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IMAGES OF ABSOLUTE NEIGHBORHOOD RETRACTS AND GENERALIZED

ABSOLUTE NEIGHBORHOOD RETRACTS UNDER REFINABLE MAPS

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INTRODUCTION

In his Ph.D. thesis of October 1929, Karol Borsuk defined and studied retracts of metric spaces [Bk 1]. This work led to the study of absolute retracts. (An absolute retract is a retract of any metric space in which it is embedded as a closed subset.) It also led to the study of absolute neighborhood retracts. (That is a space which is a retract only of some neighborhood of any metric space in which it is embedded as a closed subset.)

Among the properties that an absolute neighborhood retract (ANR) has are (a) an ANR is locally connected, (b) two maps from a metric space into an ANR which are sufficiently close are homotopic, and (c) a map from a closed subset of a metric space into an ANR can be extended to some neighborhood of the closed subset. Because of (b) and (c) the ANR's have nice homotopy and extension properties. [Bk 4; Hu]

While it is clear that the homeomorphic image of an ANR must be an ANR it is also clear that the continuous image of an ANR is not necessarily an ANR. The last part of this statement follows since by the Hahn-Mazurkiewicz theorem [Wi, p.76] every Peano continuum is the continuous image of the unit interval, and there exist Peano continua which are not ANR's. Because of these facts one can ask what kind of conditions must a map satisfy in order to guarantee that the image of an ANR be an ANR.

For finite dimensional compact metric spaces if the point inverses of a continuous map of a compact ANR onto a finite dimensional metric space are all AR's then the image is an ANR [Bk 4, p.131]. A generalization of this result involving the use of CE maps or Vietoris maps (maps so that the inverse image of a point has the shape of a point) has been recently given by Kozlowski [Ko, p.48]. Both of these results require that image must be a finite dimensional compactum.

Another approach to this problem is to use the uniform limits of ε-maps, called refinable maps, which were introduced by Ford and Rogers [Fo and Rg]. Since the image of a finite dimensional space under a refinable map must have the same finite dimension there is not the dimension raising problem with refinable maps that CE maps may have.

Among the results which give certain cases in which the image of a compact ANR is an ANR are the following:

The image of a compact 1-dimensional ANR under a refinable map is an ANR.

The image of a 2-sphere under a refinable map is a 2-sphere; moreover, the refinable map is a near homeomorphism [Fo and Rg, Theorem 4, p.8].

The image of a compact orientable 2-manifold without boundary under a refinable map is a homeomorphic 2-manifold.

The general question posed by Ford and Rogers as to whether or not the image of an ANR under a refinable map is also an ANR is unknown even in the finite dimensional case.

The connection between an affirmative answer to the Ford and Rogers question and CE images of ANR's is apparent using the following results from [Fo and Rg] which will be generalized to a more general class of spaces to be called quasi-ANR's (q-ANR's):

If r is a refinable map from the ANR (q-ANR) X onto Y and $\varepsilon > 0$, then there exist maps g_1 from X onto Y and g_2 from Y onto X such that

(i) $d(rg_2, \mathrm{id}\ Y) < \varepsilon$ and

(ii) $d(g_1g_2, \mathrm{id}\ Y) < \varepsilon$ and $d(g_2g_1, \mathrm{id}\ X) < \varepsilon$. [Fo and Rg, Corollary 3.3, p.6].

If in addition to the hypothesis given above both X and Y are ANR's then r is a fine homotopy equivalence [Ch, p.91].

Thus, by a result due to Haver r is a CE map [Ha, Ch, p. 91].

The first results (i) and (ii) cited in the previous paragraph follow from a result by Lončar and Marsdešić [Lo and Ma, Lemma 1, pp. 41-42]. In Chapter II we will show that the Lončar and Mardešić result holds for a larger class of spaces than the ANR's. In fact, among the locally connected spaces the quasi-ANR's are exactly the spaces for which the Lončar and Mardešić result holds. Hence, quasi-ANR's are spaces for which the results Ford and Rogers state in the case of an ANR must hold. A nice result in this setting is that the image of a quasi-ANR under a refinable map must be a quasi-ANR.

In the last chapter (V) we will find the position of quasi-ANR's relative to other generalizations of ANR such as the approximative ANR's due to Noguchi and the approximative ANR's due to Clapp. One of the main differences is the requirement of surjectivity of the approximative retractions in the case of the quasi-ANR's.

In this dissertation all spaces are assumed to be compact metric spaces unless otherwise indicated. All maps are continuous functions. A homeomorphism is one to one, onto, and both it and its inverse are continuous. An embedding is a map which is a homeomorphism to its image.

As a matter of notation if S is a set in some metric space X, by $N_\varepsilon(S)$ we will mean the set of points in X whose distance from S is less than ε. If $S = \{X\}$ we abuse the notation and write $N_\varepsilon(X)$.

IMAGES OF ABSOLUTE NEIGHBORHOOD RETRACTS AND GENERALIZED ABSOLUTE NEIGHBORHOOD RETRACTS UNDER REFINABLE MAPS

CHAPTER I

REFINABLE MAPS

I.1 Definition. A map f from a space X onto a space Y is said to be an ε-map for a given positive real number ε if for each y in Y, diam $f^{-1}(y) < \varepsilon$. [Al, p.103; Ma and Se, p.146]

I.2 Definition. A map r from a space X onto a space Y is said to be refinable if for every $\varepsilon > 0$ there is an ε-map f ε-near r (that is, $d(f,r) = \sup\{d(f(x),r(x)) \mid x \text{ belongs to } X\} < \varepsilon$). [Fo and Rg] Such a map f is called an ε-refinement.

The following results will be quoted from Ford and Rogers without proof.

I.3 Theorem. If r is a refinable map from X onto Y and H is a subcontinuum of Y, then there is a continuum C in X such that $r(C) = H$ and C contains $r^{-1}(\mathrm{int}(H))$, where $\mathrm{int}(H)$ denotes the topological interior of H . [Fo and Rg, Theorem 1, p.2]

I.4 Definition. The map r from X onto Y is weakly confluent if and only if for each subcontinuum H of Y some component of $r^{-1}(H)$ is mapped onto H by r . [Le]

I.3.1 Corollary. Every refinable map from X onto Y is weakly confluent. [Fo and Rg, Corollary 1.1, p.2]

I.3.2 Corollary. If r is a refinable map from X onto Y and Y is connected imKleinen at p , then $r^{-1}(p)$ is connected; hence, if Y is locally connected then r is monotone. [Fo and Rg, Corollary 1.2, p.3]

I.3.3 Corollary. If r is a refinable map from X onto Y , then X is decomposable if and only if Y is decomposable. [Fo and Rg, Corollary 1.3, p.3]

Before stating Theorem 2 of [Fo and Rg] together with a slight generalization, the following definitions are necessary.

I.5 Definition. A point x of a space X is a local cut point of X if and only if there is an open neighborhood U of x which is separated by x (i.e., $U - \{x\}$ is separated). By a theorem in [Wh, p.61] the condition given in this definition is equivalent to requiring that x be a local cut point of X provided there is an open neighborhood U of x such that if V is any open neighborhood of x contained in U then $V - \{x\}$ is separated. The preceding sentence yields the definition found in [Ma and Se, p.155].

I.6 Definition. A point x of a space X is a weak cut point of X if and only if there are distinct points h and k of X

different from x such that every continuum containing h and k must also contain x.

I.7 Definition. A point x of a space X is a local weak cut point of X if and only if there exist an open neighborhood U of x and two distinct points h and k in U different from x such that every continuum in U containing h and k must also contain x.

I.8 Theorem. Suppose r is a refinable map from X onto Y and the point q separates Y. Then some point of $r^{-1}(q)$ is a weak cut point of X. [Fo and Rg, Theorem 2, p.4]

I.9 Theorem. Suppose r is a refinable map from X onto Y and y is a local cut point of Y. Then some point of $r^{-1}(y)$ is a local weak cut point of X.

Proof: Since y is a local cut point of Y there is an open neighborhood V of y such that $V - \{y\}$ is separated. Let $U = r^{-1}(V)$. Also let $V - \{y\} = H \cup K$ where H and K are nonempty mutually disjoint open sets. Let h belong to $r^{-1}(H)$ and k belong to $r^{-1}(K)$.

Since r is a refinable map there is a sequence of $1/i$-maps f_i from X onto Y such that $d(f_i,r) < 1/i$ and $\{f_i^{-1}(y)\}$ converges in the Hausdorff metric to a nonempty closed subset of X. Since $\operatorname{diam} f_i^{-1}(y) \to 0$ as $i \to \infty$, $\{f_i^{-1}(y)\} \to \{x\}$ for some x in X.

Suppose M is a continuum in U such that h belongs to M and k belongs to M. Since $f_i(x) \to r(x)$ and $f_i(x) \to y$ we have $r(x) = y$; hence, x does not belong to $r^{-1}(H) \cup r^{-1}(K)$. If x does not belong to M there is an i large enough so that $f_i(M)$ is a subset of V , $f_i(h)$ belongs to H , $f_i(k)$ belongs to K , and M does not intersect $f_i^{-1}(y)$. Now h belongs to the intersection of M and $f_i^{-1}(H)$, and k belongs to the intersection of M and $f_i^{-1}(K)$. Since $f_i(M)$ is a subset of V and M does not intersect $f_i^{-1}(y)$, M is a subset of $f_i^{-1}(H) \cup f_i^{-1}(K)$. Thus, $f_i^{-1}(H)$ and $f_i^{-1}(K)$ separate M which is impossible since M is connected. One must conclude that x belongs to M , which shows that x must be a local weak cut point of X .

The next lemma is a standard result concerning weak cut points.

I.10 <u>Lemma</u>. If a space X is locally connected and x is a weak cut point of X then x is a cut point of X .

<u>Proof</u>: Since x is a weak cut point of X there are distinct points h and k different from x so that any continuum which contains both h and k must also contain x . If x is not a cut point then $X - \{x\}$ is connected. Since X is locally connected and compact there is a simple chain of compact connected sets $Cl(U_1), Cl(U_2), \ldots, Cl(U_n)$ such that x does not belong to any $Cl(U_i)$, each U_i is open and nonempty, and h belongs to U_1 while k belongs to U_n . Thus the union of the $Cl(U_i)$ is a continuum from h to k missing x which is impossible. It must be concluded, therefore, that x is a cut point.

I.11 <u>Lemma</u>. If x is a local weak cut point of X and X is a locally connected space, then x is a local cut point of X .

Proof: Since x is a local weak cut point of X and X is locally connected there is a closed neighborhood V of x which is locally connected, and such that x is a weak cut point of V. By Lemma I.10, x is a cut point of V. Thus, according to Definition I.5, x is a local cut point of X.

To show that in the preceding theorems (Theorems I.8 and I.11) it is not necessarily true that every point in the inverse image of a cut point under a refinable map is a weak cut point or a local weak cut point consider the following example.

I.12 Example. In the plane E^2 for $n = 1,2,\ldots$, let R_n be the closed segment with endpoint at the origin inclined at an angle of $\pi/2^n$, having length $2/2^n$. Let X = the union of the R_n and Y = the union of the R_n where n is greater than or equal to 2. Define r from X onto Y by $r(R_1)$ = the origin and $r|R_n$ = the identity on R_n for $n > 1$. Using the sequence of maps f_n defined by allowing f_n (origin) = origin, for $1 < i < n$ allowing $f_n|R_i$ = the identity on R_i, allowing $f_n|R_1$ to be a linear map of R_1 onto R_n, and for $i > n$ allowing $f_n|R_i$ to be a linear map of R_i onto R_{i+1}, it can be seen that r is the uniform limit of the f_n's and that each f_n is a homeomorphism. Hence, r is a near homeomorphism; thus, r is a refinable map.

Now r^{-1} (origin) = R_1. The free endpoint $(0,1)$ of R_1 is clearly not either a weak cut point or even a local weak cut point of X while the origin is a cut point of Y.

I.13 Definition. A locally connected space X is said to be locally cyclic provided that for every x in X and neighborhood U of x there is an open set V which is a subset of U containing x and such that $V - \{x\}$ is connected [Ma and Se]. It is clear that a locally connected locally cyclic space contains no local cut points; hence, by Lemma I.11 such a space contains no local weak cut points. Conversely a locally connected space which contains no local cut points is a locally cyclic space.

The following two corollaries are closely related. The first corollary, I.13.1, appears in [Fo and Rg, p.4]. The second corollary is an analogous version of the first corollary in the case of locally cyclic spaces. The proof given here is an adaptation of the proof of Corollary I.13.1 as it appears in Ford and Rogers.

I.13.1 Corollary. If r is a refinable map from X onto Y, and X is locally connected and has no cut point (thus by [Wh, p.79] X is cyclicly connected), then if y belongs to Y, $X - r^{-1}(y)$ is connected. Hence, Y has no cut point.

Proof: See Ford and Rogers [Fo and Rg, p.4]. Since r is a closed map, Y is locally connected; hence, by the result in [Wh, p.79], Y is cyclicly connected.

I.13.2 Corollary. If X is a locally connected locally cyclic space, then Y, the image of X under a refinable map r, is also a locally connected locally cyclic space.

Proof: We first note that since r is a closed map, hence an identification, Y is locally connected.

Suppose a point y in Y were a local cut point. By Theorem I.9 some point of $r^{-1}(y)$ would be a local weak cut point. By Lemma I.11 such a point would be a local cut point of X which is impossible since X is locally cyclic.

Thus, we must conclude that Y can have no local cut points. From the remark following Definition I.13 we may conclude that Y is locally cyclic.

Next we will show that refinable maps preserve finite dimension in the sense that if there is a refinable map between two compact (or locally compact separable) metric spaces and one of the spaces has a finite (inductive) dimension [Hz and Wa] then the other space has the same dimension. Notice that this is different from the case of a CE map where it is in general unknown if such maps can raise dimension.

I.14 Lemma. Let X be a compact metric space, and suppose $\dim Y = n$ where n is finite. If r is a refinable map from X onto Y, then $\dim X \leqq \dim Y = n$.

Proof: The following characterization of dimension will be used. Suppose X is a compact metric space. Then $\dim X \leqq n$ if and only if for every positive real number ε there is an ε-map of X into a polytope of dimension less than or equal to n (Alexandroff's theorem on approximation to compact metric spaces by polytopes [Hz and Wa, p.72]).

We are given $\dim Y = n$ and $r : X \twoheadrightarrow Y$ ($\twoheadrightarrow$ means onto) is a refinable map. Let $\varepsilon > 0$ be given. Then there is a map $f : X \twoheadrightarrow Y$ such that for all y in Y, $\operatorname{diam} f^{-1}(y) < \varepsilon/3$ and

$d(f,r) = \sup\{d(f(x),r(x)) \mid x$ is in $X\} < \varepsilon/3$. Since f is continuous and X is both compact and Hausdorff, the collection $\{f^{-1}(y) \mid y$ belongs to $Y\}$ is an upper semicontinuous decomposition of X [Ho and Yg, p.132]; hence, there is a finite open cover α of Y such that for each A in α there is a y in Y with $f^{-1}A \subset N_{\varepsilon/3}(f^{-1}(y))$.

Since $\dim Y = n$, there is an α-map $g : Y \to K$ where K is a polytope of dimension less than or equal to n . We claim that $gf : X \to K$ is an ε-map; hence, since ε may be chosen arbitrarily small $\dim X \leqq n$.

To prove the claim let k belong to K . Then $g^{-1}(k)$ is a subset of some set A belonging to α , since g is an α-map. Hence, there is a y in Y such that $f^{-1}(g^{-1}(k))$ is contained in $N_{\varepsilon/3}(f^{-1}(y))$. Since $\operatorname{diam} f^{-1}(y) < \varepsilon/3$, $\operatorname{diam} f^{-1}(g^{-1}(k)) < \varepsilon$.

I.15 Lemma. Let X be a compact finite dimensional metric space with $\dim X = n$, and let Y be a metric space. If r is a refinable map from X onto Y then $\dim Y \leqq \dim X = n$.

Proof: This proof will use the covering characterization of dimension [Hz and Wa, p.66]. That is, a compact metric space has dimension less than or equal to n provided every open cover has an open cover refining the original cover such that at most $n + 1$ elements of the refinement can have nonempty intersection. In this case the refinement is said to have order n .

Let α be an open cover of Y . Since Y is the continuous image of a compactum, Y must be compact; thus, there is an $\varepsilon > 0$

such that if $\operatorname{diam} S < 2\varepsilon$ and S is a subset of Y, then for some A in α, S is contained in A. Let α^* be a finite open cover of Y with $\operatorname{mesh} \alpha^* = \sup\{\operatorname{diam} A \mid A \text{ belongs to } \alpha^*\} < \varepsilon/3$. Then $r^{-1}(\alpha^*)$ is an open cover of the n-dimensional space X. Hence, there is a refinement β of $r^{-1}(\alpha^*)$ such that the order of β is less than or equal to n. Since X is compact there is a $\delta > 0$ such that if T is a subset of X whose diameter is less than δ then T is contained in some member of β.

Since $r : X \twoheadrightarrow Y$ is a refinable map there is a δ-map $f : X \twoheadrightarrow Y$, $\varepsilon/3$-near r. Consider the cover $\gamma = \{U_B = \text{union}\{f^{-1}(y) \mid f^{-1}(y) \subset B\} \mid B \text{ belongs to } \beta\}$. Notice that γ is a cover since each x in X belongs to $f^{-1}(y)$ for some y in Y, and since $\operatorname{diam} f^{-1}(y) < \delta$ implies the existence of some B in β which contains $f^{-1}(y)$. The cover γ is an open cover since each B in β is open [Ho and Yg, Theorem 3-32, p.133].

Each U_B is saturated; thus, $\{f(U_B) \mid B \text{ belongs to } \beta\}$ is an open cover of Y. It is now claimed that $\{f(U_B) \mid B \text{ belongs to } \beta\}$ refines α. This claim follows since $\operatorname{diam} f(U_B) \leq \varepsilon < 2\varepsilon$. To verify this last formula let u_1 and u_2 each belong to U_B. Then the following inequalities hold: $d(f(u_1),f(u_2)) \leq d(f(u_1),r(u_1)) + d(r(u_1),r(u_2)) + d(r(u_2),f(u_2)) < \varepsilon$. Notice that $d(r(u_1),r(u_2)) < \varepsilon/3$ since each of u_1 and u_2 belongs to B, and B is in β which refines $r^{-1}(\alpha^*)$.

Next it will be shown that the order of $\{f(U_B) \mid B \text{ belongs to } \beta\}$ is less than or equal to n. Suppose the intersection of

$f(U_{B_1}),\ldots,f(U_{B_k})$, where the B_i in β are distinct, is nonempty. Let y belong to the intersection. Then $f^{-1}(y)$ is a subset of each $f^{-1}(f(U_{B_i})) = U_{B_i}$ (each U_{B_i} is saturated). Thus, the intersection of the B_i is nonempty. Since the order of β is at most n it must be that $k \leqq n + 1$. Hence, the order of $\{f(U_B) \mid B \text{ belongs to } \beta\}$ is at most n .

Since α is an arbitrary open cover of Y , we conclude that $\dim Y \leqq n = \dim X$.

Putting Lemmas I.14 and I.15 together one obtains the following theorem:

I.16 Theorem. If there is a refinable map between compact metric spaces and one of the spaces has a finite dimension then both spaces have the same finite dimension.

It should be noted at this point, that the refinability of a map is not dependent on the choice of metric. In fact, in general topological spaces one may define a map r from X onto Y to be refinable provided that for every open cover α of X and every open cover β of Y there is a map f from X onto Y such that f is β-near r (i.e., for every x in X there is a B in β which contains both $f(x)$ and $r(x)$), and f is an α-map (i.e., for every y in Y there is an A in α such that $f^{-1}(y)$ is contained in A). It is clear that in the case of compact metric spaces this definition is equivalent to Definition I.2 using any pair of consistent metrics.

As a corollary of this result, a generalization to locally compact metric spaces can be obtained. If each of X and Y is a locally compact metric space, a map r of X onto Y will be said to be <u>proper refinable</u> if and only if (a) $r^{-1}(C)$ is compact for each compact subset C of Y, and (b) for every open cover α of X and every open cover β of Y there is an α-map $f : X \twoheadrightarrow Y$ β-near r. We then have the following:

I.16.1 <u>Corollary</u>. Let each of X and Y be a nonempty locally compact separable metric space. Then if one of the spaces has a finite dimension and r is a proper refinable map from X onto Y, both X and Y have the same dimension.

<u>Proof</u>: Let $X^* = X \cup \{\infty\}$ and $Y^* = Y \cup \{\infty\}$ be the one point compactifications of X and Y, respectively. Since each of X and Y is second countable each of X^* and Y^* is a compact metric space [Du, p.247].

Let $r^* : X^* \twoheadrightarrow Y^*$ be defined by $r^*|X = r$ and $r^*(\infty) = \infty$. It is claimed that r^* is a refinable map from X^* onto Y^*. Let α be an open cover of X so that in X^*, α has a mesh less than a given $\varepsilon > 0$. By Theorem 4.1 of [Ch, p.2] there is an open cover γ of Y so that any pair of maps of X into Y γ-near each other are either both proper or both not proper. Let β be an open cover of Y which refines γ and in Y^* has a mesh less than ε.

By the refinability of r there is an α-map $f : X \twoheadrightarrow Y$ β-near r. Since r is proper f is proper. Hence $f^* : X^* \twoheadrightarrow Y^*$ defined by $f^*|X = f$ and $f^*(\infty) = \infty$ is a map. By the way α and β

have been chosen, it is clear that f^* is an ε-map ε-near r^* (at ∞, f^* and r^* agree). Hence r^* is a refinable map.

Suppose one of X or Y is finite dimensional. If $\dim X = n$ (or $\dim Y = n$) then $\dim X^* = n$ (or $\dim Y^* = n$) since the dimension of a nonempty separable metric space cannot be raised by adjunction of one point [Hu and Wa, p.32]. By Theorem I.16, $\dim Y^* = n$ (or $\dim X^* = n$). Again since $\dim Y \leq \dim Y^*$ (or $\dim X \leq \dim X^*$), and the dimension cannot be raised by the adjunction of a single point, $\dim Y = n$ (or $\dim X = n$).

In the next section we will show that if either of two spaces has a finite number of components and there is a refinable map between the spaces then the refinable map induces a bijection between the components of the two spaces. Moreover, we can show that in this case the restriction of a refinable map to a component is still refinable. In general it is not known when the restriction of a refinable map to a proper subspace will still be refinable [Fo and Rg, pp.4-5].

Before proving these statements we will adopt the following notation and definition. If X is a space by $k(X)$ we will mean the set of all components of X.

I.17 Definition. A map f from a space X onto a space Y will be said to be *component preserving* provided the function φ defined by $\varphi(K) = f(K)$, where K is a component of X, is a bijection of $k(X)$ to $k(Y)$.

I.18 Lemma. A map f from a space X onto a space Y is component preserving if and only if for each component H of Y, $f^{-1}(H)$

is a component of X .

I.19 Lemma. If r from X onto Y is a refinable map between compact metric spaces and Y is connected then X is connected.

Proof: Suppose X is not connected. Let $X = U \cup V$ where U is open, V is open, $U \cap V = \emptyset$, $U \neq \emptyset$ and $V \neq \emptyset$. Then each of U and V is closed; hence, each of U and V is compact. Let $\varepsilon = d(U,V) = \inf\{d(u,v) \mid u$ belongs to U and v belongs to $V\}$. ε is a positive real number since $U \cap V = \emptyset$ and both U and V are closed. Let f from X onto Y be an ε-map ε-near r . Consider $f(U)$ and $f(V)$. Each of $f(U)$ and $f(V)$ is closed and $f(U) \cup f(V) = Y$.

We now claim that $f^{-1}f(U) = U$ and $f^{-1}f(V) = V$. Suppose y is in $f(U)$. Then $f^{-1}(y) \cap U \neq \emptyset$. Since diam $f^{-1}(y) < d(U,V)$, $f^{-1}(y) \cap V = \emptyset$. Thus $f^{-1}(y)$ is a subset of U . Hence, $f^{-1}(f(U)) = U$. Similarly it follows that $f^{-1}(f(V)) = V$. Since U and V do not intersect $f(U)$ and $f(V)$ do not intersect. Thus, $f(U)$ and $f(V)$ form a separation of Y which is impossible.

A corollary of the proof of the preceding lemma is that if P is a class of connected metric spaces and X is P-like (can be mapped, for every $\varepsilon > 0$, onto some member of P by an ε-map) then X is connected.

I.20 Lemma. Let r from X onto Y be a refinable map between compact metric spaces. Suppose a subset H of Y is both open and closed. Then $r|r^{-1}(H)$ is a refinable map from $r^{-1}(H)$ onto H .

Proof: Let $\varepsilon > 0$ be given. Let $K = Y - H$. Since H is both open and closed, $Y - H$ is both open and closed. Let $\eta = d(H,K)$. Then $\eta > 0$ since H and K do not intersect. Since r is refinable there is a map f from X onto Y so that f is a $\min\{\varepsilon,\eta\}$-refinement of r.

It is claimed that f maps $r^{-1}(H)$ onto H. For if x belongs to $r^{-1}(H)$ then since $d(f(x),r(x)) < d(H,K)$, $f(x)$ belongs to H. Hence, $f(r^{-1}(H))$ is contained in H. To show that $f(r^{-1}(H)) = H$, let y belong to H. Since f is onto Y there is an x in X such that $f(x) = y$. Clearly $r(x)$ belongs to H; thus, x belongs to $r^{-1}(H)$ ($d(r,f) < d(H,K)$).

Since $f|r^{-1}(H)$ is an ε-map ε-near $r|r^{-1}(H)$ and $\varepsilon > 0$ is arbitary, $f|r^{-1}(H)$ is the required map needed to show that $r|r^{-1}(H)$ is refinable.

I.21 Lemma. If r from a space X onto a space Y is a map and X has at most a finite number of components, then Y has at most a finite number of components.

Proof: This lemma follows from the fact that any map preserves connectedness and the fact that r is surjective.

I.22 Theorem. If r from a space X onto a space Y is a refinable map and Y has at most a finite number of components, then r is component preserving.

Proof: According to Lemma I.18 it is sufficient to show that if H is a component of Y then $r^{-1}(H)$ is a component of X.

Suppose H is a component of Y. Then H is both open and closed. By Lemma I.20, $r|r^{-1}(H)$ is a refinable map of $r^{-1}(H)$ onto H. According to Lemma I.19, $r^{-1}(H)$ is connected. It is now claimed that $r^{-1}(H)$ is a component of X. This fact follows since if K is a connected set in X which contains $r^{-1}(H)$ then $r(K)$ is a connected set in Y which contains H. In this case $H = r(K)$ since H is a component of Y; thus, $K = r^{-1}(H)$.

I.22.1 Corollary. If r is a refinable map from a compact metric space X onto a metric space Y, and X has at most a finite number of components then r is component preserving.

Proof: Apply Lemma I.21 and Theorem I.22.

I.22.2 Corollary. Suppose r is a refinable map from a compact metric space X onto a metric space Y, and X has at most a finite number of components. Then for any component K of X, $r|K$ is a refinable map.

Proof: According to Corollary I.22.1, r is component preserving. Hence, $r(K)$ is a component of Y and $K = r^{-1}(r(K))$. Thus, $r(K)$ is both open and closed; therefore, by Lemma I.20, $r|K$ is a refinable map.

I.22.3 Corollary. Suppose r from a compact metric space X onto a metric space Y is a refinable map. If $k(X)$ is countably infinite then $k(Y)$ is countably infinite.

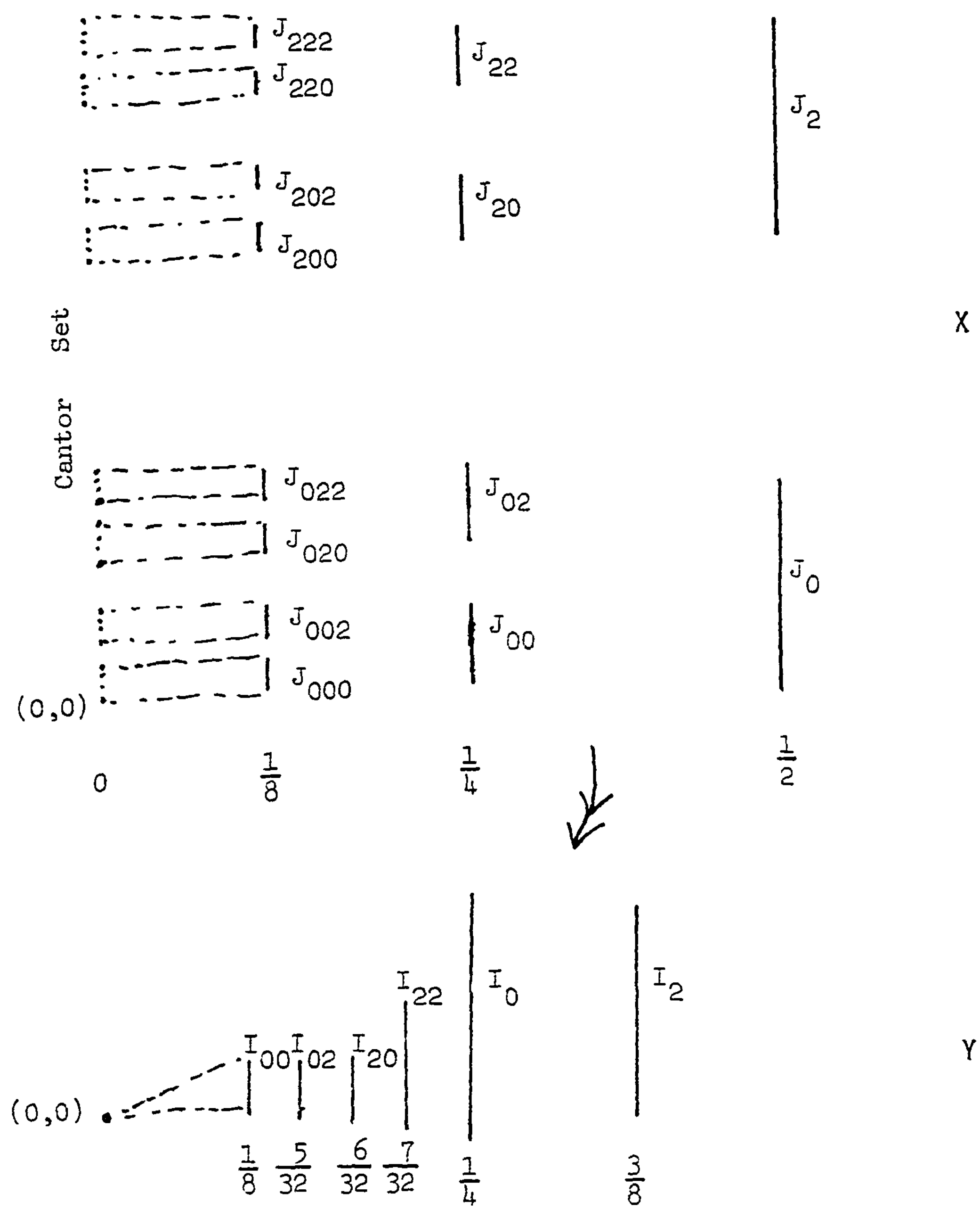
Proof: Clearly $r(k(X))$ is a countable cover of Y by connected sets. Since each set in $r(k(X))$ is contained in some component of Y, $k(Y)$ is at most countable. If $k(Y)$ were finite Theorem

I.22 would imply that r is component preserving; hence, $k(X)$ would be finite.

On the other hand, given that r is a refinable map from a space X onto a space Y and $k(Y)$ is countably infinite, it is not necessarily true that r is component preserving or even that $k(X)$ is countably infinite as the next example shows.

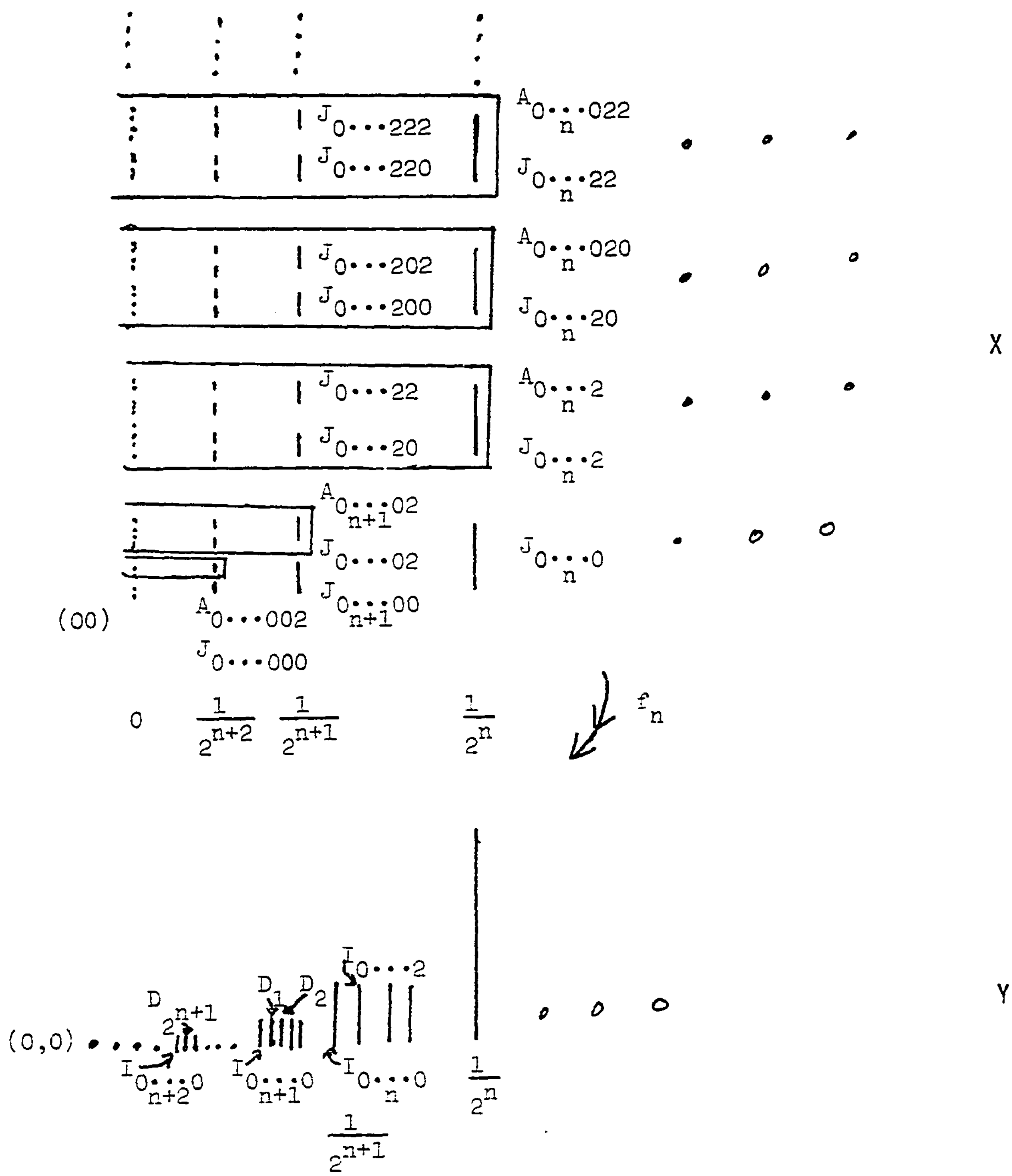
I.23 <u>Example</u>. Let X be the various stages of the middle thirds construction of the Cantor set together with the Cantor set as depicted in Figure I.23.1. Let Y be a sequence of vertical segments whose diameters go to 0 together with the origin as depicted in Figure I.23.1. Using the notation in Figure I.23.1 define r from X onto Y so that r takes $J_{a_1 \dots a_n}$ onto $I_{a_1 \dots a_n}$ linearly, and so that r (Cantor set) is the origin.

It is claimed that r is a refinable map. For a natural number n let f_n be defined so that if $k < n$ then $f_n | J_{a_1 \dots a_k} = r | J_{a_1 \dots a_k}$; for all k, $f_n | J_{0 \underset{k}{\dots} 0} = r | J_{0 \underset{k}{\dots} 0}$; if $k \geqq n$ and $\sum_{i=1}^{n} a_i \neq 0$ then $f_n | A_{a_1 \dots a_n} = r\, \Pi_{J_{a_1 \dots a_n}}$, where $\Pi_{J_{a_1 \dots a_n}}$ is the horizontal projection onto $J_{a_1 \dots a_n}$; if $k > n$ and $\sum_{i=1}^{n} a_i = 0$ let f_n map $A_{0 \underset{k}{\dots} 02}$ to D_k by first projecting onto $J_{0 \underset{k}{\dots} 02}$, then follow this by a linear map onto D_k ; finally let $f_n(0,0) = (0,0)$. For the definition of $A_{a_1 \dots a_k}$ see Figure I.23.2.



J_{222}
J_{220}
J_{22}
J_2
J_{202}
J_{200}
J_{20}
Cantor Set
X
J_{022}
J_{020}
J_{02}
J_0
J_{002}
J_{000}
J_{00}
(0,0)
0
$\frac{1}{8}$
$\frac{1}{4}$
$\frac{1}{2}$
I_{22}
I_0
I_2
I_{00} I_{02} I_{20}
Y
(0,0)
$\frac{1}{8}$ $\frac{5}{32}$ $\frac{6}{32}$ $\frac{7}{32}$ $\frac{1}{4}$ $\frac{3}{8}$

Figure I.23.1



$A_{\underset{n}{0\cdots}022}$
$J_{0\cdots222}$
$J_{0\cdots220}$
$J_{\underset{n}{0\cdots}22}$
$A_{\underset{n}{0\cdots}020}$
$J_{0\cdots202}$
$J_{0\cdots200}$
$J_{\underset{n}{0\cdots}20}$
X
$J_{0\cdots22}$
$A_{\underset{n}{0\cdots}2}$
$J_{0\cdots20}$
$J_{\underset{n}{0\cdots}2}$
$A_{\underset{n+1}{0\cdots}02}$
$J_{0\cdots02}$
$J_{\underset{n}{0\cdots}0}$
$J_{\underset{n+1}{0\cdots}00}$
(00)
$A_{0\cdots002}$
$J_{0\cdots000}$
0
$\frac{1}{2^{n+2}}$
$\frac{1}{2^{n+1}}$
$\frac{1}{2^n}$
f_n
$I_{0\cdots2}$
D_1
D_2
$D_{2^{n+1}}$
Y
(0,0)
$I_{\underset{n+2}{0\cdots}0}$
$I_{\underset{n+1}{0\cdots}0}$
$I_{\underset{n}{0\cdots}0}$
$\frac{1}{2^n}$
$\frac{1}{2^{n+1}}$

Figure I.23.2

In Y the sets D_i are the intervals $I_{a_1 \ldots a_{k+n}}$ where $\sum_{i=1}^{k+n} a_i \neq 0$ and $k \geqq 1$.

Notice that since f_n differs from r only at points (s,t) with $s < 1/2^n$, $d(f,r) < 1/2^n + 1/3^n$. Since $\operatorname{diam} A_{a_1 \ldots a_k} \leqq 1/3^k + 1/2^k \leqq 1/3^n + 1/2^n$ for $k \geqq n$ and f_n has singleton point inverses for $k < n$, it follows that $\operatorname{diam} f_n^{-1}(y) \leqq 1/3^n + 1/2^n < 1/2^{n-1}$. Thus f_n is a $1/2^{n-1}$ refinement of r which shows that r must be a refinable map. It is clear that X has uncountably many components while Y has countably many components.

An easy consequence of Theorem I.22 and Corollaries I.22.1 and I.22.2 is that if X is a compact metric space, r is a refinable map from X onto a metric space Y, and either X or Y is locally connected, then r is a component preserving map, and the restriction of r to any component of X is a refinable map. This fact is important in the sequel where the spaces to be studied will be locally connected. With this property the proofs of facts concerning these spaces can be reduced to the case where the spaces are connected.

This chapter will be concluded with some properties of products, finite compositions, and cones of refinable maps.

I.24 Theorem. For each natural number i, let each of X_i and Y_i be a compact metric space. Suppose that for each natural number i the map r_i from X_i onto Y_i is a refinable map. Then $r = \Pi r_i$ which maps the compact metric space ΠX_i onto ΠY_i is a refinable map.

Proof: Let d_i represent the metric on each of X_i and Y_i, and let $d = \Sigma d_i/2^i$ represent the metric on each of the product spaces ΠX_i and ΠY_i.

Let $\varepsilon > 0$ be given. For each i, let f_i be an $\varepsilon/2$-refinement of r_i. Then $\Pi f_i = f$ satisfies the following properties:

(a) $d(f,r) < \varepsilon$.

(b) If $f((x_i)) = (y_i) = f((z_i))$ then $d((x_i),(z_i)) \leq \varepsilon/2$.

Proof of (a): $d(f,r) = \Sigma d_i(f_i,r_i)/2^i \leq \varepsilon/(2\cdot 2^i) \leq \varepsilon/2 < \varepsilon$.

Proof of (b): If $\Pi f_i(x_i) = (y_i) = \Pi f_i(z_i)$ then each of x_i and y_i belongs to $f_i^{-1}(y_i)$. Since f_i is an $\varepsilon/2$-map, $d_i(x_i,z_i) < \varepsilon/2$. Thus, $d((x_i),(z_i)) = \Sigma d_i(x_i,z_i)/2^i \leq \varepsilon/(2\cdot 2^i) \leq \varepsilon/2$.

Property (b) shows that f is an ε-map while property (a) shows that f is ε-near r. Since $\varepsilon > 0$ is arbitrary, r is a refinable map.

I.24.1 *Corollary*. Finite products of refinable maps on products of metric compacta are refinable.

Proof: Let X_i and Y_i be single points except for finitely many i and apply Theorem I.24.

I.25 *Theorem*. The finite composition of refinable maps is refinable.

Proof: Let r from the compact metric space X onto the metric space Y be a refinable map, and let s from Y onto the metric space Z be a refinable map. Then the claim of the theorem is that sr from X onto Z is a refinable map.

Let $\varepsilon > 0$ be given. Then there is a $\delta > 0$ such that for y_1 and y_2 in Y, $d(y_1,y_2) < \delta$ implies $d(s(y_1),s(y_2)) < \varepsilon/3$. Let f from X onto Y be a $\min\{\delta/3,\varepsilon/3\}$-refinement of r.

For each y in Y there is a $\gamma_y > 0$ such that $f^{-1}(N_{\gamma_y}(y))$ is a subset of $N_{\varepsilon/3}(f^{-1}(y))$, since $\{f^{-1}(y) \mid y \text{ belongs to } Y\}$ is an upper semicontinuous decomposition of X.

Since Y is compact there are points $y_1,y_2,\ldots,y_n$ in Y such that $\{N_{(\gamma_{y_i})/2}(y_i)\}$ covers Y. Let $\gamma = \min(\gamma_{y_i})/2$. Let g from Y onto Z be a $\min\{\delta/3,\varepsilon/3\}$-refinement of s.

Clearly gf maps X onto Z. Now the following inequalities hold: $d(gf(x),sr(x)) \leq d(gf(x),sf(x)) + d(sf(x),sr(x)) < \varepsilon/3 + \varepsilon/3$ since $d(f(x),r(x)) < \delta/3 < \delta$ and g is $\varepsilon/3$-near s. Thus gf is ε-near sr. Also it is claimed that for each z in Z, $\operatorname{diam}(gf)^{-1}(z) < \varepsilon$. To verify this assume that $gf(x_1) = z = gf(x_2)$ for x_1 and x_2 in X. Then since g is a γ-map $d(f(x_1),f(x_2)) < \gamma$. Now there is a y_i such that $d(f(x_1),y_i) < \gamma_{y_i}/2$; hence, $d(f(x_2),y_i) < d(f(x_1),f(x_2)) + d(f(x_1),y_i) < \gamma_{y_i}/2 + \gamma < \gamma_{y_i}$. Thus $d(x_1,f^{-1}(y_i)) < \varepsilon/3$ and $d(x_2,f^{-1}(y_i)) < \varepsilon/3$. Since $\operatorname{diam} f^{-1}(y_i) < \varepsilon/3$ (f is an $\varepsilon/3$-map), $d(x_1,x_2) < d(x_1,f^{-1}(y_i)) + \operatorname{diam} f^{-1}(y_i) + d(x_2,f^{-1}(y_i)) < \varepsilon$. Hence, $\operatorname{diam}(gf)^{-1}(z) < \varepsilon$ for all z in Z. Thus gf is an ε-refinement of sr. Since $\varepsilon > 0$ is arbitrary, sr is a refinable map.

The next theorem will show that the cone of a refinable map between compact metric spaces is a refinable map between the cones of the spaces. In order to prove this fact we will first prove the result for the geometric cone of each of the spaces embedded in the Hilbert cube $Q = \Pi I_i$ ($I_i = [-1,1]$ for each i). To do this we will adopt the following notational conventions. If X is a compact metric space then X may be embedded in Q_2 by an embedding h_X where $Q_2 = \prod_{i \geq 2} I_i$. Let $G(X,h_X)$ be the subset of Q which is the union of all segments from the origin to points of the form $(1,h_X(x))$ where x belongs to X.

There is a homeomorphism φ_X from the cone of X, $C(X)$, to $G(X,h_X)$ which is given by sending the point $\langle x,t\rangle$ in $C(X)$ to $(1 - t,(1 - t)h_X(x))$. In this case, t belongs to $I = [0,1]$ and $\langle x,t\rangle$ is the equivalence class of (x,t) where all the points with $t = 1$ have been identified.

I.26 <u>Theorem</u>. Let each of X and Y be a compact metric space. If r from X onto Y is a refinable map then $C(r)$ from $C(X)$ to $C(Y)$ is surjective and is also a refinable map.

<u>Proof</u>: $C(r)$ is surjective since $r \times \text{id } I$ from $X \times I$ to $Y \times I$ is surjective, i.e., $\langle y,t\rangle = \langle r(x),t\rangle$ for some x in X because r is onto Y.

Since each of X and Y is a compact metric space there are embeddings h_X of X into Q_2 and h_Y of Y into Q_2. Let φ_X and φ_Y be the homeomorphisms between the cones and geometric cones of X and Y . Since homeomorphisms are obviously refinable maps and since r is a refinable map, $s = h_Y r h_X^{-1}$ from $h_X(X)$ onto $h_Y(Y)$ is a refinable map according to Theorem I.25.

We now claim that $G(s)$ from $G(X,h_X)$ onto $G(Y,h_Y)$ defined by taking $(u,uh_X(x))$ to $(u,ush_X(x))$ is a refinable map. Note that $sh_X(x) = h_Y(r(x))$ assuring the fact that $G(s)$ is onto.

Let $\varepsilon > 0$ be given. Let f from $h_X(X)$ onto $h_Y(Y)$ be an ε-refinement of s . Define $G(f)$ from $G(X,h_X)$ onto $G(Y,h_Y)$ by sending $(u,uh_X(x))$ to $(u,ufh_X(X))$. Now
$d(G(f)(u,uh_X(x)),G(s)(u,uh_X(x))) = d((u,ufh_X(x)),(u,ush_X(x))) =$
$ud(fh_X(x),sh_X(x)) \leqq d(fh_X(x),sh_X(x)) < \varepsilon$. Thus $G(f)$ is ε-near $G(s)$. Further if $G(f)(u_1,u_1h_X(x_1)) = G(f)(u_2,u_2h_X(x_2))$ then
$(u_1,u_1f(h_X(x_1))) = (u_2,u_2f(h_X(x_2)))$. Hence $u_1 = u_2$. If $u_1 \neq 0$ then $f(h_X(x_1)) = f(h_X(x_2))$ so that $d(h_X(x_1),h_X(x_2)) < \varepsilon$ since f is an ε-map. In that case $d(u_1,u_1h_X(x_1)),(u_1,u_2h_X(x_2)) =$
$u_1d(h_X(x_1),h_X(x_2)) < u_1\varepsilon < \varepsilon$. If $u_1 = 0$ then $(u_1,u_1h_X(x_1)) =$ origin $= (u_2,u_2h_X(x_2))$. In either case the distance is less than ε . Hence, $G(f)$ is an ε-map. This proves the claim.

Now $C(r) = \varphi_Y^{-1}G(s)\varphi_X$. Since $C(r)$ is the composition of refinable maps, $C(r)$ is a refinable map according to Theorem I.25.

CHAPTER II

SPACES SATISFYING THE LONČAR-MARDEŠIĆ LEMMA

In the third section of their paper on refinable maps, Ford and Rogers [Fo and Rg] prove a number of interesting results concerning the image of an ANR under a refinable map. These results are based on the following lemma which appears in a paper by Lončar and Mardešić.

Lončar-Mardešić Lemma. Let X be a metric continuum, let A be an ANR, and let f be a map of X onto A. Then for every positive real number δ there is a positive real number ε such that if g is an ε-map from X onto a metric space Y, then there is a map h from Y onto A such that $d(hg,f) < \delta$ [Lo and Ma, Lemma 1, pp.41-42].

This lemma was first proved for the case when A is a polyhedron [Ma and Se]. As will be shown later, A can be replaced by a member of an even larger class of spaces. For the present any compact metric space which can replace A in the above lemma will be called a connected Lončar-Mardešić space (connected L-M space).

In general by a Lončar-Mardešić space will be meant a compact metric space A such that if f is a component preserving map from a compact metric space X onto A, then for every $\delta > 0$ there is an $\varepsilon > 0$ such that if g is an ε-map of X onto a metric space Y, then there is a map h from Y onto A such that $d(hg,f) < \delta$.

We will now prove some properties of L-M spaces and show how the Ford and Rogers' results [Fo and Rg, Chapter 3] still hold when the domain of a refinable map is an L-M space.

II.1 Theorem. Every L-M space is locally connected.

Proof: Let A be an L-M space and f be the identity map from A onto itself. Let $\delta = 1$. Since A is an L-M space there is an $\varepsilon > 0$ such that if g from A onto a metric space Y is an ε-map then there is a map h from Y onto A such that $d(hg, \text{id } A) < 1$.

Since A is a compact metric space there is a finite polyhedron N and an ε-map g from A onto N [Hz and Wa, p.72-73]. Thus, there is a map h from N onto A. Since N is locally connected and h is an identification (A is compact), $h(N) = A$ is locally connected [Du, p.125].

Since every finite polyhedron is an ANR, and since 1 in the above proof can be replaced by an arbitrary $\delta > 0$ the following corollary holds.

II.1.1 Corollary. If A is an L-M space then for every $\delta > 0$ there exist an ANR N and maps g from A onto N, h from N onto A such that $d(hg, \text{id } A) < \delta$.

II.2 <u>Lemma</u>. Every component of an L-M space is an L-M space.

<u>Proof</u>: Let A be an L-M space, and let H be a component of A. Since A is locally connected and compact, H is both open and closed. Hence, $A - H$ is closed; therefore, it is compact. Since $A - H$ and H do not intersect, and both are compact, $\eta = d(H, A - H) > 0$.

Let $\delta > 0$ be given and let f be a map of a continuum X onto H. Let X^* be the disjoint union of X with $A - H$. Let f^* from X^* onto A be the map which when restricted to H is f and the identity otherwise.

Clearly f^* is component preserving. Since A is an L-M space there is an $\varepsilon > 0$ such that if g^* is an ε-map from X^* onto a metric space Y^*, then there is an h^* from Y^* onto A with $d(h^*g^*, f^*) < \min\{\delta, \eta\}$. Let g be an ε-map from X onto a metric space Y. By letting each of X^* and Y^* be the disjoint union of the respective space with $A - H$, and by letting g^* be the map which is g on X and the identity on $A - H$, there is a map h^* from Y^* onto A with $d(h^*g^*, f^*) < \min\{\delta, \eta\}$. Let $h = h^*|Y$. If x is in X then $d(hg(x), f(x)) < \min\{\delta, \eta\}$. Since $f(x)$ is in H and $d(hg(x), f(x)) < d(H, A - H)$, $hg(x)$ belongs to H.

All that remains is to show that h is onto H. Let y belong to H. Then for some x in X^*, $h^*g^*(x) = y$. Since $d(f^*(x), h^*g^*(x)) < \eta$, $f^*(x)$ belongs to H. Now x does not belong to $A - H$ since on $A - H$, f^* is the identity map. Thus, x belongs to X. Since $g^*(x) = g(x)$, $g(x)$ belongs to Y. Thus, $g(x)$ is an element of Y which is mapped to y by $h = h^*|Y$. Thus, it may be concluded that H is a connected L-M space.

II.3 Lemma. Let X be a compact metric space with at most a finite number of components. Then there is a positive real number η such that any η-map from X onto a metric space Y is component preserving.

Proof: Let X be as in the hypothesis of the statement of the lemma. Let $\eta = \min\{d(K,L) \mid K \neq L,\ K \text{ and } L \text{ belong to } k(X)\}$.

Suppose f is an η-map from X onto a metric space Y. By Lemma I.18 it suffices to show $f^{-1}(H)$ is a component of X whenever H is a component of Y. Since f is onto Y there is an x in X such that $f(x)$ belongs to H. Hence, there is a component K of X containing x; then, $f(K)$ is a subset of H.

Now it is claimed that $f^{-1}f(K) = K$. Suppose $f(x) = f(k)$ for some x in X and k in K. Since f is an η-map $d(x,k) < \eta \leq d(K,L)$ where L is any other component of X. Hence, x must belong to the component K.

Since f is a closed map $f(K)$ is closed. Since $K = f^{-1}f(K)$ is open, and f is an identification, $f(K)$ is also open. Thus $f(K)$ is a nonempty open and closed set contained in the component H; hence, $f(K) = H$. From these facts it follows that $K = f^{-1}(H)$. Thus, according to Lemma I.18, f is component preserving.

Before stating the next theorem, the following notation from [Fo and Rg] will be stated. If f is a map from X onto Y and $\varepsilon > 0$, then $L(f,\varepsilon) = \sup\{c \mid$ if H is a subset of X and $\operatorname{diam} H < c$, then $\operatorname{diam} f(H) < \varepsilon\}$. Since X is assumed to be compact, f is uniformly continuous; hence, $L(f,\varepsilon) > 0$ [Fo and Rg, p.5].

II.4 Theorem. Suppose r is a refinable map from a space X onto a space Y , either X or Y has a finite number of components, $\eta > 0$ and $\varepsilon > 0$. Then there is a positive real number δ such that if f is a δ-map from X onto an L-M space A , then there exist an η-map g from X onto Y ε-near r and an ε-map h from Y onto A such that $d(f,hg) < \varepsilon$ [Fo and Rg, Theorem 3, p.5].

Proof: Since either X or Y has a finite number of components, by Corollary I.22.1 or by Theorem I.22, r is a component preserving map. Furthermore, by Corollary I.22.2, $r|K$ where K is a component of X is a refinable map. Since $k(X)$ is finite, $\gamma = \min\{d(H,L) \mid H$ and L are distinct components of $X\}$ is a positive real number.

According to Lemma II.3, by choosing δ smaller than γ one can assure that a δ-map f from X onto an L-M space A is component preserving. Hence, the theorem need be proved only for each component of X . For this reason it is assumed that X is connected.

Suppose r is a refinable map from X onto Y , X is connected, $\eta > 0$ and $\varepsilon > 0$. Let $\delta = L(r,\varepsilon/4) \leqq L(r,\varepsilon)$. Suppose f from X onto A is a δ-map where A is an L-M space.

It is now claimed that there is a $c_1 > 0$ such that if $d(g,f) < c_1$ where g is a map of X onto A , then g is a δ-map. Since A is compact $\sup\{\operatorname{diam} f^{-1}(a) \mid a$ belongs to $A\} < \delta$. Let

δ' be given by the formula: $\delta' = \delta - \sup\{\text{diam } f^{-1}(a) \mid a \text{ belongs to } A\}$. Now since the collection $\{f^{-1}(a) \mid a \text{ belongs to } A\}$ is an upper semi-continuous decomposition of X there is an $\varepsilon' > 0$ such that if a_1 and a_2 belong to A , and if $d(a_1,a_2) < \varepsilon'$, then for some a in A , $f^{-1}(a_1)$ union $f^{-1}(a_2)$ is contained in $N_{\delta'/3}(f^{-1}(a))$. Hence, if x_1 belongs to $f^{-1}(a_1)$ and if x_2 belongs to $f^{-1}(a_2)$, then $d(x_1,x_2) < \delta'/3 + \text{diam } f^{-1}(a) + \delta'/3$ for some a in A . Let $c_1 = \min\{\varepsilon'/2,\varepsilon\}$. If g from X onto A is a map which is c_1-near f , then g will be a δ-map. For if $g(x_1) = g(x_2)$, then for $i = 1,2$, $d(g(x_i),f(x_i)) < \varepsilon'/2$; so that $d(f(x_1),f(x_2)) < \varepsilon'$; hence, $d(x_1,x_2) < \delta$.

Since A is an L-M space there is a $c_2 > 0$ such that if g is any c_2-map from X onto Y then there is a map h from Y onto A such that $d(f,hg) < c_1$.

Since r is a refinable map there is a $\min\{\eta,c_2\}$-map g from X onto Y $\varepsilon/4$-near r . Hence, there is a map h from Y onto A so that $d(f,hg) < c_1$. Thus hg is a δ-map. Since $c_1 \leqq \varepsilon$, $d(f,hg) < \varepsilon$. Also g is clearly an η-map ε-near r .

To complete the proof it is necessary to show that h is an ε-map. Suppose $h(y_1) = h(y_2)$ where each of y_1 and y_2 belongs to Y . Then $y_1 = g(x_1)$ for some x_1 in X , and $y_2 = g(x_2)$ for some x_2 in X . Since $hg(x_1) = hg(x_2)$, and since hg is a δ-map, $d(x_1,x_2) < \delta = L(r,\varepsilon/4)$. Thus, $d(r(x_1),r(x_2)) < \varepsilon/4$. Since $d(g,r) < \varepsilon/4$, it follows that $d(g(x_1),g(x_2)) \leqq d(g,r) + d(r(x_1),r(x_2)) + d(r,g) < 3\varepsilon/4 < \varepsilon$. Therefore, h is an ε-map from Y onto A .

Theorem II.4 is precisely the theorem that Ford and Rogers use to prove the following results which will now be stated for L-M spaces.

II.4.1 Corollary. If X is an L-M space and r is a refinable map from X onto Y, then X is Y-like and Y is X-like [Fo and Rg, Corollary 3.2, p.6].

II.4.2 Corollary. If X is an L-M space, r is a refinable map from X onto a metric space Y and $\varepsilon > 0$, then there exist ε-maps g_1 from X onto Y and g_2 from Y onto X such that:

(i) $d(r,g_1) < \varepsilon$

(ii) $d(rg_2, \mathrm{id}\ Y) < \varepsilon$

(iii) $d(g_1g_2, \mathrm{id}\ Y) < \varepsilon$ and $d(g_2g_1, \mathrm{id}\ X) < \varepsilon$

[Fo and Rg, Corollary 3.3, p.6].

Note: If $d(g_ig_j, \mathrm{id}) < \varepsilon/2$ then g_j must be an ε-map. Thus, the conclusion that g_1 and g_2 are ε-maps follows from part (iii) using $\varepsilon/2$.

These results can be pictured using "ε-commuting" diagrams. Consider the following diagram where each of the spaces is compact-metric.

$$\begin{tikzcd} X \arrow[r, two heads, "f"] \arrow[d, two heads, "g"] & Y \\ Z \arrow[ur, two heads, "h"'] & \end{tikzcd}$$

II.5 Definition. For $\varepsilon > 0$ the above diagram will be said to ε-commute if $d(f,hg) < \varepsilon$. This will be symbolized by:

$$\begin{tikzcd} X \arrow[r, two heads, "f"] \arrow[d, two heads, "g"'] \arrow[dr, phantom, "\varepsilon"] & Y \\ Z \arrow[ur, two heads, "h"'] & \end{tikzcd} .$$

Thus, Theorem II.4 states that if either X or Y has a finite number of components, and if r is a refinable map of X onto Y , i.e., $X \xrightarrow[\text{ref.}]{r} Y$, then for $\varepsilon > 0$ there is a $\delta > 0$ such that if f is a δ-map onto an L-M space A there is a map g close to r and a map h so that the following ε-commuting diagram holds:

$$\begin{tikzcd} X \arrow[r, two heads, "g"] \arrow[d, two heads, "f"'] \arrow[dr, phantom, "\varepsilon"] & Y \\ A \arrow[ur, two heads, "h"'] & \end{tikzcd} .$$

By choosing g close enough to r the following diagram is valid:

$$\begin{tikzcd} X \arrow[r, two heads, "r"] \arrow[d, two heads, "f"'] \arrow[dr, phantom, "\varepsilon"] & Y \\ A \arrow[ur, two heads, "h"'] & \end{tikzcd} .$$

Corollary II.4.2 gives the existence of the following diagrams:

$$\begin{tikzcd} X \arrow[r, two heads, "\mathrm{id}\ X"] \arrow[d, two heads, "r"'] \arrow[dr, phantom, "\varepsilon"] & X \\ Y \arrow[ur, two heads, "g_2"'] & \end{tikzcd} , \quad \begin{tikzcd} X \arrow[r, two heads, "\mathrm{id}\ X"] \arrow[d, two heads, "g_1"'] \arrow[dr, phantom, "\varepsilon"] & X \\ Y \arrow[ur, two heads, "g_2"'] & \end{tikzcd} , \quad \text{and} \quad \begin{tikzcd} Y \arrow[r, two heads, "\mathrm{id}\ Y"] \arrow[d, two heads, "g_2"'] \arrow[dr, phantom, "\varepsilon"] & Y \\ X \arrow[ur, two heads, "g_1"'] & \end{tikzcd} .$$

These diagrams are of interest for several reasons. Among these reasons is that if for two compacta like X and Y above, for every

$\varepsilon > 0$ there exist maps g_1 and g_2 such that the last two diagrams ε-commute, then X will be Y-like and Y will be X-like. This observation leads to the following definition.

II.6 Definition. Suppose each of X and Y is a compact metric space. X will be said to be (strongly) quasi-homeomorphic to Y provided that for every positive real number ε there exist maps g from X onto Y and h from Y onto X such that the following diagrams ε-commute:

$$
\begin{tikzcd}
X \arrow[r, two heads, "\mathrm{id}\ X"] \arrow[d, two heads, "g"'] \arrow[dr, phantom, "\varepsilon"] & X \\
Y \arrow[ur, two heads, "h"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \arrow[r, two heads, "\mathrm{id}\ Y"] \arrow[d, two heads, "h"'] \arrow[dr, phantom, "\varepsilon"] & Y \\
X \arrow[ur, two heads, "g"'] &
\end{tikzcd}
\quad .
$$

II.7 Definition. Let each of X and Y be a compact metric space. A sequence of ordered pairs of maps (g_i, h_i) such that for each i, g_i maps X onto Y and h_i maps Y onto X will be called a (strong) quasi-homeomorphism provided there is a null sequence of positive real numbers ε_i such that g_i and h_i ε_i-commute with $\mathrm{id}\ X$ and $\mathrm{id}\ Y$, i.e., $d(g_i h_i, \mathrm{id}\ Y) < \varepsilon_i$ and $d(h_i g_i, \mathrm{id}\ X) < \varepsilon_i$.

II.8 Definition. Let Ξ be a collection of compact metric spaces. A compact metric space is said to be quasi-Ξ if and only if for every $\varepsilon > 0$ there is a member of Ξ and there are maps which ε-commute with the identity on the space and on the member of Ξ. It is further required that the maps be surjective. Thus, a space X is quasi-Ξ if for every $\varepsilon > 0$ there exist a C in Ξ, a map g from

X onto C, and a map h from C onto X such that $d(hg, \mathrm{id}\, X) < \varepsilon$.

It is clear that if a space is quasi-Ξ then it is Ξ-like. The converse, however, is not true as the next example illustrates.

II.9 <u>Example</u>. Let X be the graph of $y = \sin(1/x)$, $0 < x \leqq 2\pi$, together with the limiting arc $\{0\} \times [-1,1]$. Let Ξ be the class consisting of the interval $[0,2\pi]$. It is clear that X is Ξ-like. For given $\varepsilon > 0$ simply project the part of $y = \sin(1/x)$ where $x \leqq x_\varepsilon$ and x_ε satisfies $0 < x_\varepsilon < \varepsilon$ and $1/x_\varepsilon$ is of the form $2\pi n + \pi/2$ for some integer n, onto $\{0\} \times [-1,1]$; then follow this by a linear map onto $[0,x_\varepsilon]$ so that $(0,-1)$ goes to 0 and $(0,1)$ goes to x_ε. On the rest of X project X onto the X-axis. Since X is not locally connected, X cannot be the image of $[0,2\pi]$. Hence X is not quasi-Ξ.

II.10 <u>Lemma</u>. A sequence of ordered pairs of maps (g_i,h_i) is a quasi-homeomorphism of X onto Y if and only if g_i maps X onto Y, h_i maps Y onto X, and for every $\varepsilon > 0$ there is an i_o such that for all $i \geqq i_o$, $d(g_ih_i, \mathrm{id}\, Y) < \varepsilon$ and $d(h_ig_i, \mathrm{id}\, X) < \varepsilon$.

<u>Proof</u>: Suppose the sequence of ordered pairs (g_i,h_i) form a quasi-homeomorphism of X onto Y. Let $\varepsilon > 0$ be given. Let ε_i be the associated null sequence. Since $\lim \varepsilon_i = 0$ there is an i_o such that $\varepsilon_{i_o} < \varepsilon$ and if $i \geqq i_o$ then $\varepsilon_i < \varepsilon_{i_o}$. If $i \geqq i_o$ then according to the definition of a quasi-homeomorphism $d(h_ig_i, \mathrm{id}\, X) < \varepsilon_i < \varepsilon$ and $d(g_ih_i, \mathrm{id}\, Y) < \varepsilon_i < \varepsilon$.

For the proof of the converse let (g_i, h_i) be a sequence of ordered pairs of maps which satisfies the conditions given in the statement of the lemma. Form the required null sequence by letting $\varepsilon_i = \max\{d(h_i g_i, \mathrm{id}\ X), d(g_i h_i, \mathrm{id}\ Y), 1/2^i\}$.

II.11 Lemma. Suppose the sequence of ordered pairs of maps (h_i, g_i) is a quasi-homeomorphism of a compact metric space X onto a metric space Y . If $\lim h_i$ exists and is a map h from X into Y then h is onto and for every positive real number ε there is a natural number i_o such that for $i \geqq i_o$, $d(h_i, h) < \varepsilon$, $d(hg_i, \mathrm{id}\ Y) < \varepsilon$, $d(h_i g_i, \mathrm{id}\ Y) < \varepsilon$, $d(g_i h_i, \mathrm{id}\ X) < \varepsilon$, each of h_i and g_i is an ε-map, and h is a refinable map.

Proof: It is well-known that if the map h is the limit of surjective maps h_i then h must be surjective.

It will now be shown that h is a refinable map. Let $\varepsilon > 0$ be given. According to Lemma II.10 there is a natural number i_1 such that for all natural numbers $i \geqq i_1$, $d(g_i h_i, \mathrm{id}\ X) < \varepsilon/2$ and $d(h_i g_i, \mathrm{id}\ Y) < \varepsilon/2$. Hence each of h_i and g_i is an ε-map. Since h is the limit of the h_i it is clear that h is a refinable map.

Since $\lim h_i = h$ there is also an i_2 such that for $i \geqq i_2$ $d(h_i, h) < \varepsilon/2$. Let $i_o = \max\{i_1, i_2\}$. If $i \geqq i_o$ then the following inequalities hold: $d(hg_i, \mathrm{id}\ Y) \leqq d(hg_i, h_i g_i) + d(h_i g_i, \mathrm{id}\ Y) < \varepsilon$. This shows that the lemma holds.

If in the above lemma, $\lim g_i$ exists and is a map g then the following situation occurs. Given $\varepsilon > 0$ there is a $\delta > 0$ such that if each of x_1 and x_2 belongs to X and $d(x_1, x_2) < \delta$ then

$d(h(x_1),h(x_2)) < \varepsilon$. Now for δ there is an i such that $d(g_i, \lim g_i = g) < \delta$ and $d(hg_i, \text{id } Y) < \varepsilon$. Thus, $d(hg, \text{id } Y) \leqq d(hg, hg_i) + d(hg_i, \text{id } Y) < 2\varepsilon$. Since $\varepsilon > 0$ is arbitrary $hg = \text{id } Y$. A symmetric argument shows that $gh = \text{id } X$. Thus, h is a homeomorphism. Thus, the following corollary holds.

II.11.1 Corollary. If the sequence of ordered pairs of maps (h_i, g_i) is a quasi-homeomorphism from a compact metric space X onto a metric space Y , and further if $\lim h_i = h$ exists and is a map and $\lim g_i = g$ exists and is a map, then $h : X \to Y$ and $g : Y \to X$ are inverse homeomorphisms.

II.12 Definition. Let Ξ be the class of compact absolute neighborhood retracts. Any space which is quasi-Ξ will be called a quasi-ANR (or q-ANR).

II.13 Lemma. Every connected quasi-ANR is an L-M space.

Proof: Let A be a connected quasi-ANR. Let f be a map from a metric continuum X onto A . Let $\varepsilon > 0$ be given. Then since A is a connected quasi-ANR there are a compact connected ANR L , a map g from A onto L , and a map h from L onto A such that $d(hg, \text{id } A) < \varepsilon/2$.

By the Lončar-Mardešić lemma [Lo and Ma, Lemma 1, p.41-42], L is an L-M space. Now gf is a map of X onto L . Since L is compact there is an $\eta > 0$ such that if $d(\ell_1, \ell_2) < \eta$ in L then $d(h(\ell_1), h(\ell_2)) < \varepsilon/2$. Since L is an L-M space there is a $\delta > 0$ such that if Π_1 is a δ-map from X onto a metric space Y then there is a map Π_2 from Y onto L such that $d(\Pi_2\Pi_1, gf) < \eta$. In this case it follows that $d(h\,\Pi_2\Pi_1, hgf) < \varepsilon/2$. Since $d(hgf,f) < \varepsilon/2$ it follows that $d(h\,\Pi_2 \circ \Pi_1, f) < \varepsilon$. Since $h\,\Pi_2$ maps Y onto A these facts show that A is an L-M space.

II.14 Lemma. If every component of a compact metric space with at most a finite number of components is an L-M space then the given space is an L-M space. (This is the converse of Lemma II.2.)

Proof: Let A be a compact metric space with at most a finite number of components so that each component is an L-M space. Let f be a component preserving map from a compact metric space X onto A . Let $\delta > 0$ be given. Further, let $\eta = \min\{d(K,L) \mid K \neq L$ and each of K and L belongs to $k(X)\}$.

Since each component of A is an L-M space, and since f is component preserving, for every K in $k(X)$ there is an $\varepsilon_K > 0$ such that given an ε_K-map g of K onto a metric space H , there is a map h from H onto $f(K)$ such that $d(hg, f|K) < \delta$. Let ε be the minimum of η and the ε_K . If g is an ε-map of X onto a metric space Y , then g is an η-map and $g|K$ is an ε_K-map.

According to Lemma II.3, g will be component preserving. Since $g|K$ is an ε_K-map, for each component K there is a map h_K from $g(K)$ onto $f(K)$ such that $d(h_K g|K, f|K) < \delta$. Define h from Y onto A by $h|g(K) = h_K$. Then clearly h is a map from Y onto A and $d(hg,f) < \delta$. Thus, A is an L-M space.

II.15 Lemma. Each component of a quasi-ANR is a quasi-ANR.

Proof: Since a quasi-ANR must be the continuous image of a compact ANR it is clear that a quasi-ANR is locally connected. Since it will also be compact, a quasi-ANR will have at most a finite number of components.

Let X be a quasi-ANR. By Lemma II.3 there is an $\eta > 0$ such that any η-map from X onto a metric space Y will be component preserving. Let $\varepsilon > 0$ be given. By the definition of a quasi-ANR there exist an ANR A, a map g from X onto A, and a map h from A onto X such that $d(hg, \mathrm{id}\ X) < \min\{\varepsilon, \eta/2\}$. It is clear that g is an η-map; hence, it is component preserving. Also, hg is an η-map; hence, hg is component preserving; furthermore, $hg(K) = K$ for all K in $k(X)$.

If K is a component of X then the following ε-commuting diagram is valid:

$$\begin{tikzcd} K \arrow[r, two heads, "\mathrm{id}\ K"] \arrow[d, two heads, "g"'] \arrow[dr, phantom, "\varepsilon"] & K \\ g(K) \arrow[ur, two heads, "h"'] & \end{tikzcd} .$$

Since $g(K)$ is a component of an ANR, $g(K)$ is an ANR. Since ε is arbitrary K must be a quasi-ANR.

II.16 <u>Theorem</u>. Let X be a compact metric space. X is an L-M space if and only if X is a quasi-ANR.

<u>Proof</u>: Suppose X is an L-M space. According to Corollary II.1.1, X satisfies the conditions for being a quasi-ANR.

On the other hand suppose X is a quasi-ANR. Then X is locally connected, and by Lemma II.15 each of its components is a connected quasi-ANR. By Lemma II.13 each component of X is an L-M space. Since X is compact and locally connected, X has at most a finite number of components. Hence, by Lemma II.14, X is an L-M space.

We now have the following corollaries.

II.16.1 Corollary. Every component of a compact metric space with at most a finite number of components is a q-ANR if and only if the entire space is a q-ANR.

II.16.2 Corollary. Suppose r is a refinable map from X onto Y, either X or Y has at most a finite number of components, $\eta > 0$ and $\varepsilon > 0$; then there is a positive real number δ such that if f is a δ-map from X onto a q-ANR A, then there exist an η-map g from X onto Y ε-near r and an ε-map h from Y onto A such that $d(f,hg) < \varepsilon$ [Fo and Rg, p.5].

II.16.3 Corollary. If X is a q-ANR, and r is a refinable map from X onto Y then X is Y-like and Y is X-like [Fo and Rg, p.6].

II.16.4 Corollary. If X is a quasi-ANR and r is a refinable map from X onto Y, then there is a (strong) quasi-homeomorphism of X to Y by a sequence of ordered pairs (g_i,h_i) where $\lim g_i = r$. (Compare this with Corollary 3.3 of [Fo and Rg, p.6].)

II.16.5 Corollary. A map r from a q-ANR X onto a space Y is refinable if and only if, for every positive real number ε, there is a map f from X onto itself such that rf is an ε-map ε-near r.

This is a form of Bing's shrinking criterion for refinable maps [Fo and Rg, p.7].

Proof: The condition, for each $\varepsilon > 0$ there is a map f from X onto itself such that rf is an ε-refinement of r, clearly implies that r is a refinable map.

On the other hand suppose r from X onto Y is a refinable map. It is now claimed that there is an $\varepsilon/2$-map h from X onto Y $\varepsilon/2$-near r, and a positive real number $\delta < \varepsilon$ such that if k from Y onto itself is a δ-map then kh is an $\varepsilon/2$-map.

Let h from X onto Y be an $\varepsilon/6$-map $\varepsilon/6$-near r. Since the collection $\{h^{-1}(y) \mid y$ belongs to $Y\}$ is an upper semicontinuous decomposition of X there is a positive real number $\delta < \varepsilon/6$ such that if for y_1 and y_2 in Y, $d(y_1,y_2) < \delta$, then each of $h^{-1}(y_1)$ and $h^{-1}(y_2)$ is contained in $N_{\varepsilon/6}(h^{-1}(y))$ for some y in Y. Now if k from Y onto itself is a δ-map then $\operatorname{diam} h^{-1}k^{-1}(y) < \varepsilon/2$ for all y in Y. For if $kh(x_1) = kh(x_2)$ then $d(h(x_1),h(x_2)) < \delta$. Thus, for some y in Y each of x_1 and x_2 belongs to $N_{\varepsilon/6}(h^{-1}(y))$; hence, $d(x_1,x_2) \leqq d(x_1,h^{-1}(y)) + \operatorname{diam} h^{-1}(y) + d(h^{-1}(y),x_2) < \varepsilon/2$.

According to Corollary II.16.4 and Lemma II.11 there is a $\delta/2$-map g from Y onto X such that $d(rg,\mathrm{id}\ Y) < \delta/2$. It follows that rg is a δ-map from Y onto itself. Hence, rgh from X onto Y is an $\varepsilon/2$-map. Also $d(r,rgh) \leqq d(r,h) + d(h,rgh) < \varepsilon/2 + \varepsilon/2 = \varepsilon$. Let $f = gh$. Then f maps X onto itself and rf is an ε-map ε-near r .

II.17 Theorem. A compact metric space which is quasi-homeomorphic to a quasi-ANR is a quasi-ANR.

Proof: Let X be a compact metric space. Let the sequence of ordered pairs of maps (g_i, h_i) be a quasi-homeomorphism of X to a quasi-ANR A. Let $\varepsilon > 0$ be given.

There is a natural number i such that for the maps g_i from X onto A and h_i from A onto X, $d(h_i g_i, \text{id } X) < \varepsilon/2$. Since h_i is uniformly continuous there is a $\delta > 0$ such that whenever the elements a_1 and a_2 belong to A and $d(a_1, a_2) < \delta$ then $d(h_i(a_1), h_i(a_2)) < \varepsilon/2$. Since A is a quasi-ANR there are a compact ANR A_δ and maps f from A onto A_δ and k from A_δ onto A such that $d(kf, \text{id } A) < \delta$.

Consider the map fg_i from X onto A_δ and the map $h_i k$ from A_δ onto X. $d(h_i kfg_i, \text{id } X) \leq d(h_i kfg_i, h_i g_i) + d(h_i g_i, \text{id } X)$. Since $d(kf, \text{id } A) < \delta$, $d(h_i kfg_i, h_i g_i) < \varepsilon/2$. Hence, $d(h_i kfg_i, \text{id } X) < \varepsilon/2 + \varepsilon/2 = \varepsilon$. Since $\varepsilon > 0$ may be chosen arbitrarily, X is a quasi-ANR.

II.17.1 Corollary. The image of a q-ANR under a refinable map is a q-ANR.

Proof: According to Corollary II.16.4 a refinable map whose domain is a quasi-ANR induces a quasi-homeomorphism onto its image. It is clear that there is a quasi-homeomorphism of the image of the refinable map onto its domain which is a q-ANR. Hence, by Theorem II.17, the image is a q-ANR.

II.17.2 Corollary. The image of an ANR under a refinable map is a q-ANR.

Proof: Every ANR is clearly a q-ANR.

II.18 Definition. Let Ξ be the class of compact absolute retracts. A quasi-Ξ space is called a quasi-AR (q-AR).

II.19 Definition. A compact metric space X is called quasi-contractible provided for every positive real number ε, there is a homotopy F_t, where $0 \leqq t \leqq 1$, from X onto X such that F_0 is a constant map, F_1 is surjective, and $d(F_1, \text{id } X) < \varepsilon$.

II.18.1 Corollary. The image of a q-AR or AR under a refinable map is a q-AR. This is a corollary of the proof of Theorem II.17 where A_δ is an AR.

II.19.1 Corollary. A quasi-contractible ANR is an AR.

Proof: Let X be a quasi-contractible ANR. It will be shown that X is contractible. Since X is an ANR there is a positive real number δ such that if f from Y into X is a map δ-near the map g from Y into X then f is homotopic to g [Hu, Theorem 1.1, p.111]. Since X is quasi-contractible there is a homotopy F_t from X into X such that F_0 is a constant map, and $d(F_1, \text{id } X) < \delta$. Since $d(F_1, \text{id } X) < \delta$, F_1 is homotopic to $\text{id } X$. Thus, $\text{id } X$ is homotopic to a constant map in X. Therefore, X is contractible.

Since every contractible ANR is an AR it follows that X is an AR.

II.19.2 Corollary. Every q-AR is quasi-contractible.

Proof: Let X be a quasi-AR. Let $\varepsilon > 0$ be given. Since X is a q-AR there are an AR A, a map f from X onto A, and a map g from A onto X such that $d(gf, \mathrm{id}\, X) < \varepsilon$.

Since A is an AR there is a homotopy H_t from A into A such that H_0 is a constant map, say $H_0(a) = a_0$ for all a in A, and such that $H_1 = \mathrm{id}\, A$. Let $F_t = gH_tf$. F_t is a homotopy from X into X. $F_1 = gf$ so that F_1 is onto X and $d(F_1, \mathrm{id}\, X) < \varepsilon$. $F_0(x) = gH_0f(x) = g(a_0)$ for all x in X; thus, F_0 is a constant map. Since $\varepsilon > 0$ is arbitrary we have shown that X is quasi-contractible.

Corollary II.18.1 together with Corollary II.19.1 implies the following result.

II.19.3 Corollary. If the image of a quasi-AR under a refinable map is an ANR then the image is an AR.

For the next result we will use the following definitions from [Hu, pp.110 and 138].

II.20 Definition. For a positive real number ε a homotopy H_t into a space X is called an ε-homotopy provided that for every element a in the domain of H_t, $\operatorname{diam} H(\{a\} \times I) < \varepsilon$.

II.21 Definition. A space X is called an ε-dominating space of Y provided that there are maps f from X into Y and g from Y into X such that gf is ε-homotopic to $\mathrm{id}\, Y$. X is ε-homotopically equivalent to Y provided that there exist maps f and g as above such that gf is ε-homotopic to $\mathrm{id}\, Y$ and fg is ε-homotopic to $\mathrm{id}\, X$.

II.21.1 <u>Corollary</u>. Let X be a quasi-ANR, and let r from X onto a metric space Y be a refinable map. Then for every positive real number ε ,

(1) if X is an ANR, Y ε-dominates X ,

(2) if Y is ANR, X ε-dominates Y ,

(3) if each of X and Y is an ANR, X is ε-homotopically equivalent to Y .

Note: The first and third results can be found in [Fo and Rg, Corollary 3.4, p.7].

<u>Proof</u>: Let X , Y and r be as in the statement of the corollary. It follows from Corollary II.16.4 and Lemma II.11 that given $\delta > 0$ there exist maps h from X onto Y and g from Y onto X such that $d(h,r) < \delta$, $d(rg,\text{id } Y) < \delta$, $d(hg,\text{id } Y) < \delta$ and $d(gh,\text{id } X) < \delta$.

Let $\varepsilon > 0$ be given. Suppose (1) X is an ANR. Then there is a $\delta > 0$ such that δ-near maps from X into itself are ε-homotopic. Let h and g be the maps of the above paragraph corresponding to δ . Then gh is ε-homotopic to $\text{id } X$; hence, Y ε-dominates X .

Suppose (2) Y is an ANR. Then there is a $\delta > 0$ such that δ-near maps of Y into itself are δ-homotopic. Choose h and g as above for this δ . Then $d(hg,\text{id } Y) < \delta$; hence, hg and $\text{id } Y$ are ε-homotopic.

Suppose (3) each of X and Y is an ANR. By choosing δ as the smaller of the δ 's in the proofs of (1) and (2) we can guarantee that both hg is homotopic to $\text{id } Y$ and gh is homotopic to $\text{id } X$

by ε-homotopies. Hence, X and Y are homotopically equivalent.

A result related to part (3) of the preceding corollary is that a refinable map between ANR's is a CE map.

II.22.1 <u>Definition</u>. A metric space Z will be said to have the <u>shape of a point</u> provided that for every map h of Z into an ANR A, h is null-homotopic in A.

II.22.2 <u>Definition</u>. A map f from a space X into a space Y is a <u>CE map</u> if and only if for each y in $f(X)$, $f^{-1}(y)$ has the shape of a point [Ko, p.8].

The method of proof to show that a refinable map between compact ANR's must be CE is to show first that a refinable map between ANR's is a fine homotopy equivalence, and then apply a result due to Haver [Ha] which states that a map between ANR's is CE if and only if it is a fine homotopy equivalence. The following definition of fine homotopy equivalence is an adaptation to the case of compacta of the definition which appears in [Ch, p.91].

II.23 <u>Definition</u>. A map f from a compactum X onto a compactum Y is a <u>fine homotopy equivalence</u> if and only if for each positive real number ε there is a homotopy F_t from X into X such that fF_t is an ε-homotopy in Y, $F_0 = \text{id } X$, and $F_1 = gf$ for some map g from Y into X.

II.24 <u>Theorem</u>. (Haver) If a map between compact ANR's is a fine homotopy equivalence then it is CE.

<u>Proof</u>: Suppose f from a compact ANR X onto an ANR Y is a fine homotopy equivalence. Let y_0 be an element of Y. It is

necessary to show that $f^{-1}(y_0)$ has the shape of a point.

Let h be a map from $f^{-1}(y_0)$ into some ANR A. Since A is also an ANE (absolute neighborhood extensor), there are a neighborhood U of $f^{-1}(y_0)$ in X and a map φ from U into A which extends h. Since $\{f^{-1}(y) \mid y \text{ belongs to } Y\}$ is an upper semi-continuous decomposition of X, there is a positive real number ε such that $f^{-1}(N_\varepsilon(y_0))$ is contained in U.

Since f is a fine homotopy equivalence there are a map g from Y into X and a homotopy F_t such that $F_0 = \text{id } X$, $F_1 = gf$, and fF_t is an ε-homotopy. It may be observed that since fF_t is an ε-homotopy, $d(fF_t(x), fF_0(x)) < \varepsilon$; hence, $F(f^{-1}(y_0) \times [0,1])$ is a subset of $f^{-1}(N_\varepsilon(y_0))$ which is contained in U, the domain of φ. A null homotopy for h may be obtained by letting $G_t = \varphi F_t | f^{-1}(y_0)$. It is clear that $G_0 = \varphi F_0 | f^{-1}(y_0) = h$, and that $G_1 = \varphi F_1 | f^{-1}(y_0) = \varphi gf | f^{-1}(y_0)$ which is the constant map whose value is $\varphi g(y_0)$. Thus, f is a CE map.

II.25 Theorem. A refinable map between compact metric ANR's is a CE map.

Proof: Let r from X onto Y be a refinable map between compact ANR's. Let $\varepsilon > 0$ be given. It will be shown that there is a map g from Y onto X such that gr is homotopic to the identity on X by a homotopy F_t of X into itself so that rF_t is a 2ε-homotopy, i.e., r is a fine homotopy equivalence.

Since Y is assumed to be an ANR, there is a positive number ε_1 with $\varepsilon_1 < \varepsilon/3$ so that any two maps from a space into Y ε_1-near each other are $\varepsilon/3$-homotopic. Let $\delta > 0$ be a uniform continuity number for ε_1 and r. Since X is assumed to be an ANR there is a positive real number δ_1 with $\delta_1 < \delta$ so that any two maps from a space into X δ_1-near each other are δ-homotopic.

By Lemma II.11 and Corollary II.16.4 there are maps h from X onto Y and g from Y onto X such that $d(h,r) < \varepsilon_1$, $d(hg,\text{id } Y) < \varepsilon_1$, $d(gh,\text{id } X) < \delta_1$, and $d(rg,\text{id } Y) < \varepsilon_1$.

Thus, h is homotopic to r by an $\varepsilon/3$-homotopy F_t, $0 \leqq t \leqq 1$, from X into Y such that $F_0 = r$ and $F_1 = h$. Also since $d(gh,\text{id } X) < \delta_1$, there is a δ-homotopy H_t, $1 \leqq t \leqq 2$, from X into itself such that $H_1 = gh$ and $H_2 = \text{id } X$.

Now for $0 \leqq t \leqq 1$ let $H_t = gF_t$. At $t = 1$, $gF_1 = gh = H_1$ as defined in the preceding paragraph. Thus, H_t where $0 \leqq t \leqq 2$ is a homotopy of X into itself. It is also true that $H_0 = gr$ and $H_2 = \text{id } X$. Furthermore, rH_t is a 2ε-homotopy. For if each of s and t is in the closed interval from 0 to 2 it is necessary to consider the following cases:

(a) $1 \leqq t \leqq 2$ and $1 \leqq s \leqq 2$;

(b) $0 \leqq t \leqq 1$ and $1 \leqq s \leqq 2$;

(c) $1 \leqq t \leqq 2$ and $0 \leqq s \leqq 1$; or

(d) $0 \leqq t \leqq 1$ and $0 \leqq s \leqq 1$.

In case (a) H_u is a δ-homotopy for $1 \leqq u \leqq 2$. Thus, $d(rH_t(x), rH_s(x)) < \varepsilon$ since δ was a uniform continuity number for ε and r.

In case (b), $H_t = gF_t$ so that $rH_t = rgF_t$. Now $d(rH_t, rH_s) \leqq d(rH_t, rgh) + d(rgh, rH_s)$. Since $d(gh = H_1, H_s) < \delta$, $d(rgh, rH_s) < \varepsilon_1 < \varepsilon/3$. Now, $d(rH_t, rgh) = d(rgF_t, rgh) \leqq d(rgF_t, F_t) + d(F_t, F_1) + d(F_1, rgF_1 = rgh) < \varepsilon_1 + \varepsilon/3 + \varepsilon_1 \leqq \varepsilon$. Hence, $d(rH_t, rH_s) < 2\varepsilon$.

In case (c), the roles of s and t are reversed so that the same argument as in (b) holds.

In case (d), $d(rH_t, rH_s) = d(rgF_t, rgF_s) \leqq d(rgF_t, F_t) + d(F_t, F_s) + d(F_s, rgF_s) < \varepsilon_1 + \varepsilon/3 + \varepsilon_1 \leqq \varepsilon$.

Thus, in any case, $d(rH_t, rH_s) < 2\varepsilon$. Hence, rH_t is a 2ε-homotopy. Therefore, r is a fine homotopy equivalence. According to Theorem II.24, r is a CE map.

Another application of the results of this chapter is that the image of a compact orientable 2-manifold without boundary is a homeomorphic 2-manifold. To give this application the following theorem [Ma and Se, Theorem 4, p.161] will be used.

II.26 <u>Theorem</u>. Let M be a compact connected orientable 2-manifold without boundary, and let X be a locally cyclic continuum which is M-like. Then X is homeomorphic to M.

II.27 <u>Theorem</u>. If M is a compact connected orientable 2-manifold without boundary and if r from M onto a metric space Y is a refinable map, then Y is homeomorphic to M.

<u>Proof</u>: Since M is an ANR, hence a c-ANR, and r from M onto Y is refinable, for every positive real number ε there are maps f from M onto Y and g from Y onto M such that

$d(gf, \mathrm{id}\ M) < \varepsilon/2$ and $d(fg, \mathrm{id}\ Y) < \varepsilon/2$ (recall M is quasi-homeomorphic to Y). Since M is a 2-manifold M is locally cyclic; hence, by Corollary I.13.2, Y is locally cyclic. Since g is an ε-map from Y onto M and $\varepsilon > 0$ has been chosen arbitrarily, Y is M-like. Applying Theorem II.26 it follows that Y is homeomorphic to M .

A stronger result has recently been given by J. Segal [Se]. He has shown that r is a refinable map of a closed (compact) 2-manifold onto a metric space Y if and only if it is a near homeomorphism. Thus, the image of any compact 2-manifold without boundary under a refinable map is homeomorphic to the 2-manifold.

This chapter will be concluded with some examples of quasi-ANR's. It is clear that every compact ANR must be a quasi-ANR; however, as the next examples show the converse is not true.

II.28 Example. (Hawaiian earring) Let H be the one-dimensional space of circles C_i which have a diameter of $1/2^i$ ($i = 0,1,\ldots$) with each circle C_i ($i = 1,2,\ldots$) tangent internally to C_0 at the same point Θ . Let A_i be the union of the C_j for $0 \leqq j \leqq i$ together with the radius R_i of C_i with length $1/2^{i+1}$ passing through the point Θ (see Figure II.28.1).

For each nonnegative integer i let f_i be the map from H onto A_i defined as follows. On the union of the sets C_k where k is a nonnegative integer less than or equal to i , let f_i be the identity map when k is less than i , and let f_i be the projection of C_k onto R_i otherwise. Notice that C_{i+1} will be mapped onto R_i by f_i .

Now for each nonnegative integer i let $\vec{g_i}$ be the map from A_i onto H defined as follows. On the union of the C_k where k is a nonnegative integer less than or equal to i , let g_i be the identity. For $i + 1, i + 2, \ldots$ where k is any nonnegative integer, a_i is the center of C_i and the distance from a_{i+k} to a_{i+k+1} is $1/2^{i+k+1}$. Let g_i map the closed segment between a_{i+k} and a_{i+k+1} onto C_{i+k+1} so that $g_i(a_{i+k}) = \theta$ for $k = 0,1,\ldots$.

It is clear that $d(f_i g_i, \mathrm{id}\, A_i) < 1/2^{i+1}$, and that $d(g_i f_i, \mathrm{id}) < \operatorname{diam} C_i = 1/2^{i+1}$. Since each A_i is an ANR it follows that H is a quasi-ANR.

II.29 Example. (The Cantor baseball diamond) In Figure II.29.1 start with an equilateral triangle $b_1 a_1 b_0$ with a_1 as the vertex on top. Let a_2 be the point on $a_1 b_1$ so that the distance from b_1 is 1/3 the distance from a_1 to b_1 . Let a_3 be the analogous point on $a_1 b_0$. Let b_2 be the point on $b_1 b_0$ so that $b_1 a_2 b_2$ forms an equilateral triangle. Continue this construction in the obvious way.

A set C_1 is obtained in the limit so that the endpoints b_i form a Cantor set. The set C_i will be that part of the construction which lies on or in the interior region of the equilateral triangle T_i in the above construction which contains a_i and b_i as two of its vertices.

For each nonnegative integer i, let A_i be the finite stage of the above construction which is the union of the T_{2^j+k} where $1 \leqq j \leqq i$ and $0 \leqq k \leqq 2^j - 1$.

Define the map f_i from C_1 onto A_i in the following manner. On T_j where $j < 2^i$, let f_i be the identity. On C_{2^i+k} where $0 \leqq k \leqq 2^i - 1$, let f_i be the projection from the barycenter of T_{2^i+k} onto T_{2^i+k}. Notice that on the polygon T_{2^i+k} the projection from the barycenter is the identity map; hence, f_i is continuous.

Now define the map g_i from A_i onto C_i in the following manner. Outside the $1/3^{i+1}$ neighborhood of the union of the T_{2^i+k} where $0 \leqq k \leqq 2^i - 1$, let g_i be the identity. It is easy to see that C_{2^i+k} where $0 \leqq k \leqq 2^i - 1$ is locally connected since there are arbitrarily small neighborhoods which are either a polygonal arc or a C_j each of which is connected.

Since C_{2^i+k} is locally connected there is a map $g_{i,k}$ from T_{2^i+k} onto C_{2^i+k}. Since C_{2^i+k} is also arcwise connected there is an arc from $g_{i,k}(v)$ to v where v is a vertex of T_{2^i+k} which is not b_1 or b_0. Let w be a boundary point of the $1/3^{i+1}$ neighborhood of T_{2^i+k} nearest v. Then the segment wv can be mapped so that v maps to $g_{i,k}(v)$, the midpoint maps to v and w maps to w. Let $g_{i,k}$ be $g_{i,k}$ as defined before together with the map of the segment wv. Then on $Cl(N_{1/3^{i+1}}(T_{2^i+k}))$ define g_i to be $g_{i,k}$.

It is clear that $d(f_i g_i(x),x)$ can be no larger than the diameter of a T_{2^i+k} plus $1/3^{i+1}$. Thus $d(g_i f_i(x),x) 1/3^i + 1/3^{i+1} 2/3^i$. Since each A_i is an ANR these facts show that C_1 is a quasi-ANR. It is clear that C_1 is not an ANR since it is not locally contractible at b_0.

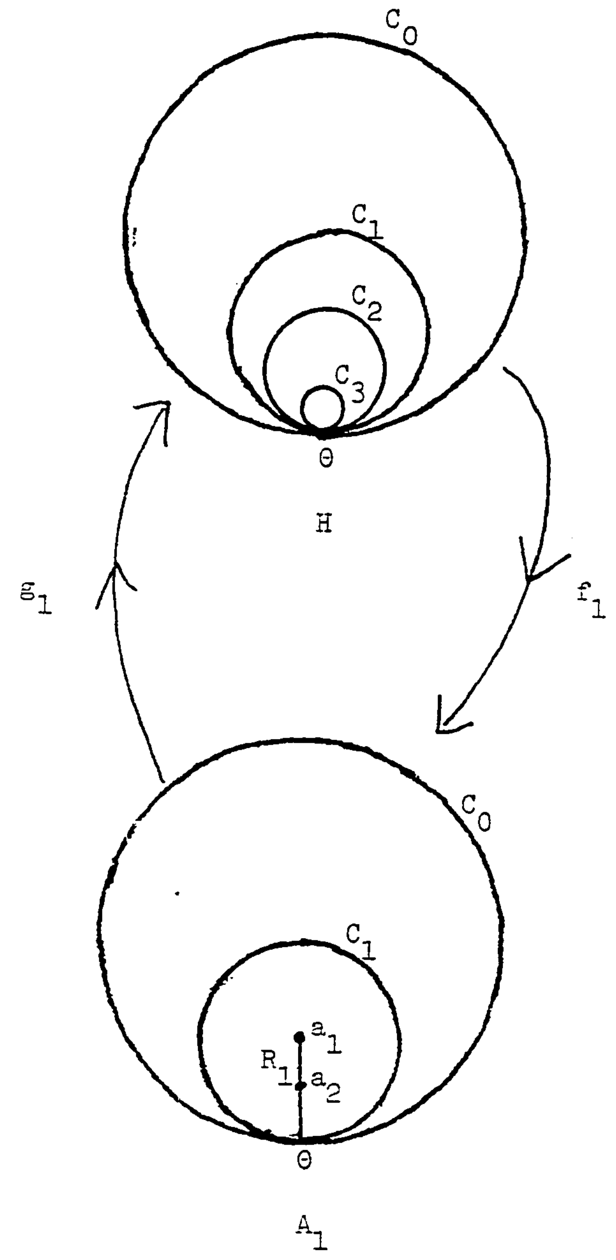


Figure II.28.1

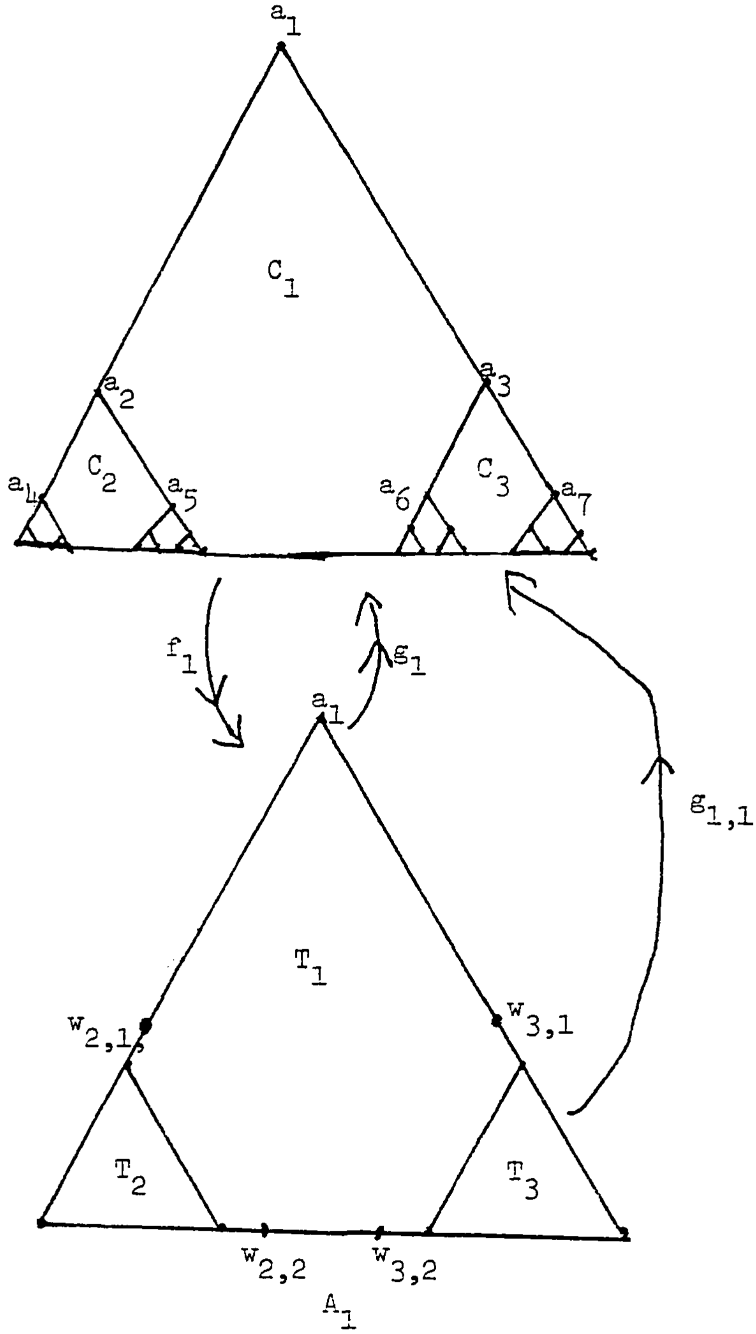


Figure II.29.1

CHAPTER III

THE IMAGE OF A 1-DIMENSIONAL ANR UNDER A REFINABLE MAP

In this chapter it will be shown that the metric image of a 1-dimensional ANR under a refinable map must be a 1-dimensional ANR. Because of the fact that a refinable map restricted to each component of a locally connected compact metric space is a refinable map (Corollary I.22.2), it is necessary only to consider connected ANR's. A 1-dimensional connected compact ANR is called an <u>ANR-curve</u>. Thus, it will first be shown that the image of an ANR-curve under a refinable map is an ANR-curve. To accomplish the proof of this result two facts will be used. The first fact is that the ANR-curves coincide with the local dendrites [Bk 4, p.138]. The second fact is that a space is a local dendrite if and only if it is a locally connected continuum containing at most a finite number of 1-spheres [Ku, p.228]. (This fact is the reason that the spaces A_i of Examples II.28 and II.29 must be ANR's.) It will then be shown that if a locally connected continuum contains at most a finite number of 1-spheres then its image under a refinable map (in fact under a monotone map) contains at most a finite number of 1-spheres. These facts will yield the desired result.

III.1 Definition. By a local dendrite is meant a continuum X so that given any point x in X and any positive real number ε there is a locally connected continuum P, and a positive real number δ such that $N_\delta(x) \subset P \subset N_\varepsilon(x)$ and P contains no 1-sphere (i.e., P is a dendrite).

III.2 Definition. If Z is a metric space containing at most a finite number of 1-spheres then $s(Z)$ will represent the number of distinct 1-spheres in Z.

III.3 Definition. A continuum X is hereditarily locally connected if and only if every subcontinuum of X is locally connected [Wh, p.89].

III.4 Definition. A continuum X is a regular curve provided that for each x in X and for each open neighborhood U of x there is an open neighborhood V of x contained in U such that the topological boundary of V is finite [Wh, p.96].

III.5 Lemma. If X is a compact 1-dimensional ANR, then X contains at most a finite number of 1-spheres.

Proof: Let H represent any homology theory over Z_2 which satisfies the Eilenberg-Steenrod axioms. By Theorem 7.3 in [Hu, p.141], $H_n(X)$ is unique up to isomorphism. Since X is a compact 1-dimensional ANR, $H_1(X)$ is either trivial or finitely generated [Hu, Corollaries 7.2 and 7.6, pp.141-145].

It is now claimed that the 1-spheres in X induce nonhomologous 1-cycles. Here we will use the homology found in [Wi], which is determined by the geometric nerves of open covers. Notice that, since $\dim X = 1$, for every open cover α of X there is a refinement β of α such that order $\beta = 1$ (i.e., if V_1, V_2 and V_3 are distinct elements of β then their common intersection is empty). Thus, for any such cover β the nerve contains no 2-simplexes. By a theorem in [Wi, Theorem 7.5, p.130], $H_1(X)$ is determined by the geometric nerves of open covers of order 1. In this case H_1 (nerve of β) = Z_1 (nerve of β).

Let C_1 and C_2 be distinct 1-spheres in X. Since C_1 and C_2 are distinct there is a cover β with order $\beta = 1$ such that at least three elements of β intersect each circle, and so that there is a B_1 in β such that $B_1 \cap C_1 \neq \emptyset$ while B_1 misses all the elements of β that intersect C_2. Similarly there is a B_2 in β so that $B_2 \cap C_2 \neq \emptyset$ and B_2 misses all the elements of β that intersect C_1. It is clear that $z_1 = \{B \text{ in } \beta \mid B \cap C_1 \neq \emptyset\}$ and $z_2 = \{B \text{ in } \beta \mid B \cap C_2 \neq \emptyset\}$ are 1-cycles. It is also clear that $z_1 \neq z_2$; hence, by [Wi, Theorem 7.5, p.130], C_1 and C_2 induce distinct 1-spheres which are not homologous.

Therefore, each 1-sphere corresponds to a distinct element of $H_1(X)$. Hence, since $H_1(X)$ has at most 2^n elements where n is the number of generators, X has at most 2^n 1-spheres.

III.6 Lemma. If X is a 1-dimensional Peano continuum containing at most a finite number of 1-spheres then X is a regular curve.

Hence, X is hereditarily locally connected. (In fact, X is a local dendrite.)

Proof: Since X contains only a finite number of 1-spheres, say $S_1, S_2, \ldots, S_n$ (if X contains no 1-spheres then it is already a dendrite), $4\varepsilon = \min\{\operatorname{diam} S_i \mid i = 1,2,\ldots,n\}$ (if X is a dendrite, let $4\varepsilon = 1$) is positive. It will now be shown that X is a local dendrite. Let $\delta > 0$ be given. Let $\eta = \min\{\varepsilon,\delta\}$. Then since X is a Peano continuum, for each x in X there is a Peano continuum P and there is a positive real number γ so that $N_\gamma(x)$ is contained in P and P is contained in $N_\eta(x)$ [Wi, Theorem 3.7, pp.79-80].

Since P is a subset of $N_\varepsilon(x)$, P contains no 1-spheres; hence, P is a dendrite. By a result in [Wh, p.99], P is a regular curve. Since P is a regular curve, by the same result in Whyburn, X is hereditarily locally connected.

III.6.1 Corollary. If X is a 1-dimensional compact locally connected metric space with at most a finite number of 1-spheres, then every subcontinuum of X is a Peano continuum.

Proof: Notice that X has a finite number of components, and each component is locally connected. Thus, each component is a 1-dimensional Peano continuum containing at most a finite number of 1-spheres.

III.6.2 Corollary. Every continuum contained in a 1-dimensional ANR is a Peano continuum.

Proof: Observe that in this case, by Lemma III.5, a 1-dimensional compact ANR is a compact 1-dimensional locally connected metric space containing at most a finite number of 1-spheres.

III.7 <u>Lemma</u>. Let X be a 1-dimensional Peano continuum containing at most a finite number of 1-spheres. Let r from X onto a metric space Y be a monotone map. Then Y is a Peano continuum containing at most a finite number of 1-spheres, and, in fact, $s(Y) \leqq s(X)$.

<u>Proof</u>: It will be shown that each 1-sphere in Y is the image of a 1-sphere in X . Thus, $s(Y) \leqq s(X)$ will easily follow.

Let S be a 1-sphere contained in Y . Consider $r^{-1}(S)$. Since r is monotone $r^{-1}(S)$ is a subcontinuum of X . By Lemma III.6, $r^{-1}(S)$ is locally connected.

Let y_1 and y_2 be distinct points in S . Let A_1 represent one of the arcs in S from y_1 to y_2 , and let A_2 represent the other arc in S from y_1 to y_2 where $\{y_1,y_2\}$ is the intersection of A_1 and A_2 . Let $A_1^o = A_1 - \{y_1,y_2\}$, and let $A_2^o = A_2 - \{y_1,y_2\}$. Since each A_i ($i = 1$, or 2) is a continuum and r is monotone $r^{-1}(A_i)$ is a continuum; hence, $r^{-1}(A_i)$ is locally connected (recall that X is hereditarily locally connected). For each i where $i = 1$, or 2 , $r^{-1}(A_i^o)$ is a connected subset open in $r^{-1}(A_i)$; hence, $r^{-1}(A_i^o)$ is a Peano space. Since $r^{-1}(A_i^o)$ is a Peano space, the set of arcwise accessible boundary points is dense in the boundary of $r^{-1}(A_i^o)$ in $r^{-1}(A_i)$ [Wi, Theorem 3.11, p.106]. Let $\mathrm{Bd}\, r^{-1}(A_i^o)$ represent the boundary of $r^{-1}(A_i^o)$ in $r^{-1}(A_i)$. Since $r^{-1}(A_i^o)$ is not all of $r^{-1}(S)$, and $r^{-1}(S)$ is connected, the boundary of $r^{-1}(A_i^o)$ in $r^{-1}(A_i)$ is nonempty. Also $r(\mathrm{Bd}\, r^{-1}(A_i^o)) = \{y_1,y_2\}$ since $\mathrm{Bd}\, r^{-1}(A_i^o)$ is contained in the union of $r^{-1}(y_1)$ and $r^{-1}(y_2)$, and since both the intersection of $r^{-1}(y_1)$ with $\mathrm{Bd}\ r^{-1}(A_i^o)$ and the intersection of $r^{-1}(y_2)$ with $\mathrm{Bd}\ r^{-1}(A_i^o)$ are nonempty. (For example, if $r^{-1}(A_i^o)$ does not intersect $r^{-1}(y_1)$ then $r^{-1}(y_1)$ and the union of $r^{-1}(A_i^o)$ with $r^{-1}(y_2)$ would be a separation of $r^{-1}(A_i)$.) Thus, there are points t_{11} and t_{12} in $\mathrm{Bd}\ r^{-1}(A_1^o)$, and there are points t_{21} and t_{22} in $\mathrm{Bd}\ r^{-1}(A_2^o)$ so that $r(t_{11}) = y_1$, $r(t_{12}) = y_2$, $r(t_{21}) = y_1$ and $r(t_{22}) = y_2$, and so that there are arcs C_1 in $\mathrm{Cl}\ r^{-1}(A_1^o)$ and C_2 in $\mathrm{Cl}\ r^{-1}(A_2^o)$ such that the intersection of C_i with $\mathrm{Bd}\ r^{-1}(A_i^o)$ is $\{t_{i1}, t_{i2}\}$. Since each $r^{-1}(y_i)$ is a Peano continuum there is an arc B_i connecting t_{1i} to t_{2i} .

It is now claimed that $C_i \cup B_i$ is an arc for each $i = 1$, or 2 . It is further claimed that the intersection of $C_1 \cup B_1$ with $C_2 \cup B_2$ is the set $\{t_{21}, t_{12}\}$. If these facts are valid it follows that $(C_1 \cup B_1) \cup (C_2 \cup B_2)$ is a 1-sphere whose image under r is S .

For these claims notice that the intersection of B_i with C_j is the singleton $\{t_{ji}\}$ (see Figure III.7.1). Also B_1 does not intersect B_2 since $r^{-1}(y_1)$ does not intersect $r^{-1}(y_2)$. Since $r^{-1}(A_1^o)$ does not intersect $r^{-1}(A_2^o)$, $C_1 - \{t_{11}, t_{12}\}$ does not intersect $C_2 - \{t_{21}, t_{22}\}$. Hence, the intersection of C_1 with C_2 equals $\{t_{21}\}$ if $t_{11} = t_{21}$ and $t_{12} \neq t_{22}$, $\{t_{12}\}$ if $t_{12} = t_{22}$ and $t_{11} \neq t_{21}$, $\{t_{12}, t_{21}\}$ if $t_{11} = t_{21}$ and $t_{12} = t_{22}$, or $\emptyset$ otherwise. In any case the intersection of C_1 with C_2 is contained in $\{t_{12}, t_{21}\}$. From the way C_1 , C_2 , B_1 and B_2 are defined the intersection of C_i and B_i is the set $\{t_{ii}\}$. Thus, $C_i \cup B_i$ is an arc. Now the following set equation holds:

$$(C_1 \cup B_1) \cap (C_2 \cup B_2)$$
$$= (C_1 \cap C_2) \cup (B_1 \cap C_2) \cup (B_2 \cap C_1) \cup (B_1 \cap B_2)$$
$$= (C_1 \cap C_2) \cup \{t_{21}\} \cup \{t_{12}\} = \{t_{21}, t_{12}\} .$$

By virtue of the claims $(C_1 \cup B_1) \cup (C_2 \cup B_2)$ is a 1-sphere. It is clear that $r(C_1) = A_1$ and that $r(C_2) = A_2$ since $r(C_i - \{t_{i1}, t_{i2}\})$ is contained in A_i^o , $r(t_{i1}) = y_1$, $r(t_{i2}) = y_2$, and $r(C_i)$ is connected. Thus, it follows that $r((C_1 \cup B_1) \cup (C_2 \cup B_2)) = S$. Thus, S is the image of a 1-sphere.

III.8 Lemma. If X is an ANR-curve and r from X onto a metric space Y is a refinable map, then Y is an ANR-curve.

Proof: By Lemma III.5, X can contain at most a finite number of distinct 1-spheres. Since X is a compact connected ANR; X is locally connected; hence, X is a 1-dimensional Peano continuum which contains at most a finite number of distinct 1-spheres. Since r is a refinable map, according to Corollary I.16.1, Y is 1-dimensional. Since r is a closed map it is an identification; hence, Y is locally connected [Du, p.125]. Since Y is locally connected and r is a refinable map, r is monotone [Fo and Rg, Corollary 1.2, p.3]. Applying Lemma III.7, Y contains at most a finite number of 1-spheres; thus, by Lemma III.6, Y is a local dendrite. Thus, given any positive real number ε there are a dendrite D contained in $N_\varepsilon(y)$ and a positive real number δ so that $N_\delta(y)$ is contained in D . Since

D is an AR [Bk 4, 13.5, p.138], $N_\delta(y)$ is an ANR neighborhood of y. Hence, Y is locally an ANR which implies that Y is an ANR [Hu, pp.97-98].

III.9 <u>Theorem</u>. The metric image of a 1-dimensional compact ANR under a refinable map is a 1-dimensional ANR.

<u>Proof</u>: Let X be a 1-dimensional compact ANR, and let r from X onto a metric space Y be a refinable map. Since X is an ANR it is locally connected; hence, X has a finite number of components $C_1, C_2, \ldots, C_n$. Now $r_i = r|C_i$ is also a refinable map by Corollary I.22.2. Since each C_i is both open and closed it is a connected compact 1-dimensional ANR (i.e., an ANR-curve). By Lemma III.8 each $r_i C_i$ is an ANR-curve. Since $Y = \cup\{r_i C_i \mid i = 1,2,\ldots,n\}$, Y must be an ANR.

III.9.1 <u>Corollary</u>. If X is a compact 1-dimensional ANR and r from X onto a metric space Y is a refinable map, then r is a CE map.

<u>Proof</u>: By Theorem III.9, Y is an ANR. According to Theorem II.25, r is a CE map.

III.10 <u>Lemma</u>. Let S be a 1-sphere in a compact 1-dimensional ANR X. Then S is not contractible in X.

<u>Proof</u>: According to [Hz and Wa, Theorem VII 3', p.151], $H_1(\text{id } S) : H_1(S) \to H_1(X)$ is an injection (here the reals modulo 1 is the coefficient group). If S is contractible in X then $H_1(\text{id } S) = 0$ where 0 is the homomorphism which sends elements of $H_1(S)$ to the 0 in $H_1(X)$. But $H_1(S) = R_1$ (the reals modulo 1) since S is a

1-sphere. It is clear that $0 : R_1 \rightarrow H_1(X)$ is not an injection; hence, S is not contractible in X.

III.10.1 Corollary. If r is a refinable map from a compact 1-dimensional ANR X, then for no y in Y does $r^{-1}(y)$ contain a 1-sphere.

Proof: Since according to Corollary III.9.1, r is a CE map, $r^{-1}(y)$ is contractible in X for every y in Y. If $r^{-1}(y)$ contained a 1-sphere, that 1-sphere would be contractible in X. By Lemma III.10 this is impossible.

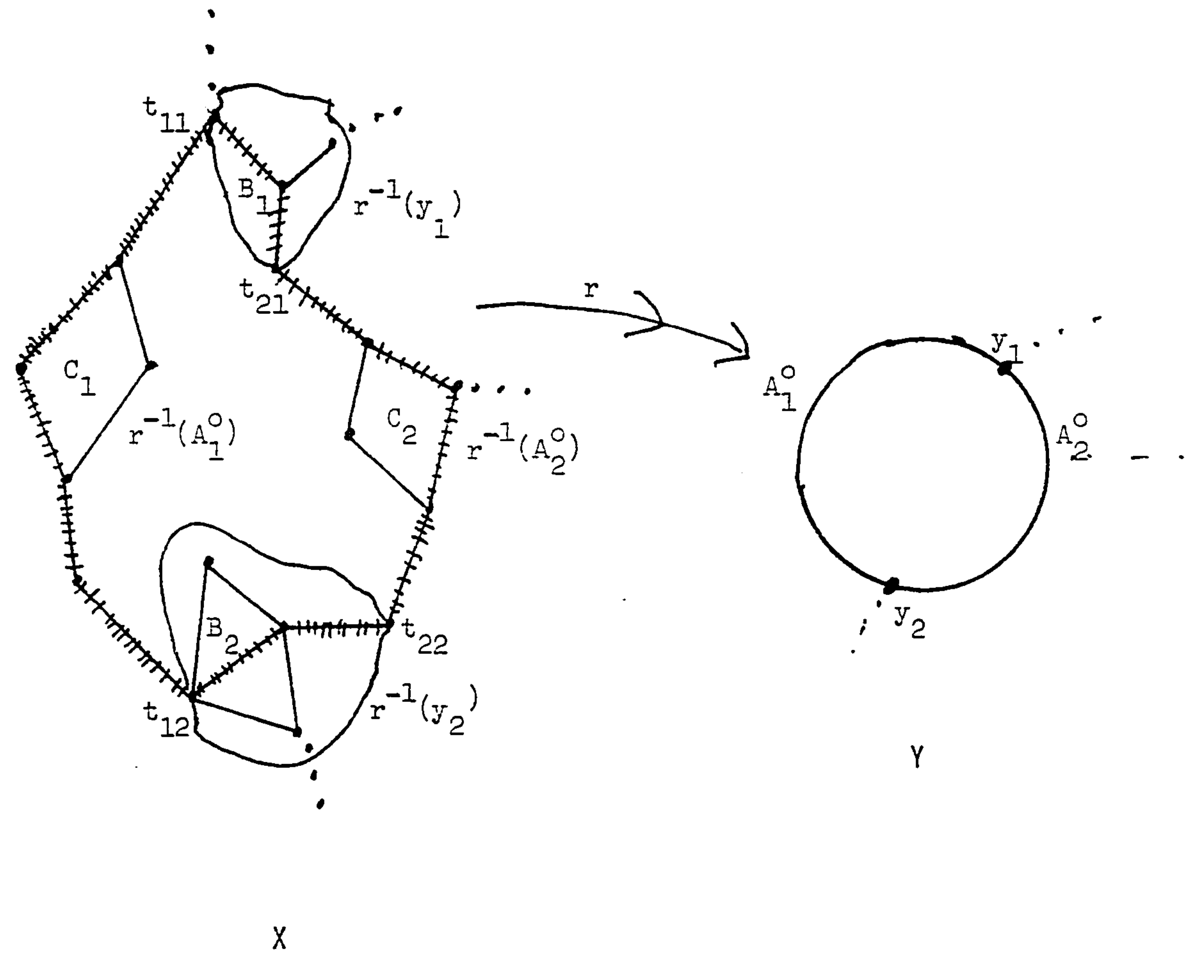


Figure III.7.1

CHAPTER IV

THE METRIC IMAGE OF A COMPACT ANR UNDER A REFINABLE MAP

According to Theorem II.25 if the image of a compact ANR under a refinable map is an ANR then the map must be a CE map. For this reason determining when a refinable map preserves ANR structure is important. In this chapter the equivalence of the following questions will be established.

IV.1 Question. Is it true that whenever X is a compact ANR and r is a refinable map from X onto a metric space Y, then Y must be an ANR [Fo and Rg, p.7]?

IV.2 Question. Is it true that whenever X is a compact AR and r from X onto A metric space Y is a refinable map, then Y must be an AR?

IV.3 Question. Is the image of the Hilbert cube under a refinable map always an ANR?

The following implications are clear since the Hilbert cube Q is a compact AR. An affirmative answer to Question IV.2 implies an affirmative answer to Question IV.3, and an affirmative answer to Question IV.1 implies an affirmative answer to Question IV.3. The following result due to Edwards will be used to prove that an affirmative answer to Question IV.3 implies an affirmative answer to Question IV.2.

IV.4 <u>Theorem</u>. (Edwards) X is an ANR if and only if $X \times Q$ is a Q-manifold (i.e., locally $X \times Q$ looks like $Q \times [0,1)$) [Ch, Theorem 44.1, p.106].

IV.5 <u>Lemma</u>. If r from a compact AR X onto a metric space Y is a refinable map, and Y is an ANR, then Y is an AR.

<u>Proof</u>: It is clear that X is a q-AR. The result follows directly from Corollary II.19.3.

IV.6 <u>Lemma</u>. If X is a nonempty compact ANR then $C(X)$ is an AR.

<u>Proof</u>: Since X is a compact ANR, $X \times [0,1]$ is also a compact ANR [Hu, p.97]. Let x_o be an element of X. Define φ from $X \times \{1\}$ to $\{x_o\}$ by $\varphi(x,1) = x_o$. Since $X \times [0,1] \cup_\varphi \{x_o\}$ identifies $X \times \{1\}$ to a point it is homeomorphic to $C(X)$. By applying a theorem of Borsuk [Bk 4, Theorem 9.1, p.116] with $X_0 = X \times \{1\}$, $X_1 = X \times [0,1]$ and $X_2 = \{x_o\}$ it follows that $X \times [0,1] \cup_\varphi \{x_o\}$ is an ANR. Hence, $C(X)$ is an ANR. It is clear that $C(X)$ contracts to the cone point; thus, $C(X)$ must be an AR.

IV.7 <u>Lemma</u>. An affirmative answer to Question IV.2 implies an affirmative answer to Question IV.1.

Proof: Let X be a compact ANR, and let r be a refinable map from X onto a metric space Y. Then by Lemma IV.6, $C(X)$ is an AR. If the image of a compact AR under a refinable map must be an AR then the image of $C(X)$ under a refinable map must be an AR. By Theorem I.26, $C(r)$ is a refinable map of $C(X)$ onto $C(Y)$. Hence, $C(Y)$ is an AR. Since $Y \times \{0\}$ contained in $C(Y)$ is a neighborhood retract of $C(Y)$, Y is an ANR.

IV.8 Lemma. An affirmative answer to Question IV.3 implies an affirmative answer to Question IV.2.

Proof: Let X be a compact AR, and let r from X onto a metric space Y be a refinable map. Suppose the metric image of the Hilbert cube Q under any refinable map is an ANR.

Now since each of X and Q is a compact AR, $X \times Q$ is a compact AR. By Corollary I.24.1, $r \times \mathrm{id}\, Q$ from $X \times Q$ onto $Y \times Q$ is a refinable map. By Edwards' result, Theorem IV.4, $X \times Q$ is a compact Q-manifold; thus, $X \times Q$ is homeomorphic to Q [Ch, Theorem 22.1, p.36]. Hence, assuming there is an affirmative answer to Question IV.3, $Y \times Q$ is ANR. Again by Edwards' result $Y \times Q \times Q$ is a Q-manifold. It is clear that $Y \times Q$ is homeomorphic to $Y \times Q \times Q$; hence, $Y \times Q$ is a Q-manifold; thus, Y is an AR. Thus, Question IV.2 has an affirmative answer if Question IV.3 does.

IV.9 Theorem. An affirmative answer to any of the Questions IV.1, IV.2 or IV.3 is equivalent to an affirmative answer to any of the Questions IV.1, IV.2 or IV.3.

Proof: It is clear that a yes answer to Question IV.1 implies a yes answer to Question IV.3. By Lemma IV.8 an affirmative answer to Question IV.3 implies an affirmative answer to Question IV.2. Finally, by Lemma IV.7 an affirmative answer to Question IV.2 implies an affirmative answer to Question IV.1. Hence the three questions are equivalent.

Notice that Question IV.3 is useful in that it reduces the question about images of ANR's under refinable maps to examining refinable maps from Q onto a q-AR. The fact that according to Corollary II.16.4 a refinable map with an AR such as Q as domain induces a quasi-homeomorphism leads to the following question.

IV.10 Question. Is it true that any compact metric space which is quasi-homeomorphic to Q must be an AR?

CHAPTER V

MORE ABOUT q-ANR'S

This chapter has two aims. The first aim is to obtain a characterization of quasi-ANR which is analogous to the embedding characterization of an ANR. The second aim is to locate q-ANR's and q-AR's among other generalized ANR's. In fact, the following implication diagram inserts the generalized ANR's discussed here into the diagrams that appear in [Bo, p.91] and [Fi, p.2].

$$\begin{tikzcd}
\mathrm{AR} \arrow[r, "1"] \arrow[d, "5"'] & \text{Uniform q-AR} \arrow[r, "2"] \arrow[ddl, "9"'] & \text{q-AR} \arrow[r, "3"] & \mathrm{SAAR} \arrow[r, "4"] \arrow[d, "6"'] & \mathrm{AAR} \arrow[d, "7"'] \\
\mathrm{ANR} \arrow[d, "8"'] & & \mathrm{ANR} \arrow[r, "11"] & \mathrm{SAANR}_N \arrow[r, "15"] \arrow[d, "13"'] & \mathrm{AANR}_N \arrow[d, "16"'] \\
\text{Uniform q-ANR} \arrow[rr, "10"] & & \text{q-ANR} \arrow[r, leftrightarrow, "12"] & \mathrm{SAANR}_C \arrow[r, "14"] & \mathrm{AANR}_C
\end{tikzcd}$$

Diagram V.1

The right side of this diagram can be completed using the diagram due to Finbow [Fi, p.2].

V.1 Theorem. A compact metric space X is a q-ANR if and only if for every positive real number ε there is a positive real number δ such that if f is a δ-map from X onto a metric space Y, then there is a map g from Y onto X such that $d(gf, \mathrm{id}\, X) < \varepsilon$.

Proof: First assume that X is a q-ANR. Then according to Theorem II.16, X is an L-M space. Since $\mathrm{id}\, X$ is a component preserving map, and X is a L-M space, given $\varepsilon > 0$, there is a $\delta > 0$ such that if f is a δ-map from X onto a metric space Y, then there is a map g from Y onto X such that $d(gf, \mathrm{id}\, X) < \varepsilon$.

Now assume that for every $\varepsilon > 0$ there is a $\delta > 0$ such that if f is a δ-map from X onto a metric space Y, then there is a map g from Y onto X such that $d(gf, \mathrm{id}\, X) < \varepsilon$. Since X is a compact metric space, for every $\delta > 0$ there exist a finite polytope K and a δ-map f from X onto K. Thus, given $\varepsilon > 0$, by letting δ be the δ corresponding to ε as above, and by letting f be the map and K be the polytope as in the preceding sentence, it follows that there is a map g from K onto X such that $d(gf, \mathrm{id}\, X) < \varepsilon$. Since K is a finite polytope, K is an ANR. Hence, X is a q-ANR.

V.2 Theorem. Suppose X is a compact metric space such that for every positive real number ε there are a q-ANR Y, a map f from X onto Y, and a map g from Y onto X such that $d(gf, \mathrm{id}\, X) < \varepsilon$. Then X is a q-ANR.

Proof: Let $\varepsilon > 0$ be given. Let Y be a q-ANR, and let f from X onto Y and g from Y onto X be maps such that $d(gf, \mathrm{id}\, X) < \varepsilon/2$. Since g is uniformly continuous there is a positive real number δ such that if for y_1 and y_2 in Y, $d(y_1,y_2) < \delta$ then $d(g(y_1),g(y_2)) < \varepsilon/2$. Now since Y is a q-ANR there exist an ANR A, a map h from A onto Y, and a map k from Y onto A such that $d(hk, \mathrm{id}\, Y) < \delta$. Thus, the following inequalities hold: $d(ghkf, \mathrm{id}\, X) \leqq d(ghkf, gf) + d(gf, \mathrm{id}\, X) < \varepsilon$. Since gh maps A onto X, and kf maps X onto A, X must be a q-ANR.

V.3 Definition. A compact metric space X is an approximative absolute neighborhood retract in the sense of Clapp [Cl, p.118] ($AANR_C$) if and only if whenever X is embedded by an embedding i into a metric space Y, then given $\varepsilon > 0$ there are a neighborhood U of $i(X)$ in Y and a map r from U into $i(X)$ so that $d(ri,i) < \varepsilon$.

V.4 Definition. A compact metric space X is an approximative absolute neighborhood retract in the sense of Noguchi [No, p.20] ($AANR_N$) if and only if whenever X is embedded by an embedding i into a metric space Y, then there is a neighborhood U of $i(X)$ in Y such that for every positive real number ε there is a map r from U into $i(X)$ such that $d(ri,i) < \varepsilon$.

V.5 Definition. A compact metric space X is an approximative absolute retract (AAR) if and only if whenever X is embedded by an embedding i into a metric space Y, then for every $\varepsilon > 0$ there is a map r from Y into $i(X)$ such that $d(ri,i)$ [Cl, p.118], [No, p.20].

We now give the following analogs of the preceding definitions. While these definitions are not given in previous work concerning approximative neighborhood retracts, they provide an alternate characterization of q-ANR's which yields results concerning the position of q-ANR's among AANR's.

V.6 Definition. A compact metric space X is a surjective approximative absolute neighborhood retract in the sense of Clapp ($SAANR_C$) if and only if it satisfies Definition V.3 with the added requirement that for each map r in the definition $r(i(X)) = i(X)$.

V.7 Definition. A compact metric space X is a surjective approximative absolute neighborhood retract in the sense of Noguchi ($SAANR_N$) if and only if it satisfies Definition V.4 with the added requirement that for each map r in the definition $r(i(X)) = i(X)$.

V.8 Definition. A compact metric space X is a surjective approximative absolute retract (SAAR) if and only if it satisfies Definition V.5 with the added requirement that for each map r in the definition $r(i(X)) = i(X)$.

In the following it will be shown that a space is a q-ANR if and only if it is a $SAANR_C$. This result will give the following nice characterization of a quasi-ANR: A compact metric space is a quasi-ANR if and only if whenever X is embedded by a map i into a metric space Y , then for every $\varepsilon > 0$ there exist a neighborhood U of $i(X)$ in Y and a map r from U into $i(X)$ such that $d(ri,i) < \varepsilon$ and $ri(X) = i(X)$.

V.9 Theorem. Every q-ANR is a $SAANR_C$; hence, every q-ANR is an $AANR_C$.

Proof: Let X be a q-ANR. Let the map i from X into a metric space Y be an embedding. Let $\varepsilon > 0$ be given. Since i is uniformly continuous (X is assumed to be compact) there is a $\delta > 0$ such that for x_1 and x_2 in X, $d(x_1,x_2) < \delta$, then $d(i(x_1),i(x_2)) < \varepsilon$. Since X is a q-ANR there are an ANR A, a map f from X onto A, and a map g from A onto X such that $d(gf,\text{id } X) < \delta$.

Consider $fi^{-1} : i(X) \twoheadrightarrow A$. Since A is an ANR, A is an ANE [Hu, p.84]. Thus, there is a neighborhood U of $i(X)$ and there is an extension $\varphi : U \twoheadrightarrow A$ of fi^{-1}.

Let $r = ig\varphi$. Notice that $ig\varphi$ maps U onto $i(X)$ and $ig\varphi(i(X)) = igfi^{-1}i(X) = igf(X) = i(X)$. Since $d(gf,\text{id } X) < \delta$, $d(igf,i) < \varepsilon$; hence, $d(ri,i) = d(ig\varphi i,i) = d(igfi^{-1}i,i) < \varepsilon$.

These facts show that X satisfies Definition V.6; hence, X is a $SAANR_C$.

V.9.1 Corollary. Every q-AR is a SAAR; hence, every q-AR is an AAR.

Proof: In the proof of Theorem V.9 the ANR A may be replaced by an AR A. Since every AR is an absolute extensor the map φ will be an extension to all of Y.

V.10 Lemma. Suppose X is a $SAANR_C$. Then given a positive real number ε there is a positive real number δ such that if f from X onto a metric space Y is a δ-map then there is a map g

from Y into X such that $d(gf, \text{id } X) < \varepsilon$ [No, Lemma 4.2, p.21; Ei, p.92].

Proof: Since X is compact it may be assumed that X is contained in the Hilbert cube Q. Let $\varepsilon > 0$ be given. Then since X is an $SAANR_C$ there exist a neighborhood U of X in Q and a map r from U onto X such that $r(X) = X$ and $d(r|X, \text{id } X) < \varepsilon/4$. Since Q is a metric space, hence normal, there is a neighborhood V of X so that $Cl(V)$ is contained in U. Since r is uniformly continuous on $Cl(V)$ there is a $\gamma > 0$ such that for the elements v_1 and v_2 in $Cl(V)$, if $d(v_1, v_2) < \gamma$, then $d(r(v_1), r(v_2)) < \varepsilon/4$. Since Q is locally convex, for each x in X there is a convex neighborhood V_x of x contained in V whose diameter is less than γ. Since X is compact there is a $\lambda > 0$ such that $\lambda < \gamma$ and if for some subset A of X, $\text{diam } A < \lambda$, then A is contained in V_x for some x in X.

Let α be a finite open λ-cover of X (i.e., if A belongs to α then $\text{diam } A < \lambda$). Let β be a finite open cover of X which star refines α. By the comments in [Hz and Wa, Section H(4), p.73] there is a quasi-barycentric β-map h from X onto a finite polyhedron P. This means that h^{-1} (open star of a vertex of P) is contained in some element of β.

Now define a map k from P into $Cl(V)$ in the following manner. Let p be a vertex of P. Then choose $k(p)$ to be an element of $h^{-1}(p)$. Now extend k linearly to the simplexes of the polyhedron P. To see that $k(P)$ is contained in $Cl(V)$, notice that if $p_0p_1\cdots p_n$ is a simplex of P then for each i, $k(p_i)$ is contained in h^{-1} (open star of p_i) which is contained in B_i where B_i is some element of β. Let p belong to the interior of $p_0p_1\cdots p_n$. Then $h^{-1}(p)$ is contained in h^{-1} (open star of p_i), which in turn is contained in B_i for each i. Thus, for each i, $k(p_i)$ belongs to the star of a point in $h^{-1}(p)$ taken in β. Hence, since β star refines α there is an A in α such that each of $k(p_0),k(p_1),\ldots,k(p_n)$ belongs to A. Thus, since $\operatorname{diam} A < \lambda$ there is an x in X such that the convex hull of $k(p_0),k(p_1),\ldots,$ $k(p_n)$ is contained in V_x where V_x is a convex neighborhood of x contained in V. Since k is linear on $p_0p_1\cdots p_n$, $k(p_0p_1\cdots p_n)$ is contained in V.

It is now claimed that for each x in X, $d(rkh(x),x) < \varepsilon/2$. There are two cases to be considered. Either $h(x)$ is in the interior of some simplex $p_0p_1\cdots p_n$ ($n \geqq 1$) of P, or $h(x)$ is a vertex of P. If the latter case occurs then $kh(x)$ belongs to $h^{-1}h(x)$ which is contained in some B_i in β. Thus, $d(kh(x),x) < \lambda < \gamma$. In the former case, $h(x)$ belongs to the open star Z_i of the vertex p_i for each i. Since h is a quasi-barycentric β-map for each i there is a B_i in β which contains $h^{-1}(Z_i)$. Since $k(p_i)$ belongs to $h^{-1}(p_i)$ which is contained in $h^{-1}(Z_i)$, which in turn is contained in B_i, $k(p_0p_1\cdots p_n)$ is in the star of x in β. Thus, for some y in X both x and $kh(x)$ belong to V_y. Hence, in this case $d(kh(x),x) < \gamma$ also. Since in either case $d(kh(x),x) < \gamma$ it is apparent that $d(rkh(x),x) < \varepsilon/4 < \varepsilon/2$.

Since $|P|$ is a compact metric space, there is an $\eta > 0$ such that if for two points p_1 and p_2 in $|P|$, $d(p_1,p_2) < \eta$, then $d(rk(p_1),rk(p_2)) < \varepsilon/2$. Since P is also an ANR, by the Lončar-Mardešić lemma [Lo and Ma, p.41-42] there is a $\delta > 0$ such that if f is a δ-map from X onto a metric space Y then there is a map Π from Y onto $|P|$ such that $d(\Pi f,h) < \eta$. In this case $d(rk\,\Pi f(x),rkh(x)) < \varepsilon/2$ for all x in X. Hence, $d(rk\,\Pi f(x),x) < \varepsilon$. To show that the conclusion of this lemma has been satisfied let $g = rk\,\Pi$.

In the following proofs let $I^\infty = \Pi[0,1/i]$ represent the version of the Hilbert cube contained in Hilbert space with the inherited metric d. The following result [Lo and Ma, Lemma 5, p.42] will be stated without proof.

V.11 Lemma. Let X be a nondegenerate continuum contained in I^∞ and let $\eta > 0$ be given. Then there is a map h from X into I^∞ with the following properties:

(i) $h(X)$ contains $N_\eta(X)$, and

(ii) $d(h(x),x) \leqq 3\eta$ for all x in X.

V.12 Lemma. Suppose X is a nondegenerate connected $SAANR_C$. Then given a positive real number ε there is a positive real number δ such that if f is a δ-map from X onto a metric space Y, then there is a map g from Y onto X such that $d(gf,\mathrm{id}\,X) < \varepsilon$.

Proof: It may be assumed that X is contained in I^∞ since X is compact. Since X is a $SAANR_C$ given a positive real number ε there is a neighborhood U of X in I^∞ and there is a map r from

$Cl(U)$ onto X such that $r(X) = X$ and $d(r|X, \mathrm{id}\, X) < \varepsilon/8$. Since r is uniformly continuous on $Cl(U)$ there is a positive real number η such that $4\eta < \varepsilon/8$, and such that if for elements u_1 and u_2 in $Cl(U)$, $d(u_1,u_2) < 4\eta$, then $d(r(u_1),r(u_2)) < \varepsilon/8$. Since X is compact η can be chosen so that $\eta < \frac{1}{2} \operatorname{diam} X$ and $N_{4\eta}(X)$ is contained in U. Let a belong to $N_{4\eta}(X)$. Then for some x in X, $d(x,a) < 4\eta$; hence, the following inequalities are valid:

$d(r(a),a) \leqq d(r(a),r(x)) + d(r(x),x) + d(x,a) < \varepsilon/8 + \varepsilon/8 + \varepsilon/8 < \varepsilon/2$. Therefore, $d(r(a),a) < \varepsilon/2$ for all a in $N_{4\eta}(X)$. By Lemma V.10 there is a positive real number γ such that if f from X onto a metric space Y is a γ-map then there is a map g_1 from Y into X such that $d(g_1f, \mathrm{id}\, X) < \eta$. This implies that X is contained in $N_\eta(g_1f(X))$ and $g_1f(X)$ is nondegenerate. Since g_1f is a map and X is connected, by Lemma V.11, there is a map h from $g_1f(X)$ into I^∞ such that $h(g_1f(X))$ contains $N_\eta(g_1f(X))$, and such that for all x in X, $d(hg_1f(x),g_1f(x)) \leqq 3\eta$. Since $d(g_1f, \mathrm{id}\, X) < \eta$, $d(hg_1f(x),x) < 4\eta$ for all x in X. Thus, $d(rhg_1f(x),r(x)) < \varepsilon/2$ for all x in X and $d(r(a),a) < \varepsilon/2$ implies $d(rhg_1f(x),x) < \varepsilon$ for all x in X. Since $hg_1f(X)$ contains $N_\eta(g_1f(X))$, $hg_1f(X)$ contains X; thus, $rhg_1f(X) = X$. Let $g = rhg_1$. Then g is a map from Y onto X which satisfies $d(gf, \mathrm{id}\, X) < \varepsilon$.

V.12.1 <u>Corollary</u>. Every connected $SAANR_C$ is a q-ANR.

<u>Proof</u>: According to Lemma V.12 a nondegenerate connected $SAANR_C$ satisfies the conditions given in Theorem V.1 for being a q-ANR. If the space is a point it is obviously an AR; hence, it is a q-ANR.

By noticing that Lemma V.10 and V.12 depend only on the ability to embed a compact metric space X in the Hilbert cube so that for every psoitive real number ε there exist a neighborhood U of X in the Hilbert cube and a map r from U onto X such that $r(X) = X$ and $d(r|X, \text{id } X) < \varepsilon$, the following corollaries hold.

V.12.2 Corollary. Suppose X is a continuum in the Hilbert cube I^∞. Suppose that for every positive real number ε there are a neighborhood U and a map r from U onto X such that $r(X) = X$ and $d(r|X, \text{id } X) < \varepsilon$. Then X is a q-ANR.

Proof: Use the proof of Lemma V.12 together with Corollary V.12.1.

V.12.3 Corollary. Let X be a SAANR_C. Then every component of X is a q-ANR.

Proof: Assume X is contained in I^∞. Let C be a component of X and $\varepsilon > 0$ be given. Since X is compact, C is compact; hence, there is a positive real number $\delta < \varepsilon$ such that $N_\delta(C)$ intersects no other component of X (see Lemma V.13). Since X is a SAANR_C there exist a neighborhood U of X in I^∞ and a map r from U onto X such that $r(X) = X$ and $d(r|X, \text{id } X) < \delta$.

It is now claimed that $r(C) = C$. Let c belong to C. Since $r(X) = X$ there is an x in X such that $r(x) = c$. Since $d(x, r(x) = c) < \delta$, x belongs to $N_\delta(C)$. Since $N_\delta(C) \cap X = C$, it follows that x belongs to C. It is also clear that since $d(c, r(c)) < \delta$ and c belongs to X, $r(c)$ belongs to C. Hence, $r(C) = C$.

Since $d(r|C, \text{id } C) < \varepsilon$ and the intersection U' of U with $N_\delta(C)$ is a neighborhood of C in I^∞, C satisfies the hypothesis of Corollary V.12.2, and it follows that C is a q-ANR.

V.13 Lemma. Every $SAANR_C$ is locally connected and compact; hence, every $SAANR_C$ has at most a finite number of components.

Proof: Let X be a $SAANR_C$. Then X is compact by definition. Thus, it may be assumed that X is contained in the Hilbert cube Q.

Since X is a $SAANR_C$ there exist a neighborhood U and a map r from U onto X such that $r(X) = X$ and $d(r|X, \text{id } X) < 1$. By a result in [Wi, Theorem 3.7, pp.79-80] there is a locally connected neighborhood (closed) K of X contained in U. $r|K$ is an identification and $r(K) = X$; hence, X is locally connected [Du, p.125].

By applying Lemma V.13, Corollary V.12.3, Corollary II.16.1, and Theorem V.9 the following theorem can be proved.

V.14 Theorem. A compact metric space is a q-ANR if and only if it is a $SAANR_C$.

In view of this theorem and Corollary V.9.1 one might ask if every SAAR is a q-AR. A partial result in this direction is the following theorem.

V.15 Theorem. Every AR-like SAAR is a q-AR.

Proof: Let X be an AR-like SAAR. It is clear that X is a $SAANR_N$; hence, it is a $SAANR_C$. According to Theorem V.14, X is a q-ANR. Since X is a q-ANR, according to Theorem V.1, given a positive real number ε there is a positive real number δ such that if f from X onto a metric space Y is a δ-map then there is a map g from

Y onto X such that $d(gf, \mathrm{id}\, X) < \varepsilon$. Since X is AR-like there are an AR A and a δ-map f from X onto A . Hence there is a map g from A onto X such that $d(gf, \mathrm{id}\, X) < \varepsilon$. Thus, X is a q-AR.

The following lemma generalizes Borsuk's Homotopy Extension Theorem [Bk 4, Theorem 8.1, p.94].

V.16 Lemma. Let A be a closed subset of a metric space Y , and let X be a q-ANR. Suppose that F from $A \times [0,1]$ to X is continuous, and that F_0 has a continuous extension f from Y into X . Then for every positive real number ε there is a map H from $Y \times [0,1]$ into X such that for (y,t) in $A \times [0,1] \cup Y \times \{0\}$, $d(H(y,t), F'(y,t)) < \varepsilon$ where

$$F'(y,t) = \begin{cases} F(y,t) & \text{if } (y,t) \text{ is in } A \times [0,1] \qquad [\text{Cl, p.128}] \\ f(y) & \text{if } t = 0 \end{cases} .$$

Furthermore, if $F_1(A) = X$ then $H_1(A) = X$.

Proof: Given F and f as in the hypothesis of this lemma define F' from $A \times [0,1] \cup Y \times \{0\}$ into X as indicated in the statement of the lemma.

Since X is compact it may be assumed to be contained in the Hilbert cube Q . Let $\varepsilon > 0$ be given. Since X is a q-ANR, by Theorem V.14 it is a SAANR_C; hence, there are an open neighborhood U of X in Q and a map r from U onto X such that $r(X) = X$ and $d(r|X, \mathrm{id}\, X) < \varepsilon$. Since U is an ANR, there is a map G from $Y \times [0,1]$ into U such that $G(y,t) = F'(y,t)$ if (y,t) belongs to $A \times [0,1] \cup Y \times \{0\}$ [Bk 4, Theorem 8.1, p.94].

Let H from $Y \times [0,1]$ into X be defined by $H_t = rG_t$. If (y,t) belongs to $A \times [0,1] \cup Y \times \{0\}$ then $H(y,t) = rG(y,t) = rF'(y,t)$. Thus, $d(H(y,t),F'(y,t)) < \varepsilon$ since $d(rF'(y,t),F'(y,t)) < \varepsilon$. If $F_1(A) = X$ then X contains $H_1(A) = rG_1(A) = rF_1'(A)$ which contains $r(X) = X$. In this case, therefore, $H_1(A) = X$.

V.17 Theorem. A quasi-contractible q-ANR is a SAAR [C1, p.128].

Proof: Let X be a quasi-contractible q-ANR. Suppose i from X into a metric space Y is an embedding. Since X is assumed to be compact $i(X)$ is closed. $i(X)$ is a $SAANR_C$ by Theorem V.14.

It is now claimed that $i(X)$ is quasi-contractible. Let $\varepsilon > 0$ be given. Since i is uniformly continuous there is a $\delta > 0$ such that if for x_1 and x_2 in X , $d(x_1,x_2) < \delta$, then $d(i(x_1),i(x_2)) < \varepsilon$. Since X is quasi-contractible there is a map F from $X \times [0,1]$ into X such that for some a in X , $F_0(X) = \{a\}$, $F_1(X) = X$, and $d(F_1,\text{id } X) < \delta$. Thus, the map $\hat{F}$ from $i(X) \times [0,1]$ into $i(X)$ defined by $\hat{F}(i(x),t) = iF(x,t)$ satisfies $\hat{F}_0(i(X)) = \{i(a)\}$, $\hat{F}_1(i(X)) = iF_1(X) = i(X)$, and $d(\hat{F}_1 i,i) = d(iF_1,i) < \varepsilon$ since $d(F_1,\text{id } X) < \delta$. Thus, $i(X)$ is quasi-contractible.

Let $\varepsilon > 0$ be given. Since $i(X)$ is quasi-contractible there is a map F from $i(X) \times [0,1]$ into $i(X)$ such that for some a in X , $F_0(i(X)) = i(a)$, $F_1(i(X)) = i(X)$ and $d(F_1 i,i) < \varepsilon/2$. Now F_0 has an extension f to all of Y by defining $f(y) = i(a)$ for all y in Y . According to Lemma V.16 there is a map H from $Y \times [0,1]$ into $i(X)$ such that $H_1(i(X)) = i(X)$ and $d(H(y,t),F'(y,t)) < \varepsilon/2$, F' as in V.16. Let $r = H_1$. Then r maps Y onto $i(X)$ where

$ri(X) = i(X)$ and the following inequalities hold: $d(ri,i) \leqq d(H_1 i, F_1' i = F_1 i) + d(F_1 i, i) < \varepsilon/2 + \varepsilon/2 = \varepsilon$. Thus, X satisfies the requirements for Definition V.8. Hence, X is a SAAR.

An immediate corollary of this result and Theorem V.15 is the following.

V.16.1 <u>Corollary</u>. An AR-like quasi-contractible q-ANR is a q-AR.

V.17 <u>Definition</u>. A compact metric space X is a <u>uniform q-ANR</u> (<u>uniform q-AR</u>) if and only if there exists an ANR (AR) A such that for every positive real number ε there are maps f from X onto A and g from A onto X such that $d(gf, \text{id } X) < \varepsilon$.

V.18 <u>Theorem</u>. A space which is quasi-homeomorphic to a uniform q-ANR is a uniform q-ANR.

<u>Proof</u>: Let (f_i, g_i) be a quasi-homeomorphism of a compact metric space X onto a uniform q-ANR Y . Let A be an ANR such that Definition V.17 is satisfied for the space Y . Let $\varepsilon > 0$ be given. Then there are maps f_i from X onto Y and g_i from Y onto X such that $d(g_i f_i, \text{id } X) < \varepsilon/2$. Since g_i is uniformly continuous there is a positive real number δ such that if for y_1 and y_2 in Y , $d(y_1, y_2) < \delta$, then $d(g_i(y_1), g_i(y_2)) < \varepsilon/2$. Since Y is a uniform q-ANR there exist maps f from Y onto A and g from A onto Y such that $d(gf, \text{id } Y) < \delta$. Hence, $d(g_i gf, g_i \text{id } Y) < \varepsilon/2$. Thus, $d(g_i gff_i, \text{id } X) \leqq d(g_i gff_i, g_i f_i) + d(g_i f_i, \text{id } X) < \varepsilon$. Therefore, X is a uniform q-ANR.

V.18.1 <u>Corollary</u>. A space which is quasi-homeomorphic to a uniform q-AR is a uniform q-AR.

<u>Proof</u>: The ANR A in the preceding proof may be replaced by an AR.

Because of Theorem V.18 and Corollary II.16.4 the following corollaries are valid.

V.18.2 <u>Corollary</u>. The metric image of a uniform q-ANR (uniform q-AR) under a refinable map is a uniform q-ANR (uniform q-AR).

V.18.3 <u>Corollary</u>. The metric image of an ANR (AR) under a refinable map is a uniform q-ANR (uniform q-AR).

V.19 <u>Lemma</u>. If each of X and Y is a q-ANR (uniform q-ANR) then $X \times Y$ is a q-ANR (uniform q-ANR).

<u>Proof</u>: Let $\varepsilon > 0$ be given. Then since X is a q-ANR there exist an ANR A (if X is a uniform q-ANR A does not depend on the choice of ε), a map f from X onto A, and a map g from A onto X such that $d(gf, \mathrm{id}\ X) < \varepsilon/2$. Similarly, for Y there exist an ANR B, a map h from Y onto B, and a map k from Y onto B such that $d(kh, \mathrm{id}\ Y) < \varepsilon/2$.

Since $A \times B$ is an ANR [Hu, p.97], $f \times h$ is a map from $X \times Y$ onto $A \times B$, and $g \times k$ is a map from $A \times B$ onto $X \times Y$ which satisfy $d((g \times k)(f \times h), \mathrm{id}\ X \times \mathrm{id}\ Y = \mathrm{id}\ X \times Y) \leqq d(gf, \mathrm{id}\ X) + d(kh, \mathrm{id}\ Y) < \varepsilon$, $X \times Y$ is a q-ANR (a uniform q-ANR if the choice of each of A and B does not depend on ε).

V.19.1 <u>Corollary</u>. If each of X and Y is a q-AR (uniform q-AR) then $X \times Y$ is a q-AR (uniform q-AR).

Proof: The ANR's A and B in the preceding proof may be replaced by AR's.

V.20 Lemma. If X is a q-ANR (uniform q-ANR) then $C(X)$ is a q-AR (uniform q-AR).

Proof: For each positive real number ε there exist an ANR A, a map f from X onto A, and a map g from A onto X such that $d(gf, \mathrm{id}\ X) < \varepsilon$. It is clear that $d(C(g)C(f), C(\mathrm{id}\ X) = \mathrm{id}\ C(X)) < \varepsilon$. The cone over A is a contractible ANR; hence, $C(A)$ is an AR. From these facts the lemma follows.

V.21 Example. Let H be the Hawaiian earring (Example II.28). By Lemma V.20, $C(H)$ is a q-AR. It is claimed that $C(H)$ is not an AR. If $C(H)$ were an AR then since $H \times \{0\}$ is a neighborhood retract of $C(H)$, H would be an ANR [Hu, p.97], which is not the case since H is not locally contractible.

H is also an example of a SAANR_C which is not a SAANR_N. Consider H as embedded in the plane E^2 as in Example II.28. Suppose H is a SAANR_N. Then there is a neighborhood U of H in E^2 such that for every positive real number ε there is a map r from U onto H which satisfies $r(H) = H$ and $d(rH, \mathrm{id}\ H) < \varepsilon$. At the point Θ there is a positive real number δ such that $N_\delta(\Theta)$ is contained in U and $S_\delta(\Theta) = \{y \text{ in } E^2 \mid d(y,\Theta) = \delta\}$ is not tangent to any of the circles C_j. It is clear that there is a least i such that C_i is contained in $N_\delta(\Theta)$. It is also clear that $H' = \cup\{C_j \mid j \geq i\}$ is a Hawaiian earring contained in $N_\delta(\Theta)$. Let H^* be the intersection of H with $\mathrm{Cl}(N_\delta(\Theta))$. Now H^* is the union of H' with the union of the finite collection of arcs $I_j = C_j \cap N_\delta(\Theta)$ where $j < i$ (see Figure V.21.1). Define the map f from H^* into H^* to be the map which first sends H' onto C_i keeping C_i fixed while also sending I_j to Θ for $j < i$, then follows this by rotating C_i by π. It is clear that f has no fixed points; furthermore, since H^* is compact and the function from H^* to the positive real numbers which sends x to $d(f(x),x)$, ($d(f(x),x) > 0$ for all x in H^* since f has no fixed points) is continuous, $\inf\{d(f(x),x) \mid x \text{ belongs to } H^*\}$ exists and is attained at some point x_0. Hence, $\gamma = d(f(x_0),x_0) > 0$. Let $\varepsilon < \min\{1/2^{i+1},\gamma/3\}$ be a positive real number such that $N_{\delta+\varepsilon}(\Theta)$ contains H', and for $j < i$, $Cl(N_{\delta+\varepsilon}(\Theta))$ does not contain C_j, and the diameter of each component of the set $C_j \cap Cl(N_{\delta+\varepsilon}(\Theta) - N_\delta(\Theta))$ is less than $\gamma/3$. Define the map s on $Cl(N_{\delta+\varepsilon}(\Theta))$ intersected with H by letting s be the identity on H^*, and by letting s map $(C_j \cap Cl(N_{\delta+\varepsilon}(\Theta))) - H^*$ to the nearest point of $C_j \cap (H^* - N_\delta(\Theta))$ for $j < i$ (notice that $C_j \cap (H^* - N_\delta(\Theta))$ contains exactly two points).

Since H is assumed to be a $SAANR_N$ there is a map r from U onto H such that $rH = H$ and $d(r|H, \text{id } H) < \varepsilon$. In particular, $r(Cl(N_\delta(\Theta)))$ is contained in the intersection of $Cl(N_{\delta+\varepsilon}(\Theta))$ with H. Thus, $sr|Cl(N_\delta(\Theta))$ is well-defined and $sr(Cl(N_\delta(\Theta))$ is contained in H^*. Thus, fsr from $Cl(N_\delta(\Theta))$ into H^* is a well-defined map. Since $Cl(N_\delta(\Theta))$ is a 2-cell it has the fixed point property. Hence, there is a point x in $Cl(N_\delta(\Theta))$ such that $fsr(x) = x$. It is clear that x belongs to H^*; in fact, by the way f has been defined x must belong to C_i. Now

$d(sr(x),x) \leqq d(sr(x),r(x)) + d(r(x),x) < \gamma/3 + \varepsilon < 2\gamma/3$; however, by the way γ has been defined $d(x = fsr(x),sr(x)) \geqq \gamma$. This yields a contradiction; hence, H cannot be a $SAANR_N$.

V.22 Example. Let $X = \cup\{\{1/n\} \times [0,1] \mid n$ is a positive integer$\} \cup \{0\} \times [0,1] \cup [0,1] \times \{0\}$. It is clear that X is not locally connected; hence, X is not a $SAANR_C$. It is now claimed that X is an AAR. For each n define r_n from $[0,1] \times [0,1]$ into X by letting r_n be the projection of $[0,1/n] \times [0,1]$ onto $\{1/n\} \times [0,1]$, and by letting r_n be a retraction of $[1/n,1] \times [0,1]$ onto $\cup\{\{1/k\} \times [0,1] \mid 1 \leqq k \leqq n\} \cup [1/n,1] \times \{0\}$. It is clear that $d(r_n|X, \mathrm{id}\ X) < 1/n$. Let i be an embedding of X into a metric space Y . Let $\varepsilon > 0$ be given. Since i is uniformly continuous there is an n such that if for x_1 and x_2 in X , $d(x_1,x_2) < 1/n$, then $d(i(x_1),i(x_2)) < \varepsilon$. i^{-1} is a map from $i(X)$ into the AR $[0,1] \times [0,1]$. Thus, there is an extension φ of i^{-1} from Y into $[0,1]^2$. Consider the map $r = ir_n\varphi$. Since $d(r_n\varphi i(x),x) < 1/n$, $d(ir_n\varphi i,i) < \varepsilon$. Hence, X satisfies the conditions for an AAR given in Definition V.5 [No, 2.8, p.20].

Example V.22 shows that arrows 4, 15, and 14 in Diagram V.1 cannot be reversed. Example V.21 shows that arrow 13, arrow 1 followed by arrow 2, and arrow 8 followed by arrow 10 in Diagram V.1 cannot be reversed. Theorem V.14 verifies the double arrow 12. The reversibility of arrows 1, 2, 8, 10 is left unanswered by the results and examples given so far. If, for example, arrow 8 (or arrow 1) is reversible then by Corollary V.18.3, the metric image of an ANR under a refinable map would be an ANR. Thus, while the general problem of whether the metric image of an ANR is an ANR has not yet been solved, further study of examples of q-ANR's and the properties of q-ANR's may help produce either a counter-example or a proof.

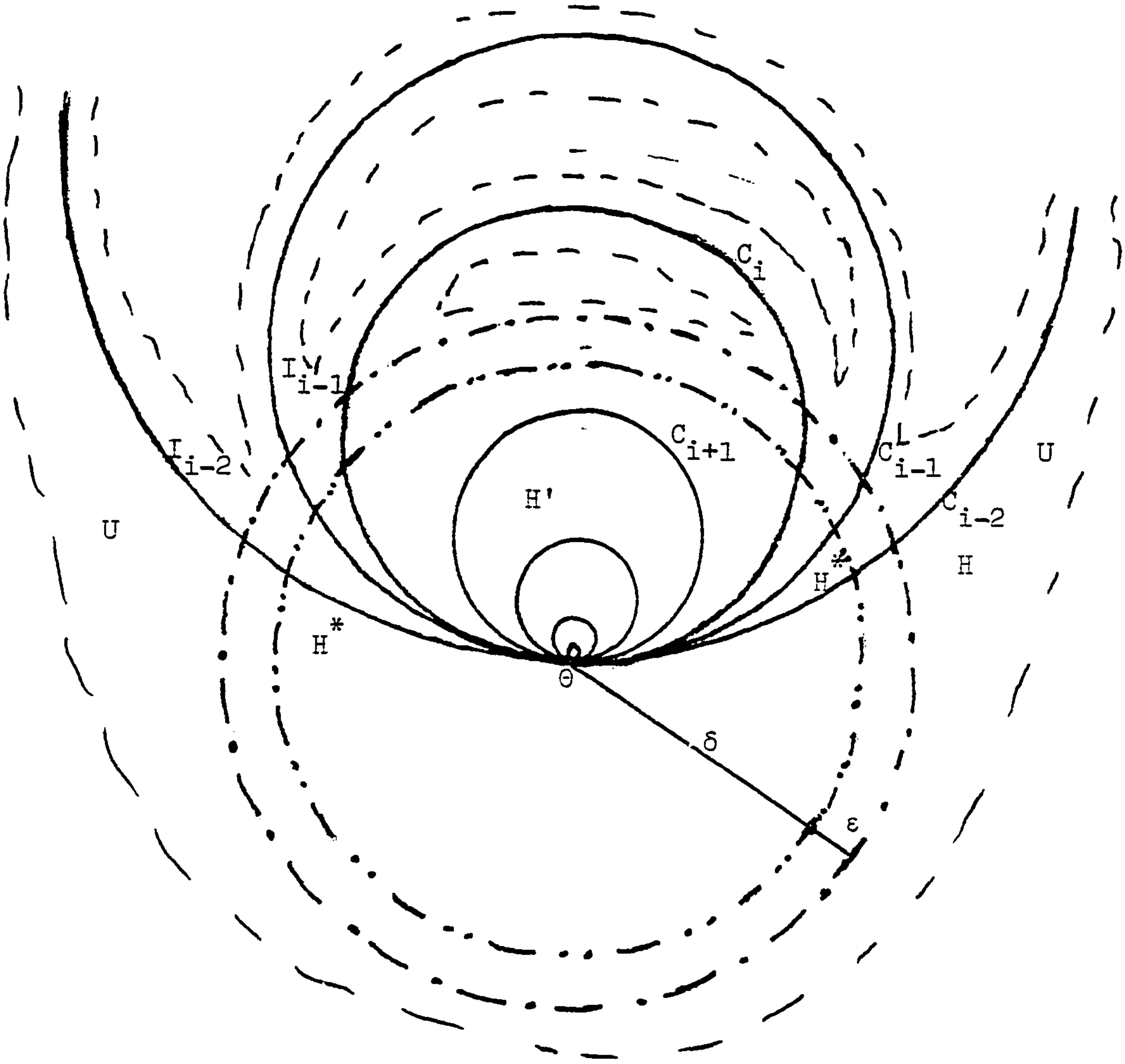


Figure V.21.1

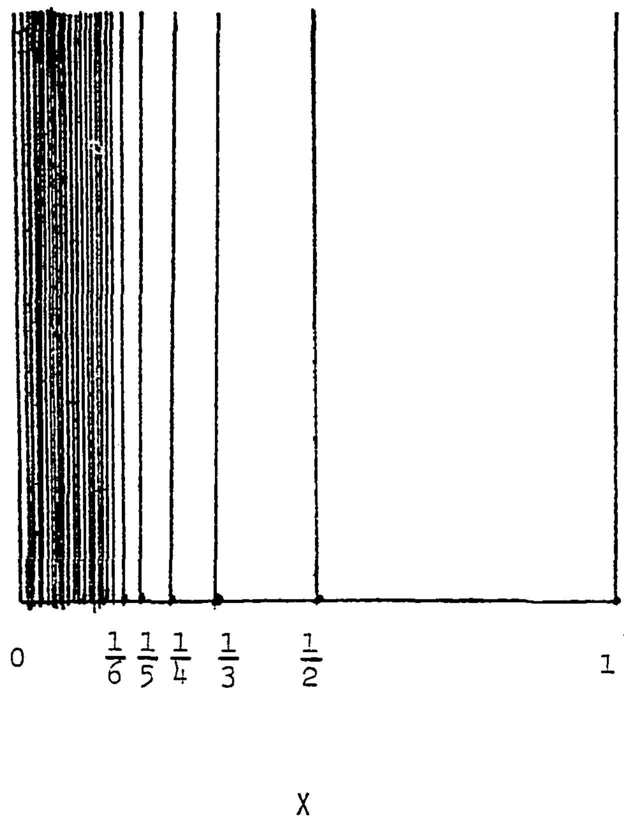


Figure V.22.1

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