SECONDARY MATHEMATICS PRESERVICE TEACHERS' CONCEPTIONS OF RATIONAL NUMBERS

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1.1 Foundation of the Problem

Many of the tasks one faces everyday involve mathematics. Whenever we determine the savings on a sale item, use a spreadsheet, or estimate the amount of our purchase to determine if we have enough money to buy it, we are relying on mathematical understanding. In today's world, those who understand and can do mathematics will have opportunities that others do not. Mathematical competence is necessary to successfully navigate the future. In order to better prepare our students for this fast-changing, mathematically-dependent world, the National Council of Teachers of Mathematics (NCTM) expressed its vision for school mathematics in its publication *Principles and Standards for School Mathematics* (2000). This vision describes a future in which all students have access to rigorous, high-quality mathematics instruction from knowledgeable teachers who have a thorough understanding of the topics they are teaching. The curriculum should be stimulating, providing students with opportunities to learn important mathematical concepts and procedures with understanding. One of the standards listed in *Principles and Standards for School Mathematics* is Number and Operation. This standard calls for instructional programs which enable all students to understand numbers, relationships among numbers, ways of representing numbers, meanings of operations, relationships among operations; and to demonstrate fluency in computation and estimatation.

In the predecessor to *Principles and Standards for School Mathematics*, the *Curriculum and Evaluation Standards for School Mathematics* (1989), NCTM expressed as an underlying goal, that students should develop rich mathematical understandings while viewing mathematics as useful for developing mathematical power. The students develop these rich understandings by developing the connections prevalent within mathematics. In order for one to develop a rich understanding of many secondary level concepts, one must have a deeper understanding of the underlying foundational mathematical concepts. In his study, Frykholm (2000) found that many mathematics education students do not possess a rich enough mathematical knowledge to promote deep mathematical understanding in the classroom. This study builds on the notion that secondary preservice teachers often do not have sufficient conceptual understanding of fundamental K-12 mathematical concepts.

The NCTM *Standards* documents state that middle school students should extend their understanding of whole number operations to fractions, decimals, integers, and rational numbers. Students should also represent fractions in a variety of meaningful situations, moving flexibly among concrete, pictoral, and symbolic representations. Thus, increased attention should be devoted to developing the meaning of fraction symbols, fostering a sense of relative size of fractions, and helping students connect their intuitive understanding to more general, formal methods.

If students are to experience mathematics in the powerful and meaningful way as suggested by the *Principles and Standards of School Mathematics*, then teachers must have a solid yet flexible knowledge of the underlying concepts of mathematics as opposed to computational fluency with algorithms. Ball (1990) found that secondary mathematics education majors believed that they knew mathematics, felt confident in their ability to do mathematics, and felt as though mathematics could be explained. However, they were no more successful than elementary education majors in providing conceptual explanations for mathematical concepts. They tended to give "rules" as explanations for concepts.

The *Standards* documents call for a pedagogy that allows students the opportunity to explore the contexts in which operations make sense as well as construct the concepts of fractions and operations, not just demonstrate the operations. Despite reforms in mathematics education, much instruction is still teacher-centered and lecture-based (Frykholm, 2000). Prospective secondary mathematics teachers typically enter the preparation process with fairly rigid and fixed conceptions of mathematics that make it difficult for them to envision classrooms in which multiple solutions are encouraged, in which the teacher relinquishes the role of the authority (Frykholm, 1999). Secondary mathematics preservice teachers are often quick to note that the old way – homework, review, lecture, practice – worked for them. Thus, they feel they do not need to handle their classrooms any differently. Therefore, beginning teachers often implement the same teacher-centered instructional strategies they encountered in high school. They do not consider the percentage of the student population for which this method does not work. Frykholm (1996) suggests that although beginning teachers report that they value reform-based teaching ideals, they lack the experience, content knowledge, and confidence to deviate from lecture based, rote instruction.

In her book, *Knowing and Teaching Elementary Mathematics*, Liping Ma (1999) discusses the profound understanding of fundamental mathematics that teachers need in order to teach effectively. She states that there are four crucial properties of understanding: connectedness, multiple perspectives, basic ideas, and longitudinal coherence. In the portion of the study dealing with division of fractions, teachers from the United States and China were given two tasks, compute $1\frac{3}{4} \div \frac{1}{2}$, and develop a representational meaning for the given expression. The process of calculation and creation of a story representation of $1\frac{3}{4} \div \frac{1}{2}$ revealed features of the teachers' procedural knowledge, their understanding of mathematics, as well as their attitudes toward mathematics. Of the 23 U. S. teachers, ten were able to solve the computation with the correct procedure and correct answer. Two used the correct procedure but had an incomplete answer. Four of the teachers used an incomplete procedure and had an incomplete answer. Six had a fragmentary memory of the algorithm and so gave no answer. One teacher had an incorrect strategy and thus had no answer. When asked to create a story to represent the expression, six teachers could not come up with a story, sixteen teachers created stories but the stories contained the misconceptions the teachers held about division of fractions, and one teacher created a correct story. Of the 72 Chinese teachers, all used a correct procedure to solve the computation and had the correct answer. They presented three alternative approaches to solving the problem: dividing by fractions using decimals, applying the distributive law, and dividing fractions without multiplying by the reciprocal of the divisor (see figure 1). Sixty-five of the teachers were able to create a correct story to represent the expression while sixteen of these created more than one story. One teacher created a story which contained misconceptions about division of fractions while six teachers could not create any story. Ma found that the U. S. teachers' procedural knowledge was weaker in operations with fractions than the other operations she examined. The U. S. teachers also lacked a solid conceptual understanding of division of fractions. Their most common misconceptions were: confusing division by $\frac{1}{2}$ with division by 2, confusing division by $\frac{1}{2}$ with multiplication by $\frac{1}{2}$, and confusing division by $\frac{1}{2}$ with both multiplication by $\frac{1}{2}$ and division by 2. They tended to use concrete wholes, such as pizza or pie, and their parts to represent a whole and a fraction. They lacked the flexibility necessary to create alternative representations. Most of the Chinese teachers represented the concepts in a more abstract way using lengths of measurement, bags of sugar, and areas of fields. Ma found that the Chinese teachers' profound understanding of the meaning of division by fractions and its connections to other models in mathematics provided them with a solid base on which to build their pedagogical content knowledge of the topic.

One reason the U. S. teachers' understanding of the meaning of division of fractions was weak may be that their knowledge lacked connections. One can represent $1\frac{3}{4} \div \frac{1}{2}$ using three different models – the measurement model, the partitive model, and as a product of factors. The measurement model would ask, "How many $\frac{1}{2}$ meter lengths are there in a rope that is $1\frac{3}{4}$ meters long?" The partitive model would ask, "If $\frac{1}{2}$ a length of rope is $1\frac{3}{4}$ meters, how long is the

$$
\begin{aligned}
1\frac{3}{4} \div \frac{1}{2} &= \frac{7}{4} \div \frac{1}{2} \\
&= \frac{7 \div 1}{4 \div 2} \\
&= \frac{7}{2} \\
&= 3\frac{1}{2}
\end{aligned}
$$

Figure 1: Dividing fractions without multiplying by the reciprocal

whole?" The product of factors model would ask, "If one side of a $1\frac{3}{4}$ square meters rectangle is $\frac{1}{2}$ meter, how long is the other side?"

Fraction concepts are introduced in the elementary grades. However, elementary teachers have been found to possess a generally low level of conceptual and factual knowledge with respect to fractions (Stevens & Wenner, 1996). Thus the topics usually receive superficial attention and are often taught in a meaningless way (Bezuk & Bieck, 1993). Therefore, teachers of middle school and high school students should work to develop student understanding rather than assume the students already understand these topics.

1.2 Significance of the Problem

Current reform movements in mathematics education call for increased attention to conceptual understanding of topics rather than rote memorization of tasks and algorithms. Research has shown the importance of conceptual knowledge in becoming well-versed in a subject. Therefore, teachers must have strong conceptions of fundamental operations that are rich and flexible. Since tomorrow's teachers are today's preservice teachers, the conceptions that preservice teachers hold of fundamental operations should be of concern to teacher educators.

Educational research has increasingly focused on the subject matter knowledge and pedagogical content knowledge of teachers and preservice teachers and their role in preparing elementary and secondary teachers (Even & Tirosh, 1995; Fischbein, et al., 1985; Shulman, 1986; Tirosh, 2000; Tirosh & Graeber, 1991). Much of this mathematical education research has been focused on elementary preservice teachers and their conceptions/misconceptions of rational numbers and their operations, but few have focused on secondary mathematics preservice teachers (Ball, 1990; Cooney, 1999; Stein, Baxter, & Leinhardt, 1990). This study investigated secondary mathematics preservice teachers' conceptions/misconceptions of rational numbers, their operations, and their representations.

An understanding of rational numbers is a cornerstone of students' mathematical development. However, the rational number domain is one that causes great difficulties for students and their teachers (Simoneaux, Gray & Golding, 1997). True understanding of rational numbers requires an understanding of each representation and the relationships among the representations. Preservice teachers should be able to model a given rational number in a variety of ways. However, in order to present a pedagogically powerful representation for a topic, a teacher should first have a comprehensive understanding of the topic.

Division of fractions is often considered the most mechanical and least understood topic in the school curriculum. Several studies have examined students' difficulties with fractions. Many of these studies have reported that students' responses to mathematical tasks are often determined by their intuitive beliefs, which are incompatible with the formal mathematical definitions and theorems (Fischbein, et al., 1979). Unfortunately, once misconceptions are established, they are difficult to "unlearn." Several researchers have shown that misconceptions established by children are not outgrown (Simoneaux, Gray & Golding, 1997; Tirosh

& Graeber, 1990). Therefore, many preservice teachers have the same misconceptions they formed as young children. The difficulties preservice teachers experience as they attempt to represent division with fractions suggest that the preservice teachers have a narrow understanding of division (Ball, 1990).

1.3 Statement of the Problem

In the early 1990s, research on teachers' mathematical knowledge began a new focus – that of studying teachers' understanding of specific mathematical topics which are included in the school curriculum (Ball, 1990; Even, 1993; Tirosh & Graeber, 1989, 1990). The reason for the increased attention to teachers' subject matter knowledge may be attributed to the heightened expectations for student learning coming from the *Curriculum and Evaluation Standards for School Mathematics* published by the National Council of Teachers of Mathematics in 1989. Students learn mathematics through the experiences and mathematical tasks that teachers provide. Therefore, teachers must know and understand the mathematics they are teaching. For the most part, research found that many teachers do not have a solid understanding of the subject matter they teach. Thus, teacher education should explicitly focus on topics included in the high school curriculum, many of which the teachers have not studied since high school. However, it can not be assumed that the teachers' subject matter knowledge is sufficiently comprehensive and articulated for teaching. Frykholm (2000) found that although secondary mathematics preservice teachers could complete procedures and algorithms for "elementary" mathematical concepts, they were unable to offer accompanying conceptual explanations. Serious misconceptions were found at the most basic levels of knowledge of rules, procedures, and concepts such as division, proof, and function. Thus, insufficient subject matter knowledge seems to be widespread with teaching consequences that should be investigated.

1.4 Research Questions

The research questions that guide this study are:

1) To what extent do secondary mathematics preservice teachers have procedural knowledge with rational numbers? Can they perform operations with rational numbers?

2) To what extent do secondary mathematics preservice teachers have conceptual knowledge of rational numbers, their representations, and their operations? Can they create a story to represent an expression? Can they determine an appropriate expression to solve a given situation? Do they know why and how the procedures and algorithms they apply work?

3) To what extent do secondary mathematics preservice teachers have pedagogical content knowledge? Do they know common difficulties students may experience? Do they know possible sources for students' misconceptions? Can they provide suggestions for correcting these misconceptions?

1.5 Theoretical Framework

Various research approaches have explored the effects of teacher knowledge on student learning. The first approaches focused on subject matter knowledge. For the most part these studies found little or no correlation between teacher subject matter knowledge and student learning (Begle & Geeslin, 1972). However, these studies focused on inaccurate measures of teacher knowledge such as number of college-level mathematics courses taken. In more recent years, research has focused more on teacher thinking, teacher knowledge, and beliefs as potentially significant influences on student learning. This section will explore how teacher knowledge and its impact on student learning is conceptualized by researchers in the field.

In her 1991 study, Lampert demonstrated that a thorough understanding of mathematics can influence what a teacher does in the classroom. She found that the impact of the teacher knowledge was demonstrated in the choice of representations, the design of activities, and the guidance of classroom discourse.

Leinhardt and Smith (1985) examined the impact of expert and novice teachers' knowledge on student learning. They suggested that teachers' knowledge impacts both the content and the processes of instruction, affecting both what they teach and how they teach. The researchers concluded that teachers with more explicit and better organized knowledge tended to provide instruction characterized by conceptual connections, appropriate and varied representations, and active and meaningful student discourse. Teachers with limited knowledge portrayed the subject as a collection of unrelated facts, provided poor or inappropriate examples and representations, and emphasized seatwork and routinized student input instead of meaningful dialogue.

Thomas Carpenter and his colleagues at University of Wisconsin focused on teachers' knowledge of students' understandings, a piece of pedagogical content knowledge (1988). They believed that the influence of this knowledge of students' understanding and misunderstanding should be evident in classroom instruction and would impact student learning. In particular, they found that teachers who have and use knowledge of their students' thinking can make more informed instructional decisions as they structure instruction so that students can connect what they are learning to the knowledge they already possess.

Shulman and Grossman (1988) found that the influence of teachers' subject matter knowledge on their classroom instruction was seen in a number of ways. They concluded that subject matter knowledge and background in a content area affect the ways in which teachers select and structure content for teaching, choose activities and assignments for students, and use textbooks and other curriculum materials. Teachers with the lowest level of knowledge were more rule-based in their teaching, often because they did not have enough mathematical knowledge to explain to their students anything but algorithms and procedures. However, teachers with greater mathematical knowledge used more conceptual teaching strategies and were more likely to explain to students why certain procedures do or do not work, to relate one concept to another and to the "big picture," and to show applications of the material studied. These teachers engaged their students in more active problem solving. The teachers' approaches to adapting lessons and activities for ability levels of their students also seemed to be related to their own knowledge level. Teachers who lacked confidence in their knowledge found few things wrong with their textbooks and were more likely to use the curricular materials without any adaptations. Teachers who had more confidence and competence in mathematics drew on their subject matter knowledge to evaluate, modify, and supplement the curricular materials.

The above research programs have revealed the benefits of a deep and broad subject matter knowledge and a rich pedagogical content knowledge. Teachers with greater mathematical knowledge applied more conceptual teaching strategies, provided explanations of why procedures worked, used appropriate and varied representations, and actively involved the students in exploring mathematical concepts more deeply through meaningful discourse. Their knowledge also allowed them to anticipate and meet the needs of their students and to supplement curriculum materials as needed. Therefore, teachers with depth and breadth to their subject matter knowledge and a richness in their pedagogical content knowledge are more likely to provide mathematics instruction focused on the development of conceptual connections, problem-solving skills, and reasoning abilities.

1.6 Purpose and Method

The purpose of this study was to determine the level of secondary mathematics preservice teachers' subject matter knowledge, including procedural knowledge and conceptual knowledge, and pedagogical content knowledge of rational numbers.

The subjects of this study were drawn from a population of secondary mathematics preservice teachers. Through convenience sampling a total of fifteen secondary mathematics preservice teachers participated in this study. These students were enrolled in a mathematics methods course at a midwestern land grant university. Demographic information concerning age, gender, high school mathematics courses, and college mathematics courses was collected. Interview participants were selected based on their responses to the ***Rational Numbers and Their Representations*** survey (see Appendix A), willingness to contribute, and scheduling concerns. Three students participated in the interviews. All participants signed a consent form before participating in any data collection.

Participants completed a survey of demographical information including the mathematics courses taken at the high school and college level, and a ***Rational Numbers and Their Representations*** survey. The survey consisted of six questions with a total of seventeen parts focusing on the symbolic representations of rational numbers including modeling rational numbers as parts of sets and regions. The survey took approximately thirty minutes to complete. The survey was evaluated to determine the preservice teachers' level of procedural and conceptual knowledge, then three preservice teachers were selected to complete an interview. The interview protocol included questions in which the preservice teachers created a story to match a given mathematical expression. The preservice teachers were asked to list common misconceptions students may have or mistakes students may make in representing and working with problems of this type. The ***Rational Numbers and Their Representations*** survey was given during a class period in the mathematics education course in which the preservice teachers were currently enrolled. The interviews were scheduled outside of classtime. The interviews were transcribed and coded to broaden the understanding of the secondary mathematics preservice teachers' conceptions of rational numbers and their representations.

The data sources for this study are the demographic survey, the ***Rational Numbers and Their Representations*** survey, and the interview transcriptions. The survey yielded quantitative data used to compare the levels of procedural and conceptual knowledge of the preservice teachers. The interview data yielded qualitative data which gives a broader understanding of the conceptual knowledge levels as well as the pedagogical content knowledge levels of the preservice teachers. Thus the data were analyzed using both quantitative and qualitative methods of data analysis to look for trends. Results from the surveys and interviews were used to determine the levels of procedural and conceptual knowledge as well as pedagogical content knowledge of the secondary mathematics preservice teachers.

The quantitative design was employed to gather information regarding the preservice teachers' procedural and conceptual knowledge of rational numbers from a global standpoint in order to yield a comparison between those preservice teachers with a higher level of conceptual knowledge and those with a lower level of conceptual knowledge. The instrument used to gather this data was the ***Rational Numbers and Their Representations*** survey. The qualitative design was employed to gather information regarding the preservice teachers' conceptual knowledge and pedagogical content knowledge. The interview transcriptions were analyzed to cultivate a deeper understanding of the secondary mathematics preservice teachers' conceptual knowledge as well as their pedagogical content knowledge.

1.7 Assumptions and Limitations

The following assumption was made regarding this study:

> The preservice teachers participated in both the survey and interview to the best of their ability. Each participant responded honestly and thoughtfully to all questions.

The following statement is a limitation regarding this study:

> Since the sample of this study involved preservice teachers enrolled in a course for secondary mathematics education majors, this is a sample of convenience. Consequently, quantitative findings may not be generalized to the entire population of secondary mathematics preservice teachers.

1.8 Definition of Terms

Conceptual knowledge – The understanding of the underlying structure of mathematics – the relationships and interconnections of ideas that explain and give meaning to mathematical procedures.

Pedagogical content knowledge – The understanding of how particular topics, principles, and strategies are comprehended and learned or miscomprehended and forgotten. Knowledge of the conceptual and procedural knowledge that students bring to the learning of a topic, the misconceptions they may have developed, and the stages of understanding that they are likely to pass through as they move toward mastery. Pedagogical content knowledge also consists of knowledge of multiple ways of representing and formulating the subject that make it comprehensible to others.

Preservice teacher – One who has declared an intention to teach, has applied to and been accepted in the Professional Education Unit of the College of Education. One who is pursuing teaching licensure and certification.

Procedural knowledge – Mastery of computational skills and knowledge of procedures for identifying mathematical components, algorithms, and definitions.

Rational number – The set of rational numbers consists of all numbers $\frac{a}{b}$ such that a and b are integers and $b \neq 0$.

Subject matter knowledge – Knowledge of key facts, concepts, principles, and explanatory frameworks of a discipline, as well as the rules of evidence used to guide inquiry in the field. It includes both conceptual knowledge and procedural knowledge.

1.9 Conclusion

Secondary mathematics teachers need a solid conceptual knowledge as well as procedural knowledge of rational numbers. They should also have a good understanding of the misconceptions their future students may have regarding rational numbers. This study was to determine the levels of conceptual and procedural knowledge as well as the pedagogical content knowledge the secondary mathematics preservice teachers had regarding rational numbers and their representations. This information can be used to determine a curriculum design which can promote solid conceptual knowledge and procedural knowledge of rational numbers, their representations, and operations as well as inform preservice teachers of misconceptions their future students may have.

2 Review of Literature

2.1 Introduction

The rational number domain is one that causes great difficulties for students and their teachers (Simoneaux, Gray & Golding, 1997). True understanding of rational numbers requires an understanding of each representation and the relationships among the representations. In order to provide students the opportunities needed to develop concepts with rational numbers the teachers need a substantial amount of knowledge themselves. The knowledge needed for teaching includes knowledge of mathematics, knowledge of the connections within the subject as well as with other subjects, and knowledge of students' understandings and misunderstandings.

2.2 Representations of Rational Numbers

There are four basic ways a rational number can be interpreted – as a measure, as a quotient or indicated division, as an operator, and as a ratio (Bezuk & Bieck, 1993; Graeber & Tenenhaus, 1993). When a rational number is interpreted as a measure, one is measuring the area of a region by partitioning it and covering it with appropriately sized units. When interpreted as a quotient, a rational number is seen as a solution to a problem of division; for example $\frac{1}{4}$ represents the amount each child gets if one cake is divided among four children. An example of a rational number being interpreted as an operator is in that of filling boxes with cookies. Each box contains 4 cookies so there are $\frac{1}{4}$ as many packages as cookies. In the ratio interpretation, $\frac{a}{b}$ refers to a relationship between two quantities. A rational number is interpreted as a ratio in the example of making orange juice from concentrate, where one can of orange juice concentrate is mixed with four cans of water. This rational number can also be represented by $\frac{1}{4}$. Proportional reasoning is one form of mathematical reasoning involving a sense of covariation, multiple comparisons, and the ability to mentally store and process several pieces of information. All proportional relationships can be represented by the function $y = mx + b$.

True understanding of rational numbers requires an understanding of each representation and the relationships among the representations. Students who encounter a wide variety of representations will exhibit greater flexibility in dealing with problems involving the application of fractions. Preservice teachers should be able to model a given rational number or an operation in a variety of ways. Consider, for example, $\frac{1}{2} + \frac{3}{4}$. Students can determine this sum by converting each fraction to a common denominator; converting each to a decimal; using a number line; representing the fractions as segments; or representing the fractions as regions within a circle or a rectangle. Thus, in order to present and accept appropriate, pedagogically powerful representations for a topic, a teacher should first have a comprehensive understanding of the topic.

2.3 Models of Division

There are two models for division – the measurement model and the partitive model. The measurement model is usually introduced first. This model is based on the idea of repeated subtraction. In this model, one is determining how many sets of a given size are contained in the total. Once the measurement model is introduced, the frequency with which it appears levels off and the partitive model then becomes dominant. The partitive model, also known as the quotitive model, focuses on finding the size of each of a given number of equal sets. The partitive model is not easily applied to situations involving non-integers. When using non-integral numbers the measurement model can be more easily understood than the partitive model. For example, compare the two models in the following statements. The measurement model would ask the following question, "How many $1\frac{1}{2}$ pound portions are there in a 12 pound bag of nuts?" An example of the partitive model is "I have 12 pounds of nuts. This fills $1\frac{1}{2}$ bags. How much is in one bag?" In this example, $1\frac{1}{2}$ represents the number of groups you already have. You want to answer the question, "How many nuts are in one whole group?" This representation corresponds directly to the invert and multiply algorithm.

Preservice teachers tend to think in terms of the partitive model rather than the measurement model (Tirosh & Graeber, 1989; Ball, 1990). In her 1990 study, Ball found that the subjects tended to consider division only in partitive terms, that is, forming a certain number of equal parts. This model of division corresponds less well to division with fractions than the measurement model. Fischbein, et al. (1985) argue that there is a primitive partitive model of division and a primitive measurement model of division. These conceptions demand that the divisor be a whole number and that the divisor be less than the dividend, respectively. Since the partitive model becomes the dominant model, misconceptions associated with division are largely due to the experience with the primitive partitive model of division. Another reason multiplication and division problems are often so complex is that many related ideas, such as understanding fractions as indicated division, rates, ratios, and proportion are involved (Sinicrope, Mick, & Kolb, 2002).

2.4 Misconceptions

Division of fractions is often considered the most mechanical and least understood topic in the school curriculum. Several studies have examined students' difficulties with fractions. Many of these studies have reported that students' responses to mathematical tasks are often determined by their intuitive beliefs, which are incompatible with the formal mathematical definitions and theorems (Fischbein, et al., 1985; Siebert, 2002). Graeber and Tirosh (1990) found that misconceptions established by children are not outgrown. Therefore, many preservice teachers have the same misconceptions as do young children (Simoneaux, Gray & Golding, 1997). Hunting (cited in Bezuk & Bieck, 1993) found that students' partitions of continuous quantities, such as pizza, differ from their partitions of discrete quantities, such as blocks. He hypothesized that the understanding of fractions in continuous contexts is a prerequisite for understanding problems in discrete contexts. Student difficulties using region, set, and number line models to illustrate fractions may result from their lack of exposure to a sufficiently wide range of models to encourage them to generalize the fraction concept (Bezuk & Bieck, 1993). Problems are also more difficult for students when they must manipulate problem conditions such as marking $\frac{2}{3}$ on a region divided into fourths (Bezuk & Bieck, 1993). However, manipulating problem conditions is an important real-world skill since problems in the real world seldom appear as neatly as they do in a school textbook. Students also have difficulty recognizing that shapes that are not congruent can still have the same area. Further, many students do not see rational numbers as an indication of division (Graeber & Baker, 1992). Students tend to treat the numerator and denominator as separate whole numbers without considering their relationship to each other or as a single rational quantity. Several studies have shown that preservice teachers see the expression $1\frac{3}{4} \div \frac{1}{2}$ as a question about fractions rather than division (Ball, 1990; Huinker, 2002; Ma, 1999). These difficulties that preservice teachers experience suggest a narrow understanding of division (Ball, 1990).

In the case of division of rational numbers, the mistakes made can be organized into three main categories: algorithmically-based mistakes, intuitively-based mistakes, and mistakes based on formal knowledge. The category of algorithmically based mistakes consists of various "bugs" in computing, including inverting the dividend instead of the divisor and inverting both the dividend and the divisor. These bugs usually are the result of the rote memorization of the algorithm. The intuitively-based mistakes result from intuitions held about division, such as the divisor must be a whole number, the divisor must be less than the dividend, and the quotient must be less than the dividend. The mistakes based on formal knowledge result from limited conceptions of the notion of fraction and inadequate knowledge related to the properties of the operations. An example of this type of mistake is overgeneralizing commutativity to include division. Difficulties with rational numbers are heightened by misconceptions that arise as students try to give meaning to the teacher-taught algorithms. Because of their misconceptions about the meaning of division by fractions, the teachers fail to create correct representations (Flores, 2002).

Some of the most common misconceptions Ma (1999) encountered were confusing division by $\frac{1}{2}$ with multiplication by 2 or division by 2. Students tend to get confused with the language and confuse dividing *in* half with dividing *by* one-half. Many times the students are not even aware of the difference even though they get different answers (Ball, 1990; Ma, 1999). Pothier and Sawada (1983) found that students often misuse the term "half". An example of this is when students say "break in half into four pieces." Awareness of such confusion is at the heart of what teachers must know if they are to help their students understand mathematics. If the students have misconceptions, which is often the case, teachers need a knowledge of the strategies most likely to correct the misconceptions (Shulman, 1986).

The development of the quantitative notion of the relative size of fractions, or the "bigness" of fractions, is very important. Students who have a good quantitative concept of fractions are able to estimate the relative size of fractions, find equivalent fractions, and estimate the location of a fraction on a number line. If students have not refined their estimation skills, they cannot recognize their faulty application of rules leading to ridiculous or inappropriate answers. Not having a conception of an operation deprives students of the ability to estimate answers. Early estimation of whether the answer to a division word problem is greater than, less than, or equal to one, can cause students to rethink incorrect procedures. Thus instruction should emphasize meaning, understanding, and reasonableness of answers.

Errors that students make can be a powerful tool to diagnose learning difficulties and thus direct remediation (Borasi, 1987). Errors can be seen as a valuable source of information about the learning process. They provide an opportunity to discover what students really know and how they constructed their knowledge. For example if students make the following mistake, $\frac{3}{4} + \frac{5}{7} = \frac{8}{11}$, the students could have confused the algorithm for multiplication with the algorithm for addition or tried to operate with fractions as they did with whole numbers. Remediation of errors is most effective if the teacher is willing and able to hypothesize about the error's possible causes and to verify which ones are relevant in the case of each individual making the mistake (Borasi, 1987).

2.5 Procedural Knowledge and Conceptual Knowledge

Research has established the importance of understanding a concept in order to become proficient in a subject. When students understand mathematics, they are able to use their knowledge flexibly. They can combine knowledge of facts, procedures, and conceptual understanding in powerful ways. Students who memorize facts or procedures without understanding are often not sure when or how to use what they know. Conceptual understanding enables students to work with novel problems and settings. Students with a strong conceptual understanding have the ability to solve problems they have not encountered before. Therefore, current reform movements in mathematics education call for increased attention to conceptual understanding of topics rather than rote memorization of tasks and algorithms. Thus, teachers must have strong conceptions of fundamental operations that are rich and flexible. The goal of teaching mathematics is for students to develop mathematical understanding. That is, the students should acquire knowledge of mathematical concepts and procedures, the relationships among them, and why the procedures work. However, understanding also implies learning about mathematical ways of knowing. Teachers must understand mathematics deeply if they are to facilitate the types of discussions that emerge when learners are engaging in authentic mathematical experiences (Ball, 1990; Frykholm, 2000). In order for students to be successful in fulfilling their personal ambitions and career goals, teachers must provide them with the best mathematical education possible, one that substantially increases the students' mathematical power as they learn to make conjectures, justify claims, and validate their own thinking. In order to promote these students' mathematical power, teachers must understand the mathematics they teach thoroughly and deeply.

The subject matter knowledge needed for teaching includes both knowledge of mathematics and knowledge about mathematics. Subject matter knowledge includes knowledge of key facts, concepts, principles and explanatory frameworks of a discipline, as well as the rules of evidence used to guide inquiry in the field (Borko, et al., 1992). Subject matter knowledge also includes algorithmic operations and the connections among different algorithmic procedures, the subset of the number system being drawn upon, the understanding of classes of student errors, and curriculum presentation (Leinhardt & Smith, 1988). Teachers' subject matter knowledge should be explicit; that is, they should be able to explain it. Simply knowing how to do something is useful when attempting to solve a problem; however, it is inadequate for teaching. Teachers must be able to talk about mathematics, about the judgments made, about the meanings and reasons for certain relationships or procedures, not just describe the steps of an algorithm. Teachers need to understand the underlying meanings and connections. Mathematics is often treated as a collection of separate facts and procedures which inhibits meaningful understanding and misrepresents the nature of the subject. Instead of considering each problem as needing a separate rule, memorized individually, teachers and students need to see the connected dynamic nature of mathematics. Teachers also need knowledge about the nature of justification within mathematics – explaining, verifying, and proving mathematical propositions.

There are two components to subject matter knowledge – procedural knowledge and conceptual knowledge. Procedural knowledge refers to mastery of computational skills as well as a knowledge of procedures for identifying mathematical components, algorithms, and definitions (Eisenhart, et al., 1993; Hiebert & Lefevre, 1986). That is, procedural knowledge is knowing how to get an answer. Procedural knowledge focuses on algorithms and procedures to solve a given problem. Procedural knowledge is characterized by a lack of relationships and connections to prior knowledge. Conceptual knowledge is knowledge that is rich in relationships. Conceptual knowledge refers to knowledge of the underlying structure of mathematics – the relationships and interconnections of ideas that explain and give meaning to mathematical procedures. Thus conceptual knowledge is a part of conceptual understanding which goes beyond knowing two things are connected to knowing "how" they are connected. Information becomes conceptual knowledge only when it is integrated into a larger network that is already in place (Hiebert & Carpenter, 1992). Students are more likely to understand why a process works if that understanding is established before students gain a routinized understanding of how the procedure works (Graeber & Tanenhaus, 1993).

Conceptual knowledge includes such ideas as the nature of fractions in general and of the particular fractions to be divided, as well as what it means to divide. Teaching division of fractions for procedural knowledge is exemplified by a step-by-step presentation of rules and algorithms as well as strategies for remembering them. Teaching division of fractions for conceptual knowledge is exemplified in the use of concrete and semi-concrete models, such as Cuisenaire rods, fraction strips, fraction circles, circular or rectangular drawings, that illustrate or represent division of fractions. Teaching for conceptual knowledge is also exemplified in the discussion of the connections among mathematical concepts. Conceptual teaching of a topic such as division of fractions is intended to help students understand the mathematical procedures used to obtain correct answers.

Poor performance with rational numbers may be a result of inadequate conceptual understanding on the part of the teacher (Simoneaux, Gray & Golding, 1997). The lack of conceptual knowledge of teachers has resulted in their delivery of a curriculum which emphasizes procedures rather than understanding. Students have memorized the algorithms, often incorrectly, but have no knowledge of the concepts underlying the procedures. There is evidence that procedural knowledge is emphasized in most schools and that teachers spend less time teaching for conceptual understanding (Eisenhart, et al., 1993). Standard school curriculum treats mathematics as though it consists of discrete bits of procedural knowledge (Ball, 1990). In most textbooks, little or no attention is given to the meaning of division of fractions, and no connections are made between division of fractions and division with whole numbers. Each is treated as a special case. However, the invert and multiply algorithm does not specifically work only with fractions. This method also applies to whole numbers. $12 \div 3$ yields the same result as $12 \times \frac{1}{3}$. However, this connection is rarely made explicit for students (Ball, 1990). To teach effectively, individuals must have knowledge of mathematics characterized by an explicit conceptual understanding of the principles and meaning underlying mathematical procedures. The knowledge must also be characterized by connectedness, rather than compartmentalization, of mathematical topics, rules, and definitions. For learners to develop strong mathematical connections, they must experience mathematics. They must make and test conjectures, explore relationships, communicate mathematically and connect concepts within mathematics as well as with other disciplines.

Unless care is taken during instruction to ensure understanding, students may be simply memorizing routines, thus learning procedural knowledge. When an algorithm is viewed as a meaningless series of steps, students may forget some of these steps or change them in ways that lead to errors. When procedures are only memorized, students can confuse the rules for whole numbers, decimals, and fractions. Students have gaps in their understandings of the meanings of the operations of multiplication and division and of the associated mathematical symbols.

Even if students remember an algorithm correctly and apply it correctly, they may not understand the underlying concept or why the algorithm works the way it does in that situation. Thus if they were faced with a similar but slightly different problem, they might not be able to solve the new problem correctly. Getting the right answer does not imply conceptual understanding. Preservice teachers' explicit statements about operations and even successful calculations can mask misconceptions about division (Tirosh & Graeber, 1991).

Both procedural and conceptual knowledge are necessary aspects of mathematical understanding. When concepts and procedures are not connected, students may have a good intuitive feel for mathematics but not be able to solve the problems, or they may generate answers but not understand what they are doing (Cramer, Post, & Currier, 1993; Steele & Widman, 1997). Students should develop relational understanding; that is, understanding both *what* to do and *why* the procedure is done in that manner as opposed to instructional learning which is learning rules without reasons (Pesek, 2000). To foster mathematical understanding in their students, teachers must be able to demonstrate conceptual and flexible representations of the concepts they are teaching. A lack of solid conceptual knowledge can lead to hesitancy and a possible inability to deliver effective instruction.

Rote-level mastery prior to an understanding of a concept creates an interference to meaningful learning of that same topic. Several mathematics education researchers have reported finding interference effects when initial procedural learning, also known as instrumental learning, is followed by conceptual learning, which is also known as relational learning (Wearne & Hiebert, 1988; Mack, 1990; Pesek, 2000) Students who previously acquire rote procedural knowledge tend to focus on symbolic manipulations and do not seem to consider the validity of their responses. The students' knowledge of rote procedures frequently interfere with their attempts to build on their informal knowledge. In her study on interference, Pesek (2000) found that students who received instrumental learning prior to relational instruction achieved no more, and most probably less, conceptual understanding than students exposed only to relational instruction. The students who received both instrumental and relational instruction were more likely to refer to formulas, operations, and fixed procedures for solving problems. However, those students who received only the relational instruction used conceptual and flexible methods of constructing solutions (Pesek, 2000).

Instruction in operations on fractions should be built on students' intuitive understanding of fractions and be based on actions on objects rather than solely on the manipulation of symbols according to a set of rules and procedures (Bezuk & Bieck, 1993). Leinhardt and Smith (1988) found that students can successfully perform operations on fractions by drawing on informal knowledge when problems were presented in the context of real life situations. Understanding mathematics involves understanding mathematics in a variety of contexts (Cooney, 1992). Teachers should emphasize the importance of context when doing mathematics. Working mathematical tasks within a context allows the students to put their answer back into the context of the problem to determine if the answer is reasonable. The habit of putting the answer back into the context of the original problem must be encouraged as the operations are extended to the domain of rational numbers. Students should be familiar with problems in context since real world problems rarely come in the perfect form found in most textbooks.

Instruction should provide students with structured learning experiences to help them acquire essential conceptual and procedural knowledge. Instruction should be meaning oriented rather than symbol oriented. Instructional procedures should encourage students to construct their own knowledge. Evidence of understanding comes when students can explain and model their conceptual knowledge. Thus, instruction should involve students in reflecting, explaining, reasoning, connecting, and communicating. Since there is no one way to represent a topic so that everyone understands it, a teacher must have a variety of alternative forms of representation. Teachers should understand the subject in sufficient depth to be able to represent it appropriately and in multiple ways including story problems, pictures, and with concrete materials. To develop, select, and use appropriate representations, teachers must understand the content they are representing, the ways of thinking and knowing associated with the content, and the students they are teaching. Prospective teachers in preservice content courses must be given the opportunity to see multiple embodiments of concepts. The teachers need to promote the improvement of the reasoning and understanding of conceptual aspects as opposed to simple skill development. In order to help someone else understand and do mathematics, it is not sufficient to simply be able to do it oneself. Teachers must be able to describe the relationships to the context and previous learning, and meanings of the procedures. Teachers must be able to generate explanations or other representations in order to respond to student questions. The teacher's solid knowledge of the meaning of a concept allows them to be more comfortable using a broad range of topics in representations.

Teaching for understanding must include both procedural and conceptual knowledge. A greater subject matter knowledge enables teachers to connect topics within a subject and to provide conceptual explanations, as opposed to purely algorithmic ones. In order to effectively teach division of fractions, one must have a substantive knowledge of the subject matter. This knowledge includes a developed conceptual understanding and procedural knowledge; that is, being able to calculate $1\frac{3}{4} \div \frac{1}{2}$. This knowledge also includes understanding the underlying principles and meanings; that is, what does it mean for $1\frac{3}{4} \div \frac{1}{2}$ to be $3\frac{1}{2}$? This substantive knowledge includes being able to appreciate and understand the connections among mathematical ideas; that is, how are fractions related to division?

2.6 Pedagogical Content Knowledge

Teacher knowledge consists of content knowledge or subject matter knowledge, which includes procedural knowledge and conceptual knowledge, and knowledge about teaching or pedagogical content knowledge. The latter includes having knowledge of students' common conceptions and misconceptions about the subject matter. Pedagogical content knowledge is the understanding of how particular topics, principles, and strategies are comprehended and learned or miscomprehended and likely to be forgotten. Pedagogical content knowledge is knowledge of the ways of representing and formulating the subject that make it comprehensible to others (Shulman, 1986). Pedagogical content knowledge includes knowledge of the conceptual and procedural knowledge that students bring to the learning of a topic, the misconceptions they may have developed, and the stages of understanding that they are likely to pass through as they move toward mastery (Carpenter, et al., 1988). Pedagogical content knowledge also includes knowledge of techniques for assessing students' understanding and diagnosing their misconceptions, knowledge of instructional strategies to connect what they are learning to the knowledge they already possess, and knowledge of instructional strategies to eliminate the misconceptions they may have developed. Thus, pedagogical content knowledge includes an understanding of what makes the learning of specific topics easy or difficult as well as the conceptions or misconceptions that students bring with them to the learning of these topics.

According to the standards found in *Professional Standards for Teaching School Mathematics* (National Council of Teachers of Mathematics, 1991), pedagogical content knowledge is essential for good teaching. Standard 2 – Knowing Mathematics and School Mathematics – states that the education of teachers of mathematics should develop their knowledge of the context and discourse of mathematics, including mathematics concepts, procedures and the connections among them. Standard 3 – Knowing Students as Learners of Mathematics – states that preservice and continuing education of teachers of mathematics should provide multiple perspectives on students as learners of mathematics. Standard 4 – Knowing Mathematical Pedagogy – states that preservice and continuing education of teachers of mathematics should develop teachers' knowledge of and ability to use and evaluate ways to represent mathematical concepts and procedures. Therefore, teachers must have knowledge about the nature and discourse of mathematics, and an understanding of what it means to know and do mathematics. Students must be asked to defend their choices and justify their answers. This information allows the teachers to make informed decisions about the students' learning. The teachers can identify the misconceptions students hold and shape what is recognized as acceptable justification (Graeber & Tanenhaus, 1993).

There are two critical components of pedagogical content knowledge of representations and a subject-specific knowledge of learners. Pedagogical content knowledge consists of an understanding of how to represent specific topics and issues in ways that are appropriate to the diverse abilities and interests of learners (Borko et al., 1992). To generate a representation, one should first know what to represent. Thus, teachers' pedagogical content knowledge is influenced by their subject matter knowledge (Even & Tirosh, 1995). Presenting a concept in different ways facilitates learning and teaches for understanding rather than for rote memorization.

Pedagogical content knowledge distinguishes between "knowing that" and "knowing why" (Tirosh, 2000). "Knowing that" refers to research-based knowledge about students' common conceptions and ways of thinking. "Knowing why" refers to general knowledge about possible sources of these conceptions and to the understanding of the sources of a specific student's reaction in a specific case. Teachers possessing a solid conceptual foundation are more equipped to "know why" a student responds in a certain way and diagnose the appropriateness of the student's responses. Students bring various intuitions and understanding to a classroom, thus they may use a variety of ways to communicate about mathematics. A teacher's conception of the topic should be rich and flexible enough to understand the students' thinking and to diagnose and correct misconceptions the students may hold. Prospective teachers need to be aware of common difficulties children experience with division of fractions as well as be able to determine the causes of the difficulties. If teachers learn more about their students' thinking, achievement can be increased (Graeber, 1999). However, studies have shown that preservice teachers' abilities to analyze the reasoning behind students' responses are poor (Ball, 1990; Even & Tirosh, 1995).

2.7 Conclusion

True understanding of rational numbers requires an understanding of the variety of representations as well as the relationships among the representations. Students who encounter a wide variety of representations will exhibit greater flexibility in dealing with problems involving the application of fractions. The goal of teaching mathematics is for students to develop a solid yet flexible mathematical understanding. That is, the students should develop knowledge of mathematical concepts and procedures, the relationships among them, and why the procedures work. However, understanding also implies learning about mathematical ways of knowing. Teachers must understand mathematics deeply if they are to facilitate the types of discussions that emerge when learners are engaging in authentic mathematical experiences. Both procedural and conceptual knowledge are necessary aspects of mathematical understanding. When concepts and procedures are not connected, students may have a good intuitive feel for mathematics but not be able to solve the problems, or they may generate answers but not understand what they are doing.

3 Method

3.1 Purpose

The purpose of this study was to determine the level of secondary mathematics preservice teachers' subject matter knowledge, including procedural and conceptual knowledge, and pedagogical content knowledge of rational numbers. Quantitative data were collected and analyzed in order to determine the levels of the preservice teachers' subject matter knowledge. Qualitative data were collected and analyzed in order to develop a deeper understanding of the preservice teachers' conceptual knowledge and pedagogical content knowledge. In this chapter, the method and procedures used to collect and analyze the data are described. The appropriateness of combining quantitative and qualitative methodologies is addressed in this chapter along with data collection and analysis procedures unique to each research paradigm.

The research questions that guided this study are:

1) To what extent do secondary mathematics preservice teachers have procedural knowledge with rational numbers? Can they perform operations with rational numbers?
2) To what extent do secondary mathematics preservice teachers have conceptual knowledge of rational numbers, their representations, and their operations? Can they create a story to represent an expression? Can they determine an appropriate expression to solve a given situation? Do they know why and how the procedures and algorithms they apply work?
3) To what extent do secondary mathematics preservice teachers have pedagogical content knowledge? Do they know common difficulties students may experience? Do they know possible sources for students' misconceptions? Can they provide suggestions for correcting these misconceptions?

3.2 Combining Qualitative and Quantitative Designs

There is an on-going debate about using both qualitative and quantitative data analysis. Cresswell (2003) reports that purists in each realm of methodology argue that neither framework should be mixed. However, Cresswell presents several models of combined designs: sequential, concurrent, and transformative. The two-phase design enables the researcher to conduct separate quantitative and qualitative phases. The advantage of this approach is that each paradigm is clearly separated from the other. The design allows for each paradigm to bring its own assumptions and analysis. This design does not suggest that one method is dominant and another less dominant. Rather, each framework contributes information, assumptions, methods of analysis unique to its paradigm to the topic under investigation.

The sequential methodology was selected for this study. In this methodology, the researcher collects data in one way then seeks to elaborate on that data using the other framework. Thus, the methodologies are combined for the purpose of expansion in order to add scope and breadth to the study. For the purposes of this study, quantitative data was collected first using the ***Rational Numbers and Their Representations*** survey; then the interviews were used to expand the findings of the survey.

3.3 Participants

The subjects of this study were drawn from a population of secondary mathematics preservice teachers. Through convenience sampling a total of fifteen secondary mathematics preservice teachers participated in this study. These students were enrolled in a mathematics methods course situated in a midwestern land grant university. Demographic information concerning age, gender, high school mathematics courses, and college mathematics courses was collected. Interview participants were selected based on their responses to the ***Rational Numbers and Their Representations*** survey, willingness to contribute, and scheduling concerns. Three students participated in the interviews. All participants signed a consent form before participating in any data collection.

3.4 Instrumentation

Quantitative data were collected through the administration of the ***Rational Numbers and Their Representations*** survey (Appendix A). Subject matter knowledge consists of two types of knowledge – procedural knowledge and conceptual knowledge. Procedural knowledge refers to mastery of computational skills and knowledge of procedures for identifying mathematical components, algorithms, and definitions (Eisenhart, et al., 1993). That is, procedural knowledge is knowing how to get an answer. Conceptual knowledge is knowledge that is rich in relationships. Conceptual knowledge refers to knowledge of the underlying structure of mathematics – the relationships and interconnections of ideas that explain and give meaning to mathematical procedures. A piece of information becomes conceptual knowledge only when it is integrated into a larger network that is already in place. Fischbein, et al. (1985), Ball (1990), and Graeber & Tirosh (1990) developed instruments to determine the subject matter knowledge of elementary preservice teachers with respect to rational numbers. The survey used in this study was constructed using the constructs of these instruments as well as constructs based on the knowledge levels developed by Kieren (Kieren, 1988; Pirie & Kieren, 1989).

The ***Rational Numbers and Their Representations*** survey consists of six open-ended questions containing seventeen parts designed to determine subject matter knowledge. Four parts directly assess procedural knowledge (5 a, b, c, d). Four parts directly assess conceptual knowledge (4 a, b, c, d). The remaining nine parts measure both procedural and conceptual knowledge including representations of rational numbers. The responses were scored on a 5 point scale from no answer (0) to correct answer and algorithm (4). See Rubric A in Table 1. Possible scores on the ***Rational Numbers and Their Representations*** survey range from a 0 to 68.

Rubric A	
0	Wrong answer; no strategy
1	Fragmentary memory of the algorithm; no answer
2	Incomplete algorithm, unsure; incomplete answer
3	Correct algorithm, incomplete answer
4	Correct algorithm, complete answer

Table 1: Rubric to evaluate ***Rational Numbers and Their Representations*** survey

The interview protocol (Appendix B) focused on certain pieces of conceptual and pedagogical content knowledge. Although conversations were guided by the interview protocol, the design remained continuous and flexible. Questions were modified to probe for more meaningful information as the interviews progressed. The interviews were audio recorded and transcribed for subsequent meaning interpretation. The first protocol question addressed computing $1\frac{3}{4} \div \frac{1}{2}$. This question was evaluated on a three point scale ranging from no answer (0) to correct answer and correct algorithm (2). See Rubric B in Table 2. The second protocol question asked the preservice teacher to create a story to represent $1\frac{3}{4} \div \frac{1}{2}$. This question was evaluated on a five point scale ranging from no story (0) to correct story with no misconceptions or pedagogical problems (4). See Rubric C in Table 3. The last three protocol questions were evaluated on a 4 point scale ranging from no response (0) to many varied, correct responses (3). See Rubric D in Table 4.

Rubric B	
0	No answer
1	Incomplete algorithm, unsure; incomplete answer
2	Correct algorithm, complete answer

Table 2: Rubric to evaluate computations

Rubric C	
0	No story
1	Incomplete story; story contains extreme misconceptions
2	Story contains misconceptions
3	Story contains correct conceptions but is pedagogically problematic
4	Story contains correct conceptions and does not pose pedagogical problems

Table 3: Rubric to evaluate stories

Rubric D	
0	No Response
1	Incorrect responses
2	One or two distinct correct responses
3	Three or more distinct correct responses

Table 4: Rubric to evaluate pedagogical content knowledge

3.5 Evidence of Validity and Reliability

Validity of the data must be evaluated within the purpose of the study. Since it is important that the measures fit the theories for which the survey was designed, construct validity should be discussed. Although construct validity cannot be definitely established, several kinds of evidence were established for the ***Rational Numbers and Their Representations*** survey in this study.

This instrument was considered to be valid for this particular study since it was used in the recent past to investigate mathematics subject matter knowledge. The items included on this instrument were developed from constructs that have been shown through research and studies to accurately determine mathematics subject matter knowledge (Fischbein, Deri, Nello & Marino, 1985; Ball, 1990; Graeber & Tirosh, 1990; Langford & Sarullo, 1993; Owens & Super, 1993; Post & Behr, 1988). Validation continues to be an ongoing process, and continued examination in terms of study specific reliability assessment and cross validation is necessary.

In order to determine the reliability of this instrument, an expert panel consisting of mathematics and mathematics education professors was invited to review the instruments. The instruments were sent to the members of the panel and their comments were collected by the researcher. The panel reviewed the instruments and found that the questions would yield data commensurate with the research questions for this study.

3.6 Procedure

Data for this study were collected during the Spring semester at a midwestern land grant university. To investigate secondary mathematics preservice teachers' subject matter knowledge, the ***Rational Numbers and Their Representations*** survey was given (see Appendix A). The data collected from this survey were investigated to answer the first research question regarding the preservice teachers' procedural knowledge. The data collected was also used to partially investigate the preservice teachers' conceptual knowledge which is addressed in the second research question. On this open-ended survey, the four parts of question five were designed to determine the preservice teachers' procedural knowledge, specifically operations with rational numbers. The four parts of question four were designed to determine the preservice teachers' conceptual knowledge of rational numbers. The remaining nine parts of the survey were designed to give further information regarding the preservice teachers' procedural and conceptual knowledge of rational numbers, particularly the representations of rational numbers.

All secondary mathematics preservice teachers enrolled in the methods course completed the ***Rational Numbers and Their Representations*** survey. The preservice teachers selected to complete the interview answered questions designed to further examine their conceptual knowledge of ***Rational Numbers and Their Representations*** as well as questions designed to determine their pedagogical content knowledge. The last two questions on the interview protocol were designed specifically to determine the preservice teachers' pedagogical content knowledge. The remaining questions on the interview protocol were designed to test both conceptual knowledge and pedagogical content knowledge. The open-ended survey and the interview protocol were data collection techniques that enabled the researcher to focus on the knowledge of the secondary mathematics preservice teachers.

The interviews were semi-structured in that the design was not highly structured but followed an interview protocol that focused on certain pieces of conceptual and pedagogical content knowledge. Although conversations were guided by the interview protocol, the design remained continuous and flexible. Questions were modified to probe for more meaningful information as the interviews progressed. The interviews were audio recorded and transcribed for subsequent meaning interpretation. Qualitative data provided the opportunity to access the rich detail necessary for gaining an understanding of the conceptions of rational numbers held by secondary mathematics preservice teachers.

3.7 Data Analysis

Quantitative data were analyzed in order to determine the levels of subject matter knowledge of the secondary mathematics preservice teachers. Qualitative data collected from interviews were analyzed to cultivate a deeper understanding of the secondary mathematics preservice teachers' conceptual knowledge and pedagogical content knowledge. The transcribed interviews were interpreted by the researcher. A coding process was used for the purpose of structuring, clarifying, and developing deeper meanings from the interview conversations. Interviews were then conducted with a sample of the preservice teachers participating. Three preservice teachers were selected for interviews based on their responses on the ***Rational Numbers and Their Representations*** survey; one who received a high score on the survey, another who received a mid-level score on the survey, and a third who received a low score on the survey.

Research Question 1: To what extent do secondary mathematics preservice teachers have procedural knowledge with rational numbers? Can they perform operations with rational numbers?

The secondary mathematics preservice teachers' procedural knowledge was examined using the ***Rational Numbers and Their Representations*** survey. Their level of procedural knowledge was determined by their score on the ***Rational Numbers and Their Representations*** survey. The question specifically addressing procedural knowledge was question five. The four parts of this question addressed operations with rational numbers. The other questions, except question four which specifically deals with conceptual knowledge, can also be used to address procedural knowledge as the preservice teachers are asked to demonstrate different representations of rational numbers. The responses were scored on a five point scale from no answer (0) to correct answer and algorithm (4). See Rubric A in Table 1. Possible scores on the Rational Numbers and Their Representation survey range from a 0 to 68.

Research Question 2: To what extent do secondary mathematics preservice teachers have conceptual knowledge of rational numbers, their representations, and their operations? Can they create a story to represent an expression? Can they determine an appropriate expression to solve a given situation? Do they know why and how the procedures and algorithms they apply work?

The secondary mathematics preservice teachers' conceptual knowledge was examined using the ***Rational Numbers and Their Representations*** survey as well as the interview process. Their level of conceptual knowledge was determined by their score on the ***Rational Numbers and Their Representations*** survey as well as coded transcriptions of their interviews. The question specifically addressing conceptual knowledge on the survey was question four. The four parts of this question addressed the idea of a unit as well as representations of rational numbers. The other questions on the survey, except question five which addressed only procedural knowledge, addressed both conceptual and procedural knowledge as the preservice teachers are asked to demonstrate representations of rational numbers as well as determine the expressions necessary to solve a given story problem. The interview questions which addressed conceptual knowledge are those which ask the preservice teachers to create a story to match a given expression and as they consider the concepts related to division of fractions. The second protocol question asked the preservice teacher to create a story to represent $1\frac{3}{4} \div \frac{1}{2}$. This question was evaluated on a five point scale ranging from no story (0) to correct story with no misconceptions or pedagogical problems (4). See Rubric C in Table 3. This rubric was based on the rubric used by Ma in her 1999 study. The third protocol question examined the idea of a "knowledge package" (Ma, 1999) as the secondary mathematics preservice teachers were asked to list other content connected with division of fractions. This question was evaluated on a four point scale ranging from no response (0) to many varied, correct responses (3). See Rubric D in Table 4.

Research Question 3: To what extent do secondary mathematics preservice teachers have pedagogical content knowledge? Do they know common difficulties students may experience? Do they know possible sources for students' misconceptions? Can they provide suggestions for correcting these misconceptions?

The secondary mathematics preservice teachers' pedagogical content knowledge was examined using the interview protocol. Their level of pedagogical content knowledge was determined based on their answers to questions that addressed alternate representations, mistakes or misconceptions their future students might have, and ways to correct these misconceptions. The last three protocol questions were evaluated on a four point scale ranging from no response (0) to many varied, correct responses (3). See Rubric D in Table 4.

3.8 Ethical Considerations

The privacy and confidentiality of the subjects was protected through the use of pseudonyms for all participants. An assurance of privacy and confidentiality was presented in writing to each participant. Since these participants were all students, they were assured that their participation in the study would in no way affect their grade or performance in the course. Confidentiality was protected. Anonymity was intended but not guaranteed.

3.9 Conclusion

The purpose of this study was to determine the level of secondary mathematics preservice teachers' subject matter knowledge, including procedural and conceptual knowledge, and pedagogical content knowledge of rational numbers. Fifteen secondary mathematics preservice teachers enrolled in a mathematics methods course at a midwestern land grant university comprised the sample population. Using a quantitative framework, data were collected using the ***Rational Numbers and Their Representations*** survey. Using qualitative techniques, data were collected through an interview process conducted on a subset of the preservice teachers. The data were analyzed using both quantitative and qualitative methods.

4 Results

4.1 Introduction

The purpose of this study was to determine the level of secondary mathematics preservice teachers' subject matter knowledge, including procedural and conceptual knowledge, and pedagogical content knowledge of rational numbers. The research questions for this study are:

1) To what extent do secondary mathematics preservice teachers have procedural knowledge with rational numbers? Can they perform operations with rational numbers?
2) To what extent do secondary mathematics preservice teachers have conceptual knowledge of rational numbers, their representations, and their operations? Can they create a story to represent an expression? Can they determine an appropriate expression to solve a given situation? Do they know why and how the procedures and algorithms they apply work?
3) To what extent do secondary mathematics preservice teachers have pedagogical content knowledge? Do they know common difficulties students may experience? Do they know possible sources for students' misconceptions? Can they provide suggestions for correcting these misconceptions?

To answer these research questions, both quantitative and qualitative data gathered from secondary mathematics preservice teachers was analyzed. The quantitative data was generated from the ***Rational Numbers and Their Representations*** survey. The mean score, standard deviation, and range of scores were determined for the entire survey as well as for each question on the survey. The qualitative data was generated from the responses of the interviews as well as the open-ended responses from the ***Rational Numbers and Their Representations*** survey. The process of content analysis was used on the responses to the open-ended questions. An inductive analysis was conducted on the interview transcriptions to locate patterns and themes around which the narrative discussion was organized.

4.2 Demographic Information

The participants in this study ranged in age from 20 to 33. There were 10 females and 5 males. All fifteen participants were Caucasian (see Table 5). The fifteen secondary mathematics preservice teachers in this study represent a group of mathematical sophisticates (see Table 6). All but two of the preservice teachers took at least four mathematics courses in high school and one third of the preservice teachers took more then six mathematics courses in high school. The preservice teachers had completed at least 22 hours of mathematics courses. Six of the fifteen completed over 30 credit hours of mathematics including College Algebra, Calculus I and II, Differential Equations, Linear Algebra, Introduction to Modern Algebra, Introduction to Modern Analysis and History of Mathematics. Thus, the preservice teachers had some knowledge of fundamental secondary school mathematics topics, since this knowledge is required for studying the advanced topics in the courses they had already completed. However, the data discussed in this chapter suggest that there are serious gaps in the preservice teachers' knowledge of some fundamental mathematics topics.

Demographic Information	
Gender	
Female	10
Male	5
Age	
20 – 25	14
26 – 30	0
31 – 35	1
Ethnicity	
African American	0
Caucasian	15
Hispanic	0
Native American	0
Other	0

Table 5: Demographic Information – Gender, Age, and Ethnicity

High School Courses			
Algebra I	14	Pre-Calculus	10
Algebra II	14	Calculus	6
Geometry	15	Statistics	1
Trigonometry	12	Other	1
College Courses			
College Algebra	10	Trigonometry	5
Calculus I	15	Calculus II	15
Differential Equations	15	Calculus of Several Variables	5
Discrete Mathematics	2	Linear Algebra	15
Intro to Modern Algebra	13	Intro to Modern Analysis	5
History of Mathematics	10	Other	10

Table 6: Demographic Information – Courses taken in High School and College

4.3 Results of *Rational Numbers and Their Representations* Instrument

The ***Rational Numbers and Their Representations*** survey generated the quantitative data. The mean score, standard deviation, and range of scores were determined for the entire survey as well as for each question on the survey. The mean score of the ***Rational Numbers and Their Representations*** survey was a 50.2 out of 68 total points with a standard deviation of 5.8. The scores ranged from 42 to 64. Table 8 gives the mean number of points out of a total of 4 points as well as the standard deviation for each question. Table 7 gives the number of students who correctly answered each question.

All of the secondary mathematics preservice teachers demonstrated competency in their knowledge of the procedures associated with operations with rational numbers. The questions regarding operations with rational numbers were among

Results of ***Rational Numbers and Their Representations*** survey					
Question	0 pts	1 pt	2 pts	3 pts	4 pts
1. What is a rational number?	3	5	2	4	1
2a. Model $\frac{3}{5}$ as part of a set	2	2	0	2	9
2b. Model $\frac{3}{5}$ as part of a region	0	1	0	2	12
2c. Model $\frac{3}{5}$ as a ratio	2	0	9	0	4
2d. Model $\frac{3}{5}$ as part of a trapezoid	1	1	8	3	2
3a. Sketch $\frac{7}{8}$ given fourths	0	0	0	2	13
3b. Sketch $\frac{2}{3}$ given fourths	0	1	6	5	3
4a. Given $\frac{2}{3}$, represent 1	0	0	0	5	10
4b. Given $\frac{5}{4}$, represent 1	0	1	1	4	9
4c. Given $\frac{5}{4}$, represent $\frac{2}{3}$	1	2	6	3	3
4d. Given $\frac{5}{3}$, represent $\frac{3}{5}$	4	1	3	1	6
5a. $1\frac{3}{4} \div \frac{1}{2}$	0	0	0	13	2
5b. $1\frac{3}{4} \times \frac{1}{2}$	0	0	1	0	14
5c. $1\frac{3}{4} - \frac{1}{2}$	0	0	1	11	3
5d. $1\frac{3}{4} + \frac{1}{2}$	0	0	0	11	4
6a. Application of division	0	3	1	6	5
6b. Application of multiplication	0	1	2	9	3

Table 7: Question-by-question results from ***Rational Numbers and Their Representations*** Instrument

Results of ***Rational Numbers and Their Representations*** survey		
Question	Mean	Standard Deviation
1. What is a rational number?	1.67	1.2
2a. Model $\frac{3}{5}$ as part of a set	2.93	1.5
2b. Model $\frac{3}{5}$ as part of a region	3.67	0.8
2c. Model $\frac{3}{5}$ as a ratio	2.27	1.2
2d. Model $\frac{3}{5}$ as part of a trapezoid	2.27	1.0
3a. Sketch $\frac{7}{8}$ given fourths	3.87	0.34
3b. Sketch $\frac{2}{3}$ given fourths	2.67	0.84
4a. Given $\frac{2}{3}$, represent 1	3.67	0.47
4b. Given $\frac{5}{4}$, represent 1	3.4	0.88
4c. Given $\frac{5}{4}$, represent $\frac{2}{3}$	2.33	1.14
4d. Given $\frac{5}{3}$, represent $\frac{3}{5}$	2.27	1.7
5a. $1\frac{3}{4} \div \frac{1}{2}$	3.13	0.34
5b. $1\frac{3}{4} \times \frac{1}{2}$	3.87	0.5
5c. $1\frac{3}{4} - \frac{1}{2}$	3.20	0.55
5d. $1\frac{3}{4} + \frac{1}{2}$	3.27	0.44
6a. Application of division	2.87	1.10
6b. Application of multiplication	2.93	0.78

Table 8: Results of ***Rational Numbers and Their Representations*** Instrument

those questions with the highest mean subscores (see Table 8). All but one of the preservice teachers were able to correctly solve all of the computation problems in question 5 of the ***Rational Numbers and Their Representations*** survey. However, only two of the preservice teachers simplified their answers. The others left their answers as improper fractions. Thus, most preservice teachers received three of four possible points (see Table 7). All of the preservice teachers used the "invert and multiply" procedure to solve the division of fractions. They correctly changed $1\frac{3}{4}$ into $\frac{7}{4}$ then found common denominators for the addition and subtraction problems.

The secondary mathematics preservice teachers represented a rational number best as part of a region in question 2 (see figure 3). Seven of the preservice teachers used a circular model (figure 3a) while the other eight used a rectangular model (figure 3b). The most common mistake was dividing the region into non-equal pieces (see figure 2). It is difficult to divide a circle into five equal pieces by simply drawing the divisions. The rectangular model is much easier to draw. The next best representation was as part of a set. The most common answer was filling in three of five pieces (see figure 4). However, two preservice teachers used set notation with $\frac{3}{5}$ being included in the set (see figure 5).

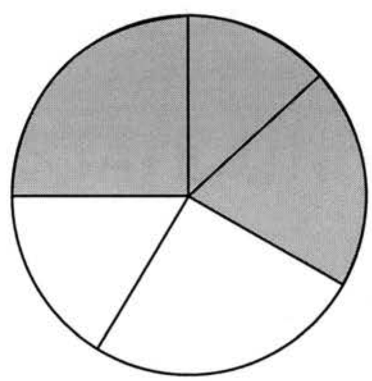

Figure 2: Representing $\frac{3}{5}$ with unequal pieces of a circle

The most common answer for a representation as a ratio was $3:5$. This answer shows that the preservice teachers can write a ratio but have no connections

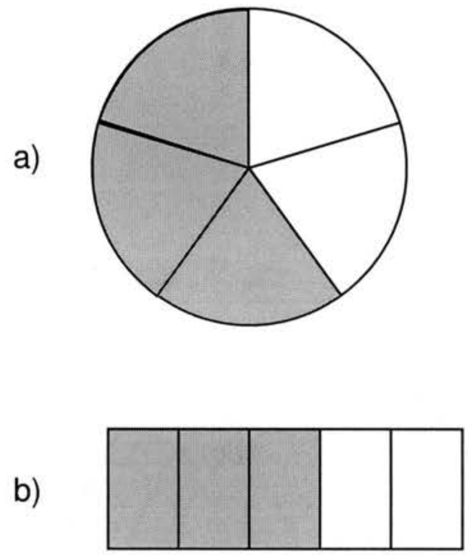

Figure 3: Representing $\frac{3}{5}$ as part of a region

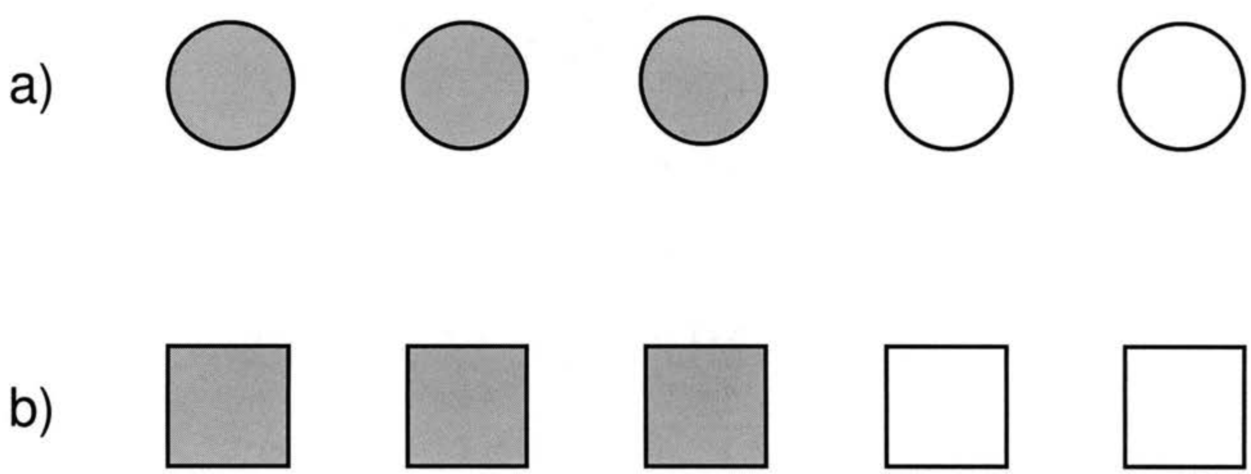

Figure 4: Representing $\frac{3}{5}$ as part of a set

$$\left\{ \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \cdots, \frac{n}{5} \right\}$$

$\frac{3}{5}$ is in the set

Figure 5: Representing $\frac{3}{5}$ as part of a set using set notation

between rational numbers and ratios. Four of the preservice teachers gave a correct answer comparing apples to bananas or boys to girls (see figure 6).

In question 2d the preservice teachers were given a trapezoid and asked to represent $\frac{3}{5}$. Two of the preservice teachers correctly divided the figure into 5 congruent triangles and shaded 3 of them; two others divided the figure into 5 close to congruent pieces then shaded 3 of them (see figure 7). Six of the preservice teachers created a rectangle with 4 congruent rectangles then two congruent triangles on either side (see figure 8a). They then shaded three of the rectangles or two rectangles with two triangles thinking that each triangle counted as a half of a rectangle. However, this is not necessarily the case. Consider the triangles created when altitudes are dropped from the vertices on the shorter base of the trapezoid to the longer base of the trapezoid. The base of this triangle need not be congruent to the bases of the rectangles created when the two triangles are removed and the remaining rectangle is divided into four congruent pieces. Therefore the two triangles are not half of the area of the rectangles. Of the remaining five preservice teachers, one did not attempt the question, three guessed that $\frac{3}{5}$ was a little more

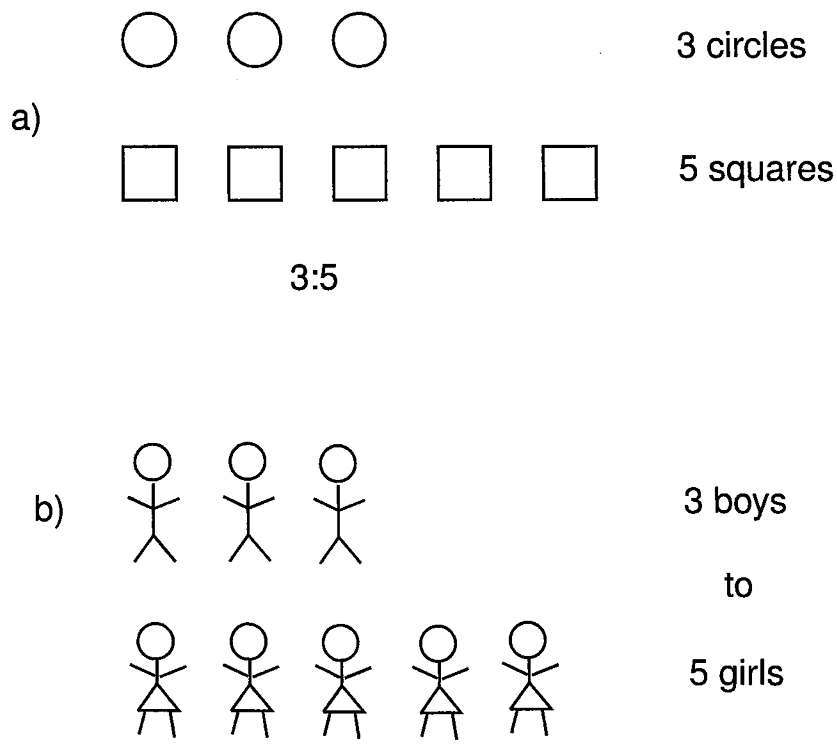

Figure 6: Representing $\frac{3}{5}$ as a ratio

than $\frac{1}{2}$ and so shaded a little more than half the trapezoid. The remaining preservice teachers divided the trapezoid into unequal pieces (see figure 8b).

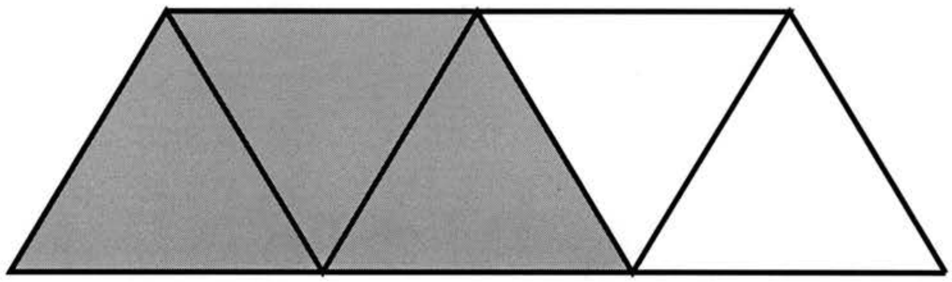

Figure 7: Representing $\frac{3}{5}$ as part of a trapezoidal region

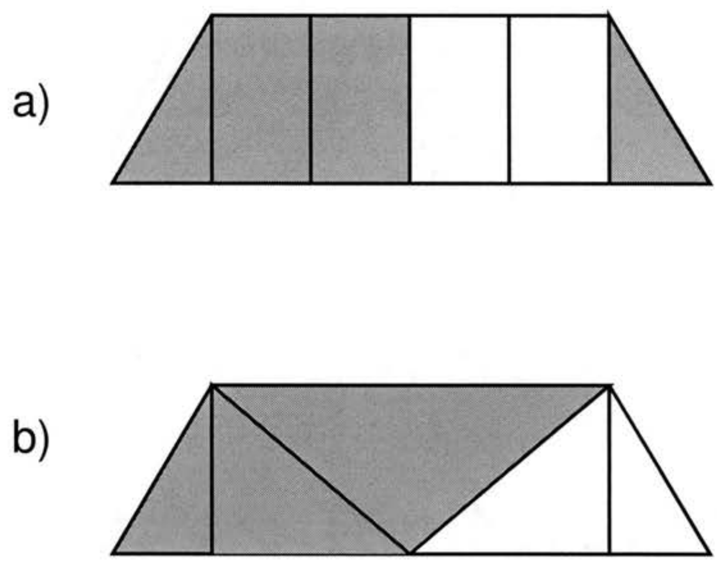

Figure 8: Most common incorrect representations of $\frac{3}{5}$ as part of a trapezoidal region

The preservice teachers did quite well on question 3a which asked them to shade $\frac{7}{8}$ of a figure divided into four pieces. Most divided each piece into two equal pieces creating eight pieces then colored in seven of the equal pieces. The second part of question 3 asks the preservice teachers to shade $\frac{2}{3}$ of the figure divided into four pieces. Three of the fifteen preservice teachers divided each piece into thirds then

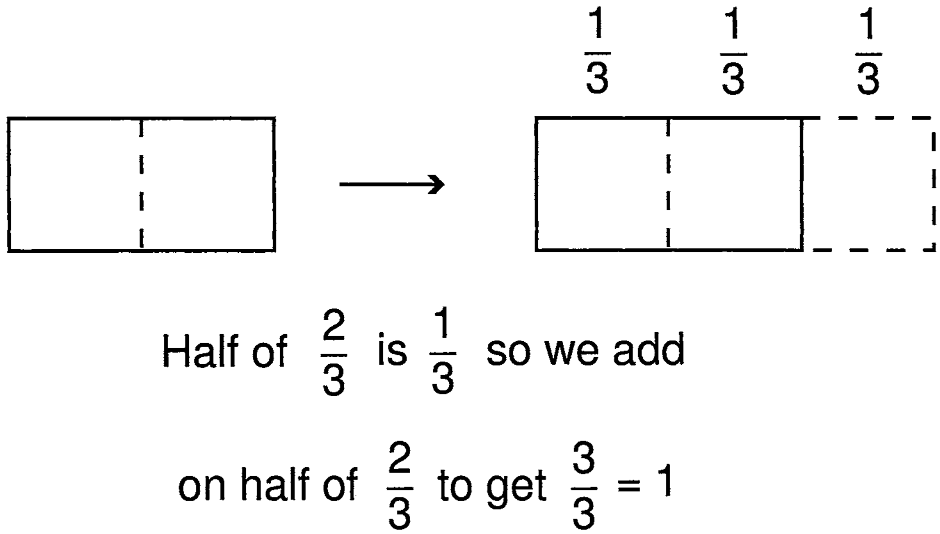

Figure 9: Relating fractional pieces to wholes

shaded the correct number of new pieces. The remaining twelve preservice teachers estimated what $\frac{2}{3}$ would look like but either shaded incorrectly or had an incorrect procedure to determine what portion to shade.

Question 4 on the ***Rational Numbers and Their Representations*** survey examines the conceptual understanding of the secondary mathematics preservice teachers. In this question, the preservice teachers are given a region that represents a fractional piece of a unit and asked to draw the unit. The preservice teachers did best when given $\frac{2}{3}$ and asked to draw one whole. All but five explained that half of the given figure was $\frac{1}{3}$ of the whole so they added that portion back to the original figure to get one whole (see figure 9). The other five preservice teachers added some area to the figure which they labeled as $\frac{1}{3}$ to make one whole but gave no reasoning as to how they selected the area to add.

The second part of question 4 gave some of the preservice teachers a little more trouble. The question tells them that the given area is $\frac{5}{4}$ of the whole and asks them to find the whole. One preservice teachers stated that she could not draw one whole because $\frac{5}{4}$ was greater than one whole. Five of the preservice teachers stated that they would take off $\frac{1}{4}$ from the $\frac{5}{4}$ to make one whole but did not say or show how they would determine what portion was $\frac{1}{4}$ or how it was related to the original figure. The remaining nine preservice teachers clearly showed how they divided the figure into five equal pieces and called each $\frac{1}{4}$ of the whole then subtracted one of the five pieces leaving the four remaining pieces to make one whole.

The remaining two parts of question 4 required the preservice teachers to complete a two-step process to find the answer. The first gives a line segment representing $\frac{5}{4}$ and asks them to find $\frac{2}{3}$. This problem requires the preservice teachers to change $\frac{5}{4}$ to one whole then find $\frac{2}{3}$ of the whole. Nine of the preservice teachers either did not attempt this question or estimated to find the whole then $\frac{2}{3}$. Three of the remaining preservice teachers got a common denominator of twelfths and completed the problem by breaking the line segment into fifteen congruent pieces then shading eight of them. The other three preservice teachers divided the line into five congruent pieces, found one whole then divided the whole into three equal pieces shading two of them.

The last part of question 4 gave the preservice teachers a figure representing $\frac{5}{3}$ and asked them to represent $\frac{3}{5}$. Seven of the preservice teachers correctly answered this question although one gave no explanation for her answer. Of the remaining eight preservice teachers, four did not attempt the problem, the other four gave incomplete or incorrect answers.

Question 6 of the ***Rational Numbers and Their Representations*** survey presented a situation then asked the preservice teachers to write the number expression they would use to solve the problem. Three of the preservice teachers wrote multiplication expressions instead of division expressions. Five of the preservice teachers wrote the correct division expression. One preservice teacher said she would "add $3\frac{3}{4}$ repetitively up to not more than 13 and find out how many times I added." The remaining six preservice teachers wrote an algebraic equation involving multiplication to solve the problem. For the second part of question 6, three preservice teachers wrote the correct multiplicative expression. Nine of the preservice teachers used a proportional equation to solve the problem. The remaining three preservice teachers had incorrect or incomplete answers.

The most surprising answers came on the first question of the ***Rational Numbers and Their Representations*** survey. This question asked for a definition of a rational number. Some of the answers given are listed below.

— A number that does not have i, it is a real number.

— A fraction or repeating number.

— Opposite of irrational, non-repeating decimal, can be written as a fraction.

— A decimal greater than 0.

— A number that is a whole number.

— A fraction.

— Not a repeating decimal.

— A number which can be represented by a fraction, integer, repeating decimal, or a decimal where its digits are finite.

— A rational number is a number that can be written in the form $\frac{p}{q}$ where p and q are integers and $q \neq 0$.

The answers given on this question show that not all of the preservice teachers have a strong understanding of what constitutes a rational number. Several of them could recite the definition but had difficulty representing a rational number in more than one or two ways. Many of them knew a fraction was a rational number but considered all rational numbers to be fractions as well. Several of the preservice teachers gave completely incorrect definitions of rational numbers leading the researcher to suspect the existence of gaps in their knowledge of rational numbers.

4.4 Initial Interview Results

Three secondary mathematics preservice teachers were interviewed. In keeping with confidentiality agreements, the pseudonyms Adam, Beth, and Carol were assigned. The interviews began with questions about integer division to determine if any difficulties the secondary mathematics preservice teachers experienced were due to division or rational numbers.

Adam was a twenty-year old Caucasian male. In high school, he completed Algebra I, Algebra II, and Geometry. He completed College Algebra, Trigonometry, Calculus I, Calculus II, Differential Equations, Linear Algebra, and Statistics at the university level. At the time of the study he was currently enrolled in Introduction to Modern Algebra and History of Mathematics. Adam made a score of 53 points out of a total of 68 points on the ***Rational Numbers and Their Representations*** survey. He did not attempt to sketch a model of $\frac{3}{5}$ as part of a set; he also struggled with the application of multiplication involving rational numbers. His answers on the survey were detailed and explicitly stated his thought processes as he worked the problems.

Beth was a twenty-two year old Caucasian female. In high school, she completed Algebra I, Algebra II, Geometry, Trigonometry, Pre-calculus, and Applied Mathematics. She completed College Algebra, Calculus I, Calculus II, Differential Equations, Linear Algebra, and Combinatorics at the university level. At the time of this study she was currently enrolled in Introduction to Modern Algebra and History of Mathematics. Beth scored 47 points out of a total of 68 points on the ***Rational Numbers and Their Representations*** survey. She defined a rational number as "a whole number." She did not divide regions into equal pieces before shading three of five equal pieces. She did not give explicit descriptions of her thought processes as she worked the problems. Several times it appeared she had guessed at answers or estimated fractional pieces.

Carol was a twenty-one year old Caucasian female. In high school, she completed Algebra I, Algebra II, Geometry, Trigonometry, Pre-calculus, and Calculus. At the university level, she completed Calculus I, Calculus II, Differential Equations, Calculus of Several Variables, Linear Algebra, Introduction to Modern Algebra, Introduction to Modern Analysis, and Number Theory. At the time of this study, she was currently enrolled in History of Mathematics and Combinatorics. Carol scored 64 points of 68 total points on the ***Rational Numbers and Their Representations*** survey. She did not seem to have any difficulty with any of the questions. She was explicit in her descriptions of the processes she used in order to solve the problems. Her answers are precise and leave little room to doubt her knowledge of rational numbers.

These three students were selected for the interview process based on their scores on the ***Rational Numbers and Their Representations*** survey. Carol was selected to represent those students whose scores seemed to exhibit a high level of knowledge of rational numbers. Beth was selected to represent those students whose scores seemed to exhibit a low level of knowledge of rational numbers. Adam was selected to represent those students who scored in the mid-range level. Since there were ten female and five male secondary mathematics preservice teachers, two females and one male were selected to participate in the interview process.

The quantitative results based on the rubrics given in Tables 2, 3, and 4 in Chapter 3 follow. There are 30 total points possible for this portion of the study. Adam scored 18 of 30 points, Beth scored 17 of 30 points and Carol scored 24 of 30 points. The three secondary mathematics preservice teachers who completed an interview were able to correctly evaluate each division expression. The preservice teachers were able to create stories for division with integers but two were unable to create a story to represent division with rational numbers. The preservice teachers were able to list mistakes their future students may make but focused their corrections of these mistakes on reteaching the algorithm.

4.5 Research Question 1

The first research question explored the secondary mathematics preservice teachers' procedural knowledge. Procedural knowledge is the mastery of computational skills and knowledge of procedures for identifying mathematical components. Thus responses on questions 2a, 2b, 2c, 3a, and 5 of the ***Rational Numbers and Their Representations*** survey were analyzed for the purpose of determining the level of the preservice teachers' procedural knowledge. The secondary mathematics preservice teachers' responses regarding the computation of the division expression with integers and the division expression with rational numbers questions in the interview protocol were also analyzed to refine the determination of their level of procedural knowledge.

4.5.1 Quantitative Results

Question 5 deals directly with operations of rational numbers. The means for the four parts of this question were 3.13, 3.87, 3.20, and 3.27 out of 4 total points respectively. All but one of the preservice teachers correctly evaluated the operations, but only two of the preservice teachers simplified their answers. All of the preservice teachers correctly changed the mixed fraction $1\frac{3}{4}$ into an improper fraction $\frac{7}{4}$. They correctly found common denominators when needed for addition and subtraction.

Questions 2a and 2b examine the most common representations of rational numbers, as part of a set and as part of a region. The means for these two questions were 2.93 and 3.67 out of 4 total points respectively. Question 2c examines representing a rational number as a ratio. The mean for this question was 2.27 out of 4 total points. The most common answer was $3:5$ which indicates that the preservice teachers know how to write a ratio but do not understand the connections between rational numbers and ratios. Question 3a asks the preservice teachers to shade $\frac{7}{8}$ of a unit divided into four equal pieces. The mean for this question was 3.87 out of 4 total points. Only two preservice teachers did not divide the unit into 8 equal pieces.

These questions examine the secondary mathematics preservice teachers' abilities to compute with rational numbers as well as identify mathematical components of rational numbers. The preservice teachers demonstrated great facility with operations on rational numbers as well as the most common representations of rational numbers. They demonstrated that they have a high level of procedural knowledge with respect to rational numbers.

4.5.2 Qualitative Results

The interview protocol included questions regarding integer division ($58 \div 7$) as well as division with rational numbers ($1\frac{3}{4} \div \frac{1}{2}$) to determine if the preservice teachers experienced the same difficulties with division of integers and division of rational numbers. This study found that the preservice teachers experienced little difficulty computing either expression.

When asked to evaluate $58 \div 7$ all of the preservice teachers responded quickly with the standard algorithm for division. Adam got an answer of 8 with a remainder of 2. Then realizing that we had been talking about rational numbers amended his answer to $8\frac{2}{7}$. Beth and Carol both continued long division and ended with a decimal answer. The preservice teachers seemed reluctant to introduce rational numbers into the solution of integer division. They all seemed much more comfortable with an answer including a remainder or a decimal than a rational number. Carol's response is given below.

> **Carol**: I would solve it using long division. In my mind, 7 would not go into 5 so I would say 7 goes into 58. I would decide how many times by doing multiplication in my mind. So I would say 8 times, which will give me 56. Subtract it, get 2. Then I'll make 58 [into] 58.0, put my decimal there [in the quotient] and carry down my 0. Then 7 will go into 20 two

times. That will be 14. Subtract it, I'll get 6. That's 60. 7 will go into 60 eight times. 7 goes into 40, let's see, 5 times. So I got 8.285.

The secondary mathematics preservice teachers were then asked to evaluate $1\frac{3}{4} \div \frac{1}{2}$. Once again, all of the preservice teachers responded quickly with the standard "invert and multiply" algorithm for division with rational numbers. All three of the preservice teachers arrived at the correct answer of $3\frac{1}{2}$. Adam's response is given below.

> **Adam**: Well, I would change this one [$1\frac{3}{4}$] to an improper fraction. That would make that $\frac{7}{4}$ and then multiply it times 2 because it's the reciprocal [of $\frac{1}{2}$]. I would get $\frac{14}{4}$ and change it back to a proper fraction...that would be $3\frac{1}{2}$.

In her response, Beth indicated, "I don't like division so I would take $\frac{7}{4}$ times 2. I like the times better." This seems as if she believes division and multiplication are different but she can solve division problems using multiplication instead of division because it is easier for her to multiply than to divide with rational numbers. She also seemed unsure of her answer for a moment and asked for verification saying, "Is that right? Gosh, I haven't done this in forever! Do you know how long it's been?"

The secondary mathematics preservice teachers' responses to the computational evaluations showed that they do have a very high level of fluency with the procedures of division with both integers and rational numbers. They used correct vocabulary to discuss the quotient, remainder, and improper fractions. They showed little hesitancy during the computations and assured themselves that their answers were correct. However, when they were asked if the could evaluate the expression in another way, they were unable to come up with a method other than the standard algorithm.

4.6 Research Question 2

The second research question explored the secondary mathematics preservice teachers' conceptual knowledge. Conceptual knowledge refers to knowledge of the underlying structure of mathematics – the relationships and interconnections of ideas that explain and give meaning to mathematical procedures. Conceptual knowledge is knowledge that is rich in relationships. A piece of information becomes conceptual knowledge only when it is integrated into a larger network that is already in place. Thus the level of conceptual knowledge can be determined by examining the connections made as well as the ability to explain why a process works. Thus questions 2d, 3b, 4a, 4b, 4c, 4d, 6a, and 6b of the ***Rational Numbers and Their Representations*** survey examine conceptual knowledge. The questions in the interview protocol which ask the secondary mathematics preservice teachers to create stories to model the expressions and discuss related concepts further examine the preservice teachers' level of conceptual knowledge.

4.6.1 Quantitative Results

Question 2d of the ***Rational Numbers and Their Representations*** survey gave the preservice teachers a trapezoidal region then asked them to sketch a model to represent $\frac{3}{5}$ as part of that region. The mean score for this question was 2.27 out of 4 total points. Several of the preservice teachers guessed that $\frac{3}{5}$ was a little more than $\frac{1}{2}$ and so colored slightly more than half of the trapezoid. Question 3b asked the preservice teachers to shade $\frac{2}{3}$ of a region divided into four equal pieces. This question requires the preservice teachers to focus on the connections between thirds and fourths in order to correctly shade the region. The mean score for this question was 2.67 out of 4 total points.

Question 4 requires the preservice teachers to focus on how fractional pieces given relate to the whole unit as well as to other fractional parts. The first two parts of this question give the preservice teacher a region representing a fractional part of a unit then asked them to draw the unit. In question 4a the region represents $\frac{2}{3}$; in question 4b the region represents $\frac{5}{4}$. The mean scores for these two parts were 3.67 and 3.40 out of 4 total points respectively. Parts c and d of this question require the preservice teachers to make two conversions in order to relate two fractional pieces. Question 4c gives a line segment representing $\frac{5}{4}$ and asks the preservice teachers to sketch a line segment representing $\frac{2}{3}$. The preservice teachers must first find the unit then find $\frac{2}{3}$ of the unit. Question 4d is similar. The mean scores for 4c was 2.33 while the mean score for 4d was 2.27 out of 4 total points. Most of the preservice teachers were unable to correctly answer these two-step problems.

Question 6 consists of two application problems. The preservice teachers were asked to write an expression they would use to solve the situation. Question 6a is an application of division while question 6b is an application of multiplication. The mean scores for these questions were 2.87 and 2.93 out of 4 total points respectively. The most common incorrect answers involved algebraic equations used to solve the problem situations rather than expressions.

The responses given by the preservice teachers on these questions exhibit a mid-range level of conceptual knowledge. The responses show some knowledge of the connections between fractional pieces and whole units, but they lack the ability to apply this knowledge to situations which require multiple steps to complete. In many of the cases, the preservice teachers were unable to explain how they got their answers even though they correctly showed the process they followed to arrive at the answers. In some cases, the preservice teachers seemed to be performing a procedure to complete the task without realizing they were applying a procedure, recognizing the procedure, or knowing why they were applying it.

4.6.2 Qualitative Results

During their interview, all three preservice teachers could give a story situation that could be modeled by $58 \div 7$ with little hesitation. Each used discrete objects such as cookies, apples, or balls. The story situations led them to an answer of 8 with 2 objects remaining. When the researcher asked how their current answer piece of a cookie. But if you ask the next question of "will there be an extra cookie left over?" then they'll say, "yeah, there will be a cookie left over."

Researcher: How many cookies will be left over?

Carol: Just one. No. Yeah. No, there would be 2.

Both Adam's story and Carol's story lend themselves to a possible fractional answer. Both needed to be led to that answer. Carol divided each cookie into 7 pieces then gave each child one piece of each cookie. Thus each child would get two one-seventh pieces. Adam split the quantity of the two apples into a total of 7 pieces and would give each child one piece which represented $\frac{2}{7}$. Beth's story follows.

Beth: You have 58 dodge balls and 7 groups. Something like that, you know, how many dodge balls . . . No, that's not right. [pause] If there is 7 dodge balls and 58 kids, how many groups would you have to pair them in, or how many . . . I don't know. 7 dodge balls and 58 kids . . . there would be 8 groups of 7 kids. One ball would go to each group. That would be 56. Then you would have 2 kids left over that didn't get a group.

Beth's story was problematic to begin with. She was unable to express her story very well. She figured out what her story situation would be asking to get the answer from the computation. Her story involved dodge balls and children. Neither of these items can be split nicely to create fractional pieces therefore her story could not be expanded to include rational numbers. When asked if she could develop a story so that there were not 2 items left over, she was unable to do so.

With regard to division of rational numbers, the preservice teachers had much more difficulty coming up with a story situation that could be modeled by $1\frac{3}{4} \div \frac{1}{2}$. Only Carol was able to come up with a story situation, although she could not related to the answer they found while evaluating the computation, the preservice teachers were still reluctant to introduce fractional pieces. Instead they focused on the remainder of 2.

> **Researcher**: Sometimes teachers try to come up with real-world situations or story problems to show the meaning or application of some particular piece of content. What story would be a good situation story for this expression?
>
> **Adam**: If I had . . . If I had 58 apples or something like that and you had 7 students, how would you divide them evenly without a kid getting upset. Each of them needs to have the same amount.
>
> **Researcher**: How much would each student have?
>
> **Adam**: Each student would have 8 and there would be 2 left over. So I guess you could say that the 2 could be the remainder.
>
> **Researcher**: How does this story relate to the computation you solved earlier?
>
> **Adam**: It's the same, 8 remainder 2. By dividing, you want to divide it equally among the kids. So you have to find how it's going to divide evenly into that. You have to find out what number would make it even. If it's not even, then you have some left over, you have a remainder. Those are the ones left over.

Carol's story was similar to Adam's story. Carol used cookies instead of apples. When Carol solved the computation, she got an answer of 8.285. When asked how her story related to her original answer she answered that each student got 8 cookies but her computed answer was a "little bit" more than 8.

> **Carol**: I got 8 and little bit more than 8. But that little bit more than 8, we can't take that cookie and . . . I mean we COULD take that cookie and divide it into .28 . . . a fraction of it and give each child that

readily relate the answer she got when she worked out the story to the answer she got from the standard algorithm.

> **Carol**: You could come up with a story, say, with fabric. You could say that you had $1\frac{3}{4}$ yards of fabric and each outfit that you were planning on making, or each vest, only requires $\frac{1}{2}$ yard of fabric. How many vests can you make?
>
> **Researcher**: How many vests can you make?
>
> **Carol**: In this case [with her computation], I came up with a fraction [$\frac{14}{4}$] so I would think most students would end up coming up with about 3 point something or other. And so they would have a decimal in the case like we had in the first problem [$58 \div 7$] and so they would realize, well, you'd have to help them through it, but they would realize that you can only make three vests and there will be some material left over.
>
> **Researcher**: Suppose that we wanted to use all of the fabric?
>
> **Carol**: Then they would say that you could make four vests. [pause] Well, they could make three and a part of another one, they would say.
>
> **Researcher**: How much of another one would they be able to make?
>
> **Carol**: Well, that depends on . . . I didn't actually divide the fraction out. We would say . . . let's see . . . we could make half of another one then.
>
> **Researcher**: So how many vests can they make?
>
> **Carol**: Three and a half.
>
> **Researcher**: How does that answer relate to the answer you got in this computation, $\frac{14}{4}$?
>
> **Carol**: Well, $\frac{14}{4}$ is $3\frac{1}{2}$ when you divide it out.

Adam and Beth had much more difficulty developing a story. Beth said she couldn't come up with a story but if she thought about it long enough she would develop one. She said the story would use “anything with fractions in it. Definitely food comes to mind, like pizza or something because of the slices.” Adam said he couldn’t think of a way to “make the problem real” saying it was difficult “because it had fractions in it. It’s kind of hard to divide when you’re starting out with fractions to begin with. I’m thinking maybe, if you can start out like a game . . . like a football game or a basketball game. Then you can show that, if you’ve already played one game and you’re $\frac{3}{4}$ of the way through the other game. . . ’Cause each game has quarters. Then somehow . . . [long pause] I just can’t come up with a story.”

Beth and Adam exhibited extreme difficulty developing a story situation that could be modeled by the division expression with rational numbers. When asked why it was harder to come up with a story in this situation as opposed to the division expression with integers, both immediately replied that the fractions made it harder. As Adam said, “it is kind of hard to divide when you’re starting out with fractions to begin with.” Beth stated that “you can show dividing by 7 easier than you can dividing by $\frac{1}{2}$. They [the students] don’t understand that you just multiply by 2. It would be easy if they understood that. But it is really hard to show multiplying by $\frac{1}{2}$ or dividing by $\frac{1}{2}$.” These statements show that these two preservice teachers do not have a solid grasp on what it means to multiply and divide when rational numbers are involved.

The secondary mathematics preservice teachers were also asked to give alternative approaches to evaluating the division expressions. None of the preservice teachers were able to evaluate the division expression involving rational numbers using a different method. Only Beth came up with an alternative approach to the integer division problem. She suggested that you draw 58 squares, grouping them together in groups of 7, leaving 2 squares out of the rectangular array (see figure 10).

A second piece of the secondary mathematics preservice teachers’ conceptual knowledge of rational numbers examined by the interview protocol was that of

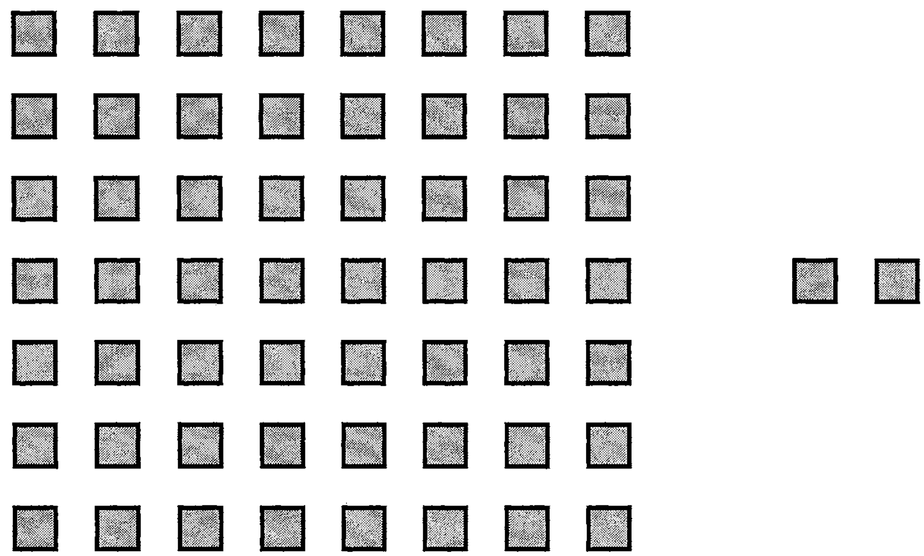

Figure 10: Representing $58 \div 7$ using a rectangular array

related and component concepts. This idea is similar to Ma's "knowledge packages" (1999). The preservice teachers were asked to list other concepts that the students needed to know in order to be able to evaluate a division expression as well as concepts that were related to division. The lists the preservice teachers gave focused on the component parts of the algorithm. They suggested that in order for students to be able to evaluate an integer division problem, they should know multiplication, division, subtraction, remainders, fractions, and decimals. Carol stated that "they should know the basic math facts, you know, like multiplication, division, and subtraction so that they can work these" types of problems. Beth suggested that students should know "fractions and decimals because the answer can come out that way if you don't do remainders." Beth also suggested that students should know "ways of solving problems. If you show them that, like, drawing a picture is one way to solve a problem. Knowing some strategies of how to solve the problem would help." Carol exhibited her higher level of conceptual knowledge when she acknowledged that "the idea of division is important so that they [the students] know that they are taking a certain amount and they are splitting it up among another amount of people."

With regard to division with rational numbers, the secondary mathematics preservice teachers focused totally on the standard "invert and multiply" algorithm when they listed concepts students needed to know in order to evaluate an expression of this type. Adam stated that the students needed "to know that when you divide you reciprocate the fraction." Students also should be able to multiply and change between improper and proper fractions. Beth said that students should know how to make improper fractions and multiplication. She stated, "most kids like multiplication better than division. To them, you memorize your multiplication tables so you know that better than division." Beth further stated that problems of this type were "not much of a visual type [of problem]. Pretty much you just have to know how to do it." Carol also focused on the algorithm in her answer. She felt students should know how to change between proper and improper fractions, multiply fractions, and reciprocate. However, she also stated that "it is important [for the students] to just have common sense and realize what kind of answer they should be getting. Knowing that $\frac{1}{2}$ is less than $1\frac{3}{4}$ in the first place, you should know that you should be able to make at least one [vest] so just being able to realize . . . common sense says whether that answer makes sense or not." Carol was focusing not only on the algorithm, but on number sense and the ability to estimate to determine if answers are reasonable.

These interview transcriptions show that these three preservice teachers have a good conception of the meaning of division when integers are involved, but have difficulty when rational numbers are introduced through the quotient or as the dividend or divisor. All of these secondary mathematics preservice teachers seem to be dependent on the algorithm and are not likely to stray very far from it. The transcriptions show that although Carol was dependent on the algorithm, she had a good conception of what it means to divide with both integers and rational numbers. She can create situations to model expressions involving both integers and rational numbers. However, her exhibited level of conceptual knowledge, while higher than the other two preservice teachers, was only moderate to high, probably because of her strong bond to the algorithms and getting decimal answers. Adam exhibited a low to moderate level of conceptual knowledge. He was able to develop a story situation to model integer division and made a start on developing a story situation for division with rational numbers even though he could not finish the story. Beth also exhibited a low level of conceptual knowledge. Her story for the integer division expression was problematic and she was unable to create a story for division with rational numbers beyond thinking she should use some type of food that was already divided into pieces.

4.7 Research Question 3

The third research question explored the secondary mathematics preservice teachers' pedagogical content knowledge. This knowledge consists of having knowledge of students' common conceptions and misconceptions about the subject matter. Pedagogical content knowledge is the understanding of how particular topics, principles, and strategies are comprehended and learned or miscomprehended and likely to be forgotten. Pedagogical content knowledge is knowledge of the ways of representing and formulating the subject that make it comprehensible to others (Shulman, 1986). Pedagogical content knowledge also includes knowledge of techniques for assessing students' understanding and diagnosing their misconceptions, knowledge of instructional strategies to connect what they are learning to the knowledge they already possess and knowledge of instructional strategies to eliminate the misconceptions they may have developed. Therefore the questions in the interview protocol that ask the preservice teachers to list mistakes students might make and misconceptions students might have as well as ways to correct these mistakes and misconceptions were examined to determine the secondary mathematics preservice teachers' pedagogical content knowledge.

The three secondary mathematics preservice teachers listed different sets of mistakes that their future students could make when evaluating a division expression involving integers. However, all three lists had the common thread of being tied to the algorithm for integer division. Adam and Beth focused on the placement of the divisor and dividend while not using those terms.

> **Adam**: They [the students] may not, like the number under the house, I guess. They may not know which number goes where. They may have the numbers switched around. Like when they are reading it, 58 divided by 7, they may put the 58 on the outside [of the division symbol] and the 7 on the inside.

> **Beth**: Maybe they [the students] will understand the wording [of a division problem] wrong. I had a lot of trouble with that when I first started out. I always had trouble knowing which number went on top. Maybe our teacher didn't explain it well enough to us or something. I always had trouble with that and it's weird now, particularly because I'm a math major. Some people could flip the numbers wrong and try to divide 7 by 58. Especially if you have a long word problem, they wouldn't understand it.

Carol and Adam both focused on division's inverse relationship with multiplication in their discussions. Carol stated, "a lot of kids, they'll think of 7 times a number and they'll come up with the wrong number and they'll think immediately of 7 times 7 is 49 and they'll think, well, they'll look at 58 and think 49 and . . . they'll think of the wrong numbers when they are doing it." Adam focused not only on multiplication, but the other arithmetic operations involved in long division.

> **Adam**: They [the students] may not multiply correctly. They may . . . they may not put enough. They may say 7 times 7 is 49 and think, well,

> that still works because 49 isn't any bigger than 58. So we have to tell them that it has to be the most that it can be without being greater than the divisor which is 58. Then they might subtract wrong. Especially if, like, . . . this [pointing to the 8 of 58] were a 0 then they may not know to make a 10 and borrow.

When asked to list some of the misconceptions students may have, all of the preservice teachers again focused on mistakes students may make with the algorithm. When asked to list some ways to correct mistakes or misconceptions their students may have, all three preservice teachers said that if the students were taught the correct way, then the students wouldn't have any trouble working the problems.

> **Adam**: I would tell them [the students] to be careful when you read it. Because 7 divided INTO 58, means the 7 goes on the outside. You divide 58 seven times. So that would be one way [to explain it to the students.] Another one, I guess, just make sure they know how to read it [the problem] correctly. It's important to know how to read and understand what you're reading.

> **Beth**: When I was little, the teacher showed us the wrong way first. And I think that's what confused me. So if you didn't show the wrong way, if you just taught them the right way, maybe that would work. It really screws you up to see the wrong way first. So if I started out teaching them this is the way it goes, this is how you say it, this is the way you write it, and don't tell them the wrong way then they won't get confused.

> **Carol**: A lot of [getting problems right] is [the students] needing to memorize their mathematics facts. And some of it is also, it's our job to help them. We can't necessarily make them memorize something. That

is something they have to do on their own. So helping our students just be able to come in contact with it and to get used to where it is in their daily practice. Then it will become more familiar to them.

When the secondary mathematics preservice teachers were asked to list mistakes their future students may make or misconceptions the students may have with regard to division with rational numbers, the preservice teachers once again focused on mistakes students could make with the standard "invert and multiply" algorithm and excluded any possible misconceptions their future students may have from their lists.

Adam: The students would probably forget to flip that one [pointing to the divisor] over. They may multiply straight across [without changing the first one into an improper fraction.] They may not know how to change it back to a proper fraction. They might mess up changing it to an improper fraction. They might add first then multiply.
Researcher: Anything else?
Adam: They may have it set up right with $\frac{7}{4} \times \frac{1}{2}$ then think they need to get a common denominator like they do in addition.

Beth: First they have to change it to an improper fraction. I still have problems with that. Of course, I don't deal with these every day any more. But, making an improper fraction can really throw you off if you make the wrong one. Or the students, . . . whenever you flip it, the $\frac{1}{2}$, if they don't multiply by the reciprocal, they could mess that up. Then changing it back. A lot of teachers don't want you to change it back, but if you're supposed to, that could cause problems.
Researcher: Anything else?
Beth: Basically the whole procedure could give them problems. They could mess up on any little piece of it.

Carol: Well, I remember when I was learning how to change mixed fractions to improper, I always got confused as to whether we multiplied 1 times 4 then added 3 or add 4 and 1 then multiply by 3 or what. So that right there, that operation is one that you can get confused on real easily. And then also, remembering that when you flip it over, whenever you are dividing a fraction you have to flip over the second fraction. My teacher actually came up with a song for that one so ever since eighth grade I have had this song in my mind. It helps, especially at the beginning when you are learning it. Also, multiplying, like I said earlier, keeping that bottom and just multiplying straight across rather than just keeping it the same. So those types of errors are probably very common. Just the arithmetic of it. And then the end, coming up and seeing something like the $\frac{14}{4}$, if they are not thinking that the 14 is bigger than 4 they could say, well, this is a fraction right here so you can only make part of a vest. Not realizing that the 14 is larger than the 4 so its actually going to be more than 1. So those types of errors, that in itself is like an explanation error. Those are the types of errors I can see them making.

Although Carol focused on the algorithm as she listed mistakes students could make, she also mentioned the concept of number sense again. The students with a poor number sense might think that since you ended up with a fractional answer that the answer is less than one instead of realizing that the fraction is actually greater than one since the numerator is larger than the denominator.

None of the secondary mathematics preservice teachers interviewed mentioned any misconceptions their students may have other than Carol's number sense explanation. None of them used correct mathematical terms like dividend, divisor, or quotient. They did appropriately use the term improper fraction although they used the term proper fraction to include a mixed number. All of the explanations given focused on arithmetic mistakes students may make as they go through a meaningless procedure. When the researcher asked the preservice teachers what it meant to "divide by a half," Adam responded similarly to Beth, "to me, it means to multiply by 2." Carol showed a measurement model of division in her story but was unable to carry that idea through to understanding how the procedure works. None of the preservice teachers were able to explain why the standard "invert and multiply" algorithm works nor were they able to give some justification for why it works. They all stated firmly, "it works." The researcher then asked what they might say to students who ask why they are multiplying when they started out dividing in the problem. Adam's response showed he had some idea of the relationship between multiplication, division, and reciprocals, but his understanding was tenuous at best and had some big gaps.

> **Adam**: I don't know [why you multiply in a division problem]. I would try to tell them . . . I would ask, "what is the opposite of addition?" They would say, "subtraction." So then I would tell them, when we added things, or when we subtracted things, we just added the opposite. So when we are dividing, we will multiply by the opposite which is the reciprocal. [He then got confused trying to make the same relationship hold in the opposite direction – going from multiplication to division.] But if you are going to use multiplication, I don't know how you would divide it. I don't know how to switch between these two. Like if you had a multiplication, I don't know how you would solve it with division.

Once the secondary mathematics preservice teachers listed the mistakes their future students might make, they were asked to explain how they might correct these mistakes or misconceptions. Once again, the preservice teachers focused on telling the students the correct method for evaluating an expression of this type. Beth stated that she would simply show her students the correct procedure.

> **Beth**: You just show them [how to work the problem] basically. I mean, this is a basic problem that you just do. You can't really make anything . . . like show them anything visual for it. It's just basically a procedure problem where you just have to show them the mistakes and maybe work through a couple more problems and say . . . show them the parts they could get wrong or that they did get wrong on the first one. I think that would be the only way.

Adam thought the best way to correct mistakes the students might make would be to break the process down into its component parts.

> **Adam**: Before the lesson, we would go over how to change from a proper fraction to an improper fraction. I would stress the importance of reciprocating the fraction and changing it from division to multiplication. Then I would show them that you multiply straight across, across the top and then across the bottom. I would make sure they understand that.

Carol went back to her story situation to correct the mistakes her future students could make. She felt if the students could see the operation, like with the vests, and had a strong number sense that they should have no problem solving problems of this type.

> **Carol**: I am a hands-on person so I would actually get $1\frac{3}{4}$ yards of fabric and actually cut it up. Or whatever the scenario was that I was doing so that they could actually cut it up and see how many pieces are left over. You could even do it with sticks. Give them a foot and $\frac{3}{4}$ of a foot and just tell them to split it up evenly and see what was left over. So I would do that kind of thing, where they can actually see it and actually be able to hands-on, be able to affect what the answer is, actually discover the answer, I think that helps the student.

However, Carol was unable to say exactly how the story and hands-on activities would help the students to correctly change a mixed number to an improper fraction, find the reciprocal, and multiply rational numbers which she listed as mistakes the students might make.

The secondary mathematics preservice teachers were asked how they would explain to middle school students a division problem with integers. Each preservice teacher developed three methods for introducing division with integers. Adam's first method focused on the standard algorithm for division with integers. Beth also mentioned this method but stated that she would not introduce this method to her students first. Beth and Carol both said they would start with easier problems given in words using items with which the students would be familiar, perhaps even having objects for the students to manipulate. Both Carol and Adam presented an alternative approach which focused on multiplication, asking, "what times 7 gives 58?"

However, when the topic turned to division with rational numbers, only Carol stated that she would use an approach other than directly teaching the standard "invert and multiply" algorithm. Carol said she would use her hands-on approach using her vest example or "pie or something like that that you can cut up in pieces." Beth stated that she could not teach division with rational numbers to middle school students. She felt that students at that age were not ready for a topic "that hard. They would get so scared. I just don't know that they could do it." When pressed for a way to teach it to older students who should be ready for the topic she simply focused on the algorithmic way of solving problems of this type.

> **Beth**: They should already know how to do improper fraction and so I would tell them that when they divide by a fraction, this is what you do. You multiply by the reciprocal. Tell them what a reciprocal is. And they should know multiplication and just tell them that way. Just show them,

really. It's kind of a memorization thing, basically. I don't think there is any other way to explain it.

Adam also focused his explanation on the standard "invert and multiply" algorithm. He did try to give an alternative approach using a "tactile approach" with colored chips but the method ended up simply following the standard "invert and multiply" approach where you would multiply chips that were the same color.

> **Adam**: I would tell them how I did it. And show them that you need to change this one [$1\frac{3}{4}$] to an improper fraction. And then . . . well, make sure it is in a horizontal form first [$1\frac{3}{4} \div \frac{1}{2}$]. That way they don't get confused. Then you change the proper fractions to improper fractions. Whenever you are dividing, just think multiply by the opposite, multiply by the reciprocal and multiply the top then multiply the bottom. That's the way I would tell them.
>
> **Researcher**: Is there any other way to explain it?
>
> **Adam**: [long pause] Yeah, I think I would use a more tactile approach, have some chips or something and show them how you make $\frac{7}{4}$ out of $1\frac{3}{4}$ by like having each one be a quarter. Each chip would be $\frac{1}{4}$. Then have 7 of those. And then, you would still have to explain that you multiply by the reciprocal so you have $\frac{7}{4}$ times 2. And you have like . . . I would have different color ones. Have 7 on top and 4 on the bottom.
>
> **Researcher**: So you would have 7 of one color and 4 of the other?
>
> **Adam**: Yeah. And I would have another color on the bottom. Then you would times . . . reciprocate, then change it to multiplication, 2 of one color on top and 1 of the other color on the bottom. Have the 1 and the 4 the same color and the 7 and the 2 the same. That way you know which ones to multiply and multiply across.

These interview transcriptions show that, for the most part, the secondary mathematics preservice teachers' pedagogical content knowledge is limited to teaching the algorithm. Although all three were able to give alternative approaches to teaching division with integers, only Carol was able to give an alternative approach to teaching division with rational numbers. None of the preservice teachers were aware of any misconceptions their future students may hold or how to correct these misconceptions. The only mistakes they thought their future students may make were based on the arithmetic components of the algorithms for division with both integers and rational numbers. Carol exhibited a low to moderate level of pedagogical content knowledge since she wanted her future students to focus on number sense and estimation skills to determine the reasonability of answers. However, other than her alternative teaching approaches she seemed very tied to the algorithm. The other two preservice teachers, Adam and Beth, exhibited a low level of pedagogical content knowledge. Both were dependent on the standard algorithms for division with both integers and rational numbers and were unable to offer any alternative approaches to teaching division with rational numbers.

4.8 Conclusion

This study investigated the teacher knowledge – procedural knowledge, conceptual knowledge, and pedagogical content knowledge of secondary mathematics preservice teachers. Quantitative data were collected and analyzed to determine the levels of procedural knowledge and conceptual knowledge of the secondary mathematics preservice teachers. Qualitative data were collected and analyzed to foster and refine the determination of the levels of procedural and conceptual knowledge and to determine the level of pedagogical content knowledge of the preservice teachers. Fifteen secondary mathematics preservice teachers enrolled in a mathematics methods course took the ***Rational Numbers and Their Representations*** survey which provided information for quantitative analysis. Three of these fifteen preservice teachers were selected to complete an interview to further determine the levels of teacher knowledge. The interviews were transcribed and coded for the qualitative analysis. This study found that the secondary mathematics preservice teachers exhibited a high level of procedural knowledge of operations with rational numbers as well as common representations of rational numbers. The preservice teachers had difficulty representing a rational number as a ratio and as part of a region when given a region which they must divide into equal pieces. The preservice teachers exhibited a moderate to high level of conceptual knowledge on the ***Rational Numbers and Their Representations*** survey as they related fractional pieces to wholes. However, the secondary mathematics preservice teachers who participated in the interview process exhibited a moderate level of conceptual knowledge of rational numbers. Only one preservice teacher, Carol, was able to create a story situation which could model a division expression involving rational numbers. All of the preservice teachers interviewed were dependent on the algorithm and based many of their explanations on the algorithm. The three preservice teachers included in the interview process exhibited a low level of pedagogical content knowledge. They could list no misconceptions their future students may have. Their main approach to correcting mistakes students could make was to refer the students back to the correct procedure. Only Carol had an alternative approach to teaching division involving rational numbers although the link between the algorithm upon which she was dependent and the alternative approach was tenuous.

5 Summary, Conclusions and Recommendations

5.1 Summary

The National Council of Teachers of Mathematics (NCTM) promotes its vision of mathematics classrooms in its publications, *Curriculum and Evaluations Standards for School Mathematics* (1989) and *Principles and Standards for School Mathematics* (2000). This vision includes mathematics instruction that is aimed at helping students build their mathematical power by learning how to formulate and solve problems, to reason and communicate mathematically, and to connect the ideas and applications of mathematics. Developing mathematical power also involves helping students make sense of mathematics and helping them learn to rely on themselves to determine whether something is mathematically correct. This new focus of mathematics instruction on reasoning, understanding, and explaining represents a radical departure from the emphasis on memorization and imitation found in conventional mathematics instruction.

This new vision of mathematics instruction is based on a fundamental rethinking of what "understanding mathematics" means and a new understanding of how students learn mathematics. Students are no longer seen as passive recipients of knowledge but rather as active participants in the learning process as they construct their own understanding of mathematical ideas and concepts. In order to foster an appropriate learning environment which encourages the students to build their own mathematical power, the teachers need to have a solid, yet flexible content knowledge (Committee on the Mathematical Education of Teachers, 1991) and pedagogical content knowledge.

Since today's preservice teachers are tomorrow's teachers, their conceptions of rational numbers are important since their conceptions influence what and how they teach. Little research has been done on secondary mathematics preservice teachers' knowledge of fundamental concepts such as rational numbers. Therefore, the purpose of this study was to determine the level of secondary mathematics

preservice teachers' subject matter knowledge, including procedural knowledge and conceptual knowledge, and pedagogical content knowledge of rational numbers.
The research questions that guide this study are:

1) To what extent do secondary mathematics preservice teachers have procedural knowledge with rational numbers? Can they perform operations with rational numbers?
2) To what extent do secondary mathematics preservice teachers have conceptual knowledge of rational numbers, their representations, and their operations? Can they create a story to represent an expression? Can they determine an appropriate expression to solve a given situation? Do they know why and how the procedures and algorithms they apply work?
3) To what extent do secondary mathematics preservice teachers have pedagogical content knowledge? Do they know common difficulties students may experience? Do they know possible sources for students' misconceptions? Can they provide suggestions for correcting these misconceptions?

The subjects of this study were drawn from a population of secondary mathematics preservice teachers. Through convenience sampling a total of fifteen secondary mathematics preservice teachers participated in this study. These students were enrolled in a mathematics methods course at an midwestern land grant university.

Participants completed a survey of demographical information including questions regarding their age, gender, ethnicity, and the mathematics courses taken at the high school and college level. The secondary mathematics preservice teachers then completed the ***Rational Numbers and Their Representations*** survey. The survey consisted of six questions with a total of seventeen parts focusing on the symbolic representations of rational numbers including modeling rational numbers as parts of sets and regions. The survey was evaluated to determine the preservice teachers' level of procedural and conceptual knowledge, then three preservice teachers were selected to complete interviews. The interview protocol included questions in which the preservice teachers created a story to match a given mathematical expression. The preservice teachers were asked to list common mistakes students might make in representing and working with problems of this type. The survey yielded quantitative data used to determine the levels of procedural and conceptual knowledge of the preservice teachers. The interview transcriptions yielded qualitative data which gave a broader understanding of the conceptual knowledge levels as well as the pedagogical content knowledge of the preservice teachers. Thus the data were analyzed using both quantitative and qualitative methods of data analysis to look for trends. Results from the surveys and interviews were used to determine the levels of procedural and conceptual knowledge as well as pedagogical content knowledge of the secondary mathematics preservice teachers.

The results of the data analysis provide a description of some aspects of secondary mathematics preservice teachers' conceptions of rational numbers. The preservice teachers scored higher on questions regarding operations of rational numbers and shading fractional pieces of a whole region. The preservice teachers had more difficulty relating fractional pieces to wholes and to other fractional pieces. This study found that the secondary mathematics preservice teachers exhibited a high level of procedural knowledge of operations with rational numbers as well as common representations of rational numbers. Some difficulties the preservice teachers had were in representing a rational number as a ratio and as part of a region when given a region which they must divide into equal pieces. The preservice teachers exhibited a moderate to high level of conceptual knowledge on the ***Rational Numbers and Their Representations*** survey as they related fractional pieces to wholes. However, the secondary mathematics preservice teachers who participated in the interview process exhibited a moderate level of conceptual knowledge of rational numbers. The three preservice teachers who completed an interview were dependent on the algorithm and based many of their explanations on the algorithm. The three preservice teachers included in the interview process exhibited a low level of pedagogical content knowledge. They could list no misconceptions their future students may have. Their main approach to correcting mistakes their future students could make was to refer the students back to the correct procedure or standard algorithm.

5.2 Research Question 1 – Summary and Conclusions

The first research question explored the secondary mathematics preservice teachers' procedural knowledge. Procedural knowledge is the mastery of computational skills and knowledge of procedures for identifying mathematical components. The means and standard deviation of selected questions from the ***Rational Numbers and Their Representations*** survey as well as the secondary mathematics preservice teachers' responses regarding the computation of the division with integers and division with rational numbers questions in the interview protocol were analyzed to determine their level of procedural knowledge.

The questions on the ***Rational Numbers and Their Representations*** survey examined the secondary mathematics preservice teachers' abilities to compute with rational numbers as well as identify mathematical components of rational numbers. The preservice teachers demonstrated great facility with operations on rational numbers as well as the most common representations of rational numbers. They demonstrated that they have a high level of procedural knowledge with respect to rational numbers. During the interview process, the secondary mathematics preservice teachers' responses to the computational evaluations strengthened the determination that the secondary mathematics preservice teachers do have a very high level of fluency with the procedures of division with both integers and rational numbers. They used correct vocabulary to discuss the quotient, remainder, and improper fractions. They showed little hesitancy during the computations and assured themselves that their answers were correct.

One would expect results such as these. Since the subjects are secondary mathematics preservice teachers, one would expect the preservice teachers to be able to add, subtract, multiply, and divide – even with rational numbers – quite easily. It makes sense that secondary mathematics preservice teachers would exhibit a high level of procedural knowledge. One would expect that the preservice teachers would be able to represent rational numbers using several different representations. One could also expect that secondary mathematics preservice teachers would be able to state what a rational number is; however, that was not the case for more than half of the preservice teachers in this study. In her 1990 study, Ball found that almost all of the secondary mathematics preservice teachers could compute expressions involving rational numbers; however, few could create an accurate representation. Frykholm (2000) found that few secondary preservice teachers were able to give mathematically sound explanations for mathematical concepts. Many struggled to find ways to explain a rule they had always accepted at face value. Since many preservice teachers learned mathematics in a traditional way – lecture, examples, homework – they did not develop the conceptual knowledge necessary to create explanations for the standard procedures they perform. Cooney (1999) summarized results of this type in his study stating that even experienced secondary mathematics teachers are limited in their ability to translate mathematical knowledge into appropriate explanations and tasks.

5.3 Research Question 2 – Summary and Conclusions

The second research question explored the secondary mathematics preservice teachers' conceptual knowledge. Conceptual knowledge refers to knowledge of the underlying structure of mathematics – the relationships and interconnections of ideas that explain and give meaning to mathematical procedures. Conceptual knowledge is knowledge that is rich in relationships. A piece of information becomes conceptual knowledge only when it is integrated into a larger network that is already in place. Thus the level of conceptual knowledge can be determined by examining the connections made as well as the ability to explain why a process works. The questions in the interview protocol which ask the secondary mathematics preservice teachers to create stories to model the expressions and discuss related concepts further examine the preservice teachers' level of conceptual knowledge.

The responses given by the preservice teachers on these questions on the ***Rational Numbers and Their Representations*** survey exhibit a mid-range level of conceptual knowledge. The average score on the survey was 50.2 out of 68 total points. The responses show some knowledge of the connections between fractional pieces and whole units, but the preservice teachers seemed to lack the ability to apply this knowledge to situations which require multiple steps to complete. In many of the cases, the preservice teachers were unable to explain how they got their answers even though they correctly showed the process they followed to arrive at the answers. In other cases, the preservice teachers seemed to be performing a procedure to complete the task without realizing they were applying a procedure, recognizing the procedure, or knowing why they were applying it. Therefore the preservice teachers need more experience dissecting procedures and tasks into their component parts and relating these components to the operations and to the context.

During the interview process, with little hesitation, all three preservice teachers could give a story situation that could be modeled by $58 \div 7$. Each used discrete objects such as cookies, apples, or balls. Their stories led them to an answer of 8 with 2 objects remaining which was not the same as the decimal answers the preservice teachers got when they first computed $58 \div 7$. Even when pressed to relate the answers, the preservice teachers were reluctant to introduce fractional pieces to the problem. Since the original problem contained integers, it was logical to assume the answer should contain integers so the preservice teachers were more comfortable with the 8 with a remainder of 2 answer. Two of the preservice teachers declared that fractions made problems harder. The impression of rational numbers being harder could foster the reluctance to introduce rational numbers into a problem situation.

With regard to division of rational numbers, the preservice teachers had much more difficulty coming up with a story situation that could be modeled by $1\frac{3}{4} \div \frac{1}{2}$. Only Carol was able to come up with a story situation, although she could not readily relate the answer she got when she worked out the situation she created to the answer she got from the standard algorithm. When asked by the researcher why it was more difficult to come up with a story to model this division expression than the one involving integers, both Adam and Beth said it was because of the fractions. Their statements showed that they did not have a solid grasp on the meanings of multiplication and division when rational numbers were involved.

A second piece of the secondary mathematics preservice teachers' conceptual knowledge of rational numbers examined by the interview protocol was that of related and component concepts. The preservice teachers were asked to list other concepts that the students needed to know in order to be able to evaluate a division expression as well as concepts that were related to division. The lists the preservice teachers gave focused on the component parts of the algorithm. With regard to division with rational numbers, the secondary mathematics preservice teachers focused totally on the standard "invert and multiply" algorithm when they listed concepts students needed to know in order to evaluate an expression of this type. The secondary mathematics preservice teachers focused on the algorithm as they computed the expressions and were unable to create alternative representations of division with rational numbers. This narrow focus suggests a restricted understanding of division of rational numbers limited to the standard algorithm thus the preservice teachers listed only components of this algorithm in their list of related concepts.

These interview transcriptions show that all of the preservice teachers have a good conception of the meaning of division when integers are involved but have difficulty when rational numbers are introduced through the quotient or as the dividend or divisor. One possible reason for this difficulty is similar to the results of Ball (1990) who found that the subjects focused on the fractions rather than the operation of division. Two of the preservice teachers in the current study said that fractions made problems more difficult because it was hard to divide when you already had fractions.

All of the secondary mathematics preservice teachers who participated in the interviews were dependent on the algorithm and are not likely to stray very far from it. The analysis of both the ***Rational Numbers and Their Representations*** survey and the interview transcriptions found that most of the preservice teachers exhibited a moderate level of conceptual knowledge. The preservice teachers were able to create alternative representations for rational numbers and relate fractional pieces to wholes, but were unable to create stories to model a division expression involving rational numbers. These results are similar to those of Liping Ma (1999). In her study, 22 of 23 American teachers were unable to create a story to represent division of fractions. Only one teacher created a conceptually correct representation, but the representation was pedagogically problematic, much as Carol's story was.

5.4 Research Question 3 – Summary and Conclusions

The third research question explored the secondary mathematics preservice teachers' pedagogical content knowledge. This knowledge consists of having knowledge of students' common conceptions and misconceptions about the subject matter. Pedagogical content knowledge is the understanding of how particular topics, principles, and strategies are comprehended and learned or miscomprehended and likely to be forgotten or confused. Pedagogical content knowledge is knowledge of the ways of representing and formulating the subject that make it comprehensible to others (Shulman, 1986). Pedagogical content knowledge also includes knowledge of techniques for assessing students' understanding and diagnosing their misconceptions, knowledge of instructional strategies to connect what they are learning to the knowledge they already possess and knowledge of instructional strategies to eliminate the misconceptions they may have developed. Therefore the questions in the interview protocol that ask the preservice teachers to list mistakes students might make and misconceptions students might have as well as ways to correct these mistakes and misconceptions were examined to determine the secondary mathematics preservice teachers' pedagogical content knowledge.

The three secondary mathematics preservice teachers listed different sets of mistakes that their future students could make when evaluating a division expression involving integers. However, all three lists had the common thread of being tied to the algorithm for integer division. When asked to list some of the misconceptions students may have, all of the preservice teachers again focused on mistakes students may make with the algorithm. When asked to list some ways to correct mistakes or misconceptions their students may have, all three preservice teachers said that if the students were taught the correct way, then the students wouldn't have any trouble working the problems.

This line of reasoning was also evident in Cooney's 1999 study where he found that most preservice teachers equate good teaching with good telling. When the secondary mathematics preservice teachers in the current study were asked to list mistakes their future student may make or misconceptions the students may have with regard to division with rational numbers, the preservice teachers once again focused on mistakes students could make with the standard "invert and multiply" algorithm and excluded any possible misconceptions their future students may have from their lists. None of the secondary mathematics preservice teachers mentioned any misconceptions their future students may have when evaluating division expressions involving rational numbers. All of the explanations given by the preservice teachers focused on arithmetic mistakes students may make as they go through a meaningless procedure.

Even and Tirosh (1995) found that many of the secondary mathematics teachers in their study made no attempt at understanding the sources of students' responses or determining the causes of misconceptions. They suggested that teachers learn to focus on student responses and study current research in order to predict common misconceptions. However, in order to predict students' misconceptions, one must have a strong knowledge, both procedural and conceptual, of the current topic. The current study found that the secondary mathematics preservice teachers lacked this strong teachers' knowledge as they simply focused on the procedural aspects of computation.

When asked to list other topics or concepts that students should know in order to evaluate division expressions, the secondary mathematics preservice teachers listed items such as multiplication, subtraction, and changing from proper to improper fractions. Their lists focused on the standard algorithms for long division and "invert and multiply." In Ma's (1999) study, the Chinese teachers were able to create a knowledge package, including topics such as the meaning of whole number multiplication, the concept of division as the inverse of multiplication, models of whole number division, the meaning of multiplication with fractions, the concept of a fraction, the concept of a unit, and so on. The teachers in Ma's study knew that these topics were connected in such a way that the students needed to understand the previous concepts and how they applied to the current concept. The teachers also knew the content well enough to know when the current topic would be used again as a building block for future topics.

These interview transcriptions show that, for the most part, the secondary mathematics preservice teachers' pedagogical content knowledge is limited to teaching the algorithm. Although all three were able to give alternative approaches to teaching division with integers, only Carol was able to give an alternative approach to teaching division with rational numbers. None of the preservice teachers were aware of any misconceptions their future students may hold or how to correct these misconceptions. The only mistakes they thought their future students may make were based on the arithmetic components of the algorithms for division with both integers and rational numbers. Thus the secondary mathematics preservice teachers exhibited a low level of pedagogical content knowledge. It would seem reasonable that preservice teachers would not necessarily exhibit a high level of pedagogical content knowledge since they have had little classroom experience. However, they should have at least a moderate level of pedagogical content knowledge on which to build once they begin their teaching experiences. Thus the courses which prepare secondary mathematics preservice teachers should include activities which cause the preservice teachers to explore the thought processes of secondary mathematics students as well as opportunities to discuss and reflect on student learning. The courses should also provide experiences which involve the preservice teachers with secondary students so that the preservice teachers gain first-hand experience as they prepare lessons, evaluate learning, and modify instruction as needed.

5.5 Implications for Related Research

The results of this study are consistent with the results of studies conducted with elementary teachers and preservice teachers with regard to their conceptions of rational numbers. In their 1985 study, Leinhardt and Smith found that many elementary teachers were able to perform operations with rational numbers quite well. They also found that while some teachers displayed a relatively rich conceptual knowledge of fractions, others relied heavily on precise knowledge of algorithms. In her 2000 study on elementary teachers' pedagogical content knowledge with respect to division of fractions, Tirosh found that most of the elementary preservice teachers in her study knew how to divide fractions but could not explain the procedure. The elementary preservice teachers could not state why the standard "invert and multiply" algorithm is used or why it works. When they were presented with alternative procedures, they did not accept the procedures as correct even if they produced correct answers. Tirosh also found that the elementary preservice teachers were unaware of major sources of students' incorrect responses to problems of this type. When asked why students gave incorrect answers, most preservice teachers attributed the mistakes to algorithmically-based errors. Few of the preservice teachers suggested the errors had basis in algorithmically-based and intuitively-based sources. These results correspond with the findings of the current study. The secondary mathematics preservice teachers who were interviewed listed only algorithmically-based errors rather than intuitively-based misconceptions.

In other studies conducted by Tirosh and Graeber (1989, 1991), the researchers found that elementary preservice teachers were familiar with the partitive model of division but had limited access to the measurement model of division. The researchers suggested that teachers' limited access could be one reason the preservice teachers had difficulties explaining division expressions when rational numbers were involved. Preservice teachers should also be proficient with both the partitive and the measurement models so that they can better utilize their students' existing conceptions during instruction (Tirosh & Graeber, 1991). As the researchers studied the relationship with preservice teachers' beliefs and performance, they suggested that the preservice teachers' procedural knowledge dominated but was not linked to correct conceptual knowledge (Tirosh & Graeber, 1989). Therefore when preservice teachers were presented with problems that they found difficult to give meaning to, the preservice teachers were more likely to rely on their procedural knowledge. The current study also found that the secondary mathematics preservice teachers relied on the algorithm as procedural knowledge dominated conceptual knowledge. Since the preservice teachers did not have a rich conceptual knowledge of rational numbers but did have a fluency with the procedures, the preservice teachers were much more likely to fall back on their knowledge of the procedures. Therefore, the preservice teachers believed that simply teaching the algorithm would be sufficient for all students to learn the concept. Since the preservice teachers had insufficient conceptual knowledge with regard to rational numbers, they were unable to create any activities which would tie the procedures to the underlying concepts.

Eisenhart, et al. (1993) studied the conceptual knowledge of one elementary preservice teacher as she struggled with teaching division of rational numbers. The researchers found that the preservice teacher believed in the importance of teaching mathematics for understanding and in the need to teach for both procedural and conceptual knowledge in order to achieve understanding. However, her knowledge of both content and pedagogy limited her ability to articulate how she would teach for conceptual knowledge. The results of the current study of secondary mathematics preservice teachers closely follow the 1993 study. With their exhibited levels of conceptual and pedagogical content knowledge, the secondary mathematics preservice teachers in the current study are unprepared to teach concepts of rational numbers in a way consistent with the vision of the National Council of Teachers of Mathematics since they are unable to create appropriate mathematical tasks which encourage students to create connections or to move flexibly between representations. The preservice teachers are unable to create or describe the connections between the standard algorithms and the underlying concepts thus it will be difficult for them to teach their students in a meaningful way.

A few studies have included secondary preservice teachers in their population. Ball (1990) studied both elementary and secondary preservice teachers. She found that although many of the preservice teachers were able to correctly solve division problems involving both whole numbers and rational numbers, several could not solve these problems. Few of the preservice teachers were able to give mathematical explanations for the underlying principles and meanings of division. Ball found that the preservice teachers' knowledge was generally fragmented, keeping division with whole numbers separate from division with rational numbers. She found that the difficulties the preservice teachers experienced indicated a narrow understanding of division. Thus the preservice teachers' understandings of mathematics was rule bound and compartmentalized. These findings correspond with the findings of the current study. The secondary mathematics teachers in the current study were able to compute division expressions but were unable to give a mathematical explanation for the underlying concepts regarding division and rational numbers. The preservice teachers also exhibited fragmented, incomplete knowledge of rational numbers as well as tenuous links between integers and rational numbers. The secondary mathematics preservice teachers also tended to separate division of integers from division of rational numbers. This study found the secondary mathematics preservice teachers' knowledge with regard to rational numbers contained large gaps.

In the 1993 study of secondary mathematics preservice teachers, Even found that the preservice teachers' subject matter knowledge is inextricably linked with their pedagogical content knowledge. The researcher found that the preservice teachers' limited conceptions of mathematical concepts, such as functions, influenced their pedagogical thinking. The preservice teachers were limited in the explanations given and tasks assigned. Many provided their students with a rule to be followed seemingly without concern for the students' understanding. This study found that the results hold for rational numbers as well. The secondary mathematics preservice teachers of this study focused solely on the procedures and algorithms stating that they would simply reteach the procedure if their students were having difficulty learning the concepts. These results are due to the fact that the secondary mathematics preservice teachers do not have a sufficient level of conceptual knowledge which is a key component of subject matter knowledge. Since the preservice teachers are unable to draw the connections between the standard algorithm and the underlying concepts, they are forced to fall back on simply reteaching the rules which hold little meaning for their students instead of creating appropriate mathematical tasks which engage the students and create opportunities for the students to create the connections for themselves.

In his 2000 study, Frykholm examined the knowledge of secondary mathematics preservice teachers. He found that many of the preservice teachers lacked a rich understanding of fundamental mathematical concepts and were unable to articulate what knowledge they did have. Most preservice teachers experience the fundamental concepts as students in elementary and secondary school themselves when they are not mathematically mature enough to fully examine the concepts for component pieces and relationships. Therefore, the information that secondary teachers are teaching is information they have not seen in a classroom setting since they were in high school themselves. Thus, Frykholm and others suggest that these fundamental mathematical concepts be revisited from an advanced standpoint in the courses which prepare secondary mathematics teachers.

Many studies have shown that beginning teachers tend to teach as they were taught. For most secondary mathematics teachers, this learning experience consisted of a teacher telling them the rules, showing a few examples, perhaps giving a mnemonic device to help students remember the procedure, then assigning the homework. This instructional strategy leaves little room for conceptual learning. Since the last time many of the preservice teachers studied many of the topics they will be teaching was when they were in high school themselves, many of the preservice teachers have an incomplete knowledge of these mathematical topics (Cooney, 1999). In the case of division with rational numbers, the secondary mathematics preservice teachers' knowledge is limited to the algorithm thus they have no background from which to create alternative representations or list common misconceptions other than possible mistakes made in computing the algorithm.

5.6 Implications for Teacher Preparation Programs

Many of the studies conducted since the publication of the National Council of Teachers of Mathematics' *Curriculum and Evaluations Standards for School*

Mathematics in 1989 have found that teachers do not have the knowledge base with which to teach in the manner suggested by the *Standards* documents (Ball, 1990; Borko, et al., 1992; Carpenter, et al., 1988; Even, 1993; Even & Tirosh, 1995; Graeber, 1999; Lampert, 1991; Leinhardt & Smith, 1985; Post, et al., 1991; Simoneaux, et al., 1997; Stevens & Wenner, 1996; Tirosh & Graeber, 1989, 1990, 1991). Thus several of these studies have suggested a revamping of the teacher preparation program. The traditional mathematics courses do not necessarily develop the kinds of understanding necessary to teach mathematics for understanding, since many times success in these courses comes from memorizing formulas and performing procedures (Ball, 1990). Therefore, there is value in broadening the experiences that typically mark the secondary mathematics preparation process (Frykholm, 2000; Usiskin, 2001, 2002). Courses that prepare secondary mathematics preservice teachers should allow them an opportunity revisit topics included in the high school curriculum to strengthen their subject matter knowledge of these concepts (Borko, et al., 1992; Even & Tirosh, 1995; Frykholm, 2000). This is not to say that we should simply increase the number of mathematics courses the preservice teachers are required to take, but we should alter the courses so that they focus on the conceptual development of topics rather than rote memorization of facts and procedures (Borko, et al., 1992; Even, 1993; Simoneaux, et al., 1997; Tirosh & Graeber, 1989). As preservice teachers increase their conceptual knowledge they become more flexible and able to connect their knowledge to lesson presentations thus creating more opportunities for their students' mathematical abilities to grow (Leinhardt & Smith, 1985).

For the case of rational numbers, if both elementary and secondary mathematics preservice teachers are provided opportunities to experience rational numbers in a concrete, process-oriented way, as prescribed by Harrison, Brindley, and Bye (1989), then they may exhibit the same significantly improved achievement in representations and operations with rational numbers. This method includes using concrete manipulatives to model representations and operations. The method of instruction was designed to engage the students in investigations of relatively broad mathematical tasks or problems that could be initially approached at a concrete level, but would lead by abstraction and generalization to the development of general mathematical concepts and strategies. The instructional strategy includes posing well-motivated problems in which actions with concrete objects facilitate understanding of the mathematical ideas involved. Students carry out a series of mathematical investigations beginning with exploring using concrete materials, leading to a systematic experimentation, recording results, formulating questions, writing accounts of results, and applying the results to practical situation. Although this study was conducted with seventh grade students, it has been shown that secondary mathematics preservice teachers may have the same conceptions as the seventh graders. Thus, it would seem the same results of improved achievement would apply to the secondary mathematics preservice teachers. These experiences would also better prepare the preservice teachers to create tasks to present the material to their future students.

Good subject matter preparation is necessary but not sufficient. Teachers tend to teach the way they were taught. Preservice teachers will be quick to note that the traditional method worked for them and thus can be reluctant to adopt new instructional strategies (Graeber, 1999). Therefore, the courses that the secondary mathematics preservice teachers experience need to allow opportunities for them to experience and develop a wider repertoire of teaching skills. University course work needs to provide preservice teachers with opportunities to strengthen their pedagogical content knowledge by offering opportunities to develop the concepts and language to draw connections between representations and applications on the one hand and algorithms and procedures on the other. Preservice teachers also need opportunities to practice and reflect upon the components of their pedagogical content knowledge base (Borko, et al., 1992; Even, 1993, Simoneaux, et al., 1997).

Tirosh (2000) found that preservice teachers' naive beliefs about learning and teaching mathematics affected their knowledge of and attitudes toward fundamental mathematics. She suggested that teacher education programs should familiarize preservice teachers with the various, and sometimes erroneous, common types of cognitive processes and how these processes could lead to various ways of thinking. The courses which prepare preservice teachers need to encourage the preservice teachers to consult current research as well as observe and participate in secondary mathematics classes so that the preservice teachers can gain a better knowledge base regarding student learning. This experience should provide them with opportunities to work with students in small groups, prepare lessons, evaluate student learning, and modify instructional strategies. Teacher preparation courses at the university level should also include opportunities for preservice teachers to listen and evaluate student responses, and to modify instruction accordingly.

Preservice teachers need time to both experience activities involving fundamental concepts of mathematics as learners and to design lessons that incorporate good instructional strategies and opportunities for their students to explain their thinking and justify their procedures and answers. Preservice teachers must realize that simply executing an algorithm or getting a correct answer does not imply conceptual knowledge. Therefore, preservice teachers also need experiences creating assessments which examine both procedural knowledge and conceptual knowledge. Preservice teachers can consult research or recent studies such as the NAEP or TIMSS studies to get ideas for questions which can evaluate both procedural knowledge and conceptual knowledge. Preservice teachers should create these questions, discuss them in class, and evaluate the questions' abilities to assess both conceptual and procedural knowledge.

Courses to prepare secondary mathematics preservice teachers need to consist of a variety of appropriate mathematical tasks which allow the preservice teachers to explore the mathematical concepts in order to discover the "why" behind the rules they accept so readily. The secondary mathematics preservice teachers should be exposed to diverse solution strategies and approaches and be able to discuss the merits of each strategy. The courses should allow ample opportunities for the preservice teachers to give written responses. These written responses should include examinations of procedures, thoughts as the preservice teachers experience a new concept or instructional strategy, reflections on student responses, as well as many other things. The written responses should be used by the instructor to discover what the preservice teachers understand about mathematical concepts as well as student learning.

An analysis of many sources shows that there exists a dichotomy between elementary mathematics courses and secondary mathematics courses. Most elementary mathematics courses cover the material the preservice teachers will be teaching as well as more advanced topics. These courses tend to engage the preservice teachers in activities which aim to expand the preservice teachers' knowledge of the concepts. Secondary mathematics courses tend to focus on issues in mathematics education and student learning styles while introducing many new instructional strategies. The content of these courses does not seem to focus on the mathematical concepts the preservice teachers will be teaching. Several studies (Frykholm, 2000; Simoneaux, Gray & Golding, 1997) have shown that secondary mathematics preservice teachers who have taken an elementary mathematics course show greater growth in conceptual knowledge and pedagogical content knowledge than those secondary mathematics preservice teachers who have not. Frykholm suggested that the format of the class as well as exposure to students (elementary preservice teachers) who were not "experts" in mathematics combined to create an environment which fostered reflections on the mathematical concepts and how students learn. The secondary mathematics preservice teachers enrolled in the elementary mathematics courses were exposed to diverse solution strategies and approaches as they participated in group activities in the elementary courses. The secondary mathematics preservice teachers noted that in the secondary mathematics courses, the group members were more likely to come to a consensus and all agree on the same method rather than discussing alternative strategies. Therefore, courses which prepare secondary mathematics preservice teachers should focus on activites to engage the preservice teachers and extend their knowledge of the concepts they will be teaching as well as their pedagogical content knowledge. Activities such as those described by Harrison, Brindley, and Bye (1989) which begin with explorations using concrete manipulatives then move to abstract generalizations should be used in these courses.

An activity to reinforce the part-whole relationship of rational numbers would begin with Cuisenaire rods. The preservice teachers would line up the rods from least to greatest. The instructor would then tell the preservice teachers that the largest rod is "one" and ask the preservice teachers to name each piece based on the information given. Once the pieces have been named through class discussion, the instructor selects another rod and tells the preservice teachers that this piece is now "one." The preservice teachers then name all of the pieces with regard to the new whole. This activity continues through several selections of new "wholes." Once the preservice teachers are comfortable with relating parts to wholes, they are ready to explore relating wholes to parts. The instructor gives the preservice teachers a rectangle and tells the preservice teachers that this rectangle represents $\frac{1}{2}$. The preservice teachers then need to draw what one whole looks like. This activity continues with rational numbers less than one as well as greater than one as the preservice teachers determine the size of one whole in each case. Group discussion as well as class discussion guides the activity. As an assessment, the preservice teachers are then asked to create their own question to determine a student's knowledge of the relationship between parts and wholes. The questions are distributed to the group then discussed to determine the appropriateness of the question. The preservice teachers then reflect on the relationships between parts and wholes, activities which can reinforce these relationships, and assessment tools to evaluate student learning.

A powerful mathematically-centered pedagogical preparation based on meaningful and comprehensive subject matter knowledge would prepare teachers to teach in the manner outlined in the vision of the National Council of Teachers of Mathematics and the Mathematical Association of America. Teachers with this type of preparation would be able to create learning environments for their students which foster the development of the students' mathematical power. The teachers would be prepared to create mathematically appropriate tasks for their students that are based on sound and significant mathematics as well as help their students develop a coherent framework for the students' mathematical ideas (Borko, et al., 1992; Cooney, 1999; Even, 1993).

5.7 Recommendations for Future Research

Although this study has explored the research questions in considerable depth and detail, there are questions left unanswered and additional questions raised. Further research is needed to address these questions. The limitation of this study also suggests further needs for additional research. Finally the implications of this study's findings propose other potential avenues for further investigation.

The size and nature of the sample restricted generalizability of the results. Further research is therefore needed to explore the conjectures generated by this study and to confirm or extend its findings. What are the levels of secondary mathematics preservice teachers in general? How widespread is the teachers' lack of pedagogical content knowledge? A mixed methods paradigm is suggested using both quantitative and qualitative methods to inform the study.

Recommendations for future research include a longitudinal study of preservice teachers as they move into their own classrooms. This research would focus on the growth in conceptual knowledge and pedagogical content knowledge levels of the new teachers and sources of this growth. It would seem that as the teachers' experience increases, their knowledge levels may increase as well. A qualitative research paradigm is suggested so that an in-depth examination of these knowledge levels might emerge.

Another recommendation for future research is an examination of the impact of the teachers' level of pedagogical content knowledge on their students. This research would employ a mixed methodology. The study would use qualitative methods to determine the teachers' level of pedagogical content knowledge. Then the study would gather student achievement scores and quantitatively analyze the scores to determine if the teachers' pedagogical content knowledge impacted student learning.

A final recommendation for future research is to examine instructional strategies and courses which can affect preservice teachers' levels of conceptual knowledge and/or pedagogical content knowledge. This study would be a primarily quantitative study with qualitative parts to further inform the study. The study would compare instructional strategies to determine which strategies best affect the conceptual knowledge and/or the pedagogical content knowledge of secondary mathematics preservice teachers. Instruments to determine the level of knowledge being examined would be administered before and after the instruction. Scores on the pre-test and post-test would be analyzed for significance. Alternatively a course could be developed with the intention of increasing preservice teachers' conceptual knowledge and/or pedagogical content knowledge. Instruments can be administered before the course and after the course to determine if the course significantly impacted the teachers' knowledge levels.

5.8 Conclusion

The conclusions of this study imply fostering a deeper understanding of the procedural knowledge, conceptual knowledge, and pedagogical content knowledge of secondary mathematics preservice teachers. This study found that the secondary mathematics preservice teachers involved exhibited a high level of procedural knowledge, a moderate level of conceptual knowledge, and a low level of pedagogical content knowledge with respect to rational numbers. The preservice teachers demonstrated great facility with operations on rational numbers as well as the most common representations of rational numbers. They demonstrated that they have a high level of procedural knowledge with respect to rational numbers.

The secondary mathematics preservice teachers exhibited some knowledge of the connections between fractional pieces and whole units, but they lacked the ability to apply this knowledge to situations which required multiple steps to complete. In many of the cases, the preservice teachers were unable to explain how they got their answers even though they correctly showed the process they followed to arrive at the answers. The analysis showed that all of the preservice teachers have a good conception of the meaning of division when integers are involved but have difficulty when rational numbers are introduced through the quotient or as the dividend or divisor. All of the secondary mathematics preservice teachers are dependent on the algorithm and are not likely to stray very far from it.

The data analysis showed that the secondary mathematics preservice teachers' pedagogical content knowledge is limited to teaching the algorithm. None of the preservice teachers were aware of any misconceptions their future students may hold or how to correct these misconceptions. The only mistakes they thought their future students may make were based on the arithmetic components of the algorithms for division with both integers and rational numbers. Thus the secondary mathematics preservice teachers exhibited a low level of pedagogical content knowledge.

Studies have shown that beginning teachers tend to teach as they were taught. Most secondary mathematics preservice teachers were taught in a very traditional manner that focused on procedures rather then connections among concepts. Since this manner of instruction worked for them, they are more likely to employ these instructional strategies in their own classrooms. However, the instructional strategies the preservice teachers encountered left little room for conceptual learning. Therefore, secondary mathematics teachers have a higher level of procedural knowledge than conceptual knowledge. In the case of division with rational numbers, the secondary mathematics preservice teachers' knowledge is limited to the algorithm thus they have no background from which to create alternative representations. Since secondary mathematics preservice teachers have had little classroom experience, they have not developed the pedagogical content knowledge that classroom experience can bring. The preservice teachers are unfamiliar with how students learn and what students think. Thus, they are unable to predict common misconceptions or pinpoint errors in students' thought processes. Since the preservice teachers' knowledge is confined to knowledge of the procedures of division with rational numbers, they are unable to list common misconceptions other than those made in computing the algorithm.

These conclusions have implications for related research, teacher preparation, and future research. The results of this study corroborated and supplemented earlier studies conducted with elementary teachers. The results of this study furthered the knowledge base of secondary mathematics preservice teachers' levels of conceptual knowledge and pedagogical content knowledge with respect to rational numbers. The conclusions of this study suggest reforms to current teacher preparation programs in order for these programs to better prepare teachers to teach in a manner congruent with the vision of the National Council of Teachers of Mathematics. Future research was suggested by the findings of this study. The recommendations included a longitudinal study of the impact of experience on the knowledge levels as preservice teachers become teachers in their own classrooms.

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A Rational Numbers and Their Representations Survey

The following pages contain the ***Rational Numbers and Their Representations*** survey. This survey consists of six open-ended questions containing seventeen parts designed to determine subject matter knowledge. Four parts directly assess procedural knowledge (5 a, b, c, d). Four parts directly assess conceptual knowledge (4 a, b, c, d). The remaining nine parts measure both procedural and conceptual knowledge including representations of rational numbers. Possible scores on the Rational Numbers and Their Representation survey range from a 0 to 68.

Rational Numbers and Their Representations

1. What is a rational number?

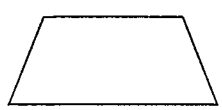
2. Sketch a model to represent the fraction $\frac{3}{5}$ in the following ways:

a) as part of a set

b) as part of a region

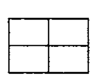
c) as a ratio

d) as part of the following region:

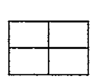


3. Shade the given figure as indicated.

a) $\frac{7}{8}$ of the whole



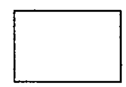
b) $\frac{2}{3}$ of the whole



4. Complete the following. **EXPLAIN YOUR REASONING.**

a) If the following figure represents $\frac{2}{3}$, sketch 1 whole.

b) If the following figure represents $\frac{5}{4}$, sketch 1 whole.



c) If ________ represents $\frac{5}{4}$, sketch $\frac{2}{3}$.

d) If the figure below represents $\frac{5}{3}$, sketch $\frac{3}{5}$

5. Solve. Show your work.

a) $1\frac{3}{4} \div \frac{1}{2}$

b) $1\frac{3}{4} \times \frac{1}{2}$

c) $1\frac{3}{4} - \frac{1}{2}$

d) $1\frac{3}{4} + \frac{1}{2}$

6. For the following stories write an expression you would use to solve the situation.

a) A paper-hanger needs $3\frac{3}{4}$ rolls of wallpaper to do a room. How many similar rooms can the paper-hanger do with 13 rolls of the same wallpaper?

b) On the highway a car travels 2 miles in 3 minutes. If the speed of the car is constant, how far does it travel in 20 minutes?

B Interview Protocol

For each of the following expressions, $58 \div 7$, $1\frac{3}{4} \div \frac{1}{2}$, the researcher will ask the following questions.

1. How would you solve this problem?

2a. Sometimes teachers try to come up with real-world situations or story problems to show the meaning or application of some particular piece of content. What would you say would be a good situation story for the expression?

2b. *After the participant describes a story, the researcher will ask* How does the story fit with the solution you came up with before?

2c. *If the participant notices that the answer to the story or other representation does not match the original answer obtained, the researcher will ask* Why did that come out differently?

3. When we consider this problem, what other facts or concepts are necessary for a student to understand in order for him/her to understand this problem?

4. What mistakes might your students make when solving a problem like this? Why might they make these mistakes? How could we correct these mistakes/misconceptions?

5. How would you explain this problem to a sixth grade student? Is there another way to explain it? *Prompt for as many ways as they can provide.*

C Institutional Review Board Approval

The following page contains the Oklahoma State University Institutional Review Board approval. This study was approved on Friday, November 15, 2002. The title of the study is *Secondary Mathematics Preservice Teachers' Conceptions of Rational Numbers.* The principal investigator is Ellen Eileen Durand Faulkenberry.

Oklahoma State University
Institutional Review Board

Protocol Expires: 11/14/2003

Date: Friday, November 15, 2002 IRB Application No ED0344

Proposal Title: SECONDARY MATHEMATICS PRESERVICE TEACHERS' CONCEPTIONS OF RATIONAL NUMBERS

Principal
Investigator(s):

Eileen Durand Faulkenberry	Dr. Patricia Lamphere-Jordan
401 Math Science	247 Willard
Stillwater, OK 74078	Stillwater, OK 74078

Reviewed and
Processed as: Exempt

Approval Status Recommended by Reviewer(s): Approved *

Dear PI :

Your IRB application referenced above has been approved for one calendar year. Please make note of the expiration date indicated above. It is the judgment of the reviewers that the rights and welfare of individuals who may be asked to participate in this study will be respected, and that the research will be conducted in a manner consistent with the IRB requirements as outlined in section 45 CFR 46.

As Principal Investigator, it is your responsibility to do the following:

1. Conduct this study exactly as it has been approved. Any modifications to the research protocol must be submitted with the appropriate signatures for IRB approval.
2. Submit a request for continuation if the study extends beyond the approval period of one calendar year. This continuation must receive IRB review and approval before the research can continue.
3. Report any adverse events to the IRB Chair promptly. Adverse events are those which are unanticipated and impact the subjects during the course of this research; and
4. Notify the IRB office in writing when your research project is complete.

Please note that approved projects are subject to monitoring by the IRB. If you have questions about the IRB procedures or need any assistance from the Board, please contact Sharon Bacher, the Executive Secretary to the IRB, in 415 Whitehurst (phone: 405-744-5700, sbacher@okstate.edu).

Sincerely,

Carol Olson, Chair
Institutional Review Board

*NOTE: Please change Room of IRB office to 415 Whitehurst

2

Vita

Ellen Eileen Durand Faulkenberry
Candidate for the degree of
Doctor of Philosophy

Thesis: SECONDARY MATHEMATICS PRESERVICE TEACHERS' CONCEPTIONS OF RATIONAL NUMBERS

Major Field: Education, concentration in Mathematics Education

Biographical Information:

Personal Data: Born in Arkadelphia, Arkansas, November 6, 1973 to William and Elaine Durand. Married to Thomas Faulkenberry of Marietta, Oklahoma, July 20, 2002.

Education: Graduated Valedictorian from Arkadelphia High School, Arkadelphia, Arkansas in May 1992; received Bachelor of Science degree in Mathematics from Henderson State University, Arkadelphia Arkansas in December 1996; received Master of Science degree in Mathematics from Oklahoma State University, Stillwater, Oklahoma, in May 2000. Completed the requirements for the Doctor of Philosophy degree in Professional Educational Studies with a specialization in Mathematics Education at Oklahoma State University, Stillwater, Oklahoma in August, 2003.

Experience: Mathematics tutor, Henderson State University, 1993 - 1996; Graduate Assistant, Henderson State University, 1997; Graduate Assistant, Oklahoma State University, 1997 - 2003.

Professional Memberships: National Council of Teachers of Mathematics, Arkansas Council of Teachers of Mathematics, Mathematical Association of America, American Mathematical Society, National Council of Supervisors of Mathematics.