78-17,903

KAO, Shaw-Yi
SEPARABLE $C_0(X)$-ALGEBRAS.

The University of Oklahoma,
Ph.D., 1978

University Microfilms International, Ann Arbor, Michigan 48106

THE UNIVERSITY OF OKLAHOMA

GRADUATE COLLEGE

SEPARABLE $C_o(X)$-ALGEBRAS

A DISSERTATION

SUBMITTED TO THE GRADUATE FACULTY

in partial fulfillment of the requirements for the

degree of

DOCTOR OF PHILOSOPHY

BY

SHAW-YI KAO

Norman, Oklahoma

1978

SEPARABLE $C_o(X)$-ALGEBRAS

APPROVED BY

DISSERTATION COMMITTEE

ACKNOWLEDGEMENTS

I wish to express my deepest appreciation to Dr. Andy R. Magid for his patient assistance, genuine guidance and continual encouragement during my academic program and in the preparation of this manuscript. My sincere appreciation is extended to Dr. Harold V. Huneke, Dr. Robert A. Morris, Dr. Leonard R. Rubin and Dr. Li Pi Su who served on my dissertation committee.

For her patience, sacrifices and support, I am deeply grateful to my wife Jeanie.

TABLE OF CONTENTS

SEPARABLE $C_o(X)$-ALGEBRAS

INTRODUCTION

Let X be a compact Hausdorff space and C(X) the $\mathbb{C}$-algebra of all complex-valued continuous functions on X. It is known that the category of faithful separable C(X)-algebras that are finitely generated projective C(X)-modules and the category of finite-fibered covering spaces of X are contravariantly equivalent [C, Theorem 2 and M1, Corollary 12]. Let X be a locally compact Hausdorff space and $C_o(X)$ the $\mathbb{C}$-algebra of all complex-valued continuous functions on X vanishing at infinity. Our study here is to explore the relation between the separable $C_o(X)$-algebra extensions and the locally compact Hausdorff finite-fibered covering spaces of X. Childs proved that if X is a compact Hausdorff space and S is a finitely generated projective separable C(X)-algebra, then for each maximal ideal M of C(X) there is an h not in M such that $S_h = (C(X)_h)^n$, a product as rings of n copies of $C(X)_h$ [C, p. 32]. The converse is also true. The above motivates our definition for separable $C_o(X)$-algebras without identity.

Let X be a locally compact Hausdorff space and S a commutative $\mathbb{C}$-algebra without identity. We call S a $C_o(X)$-algebra if there is a $\mathbb{C}$-algebra homomorphism θ: $C_o(X) \to S$, and call S a $C_o(X)$-algebra extension if θ is injective. Let S be a $C_o(X)$-algebra extension. If $f \in C_o(X)$, we let Z(f) denote the zero set of f and $\bar{f} = \theta(f)$. S is called a separable $C_o(X)$-algebra if (i) for each regular maximal ideal N of $C_o(X)$ there is an f in $C_o(X)$ but not in N and a positive integer n such that $X - Z(f) \subseteq F$, a compact subset of X, and $S_{\bar{f}} = (C_o(X)_f)^n$ [see Chapter II], a product as rings of n copies of $C_o(X)_f$, and (ii) no regular maximal ideal of S contains all $\bar{f}$, where f satisfies (i). We prove that if Y is a locally compact Hausdorff finite-fibered covering space of X, then $C_o(Y)$ is a separable $C_o(X)$-algebra extension, and conversely, if S is a separable $C_o(X)$-algebra extension, then the structure space Rm S of S is a locally compact Hausdorff finite-fibered covering space of X. The set of all locally compact Hausdorff finite-fibered covering spaces of X together with the proper maps [B2, p. 97] between two covering spaces which commute with covering projections, and the set of all separable $C_o(X)$-algebra extensions together with the $\mathbb{C}$-algebra homomorphisms between separable $C_o(X)$-algebra extensions which commute with $C_o(X)$-algebra homomorphisms form two categories, and the map $Y \to C_o(Y)$ is a full and faithful functor. We also give an example to show in general the functor is not a contravariant equivalence. However, if we restrict the latter category to all $\mathbb{C}^*$-Banach algebras that are separable $C_0(X)$-algebra extensions, then the functor $Y \to C_0(Y)$ is a contravariant equivalence.

CHAPTER I

COMMUTATIVE $\mathbb{C}$-ALGEBRAS WITHOUT IDENTITY AND STRUCTURE SPACES

The principal result in this chapter is that the structure space of a nonradical commutative $\mathbb{C}$-algebra without identity is a locally compact space, and is a locally compact Hausdorff space if the $\mathbb{C}$-algebra is the ring of all complex-valued continuous functions on a locally compact Hausdorff space vanishing at infinity. We begin by reviewing some definitions related to the commutative $\mathbb{C}$-algebras without identity.

Throughout this chapter R denotes a commutative $\mathbb{C}$-algebra without identity. Let $\tilde{R} = \mathbb{C} \times R = \{(k,r) | k \in \mathbb{C}, r \in R\}$. Then $\tilde{R}$ forms a commutative $\mathbb{C}$-algebra with identity $e = (1,0)$ if the addition and scalar multiplication are defined componentwise and the multiplication is defined by $(k,r)(p,s) = (kp, ks + pr + rs)$. It is clear that R can be identified with the maximal ideal $(0,R)$ of $\tilde{R}$, and $\tilde{R}/R$ is isomorphic to $\mathbb{C}$. We call a $\mathbb{C}$-algebra ideal I a <u>regular ideal</u> if there exists an element u in R such that $ur - r \in I$ for all $r \in R$, or equivalently the quotient ring R/I is a commutative $\mathbb{C}$-algebra with identity $\bar{u} = u + I$. Call a regular ideal a _regular maximal ideal_ if it is a maximal ideal, or equivalently the quotient ring is a field. The intersection of all regular maximal ideals is the _radical_. R is called a _semi-simple_ $\mathbb{C}$-algebra if the radical is 0, a _nonradical_ $\mathbb{C}$-algebra if R has at least one regular maximal ideal and a _radical_ $\mathbb{C}$-algebra if R has no regular maximal ideal. Since R has no identity, it may be a radical $\mathbb{C}$-algebra [M2, p. 28]. Henceforth, we shall _assume all $\mathbb{C}$-algebras to be nonradical_.

Lemma 1.1: If J is a $\mathbb{C}$-algebra ideal of R containing a regular ideal I, then J itself is a regular ideal.
Proof: If $\bar{u}$ is the identity mod I, then clearly $\bar{u}$ is the identity mod J.

Lemma 1.2: Every regular ideal of R is contained in a regular maximal ideal.
Proof: Let I be a regular ideal of R. By definition R/I is a commutative $\mathbb{C}$-algebra with identity, hence contains a maximal ideal M/I, where M is an ideal of R containing I. It is clear that M is a maximal ideal. By Lemma 1.1 M is also a regular ideal.

Lemma 1.3: I is a regular ideal of R if and only if there exists an ideal $\tilde{I}$ of $\tilde{R}$ such that $\tilde{I} \not\subseteq R$ and $\tilde{I} \cap R = I$. Moreover, I is regular maximal if and only if $\tilde{I}$ is maximal.
Proof: Suppose that I is a regular ideal of R. Then there exists an element u in R such that $ur - r \in I$ for all r in

R. Let $\tilde{I} = \{(k,r) \mid ku + r \in I\}$. Clearly, $\tilde{I}$ is an ideal of $\tilde{R}$ containing I. Since $(1,-u)$ is in $\tilde{I}$ but not in R, $\tilde{I} \not\subseteq R$. If $(k,r) \in \tilde{I} \cap R$, then $k = 0$ and $r \in I$. Thus $\tilde{I} \cap R \subseteq I$. On the other hand, both $\tilde{I}$ and R contain I, so $I \subseteq \tilde{I} \cap R$. Hence $I = \tilde{I} \cap R$. Now we assume further that I is a regular maximal ideal of R. Clearly, the map α: $\tilde{R} \to R/I$ defined by $\alpha(k,r) = (ku + r) + I$ is a surjective $\mathbb{C}$-algebra homomorphism with kernel $\tilde{I}$. Thus $\tilde{R}/\tilde{I} \cong R/I$. Since I is regular maximal, R/I is a field. Therefore, $\tilde{R}/\tilde{I}$ is a field. It follows that $\tilde{I}$ is a maximal ideal of $\tilde{R}$.

Conversely, since $\tilde{I} \not\subseteq R$, there exists an element s in $\tilde{I}$ but not in R. R is a maximal ideal of $\tilde{R}$ and $s \notin R$. Thus the ideal generated by R and s is the whole ring $\tilde{R}$. It follows that there exists an element m in $\tilde{R}$ such that $e = sm + u$ for some u in R. For any r in R, we have $ur - r = (e - sm)r - r = er - smr - r = r - smr - r = -smr \in R \cap \tilde{I} = I$. Thus I is a regular ideal of R. Now suppose that $\tilde{I}$ is a maximal ideal of $\tilde{R}$. Clearly, the map β: $R/I \to \tilde{R}/\tilde{I}$ defined by sending $r + I$ onto $r + \tilde{I}$ is a well-defined, injective $\mathbb{C}$-algebra homomorphism. The image of R/I under β is an ideal of $\tilde{R}/\tilde{I}$ and $\tilde{R}/\tilde{I}$ is a field. Thus β is either 0 or surjective. But we know that β is injective, so β must be a surjective map. Thus β is a $\mathbb{C}$-algebra isomorphism, and hence R/I is a field. By definition I is a regular maximal ideal of R.

Let $\mathrm{Rm}\, R$ denote the set of all regular maximal ideals of R and $m\tilde{R} - \{R\}$ the set of all maximal ideals of

$\tilde{R}$ not equal to R. We have:

Lemma 1.4: There is a bijection between Rm R and $m\tilde{R} - \{R\}$.

Proof: Define a map α: $m\tilde{R} - \{R\} \to \text{Rm } R$ by $\alpha(M) = M \cap R$. By Lemma 1.3 α is well-defined and surjective. We need only to show that α is injective. Let $P, Q \in m\tilde{R} - \{R\}$ and $P \neq Q$. Since no two of P, Q, R are equal, $R \neq P \cap Q$. It follows that there exists $y \in P - Q$, $z \in Q - P$ and $x \in R$ with $x \notin P \cap Q$. Since P, Q and R are ideals, $xy \in P \cap R$ and $xz \in Q \cap R$. We claim that either $xy \notin Q \cap R$ or $xz \notin P \cap R$. If $xy \in Q \cap R$ and $xz \in P \cap R$, then $xy \in Q$ and $xz \in P$. P and Q are prime ideals, so $x \in P$ and $x \in Q$. Thus, $x \in P \cap Q$, a contradiction. Hence $P \cap R \neq Q \cap R$, i.e., α is injective.

For each r in R, let $F_r = \{N \in \text{Rm } R \mid r \in N\}$. Then Rm R can be made into a topological space by taking the family of all F_r as a base for the closed sets. The space is called the structure space and the topology is called the hull-kernel topology. Endow $m\tilde{R}$ with the hull-kernel topology. We have the following theorem.

Theorem 1.5: Regarding $m\tilde{R} - \{R\}$ as a subspace of $m\tilde{R}$, it is homeomorphic to Rm R.

Proof: By Lemma 1.4 there is a bijection α: $m\tilde{R} - \{R\} \to \text{Rm } R$ that sends M onto $M \cap R$. We need only to show that α is a continuous open map. Let $r \in R$ and $F_r = \{N \in \text{Rm } R \mid r \in N\}$. Then $\alpha^{-1}(F_r) = \{M \in m\tilde{R} - \{R\} \mid r \in M\}$ which is a basic closed set in $m\tilde{R} - \{R\}$. Thus α is continuous. Let $\tilde{r} \in \tilde{R}$ and $U_{\tilde{r}} = \{M \in m\tilde{R} - \{R\} | \tilde{r} \notin M\}$. Then $U_{\tilde{r}}$ is a basic open set in $m\tilde{R} - \{R\}$. We show that $\alpha(U_{\tilde{r}})$ is open in Rm R as follows: Let $N \in \alpha(U_{\tilde{r}})$ and $M \in U_{\tilde{r}}$ such that $\alpha(M) = M \cap R = N$. Since $M \in U_{\tilde{r}}$, then $\tilde{r} \notin M$. Since $M \neq R$ and both R and M are prime ideals of $\tilde{R}$, there exists an element a in R but not in M such that $a\tilde{r} \in R$ and $a\tilde{r} \notin M$. Let $r = a\tilde{r}$ and $V_r = \{N' \in \text{Rm } R | r \notin N'\}$. Since $r = a\tilde{r} \notin M \cap R = N$, N is in the basic open set V_r. Let $N' \in V_r$ and $M' \in m\tilde{R} - \{R\}$ such that $M' \cap R = N'$. Then since $a\tilde{r} = r \notin N' = M' \cap R$, we have $\tilde{r} \notin M'$. Thus $M' \in U_{\tilde{r}}$, and hence $N' = M' \cap R \in \alpha(U_{\tilde{r}})$. Therefore, $N \in V_r \subseteq \alpha(U_{\tilde{r}})$ which implies that $\alpha(U_{\tilde{r}})$ is open, and hence α is an open map.

Corollary 1.6: Rm R is a locally compact space.

Proof: Since $\tilde{R}$ is a commutative $\mathbb{C}$-algebra with identity, the structure space $m\tilde{R}$ is a compact T_1 space [GJ, p. 111]. Now $m\tilde{R} - \{R\}$ is open in $m\tilde{R}$. Thus $m\tilde{R} - \{R\}$, hence Rm R, is a locally compact space.

If R is the $\mathbb{C}$-algebra of all complex-valued continuous functions on a locally compact Hausdorff space vanishing at infinity, then we know the exact form of the regular maximal ideals of R and the structure space is a locally compact Hausdorff space. We first define the following notations.

X = a locally compact Hausdorff space.

$\hat{X}$ = the one point compactification of X.

$C_o(X)$ = the $\mathbb{C}$-algebra of all complex-valued continuous functions on X vanishing at infinity, that is, $f \in C_o(X)$ if and only if f is continuous on X and for each $\varepsilon > 0$ there exists a compact subset K_ε of X such that $|f(x)| < \varepsilon$ for all $x \in X - K_\varepsilon$.

$C(\hat{X})$ = the $\mathbb{C}$-algebra of all complex-valued continuous functions on $\hat{X}$.

$\widetilde{C_o(X)}$ = the $\mathbb{C}$-algebra obtained by adjoining the complex identity to $C_o(X)$.

$Z(f) = \{x \in X \text{ (or } \hat{X}) \mid f(x) = 0,\ f \in C_o(X) \text{ (or } C(\hat{X}))\}$.

$Z(I) = \{Z(f) \mid f \in I\}$.

<u>Lemma 1.7</u>: $\widetilde{C_o(X)}$ is isomorphic to $C(\hat{X})$.

<u>Proof</u>: $\widetilde{C_o(X)}$ is isomorphic to $C(\hat{X})$ under the map $(k,f) \to k + f$.

<u>Lemma 1.8</u>: The family $Z(C_o(X)) = \{Z(f) \mid f \in C_o(X)\}$ is a base for the closed sets of X.

<u>Proof</u>: Let F be a closed set in X and $x \notin X$. Then $F \cup \{\infty\}$ is a closed set in $\hat{X}$ and $x \notin F \cup \{\infty\}$. Since $\hat{X}$ is completely regular, there exists a continuous function $g\colon \hat{X} \to \mathbb{C}$ such that $g(x) = 1$ and $g(F \cup \{\infty\}) = 0$. Let $f = g|_X$. Then $f \in C_o(X)$, $F \subseteq Z(f)$ and $x \notin Z(f)$. Thus $Z(C_o(X))$ is a base for the closed sets of X.

<u>Lemma 1.9</u>: For each $x \in X$ the set $N_x = \{f \in C_o(X) \mid f(x) = 0\}$ is a regular maximal ideal of $C_o(X)$.

<u>Proof</u>: For each $x \in X$, define a map $\alpha\colon C_o(X) \to \mathbb{C}$ by

$\alpha(f) = f(x)$. It is clear that α is a $\mathbb{C}$-algebra epimorphism with kernel N_x. Thus $C_o(X)/N_x \cong \mathbb{C}$ and hence N_x is a regular maximal ideal of $X_o(X)$.

The next lemma is the converse of Lemma 1.9.

<u>Lemma 1.10</u>: If N is a regular maximal ideal of $C_o(X)$, then there exists $x \in X$ such that $N = N_x$.

<u>Proof</u>: By Lemma 1.4 there exists a maximal ideal $M \neq C_o(X)$ in $C(\hat{X})$ such that $M \cap C_o(X) = N$. Since $\hat{X}$ is a compact Hausdorff space, $M = M_x$ for some $x \in X$ [GJ, p. 56]. Since $M \neq C_o(X)$, $x \neq \infty$. Thus $x \in X$. But then $N = M \cap C_o(X)$ $= M_x \cap C_o(X) = \{f \in C_o(X) | f(x) = 0\} = N_x$.

Lemma 1.10, together with Lemma 1.9, gives us the following theorem.

<u>Theorem 1.11</u>: The set of regular maximal ideals of $C_o(X)$ is precisely the set of all $N_x = \{f \in C_o(X) | f(x) = 0\}$, for $x \in X$. The ideals N_x are distinct for distinct x. For each x, $C_o(X)/N_x \cong \mathbb{C}$

<u>Proof</u>: We need only to show the second assertion. It follows immediately from the fact that X is completely regular.

Now we prove the main result in this chapter.

<u>Theorem 1.12</u>: The structure space Rm $C_o(X)$ of $C_o(X)$ is homeomorphic to X, and hence Rm $C_o(X)$ is a locally compact Hausdorff space.

Proof: By Theorem 1.11, the map α: $X \to \mathrm{Rm}\ C_o(X)$ defined by $\alpha(x) = N_x$ is well-defined and bijective. Since $\alpha(Z(f)) = \{N_x | f \in N_x\}$ and $\{Z(f) | f \in C_o(X)\}$, $\{N_x | f \in N_x\}$ are bases for the closed sets of X and $\mathrm{Rm}\ C_o(X)$ respectively, then α is a continuous closed map. Hence, it is a homeomorphism from X onto $\mathrm{Rm}\ C_o(X)$.

Corollary 1.13: $mC(\hat{X})$ and $\widehat{\mathrm{Rm}\ C_o(X)}$, the one point compactification of $\mathrm{Rm}\ C_o(X)$, are homeomorphic.

CHAPTER II

LOCALIZATION OF COMMUTATIVE $\mathbb{C}$-ALGEBRAS WITHOUT IDENTITY

Throughout this chapter R denotes a commutative $\mathbb{C}$-algebra without identity and S denotes a multiplicative closed set in R in the sense that $0 \notin S$ and $st \in S$ whenever $s \in S$ and $t \in S$.

Let $R_S = \{(r,s) | r \in R,\ s \in S\}$. Define a relation $\sim$ on R_S by $(r_1,s_1) \sim (r_2,s_2)$ if and only if there exists $s \in S$ such that $s(r_1s_2 - r_2s_1) = 0$.

Lemma 2.1: $\sim$ is an equivalence relation on R_S.

Proof: Clearly, $\sim$ is reflexive and symmetric. Let $(r_1,s_1) \sim (r_2,s_2)$ and $(r_2,s_2) \sim (r_3,s_3)$. There exist t, $s \in S$ such that $t(r_1s_2 - r_2s_1) = 0$ and $s(r_2s_3 - r_3s_2) = 0$. Thus

$$tr_1s_2 - tr_2s_1 = 0 \qquad (1)$$

and

$$sr_2s_3 - sr_3s_2 = 0 \qquad (2)$$

Since $ss_3(1) + ts_2(2) = 0$, we have $(tss_2)(r_1s_3 - r_2s_1) = 0$. Since $tss_2 \in S$, $(r_1,s_1) \sim (r_3,s_3)$. Thus $\sim$ is transitive and hence $\sim$ is an equivalence relation on R_S.

Our notation for the equivalence class of (r,s) will be r/s. We can make the equivalence classes of R_S into a

$\mathbb{C}$-algebra by defining $r_1/s_1 + r_2 s_2 = (r_1 s_2 + r_2 s_1)/s_1 s_2$, $k(r_1/s_1) = kr_1/s_1$ and $(r_1/s_1)(r_2/s_2) = r_1 r_2/s_1 s_2$, where r_1 and r_2 in R, s_1 and s_2 in S and k in $\mathbb{C}$. Straightforward computations show that the above operations are well-defined and show the following lemmas. We use R_S to denote the set of equivalence classes.

Lemma 2.2: R_S is a commutative $\mathbb{C}$-algebra with identity s/s for any $s \in S$ and the map $r \to rs/s$ is a $\mathbb{C}$-algebra homomorphism from R into R_S.

Lemma 2.3: The set $S' = S \cup \{e\}$ is a multiplicative closed set in $\tilde{R}$ ($\tilde{R}$ is the $\mathbb{C}$-algebra obtained by adjoining the complex identity to R and e is the identity of $\tilde{R}$) and $\tilde{R}_{S'} \cong R_S$.

If r is a non-nilpotent element in R, i.e., $r^n \neq 0$ for $n \geq 1$, then the set $S = \{r^n | n \geq 1\}$ is clearly a multiplicative closed set in R. For convenience we denote R_S by R_r.

We need the following lemmas to establish a correspondence between the prime ideals of R_S and the prime ideals of R disjoint from S.

Lemma 2.4: If P is a prime ideal of R disjoint from S, then $P_S = \{g/s | g \in P,\ s \in S\}$ is a prime ideal of R_S.

Proof: Clearly, P_S is an ideal of R_S. Let $g/s,\ h/t \in R_S$ and $g/s \cdot h/t \in P_S$. Then $gh/st = k/u$ for some $k \in P$. It follows that there exists $s' \in S$ such that $s'(ugh - stk) = 0$. Thus $s'ugh = s'stk \in P$ since $k \in P$. Since $s'u \notin P$,

$gh \in P$. P is a prime ideal, so either $g \in P$ or $h \in P$. Thus $g/s \in P_S$ or $h/t \in P_S$. Hence, P_S is prime.

Lemma 2.5: If J is a prime ideal of R_S, then the set $I = \{x | xs/s \in J\}$ is a prime ideal of R disjoint from S and $I_S = J$.

Proof: (1) I is an ideal of R. Let x and y be in I. Then xs/s and ys/s are in J. J is an ideal, so $(x + y)s/s = xs/s + ys/s \in J$. Thus $x + y \in I$. Let $t \in R$ and $x \in I$. Then $ts/s \in R_S$ and $xs/s \in J$. Thus $txs/s = txs^2/s^2 = ts/s \cdot xs/s \in J$ and hence $tx \in I$

(2) I is prime. If $xy \in I$ then $xs/s \cdot ys/s = xyss/ss = xys/s \in J$. J is prime, so either $xs/s \in J$ or $ys/s \in J$. Thus either $x \in I$ or $y \in I$. Hence I is prime.

(3) $S \cap I = \phi$. If $s \in S \cap I$ then $ss/s \in J$. Thus $s/s = sss/sss = ss/s \cdot s/ss \in J$. Hence $J = R_S$, a contradiction. Therefore, $S \cap I = \phi$.

(4) $I_S = J$. Let $x/s \in I_S$ where $x \in I$. Then $xs/s \in J$ and hence $x/s \cdot ss/s = xs/s \in J$. Since $ss/s \notin J$ and J is prime, $x/s \in J$. Thus $I_S \subseteq J$. On the other hand, let $j = x/s \in J$. Then $xs/s = x/s \cdot ss/s \in J$. Thus $x \in I$ and $x/s \in I_S$. Hence $J \subseteq I_S$. So $I_S = J$.

Lemma 2.6: If P and Q are two distinct prime ideals of R disjoint from S, then P_S and Q_S are two distinct prime ideals in R_S.

Proof: Since $P \neq Q$, there exists $x \in P$ with $x \notin Q$. Then $x/s \in P_S$ for $s \in S$. If $x/s \in Q_S$, then $x/s = g/t$ for some

$g \in Q$. It follows that there exists $s' \in S$ such that $s'(tx - sg) = 0$. Thus $s'tx = s'sg \in Q$. Q is a prime ideal and $s't \notin Q$, so $x \in Q$, a contradiction. Thus, $x/s \notin Q_S$ and hence $P_S \neq Q_S$.

We summarize Lemma 2.4, Lemma 2.5 and Lemma 2.6 as follows.

Theorem 2.7: There is an order preserving bijection between the set of all prime ideals of R_S and the set of all prime ideals of R disjoint from S.

Corollary 2.8: There is a bijection between the set of all maximal ideals of R_S and the set of all maximal prime ideals of R disjoint from S.

Proof: The map α defined by sending maximal prime ideals P of R disjoint from S onto P_S is the desired bijection.

Corollary 2.8 plays a very important role in Chapter V.

CHAPTER III

FINITE-FIBERED COVERING SPACES

Let X and Y be two locally compact Hausdorff spaces. In this chapter we give the algebraic necessary conditions for Y to be a finite-fibered covering space of X. First recall:

Definition 3.1: Let p be a continuous function from a topological space Y onto a space X. If each $x \in X$ has an open neighborhood U_x such that $p^{-1}(U_x)$ is a finite disjoint union of open sets U, each of which is homeomorphic to U_x under the map $p|_U$, then p is called a covering projection with finite fibers, X is called the base space, and Y is a finite-fibered covering space of X.

Throughout this chapter X denotes the base space, Y a finite-fibered covering space of X and p a covering projection from Y onto X.

Lemma 3.2: p is an open map.

Proof: [S, p. 63].

Lemma 3.3: Y is a Hausdorff space if X is.

Proof: Let y and z be two distinct points in Y. It is clear that y and z can be separated by open sets in Y if $p(y) \neq p(z)$. Now suppose that $p(y) = p(z) = x$. There exists an open neighborhood U_x of x such that $p^{-1}(U_x) = \bigcup_{i=1}^{n} U_i$ where U_i is open in Y, $U_i \cap U_j = \phi$ for $i \neq j$ and $U_i \cong U_x$ under the map $p|_{U_i}$. Since $y \in p^{-1}(U_x)$, $y \in U_i$ for some i. Similarly, $z \in U_j$ for some j. Since $y \neq z$ and $U_i \cong U_x$, y and z cannot be in the same U_i. $U_i \cap U_j = \phi$ implies that Y is Hausdorff.

If we assume further that X and Y both are locally compact Hausdorff spaces, then p has additional properties.

Lemma 3.4: p is a closed map.

Proof: Let F be a closed set in Y and $x \in X - p(F)$. Then there exists an open neighborhood U_x of x such that $p^{-1}(U_x) = \bigcup_{i=1}^{n} U_i$, where U_i are open, $U_i \cap U_j = \phi$ for $i \neq j$ and $U_i \cong U_x$ under $p|_{U_i}$. Let $p^{-1}(x) = \{y_1, y_2, \cdots, y_n\}$ and $y_i \in U_i$. Then, for each y_i, there exists an open neighborhood U_{y_i} of y_i such that $U_{y_i} \cap F = \phi$ since Y is regular. For each i, let $V_i = U_{y_i} \cap U_i$. Then V_i is open, $y_i \in V_i$, $V_i \cap F = \phi$ and $V_i \cap V_j = \phi$ for $i \neq j$. By Lemma 3.2, $p(v_i)$ is open in X and $x \in p(V_i)$ for all i. Thus $x \in \bigcap_{i=1}^{n} p(V_i) = V$ and V is open. Clearly, $V \subseteq U_x$ and hence $p^{-1}(V) = p^{-1}(V) \cap p^{-1}(U_x) = p^{-1}(V) \cap (\bigcup_{i=1}^{n} U_i) = \bigcup_{i=1}^{n} (p^{-1}(V) \cap U_i)$. For each $i = 1, 2, \cdots, n$, if $w \in p^{-1}(V) \cap U_i$, then $w \in U_i$ and $p(w) \in p(V_i) \subseteq U_x$. Since U_i is homeomorphic to U_x, $w \in V_i$. Thus $p^{-1}(V) \cap U_i \subseteq V_i$ for all i. Hence

$p^{-1}(V) = \bigcup_{i=1}^{n} (p^{-1}(V) \cap U_i) \subseteq \bigcup_{i=1}^{n} V_i$. We claim that $V \cap p(F) = \phi$. Let $z \in V \cap p(F)$. Since $z \in p(F)$, there exists $y \in F$ such that $p(y) = z$. Since $z \in V$, $y \in p^{-1}(z) \subseteq p^{-1}(V)$ and hence $y \in \bigcup_{i=1}^{n} V_i$. Thus $\bigcup_{i=1}^{n} V_i \cap F \neq \phi$, a contradiction. So $V \cap p(F) = \phi$ which implies that $X - p(F)$ is open. Thus $p(F)$ is closed.

Lemma 3.5: p is a proper map.

Proof: Let $y \in Y$. By definition of p, $p^{-1}(y)$ is finite and hence it is compact. p is a closed. Thus p is a proper map [B2, Theorem 1, p. 101].

We need the next lemma not only for proving Theorem 3.7, but also for later use.

Lemma 3.6: Let A and B be two locally compact Hausdorff spaces. Suppose $q: A \to B$ is a proper map. Then q induces a $\mathbb{C}$-algebra homomorphism q^* from $C_o(B)$ into $C_o(A)$.

Proof: Define $q^*: C_o(B) \to C_o(A)$ by $q^*(f) = f \circ q$. We show that q^* is well-defined. Let $f \in C_o(B)$ and $\varepsilon > 0$. Then there exists a compact subset K of B such that $|f(b)| < \varepsilon$ for $b \in B - K$. Since q is a proper map, $q^{-1}(K)$ is a compact subset of A [B2, Proposition 7, p. 104]. Let $a \in A - q^{-1}(K)$. Then $q(a) \notin K$ and hence $|f \circ q(a)| = |f(q(a))| < \varepsilon$. Therefore $f \circ p$ is in $C_o(Y)$. Clearly, q^* is a $\mathbb{C}$-algebra homomorphism

Theorem 3.7: The covering projection p induces a $\mathbb{C}$-algebra monomorphism p^* from $C_o(X)$ into $C_o(Y)$.

Proof: Because of Lemma 3.5 and Lemma 3.6 we need only to show the map p^*: $C_c(X) \to C_o(Y)$ defined by $p^*(f) = f \circ p$ is injective. Suppose $f \circ p = 0$. Since p is surjective, for each $x \in X$ there exists $y \in Y$ such that $p(y) = x$. Thus $f(x) = f(p(y)) = f \circ p(y) = 0$. p^* is injective.

If $f \in C_o(X)$, we let f^* denote the image of f under p^*. Note that if f is not the zero map, then f is a non-nilpotent element in $C_o(X)$ and f^* is a non-nilpotent element in $C_o(Y)$ since p^* is injective. Thus $C_o(X)_f$ and $C_o(Y)_{f^*}$ [Chapter II] are two commutative $\mathbb{C}$-algebras with identity. The following theorem gives the algebraic necessary conditions for a locally compact Hausdorff space to be a finite-fibered covering space of a locally compact Hausdorff space.

Theorem 3.8: Let X and Y be two locally compact Hausdorff spaces. If Y is a finite-fibered covering space of X, then (i) for each regular maximal ideal N of $C_o(X)$ there is an $f \in C_o(X) - N$ and a positive integer n such that $X - Z(f) \subseteq F$, a compact subset of X, and $C_o(Y)_{f^*} = (C_o(X)_f)^n$, a product as rings of n copies of $C_o(X)_f$, and (ii) no regular maximal ideal of $C_o(Y)$ contains all f^*, where f satisfies (i).

Proof: (i) Let N be a regular maximal ideal of $C_o(X)$. By Lemma 1.10, there exists $x \in X$ such that $N = N_x$. Since Y is a finite-fibered covering space of X, there exists an open neighborhood U_x of x such that $p^{-1}(U_x) = \bigcup_{i=1}^{n} U_i$ where U_i are open, $U_i \cap U_j = \phi$ for $i \neq j$ and $U_i \cong U_x$ under $p|_{U_i}$. Since X is locally compact, there exists a compact subset

F of X such that $x \in \text{int}(F) \subseteq F \subseteq U_x$. Since X is also completely regular, there exists a $f \in C_o(X)$ such that $x \in W = X - Z(f) \subseteq \text{int}(F) \subseteq F \subseteq U_x$. Then $f \notin N_x$ and $p^{-1}(W) = \bigcup_{i=1}^{n} W_i$ where W_i are open, $W_i \subseteq U_i$, $W_i \cap W_j = \phi$ for $i \neq j$ and $W_i \cong W$ under $p|_{U_i}$. Furthermore, let $\bar{W}$ be the closure of W in X, $\bar{W}_i$ the closure of W_i in Y, bdry W the boundary of W and bdry W_i the boundary of W_i. Then we have $p^{-1}(W) = \bigcup_{i=1}^{n} \bar{W}_i$, $\bar{W}_i \cap \bar{W}_j = \phi$ for $i \neq j$ and $\bar{W}_i \cong \bar{W}$ under $p|_{U_i}$ and $p^{-1}(\text{bdry } W) = \bigcup_{i=1}^{n} (\text{bdry } W_i)$, $(\text{bdry } W_i) \cap (\text{bdry } W_j) = \phi$ for $i \neq j$ and bdry $W_i \cong$ bdry W under the map $p|_{U_i}$. Let f_i, $i = 1, 2, \cdots, n$, be such that

$$f_i(y) = \begin{cases} f^*(y) & \text{if } y \in \bar{W}_i \\ 0 & \text{if } y \in \bar{Y} - W_i \end{cases}.$$

Then clearly, $f_i \in C_o(Y)$ for each i. Let $e_i = f_i/f^*$. Then $e_i \in C_o(Y)_{f^*}$.

Claim 1: $e_i e_j = 0$ for $i \neq j$.

If $y \notin \bar{W}_i$, then $f_i(y) = 0$. If $y \notin \bar{W}_j$, then $f_j(y) = 0$. Since $\bar{W}_i \cap \bar{W}_j = \phi$, $X - \bar{W}_i \cup X - \bar{W}_j = Y$. Thus $f^*(f_i f_j)(y) = 0$ for all $y \in Y$. Hence $f_i/f^* \cdot f_j/f^* = 0$, i.e., $e_i e_j = 0$.

Claim 2: $e_1 + e_2 + \cdots + e_n = f^*/f^*$.

If $y \in \bigcup_{i=1}^{n} \bar{W}_i$, then y is in one and only one $\bar{W}_i$. Thus

$$\begin{aligned} & f^*(f^*f_1 + f^*f_2 + \cdots + f^*f_n - f^*f^*)(y) \\ &\quad = f^*(y)(f^*(y)f_i(y) - f^*(y)f^*(y)) \\ &\quad = f^*(y)(f^*(y)f^*(y) - f^*(y)f^*(y)) = 0. \end{aligned}$$

If $y \notin \bigcup_{i=1}^{n} \bar{W}_i$, then $f^*(y) = f(p(y)) = 0$ since $p(y) \notin \bar{W}$. Thus for any $y \in Y$, $f^*(f^*f_1 + f^*f_2 + \cdots + f^*f_n - f^*f^*) = 0$.

Hence, $e_1 + e_2 + \cdots + e_n = 0$. By Claim 1 and Claim 2 we have $C_o(Y)_{f*} = \bigoplus_{i=1}^{n} C_o(Y)_{f*}\, e_i$.

Claim 3: $C_o(X)_f = C_o(Y)_{f*}\cdot e_i$ for each i.

Define a map α: $C_o(X)_f \to C_o(Y)_{f*}\cdot e_i$ by $\alpha(h/f^n) = h^*/f^{*n}\cdot e_i$. α is well-defined. For if $h/f^n = k/f^m$, then there exists f^q such that $f^q(hf^m - kf^n) = 0$. Thus $f^{*q}(h^*f^{*m+1}f_i - k^*f^{*n+1}f_i) = 0$ and hence $h^*/f^{*n}\cdot f_i/f^* = k^*/f^{*n}\cdot f_i/f^*$. Clearly, α is a $\mathbb{C}$-algebra homomorphism. α is injective. For if $h^*/f^{*n}\cdot e_i = h^*/f^{*n}\cdot f_i/f^* = 0$, then there exists f^{*r} such that $f^{*r}h^*f_i = 0$. Let $x \in \bar{W}$. Then there exists $y \in \bar{W}_i$ such that $p(y) = x$. Thus $f^{r+1}(x)h(x) = f^r(x)h(x)f(x) = f^r(p(y))h(p(y))f(p(y)) = f^{*r}(y)h^*(y)f_i(y) = 0$. If $x \notin \bar{W}$, then $f(x) = 0$. Thus $f^{r+1}h(x) = 0$ for all $x \in X$. Therefore, $h/f^n = 0$. We show next that α is also surjective: Let $g/f^{*n}\cdot e_i = g/f^{*n}\cdot f^i/f^* = gf_i/f^{*n+1} \in C_o(Y)_{f*}\cdot e_i$. Then

$$gf_i(y) = \begin{cases} g(y)f^*(y) & \text{if } y \in \bar{W}_i \\ 0 & \text{if } y \in Y - W_i \end{cases}$$

Since $\bar{W}_i \cong \bar{W}$, there exists a function $h \in C_o(X)$ such that $h^*|_{\bar{W}_i} = gf_i|_{\bar{W}_i} = gf^*|_{\bar{W}_i}$. We claim that $\alpha(h/f^{n+1}) = g/f^{*n}\cdot e_i$, i.e. $h^*/f^{*n+1}\cdot e_i = f/f^{*n}\cdot e_i$; for if $y \in Y - W_i$, then $f_i(y) = 0$. Thus

$$f^*(f^{*n+1}h^*f_i - f^{*n+2}gf_i)(y) = 0.$$

If $y \in \bar{W}_i$, then $h^*(y) = gf_i(y) = gf^*(y)$. Thus

$$\begin{aligned} &f^*(f^{*n+1}h^*f_i - f^{*n+2}gf_i)(y) \\ &\quad = f^*(y)(f^{*n+3}(y)g(y) - f^{*n+3}(y)g(y)) = 0. \end{aligned}$$

Hence $f^*(f^{*n+1}h^*f_i - f^{*n+2}gf_i)(y) = 0$ for all y, i.e. $h^*/f^{*n+1}\cdot e_i = g/f^{*n}\cdot e_i$. Therefore, α is surjective. By Claim 3 we have $C_o(Y)_{f^*} \cong (C_o(X)_f)^n$.

(ii) Let M be a regular maximal ideal of $C_o(Y)$. Then, by Lemma 1.10, there exists $y \in Y$ such that $M = M_y$. Since $M_y \cap p^*(C_o(X)) = N_{p(y)} = \{g \in C_o(X) | g(p(y)) = 0\}$ is a regular maximal ideal of $C_o(X)$, there exists $f \notin N_{p(y)}$ satisfying (i). Since $f \notin N_{p(y)}$, then $f^* \notin M_y \cap p(C_o(X))$. But $f^* \in p(C_o(X))$. Hence $f^* \notin M_y$.

CHAPTER IV

SEPARABLE $C_o(X)$-ALGEBRAS

Let S be a commutative $\mathbb{C}$-algebra without identity. Recall that S is a $C_o(X)$-algebra if there exists a $\mathbb{C}$-algebra homomorphism θ: $C_o(X) \to S$, S is a $C_o(X)$-algebra extension if θ is injective and for simplicity we regard $C_o(X)$ as a $\mathbb{C}$-subalgebra of S. S is a separable $C_o(X)$-algebra extension if S is a $C_o(X)$-algebra extension and satisfies: (i) for each regular maximal ideal N of $C_o(X)$ there exists $f \in C_o(X) - N$ and a positive integer n such that $X - Z(f) \subseteq F$, a compact subset of X, and $S_f = (C_o(X)_f)^n$, a product as rings of n copies of $C_o(X)_f$, and (ii) no regular maximal ideal of S contains all f where f satisfies (i). Theorem 3.8 proves that if Y and X are two locally compact Hausdorff spaces and Y is a finite-fibered covering of X, then $C_o(Y)$ is a separable $C_o(X)$ extension. In this chapter we first deal with the converse question: Is Y a finite-fibered covering space of X if $C_o(Y)$ is a separable $C_o(X)$-algebra extension? We prove that the answer is positive by showing a strong version of this question: If S is a separable $C_o(X)$-algebra extension, then the structure space Rm S of S is a finite-fibered covering space of X. Since

$Rm\ C_o(Y) \cong Y$ (Theorem 1.12), the converse of Theorem 3.8 follows immediately.

Throughout the following lemmas S denotes a separable $C_o(X)$-algebra extension.

<u>Lemma 4.1</u>: If M is a regular maximal ideal of S, then $M \cap C_o(X) = N$ is a regular maximal ideal of $C_o(X)$.

<u>Proof</u>: Let θ: $C_o(X) \to S$ be the injective $\mathbb{C}$-algebra homomorphism. Clearly, the map $\tilde{\theta}$: $\widetilde{C_o(X)} = C(\hat{X}) \to \hat{S}$ defined by $\tilde{\theta}(k,f) = (k, \theta(f))$ is an injective $\mathbb{C}$-algebra homomorphism and we have the following commutative diagram:

$$\begin{tikzcd} C_o(X) \arrow[r, "\theta"] \arrow[d, hook] & S \arrow[d, hook] \\ C(\hat{X}) \arrow[r, "\tilde{\theta}"] & \tilde{S} \end{tikzcd}$$

M is a regular maximal ideal, so by Lemma 1.4, there exists a maximal ideal $\tilde{M} \neq S$ of $\tilde{S}$ such that $\tilde{M} \cap S = M$. Clearly, $\tilde{M} \cap C(\hat{X})$ is a prime ideal of $C(\hat{X})$. Suppose $\tilde{M} \cap C(\hat{X}) \subseteq C_o(X)$. Then $\tilde{M} \cap C(\hat{X})$ contains all continuous functions in $C(\hat{X})$ vanishing at a neighborhood of infinity [GJ, p. 62]. Since $\tilde{M} \cap C(\hat{X}) \cap S \subseteq \tilde{M} \cap C_o(X) \subseteq \tilde{M} \cap S = M$, M contains all continuous functions in $C(\hat{X})$ vanishing at a neighborhood of infinity, in particular all f satisfying (i), a contradiction. Hence $\tilde{M} \cap C(\hat{X}) \not\subseteq C_o(X)$. By Lemma 1.3, $\tilde{M} \cap C(\hat{X}) \cap C_o(X)$ $= \tilde{M} \cap C_o(X)$ is a regular ideal of $C_o(X)$. But $N = M \cap C_o(X)$ $= \tilde{M} \cap S \cap C_o(X) = \tilde{M} \cap C_o(X)$, so N is a regular ideal and hence is contained in a regular maximal ideal N' of $C_o(X)$.

Since S is a separable $C_o(X)$-algebra, there exists $f \in C_o(X) - N'$ such that $S_f = (C_o(X)_f)^n$. Since $f \notin N'$, $f \notin N = M \cap C_o(X)$. Thus $f \notin M$. By Corollary 2.8, M_f is a maximal ideal of $C_o(X)_f$. Since S_f is integral over $C_o(X)_f$, $M_f \cap C_o(X)_f$ is a maximal ideal of $C_o(X)_f$. But $M_f \cap C_o(X)_f = (M \cap C_o(X))_f = N_f$. Thus N_f is a maximal ideal of $C_o(X)_f$. By Corollary 2.8, N is a maximal prime ideal of $C_o(X)$ disjoint from $\{f^n | n \geq 1\}$. But $N \subseteq N'$, $f \notin N'$ and N' is a prime ideal of $C_o(X)$. Thus, by the maximality of N, $N = N'$. Hence $M \cap C_o(X) = N$ is a regular maximal ideal.

We would like to know whether or not the regular maximal ideals of $C_o(X)$ are contractions of the regular maximal ideals of S. The following lemma answers this question.

Lemma 4.2: If N is a regular maximal ideal of $C_o(X)$, then there exists a regular maximal ideal M of S such that $N = M \cap C_o(X)$.

Proof: By assumption, there exists $f \in C_o(X) - N$ such that $X - Z(f) \subseteq F$, a compact subset of X and $S_f = (C_o(X)_f)^n$. By Corollary 2.8, N_f is a maximal ideal of $C_o(X)_f$. By Corollary 2.8 together with the fact that S_f is integral over $C_o(X)_f$, we have a maximal prime ideal M' of S disjoint from $\{f^n | N \geq 1\}$ such that $M'_f \cap C_o(X)_f = N_f$. But $M'_f \cap C_o(X)_f = (M' \cap C_o(X))_f = N_f$. Thus $M' \cap C_o(X) = N$ (Theorem 2.7). We now show that M' is a regular ideal. Since M' is a prime ideal disjoint from $\{f^n | n \geq 1\}$, M' is an ideal of $\tilde{S}$ disjoint from $\{f^n | n \geq 1\}$. Thus there exists a maximal prime ideal $\tilde{M}$ of $\tilde{S}$ such that $M' \subseteq \tilde{M}$ and $M \cap \{f^n | n \geq 1\} = \emptyset$. If $\tilde{M} \subseteq S$, then $\tilde{M} \cap C(\hat{X}) \subseteq S \cap C(\hat{X}) = C_o(X)$. $\tilde{M} \cap C(\hat{X})$ is a prime ideal of $C(\hat{X})$, so $f \in M \cap C(\hat{X})$ [GJ, p. 62], a contradiction. Thus $\tilde{M} \not\subseteq S$. By Lemma 1.3, $\tilde{M} \cap S$ is a regular prime ideal of S. Since $M' \subseteq \tilde{M} \cap S$, $f \notin \tilde{M} \cap S$ and M' is a maximal prime ideal disjoint from $\{f^n | n \geq 1\}$, $M' = \tilde{M} \cap S$ and hence M' is a regular ideal of S. By Lemma 1.2, M' is contained in a regular maximal ideal M of S. Thus $N = M' \cap C_o(X) \subseteq M \cap C_o(X)$ is an ideal of $C_o(X)$ and N is a maximal ideal of $C_o(X)$, $N = M \cap C_o(X)$ or $M \cap C_o(X) = C_o(X)$. If $M \cap C_o(X) = C_o(X)$, then $C_o(X) \subseteq M$, a contradiction. Thus $N = M \cap C_o(X)$.

The next lemma shows that the only maximal ideals of S which contract to regular maximal ideals of $C_o(X)$ are the regular maximal ideals of S. We need it for later use.

<u>Lemma 4.3</u>: Let $f \in C_o(X)$ satisfy (i) and M be a maximal prime ideal of S disjoint from $\{f^n | n \geq 1\}$. Then $M \cap C_o(X)$ is a regular maximal ideal of $C_o(X)$ if and only if M is a regular maximal ideal of S.

<u>Proof</u>: The sufficient condition follows from Lemma 4.1. We prove the necessary condition. Since M is a prime ideal of S disjoint from $\{f^n | n \geq 1\}$, M is an ideal of $\tilde{S}$ disjoint from $\{f^n | n \geq 1\}$. There exists a maximal prime ideal P of $\tilde{S}$ containing M with $f^n \notin P$ for all $n \geq 1$. $\tilde{S}$ is a commutative ring with identity, so P is contained in a maximal ideal of $\tilde{S}$. If $P \subseteq S$, then $P \cap C(\hat{X}) \subseteq S \cap C(\hat{X}) = C_o(X)$. Clearly,

$P \cap C(\hat{X})$ is a prime ideal of $C(\hat{X})$, so $f \in P \cap C(\hat{X})$ [GJ, p. 62], a contradiction. Hence, $P \subseteq \tilde{M}$, $\tilde{M}$ maximal ideal of $\tilde{S}$ not equal to S. We next show that P is actually equal to $\tilde{M}$. If $P \subsetneq \tilde{M}$, then $f^n \in \tilde{M}$ and hence $P \cap S \subsetneq \tilde{M} \cap S$. Since $P \cap S$ is a prime ideal of S disjoint from $\{f^n | n \geq 1\}$ and $M \subseteq P \cap S$, $M = P \cap S \subsetneq \tilde{M} \cap S$. Thus $M \cap C_o(X) \subsetneq (\tilde{M} \cap S) \cap C_o(X)$ since $f^n \in M \cap C_o(X)$ and $f^n \notin (\tilde{M} \cap S) \cap C_o(X)$. If $(\tilde{M} \cap S) \cap C_o(X) = C_o(X)$, then $C_o(X) \subseteq \tilde{M} \cap S$ which is a regular maximal ideal of S (Lemma 1.3), a contradiction. Hence $(\tilde{M} \cap S) \cap C_o(X)$ is a proper ideal of $C_o(X)$ containing $M \cap C_o(X)$. But that contradicts the assumption that $M \cap C_o(X)$ is a regular maximal ideal of $C_o(X)$. Thus $P = \tilde{M}$, i.e., P is a regular maximal ideal of $\tilde{S}$ not equal to S. Now since $M \subseteq P$ and $f^n \notin P$, $M \subseteq P \cap S$ and $f^n \notin P \cap S$. By the maximality of M and the fact that $P \cap S$ is a regular maximal ideal of S, we conclude that $M = P \cap S$ and hence M is a regular maximal ideal of S.

Lemma 4.4: There is a continuous open map from the structure space Rm S onto X.

Proof: By Theorem 1.12, X is homeomorphic to Rm $C_o(X)$. Hence it is sufficient to show there is a continuous open map from Rm S onto Rm $C_o(X)$. By Lemma 4.1 and Lemma 4.2, the map p: Rm S $\to$ Rm $C_o(X)$ defined by $p(M) = M \cap C_o(X)$ is well-defined and surjective. We first show that p is continuous. Let $a \in C_o(X)$ and $F_a = \{N \in \text{Rm } C_o(X) | a \in N\}$, a basic closed set. Then $p^{-1}(F_a) = \{M \in \text{Rm } S | a \in M\}$ is a basic closed set of Rm S. Hence, p is continuous. Let $x \in S$ and $U_x = \{M \in \mathrm{Rm}\, S \mid x \notin M\}$ be a basic open set of Rm S. We show that $p(U_x)$ is open in Rm $C_o(X)$. Let $N \in p(U_x)$ and $M \in U_x$ such that $M \cap C_o(X) = N$. Since S is a separable $C_o(X)$-algebra extension, there exists $f \in C_o(X) - N$ such that $S_f = (C_o(X)_f)^n$. Since $f \notin N = C_o(X) \cap M$ and M is a regular maximal ideal of S, M is a maximal prime ideal of S disjoint from $\{f^n \mid n \geq 1\}$. By Corollary 2.8, M_f is a maximal ideal of S_f. Since $M_f \cap C_o(X)_f = (M \cap C_o(X))_f = N_f$ and $S_f = \bigoplus_{i=1}^{n} C_o(X)_f$, $M_f = C_o(X)_f \oplus C_o(X)_f \oplus \cdots \oplus N_f \oplus \cdots \oplus C_o(X)_f$ where N_f is on the i'th component. Since $x \in S - M$, $x/f = (r_1/f^{t_1}, r_2/f^{t_2}, \cdots, r_i/f^{t_i}, \cdots r_n/f^{t_n}) \in S_f - M_f$ where $r_i \in C_o(X)$, t_i positive integer, $i = 1, 2, \cdots, n$. Thus $r_i/f^{t_i} \notin N_f$ and hence $r_i \notin N$. Let $V = \{N' \in \mathrm{Rm}\, C_o(X) \mid fr_i \notin N'\}$. Since V is open in Rm $C_o(X)$, we complete the proof if we can show $N \in V \subseteq p(U_x)$. Clearly, $N \in V$. Let $N' \in V$. Then $fr_i \notin N'$. Since N' is an ideal of $C_o(X)$, $f \notin N'$ and $r_i \notin N'$. By Corollary 2.8, $f \notin N'$ implies N'_f is a maximal ideal of $C_o(X)_f$. Since $S_f = \bigoplus_{i=1}^{n} C_o(X)_f$, $I = C_o(X)_f \oplus \cdots \oplus N'_f \oplus \cdots \oplus C_o(X)_f$, where N'_f is the i'th component, is a maximal ideal of S_f and $I \cap C_o(X)_f = N'_f$. By Corollary 2.8, there exists a unique maximal prime ideal Q of S disjoint from $\{f^n \mid n \geq 1\}$ such that $Q_f = I$. Thus $N'_f = I \cap C_o(X)_f = Q_f \cap C_o(X)_f = (Q \cap C_o(X))_f$. By Theorem 2.7, $N' = Q \cap C_o(X)$. By Lemma 4.3, Q is a regular maximal ideal of S. Because $r_i \notin N'$ implies $r_i/f^{t_i} \notin N'_f$, $x/f \notin Q_f$. Thus $x \notin Q$ and hence $Q \in U_x$. Therefore, $p(Q) = Q \cap C_o(X) = N' \in p(U_x)$. Hence

$N \in V \subseteq p(U_x)$.

Let $N_o \in \text{Rm } C_o(X)$ and $f \in C_o(X) - N_o$ be such that $S_f = (C_o(X)_f)^n$. Let $V_f = \{N \in \text{Rm } C_o(X) \mid f \notin N\}$ and $A_i = \{M \in \text{Rm } S \mid f \notin M,\ M_f = C_o(X)_f \oplus \cdots \oplus N_f \oplus \cdots \oplus C_o(X)_f,\ N_f$ on the i'th component$\}$, $i = 1,2,\cdots,n$.

<u>Lemma 4.5</u>: $p^{-1}(V_f) = \bigcup_{i=1}^{n} A_i$.

<u>Proof</u>: Let $M \in p^{-1}(V_f)$. Then $p(M) = M \cap C_o(X) = N \in V_f$ and hence $f \notin N = M \cap C_o(X)$. Thus M_f is a maximal ideal of S_f. Since $S_f = (C_o(X)_f)^n = \bigoplus_{i=1}^{n} C_o(X)_f$, $M_f = C_o(X)_f \oplus \cdots \oplus I \oplus \cdots \oplus C_o(X)_f$, where I is a maximal ideal of $C_o(X)_f$ and is on the i'th component. Since $I = M_f \cap C_o(X)_f = (M \cap C_o(X))_f = N_f$, $M_f = C_o(X)_f \oplus \cdots \oplus N_f \oplus \cdots \oplus C_o(X)_f$ and hence $M \in A_i$. Thus $p^{-1}(V_f) \subseteq \bigcup_{i=1}^{n} A_i$. On the other hand, if $M \in \bigcup_{i=1}^{n} A_i$, then $M \in A_i$ for some i. Hence $f \notin M$ and $M_f = C_o(X)_f \oplus \cdots \oplus N_f \oplus \cdots \oplus C_o(X)_f$ for some $N \in V_f$. But $M_f \cap C_o(X)_f = (M \cap C_o(X))_f = N_f$. Hence, by Theorem 2.7, $M \cap C_o(X) = N \in V_f$. Thus $M \in p^{-1}(V_f)$ and hence $\bigcup_{i=1}^{n} A_i \subseteq p(V_f)$. Therefore, $p^{-1}(V_f) = \bigcup_{i=1}^{n} A_i$.

<u>Lemma 4.6</u>: (i) Each A_i is open in Rm S.

(ii) $A_i \cap A_j = \emptyset$ for $i \neq j$.

(iii) $p|_{A_i}$ is a homeomorphism from A_i onto V_f.

<u>Proof</u>: (i) Let $M \in A_i$. Then $M_f = C_o(X)_f \oplus \cdots \oplus N_f \oplus \cdots \oplus C_o(X)_f$ where $N \in V_f$. Let $(0,0,\cdots,f/f,0,\cdots,0) = a/f^t$. Since $f/f \notin N_f$, $a/f^t \notin M_f$. Thus $a \notin M$. Let $U = \{M' \in \text{Rm } S \mid fa \notin M'\}$. Then $M \in U$ and U is open. We complete the

proof by showing that $U \subseteq A_i$. Let $M' \in U$. Then $fa \notin M'$ and hence $f \notin M'$ and $a \notin M'$ since M' is an ideal. M'_f is a maximal ideal of S_f since $f \notin M'$. Thus $M'_f = C_o(X)_f \oplus \cdots \oplus I \oplus \cdots \oplus C_o(X)_f$ where I is a maximal ideal of $C_o(X)_f$ and is on the j'th component. By Corollary 2.8, there exists a maximal prime ideal J of $C_o(X)$ disjoint from $\{f^n \mid n \geq 1\}$ such that $J_f = I$. Since $(M' \cap C_o(X))_f = M'_f \cap C_o(X)_f = I = J_f$, $M' \cap C_o(X) = J$. By Lemma 4.1, $M' \cap C_o(x)$, hence J, is a regular maximal ideal of $C_o(X)$ and $f \notin J$. Thus $J \in V_f$. Also $a \notin M'$ implies $a/f^t = (0,0,\cdots,f/f,\cdots,0) \notin M'_f = C_o(X)_f \oplus \cdots \oplus J_f \oplus \cdots \oplus C_o(X)_f$. Thus J_f must be on the i'th component and hence $M' \in A_i$. Therefore $M \in U \subseteq A_i$.

(ii) Trivial.

(iii) Clearly, $p|_{A_i}$ is a continuous bijection from A_i onto V_f. We need only to show that $p|_{A_i}$ is open. It follows from the fact that p is an open map (Lemma 4.4) and A_i is an open set of Rm S.

We summarize Lemma 4.4, Lemma 4.5 and Lemma 4.6 as follows:

<u>Theorem 4.7</u>: If S is a separable $C_o(X)$-algebra extension, then the structure space Rm S of S is a finite-fibered covering space of X.

Note that, since the composition of two proper maps is a again a proper map [B2, p. 99] and the composition of two $\mathbb{C}$-algebra homomorphisms is a $\mathbb{C}$-algebra homomorphism, the set of all locally compact Hausdorff finite-fibered covering spaces of a fixed locally compact space X together with the proper maps between two covering spaces which commute with covering projections, and the set of all separable $C_o(X)$-algebra extensions together with the $\mathbb{C}$-algebra homomorphisms between two separable $C_o(X)$-algebra extensions which commute with $C_o(X)$-algebra homomorphisms form two categories. We prove that, in the second half of this chapter, the map $Y \to C_o(Y)$ is a full and faithful functor and an example is given to show that the functor in general is not a contravariant equivalence.

Let G denote the category of locally compact Hausdorff finite-fibered covering spaces of a fixed locally compact Hausdorff space X, and let H denote the category of separable $C_o(X)$-algebra extensions.

Theorem 4.8: The map ϕ: $G \to H$ defined by $\phi(Y) = C_o(Y)$ and $\phi(q) = q^*$ is a full and faithful contravariant functor.

Proof: We first show that ϕ is a functor. (i) Let Y be an object in G. Then, by Theorem 3.8 $\phi(Y) = C_o(Y)$ is an object in H. Let

$$\begin{tikzcd} Y \arrow[rr, "q"] \arrow[dr, "p_1"'] & & Z \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

be a commutative diagram in G, where q is a proper map and p_1, p_2 are covering projections. Then, by Lemma 3.6, $\phi(q) = q^*$: $C_o(X) \to C_o(Y)$ is a $\mathbb{C}$-algebra homomorphism and

$q^*p_2^*(f) = (f\circ p_2)(q) = f(p_2\circ q) = f\circ p_1 = p_1^*(f)$ for all $f \in C_o(X)$. Hence the diagram

$$\begin{tikzcd} C_o(Z) \arrow[rr, "q^*"] & & C_o(Y) \\ & C_o(X) \arrow[ul, "p_2^*"] \arrow[ur, "p_1^*"'] & \end{tikzcd}$$

commutes. Therefore, $\phi(q) = q^*$ is a morphism of G.
(ii) Since $(1_Y)^*(g) = g\circ 1_Y = g$ for all $g \in C_o(Y)$, $\phi(1_Y) = (1_Y)^* = 1_{\phi(Y)}$. (iii) Let Y, Z and W be in H and $q\colon\ Y \to Z$, $r\colon\ Z \to W$ be two proper maps. Since for each $g \in C_o(Y)$ we have $(r\circ q)^*(g) = g\circ(r\circ q) = (g\circ r)(q) = q^*(g\circ r) = q^*(r^*(g)) = (q^*\circ r^*)(g)$, then $(rq)^* = q^*r^*$. Therefore, $\phi(rq) = \phi(q)\phi(r)$. By (i), (ii) and (iii) ϕ is a functor. Next we show that ϕ is faithful. Let $q\colon\ Y \to Z$ and $q'\colon\ Y \to Z$ be two proper maps. If $q \neq q'$, then there exists a $y \in Y$ such that $q(y) \neq q'(y)$. Since Z is completely regular, there exists a continuous function $h \in C_o(Z)$ such that $h(q(y)) \neq h(q'(Y))$. Thus $h\circ q \neq h\circ q'$ and hence $q^* \neq q'^*$. Finally, we show ϕ is full. Let

$$\begin{tikzcd} C_o(Z) \arrow[rr, "\theta"] & & C_o(Y) \\ & C_o(X) \arrow[ul, "p_2^*"] \arrow[ur, "p_1^*"'] & \end{tikzcd}$$

be a commutative diagram, where ϕ is a $\mathbb{C}$-algebra homomorphism and p_1^* and p_2^* are the $\mathbb{C}$-algebra homomorphisms induced by the covering projections $p_1\colon\ Y \to X$ and $p_2\colon\ Z \to X$ respectively. Let M_y be a regular maximal ideal of $C_o(Y)$.

Then $\theta^{-1}(M_y)$ is an ideal of $C_o(Z)$ and the map $\bar{\theta}$: $C_o(X)/\theta^{-1}(M_y) \to C_o(Y)/M_y$ defined by $h + \theta^{-1}(M_y) \to \theta(h) + M_y$ is an injective $\mathbb{C}$-algebra homomorphism. Since $C_o(Y)/M_y \cong \mathbb{C}$, $\bar{\theta}$ is also surjective and hence $\bar{\theta}$ is a $\mathbb{C}$-algebra isomorphism. Thus $C_o(Z)/\theta^{-1}(M_y) \cong C$ and hence $\theta^{-1}(M_y)$ is a regular maximal ideal. Define a map α: $\text{Rm } C_o(Y) \to \text{Rm } C_o(Z)$ by $\alpha(M_y) = \theta^{-1}(M_y)$. α is well-defined.

Claim 1: α is a proper map. The continuity of α follows from the fact that $\alpha^{-1}(\{N \in \text{Rm } C_o(Z) \mid h \in N\}) = \{M \in \text{Rm } C_o(Y) \mid \theta(h) \in M\}$. Note that we have the following diagram:

$$\begin{tikzcd}
\text{Rm } C_o(Y) \arrow[rr, "\alpha"] \arrow[dr, "p_1"'] & & \text{Rm } C_o(Z) \arrow[dl, "p_2"] \\
& \text{Rm } C_o(X) &
\end{tikzcd}$$

Since p_1 and p_2 are proper maps, it is sufficient to show that the above diagram commutes, i.e., $p_2 \circ \alpha(M_y) = p_1(M_y)$ for all $M_y \in \text{Rm } C_o(Y)$ [B2, p. 99]. Since

$$\begin{aligned}
f \in p_1(M_y) &\Leftrightarrow f \circ p_1(y) = 0 \\
&\Leftrightarrow (f \circ p_2)(y) = 0 \quad (\text{since } \theta \cdot p_2^* = p_1^*) \\
&\Leftrightarrow \theta(f \circ p_2) \in M \\
&\Leftrightarrow f \circ p_2 \in \theta^{-1}(M_y) \\
&\Leftrightarrow f \in p_2(\theta^{-1}(M_y)) \\
&\Leftrightarrow f \in p_2(\alpha(M_y))
\end{aligned}$$

Then $p_1 = p_2 \circ \alpha$. Hence α is a proper map.

Claim 2: $\phi(\alpha) = \theta$. Let $h \in C_o(Z)$ and $y \in Y$. Then

$(\phi(\alpha)(h))(y) = (h\circ\alpha)(y) = h(\alpha(y)) = h(\theta^{-1}(M_y)) = h + \theta^{-1}(M_y)$ $= \theta(h) + M_y = \theta(h)(y)$. Thus $\phi(\alpha)(h) = \theta(h)$ and hence $\phi(\alpha) = \theta$. By claim 1 and claim 2 ϕ is full.

Next, we give an example to show that ϕ is not a contravariant equivalence. Let $X = [-1,1)$, $y = [-2,0) \cup (0,2]$ and $p\colon\ Y \to X$ be defined by

$$p(y) = \begin{cases} y + 1 & \text{if } y \in [-2,0) \\ -y + 1 & \text{if } y \in (0,2] \end{cases}$$

Then X and Y both are locally compact Hausdorff spaces and Y is a covering space of X with finite fibers. If $g \in C_o(Y)$ we let $g_1 = g|_{[-2,0)}$ and $g_2 = g|_{(0,2]}$. Let $S = \{g \in C_o(Y) \mid$ there exists β, $-2 \leq \beta < 0$, such that $g_1(y) = g_2(-y)$ for $y \in (\beta,0)\}$. Then clearly S is a $\mathbb{C}$-algebra (commutative) without identity, $S \subseteq C_o(Y)$ and the map $p^*\colon\ C_o(X) \to S$ defined by $p^*(f) = f\circ p$ is an injective homomorphism. Hence S is a $C_o(X)$-algebra extension. Let $\hat{X} = [-1,1]$ and $\hat{Y} = [-2,2]$ be the one point compactifications of X and Y respectively, $g_1 = g|_{[-2,0]}$, $g_2 = g|_{[0,2]}$ for $g \in C(\hat{Y})$ and $\hat{S} = \{g \in C(\hat{Y}) \mid$ there exists β, $-2 \leq \beta < 0$, such that $g_1(y) = g_2(-y)$ for $y \in (\beta,0]\}$. Then $\hat{S}$ is a commutative $\mathbb{C}$-algebra with identity, $\hat{S} \subseteq C(\hat{Y})$ and $\hat{S}$ is isomorphic to $\tilde{S}$, the $\mathbb{C}$-algebra obtained by adjoining the complex identity to S under the map $g \to (g(0),\ g - g(0))$. We identify $\hat{S}$ with $\tilde{S}$. Our first goal is to show that the structure space Rm S of S is homeomorphic to Y, and we prove it by showing that the structure space Rm $\hat{S}$ is homeomorphic to $\hat{Y}$. We begin with the following claims.

(4.9) Every ideal of $\hat{S}$ is fixed [GJ, p. 54].

Proof: Let I be an ideal of $\hat{S}$. Then $Z(I) = \{Z(f) | f \in I\}$ is a family of closed sets. Let $f_1, f_2, \cdots, f_n \in I$. Clearly, $\bigcap_{i=1}^{n} Z(f_i) = Z(\sum_{i=1}^{n} |f_i|^2)$. If $\bigcap_{n=1}^{n} Z(f_i) = \emptyset$, then $Z(\sum_{i=1}^{n} |f_i|^2) = \emptyset$ and hence $\sum_{i=1}^{n} |f_i|^2$ is a unit in $\hat{S}$. $\hat{S}$ is closed under the complex conjugation, i.e. if $f \in S$, then the complex conjugate $\bar{f} \in S$. Since $\sum_{i=1}^{n} |f_i|^2 = \sum_{i=1}^{n} f_i\bar{f}_i$ and I is an ideal of $\hat{S}$, $\sum_{i=1}^{n} |f_i|^2 \in I$. Thus the identity is in I and hence $I = \hat{S}$, a contradiction. Thus $\bigcap_{i=1}^{n} Z(f_i) \neq \emptyset$. Since $\hat{Y}$ is compact and Z(I) has the finite intersection property, $\cap Z(I) \neq \emptyset$, i.e. I is a fixed ideal.

(4.10) N is a maximal ideal of $\hat{S}$ if and only if there exists a $y \in \hat{Y}$ such that $N = N_y = \{g \in \hat{S} | g(y) = 0\}$. Furthermore, $N_y = S$ if and only if $y = 0$ and N_y are distinct for distinct y.

Proof: Let N be a maximal ideal of $\hat{S}$. By (4.5) $\cap Z(N) \neq \emptyset$. Let $y \in \cap Z(N)$. Then $N \subseteq N_y \subsetneq \hat{S}$. Since N is maximal, $N = N_y$. Conversely, if $N = N_y$ for some $y \in \hat{Y}$, then the map γ: $S \to C$ defined by $\gamma(g) = g(y)$ is clearly a **C**-algebra homomorphism with kernel N_y. Note that γ is also surjective. Hence N_y is a regular maximal ideal of S. Now for the second assertion, if $y = 0$, clearly $N_y = S$. Suppose $y \neq 0$. Let $\delta = \frac{1}{2}|y| > 0$. Since Y is perfectly normal, there exists a $g \in C(\hat{Y})$ such that $Z(g) = [-\delta, \delta]$. Hence $g \in \hat{S}$ and $g \in S$,

$g \notin N_y$. Thus $S \neq N_y$. Finally, if $y_1 \neq y_2 \neq 0$. Let $\delta = \frac{1}{2}\min\{|y_1|,|y_2|\} > 0$. By the perfect normality of $\hat{Y}$, there exists a $g \in C(\hat{Y})$ such that $Z(g) = [-\delta,\delta] \cup \{y_1\}$. Hence $g \in S$, $g \in N_{y_1}$ and $g \notin N_{y_2}$. Thus $N_{y_1} \neq N_{y_2}$.

(4.11) $m\,\hat{S}$ is homeomorphic to $\hat{Y}$ and hence Rm S is homeomorphic to Y.

Proof: The map α: $\hat{Y} \to m\,\hat{S}$ defined by $\alpha(y) = N_y$ is a bijection (4.10). Since $\alpha(Z(f)) = \{N_y | f \in N_y\}$ and $Z(f)$ and $\{N_y | f \in N_y\}$ are bases for the closed sets of $\hat{Y}$ and $m\,\hat{S}$ respectively, then α is a homeomorphism. Now, by Theorem 1.5, Rm S is homeomorphic to $m\,\hat{S} - \{S\}$. But $m\,\hat{S} - \{S\}$ is homeomorphic to $\hat{Y} - \{\alpha(S)\} = \hat{Y} - \{0\} = Y$. Thus Rm S is homeomorphic to Y.

Our second goal is to show that S is a separable $C_o(X)$-algebra extension.

(4.12) S is a separable $C_o(X)$-algebra extension.

Proof: (i) Let N_{x_o} be a regular maximal ideal of $C_o(X)$, $\delta = \frac{1}{2}\min\{1 - x_o, x_o + 1\}$ and $N(x,\delta) = (x_o - \delta, x_o + \delta)$. Since X is perfectly normal, we can choose a function f: $X \to \mathbb{C}$ such that $f(x) = 0$ if and only if $x \in X - N(x_o,\delta)$. Clearly, $p^{-1}(N(x_o,\delta)) = U_1 \cup U_2$ where U_1, U_2 are open in Y and $U_1 \subseteq [-2,0)$, $U_2 \subseteq (0,2]$. Define h_i: $Y \to \mathbb{C}$ by

$$h_i(y) = \begin{cases} f(p(y)) & \text{if } y \in U_i \\ 0 & \text{otherwise} \end{cases},$$

$i = 1,2$. Then clearly, $h_i \in S$, $i = 1,2$. Hence $h_i/f \in S_f$,

$i = 1,2$. Since if $y \in U_i$, $i = 1,2$, then

$$\begin{aligned} f(p(y))(f(p(y))h_1(y) + f(p(y))h_2(y) - f(p(y))f(p(y))) \\ &= f(p(y))(f(p(y))h_i(y) - f(p(y))) \\ &= f(p(y))(f(p(y))f(p(y)) - f(p(y))f(p(y))) \\ &= 0 \end{aligned}$$

and if $y \notin U_i$, then $p(y) \notin N(x_o,\delta)$ and hence $f(p(y)) = 0$. Thus $(f\circ p)((f\circ p)h_1 + (f\circ p)h_2 - (f\circ p)(f\circ p)) = 0$. Hence $h_1/f + h_2/f = f/f$. Since if $y \in U_1$, then $h_2(y) = 0$; if $y \in U_2$, then $h_1(y) = 0$ and if $y \notin U_1 \cup U_2$, then $f(p(y)) = 0$, so $f(p(y))(h_1(y)h_2(y)) = 0$ for all y. Hence $h_1/f \cdot h_2/f = 0$. Therefore, $S_f = S_f \cdot h_1/f \oplus S_f \cdot h_2/f$. Next, we show that $C_o(X)_f$ is isomorphic to $S_f \cdot h_i/f$. Define a map α: $C_o(X)_f \to S_f \cdot h_i/f$ by $\alpha(r/f^n) = r/f^n \cdot h_i/f$. α is well-defined. For if $r/f^n = s/f^m$, then there exists f^q such that $f^q(rf^m - sf^n) = 0$. Thus $f^q(rf^{m+1}h_i - sf^{n+1}h_i) = 0$ and hence $r/f^n \cdot h_i/f = s/f_m \cdot h_i/f$. α is clearly a $\mathbb{C}$-algebra homomorphism. α is injective. For if $r/f^n \cdot h_i/f = 0$ then there exists $t > 0$ such that $f^t rh_i = 0$. If $x \in X - N(x_o,\delta)$ then $f(x) = 0$. If $x \in N(x_o,\delta)$ then there exists $y \in U_i$ such that $p(y) = x$. Thus

$$\begin{aligned} f^t(x)r(x) &= f^t(x)r(x)f(x) \\ &= f^t(p(y))r(p(y))f(p(y)) \\ &= f^t(p(y))r(p(y))h_i(y) \\ &= (f^t rh_i)(y) = 0 \end{aligned}$$

and hence $f^{t+1}(x)r(x) = 0$ for all $x \in X$. Thus $r/f^n = 0$. α is also surjective; let $s/f^n \cdot h_i/f \in S_f \cdot h_i/f$. Define a map $r\colon X \to \mathbb{C}$ by

$$r(x) = \begin{cases} s((p|_{U_i})^{-1})(x))f(x) & \text{if } x \in N(x_o,\delta) \\ 0 & \text{otherwise} \end{cases}$$

Then clearly $r \in C_o(X)$. If $y \in U_i$ then $p(y) = x \in N(x_o,\delta)$. Thus

$$f(p(y))(f^{n+1}(p(y))r(p(y))h_i(y) - f^{n+2}(p(y))s(y)h_i(y))$$
$$= f(x)(f^{n+1}(x)r(x)f(x) - f^{n+2}(x)s(y)f(x))$$
$$= f(x)(f^{n+1}(x)s(y)f(x)f(x) - f^{n+2}(x)s(y)f(x))$$
$$= 0.$$

If $y \notin U_i$ then $h_i(y) = 0$. Hence

$$f(p(y))(f^{n+1}(p(y))r(p(y))h_i(y) - f^{n+2}(p(y))s(y)h_i(y)) = 0 \quad \text{for all } y \in Y.$$

Thus $\alpha(r/f^{n+1}) = r/f^n \cdot h_i/f$ and hence α is surjective.

(ii) Let M be a regular maximal ideal of S. Then by (4.10) there exists $y \in Y$ such that $M = M_y$. Since $M_y \cap C_o(X) = N_{p(y)}$ is a regular maximal ideal of $C_o(X)$, there exists $f \notin N_{p(y)}$ satisfying (i). Since $f \notin N_{p(y)}$, $f \notin M$. Therefore, S is a separable $C_o(X)$-algebra extension.

Next we show that S and $C_o(Y)$ are not isomorphic as $\mathbb{C}$-algebras. First note that $C_o(Y)$ is a semi-simple Banach algebra with respect to the sup norm, i.e. $\|f\| = \sup_{y\in Y} f(y)$, and S is not a Banach algebra with respect to the sup norm since the sequence $\{f_n\}$, where

$$f_n(y) = \begin{cases} y & \text{if } 0 < y \leq 2 \\ -y & \text{if } -\frac{1}{n} \leq y < 0 \\ y + \frac{2}{n} & \text{if } -2 \leq y < -\frac{1}{n} \end{cases}$$

is a Cauchy sequency in S converging to no element in S. Also note that S separates Y and for each $y \in Y$ there exists $f \in S$ such that $f(y) \neq 0$. Hence, by Weierstrass' theorem, S is dense in $C_o(Y)$.

(4.12) S and $C_o(Y)$ are not isomorphic as $\mathbb{C}$-algebras.

Proof: Suppose there exists a $\mathbb{C}$-algebra isomorphism ϕ: $C_o(Y) \to S$. Define $\|s\|_\phi = \|\phi^{-1}(s)\|$ for $s \in S$. Then clearly S is a Banach algebra with respect to the norm $\|s\|_\phi$. By [N, Theorem 1, p. 210], $\|s\| \leq K\|s\|_\phi$ for all $s \in S$, where K is a constant. Clearly, the map $*$: $S \to S$ defined by $s^* = \phi(\overline{\phi^{-1}(s)})$ is an involution and $\|s^*s\|_\phi = \|\phi^{-1}(s^*s)\|$ $= \|\phi^{-1}(s^*)\phi^{-1}(s)\| = \|\phi^{-1}(\phi(\overline{\phi^{-1}(s)}))\phi^{-1}(s)\| = \|\overline{\phi^{-1}(s)}\phi^{-1}(s)\|$ $= \|\phi^{-1}(s)\|^2 = \|s\|_\phi^{\,2}$. Thus S is a $\mathbb{C}^*$-algebra. By [S, Theorem 1.2.4, p. 5], $\|s\|_\phi \leq \|s\|$ for all $s \in S$. Hence $\|s\|$ and $\|s\|_\phi$ are equivalent. Therefore, S is a Banach algebra with respect to the sup norm, which is a contradiction.

Now we are ready to show:

(4.13) The functor defined in Theorem 4.8 is not a contravariant equivalence.

Proof: Suppose the functor is a contravariant equivalence. Then there exists a finite-fibered covering space Z of X such that $C_o(Z) \cong S$. By Theorem 1.12 Z is homeomorphic to Rm $C_o(Z) \cong$ Rm S. By (4.11) Y is homeomorphic to Rm S. Thus Z is homeomorphic to Y. Hence $S \cong C_o(Z)$ is isomorphic to $C_o(Y)$ which contradicts (4.12). Thus S does not come from a covering space, i.e., the functor is not a contravariant equivalence.

We end this chapter by noting that:

Theorem 4.14: The category of locally compact Hausdorff finite-fibered covering spaces of a fixed locally compact Hausdorff space X and the category of C*-algebras that are separable $C_o(X)$-algebra extensions are contravariantly equivalent.

Proof: We need only to show that any C*-algebra S that is a separable $C_o(X)$-algebra extension comes from a finite-fibered covering space of X, i.e. there exists a finite-fibered covering space Z of X such that $C_o(Z)$ is isomorphic to S. Since S is a separable extension of $C_o(X)$, by Theorem 4.7, Z = Rm S is a finite-fibered covering space of X. Since S is also a C*-algebra, by [M2, p. 56], $C_o(Z) = C_o(\text{Rm } S)$ is isomorphic to S.

BIBLIOGRAPHY

[B1] Bourbaki, N., Commutative Algebra, Addison-Wesley, Reading, Massachusetts, 1972.

[B2] Bourbaki, N., General Topology Part I, Addison-Wesley, Reading, Massaxhusetts, 1966.

[C] Childs, L. N., On Covering Spaces and Galois Extensions, Pacific J. Math. 37 (1971), 29-33.

[DI] DeMeyer, F. and Ingraham, E., Separable Algebras Over Commutative Rings, Math Lecture Notes #181, Springer-Verlag, New York, 1971.

[GJ] Gillman, L. and Jerison, M., Rings of Continuous Functions, D. Van Nostrand, Inc., Princeton, New Jersey, 1960.

[K] Kaplansky, I., Commutative Rings, Allyn and Bacon, Inc., Boston, 1970.

[M1] Magid, A. R., Algebraically Separable Extensions of Banach Algebras, Mich. Math. J. 21 (1974), 137-143.

[M2] Mosak, R., Banach Algebras, The University of Chicago Press, Chicago and London, 1975.

[N] Naimark, M. A., Normed Rings, Wolters-Noordkoff, Groningen, 1970.

[S1] Sakai, S., C*-Algebras and W*-Algebras, Springer-Verlag, New York, 1971.

[S2] Spanier, E. H., Algebraic Topology, McGraw-Hill, Inc., New York, 1966.