4} - \frac{r^4}{L^4} + \frac{3\eta r^2}{2\epsilon L^2} - \frac{8\eta r^2}{\epsilon L^2}\ln\frac{r}{R} - \frac{r^2}{L^2}\ln\frac{1-r^2/L^2}{1-R^2/L^2}\right]\left(1+\frac{P}{\epsilon}\right)$$

$$= \gamma\epsilon^2 r\left[\frac{4\eta}{\epsilon} - \frac{r^2}{2L^2} + \frac{r^2R^2}{2L^4} - \frac{r^4}{L^4} + \frac{3\eta r^2}{2\epsilon L^2} - \frac{8\eta r^2}{\epsilon L^2}\ln\frac{r}{R} - \frac{r^2}{L^2}\ln\frac{1-r^2/L^2}{1-R^2/L^2}\right.$$

$$\left. + \left(\frac{4\eta}{\epsilon} - \frac{r^2}{2L^2}\right)\frac{P}{\epsilon}\right] \qquad (3\text{-}44)$$

$$\gamma P(\epsilon+P)\left[2r + \frac{\lambda' r^2}{2} + \frac{\nu' r^2}{2}\right] = \gamma\epsilon^2 r\left[2 + \frac{\lambda' r}{2} + \frac{\nu' r}{2}\right]\frac{P}{\epsilon}\left(1+\frac{P}{\epsilon}\right)$$

$$= \gamma\epsilon^2 r\left[2 + \frac{r^2}{L^2(1-r^2/L^2)} + \frac{3(\epsilon+P)r^2}{2L^2(\epsilon+\eta)(1-r^2/L^2)} - \frac{r^2}{L^2(1-r^2/L^2)}\right]$$

$$\cdot \frac{P}{\epsilon}\left(1+\frac{P}{\epsilon}\right)$$

$$= \gamma\epsilon^2 r\,(1-r^2/L^2)\left[2 + \frac{3r^2(1+P/\epsilon)}{2L^2(1+\eta/\epsilon)(1-r^2/L^2)}\right]$$

$$\cdot\left[\frac{4\eta}{\epsilon}\ln\frac{r}{R} + \left(\frac{1}{2} - \frac{15\eta}{8\epsilon}\right)\ln\frac{1-r^2/L^2}{1-R^2/L^2} + \left(\frac{1}{4} + \frac{7\eta}{8\epsilon}\right)\left(\frac{r^2}{L^2} - \frac{R^2}{L^2}\right)\right]\left(1+\frac{P}{\epsilon}\right)$$

$$= \gamma\epsilon^2 r\left[2 - \frac{2r^2}{L^2} + \frac{r^2}{L^2}\left(\frac{3}{2} + \frac{3P}{2\epsilon} - \frac{3\eta}{2\epsilon}\right)\right]\left[\frac{4\eta}{\epsilon}\ln\frac{r}{R}\right.$$

$$\left. + \left(\frac{1}{2} - \frac{15\eta}{8\epsilon}\right)\ln\frac{1-r^2/L^2}{1-R^2/L^2} + \frac{r^2}{4L^2} - \frac{R^2}{4L^2} + \frac{7\eta r^2}{8\epsilon L^2} - \frac{7\eta R^2}{8\epsilon L^2}\right]\left(1+\frac{P}{\epsilon}\right)$$

Continued on the next page.

$$= \gamma \epsilon^2 r \left[2 - \frac{r^2}{2L^2}\right]\left[\frac{4\eta}{\epsilon} \ln \frac{r}{R} + \left(\frac{1}{2} - \frac{15\eta}{8\epsilon}\right) \ln \frac{1 - r^2/L^2}{1 - R^2/L^2} + \frac{r^2}{4L^2} - \frac{R^2}{4L^2}\right.$$

$$\left. + \frac{7\eta r^2}{8\epsilon L^2} - \frac{7\eta R^2}{8\epsilon L^2}\right]\left(1 + \frac{P}{\epsilon}\right)$$

$$= \gamma \epsilon^2 r \left[\left(\frac{8\eta}{\epsilon} - \frac{2\eta r^2}{\epsilon L^2}\right) \ln \frac{r}{R} + \left(1 - \frac{15\eta}{4\epsilon} - \frac{r^2}{4L^2}\right) \ln \frac{1 - r^2/L^2}{1 - R^2/L^2}\right.$$

$$+ \frac{r^2}{2L^2} - \frac{R^2}{2L^2} + \frac{r^2 R^2}{8L^4} - \frac{r^4}{8L^4} + \frac{7\eta r^2}{4\epsilon L^2} - \frac{7\eta R^2}{4\epsilon L^2}$$

$$\left. + \left(\frac{8\eta}{\epsilon} \ln \frac{r}{R} + \ln \frac{1 - r^2/L^2}{1 - R^2/L^2} + \frac{r^2}{2L^2} - \frac{R^2}{2L^2}\right)\frac{P}{\epsilon}\right] \qquad (3\text{-}45)$$

$$\gamma P'(\epsilon + P)\left[2r - \frac{\nu' r^2}{2}\right] = \gamma \epsilon^2 \frac{P'}{\epsilon}\left[2r - \frac{\nu' r^2}{2}\right]\left(1 + \frac{P}{\epsilon}\right)$$

$$= \gamma \epsilon^2 \left\{\left(\left(1 - r^2/L^2\right)\left[\frac{4\eta}{\epsilon} - \frac{2r^2}{L^2}\frac{\left(\frac{1}{2} - \frac{15\eta}{8\epsilon}\right)}{\left(1 - r^2/L^2\right)} + \frac{2r^2}{L^2}\left(\frac{1}{4} + \frac{7\eta}{8\epsilon}\right)\right.\right.\right.$$

$$\left.\left. - \frac{2r^2}{L^2}\left[\frac{4\eta}{\epsilon} \ln \frac{r}{R} + \left(\frac{1}{2} - \frac{15\eta}{8\epsilon}\right) \ln \frac{1 - r^2/L^2}{1 - R^2/L^2} + \left(\frac{1}{4} + \frac{7\eta}{8\epsilon}\right)\left(\frac{r^2}{L^2} - \frac{R^2}{L^2}\right)\right]\right]\right)$$

$$\left. \cdot \left[2 - \frac{r^2}{L^2\left(1 - r^2/L^2\right)}\left(\frac{3(\epsilon + P)}{2(\epsilon + \eta)} - 1\right)\right]\left(1 + \frac{P}{\epsilon}\right)\right\}$$

Continued on the next page.

$$= \gamma\epsilon^2\left\{\left[\frac{4\eta}{\epsilon} - \frac{4\eta r^2}{\epsilon L^2} - \frac{r^2}{L^2} + \frac{15\eta r^2}{4\epsilon L^2} + \frac{r^2}{2L^2} + \frac{7\eta r^2}{4\epsilon L^2} - \frac{r^4}{2L^4}\right.\right.$$

$$\left. - \frac{8\eta r^2}{\epsilon L^2}\ln\frac{r}{R} - \frac{r^2}{L^2}\ln\frac{1-r^2/L^2}{1-R^2/L^2} - \frac{r^4}{2L^4} + \frac{r^2R^2}{2L^4}\right]$$

$$\left.\cdot\left[2 - \frac{3r^2}{2L^2} + \frac{r^2}{L^2}\right]\left(1+\frac{P}{\epsilon}\right)\right\}$$

$$= \gamma\epsilon^2\left\{\left[\frac{4\eta}{\epsilon} - \frac{r^2}{2L^2} - \frac{r^4}{L^4} + \frac{r^2R^2}{2L^4} + \frac{3\eta r^2}{2\epsilon L^2} - \frac{8\eta r^2}{\epsilon L^2}\ln\frac{r}{R}\right.\right.$$

$$\left.\left. - \frac{r^2}{L^2}\ln\frac{1-r^2/L^2}{1-R^2/L^2}\right]\left[2 - \frac{r^2}{2L^2}\right]\left(1+\frac{P}{\epsilon}\right)\right\}$$

$$= \gamma\epsilon^2\left\{\left[\frac{8\eta}{\epsilon} - \frac{r^2}{L^2} - \frac{3r^4}{L^4} + \frac{r^2R^2}{L^4} + \frac{3\eta r^2}{\epsilon L^2} - \frac{16\eta r^2}{\epsilon L^2}\ln\frac{r}{R}\right.\right.$$

$$\left.\left. - \frac{2r^2}{L^2}\ln\frac{1-r^2/L^2}{1-R^2/L^2} - \frac{2\eta r^2}{\epsilon L^2} + \frac{r^4}{4L^4}\right]\left(1+\frac{P}{\epsilon}\right)\right\}$$

$$= \gamma\epsilon^2\left\{\left[\frac{8\eta}{\epsilon} - \frac{r^2}{L^2} + \frac{r^2R^2}{L^4} - \frac{7r^4}{4L^4} + \frac{\eta r^2}{\epsilon L^2} - \frac{16\eta r^2}{\epsilon L^2}\ln\frac{r}{R}\right.\right.$$

$$\left.\left. - \frac{2r^2}{L^2}\ln\frac{1-r^2/L^2}{1-R^2/L^2}\right] + \left[\frac{8\eta}{\epsilon} - \frac{r^2}{L^2}\right]\frac{P}{\epsilon}\right\} \qquad (3\text{-}46)$$

$$\gamma P(\epsilon+P)\left[\lambda' r + 2\nu' r - \frac{\nu'' r^2}{2} - \frac{\nu'\lambda' r^2}{4} - \frac{(\nu' r)^2}{2} - 2\right]$$

Since $\frac{\nu'\lambda' r^2}{4}$ and $\frac{(\nu' r)^2}{2}$ are second order, we get

$$\gamma\epsilon^2\left[\lambda' r + 2\nu' r - \frac{\nu'' r^2}{2} - 2\right]\frac{P}{\epsilon}\left(1+\frac{P}{\epsilon}\right)$$

$$= \gamma\epsilon^2\left[\frac{2r^2}{L^2(1-r^2/L^2)} + \frac{6r^2(\epsilon+P)}{L^2(\epsilon+n)(1-r^2/L^2)} - \frac{4r^2}{L^2(1-r^2/L^2)}\right.$$

$$- \frac{3r^2}{2L^2(\epsilon+n)}\left(\frac{2\epsilon r^2}{L^2(1-r^2/L^2)^2} + \frac{\epsilon}{1-r^2/L^2} + \frac{2r^2 P}{L^2(1-r^2/L^2)^2}\right.$$

$$\left.\left. + \frac{rP'}{1-r^2/L^2} + \frac{P}{1-r^2/L^2}\right) + \frac{2r^4}{L^4(1-r^2/L^2)^2} + \frac{r^2}{L^2(1-r^2/L^2)} - 2\right]\frac{P}{\epsilon}\left(1+\frac{P}{\epsilon}\right)$$

$$= \gamma\epsilon^2\left[\frac{6r^2(1+P/\epsilon - n/\epsilon)}{L^2(1-r^2/L^2)} - \frac{2r^2}{L^2(1-r^2/L^2)} - \frac{3r^2}{2L^2(1-r^2/L^2)}\right.$$

$$\left. - \frac{3r^3 P'}{2L^2(\epsilon+n)(1-r^2/L^2)} + \frac{r^2}{L^2(1-r^2/L^2)} - 2\right]\frac{P}{\epsilon}\left(1+\frac{P}{\epsilon}\right)$$

Continued on the next page.

$$= \gamma\epsilon^2\left[\frac{7r^2}{2L^2} - 2 - \frac{3r^3P'}{2\epsilon L^2}\right]\left(1+\frac{P}{\epsilon}\right)\frac{P}{\epsilon}$$

$$= \gamma\epsilon^2\left[\frac{7r^2}{2L^2} - 2 - \frac{3r^2}{2L^2}\left\{\left(1-r^2/L^2\right)\left[\frac{4\eta}{\epsilon} - \frac{2r^2}{L^2}\frac{\left(1/2-15\eta/8\epsilon\right)}{\left(1-r^2/L^2\right)}\right.\right.\right.$$

$$\left. + \frac{2r^2}{L^2}\left(\frac{1}{4}+\frac{7\eta}{8\epsilon}\right)\right] - \frac{2r^2}{L^2}\left[\frac{4\eta}{\epsilon}\ln\frac{r}{R} + \left(\frac{1}{2}-\frac{15\eta}{8\epsilon}\right)\ln\frac{1-r^2/L^2}{1-R^2/L^2}\right.$$

$$\left.\left.\left. + \left(\frac{1}{4}+\frac{7\eta}{8\epsilon}\right)\left(\frac{r^2}{L^2}-\frac{R^2}{L^2}\right)\right]\right\}\right]\left(1+\frac{P}{\epsilon}\right)\frac{P}{\epsilon}$$

$$= \gamma\epsilon^2\left[\frac{7r^2}{2L^2} - 2\right]\left(1+\frac{P}{\epsilon}\right)\frac{P}{\epsilon}$$

$$= \gamma\epsilon^2\left[\frac{7r^2}{2L^2} - 2 - \frac{2P}{\epsilon}\right]\frac{P}{\epsilon} \qquad (3\text{-}47$$

$$8\eta r P' = \epsilon^2 8\eta r \frac{P'}{\epsilon^2}$$

$$= \epsilon^2\frac{8\eta}{\epsilon}\left\{\left(1-r^2/L^2\right)\left[\frac{4\eta}{\epsilon} - \frac{2r^2\left(1/2-15\eta/8\epsilon\right)}{L^2\left(1-r^2/L^2\right)} + \frac{2r^2}{L^2}\left(\frac{1}{4}+\frac{7\eta}{8\epsilon}\right)\right]\right.$$

$$\left. - \frac{2r^2}{L^2}\left[\frac{4\eta}{\epsilon}\ln\frac{r}{R} + \left(\frac{1}{2}-\frac{15\eta}{8\epsilon}\right)\ln\frac{1-r^2/L^2}{1-R^2/L^2} + \left(\frac{1}{4}+\frac{7\eta}{8\epsilon}\right)\left(\frac{r^2}{L^2}-\frac{R^2}{L^2}\right)\right]\right\}$$

Continued on the next page.

$$= \epsilon^2\left\{\left(1-r^2/L^2\right)\left[\frac{32\eta^2}{\epsilon^2} - \frac{16\eta r^2}{\epsilon L^2}\left(\frac{1}{2}-\frac{15\eta}{8\epsilon}\right)\left(1+\frac{r^2}{L^2}\right) + \frac{16\eta r^2}{\epsilon L^2}\left(\frac{1}{4}+\frac{7\eta}{8\epsilon}\right)\right]\right.$$

$$\left. - \frac{16\eta r^2}{\epsilon L^2}\left[\frac{4\eta}{\epsilon}\ln\frac{r}{R} + \left(\frac{1}{2}-\frac{15\eta}{8\epsilon}\right)\ln\frac{1-r^2/L^2}{1-R^2/L^2} + \left(\frac{1}{4}+\frac{7\eta}{8\epsilon}\right)\left(\frac{r^2}{L^2}-\frac{R^2}{L^2}\right)\right]\right\}$$

$$= \epsilon^2\left\{\left(1-r^2/L^2\right)\left[\frac{32\eta^2}{\epsilon^2} - \frac{4\eta r^2}{\epsilon L^2}\right]\right\}$$

$$= \epsilon^2\left[\frac{32\eta^2}{\epsilon^2} - \frac{4\eta r^2}{\epsilon L^2}\right] \tag{3-48}$$

$$\left[rP'\right]^2 = \left\{r\left(1-r^2/L^2\right)\left[\frac{4\eta}{r} - \frac{2\epsilon r\left(1/2 - 15\eta/8\epsilon\right)}{L^2\left(1-r^2/L^2\right)} + \frac{2\epsilon r}{L^2}\left(\frac{1}{4}+\frac{7\eta}{8\epsilon}\right)\right]\right.$$

$$\left. - \frac{2r}{L^2}\left[4\eta\ln\frac{r}{R} + \epsilon\left(\frac{1}{2}-\frac{15\eta}{8\epsilon}\right)\ln\frac{1-r^2/L^2}{1-R^2/L^2} + \epsilon\left(\frac{1}{4}+\frac{7\eta}{8\epsilon}\right)\left(\frac{r^2}{L^2}-\frac{R^2}{L^2}\right)\right]\right\}^2$$

$$= \epsilon^2\left\{\left(1-r^2/L^2\right)\left[\frac{4\eta}{\epsilon} - \frac{r^2}{L^2}\left(1-\frac{15\eta}{4\epsilon}\right)\left(\frac{1}{1-r^2/L^2}\right) + \frac{r^2}{L^2}\left(\frac{1}{2}+\frac{7\eta}{4\epsilon}\right)\right]\right.$$

$$\left. - \frac{8\eta r^2}{\epsilon L^2}\ln\frac{r}{R} - \left(\frac{r^2}{L^2} - \frac{15\eta r^2}{4\epsilon L^2}\right)\ln\frac{1-r^2/L^2}{1-R^2/L^2} - \left(\frac{1}{2}+\frac{7\eta}{4\epsilon}\right)\left(\frac{r^4}{L^4}-\frac{r^2R^2}{L^4}\right)\right\}^2$$

Continued on the next page.

$$\epsilon^2\left[\frac{4\eta}{\epsilon}-\frac{4\eta r^2}{\epsilon L^2}-\frac{r^2}{L^2}+\frac{15\eta r^2}{4\epsilon L^2}+\frac{r^2}{2L^2}+\frac{7\eta r^2}{4\epsilon L^2}-\frac{r^4}{2L^4}-\frac{8\eta r^2}{\epsilon L^2}\ln\frac{r}{R}\right.$$

$$\left.-\frac{r^2}{L^2}\ln\frac{1-r^2/L^2}{1-R^2/L^2}-\frac{r^4}{2L^4}+\frac{r^2R^2}{2L^4}\right]^2$$

$$=\epsilon^2\left[\frac{4\eta}{\epsilon}-\frac{r^2}{2L^2}+\frac{r^2R^2}{2L^4}-\frac{r^4}{L^4}+\frac{3\eta r^2}{2\epsilon L^2}-\frac{8\eta r^2}{\epsilon L^2}\ln\frac{r}{R}-\frac{r^2}{L^2}\ln\frac{1-r^2/L^2}{1-R^2/L^2}\right]^2$$

$$=\epsilon^2\left[\frac{16\eta^2}{\epsilon^2}+\frac{r^4}{4L^4}-\frac{4\eta r^2}{\epsilon L^2}\right] \tag{3-49}$$

$$16\eta^2=\epsilon^2\left[\frac{16\eta^2}{\epsilon^2}\right] \tag{3-50}$$

$$(\lambda'+\nu')(\epsilon+P)(\eta+P)r=\epsilon^2 r\left[(\lambda'+\nu')\left(1+\frac{P}{\epsilon}\right)\left(\frac{\eta}{\epsilon}+\frac{P}{\epsilon}\right)\right]$$

$$=\epsilon^2 r\left[(\lambda'+\nu')\left(\frac{\eta}{\epsilon}+\frac{P}{\epsilon}+\frac{\eta P}{\epsilon^2}+\frac{P^2}{\epsilon^2}\right)\right]$$

$$=\epsilon^2\left[\frac{2r^2}{L^2(1-r^2/L^2)}+\frac{3r^2(\epsilon+P)}{L^2(\epsilon+\eta)(1-r^2/L^2)}-\frac{2r^2}{L^2(1-r^2/L^2)}\right]$$

$$\cdot\left(\frac{\eta}{\epsilon}+\frac{P}{\epsilon}+\frac{\eta P}{\epsilon^2}+\frac{P^2}{\epsilon^2}\right)$$

Continued on the next page.

$$= \epsilon^2 \left[\frac{3r^2}{L^2}\left(1+\frac{r^2}{L^2}\right)\left(1+\frac{P}{\epsilon}-\frac{\eta}{\epsilon}\right)\right]\left(\frac{\eta}{\epsilon}+\frac{P}{\epsilon}+\frac{\eta P}{\epsilon^2}+\frac{P^2}{\epsilon^2}\right)$$

$$= \epsilon^2\left[\frac{3\eta r^2}{\epsilon L^2}+\frac{3Pr^2}{\epsilon L^2}\right] \qquad (3\text{-}51)$$

$$4rP'(\epsilon+P) = \epsilon^2 4r \frac{P'}{\epsilon}\left[1+\frac{P}{\epsilon}\right]$$

$$=\epsilon^2 4r\left\{(1-r^2/L^2)\left[\frac{4\eta}{\epsilon r} - \frac{2r(1/2 - 15\eta/8\epsilon)}{L^2(1-r^2/L^2)} + \frac{2r}{L^2}\left(\frac{1}{4}+\frac{7\eta}{8\epsilon}\right)\right]\right.$$

$$\left.- \frac{2r}{L^2}\left[\frac{4\eta}{\epsilon}\ln\frac{r}{R} + \left(\frac{1}{2}-\frac{15\eta}{8\epsilon}\right)\ln\frac{1-r^2/L^2}{1-R^2/L^2} + \left(\frac{1}{4}+\frac{7\eta}{8\epsilon}\right)\left(\frac{r^2}{L^2}-\frac{R^2}{L^2}\right)\right]\right\}\left[1+\frac{P}{\epsilon}\right]$$

$$=\epsilon^2 4\left\{(1-r^2/L^2)\left[\frac{4\eta}{\epsilon} - \frac{r^2}{L^2}\left(1-\frac{15\eta}{4\epsilon}\right)\left(\frac{1}{1-r^2/L^2}\right)\right) + \frac{r^2}{L^2}\left(\frac{1}{2}+\frac{7\eta}{4\epsilon}\right)\right]\right.$$

$$\left.- \frac{8\eta r^2}{\epsilon L^2}\ln\frac{r}{R} - \left(\frac{r^2}{L^2}-\frac{15\eta r^2}{4\epsilon L^2}\right)\ln\frac{1-r^2/L^2}{1-R^2/L^2} - \left(\frac{1}{2}+\frac{7\eta}{4\epsilon}\right)\left(\frac{r^4}{L^4}-\frac{r^2R^2}{L^4}\right)\right\}\left[1+\frac{P}{\epsilon}\right]$$

$$=\epsilon^2 4\left\{\frac{4\eta}{\epsilon} - \frac{4\eta r^2}{\epsilon L^2} - \frac{r^2}{L^2} + \frac{15\eta r^2}{4\epsilon L^2} + \frac{r^2}{2L^2} + \frac{7\eta r^2}{4\epsilon L^2} - \frac{r^4}{2L^4}\right.$$

$$\left.-\frac{8\eta r^2}{\epsilon L^2}\ln\frac{r}{R} - \frac{r^2}{L^2}\ln\frac{1-r^2/L^2}{1-R^2/L^2} - \frac{r^4}{2L^4} + \frac{r^2R^2}{2L^4}\right\}\left[1+\frac{P}{\epsilon}\right]$$

Continued on the next page.

$$= \epsilon^2 4 \left\{ \frac{4\kappa}{\epsilon} - \frac{r^2}{2L^2} + \frac{r^2R^2}{2L^4} - \frac{r^4}{L^4} + \frac{3\kappa r^2}{2\epsilon L^2} - \frac{8\kappa r^2}{\epsilon L^2} \ln \frac{r}{R} \right.$$

$$\left. - \frac{r^2}{L^2} \ln \frac{1 - r^2/L^2}{1 - R^2/L^2} \right\} \left[1 + \frac{P}{\epsilon} \right]$$

$$= \epsilon^2 \left\{ \frac{16\kappa}{\epsilon} - \frac{2r^2}{L^2} + \frac{2r^2R^2}{L^4} - \frac{4r^4}{L^4} + \frac{6\kappa r^2}{\epsilon L^2} - \frac{32\kappa r^2}{\epsilon L^2} \ln \frac{r}{R} \right.$$

$$\left. - \frac{4r^2}{L^2} \ln \frac{1 - r^2/L^2}{1 - R^2/L^2} + \left(\frac{16\kappa}{\epsilon} - \frac{2r^2}{L^2} \right) \frac{P}{\epsilon} \right\} \tag{3-52}$$

$$\sigma^2 e^{\lambda - \nu} r^2 (\epsilon + P)^2 = \sigma^2 \epsilon^2 r^2 e^{\lambda - \nu} \left(1 + 2P/\epsilon \right)$$

$$= \sigma^2 \epsilon^2 \left[r^2 \left(1 + \frac{r^2}{L^2} \right) \left(1 - \frac{r^2}{2L^2} + \frac{3R^2}{2L^2} \right) \right] \left[1 + \frac{2P}{\epsilon} \right]$$

$$= \sigma^2 \epsilon^2 \left[r^2 \left(1 - \frac{r^2}{2L^2} + \frac{3R^2}{2L^2} + \frac{r^2}{L^2} \right) \right] \left[1 + \frac{2P}{\epsilon} \right]$$

$$= \sigma^2 \epsilon^2 \left[r^2 \left(1 + \frac{r^2}{2L^2} + \frac{3R^2}{2L^2} \right) \right] \left[1 + \frac{2P}{\epsilon} \right]$$

$$= \sigma^2 \epsilon^2 \left[r^2 \left(1 + \frac{r^2}{2L^2} + \frac{3R^2}{2L^2} + \frac{2P}{\epsilon} \right) \right] \tag{3-53}$$

Now we substitute (3-43) - (3-53) into (3-42).

$$\xi''\Big\{\gamma\epsilon^2 r^2\Big[\Big(\frac{4\eta}{\epsilon}-\frac{2\eta r^2}{\epsilon L^2}-\frac{2\eta R^2}{\epsilon L^2}\Big)\ln\frac{r}{R}+\frac{16\eta^2}{\epsilon^2}\ln^2\frac{r}{R}+\frac{1}{4}\ln^2\frac{1-r^2/L^2}{1-R^2/L^2}$$

$$+\Big(\frac{1}{2}-\frac{15\eta}{8\epsilon}-\frac{r^2}{4L^2}-\frac{R^2}{4L^2}+\frac{4\eta}{\epsilon}\ln\frac{r}{R}\Big)\ln\frac{1-r^2/L^2}{1-R^2/L^2}$$

$$+\frac{r^2}{4L^2}-\frac{R^2}{4L^2}+\frac{r^2R^2}{8L^4}-\frac{3r^4}{16L^4}+\frac{R^4}{16L^4}+\frac{7\eta r^2}{8\epsilon L^2}-\frac{7\eta R^2}{8\epsilon L^2}\Big]\Big\}$$

$$+\xi'\Big\{\gamma\epsilon^2 r\Big[\frac{4\eta}{\epsilon}-\frac{r^2}{2L^2}+\frac{r^2R^2}{2L^4}-\frac{r^4}{L^4}+\frac{3\eta r^2}{2\epsilon L^2}-\frac{8\eta r^2}{\epsilon L^2}\ln\frac{r}{R}$$

$$-\frac{r^2}{L^2}\ln\frac{1-r^2/L^2}{1-R^2/L^2}+\Big(\frac{4\eta}{\epsilon}-\frac{r^2}{2L^2}\Big)\frac{P}{\epsilon}+\Big(\frac{8\eta}{\epsilon}-\frac{2\eta r^2}{\epsilon L^2}\Big)\ln\frac{r}{R}$$

$$+\Big(1-\frac{15\eta}{4\epsilon}-\frac{r^2}{4L^2}\Big)\ln\frac{1-r^2/L^2}{1-R^2/L^2}+\frac{r^2}{2L^2}-\frac{R^2}{2L^2}+\frac{r^2R^2}{8L^4}-\frac{r^4}{8L^4}$$

$$+\frac{7\eta r^2}{4\epsilon L^2}-\frac{7\eta R^2}{4\epsilon L^2}+\Big(\frac{8\eta}{\epsilon}\ln\frac{r}{R}+\ln\frac{1-r^2/L^2}{1-R^2/L^2}+\frac{r^2}{2L^2}-\frac{R^2}{2L^2}\Big)\frac{P}{\epsilon}\Big]\Big\}$$

$$+\xi\Big\{\gamma\epsilon^2\Big[\frac{8\eta}{\epsilon}-\frac{r^2}{L^2}+\frac{r^2R^2}{L^4}-\frac{7r^4}{4L^4}+\frac{\eta r^2}{\epsilon L^2}-\frac{16\eta r^2}{\epsilon L^2}\ln\frac{r}{R}$$

$$-\frac{2r^2}{L^2}\ln\frac{1-r^2/L^2}{1-R^2/L^2}+\Big(\frac{8\eta}{\epsilon}-\frac{r^2}{L^2}\Big)\frac{P}{\epsilon}+\Big(\frac{7r^2}{2L^2}-2-\frac{2P}{\epsilon}\Big)\frac{P}{\epsilon}\Big]$$

Continued on the next page.

$$+\epsilon^2\left[\frac{4\eta r^2}{\epsilon L^2}-\frac{32\eta^2}{\epsilon^2}+\frac{16\eta^2}{\epsilon^2}+\frac{r^4}{4L^4}-\frac{4\eta r^2}{\epsilon L^2}+\frac{16\eta^2}{\epsilon^2}-\frac{3\eta r^2}{\epsilon L^2}\right.$$

$$-\frac{3r^2P}{\epsilon L^2}-\frac{16\eta}{\epsilon}+\frac{2r^2}{L^2}-\frac{2r^2R^2}{L^4}+\frac{4r^4}{L^4}-\frac{6\eta r^2}{\epsilon L^2}$$

$$+\frac{32\eta^2r^2}{\epsilon L^2}\ln\frac{r}{R}+\frac{4r^2}{L^2}\ln\frac{1-r^2/L^2}{1-R^2/L^2}-\left(\frac{16\eta}{\epsilon}-\frac{2r^2}{L^2}\right)\frac{P}{\epsilon}$$

$$\left.\left.+\sigma^2r^2\left(1+\frac{r^2}{2L^2}+\frac{3R^2}{2L^2}+\frac{2P}{\epsilon}\right)\right]\right]\Bigg\}=0 \qquad (3\text{-}54)$$

Dividing by ϵ^2 and collecting terms this becomes

$$\xi''\left\{\gamma r^2\left[\left(\frac{4\eta}{\epsilon}-\frac{2\eta r^2}{\epsilon L^2}-\frac{2\eta R^2}{\epsilon L^2}\right)\ln\frac{r}{R}+\frac{16\eta^2}{\epsilon^2}\ln^2\frac{r}{R}\right.\right.$$

$$+\frac{1}{4}\ln^2\frac{1-r^2/L^2}{1-R^2/L^2}+\left(\frac{1}{2}-\frac{15\eta}{8\epsilon}-\frac{r^2}{4L^2}-\frac{R^2}{4L^2}+\frac{4\eta}{\epsilon}\ln\frac{r}{R}\right)\ln\frac{1-r^2/L^2}{1-R^2/L^2}$$

$$\left.\left.+\frac{r^2}{L^2}-\frac{R^2}{4L^2}+\frac{r^2R^2}{8L^4}-\frac{3r^4}{16L^4}+\frac{R^4}{16L^4}+\frac{7\eta r^2}{8\epsilon L^2}-\frac{7\eta R^2}{8\epsilon L^2}\right]\right\}$$

$$+\xi'\left\{\gamma r\left[\frac{4\eta}{\epsilon}-\frac{R^2}{2L^2}+\frac{5r^2R^2}{8L^4}-\frac{9r^4}{8L^4}+\frac{13\eta r^2}{4\epsilon L^2}-\frac{7\eta R^2}{4\epsilon L^2}\right.\right.$$

Continued on the next page.

$$+\left(\frac{8\gamma}{\epsilon}-\frac{10\gamma r^2}{\epsilon L^2}\right)\ln\frac{r}{R}+\left(1-\frac{15\gamma}{4\epsilon}-\frac{5r^2}{4L^2}\right)\ln\frac{1-r^2/L^2}{1-R^2/L^2}$$

$$+\left(\frac{4\gamma}{\epsilon}-\frac{R^2}{2L^2}+\frac{8\gamma}{\epsilon}\ln\frac{r}{R}+\ln\frac{1-r^2/L^2}{1-R^2/L^2}\right)\frac{P}{\epsilon}\Bigg]\Bigg]\Bigg\}$$

$$+\bar{\xi}\Bigg\{\gamma\Bigg[\frac{8\gamma}{\epsilon}-\frac{r^2}{L^2}+\frac{r^2R^2}{L^4}-\frac{7r^4}{4L^4}+\frac{\gamma r^2}{\epsilon L^2}-\frac{16\gamma r^2}{\epsilon L^2}\ln\frac{r}{R}$$

$$-\frac{2r^2}{L^2}\ln\frac{1-r^2/L^2}{1-R^2/L^2}+\left(\frac{5r^2}{2L^2}-2-\frac{2P}{\epsilon}+\frac{8\gamma}{\epsilon}\right)\frac{P}{\epsilon}\Bigg]$$

$$+\Bigg[\frac{3r^2}{L^2}-\frac{2r^2R^2}{L^4}+\frac{17r^4}{4L^4}-\frac{16\gamma}{\epsilon}-\frac{9\gamma r^2}{\epsilon L^2}+\frac{32\gamma r^2}{\epsilon L^2}\ln\frac{r}{R}$$

$$+\frac{4r^2}{L^2}\ln\frac{1-r^2/L^2}{1-R^2/L^2}-\left(\frac{16\gamma}{\epsilon}+\frac{r^2}{L^2}\right)\frac{P}{\epsilon}$$

$$+\sigma^2r^2\left(1+\frac{r^2}{2L^2}+\frac{3R^2}{2L^2}+\frac{2P}{\epsilon}\right)\Bigg]\Bigg]\Bigg\}=0 \qquad (3\text{-}55)$$

Substituting for P , (3-55) becomes

$$\Xi''\Big\{\gamma r^2\Big[\Big(\frac{4\eta}{\epsilon}-\frac{2\eta r^2}{\epsilon L^2}-\frac{2\eta R^2}{\epsilon L^2}\Big)\ln\frac{r}{R}+\frac{16\eta^2}{\epsilon^2}\ln^2\frac{r}{R}$$

$$+\frac{1}{4}\ln^2\frac{1-r^2/L^2}{1-R^2/L^2}+\Big(\frac{1}{2}-\frac{15\eta}{8\epsilon}-\frac{r^2}{4L^2}-\frac{R^2}{4L^2}+\frac{4\eta}{\epsilon}\ln\frac{r}{R}\Big)\ln\frac{1-r^2/L^2}{1-R^2/L^2}$$

$$+\frac{r^2}{4L^2}-\frac{R^2}{4L^2}+\frac{r^2R^2}{8L^4}-\frac{3r^4}{16L^4}+\frac{R^4}{16L^4}+\frac{7\eta r^2}{8\epsilon L^2}-\frac{7\eta R^2}{8\epsilon L^2}\Big]\Big\}$$

$$+\Xi'\Big\{\gamma r\Big[\frac{4\eta}{\epsilon}-\frac{R^2}{2L^2}+\frac{5r^2R^2}{8L^4}-\frac{9r^4}{8L^4}+\frac{13\eta r^2}{4\epsilon L^2}-\frac{7\eta R^2}{4\epsilon L^2}$$

$$+\Big(\frac{8\eta}{\epsilon}-\frac{16\eta r^2}{\epsilon L^2}\Big)\ln\frac{r}{R}+\Big(1-\frac{15\eta}{4\epsilon}-\frac{5r^2}{4L^2}\Big)\ln\frac{1-r^2/L^2}{1-R^2/L^2}$$

$$+\Big(\frac{4\eta}{\epsilon}-\frac{R^2}{2L^2}+\frac{8\eta}{\epsilon}\ln\frac{r}{R}+\ln\frac{1-r^2/L^2}{1-R^2/L^2}\Big)\Big(1-\frac{r^2}{L^2}\Big)\Big(\frac{4\eta}{\epsilon}\ln\frac{r}{R}$$

$$+\Big(\frac{1}{2}-\frac{15\eta}{8\epsilon}\Big)\ln\frac{1-r^2/L^2}{1-R^2/L^2}+\Big(\frac{1}{4}+\frac{7\eta}{8\epsilon}\Big)\Big(\frac{r^2}{L^2}-\frac{R^2}{L^2}\Big)\Big)\Big)\Big]\Big\}$$

$$+\Xi\Big\{\gamma\Big[\frac{8\eta}{\epsilon}-\frac{r^2}{L^2}+\frac{r^2R^2}{L^4}-\frac{7r^4}{4L^4}+\frac{\eta r^2}{\epsilon L^2}-\frac{16\eta r^2}{\epsilon L^2}\ln\frac{r}{R}$$

$$-\frac{2r^2}{L^2}\ln\frac{1-r^2/L^2}{1-R^2/L^2}+\Big(\frac{5r^2}{2L^2}-2+\frac{8\eta}{\epsilon}-2(1-r^2/L^2)\Big(\frac{4\eta}{\epsilon}\ln\frac{r}{R}$$

$$+\Big(\frac{1}{2}-\frac{15\eta}{8\epsilon}\Big)\ln\frac{1-r^2/L^2}{1-R^2/L^2}+\Big(\frac{1}{4}+\frac{7\eta}{8\epsilon}\Big)\Big(\frac{r^2}{L^2}-\frac{R^2}{L^2}\Big)\Big)\Big)$$

Continued on the next page.

$$\cdot\left(\left(1-r^2/L^2\right)\left(\frac{4\eta}{\epsilon}\ln\frac{r}{R}+\left(\frac{1}{2}-\frac{15\eta}{8\epsilon}\right)\ln\frac{1-r^2/L^2}{1-R^2/L^2}\right.\right.$$

$$\left.\left.\left.+\left(\frac{1}{4}+\frac{7\eta}{8\epsilon}\right)\left(\frac{r^2}{L^2}-\frac{R^2}{L^2}\right)\right)\right)\right]+\left[\frac{2r^2}{L^2}-\frac{2r^2R^2}{L^4}+\frac{17r^4}{4L^4}-\frac{16\eta}{\epsilon}\right.$$

$$-\frac{9\eta r^2}{\epsilon L^2}+\frac{32\eta r^2}{\epsilon L^2}\ln\frac{r}{R}+\frac{4r^2}{L^2}\ln\frac{1-r^2/L^2}{1-R^2/L^2}$$

$$-\left(\frac{16\eta}{\epsilon}+\frac{r^2}{L^2}\right)\left(1-\frac{r^2}{L^2}\right)\left(\frac{4\eta}{\epsilon}\ln\frac{r}{R}+\left(\frac{1}{2}-\frac{15\eta}{8\epsilon}\right)\ln\frac{1-r^2/L^2}{1-R^2/L^2}\right.$$

$$\left.+\left(\frac{1}{4}+\frac{7\eta}{8\epsilon}\right)\left(\frac{r^2}{L^2}-\frac{R^2}{L^2}\right)\right)+\sigma^2r^2\left(1+\frac{r^2}{2L^2}+\frac{3R^2}{2L^2}\right.$$

$$+2\left(1-\frac{r^2}{L^2}\right)\left(\frac{4\eta}{\epsilon}\ln\frac{r}{R}+\left(\frac{1}{2}-\frac{15\eta}{8\epsilon}\right)\ln\frac{1-r^2/L^2}{1-R^2/L^2}\right.$$

$$\left.\left.\left.\left.+\left(\frac{1}{4}+\frac{7\eta}{8\epsilon}\right)\left(\frac{r^2}{L^2}-\frac{R^2}{L^2}\right)\right)\right)\right]\right\}=0 \tag{3-56}$$

Expanding and simplifying we get

$$\Xi''\left\{\gamma r^2\left[\left(\frac{4\eta}{\epsilon}-\frac{2\eta r^2}{\epsilon L^2}-\frac{2\eta R^2}{\epsilon L^2}\right)\ln\frac{r}{R}+\frac{16\eta^2}{\epsilon^2}\ln^2\frac{r}{R}\right.\right.$$

$$+\frac{1}{4}\ln^2\frac{1-r^2/L^2}{1-R^2/L^2}+\left(\frac{1}{2}-\frac{15\eta}{8\epsilon}-\frac{r^2}{4L^2}-\frac{R^2}{4L^2}+\frac{4\eta}{\epsilon}\ln\frac{r}{R}\right)\ln\frac{1-r^2/L^2}{1-R^2/L^2}$$

$$\left.\left.+\frac{r^2}{4L^2}-\frac{R^2}{4L^2}+\frac{r^2R^2}{8L^4}-\frac{3r^4}{16L^4}+\frac{R^4}{16L^4}+\frac{7\eta r^2}{8\epsilon L^2}-\frac{7\eta R^2}{8\epsilon L^2}\right]\right\}$$

$$+\Xi'\left\{\gamma r\left[\frac{4\eta}{\epsilon}-\frac{R^2}{2L^2}+\frac{5r^2R^2}{8L^4}-\frac{9r^4}{8L^4}+\frac{13\eta r^2}{4\epsilon L^2}-\frac{7\eta R^2}{4\epsilon L^2}\right.\right.$$

$$+\left(\frac{8\eta}{\epsilon}-\frac{10\eta r^2}{\epsilon L^2}\right)\ln\frac{r}{R}+\left(1-\frac{15\eta}{4\epsilon}-\frac{5r^2}{4L^2}\right)\ln\frac{1-r^2/L^2}{1-R^2/L^2}$$

$$+\frac{16\eta^2}{\epsilon^2}\ln\frac{r}{R}+\frac{2\eta}{\epsilon}\ln\frac{1-r^2/L^2}{1-R^2/L^2}+\frac{\eta r^2}{\epsilon L^2}-\frac{\eta R^2}{\epsilon L^2}-\frac{2\eta R^2}{\epsilon L^2}\ln\frac{r}{R}$$

$$-\frac{R^2}{4L^2}\ln\frac{1-r^2/L^2}{1-R^2/L^2}-\frac{r^2R^2}{8L^4}+\frac{R^4}{8L^4}+\frac{32\eta^2}{\epsilon^2}\ln^2\frac{r}{R}$$

$$+\frac{4\eta}{\epsilon}\left(\ln\frac{r}{R}\right)\ln\frac{1-r^2/L^2}{1-R^2/L^2}+\frac{2\eta r^2}{\epsilon L^2}\ln\frac{r}{R}-\frac{2\eta R^2}{\epsilon L^2}\ln\frac{r}{R}$$

$$+\frac{4\eta}{\epsilon}\left(\ln\frac{r}{R}\right)\ln\frac{1-r^2/L^2}{1-R^2/L^2}+\frac{1}{2}\ln^2\frac{1-r^2/L^2}{1-R^2/L^2}$$

Continued on the next page.

$$+\frac{r^2}{4L^2}\ln\frac{1-r^2/L^2}{1-R^2/L^2}-\frac{R^2}{4L^2}\ln\frac{1-r^2/L^2}{1-R^2/L^2}\Bigg]\Bigg\}$$

$$+\xi\Bigg\{\gamma\Bigg[\frac{8\eta}{\epsilon}-\frac{r^2}{L^2}+\frac{r^2R^2}{L^4}-\frac{7r^4}{4L^4}+\frac{\eta r^2}{\epsilon L^2}-\frac{16\eta r^2}{\epsilon L^2}\ln\frac{r}{R}$$

$$-\frac{2r^2}{L^2}\ln\frac{1-r^2/L^2}{1-R^2/L^2}+\Bigg(\frac{2r^2}{L^2}+\frac{R^2}{2L^2}+\frac{8\eta}{\epsilon}-\frac{8\eta}{\epsilon}\ln\frac{r}{R}$$

$$-\ln\frac{1-r^2/L^2}{1-R^2/L^2}-2\Bigg)\Bigg(\frac{4\eta}{\epsilon}\ln\frac{r}{R}+\frac{1}{2}\ln\frac{1-r^2/L^2}{1-R^2/L^2}+\frac{r^2}{4L^2}-\frac{R^2}{4L^2}$$

$$-\frac{4\eta r^2}{\epsilon L^2}\ln\frac{r}{R}-\frac{15\eta}{8\epsilon}\ln\frac{1-r^2/L^2}{1-R^2/L^2}-\frac{r^2}{2L^2}\ln\frac{1-r^2/L^2}{1-R^2/L^2}$$

$$-\frac{r^4}{4L^4}+\frac{r^2R^2}{4L^4}+\frac{7\eta r^2}{8\epsilon L^2}-\frac{7\eta R^2}{8\epsilon L^2}\Bigg)\Bigg]$$

$$+\Bigg[\frac{2r^2}{L^2}-\frac{2r^2R^2}{L^4}+\frac{17r^4}{4L^4}-\frac{16\eta}{\epsilon}-\frac{9\eta r^2}{\epsilon L^2}+\frac{32\eta r^2}{\epsilon L^2}\ln\frac{r}{R}$$

$$+\frac{4r^2}{L^2}\ln\frac{1-r^2/L^2}{1-R^2/L^2}-\frac{64\eta^2}{\epsilon^2}\ln\frac{r}{R}-\frac{8\eta}{\epsilon}\ln\frac{1-r^2/L^2}{1-R^2/L^2}$$

$$-\frac{4\eta r^2}{\epsilon L^2}+\frac{4\eta R^2}{\epsilon L^2}-\frac{4\eta r^2}{\epsilon L^2}\ln\frac{r}{R}-\frac{r^2}{2L^2}\ln\frac{1-r^2/L^2}{1-R^2/L^2}-\frac{r^4}{4L^4}+\frac{r^2R^2}{4L^4}$$

$$+\sigma^2r^2\Bigg(1+\frac{r^2}{2L^2}+\frac{3R^2}{2L^2}+\frac{8\eta}{\epsilon}\ln\frac{r}{R}+\ln\frac{1-r^2/L^2}{1-R^2/L^2}+\frac{r^2}{2L^2}-\frac{R^2}{2L^2}\Bigg)\Bigg]\Bigg\}=0 \tag{3-57}$$

Collecting terms and simplifying, gives

$$\xi''\Bigg\{\gamma r^2\Bigg[\left(\frac{4\eta}{\epsilon}-\frac{2\eta r^2}{\epsilon L^2}-\frac{2\eta R^2}{\epsilon L^2}\right)\ln\frac{r}{R}+\frac{16\eta^2}{\epsilon^2}\ln^2\frac{r}{R}$$

$$+\frac{1}{4}\ln^2\frac{1-r^2/L^2}{1-R^2/L^2}+\left(\frac{1}{2}-\frac{15\eta}{8\epsilon}-\frac{r^2}{4L^2}-\frac{R^2}{4L^2}+\frac{4\eta}{\epsilon}\ln\frac{r}{R}\right)\ln\frac{1-r^2/L^2}{1-R^2/L^2}$$

$$+\frac{r^2}{4L^2}-\frac{R^2}{4L^2}+\frac{r^2R^2}{8L^4}-\frac{3r^4}{16L^4}+\frac{R^4}{16L^4}+\frac{7\eta r^2}{8\epsilon L^2}-\frac{7\eta R^2}{8\epsilon L^2}\Bigg]\Bigg\}$$

$$+\xi'\Bigg\{\gamma r\Bigg[\frac{4\eta}{\epsilon}-\frac{R^2}{2L^2}+\frac{r^2R^2}{2L^4}-\frac{9r^4}{8L^4}+\frac{R^4}{8L^4}+\frac{17\eta r^2}{4\epsilon L^2}-\frac{11\eta R^2}{4\epsilon L^2}$$

$$+\left(\frac{8\eta}{\epsilon}-\frac{8\eta r^2}{\epsilon L^2}-\frac{4\eta R^2}{\epsilon L^2}+\frac{16\eta^2}{\epsilon^2}\right)\ln\frac{r}{R}+\left(1-\frac{7\eta}{4\epsilon}-\frac{r^2}{L^2}-\frac{R^2}{2L^2}\right)\ln\frac{1-r^2/L^2}{1-R^2/L^2}$$

$$+\frac{32\eta^2}{\epsilon^2}\ln^2\frac{r}{R}+\frac{1}{2}\ln^2\frac{1-r^2/L^2}{1-R^2/L^2}+\frac{8\eta}{\epsilon}\left(\ln\frac{r}{R}\right)\ln\frac{1-r^2/L^2}{1-R^2/L^2}\Bigg]\Bigg\}$$

$$+\xi\Bigg\{\gamma\Bigg[\frac{8\eta}{\epsilon}-\frac{r^2}{L^2}+\frac{r^2R^2}{L^4}-\frac{7r^4}{4L^4}+\frac{\eta r^2}{\epsilon L^2}-\frac{16\eta r^2}{\epsilon L^2}\ln\frac{r}{R}$$

$$-\frac{2r^2}{L^2}\ln\frac{1-r^2/L^2}{1-R^2/L^2}+\frac{8\eta r^2}{\epsilon L^2}\ln\frac{r}{R}+\frac{r^2}{L^2}\ln\frac{1-r^2/L^2}{1-R^2/L^2}$$

$$+\frac{r^4}{2L^4}-\frac{r^2R^2}{2L^4}+\frac{2\eta R^2}{\epsilon L^2}\ln\frac{r}{R}+\frac{R^2}{4L^2}\ln\frac{1-r^2/L^2}{1-R^2/L^2}$$

Continued on the next page.

$$+\frac{R^2r^2}{8L^4}-\frac{R^4}{8L^4}+\frac{32\eta^2}{\epsilon^2}\ln\frac{r}{R}+\frac{4\eta}{\epsilon}\ln\frac{1-r^2/L^2}{1-R^2/L^2}+\frac{2\eta r^2}{\epsilon L^2}-\frac{2\eta R^2}{\epsilon L^2}$$

$$-\frac{32\eta^2}{\epsilon^2}\ln^2\frac{r}{R}-\frac{4\eta}{\epsilon}\left(\ln\frac{r}{R}\right)\ln\frac{1-r^2/L^2}{1-R^2/L^2}-\frac{2\eta r^2}{\epsilon L^2}\ln\frac{r}{R}$$

$$+\frac{2\eta R^2}{\epsilon L^2}\ln\frac{r}{R}-\frac{4\eta}{\epsilon}\left(\ln\frac{r}{R}\right)\ln\frac{1-r^2/L^2}{1-R^2/L^2}-\frac{1}{2}\ln^2\frac{1-r^2/L^2}{1-R^2/L^2}$$

$$-\frac{r^2}{4L^2}\ln\frac{1-r^2/L^2}{1-R^2/L^2}+\frac{R^2}{4L^2}\ln\frac{1-r^2/L^2}{1-R^2/L^2}-\frac{8\eta}{\epsilon}\ln\frac{r}{R}$$

$$-\ln\frac{1-r^2/L^2}{1-R^2/L^2}-\frac{r^2}{2L^2}+\frac{R^2}{2L^2}+\frac{8\eta r^2}{\epsilon L^2}\ln\frac{r}{R}+\frac{15\eta}{4\epsilon}\ln\frac{1-r^2/L^2}{1-R^2/L^2}$$

$$+\frac{r^2}{L^2}\ln\frac{1-r^2/L^2}{1-R^2/L^2}+\frac{r^4}{2L^4}-\frac{r^2R^2}{2L^4}-\frac{7\eta r^2}{4\epsilon L^2}+\frac{7\eta R^2}{4\epsilon L^2}\Bigg]$$

$$+\Bigg[\frac{2r^2}{L^2}-\frac{7r^2R^2}{4L^4}+\frac{4r^4}{L^4}-\frac{16\eta}{\epsilon}-\frac{13\eta r^2}{\epsilon L^2}+\frac{4\eta R^2}{\epsilon L^2}$$

$$+\left(\frac{28\eta r^2}{\epsilon L^2}-\frac{64\eta^2}{\epsilon^2}\right)\ln\frac{r}{R}+\left(\frac{7r^2}{2L^2}-\frac{8\eta}{\epsilon}\right)\ln\frac{1-r^2/L^2}{1-R^2/L^2}$$

$$+\sigma^2r^2\left(1+\frac{r^2}{L^2}+\frac{R^2}{L^2}+\ln\frac{1-r^2/L^2}{1-R^2/L^2}+\frac{8\eta}{\epsilon}\ln\frac{r}{R}\right)\Bigg]\Bigg]\Bigg\}=0 \qquad (3\text{-}58)$$

Continuing to simplify, we get

$$\xi''\left\{\gamma r^2\left[\left(\frac{4\eta}{\epsilon}-\frac{2\eta r^2}{\epsilon L^2}-\frac{2\eta R^2}{\epsilon L^2}\right)\ln\frac{r}{R}+\frac{16\eta^2}{\epsilon^2}\ln^2\frac{r}{R}\right.\right.$$

$$+\frac{1}{4}\ln^2\frac{1-r^2/L^2}{1-R^2/L^2}+\left(\frac{1}{2}-\frac{15\eta}{8\epsilon}-\frac{r^2}{4L^2}-\frac{R^2}{4L^2}+\frac{4\eta}{\epsilon}\ln\frac{r}{R}\right)\ln\frac{1-r^2/L^2}{1-R^2/L^2}$$

$$\left.\left.+\frac{r^2}{4L^2}-\frac{R^2}{4L^2}+\frac{r^2R^2}{8L^4}-\frac{3r^4}{16L^4}+\frac{R^4}{16L^4}+\frac{7\eta r^2}{8\epsilon L^2}-\frac{7\eta R^2}{8\epsilon L^2}\right]\right\}$$

$$+\xi'\left\{\gamma r\left[\frac{4\eta}{\epsilon}-\frac{R^2}{2L^2}+\frac{r^2R^2}{2L^4}-\frac{9r^4}{8L^4}+\frac{R^4}{8L^4}+\frac{17\eta r^2}{4\epsilon L^2}-\frac{11\eta R^2}{4\epsilon L^2}\right.\right.$$

$$+\left(\frac{8\eta}{\epsilon}-\frac{8\eta r^2}{\epsilon L^2}-\frac{4\eta R^2}{\epsilon L^2}+\frac{16\eta^2}{\epsilon^2}\right)\ln\frac{r}{R}+\left(1-\frac{7\eta}{4\epsilon}-\frac{r^2}{L^2}-\frac{R^2}{2L^2}\right)\ln\frac{1-r^2/L^2}{1-R^2/L^2}$$

$$\left.\left.+\frac{32\eta^2}{\epsilon^2}\ln^2\frac{r}{R}+\frac{1}{2}\ln^2\frac{1-r^2/L^2}{1-R^2/L^2}+\frac{8\eta}{\epsilon}\left(\ln\frac{r}{R}\right)\ln\frac{1-r^2/L^2}{1-R^2/L^2}\right]\right\}$$

$$+\xi\left\{\gamma\left[\frac{8\eta}{\epsilon}-\frac{3r^2}{2L^2}+\frac{R^2}{2L^2}+\frac{r^2R^2}{8L^4}-\frac{3r^4}{4L^4}-\frac{R^4}{8L^4}+\frac{5\eta r^2}{4\epsilon L^2}-\frac{\eta R^2}{8\epsilon L^2}\right.\right.$$

$$-\left(\frac{2\eta r^2}{\epsilon L^2}-\frac{4\eta R^2}{\epsilon L^2}+\frac{8\eta}{\epsilon}-\frac{32\eta^2}{\epsilon^2}\right)\ln\frac{r}{R}-\left(1+\frac{r^2}{4L^2}-\frac{R^2}{2L^2}-\frac{3\eta}{4\epsilon}\right)\ln\frac{1-r^2/L^2}{1-R^2/L^2}$$

$$\left.-\frac{32\eta^2}{\epsilon^2}\ln^2\frac{r}{R}-\frac{8\eta}{\epsilon}\left(\ln\frac{r}{R}\right)\ln\frac{1-r^2/L^2}{1-R^2/L^2}-\frac{1}{2}\ln^2\frac{1-r^2/L^2}{1-R^2/L^2}\right]$$

Continued on the next page.

$$+\left[\frac{2r^2}{L^2}-\frac{7r^2R^2}{4L^4}+\frac{4r^4}{L^4}-\frac{16\eta}{\epsilon}-\frac{13\eta r^2}{\epsilon L^2}+\frac{4\eta R^2}{\epsilon L^2}\right.$$

$$+\left(\frac{28\eta r^2}{\epsilon L^2}-\frac{64\eta^2}{\epsilon^2}\right)\ln\frac{r}{R}+\left(\frac{7r^2}{2L^2}-\frac{8\eta}{\epsilon}\right)\ln\frac{1-r^2/L^2}{1-R^2/L^2}$$

$$\left.+\alpha^2r^2\left(1+\frac{r^2}{L^2}+\frac{R^2}{L^2}+\ln\frac{1-r^2/L^2}{1-R^2/L^2}+\frac{8\eta}{\epsilon}\ln\frac{r}{R}\right)\right]\Bigg\}=0 \qquad (3\text{-}59)$$

In order to simplify (3-59), we proceed as follows:

$$\ln(1 - r^2/L^2) \simeq - r^2/L^2 - \frac{r^4}{2L^4} \tag{3-60}$$

Unfortunately, a series expansion for $\ln r/R$ would require very many terms in order to be valid both near the center and near the surface. In order to avoid this difficulty, consider the function

$$F(\delta) = r^{1+\delta}$$

$$\frac{dF}{d\delta} = (r^{1+\delta})\ln r, \quad \frac{d^2F}{d\delta^2} = (r^{1+\delta})\ln^2 r$$

Then a series expansion of $F(\delta)$ around $\delta = 0$ gives

$$r^{1+\delta} = r + r\delta\ln r + \frac{r(\delta\ln r)^2}{2!} + \cdots + r\frac{(\delta\ln r)^{n-1}}{(n-1)!} + \cdots$$

If we now let $\delta = \eta/\epsilon$, then to first order

$$\frac{\eta}{\epsilon} r\ln r \simeq r^{1+\eta/\epsilon} - r \tag{3-61}$$

Putting (3-60) and (3-61) into (3-59), we get

$$\xi''\left\{\gamma r^2\left[\left(4-\frac{2r^2}{L^2}-\frac{2R^2}{L^2}\right)\left(\frac{r^{n/\epsilon}}{R^{n/\epsilon}}-1\right)+\frac{16\eta}{\epsilon}\left(\frac{r^{2n/\epsilon}}{R^{2n/\epsilon}}-\frac{2r^{n/\epsilon}}{R^{n/\epsilon}}+1\right)\right.\right.$$

$$+\left(\frac{1}{2}-\frac{15\eta}{8\epsilon}-\frac{r^2}{4L^2}-\frac{R^2}{4L^2}+\frac{4r^{n/\epsilon}}{R^{n/\epsilon}}-4\right)\left(\frac{R^2}{L^2}-\frac{r^2}{L^2}+\frac{R^4}{2L^4}-\frac{r^4}{2L^4}\right)$$

$$\left.\left.+\frac{R^4}{4L^4}-\frac{r^2R^2}{2L^4}+\frac{r^4}{4L^4}+\frac{r^2}{4L^2}-\frac{R^2}{4L^2}+\frac{r^2R^2}{8L^4}-\frac{3r^4}{16L^4}+\frac{R^4}{16L^4}+\frac{7\eta r^2}{8\epsilon L^2}-\frac{7\eta R^2}{8\epsilon L^2}\right]\right\}$$

$$+\xi'\left\{\gamma r\left[\frac{4\eta}{\epsilon}-\frac{R^2}{2L^2}+\frac{r^2R^2}{2L^4}-\frac{9r^4}{8L^4}+\frac{R^4}{8L^4}+\frac{17\eta r^2}{4\epsilon L^2}-\frac{11\eta R^2}{4\epsilon L^2}\right.\right.$$

$$+\left(8-\frac{8r^2}{L^2}-\frac{4R^2}{L^2}+\frac{16\eta}{\epsilon}\right)\left(\frac{r^{n/\epsilon}}{R^{n/\epsilon}}-1\right)$$

$$+\left(1-\frac{7\eta}{4\epsilon}-\frac{r^2}{L^2}-\frac{R^2}{2L^2}\right)\left(\frac{R^2}{L^2}-\frac{r^2}{L^2}+\frac{R^4}{2L^4}-\frac{r^4}{2L^4}\right)$$

$$+\frac{32\eta}{\epsilon}\left(\frac{r^{2n/\epsilon}}{R^{2n/\epsilon}}-\frac{2r^{n/\epsilon}}{R^{n/\epsilon}}+1\right)+\frac{R^4}{2L^4}-\frac{r^2R^2}{L^4}+\frac{r^4}{2L^4}$$

$$\left.\left.8\left(\frac{r^{n/\epsilon}}{R^{n/\epsilon}}-1\right)\left(\frac{R^2}{L^2}-\frac{r^2}{L^2}-\frac{r^4}{2L^4}+\frac{R^4}{2L^4}\right)\right]\right]\Bigg\}$$

$$+\xi\left\{\gamma\left[\frac{8\eta}{\epsilon}-\frac{3r^2}{2L^2}+\frac{R^2}{2L^2}+\frac{r^2R^2}{8L^4}-\frac{3r^4}{4L^4}-\frac{R^4}{8L^4}+\frac{5\eta r^2}{4\epsilon L^2}-\frac{\eta R^2}{8\epsilon L^2}\right.\right.$$

$$-\left(\frac{2r^2}{L^2}-\frac{4R^2}{L^2}+8-\frac{32\eta}{\epsilon}\right)\left(\frac{r^{n/\epsilon}}{R^{n/\epsilon}}-1\right)-\frac{32\eta}{\epsilon}\left(\frac{r^{2n/\epsilon}}{R^{2n/\epsilon}}-\frac{2r^{n/\epsilon}}{R^{n/\epsilon}}+1\right)$$

$$-\left(1+\frac{r^2}{4L^2}-\frac{R^2}{2L^2}-\frac{31\eta}{4\epsilon}\right)\left(\frac{R^2}{L^2}-\frac{r^2}{L^2}-\frac{r^4}{2L^4}+\frac{R^4}{2L^4}\right)$$

Continued on the next page.

$$- 8\left(\frac{r^{n/\epsilon}}{R^{n/\epsilon}} - 1\right)\left(\frac{R^2}{L^2} - \frac{r^2}{L^2} - \frac{r^4}{2L^4} + \frac{R^4}{2L^4}\right) - \left(\frac{R^4}{2L^4} - \frac{r^2R^2}{L^4} + \frac{r^4}{2L^4}\right)\Bigg]$$

$$+ \Bigg[\frac{2r^2}{L^2} - \frac{7r^2R^2}{4L^4} + \frac{4r^4}{L^4} - \frac{16\eta}{\epsilon} - \frac{13\eta r^2}{\epsilon L^2} + \frac{4\eta R^2}{\epsilon L^2}$$

$$+ \left(\frac{28r^2}{L^2} - \frac{64\eta}{\epsilon}\right)\left(\frac{r^{n/\epsilon}}{R^{n/\epsilon}} - 1\right) + \left(\frac{7r^2}{2L^2} - \frac{8\eta}{\epsilon}\right)\left(\frac{R^2}{L^2} - \frac{r^2}{L^2} - \frac{r^4}{2L^4} + \frac{R^4}{2L^4}\right)$$

$$+ \sigma^2 r^2\left(1 + \frac{r^2}{L^2} + \frac{R^2}{L^2} + \frac{R^2}{L^2} - \frac{r^2}{L^2} + \frac{8r^{n/\epsilon}}{R^{n/\epsilon}} - 8\right)\Bigg]\Bigg\} = 0 \qquad (3\text{-}62)$$

Expanding and collecting terms.

$$\xi''\Bigg\{\gamma r^2\Bigg[\frac{r^2}{4L^2} - \frac{R^2}{4L^2} - \frac{3r^2R^2}{8L^4} + \frac{r^4}{16L^4} + \frac{5R^4}{16L^4} + \frac{7\eta r^2}{8\epsilon L^2} - \frac{7\eta R^2}{8\epsilon L^2}$$

$$+ \frac{R^2}{2L^2} - \frac{r^2}{2L^2} + \frac{R^4}{4L^4} - \frac{r^4}{4L^4} + \frac{15\eta r^2}{8\epsilon L^2} - \frac{15\eta R^2}{8\epsilon L^2} - \frac{r^2R^2}{4L^4}$$

$$+ \frac{r^4}{4L^4} - \frac{R^4}{4L^4} + \frac{R^2r^2}{4L^4} + \frac{4r^{n/\epsilon}R^2}{R^{n/\epsilon}L^2} - \frac{4r^2r^{n/\epsilon}}{L^2R^{n/\epsilon}} + \frac{2R^4r^{n/\epsilon}}{L^4R^{n/\epsilon}}$$

$$- \frac{2r^4r^{n/\epsilon}}{L^4R^{n/\epsilon}} - \frac{4R^2}{L^2} + \frac{4r^2}{L^2} - \frac{2R^4}{L^4} + \frac{2r^4}{L^4} + \frac{4r^{n/\epsilon}}{R^{n/\epsilon}} - 4$$

$$- \frac{2r^2r^{n/\epsilon}}{L^2R^{n/\epsilon}} + \frac{2r^2}{L^2} - \frac{2R^2r^{n/\epsilon}}{L^2R^{n/\epsilon}} + \frac{2R^2}{L^2} + \frac{16\eta r^{2n/\epsilon}}{\epsilon R^{2n/\epsilon}}$$

$$- \frac{32\eta r^{n/\epsilon}}{\epsilon R^{n/\epsilon}} + \frac{16\eta}{\epsilon}\Bigg]\Bigg\}$$

Continued on the next page.

$$+\xi'\left\{\gamma r\left[\frac{4\eta}{\epsilon}-\frac{R^2}{2L^2}-\frac{r^2R^2}{2L^4}-\frac{5r^4}{8L^4}+\frac{5R^4}{8L^4}+\frac{17\eta r^2}{4\epsilon L^2}-\frac{11\eta R^2}{4\epsilon L^2}\right.\right.$$

$$+\frac{R^2}{L^2}-\frac{r^2}{L^2}+\frac{R^4}{2L^4}-\frac{r^4}{2L^4}-\frac{7\eta R^2}{4\epsilon L^2}+\frac{7\eta r^2}{4\epsilon L^2}-\frac{r^2R^2}{L^4}+\frac{r^4}{L^4}$$

$$-\frac{R^4}{2L^4}+\frac{R^2r^2}{2L^4}+\frac{8r^{n/\epsilon}}{R^{n/\epsilon}}-\frac{16r^2r^{n/\epsilon}}{L^2R^{n/\epsilon}}+\frac{4R^2r^{n/\epsilon}}{L^2R^{n/\epsilon}}$$

$$-\frac{4r^4r^{n/\epsilon}}{L^4R^{n/\epsilon}}+\frac{4R^4r^{n/\epsilon}}{L^4R^{n/\epsilon}}+\frac{16\eta r^{n/\epsilon}}{\epsilon R^{n/\epsilon}}-8+\frac{16r^2}{L^2}-\frac{4R^2}{L^2}$$

$$\left.\left.+\frac{4r^4}{L^4}-\frac{4R^4}{L^4}-\frac{16\eta}{\epsilon}+\frac{32\eta r^{2n/\epsilon}}{\epsilon R^{2n/\epsilon}}-\frac{64\eta r^{n/\epsilon}}{\epsilon R^{n/\epsilon}}+\frac{32\eta}{\epsilon}\right]\right\}$$

$$+\xi\left\{\gamma\left[\frac{8\eta}{\epsilon}-\frac{3r^2}{2L^2}+\frac{R^2}{2L^2}+\frac{r^2R^2}{8L^4}-\frac{3r^4}{4L^4}-\frac{R^4}{8L^4}+\frac{5\eta r^2}{4\epsilon L^2}\right.\right.$$

$$-\frac{\eta R^2}{8\epsilon L^2}-\frac{R^4}{2L^4}+\frac{r^2R^2}{L^4}-\frac{r^4}{2L^4}-\frac{R^2}{L^2}+\frac{r^2}{L^2}+\frac{r^4}{2L^4}-\frac{R^4}{2L^4}$$

$$-\frac{r^2R^2}{4L^4}+\frac{r^4}{4L^4}+\frac{R^4}{2L^4}-\frac{R^2r^2}{2L^4}+\frac{31\eta R^2}{4\epsilon L^2}-\frac{31\eta r^2}{4\epsilon L^2}$$

$$+\frac{6r^2r^{n/\epsilon}}{L^2R^{n/\epsilon}}-\frac{4R^2r^{n/\epsilon}}{L^2R^{n/\epsilon}}+\frac{32\eta r^{n/\epsilon}}{\epsilon R^{n/\epsilon}}-\frac{8r^{n/\epsilon}}{R^{n/\epsilon}}+\frac{4r^4r^{n/\epsilon}}{L^4R^{n/\epsilon}}$$

$$-\frac{4R^4r^{n/\epsilon}}{L^4R^{n/\epsilon}}-\frac{6r^2}{L^2}+\frac{4R^2}{L^2}-\frac{32\eta}{\epsilon}+8-\frac{4r^4}{L^4}+\frac{4R^4}{L^4}$$

$$\left.-\frac{32\eta r^{2n/\epsilon}}{\epsilon R^{2n/\epsilon}}+\frac{64\eta r^{n/\epsilon}}{\epsilon R^{n/\epsilon}}-\frac{32\eta}{\epsilon}\right]$$

Continued on the next page.

$$+\left[\frac{2r^2}{L^2}-\frac{7r^2R^2}{4L^4}+\frac{4r^4}{L^4}-\frac{16\eta}{\epsilon}-\frac{13\eta r^2}{\epsilon L^2}+\frac{4\eta R^2}{\epsilon L^2}+\frac{28r^2r^{n/\epsilon}}{L^2R^{n/\epsilon}}\right.$$

$$-\frac{28r^2}{L^2}-\frac{64\eta r^{n/\epsilon}}{\epsilon R^{n/\epsilon}}+\frac{64\eta}{\epsilon}+\frac{7r^2R^2}{2L^4}-\frac{7r^4}{2L^4}-\frac{8\eta R^2}{\epsilon L^2}$$

$$\left.\left.+\frac{8\eta r^2}{\epsilon L^2}+\sigma^2r^2\left(1+\frac{2R^2}{L^2}+\frac{8r^{n/\epsilon}}{R^{n/\epsilon}}-8\right)\right]\right\}=0 \qquad (3\text{-}63)$$

Continuing to collect terms, we get

$$\xi''\Big\{\gamma r^2\Big[\frac{23r^2}{4L^2}-\frac{7R^2}{4L^2}+\frac{2R^2r^{n/\epsilon}}{L^2R^{n/\epsilon}}-\frac{6r^2r^{n/\epsilon}}{L^2R^{n/\epsilon}}+\frac{4r^{n/\epsilon}}{R^{n/\epsilon}}$$

$$+\frac{16\eta}{\epsilon}-\frac{32\eta r^{n/\epsilon}}{\epsilon R^{n/\epsilon}}+\frac{16\eta r^{2n/\epsilon}}{\epsilon R^{2n/\epsilon}}-\frac{3r^2R^2}{8L^4}+\frac{33r^4}{16L^4}$$

$$-\frac{27R^4}{16L^4}+\frac{2R^4r^{n/\epsilon}}{L^4R^{n/\epsilon}}-\frac{2r^4r^{n/\epsilon}}{L^4R^{n/\epsilon}}+\frac{11\eta r^2}{4\epsilon L^2}-\frac{11\eta R^2}{4\epsilon L^2}-4\Big]\Big\}$$

$$+\xi'\Big\{\gamma r\Big[\frac{15r^2}{L^2}-\frac{7R^2}{2L^2}+\frac{4R^2r^{n/\epsilon}}{L^2R^{n/\epsilon}}-\frac{16r^2r^{n/\epsilon}}{L^2R^{n/\epsilon}}+\frac{8r^{n/\epsilon}}{R^{n/\epsilon}}$$

$$+\frac{20\eta}{\epsilon}-\frac{48\eta r^{n/\epsilon}}{\epsilon R^{n/\epsilon}}+\frac{32\eta r^{2n/\epsilon}}{\epsilon R^{2n/\epsilon}}-\frac{r^2R^2}{L^4}+\frac{31r^4}{8L^4}$$

$$-\frac{27R^4}{8L^4}+\frac{4R^4r^{n/\epsilon}}{L^4R^{n/\epsilon}}-\frac{4r^4r^{n/\epsilon}}{L^4R^{n/\epsilon}}+\frac{6\eta r^2}{\epsilon L^2}-\frac{9\eta R^2}{2\epsilon L^2}-8\Big]\Big\}$$

$$+\xi\Big\{\gamma\Big[\frac{7R^2}{2L^2}-\frac{13r^2}{2L^2}-\frac{4R^2r^{n/\epsilon}}{L^2R^{n/\epsilon}}+\frac{6r^2r^{n/\epsilon}}{L^2R^{n/\epsilon}}-\frac{8r^{n/\epsilon}}{R^{n/\epsilon}}$$

$$-\frac{56\eta}{\epsilon}+\frac{96\eta r^{n/\epsilon}}{\epsilon R^{n/\epsilon}}-\frac{32\eta r^{2n/\epsilon}}{\epsilon R^{2n/\epsilon}}+\frac{3r^2R^2}{8L^4}-\frac{9r^4}{2L^4}$$

$$+\frac{27R^4}{8L^4}-\frac{4R^4r^{n/\epsilon}}{L^4R^{n/\epsilon}}+\frac{4r^4r^{n/\epsilon}}{L^4R^{n/\epsilon}}-\frac{13\eta r^2}{2\epsilon L^2}+\frac{61\eta R^2}{8\epsilon L^2}+8\Big]$$

Continued on the next page.

$$-\left[\frac{26r^2}{L^2} - \frac{28r^2 r^{n/\epsilon}}{L^2 R^{n/\epsilon}} - \frac{48\eta}{\epsilon} + \frac{64\eta r^{n/\epsilon}}{\epsilon R^{n/\epsilon}} - \frac{7r^2R^2}{4L^4} - \frac{r^4}{2L^4}\right.$$

$$\left.+\frac{5\eta r^2}{\epsilon L^2} + \frac{4\eta R^4}{\epsilon L^4} - \lambda^2 r^2\left(1 + \frac{2R^2}{L^2} + \frac{8r^{n/\epsilon}}{R^{n/\epsilon}} - 8\right)\right]\Bigg\} = 0 \qquad (3\text{-}64)$$

We try for a series solution.

$$\xi = \sum a_k r^k \qquad (3\text{-}65)$$

In order that the solution proposed by (3-65) be feasible, we make the following approximations

$$r^{n/\epsilon} \rightarrow \overline{r^{n/\epsilon}} = R^{n/\epsilon}\left(1 - n/\epsilon\right) \qquad (3\text{-}66)$$

$$r^{4+n/\epsilon} \rightarrow r^2\,\overline{r^{2+n/\epsilon}} = r^2\frac{R^{2+n/\epsilon}}{3}\left(1 - n/3\epsilon\right)$$

where the bar indicates, the average value of that quantity over the interval $r = 0 \rightarrow R$.

Then (3-64) becomes

$$
\begin{aligned}
&\sum k\,(k-1)\,\hat{a}_k\, r^k \Bigg\{ \gamma \Bigg[\frac{23r^2}{4L^2} - \frac{7R^2}{4L^2} + \frac{2R^2}{L^2} - \frac{2\eta R^2}{\epsilon L^2} - \frac{6r^2}{L^2} \\
&+ \frac{6\eta r^2}{\epsilon L^2} + 4 - \frac{4\eta}{\epsilon} + \frac{16\eta}{\epsilon} - \frac{32\eta}{\epsilon} + \frac{32\eta^2}{\epsilon^2} + \frac{16\eta}{\epsilon} - \frac{32\eta^2}{\epsilon^2} \\
&- \frac{3r^2R^2}{8L^4} + \frac{11r^2R^2}{16L^4} - \frac{27R^4}{16L^4} + \frac{2R^4}{L^4} - \frac{2r^2R^2}{3L^4} + \frac{11\eta r^2}{4\epsilon L^2} \\
&- \frac{11\eta R^2}{4\epsilon L^2} - 4 \Bigg]\Bigg\} + \sum k\, a_k\, r^k \Bigg\{ \gamma \Bigg[\frac{15r^2}{L^2} - \frac{7R^2}{2L^2} + \frac{4R^2}{L^2} - \frac{4\eta R^2}{\epsilon L^2} \\
&- \frac{16r^2}{L^2} + \frac{16\eta r^2}{\epsilon L^2} + 8 - \frac{8\eta}{\epsilon} + \frac{20\eta}{\epsilon} - \frac{48\eta}{\epsilon} + \frac{48\eta^2}{\epsilon^2} + \frac{32\eta}{\epsilon} \\
&- \frac{64\eta^2}{\epsilon^2} - \frac{r^2R^2}{L^4} + \frac{31r^2R^2}{24L^4} - \frac{27R^4}{8L^4} + \frac{4R^4}{L^4} - \frac{4r^2R^2}{3L^4} \\
&+ \frac{6\eta r^2}{\epsilon L^2} - \frac{9\eta R^2}{2\epsilon L^2} - 8 \Bigg]\Bigg\} - \sum a_k\, r^k \Bigg\{ \gamma \Bigg[\frac{13r^2}{2L^2} - \frac{7R^2}{2L^2} \\
&+ \frac{4R^2}{L^2} - \frac{4\eta R^2}{\epsilon L^2} - \frac{6r^2}{L^2} + \frac{6\eta r^2}{\epsilon L^2} + 8 - \frac{8\eta}{\epsilon} + \frac{56\eta}{\epsilon} \\
&- \frac{96\eta}{\epsilon} + \frac{96\eta^2}{\epsilon^2} + \frac{32\eta}{\epsilon} - \frac{64\eta^2}{\epsilon^2} - \frac{3r^2R^2}{8L^4} + \frac{3r^2R^2}{2L^4} \\
&- \frac{27R^4}{8L^4} + \frac{4R^4}{L^4} - \frac{4r^2R^2}{3L^4} + \frac{13\eta r^2}{2\epsilon L^2} - \frac{61\eta R^2}{8\epsilon L^2} - 8 \Bigg]
\end{aligned}
$$

Continued on the next page.

$$+\left[\frac{26r^2}{L^2}-\frac{28r^2}{L^2}+\frac{28\eta r^2}{\epsilon L^2}-\frac{48\eta}{\epsilon}+\frac{64\eta}{\epsilon}-\frac{64\eta^2}{\epsilon^2}\right.$$

$$-\frac{7r^2R^2}{4L^4}-\frac{r^2R^2}{6L^4}+\frac{5\eta r^2}{\epsilon L^2}+\frac{4\eta R^2}{\epsilon L^2}$$

$$\left.\left.-\sigma^2 r^2\left(1+\frac{2R^2}{L^2}-\frac{8\eta}{\epsilon}\right)\right]\right\}=0$$

Collecting terms.

$$\sum k(k-1)a_k r^k\left\{\gamma\left[\frac{R^2}{4L^2}-\frac{r^2}{4L^2}-\frac{17r^2R^2}{48L^4}+\frac{5R^4}{16L^4}-\frac{4\eta}{\epsilon}\right.\right.$$

$$\left.\left.+\frac{35\eta r^2}{4\epsilon L^2}-\frac{19\eta R^2}{4\epsilon L^2}\right]\right\}+\sum k a_k r^k\left\{\gamma\left[\frac{R^2}{2L^2}-\frac{r^2}{L^2}\right.\right.$$

$$\left.\left.-\frac{25r^2R^2}{24L^4}+\frac{5R^4}{8L^4}-\frac{4\eta}{\epsilon}+\frac{22\eta r^2}{\epsilon L^2}-\frac{17\eta R^2}{2\epsilon L^2}-\frac{16\eta^2}{\epsilon^2}\right]\right\}$$

Continued on the next page.

$$
-\sum a_k r^k \left\{ \gamma \left[\frac{R^2}{2L^2} + \frac{r^2}{2L^2} - \frac{5r^2R^2}{24L^4} + \frac{5R^4}{8L^4} - \frac{16\eta}{\epsilon} + \frac{25\eta r^2}{2\epsilon L^2} \right. \right.
$$

$$
\left. - \frac{93\eta R^2}{8\epsilon L^2} + \frac{32\eta^2}{\epsilon^2} \right] - \left[\frac{2r^2}{L^2} + \frac{23r^2R^2}{12L^4} - \frac{16\eta}{\epsilon} - \frac{33\eta r^2}{\epsilon L^2} \right.
$$

$$
\left.\left. - \frac{4\eta R^2}{\epsilon L^2} + \frac{64\eta^2}{\epsilon^2} + \sigma^2 r^2 \left(1 + \frac{2R^2}{L^2} - \frac{8\eta}{\epsilon} \right) \right] \right\} = 0 \qquad (3\text{-}67)
$$

Rearranging, this becomes

$$
\sum \left\{ \gamma \left[\frac{R^2}{4L^2} + \frac{5R^4}{16L^4} - \frac{4\eta}{\epsilon} - \frac{19\eta R^2}{4\epsilon L^2} \right] (k^2 - k) \right.
$$

$$
+ \gamma \left[\frac{R^2}{2L^2} + \frac{5R^4}{8L^4} - \frac{4\eta}{\epsilon} - \frac{17\eta R^2}{2\epsilon L^2} - \frac{16\eta^2}{\epsilon^2} \right] k
$$

$$
\left. - \gamma \left[\frac{R^2}{2L^2} + \frac{5R^4}{8L^4} - \frac{16\eta}{\epsilon} - \frac{93\eta R^2}{8\epsilon L^2} + \frac{32\eta^2}{\epsilon^2} \right] - \frac{16\eta}{\epsilon} - \frac{4\eta R^2}{\epsilon L^2} + \frac{64\eta^2}{\epsilon^2} \right\} a_k r^k
$$

$$
- \sum \left\{ \gamma \left[\frac{1}{4L^2} + \frac{17R^2}{48L^4} - \frac{35\eta}{4\epsilon L^2} \right] (k^2 - k) + \gamma \left[\frac{1}{L^2} + \frac{25R^2}{24L^4} - \frac{22\eta}{\epsilon L^2} \right] k \right.
$$

$$
+ \gamma \left[\frac{1}{2L^2} - \frac{5R^2}{24L^4} + \frac{25\eta}{2\epsilon L^2} \right] - \left[\frac{2}{L^2} + \frac{23R^2}{12L^4} - \frac{33\eta}{\epsilon L^2} \right]
$$

$$
\left. - \sigma^2 \left[1 + \frac{2R^2}{L^2} - \frac{8\eta}{\epsilon} \right] \right\} a_k r^{k+2} = 0 \qquad (3\text{-}68)
$$

Collecting terms.

$$\sum\Big\{\gamma\left[\frac{R^2}{4L^2}+\frac{5R^4}{16L^4}-\frac{4\eta}{\epsilon}-\frac{19\eta R^2}{4\epsilon L^2}\right]k^2+\gamma\left[\frac{R^2}{4L^2}+\frac{5R^4}{16L^4}-\frac{15\eta R^2}{4\epsilon L^2}-\frac{16\eta^2}{\epsilon^2}\right]k$$

$$-\gamma\left[\frac{R^2}{2L^2}+\frac{5R^4}{8L^4}-\frac{16\eta}{\epsilon}-\frac{93\eta R^2}{8\epsilon L^2}+\frac{32\eta^2}{\epsilon^2}\right]-\frac{16\eta}{\epsilon}-\frac{4\eta R^2}{\epsilon L^2}+\frac{64\eta^2}{\epsilon^2}\Big\}a_k r^k$$

$$-\sum\Big\{\gamma\left[\frac{1}{4L^2}+\frac{17R^2}{48L^4}-\frac{35\eta}{4\epsilon L^2}\right]k^2+\gamma\left[\frac{3}{4L^2}+\frac{11R^2}{16L^4}-\frac{53\eta}{4\epsilon L^2}\right]k$$

$$+\gamma\left[\frac{1}{2L^2}-\frac{5R^2}{24L^4}+\frac{25\eta}{2\epsilon L^2}\right]-\left[\frac{2}{L^2}+\frac{23R^2}{12L^4}-\frac{33\eta}{\epsilon L^2}\right]$$

$$-\sigma^2\left[1+\frac{2R^2}{L^2}-\frac{8\eta}{\epsilon}\right]\Big\}a_k r^{k+2}=0 \qquad (3\text{-}69)$$

In the first summation let k = l, in the second let k = l - 2.

$$\sum\Big\{\gamma\left[\frac{R^2}{4L^2}+\frac{5R^4}{16L^4}-\frac{4\eta}{\epsilon}-\frac{19\eta R^2}{4\epsilon L^2}\right]l^2+\gamma\left[\frac{R^2}{4L^2}+\frac{5R^4}{16L^4}-\frac{15\eta R^2}{4\epsilon L^2}-\frac{16\eta^2}{\epsilon^2}\right]l$$

$$-\gamma\left[\frac{R^2}{2L^2}+\frac{5R^4}{8L^4}-\frac{16\eta}{\epsilon}-\frac{93\eta R^2}{8\epsilon L^2}+\frac{32\eta^2}{\epsilon^2}\right]-\frac{16\eta}{\epsilon}-\frac{4\eta R^2}{\epsilon L^2}+\frac{64\eta^2}{\epsilon^2}\Big\}a_l r^l$$

$$-\sum\Big\{\gamma\left[\frac{1}{4L^2}+\frac{17R^2}{48L^4}-\frac{35\eta}{4\epsilon L^2}\right](l-2)^2+\gamma\left[\frac{3}{4L^2}+\frac{11R^2}{16L^4}-\frac{53\eta}{4\epsilon L^2}\right](l-2)$$

$$+\gamma\left[\frac{1}{2L^2}-\frac{5R^2}{24L^4}+\frac{25\eta}{2\epsilon L^2}\right]-\left[\frac{2}{L^2}+\frac{23R^2}{12L^4}-\frac{33\eta}{\epsilon L^2}\right]$$

$$-\sigma^2\left[1+\frac{2R^2}{L^2}-\frac{8\eta}{\epsilon}\right]\Big\}a_{l-2}r^l=0 \qquad (3\text{-}70)$$

Expanding the second summation, we get

$$\sum\Big\{\gamma\left[\frac{R^2}{4L^2}+\frac{5R^4}{16L^4}-\frac{4\eta}{\epsilon}-\frac{19\eta R^2}{4\epsilon L^2}\right]\ell^2+\gamma\left[\frac{R^2}{4L^2}+\frac{5R^4}{16L^4}-\frac{15\eta R^2}{4\epsilon L^2}-\frac{16\eta^2}{\epsilon^2}\right]\ell$$

$$-\gamma\left[\frac{R^2}{2L^2}+\frac{5R^4}{8L^4}-\frac{16\eta}{\epsilon}-\frac{93\eta R^2}{8\epsilon L^2}+\frac{32\eta^2}{\epsilon^2}\right]-\frac{16\eta}{\epsilon}-\frac{4\eta R^2}{\epsilon L^2}+\frac{64\eta^2}{\epsilon^2}\Big\}a_\ell r^\ell$$

$$-\sum\Big\{\gamma\left[\frac{1}{4L^2}+\frac{17R^2}{48L^4}-\frac{35\eta}{4\epsilon L^2}\right]\ell^2-\gamma\left[\frac{1}{L^2}+\frac{17R^2}{12L^4}-\frac{35\eta}{\epsilon L^2}\right]\ell$$

$$+\gamma\left[\frac{1}{L^2}+\frac{17R^2}{12L^4}-\frac{35\eta}{4\epsilon L^2}\right]+\gamma\left[\frac{3}{4L^2}+\frac{11R^2}{16L^4}-\frac{53\eta}{4\epsilon L^2}\right]\ell$$

$$-\gamma\left[\frac{3}{2L^2}+\frac{11R^2}{8L^4}-\frac{53\eta}{2\epsilon L^2}\right]+\gamma\left[\frac{1}{2L^2}-\frac{5R^2}{24L^4}+\frac{25\eta}{2\epsilon L^2}\right]$$

$$-\left[\frac{2}{L^2}+\frac{23R^2}{12L^4}-\frac{33\eta}{\epsilon L^2}\right]-\sigma^2\left[1+\frac{2R^2}{L^2}-\frac{8\eta}{\epsilon}\right]\Big\}a_{\ell-2}r^\ell=0 \qquad (3\text{-}71)$$

Collecting terms, this becomes

$$\sum\Big\{\gamma\left[\frac{R^2}{4L^2}+\frac{5R^4}{16L^4}-\frac{4\eta}{\epsilon}-\frac{19\eta R^2}{4\epsilon L^2}\right]\ell^2+\gamma\left[\frac{R^2}{4L^2}+\frac{5R^4}{16L^4}-\frac{15\eta R^2}{4\epsilon L^2}-\frac{16\eta^2}{\epsilon^2}\right]\ell$$

$$-\gamma\left[\frac{R^2}{2L^2}+\frac{5R^4}{8L^4}-\frac{16\eta}{\epsilon}-\frac{93\eta R^2}{8\epsilon L^2}+\frac{32\eta^2}{\epsilon^2}\right]-\frac{16\eta}{\epsilon}-\frac{4\eta R^2}{\epsilon L^2}+\frac{64\eta^2}{\epsilon^2}\Big\}a_\ell r^\ell$$

$$-\sum\Big\{\gamma\left[\frac{1}{4L^2}+\frac{17R^2}{48L^4}-\frac{35\eta}{4\epsilon L^2}\right]\ell^2-\gamma\left[\frac{1}{4L^2}+\frac{35R^2}{48L^4}-\frac{87\eta}{4\epsilon L^2}\right]\ell$$

$$-\gamma\left[\frac{R^2}{6L^4}-\frac{121\eta}{4\epsilon L^2}\right]-\left[\frac{2}{L^2}+\frac{23R^2}{12L^4}-\frac{33\eta}{\epsilon L^2}\right]-\sigma^2\left[1+\frac{2R^2}{L^2}-\frac{8\eta}{\epsilon}\right]\Big\}a_{\ell-2}r^\ell=0 \qquad (3\text{-}72)$$

Then the recurrence relation between the cooefficients is given by

$$a_l = \frac{\gamma\left[\frac{1}{4L^2}+\frac{17R^2}{48L^4}-\frac{35\eta}{4\epsilon L^2}\right]l^2-\gamma\left[\frac{1}{4L^2}+\frac{35R^2}{48L^4}-\frac{87\eta}{4\epsilon L^2}\right]l -\gamma\left[\frac{R^2}{6L^4}-\frac{121\eta}{4\epsilon L^2}\right]-\left[\frac{2}{L^2}+\frac{23R^2}{12L^4}-\frac{33\eta}{\epsilon L^2}\right]-\sigma^2\left[1+\frac{2R^2}{L^2}-\frac{8\eta}{\epsilon}\right]}{\gamma\left[\frac{R^2}{4L^2}+\frac{5R^4}{16L^4}-\frac{4\eta}{\epsilon}-\frac{19\eta R^2}{4\epsilon L^2}\right]l^2+\gamma\left[\frac{R^2}{4L^2}+\frac{5R^4}{16L^4}-\frac{15\eta R^2}{4\epsilon L^2}-\frac{16\eta^2}{\epsilon^2}\right]l -\gamma\left[\frac{R^2}{2L^2}+\frac{5R^4}{8L^4}-\frac{16\eta}{\epsilon}-\frac{93\eta R^2}{8\epsilon L^2}+\frac{32\eta^2}{\epsilon^2}\right]-\left[\frac{16\eta}{\epsilon}-\frac{4\eta R^2}{\epsilon L^2}+\frac{64\eta^2}{\epsilon^2}\right]}\, a_{l-2} \tag{3-73}$$

For very large values of l the expansion coefficients approach the same constant ratio causing the series to diverge at the surface of the sphere. To see this more clearly, we take the ratio of succeeding terms for large l. We get

$$\left[\frac{\frac{R^2}{4L^2}+\frac{17R^4}{48L^4}-\frac{35\eta R^2}{4\epsilon L^2}}{\frac{R^2}{4L^2}+\frac{5R^4}{16L^4}-\frac{4\eta}{\epsilon}-\frac{19\eta R^2}{4\epsilon L^2}}\right]\frac{r^2}{R^2}$$

where we have multiplied and divided the numerator by R^2. It is clear that the quantity within the brackets is always greater than one, and at $r = R$, the ratio of succeeding terms is always greater than one. Hence, the series diverges at the surface of the sphere. We must, therefore, insist that the series terminate. If the highest term is r^m, where m is a positive integer, the frequencies of the permitted pulsation modes are

$$\sigma^2 = \frac{\gamma\left[\frac{1}{4L^2} + \frac{17R^2}{48L^4} - \frac{35\eta}{4\epsilon L^2}\right]m^2 - \gamma\left[\frac{1}{4L^2} + \frac{35R^2}{48L^4} - \frac{87\eta}{4\epsilon L^2}\right]m - \gamma\left[\frac{R^2}{6L^4} - \frac{121\eta}{4\epsilon L^2}\right] - \left[\frac{2}{L^2} + \frac{23R^2}{12L^4} - \frac{33\eta}{\epsilon L^2}\right]}{1 + \frac{2R^2}{L^2} - \frac{8\eta}{\epsilon}} \qquad (3\text{-}74)$$

$$m = 3, 5, 7 \cdots$$

The modes are determined by the requirement that the frequency not be negative. The even-numbered modes are eliminated by the requirement that the center of the sphere remains fixed during the oscillations.

Using $m = 3$, the sphere will be stable if

$$\gamma\left[\frac{3}{2L^2} + \frac{5R^2}{6L^4} + \frac{67\eta}{4\epsilon L^2}\right] \geq \frac{2}{L^2} + \frac{23R^2}{12L^4} - \frac{33\eta}{\epsilon L^2}$$

$$\gamma\left[\frac{3}{2}\left(1 + \frac{5R^2}{9L^2} + \frac{67\eta}{6\epsilon}\right)\right] \geq \left[2\left(1 + \frac{23R^2}{24L^2} - \frac{33\eta}{2\epsilon}\right)\right]$$

$$\gamma \geq \frac{4}{3}\left(1 + \frac{23R^2}{24L^2} - \frac{33\eta}{2\epsilon}\right)\left(1 - \frac{5R^2}{9L^2} - \frac{67\eta}{6\epsilon}\right)$$

$$\gamma \geq \frac{4}{3}\left(1 - \frac{5R^2}{9L^2} - \frac{67\eta}{6\epsilon} + \frac{23R^2}{24L^2} - \frac{33\eta}{2\epsilon}\right)$$

$$\gamma \geq \frac{4}{3}\left(1 + \frac{29R^2}{72L^2} - \frac{83\eta}{3\epsilon}\right)$$

$$\gamma \geq \frac{4}{3} + \frac{29R^2}{54L^2} - \frac{332\eta}{9\epsilon} \qquad (3\text{-}75)$$

The last two terms on the right are the first order, general relativistic corrections to the Newtonian result.

Since:

$$\frac{R^2}{L^2} = \frac{1}{R}\left[\frac{4\pi\epsilon R^3}{3c^2}\left(1+\frac{\eta}{\epsilon}\right)\frac{2G}{c^2}\right] = \frac{1}{R}\left[\frac{2GM}{c^2}\left(1+\frac{\eta}{\epsilon}\right)\right] \tag{3-76}$$

Then (3-75) becomes

$$R \geqslant \frac{29}{54\left(\gamma - 4/3 + \frac{332\eta}{9\epsilon}\right)}\left[\frac{2GM}{c^2}\left(1+\frac{\eta}{\epsilon}\right)\right] \tag{3-77}$$

This is the minimum radius at which dynamical stability is possible.

Defining the minimum radius of the uncharged model by

$$R_0 \geqslant \frac{29}{54(\gamma - 4/3)}\left[\frac{2GM}{c^2}\right] \tag{3-78}$$

We form the ratio of the minimum radii.

$$\frac{R}{R_0} = \frac{\dfrac{29}{54\left(\gamma - 4/3 + \frac{332\eta}{9\epsilon}\right)}\left[\dfrac{2GM}{c^2}\left(1+\dfrac{\eta}{\epsilon}\right)\right]}{\dfrac{29}{54(\gamma - 4/3)}\left[\dfrac{2GM}{c^2}\right]}$$

$$= \frac{\gamma - 4/3}{(\gamma - 4/3)\left(1 + \frac{332\eta}{9\epsilon(\gamma - 4/3)}\right)}\left[1+\frac{\eta}{\epsilon}\right]$$

$$= \left(1 - \frac{332\eta}{9\epsilon(\gamma - 4/3)}\right)\left[1+\frac{\eta}{\epsilon}\right] \tag{3-79}$$

This becomes

$$\frac{R}{R_c} = 1 + \frac{\eta}{6} - \frac{332\,\eta}{96\,(\gamma - 4/3)} \tag{3-80}$$

Since the last term is larger than the middle term for any physically conceivable value of γ , we have

$$\frac{R}{R_c} < 1 \tag{3-81}$$

We conclude, therefore, from (3-81) and (3-74) that the physical effects of electric charge upon dynamical stability are to decrease the minimum radius at which dynamical stability is possible, and to increase the frequency of the fundamental mode of the permitted pulsation modes.

Discussion

We have shown in the appendix, that dynamical stability in the Newtonian limit does not depend upon charge. What then, is the physical origin of the increased stability revealed by the general relativistic analysis? It seems plausible, that since charge contributes to gravitational mass, the increased stability may be the result of increased gravitational binding, due in turn to the increase in mass contributed by the charge.

CHAPTER IV

CHARGED DUST

In studying questions of gravitational collapse, an important and frequently used model is charged dust. This is a "star" with no internal pressure, equilibrium being maintained under a balance of gravitational attraction and electrical repulsion. The equilibrium solution is due to Bonner. This model has shown itself to be useful in dealing with such questions as to whether and under what circumstances forces of electrical repulsion might prevent or halt gravitational collapse. The subject of gravitational collapse is outside the scope of this investigation, however it is of interest to determine the stability of this model.

Letting $c = G = 1$, Bonner's solution is given by

$$ds^2 = f^2 dt^2 - f^{-2}\left(d\bar{r}^2 + \bar{r}^2 d\theta^2 + \bar{r}^2 \sin^2\theta \, d\phi^2\right) \tag{4-1}$$

where $\bar{r}$ is a radial coordinate, and

$$f^{-2} = \frac{a^3 + m\bar{r}^2}{(a+m)^3} \tag{4-2}$$

where m is the gravitational mass of the source, and a is its radius.

Before we can use Bonner's solution, we must transform his coordinate system into ours. Comparing (2-1) and (4-1), we see that

$$f^2 = e^{\nu}, \quad r^2 = f^{-2}\bar{r}^2, \quad e^{\lambda}dr^2 = f^{-2}d\bar{r}^2 \tag{4-3}$$

In (4-2) we substitute for $\bar{r}^2$ from (4-3).

$$f^2 = \frac{a^3 + m r^2 f^2}{(a+m)^3}$$

$$f\left[1 - \frac{m r^2}{(a+m)^3}\right] = \frac{a^3}{(a+m)^3}$$

$$\therefore f^2 = e^{\nu} = \frac{a^3}{(a+m)^3 - m r^2} \qquad (4\text{-}4)$$

From (4-2) and (4-3) we also have

$$r^2 = f^{-2}\bar{r}^2 = \frac{(a+m)^3\,\bar{r}^2}{R^3 + m\bar{r}^2} \qquad (4\text{-}5)$$

When $\bar{r} = a$ then $r = R$.

$$R^2 = \frac{(a+m)^3 a^2}{a^3 + m a^2} = (a+m)^2 \qquad (4\text{-}6)$$

Using (4-5) in (4-4) we have

$$e^{\nu} = \frac{a^3}{R^3 - m r^2} \qquad (4\text{-}7)$$

$$\nu' = \frac{2mr}{R^3 - m r^2} \qquad (4\text{-}8)$$

$$\nu'' = \frac{2mr}{R^3 - m r^2} + \frac{4m^2 r^2}{(R^3 - m r^2)^2} \qquad (4\text{-}9)$$

Putting (4-6) into (4-5) we have

$$r = \frac{\bar{r}\, R^{3/2}}{(a^3 + m\bar{r}^2)^{1/2}} \tag{4-10}$$

$$\frac{dr}{d\bar{r}} = \frac{R^{3/2}}{(a^3 + m\bar{r}^2)^{1/2}} - \frac{R^{3/2}\, m\bar{r}^2}{(a^3 + m\bar{r}^2)^{3/2}}$$

$$= \frac{R^{3/2}(a^3 + m\bar{r}^2) - R^{3/2}\, m\bar{r}^2}{(a^3 + m\bar{r}^2)^{3/2}}$$

$$\therefore \quad \frac{dr}{d\bar{r}} = \frac{a^3 R^{3/2}}{(a^3 + m\bar{r}^2)^{3/2}} \tag{4-11}$$

Referring to (4-3) and using (4-11) we have

$$e^{\lambda} = f^{-2}\left(\frac{d\bar{r}}{dr}\right)^2 = \left(\frac{R^3}{a^3 + m\bar{r}^2}\right)\left[\frac{(a^3 + m\bar{r}^2)^{3/2}}{a^3 R^{3/2}}\right]^2$$

$$= \left[\frac{a^3 + m\bar{r}^2}{a^3}\right]^2 = \left[\frac{a^3 + m r^2 f^2}{a^3}\right]^2$$

$$= \left[1 + \frac{mr^2}{a^3}\left(\frac{a^3}{R^3 - mr^2}\right)\right]^2$$

$$\therefore \quad e^{\lambda} = \left[\frac{R^3}{R^3 - mr^2}\right]^2 \tag{4-12}$$

$$\lambda' = \frac{4mr}{R^3 - mr^2} \tag{4-13}$$

From the field equations, we get

$$8\pi(T_0^{\,0}-T_1^{\,1}) = 8\pi\epsilon = e^{-\lambda}\left[\frac{\lambda'}{r}+\frac{\nu'}{r}\right]$$

$$8\pi\epsilon = \left[\frac{R^3-mr^2}{R^3}\right]\left(\frac{6m}{R^3-mr^2}\right)$$

$$\epsilon = \frac{3}{4\pi}\left[\frac{m(R^3-mr^2)}{R^6}\right] \tag{4-14}$$

We repeat $T_1^{\,1}$ component of the field equations.

$$8\pi T_1^{\,1} = 8\pi(\eta-P) = -e^{-\lambda}\left[\frac{\nu'}{r}+\frac{1}{r^2}\right]+\frac{1}{r^2}$$

Since $P=0$ this becomes

$$8\pi\eta = -\left[\frac{R^3-mr^2}{R^3}\right]^2\left[\frac{2m}{R^3-mr^2}+\frac{1}{r^2}\right]+\frac{1}{r^2}$$

$$= -\left[\frac{R^3-mr^2}{R^3}\right]^2\left[\frac{2mr^2+R^3-mr^2}{r^2(R^3-mr^2)}\right]+\frac{1}{r^2}$$

$$= -\left[\frac{R^3-mr^2}{R^6}\right]\left[\frac{mr^2+R^3}{r^2}\right]+\frac{1}{r^2}$$

$$= -\left[\frac{R^6-m^2r^4}{r^2R^6}\right]+\frac{1}{r^2}$$

$$= -\left[\frac{R^6-m^2r^4-R^6}{r^2R^6}\right]$$

$\therefore$

$$\eta = \frac{1}{8\pi}\left[\frac{m^2r^2}{R^6}\right] \tag{4-15}$$

From (3-47) we have

$$\eta^{1/2} = \frac{\sqrt{2\pi}}{r^2}\int_0^r \rho r^2 e^{\lambda/2} dr$$

Using (4-15) this produces

$$\frac{mr}{\sqrt{8\pi}\,R^3} = \frac{\sqrt{2\pi}}{r^2}\int_0^r \rho r^2 e^{\lambda/2} dr$$

$$\frac{mr^3}{4\pi R^3} = \int_0^r \rho r^2 e^{\lambda/2} dr$$

By inspection we get

$$\rho = \frac{3m\, e^{-\lambda/2}}{4\pi R^3} \tag{4-16}$$

Substituting (4-12) into (4-16) gives the charge density of the source.

$$\rho = \frac{3m}{4\pi}\left(\frac{R^3 - mr^2}{R^6}\right) \tag{4-17}$$

Comparing (4-14) with (4-17) see that

$$\rho = \epsilon \tag{4-18}$$

Finally, we check our solution against the $T_2^{\,2}$ component of the field equations.

$$8\pi T_2^{\,2} = -8\pi\eta = -\frac{e^{-\lambda}}{2}\left[\nu'' + \frac{(\nu')^2}{2} + \frac{\nu' - \lambda'}{r} - \frac{\nu'\lambda'}{2}\right]$$

Using (4-8), (4-9), (4-12), (4-13), we obtain

$$8\pi\eta = \frac{(R^3-mr^2)^2}{2R^6}\left[\frac{2m}{R^3-mr^2} + \frac{4m^2r^2}{(R^3-mr^2)^2} + \frac{2m^2r^2}{(R^3-mr^2)^2}\right.$$

$$\left. + \frac{2m}{R^3-mr^2} - \frac{4m}{R^3-mr^2} - \left(\frac{mr}{R^3-mr^2}\right)\left(\frac{4mr}{R^3-mr^2}\right)\right]$$

$$= \frac{(R^3-mr^2)^2}{2R^6}\left[\frac{2m^2r^2}{(R^3-mr^2)^2}\right]$$

$$= \frac{m^2r^2}{R^6} \tag{4-19}$$

which agrees with (4-15). Hence, our solution satisfies all of the field equations.

From the appendix we have

$$\delta\eta = -\left[\frac{d\eta}{dr} + \frac{4\eta}{r}\right]\xi$$

$$= -\frac{1}{8\pi}\left[\frac{2m^2r}{R^6} + \frac{4m^2r}{R^6}\right]\xi$$

$$= -\frac{3m^2r}{4\pi R^6}\xi \tag{4-20}$$

We are now ready for the pulsation equation.

In (2-86) we set $P = P' = 0$.

$$\sigma^2 e^{\lambda-\nu}\epsilon\xi = \xi\left[8\pi\eta\epsilon e^{\lambda} - \frac{8\eta\eta'}{\epsilon r} - \frac{(\eta')^2}{\epsilon} - \frac{16\eta^2}{\epsilon r^2}\right] \tag{4-21}$$

Next we expand the terms in (4-21).

$$\sigma^2 e^{\lambda-\nu}\epsilon = \sigma^2\left[\left(\frac{R^3}{R^3-mr^2}\right)^2\left(\frac{R^3-mr^2}{a^3}\right)\left(\frac{3m(R^3-mr^2)}{4\pi R^6}\right)\right]$$

$$= \sigma^2\left[\frac{3m}{4\pi a^3}\right] \tag{4-22}$$

$$8\pi e^{\lambda}\eta\epsilon = \left[\frac{8\pi R^6}{(R^3-mr^2)^2}\left(\frac{m^2r^2}{8\pi R^6}\right)\left(\frac{3m(R^3-mr^2)}{4\pi R^6}\right)\right]$$

$$= \frac{3m^3r^2}{4\pi R^6(R^3-mr^2)} \tag{4-23}$$

$$\frac{8\eta\eta'}{\epsilon r} = \frac{8}{r}\left(\frac{1}{8\pi}\right)\left(\frac{m^2r^2}{R^6}\right)\left(\frac{m^2r}{4\pi R^6}\right)\left(\frac{4\pi R^6}{3m(R^3-mr^2)}\right)$$

$$= \frac{r^2m^5}{3\pi R^6(R^3-mr^2)} \tag{4-24}$$

$$\frac{(\eta')^2}{\epsilon} = \frac{m^4r^2\,4\pi R^6}{16\pi^2R^{12}\,3m(R^3-mr^2)}$$

$$= \frac{r^2m^3}{12\pi R^6(R^3-mr^2)} \tag{4-25}$$

$$\frac{16\chi^2}{\epsilon r^2} = \frac{16}{r^2}\left(\frac{1}{64\pi^2}\right)\left(\frac{m^4 r^4}{R^{12}}\right)\left(\frac{4\pi}{3}\right)\left(\frac{R^6}{m\,(R^3 - m r^2)}\right)$$

$$= \frac{r^2 m^3}{3\pi R^6 (R^3 - m r^2)} \tag{4-26}$$

Substituting (4-22) - (4-26) into (4-21), we immediately obtain

$$\sigma^2 = 0 \tag{4-27}$$

So, Bonner's model is neutrally stable. Physically, this means that Bonner's model will neither expand nor contract by itself.

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APPENDIX A

Consider a charged fluid of radius R . Let the mass density ρ_m and the electrical energy density η be constants. It is easy to show that the charge density which corresponds to a constant electrical energy density is equal to $\rho_e = \left(\frac{2\eta}{\pi r^2}\right)^{1/2}$. We have, in fact, performed this calculation relativistically in (3-38).

If we require the sphere to be in gravitational equilibrium then

$$- \frac{G M \rho_m 4\pi r^2 dr}{r^2} + \frac{Q \rho_e 4\pi r^2 dr}{r^2} = 4\pi r^2 dP \qquad \text{(A-1)}$$

where $M(r)$ and $Q(r)$ are the mass and charge contained within a sphere of radius r .

Since:

$$Q(r) = 2r^2 (2\eta\pi)^{1/2} \qquad \text{(A-2)}$$

$$\frac{dP}{dr} = - \frac{G 4\pi r \rho_m^2}{3} + \frac{4\eta}{r} \qquad \text{(A-3)}$$

$$P = - \frac{2\pi G r^2 \rho_m^2}{3} + 4\eta \ln r + const \qquad \text{(A-4)}$$

When $r = R$ then $P = 0$.

$$P = 4\eta \ln \frac{r}{R} + \frac{2\pi G}{3} \rho_m^2 (R^2 - r^2) \qquad \text{(A-5)}$$

Now, referring to (3-26), we have

$$P = (1 - r^2/L^2)\left[4\eta \ln \frac{r}{R} + \epsilon\left(\frac{1}{2} - \frac{15\eta}{8\epsilon}\right)\ln \frac{1 - r^2/L^2}{1 - R^2/L^2} + \left(\frac{1}{4} + \frac{7\eta}{8\epsilon}\right)\left(\frac{r^2}{L^2} - \frac{R^2}{L^2}\right)\right] \tag{A-6}$$

We are seeking the Newtonian limit of the pressure. Expanding the center term to lowest order, and dropping higher order terms, we get

$$P = 4\eta \ln \frac{r}{R} - \frac{\epsilon r^2}{2L^2} + \frac{\epsilon R^2}{2L^2} + \frac{\epsilon}{4}\left(\frac{r^2}{L^2} - \frac{R^2}{L^2}\right)$$

or

$$P = 4\eta \ln \frac{r}{R} + \frac{\epsilon}{4L^2}(R^2 - r^2) \tag{A-7}$$

Since:

$$L^2 = \frac{3c^4}{8\pi G(\epsilon + \eta)}$$

$$P = 4\eta \ln \frac{r}{R} + \frac{\epsilon(2\pi G)(\epsilon + \eta)(R^2 - r^2)}{3c^4} \tag{A-8}$$

$$P = 4\eta \ln \frac{r}{R} + \left(\frac{2\pi G \rho_m}{3c^2}\right)\left(\rho_m c^2 + \eta\right)\left(R^2 - r^2\right) \tag{A-9}$$

Since $\rho_m c^2 \gg \tau$

$$P = 4\eta \ln \frac{r}{R} + \frac{2\pi G}{3} \rho_m^2 (R^2 - r^2) \qquad \text{(A-10)}$$

This is the Newtonian limit of (A-6), and we see that it agrees with (A-5) as indeed it must.

APPENDIX B

From (2-9) and (2-11), we have

$$\eta_0 = \frac{e^{-(\nu_c+\lambda_c)}}{8\pi} (F_{1c})^2 \tag{B-1}$$

Now F_{1c} can be written as

$$F_{1c} = -\frac{e^{\frac{\nu_c+\lambda_c}{2}}}{r^2} \left[4\pi \int_0^r \rho_c r^2 e^{\lambda_c/2} dr \right]$$

$$= -\frac{e^{\nu_c+\lambda_c}}{r^2} q_c(r) \tag{B-2}$$

where $q(r)$ is the charge contained within the sphere of radius r.

Then (B-1) becomes

$$\eta_c = \frac{q_c^2}{8\pi r^4} \tag{B-3}$$

For the time dependent case, we have

$$\eta = \eta_c + \delta\eta = \frac{(q_c + \delta q)^2}{8\pi r^4} = \frac{q_c^2}{8\pi r^4} + \frac{q_c\,\delta q}{4\pi r^4}$$

$$\therefore \quad \delta\eta = \frac{q_c\,\delta q}{4\pi r^4} \tag{B-4}$$

But we have previously shown (see 2-54) that

$$\delta\eta = F_{1c}\,\rho_c\, e^{-\lambda_1/2}\,\xi = -\frac{e^{\frac{\nu_c+\lambda_c}{2}}}{r^2}\, g_c\,\rho_c\, e^{-\lambda_c/2}\,\xi$$

$$\therefore \quad \delta\eta = -g_c\,\rho_c\,\frac{e^{\lambda_c/2}}{r^2}\,\xi \qquad \text{(B-5)}$$

Equating (B-4) to (B-5) produces

$$\delta g = -4\pi r^2 \rho_c\, e^{\lambda_c/2}\,\xi = -\frac{dg_c}{dr}\,\xi \qquad \text{(B-6)}$$

Next we form from (B-3)

$$\frac{d\eta_c}{dr} = \frac{1}{8\pi}\left[\frac{2g_c}{r^4}\frac{dg_c}{dr} - \frac{4g_c^2}{r^5}\right]$$

$$= \frac{1}{8\pi}\left[\frac{2g_c}{r^4}\left(4\pi r^2 \rho_c\, e^{\lambda_c/2}\right) - \frac{4}{r^5}\left(8\pi\eta_c r^4\right)\right]$$

$$= g_c\,\rho_c\,\frac{e^{\lambda/2}}{r^2} - \frac{4\eta_0}{r} \qquad \text{(B-7)}$$

Comparing (B-5) and (B-7) we see that

$$\delta\eta = -\left[\frac{d\eta_0}{dr} + \frac{4\eta_c}{r}\right]\xi \qquad \text{(B-8)}$$

APPENDIX C

NON-RELATIVISTIC ANALYSIS OF THE CHARGED HOMOGENEOUS MODEL

In (3-42) we set $e^{\lambda-\nu}=1$, $\nu'=\lambda'=0$, and $\epsilon \gg P$,

$$\xi''\gamma r^2 P + \xi'\gamma\left[r^2P' + 2rP\right] + \xi\left[2\gamma r P' - 2\gamma P\right.$$

$$\left. - \frac{8\eta r P'}{\epsilon} + \frac{r^2(P')^2}{\epsilon} + \frac{16\eta^2}{\epsilon} - 4rP' + \sigma^2\epsilon r^2\right] = 0 \qquad \text{(C-1)}$$

From appendix A we have

$$P = 4\eta \ln\frac{r}{R} + \frac{2\pi G\epsilon^2}{3c^4}(R^2 - r^2)$$

$$P' = \frac{4\eta}{r} - \frac{4\pi G\epsilon^2 r}{3c^4}$$

$$(P')^2 = \frac{16\eta^2}{r^2} + \frac{16\pi^2 G^2\epsilon^4 r^2}{9c^8} - \frac{32\pi G\eta\epsilon^2}{3c^4}$$

Next we evaluate C-1 term by term

$$\gamma r^2 P = \gamma r^2\left[4\eta \ln\frac{r}{R} + \frac{2\pi G\epsilon^2}{3c^4}(R^2 - r^2)\right] \qquad \text{(C-2)}$$

$$\gamma\left[r^2P' + 2rP\right] = \gamma r\left[4\eta - \frac{4\pi G\epsilon^2 r^2}{3c^4} + 8\eta\ln\frac{r}{R} + \frac{4\pi G\epsilon^2}{3c^4}(R^2 - r^2)\right]$$

$$= \gamma r\left[4\eta + 8\eta\ln\frac{r}{R} + \frac{4\pi G\epsilon^2}{3c^4}(R^2 - r^2)\right] \qquad \text{(C-3)}$$

$$\gamma\left[2rP'\right] = \gamma\left[8\eta - \frac{8\pi G\epsilon^2 r^2}{3c^4}\right] \tag{C-4}$$

$$\gamma\left[2P\right] = \gamma\left[8\eta \ln\frac{r}{R} + \frac{4\pi G\epsilon^2}{3c^4}(R^2 - r^2)\right] \tag{C-5}$$

$$\frac{8\eta r P'}{\epsilon} = \frac{32\eta^2}{\epsilon} - \frac{32\pi G\eta\epsilon r^2}{3c^4} \tag{C-6}$$

$$\frac{r^2(P')^2}{\epsilon} = \frac{r^2}{\epsilon}\left[\frac{16\eta^2}{r^2} + \frac{16\pi^2 G^2\epsilon^4 r^2}{9c^8} - \frac{32\pi G\eta\epsilon^2}{3c^4}\right] \tag{C-7}$$

$$4rP' = 16\eta - \frac{16\pi G\epsilon^2 r^2}{3c^4} \tag{C-8}$$

Substituting all of the above into C-1, we obtain

$$\xi''\left\{\gamma r^2\left[4\eta \ln\frac{r}{R} + \frac{2\pi G\epsilon^2}{3c^4}(R^2 - r^2)\right]\right\}$$

$$+\,\xi'\left\{\gamma r\left[4\eta + 8\eta \ln\frac{r}{R} + \frac{4\pi G\epsilon^2}{3c^4}(R^2 - 2r^2)\right]\right\}$$

$$+\,\xi\left\{\gamma\left[8\eta - \frac{8\pi G\epsilon^2 r^2}{3c^4} - 8\eta\ln\frac{r}{R} - \frac{4\pi G\epsilon^2}{3c^4}(R^2 - r^2)\right]\right.$$

$$-\frac{32\eta^2}{\epsilon} + \frac{32\pi G\eta\epsilon r^2}{3c^4} + \frac{16\eta^2}{\epsilon} + \frac{16\pi^2 G^2\epsilon^3 r^4}{9c^8}$$

$$\left.-\frac{32\pi G\eta\epsilon r^2}{3c^4} + \frac{16\eta^2}{\epsilon} - 16\eta + \frac{16\pi G\epsilon^2 r^2}{3c^4} + \sigma^2\epsilon r^2\right\} = 0 \tag{C-9}$$

Collecting terms

$$\xi''\left\{\gamma r^2\left[4\eta \ln\frac{r}{R}+\frac{2\pi G\epsilon^2}{3c^4}(R^2-r^2)\right]\right\}$$

$$+\xi'\left\{\gamma r\left[4\eta+8\eta\ln\frac{r}{R}+\frac{4\pi G\epsilon^2}{3c^4}(R^2-2r^2)\right]\right\}$$

$$-\xi\left\{\gamma\left[\frac{4\pi G\epsilon^2}{3c^4}(R^2+r^2)+8\eta\ln\frac{r}{R}-8\eta\right]\right.$$

$$\left.+16\eta-\frac{16\pi G\epsilon^2 r^2}{3c^4}-\frac{16\pi^2 G^2\epsilon^3 r^4}{9c^8}-\sigma^2\epsilon r^2\right\}=0 \qquad \text{(C-10)}$$

We will neglect $\frac{16\pi^2 G^2\epsilon^3 r^4}{9c^8}$ compared to $\sigma^2\epsilon r^2$.

In so doing, we neglect very long term instabilities.

In order to simplify (C-10), we make use of (3-61) and (3-66). We get

$$\xi''\left\{\gamma r^2\left[\frac{2\pi G\epsilon^2}{3c^4}(R^2-r^2)-4\eta\right]\right\}$$

$$+\xi'\left\{\gamma r\left[\frac{4\pi G\epsilon^2}{3c^4}(R^2-2r^2)-4\eta\right]\right\}$$

$$-\xi\left\{\gamma\left[\frac{4\pi G\epsilon^2}{3c^4}(R^2+r^2)-16\eta\right]\right.$$

$$\left.+16\eta-\frac{16\pi G\epsilon^2 r^2}{3c^4}-\sigma^2\epsilon r^2\right\}=0 \qquad \text{(C-11)}$$

We assume a series solution: $\xi = \sum a_k r^k$

$$\sum k(k-1)a_k r^k \left\{ \gamma\left[\frac{2\pi G\epsilon^2}{3c^4}(R^2-r^2) - 4\eta\right]\right\}$$

$$+ \sum k a_k r^k \left\{\gamma\left[\frac{4\pi G\epsilon^2}{3c^4}(R^2-2r^2) - 4\eta\right]\right\}$$

$$- \sum a_k r^k \left\{\gamma\left[\frac{4\pi G\epsilon^2}{3c^4}(R^2+r^2) - 16\eta\right] + 16\eta - \frac{16\pi G\epsilon^2 r^2}{3c^4} - \sigma^2\epsilon r^2\right\} = 0 \qquad \text{(C-12)}$$

Rearranging, this becomes

$$\sum \left\{ \gamma\left[\frac{2\pi G\epsilon^2 R^2}{3c^4} - 4\eta\right](k^2-k) + \gamma\left[\frac{4\pi G\epsilon^2 R^2}{3c^4} - 4\eta\right]k\right.$$

$$\left. - \gamma\left[\frac{4\pi G\epsilon^2 R^2}{3c^4} - 16\eta\right] - 16\eta\right\} a_k r^k$$

$$- \sum \left\{ \gamma\left[\frac{2\pi G\epsilon^2}{3c^4}\right](k^2-k) + \gamma\left[\frac{8\pi G\epsilon^2}{3c^4}\right]k + \gamma\left[\frac{4\pi G\epsilon^2}{3c^4}\right]\right.$$

$$\left. - \frac{16\pi G\epsilon^2}{3c^4} - \sigma^2\epsilon\right\} a_k r^{k+2} = 0 \qquad \text{(C-13)}$$

Collecting terms

$$\sum \left\{ \gamma \left[\frac{2\pi G\epsilon^2 R^2}{3c^4} - 4\eta \right] k^2 + \gamma \left[\frac{2\pi G\epsilon^2 R^2}{3c^4} \right] k \right.$$

$$\left. - \gamma \left[\frac{4\pi G\epsilon^2 R^2}{3c^4} - 16\eta \right] - 16\eta \right\} a_k r^k$$

$$- \sum \left\{ \gamma \left[\frac{2\pi G\epsilon^2}{3c^4} \right] k + \gamma \left[\frac{2\pi G\epsilon^2}{c^4} \right] k + \gamma \left[\frac{4\pi G\epsilon^2}{3c^4} \right] \right.$$

$$\left. - \frac{16\pi G\epsilon^2}{3c^4} - \sigma^2 \epsilon \right\} a_k r^{k+2} = 0 \qquad \text{(C-14)}$$

In the first summation let $k = l$, in the second $k = l-2$.

$$\sum \left\{ \gamma \left[\frac{2\pi G\epsilon^2 R^2}{3c^4} - 4\eta \right] l^2 + \gamma \left[\frac{2\pi G\epsilon^2 R^2}{3c^4} \right] l \right.$$

$$\left. - \gamma \left[\frac{4\pi G\epsilon^2 R^2}{3c^4} - 16\eta \right] - 16\eta \right\} a_l r^l$$

$$- \sum \left\{ \gamma \left[\frac{2\pi G\epsilon^2}{3c^4} \right] (l-2)^2 + \gamma \left[\frac{2\pi G\epsilon^2}{c^4} \right] (l-2) + \gamma \left[\frac{4\pi G\epsilon^2}{3c^4} \right] \right.$$

$$\left. - \frac{16\pi G\epsilon^2}{3c^4} - \sigma^2 \epsilon \right\} a_{l-2} r^l = 0 \qquad \text{(C-15)}$$

Expanding the second summation

$$\sum\left\{\gamma\left[\frac{2\pi G\epsilon^2R^2}{3c^4}-4\eta\right]\ell^2+\gamma\left[\frac{2\pi G\epsilon^2R^2}{3c^4}\right]\ell-\gamma\left[\frac{4\pi G\epsilon^2R^2}{3c^4}-16\eta\right]\right.$$

$$\left.-16\eta\right\}a_\ell r^\ell-\sum\left\{\gamma\left[\frac{2\pi G\epsilon^2}{3c^4}\right]\ell^2-\gamma\left[\frac{8\pi G\epsilon^2}{3c^4}\right]\ell+\gamma\left[\frac{8\pi G\epsilon^2}{3c^4}\right]\right.$$

$$+\gamma\left[\frac{2\pi G\epsilon^2}{c^4}\right]\ell-\gamma\left[\frac{4\pi G\epsilon^2}{c^4}\right]+\gamma\left[\frac{4\pi G\epsilon^2}{3c^4}\right]-\frac{16\pi G\epsilon^2}{3c^4}$$

$$\left.-\sigma^2\epsilon\right\}a_{\ell-2}r^\ell=0 \tag{C-16}$$

Collecting terms

$$\sum\left\{\gamma\left[\frac{2\pi G\epsilon^2R^2}{3c^4}-4\eta\right]\ell^2+\gamma\left[\frac{2\pi G\epsilon^2R^2}{3c^4}\right]\ell-\gamma\left[\frac{4\pi G\epsilon^2R^2}{3c^4}-16\eta\right]-16\eta\right\}a_\ell r^\ell$$

$$-\sum\left\{\gamma\left[\frac{2\pi G\epsilon^2}{3c^4}\right]\ell^2-\gamma\left[\frac{2\pi G\epsilon^2}{3c^4}\right]\ell-\frac{16\pi G\epsilon^2}{3c^4}-\sigma^2\epsilon\right\}a_{\ell-2}r^\ell=0 \tag{C-17}$$

Then the recurrence relation is

$$a_\ell=\frac{\gamma\left[\frac{2\pi G\epsilon^2}{3c^4}\right]\ell^2-\gamma\left[\frac{2\pi G\epsilon^2}{3c^4}\right]-\frac{16\pi G\epsilon^2}{3c^4}-\sigma^2\epsilon}{\gamma\left[\frac{2\pi G\epsilon^2R^2}{3c^4}-4\eta\right]\ell^2+\gamma\left[\frac{2\pi G\epsilon^2R^2}{3c^4}\right]\ell-\gamma\left[\frac{4\pi G\epsilon^2R^2}{3c^4}-16\eta\right]-16\eta}\,a_{\ell-2} \tag{C-18}$$

As with (3-73), the series diverges at the surface of the sphere. We must insist that it terminate. If the highest term is r^m where m is a positive integer, the frequencies of the permitted pulsation modes are

$$\sigma^2 = \gamma\left[\frac{2\pi G\epsilon}{3c^4}\right]m^2 - \gamma\left[\frac{2\pi G\epsilon}{3c^4}\right]m - \frac{16\pi G\epsilon}{3c^4} \tag{C-19}$$

$$m = 3, 5, 7, \ldots$$

Using $m = 3$, the sphere will be stable if

$$\gamma\left[\frac{18\pi G\epsilon}{3c^4}\right] - \gamma\left[\frac{6\pi G\epsilon}{3c^4}\right] - \frac{16\pi G\epsilon}{3c^4} \geq 0$$

or

$$\gamma \geq \frac{4}{3} \tag{C-20}$$

Hence, dynamical stability in the Newtonian limit, does not depend upon charge.