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THE THEORY AND APPLICATION OF PRIMAL-DUAL ANALYSIS IN COMPARATIVE STATICS

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CHAPTER I

INTRODUCTION

The Origins, Purpose, and History of Comparative Statics

This research was undertaken to develop some new theorems in comparative statics (CS), and some new applications analyzing the CS properties of specific models in several fields. The project was chosen partly with the aim of promoting wider knowledge of an under-appreciated method for CS analysis, described by its developer nearly 30 years ago as ". . . simply a better way to do comparative statics."[1]

Textbooks describe comparative statics in general as the analysis of the response of a system to a change in its parameters. Under the name of "sensitivity analysis," similar procedures are used in most disciplines which employ optimization techniques. In economics, the term usually refers to changes in the optimized value of an objective function and of its choice variables, most often in the presence of one or more constraints, in response to a change in the value of a parameter.

Slutsky, a Russian statistician, was apparently the first to apply partial derivatives in this way as a tool of analysis to economic problems. His paper, recognized as seminal

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[1] Silberberg, 1974 (1), p. 171.

by most authorities, has been difficult to find in English until recently.[2] Pareto, however, laid some of the groundwork using differentials several years earlier.[3] Hotelling was using most of the tools of modern textbook comparative statics – partial derivatives, Lagrangian functions, solutions by Cramer's Rule, Jacobian determinants, bordered determinants, etc. – in the early 1930s,[4] but he used them in an *ad hoc* fashion to make particular points, rather than as parts of a coherent theory.

Hicks, acknowledging his debts to Slutsky and Pareto, may have been the first to employ the mathematical methods of CS as an essential part of a broad economic argument,[5] but he did not use the term, and he relegated the mathematics to an appendix. In this regard, Samuelson noted that:

> Hicks, like [Marshall], has succeeded in keeping formidable mathematical analysis below the surface and locked up in appendices, thereby securing for his work a much wider audience than would otherwise be possible. This *tour de force* was made possible to a considerable degree by the repeated use of the already mentioned theorem relating to the demand for a group of commodities when all their prices change in the same proportion.[6]

When Samuelson's *Foundations of Economic Analysis* was published in 1947,[7] the use of the term "comparative statics" was taken for granted. Since the values of the variables of interest in economics are often the result of an optimization procedure

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[2] Slutsky, 1915.

[3] Pareto, 1909, cited in Samuelson, 1983, p. 212n, and also in Hicks, 1946, p. 3.

[4] Hotelling, 1932, and Hotelling, 1935. The first occurrance of what is known today as "Hotelling's Lemma" appears to be on p. 103 of the *Collected Economic Articles* . . . ."

[5] In *Value and Capital*, first published in 1939, although the latter begins by expanding upon Hicks and Allen, 1934.

[6] Samuelson, 1983, p. 141. Samuelson is referring to Hicks' theorem that when the prices of all of a group of goods change proportionately, the demand for them behaves as if they were a single composite good. Samuelson refers to the first edition of *Value and Capital*. In the second edition, the theorem is Theorem 10, pp. 312-313, the numbering of the theorems being somewhat different between the two editions. Hicks also notes the importance of the theorem and its extensive use in the text. It is worth noting that both Hicks and Hotelling used what amount to bordered Hessian determinants, without making reference to the matrices from which they are derived.

[7] The original version had been published earlier as the winner of the David A. Wells prize for the year 1941-42.

believed to represent the economic concept of "equilibrium," Samuelson further explains the meaning as ". . . the investigation of changes in a system from one position of equilibrium to another without regard to the transitional process involved in the adjustment."[8] (It is a central principle, however, of Samuelson's analysis that such changes can only be fully understood in terms of that "transitional process," i.e., the dynamics of the system.)

Silberberg gives a much more technical definition (emphasis in original):

> The logical simulation, usually with mathematics, of the testing of theories in economics is called the *theory of comparative statics* . . . . The use of the term comparative statics refers to the absence of a prediction about the *rate* of change of variables over time, as opposed to the *direction* of change. . . . *Comparative statics is that mathematical technique by which an economic model is investigated to determine if refutable hypotheses are forthcoming.*[9]

While he goes on to formalize and make rigorous this notion of "refutable hypotheses" as the fundamental criterion of what constitutes ". . . theories [that] are useful in empirical science,"[10] Silberberg means exactly what Samuelson meant 30 years earlier by the term, "meaningful theorems:"

> By a *meaningful theorem* I mean simply a hypothesis about empirical data which could conceivably be refuted, if only under ideal conditions.[11]

What we are concerned with is the source of such theorems or hypotheses. Samuelson stated the matter as if entirely obvious (emphasis added):

> They proceed almost wholly from two types of very general hypotheses. The first is that *the conditions of equilibrium are equivalent to the maximization (minimization) of some magnitude.*[12]

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[8] Samuelson, 1983, p. 8.
[9] Silberberg, 1990, pp. 15-16.
[10] Silberberg, 1990, p. 13.
[11] Samuelson, 1983, p. 4.
[12] Samuelson, 1983, p. 5. The second type of hypothesis referred to is that the equilibrium is "stable," given the dynamic properties of the system.

Silberberg presents a formal argument to show that, under standard assumptions on the regularity of the functions involved, certain conditions on one or more of the "comparative statics derivatives" (the partial derivatives of the endogenous decision variables with respect to the parameters) are logically entailed by the behavioral assumption with which the process begins. These conditions, which may be relations between two or more of the derivatives, or simply restrictions on their signs, constitute hypotheses about events in the physical world which may or may not occur, and hence they are refutable. If these predictions of the CS process are contrary to empirical observation, the behavioral assumption must itself be false.

As a result, this process can help us to choose between two theories (behavioral assumptions) as explanations of the behavior of an actual economic agent if they yield different predictions (entail different conditions on the CS derivatives). For example, Kalman and Intriligator present an example in which it is shown by means of CS that certain observable phenomena are consistent with a producer behaving according to the Baumol model (maximizing sales revenue subject to a profit constraint) but not with a producer maximizing profit according to the classical model, and vice-versa.[13] If phenomena inconsistent with one model (say model "B") and not those inconsistent with the other (say model "C") are observed in the case of an actual producer, it is inferred that this producer *cannot* be behaving according to model "B," but *may* be behaving according to model "C." Note that it can never be "proved" that an agent is in fact behaving according to a given model, only that one is *not* so behaving: The hypotheses are refutable, but are supportable only in the sense of their not being refuted.

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[13] Kalman and Intriligatgor, 1973.

Between Pareto and Silberberg, a number of other investigators made significant contributions to the theory and methodology, and also expanded and clarified its application to specific problems. Samuelson mentions, in addition to those above, W.E. Johnson, Rene Roy, Ronald Shephard, Nicholas Georgescu-Roegen, and others.[14] Of these, Shephard's contributions to the theory of duality in production are perhaps the best known and most widely cited today,[15] although Samuelson himself cites Hotelling (1932) as the inspiration for his own investigations of duality.[16]

Closely related to the CS derivatives, and intimately bound to the concept of duality, is the so-called "envelope theorem," which states that if a function is optimized for given values of the parameters and then the value of a parameter is changed by an infinitesimal amount, the change in the value of the original function is the same to the first order regardless of whether the decision variables are allowed to adjust to their optimum values for the new value of the parameter. The theorem originated, famously, in Jacob Viner's dispute with his draftsman Dr. Wong, a mathematician, over the relation between the long-run and short-run average cost curves of the firm.[17] In Samuelson's

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[14] Samuelson, 1983, pp. 95, 351, 453, and elsewhere.
[15] Shephard, 1970.
[16] Samuelson, 1983, p. 453.
[17] cf. Silberberg, 1999, Silberberg, 1990 (p. 190), Viner, 1932. It is perhaps instructive that the correct relation was "seen" graphically by Wong, while Viner, one of the great economists of his day, was unable to understand why the envelope curve could not pass through the minimum points of the short-run curves. This despite the fact that Viner recognized that the long-run curve could never be above the short-run curve, and despite his inclusion of various related derivatives in his argument. According to Silberberg, Viner was fooled by the same two paradoxes noted in textbooks that still fool many students today. His intuition led him correctly to believe that the long-run curve should be an envelope to the short-run curves, but incorrectly to doubt that the slopes of the curves could be the same whether capital is fixed or variable. The greater paradox then and now, however, is that the minimum-cost output for any given plant can (almost) always be produced at lower cost by a slightly different plant, while the output that a given plant can produce at lower cost than any other possible plant is not the output that it can produce at minimum cost. It is this latter paradox that would not be apparent except for the mathematical analysis.

derivation of this theorem,[18] this important result, which never seems quite intuitive (at least if the authors of introductory microeconomics textbooks are to be believed) follows from the fact that *at the optimum,* the instantaneous rate of change of the objective function with respect to each of the decision variables is zero (the first-order condition for an optimum). In the formulation of CS to be discussed, it is seen to emerge directly as a part of the first-order conditions themselves.

Duality

The term "duality" takes on at least three distinct but related meanings, depending upon the context. In broadest terms, it has to do with the fact that there are two different formulations of any constrained optimization problem, both of which describe the same set of relations, such that the solution to one formulation is part of the other formulation and vice-versa.

The context most familiar from all introductory microeconomics texts which employ calculus is that of the utility-maximization model. An objective function (of unspecified form) is optimized subject to a single equality constraint, consisting of a function of the choice variables and the parameters of the problem set equal to another parameter representing the "level" of the constraint. In this context, the primal constraint function becomes the objective function in the dual problem, and the primal objective function becomes the constraint function in the dual. The dual problem then is to determine the optimum value of the dual objective function (the primal constraint),

[18] Samuelson (1983), p. 34. Samuelson derives the theorem for the unconstrained case, noting only that a similar result entails for constrained optimization problems.

subject to the original (primal) objective function being fixed at a specified level (for sake of argument, the value found as the solution to the primal problem).

If the primal problem is formulated as a maximization, the dual is in general formulated as a minimization, and vice-versa. For example, in the utility-maximization model the primal problem is to choose the levels of a vector of consumption goods so as to maximize the resulting utility given a specified budget, and given the prices of the goods. The dual is to choose the levels of the same vector of goods to minimize the expenditure necessary to provide a specified level of utility, given the same vector of prices. If the specified level of utility in the dual problem is the value found as the solution to the primal problem, the minimum expenditure found as the solution to the dual is the specified budget for the primal problem, and vice-versa.

In practice, some problems require a certain amount of manipulation to get them into a form amenable to standard Lagrangian methods of optimization. In such cases, and perhaps in others, the features of the problem which require such manipulation may also require that the optimization in the dual problem be in the same direction as in the primal problem. Such an example is presented in Chapter IV.

The optimized value of either objective function is called its "value function," a function *only* of the parameters of the problem. In economics the value function is sometimes (but not always) referred to as the "indirect" function of the corresponding "direct" objective function. This indirect function is also sometimes referred to as the "dual function," insofar as important aspects of the duality relation are inherent in it. For example, Beattie and Taylor base their discussion of duality in production on the optimized value of the function $\pi = R(\boldsymbol{y}, \boldsymbol{p}) - C(\boldsymbol{x}, \boldsymbol{w})$, subject to $\boldsymbol{y} = f(\boldsymbol{x})$.[19] They call this optimized value of π the "indirect profit function," whereas most books refer to it as "*the* profit function."[20] Beattie and Taylor employ this "dual" function only as a simpler way to derive the supply and factor demand (conditional and unconditional) equations than the "ground-up" method from the function above, which they refer to as the "direct" profit function.

The true dualities in production, however, are more complex and can well be considered a context distinct from that above. The authoritative work on production theory is that by Shephard.[21] Building on the approach of Samuelson,[22] Shephard analyzes the duality between the cost and production "structures," and the relationship between the cost function and the revenue function by way of their respective dualities with "distance functions" of the input and output sets of the production function, rather than optimizing a profit function directly. (While Samuelson does refer to "profit maximization," he makes it quite clear that this involves merely the addition of the product of output, the essential features of which are already embodied in his analysis of cost, and output price – not necessarily given – to the negative of cost.)

Varian develops the main results of Shephard's theory primarily in the familiar terms of vector calculus,[23] whereas Shephard employs all the tools of modern mathematical analysis. Mas-Colell and Whinston develop duality theory for demand briefly, with the emphasis strongly in terms of sets and mappings.[24] In this context, the

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[19] Beattie and Taylor, 1993, pp. 223-231.
[20] Varian, 1992, p. 25, Silberberg, 1990, p. 192.
[21] Shephard, 1970.
[22] Samuelson, 1983, 57-89.
[23] Varian, 1992, Ch. 1-6.
[24] Mas-Colell and Whinston, 1995, pp. 63-75.

variables in the primal and dual problems are not the same, but it has in common with the first context the fact that the solution set for the primal problem is part of the formulation of the dual, and vice-versa.

Still more general is the context of mathematical programming (including but not limited to linear programming), in which multiple constraints are typical, both equality and inequality constraints are permitted, and restrictions on the signs of variables, the senses of the inequalities, and the correspondence between the number of variables and the number of constraints are relaxed. In the linear programming context, if the primal problem consists of m constraints in n variables, the dual consists of n constraints in m variables. The levels of the constraints in the primal problem become the coefficients of the variables in the objective function of the dual, and vice-versa, while the matrix of coefficients in the constraints of the dual problem is the transpose of that in the primal problem. Such problems are solved not by analytic methods but by iterative algorithms.

This differences between this context and the others are obvious, although there appears to be no reason its general features could not be stated in the set-theoretic framework. (It is also homologous with certain parts of game theory.) Nevertheless it has in common with the first two contexts that the solutions to the primal and dual problems each imply a specific solution to the other (that is, an optimum value of the associated objective function, and the associated values of the choice variables). And under some broad assumptions that could come under the heading of "regularity," the two objective functions – of different sets of variables – have the same value. Furthermore, Samuelson was able to show a deeper homology between the duality of linear programming and that of Newtonian calculus (the first context above), including the envelope theorem,[25] as well as the related contexts in non-linear and dynamic programming.[26]

The version of duality under consideration in this paper is primarily that of the first context, although the specific sense of "dual" in the term "Primal-Dual" is that of the "dual function," or "value function," the optimized value of the primal objective function, which is a function only of the parameters. In each model that follows, however, we will also analyze the *dual problem* (in the first context, and using the Primal-Dual method), because in all of these contexts the dual problem can lead to insights which are not readily available from the primal problem if they are available at all.

Primal-Dual Analysis

Samuelson yet again, in exploring the implications of a 1960 paper by Houthakker which examined additive utility functions and additive indirect utility functions, discovered that it was possible to derive useful theorems regarding functions of *both* quantities and price-income ratios by utilizing the *difference* between the direct (or "primal") and indirect (or "dual") objective functions. He conjectured that this was "a technique that can be useful in a great number of more difficult problems,"[27] that just mentioned being but one.

The truth of this conjecture was later demonstrated by Silberberg.[28] It is doubtful whether even Samuelson realized the full implications of his discovery at the time. For what Silberberg showed was that when *any* optimization problem is formulated in this

[25] Samuelson, 1983, Mathematical Appendix C1.
[26] Samuelson, 1983, Mathematical Appendices C2 and C3.
[27] Samuelson, 1965, p. 781
[28] Silberberg, 1974 (1).

way,[29] *all* the CS relations, and hence all the refutable hypotheses implied by maximization, are contained in the sign definiteness of the Hessian matrix of this "Primal-Dual" function with respect to the parameters. If there are constraints in the original problem, the matrix is the appropriate bordered Hessian. The summary below follows the slight reformulation published a few years later,[30] which clarified the mechanics of the process.

Consider the problem, maximize a function $f(\boldsymbol{x}, \boldsymbol{\alpha})$, where $\boldsymbol{x}$ is a vector of n decision variables and $\boldsymbol{\alpha}$ is a vector of m parameters, subject to a vector of constraints $g(\boldsymbol{x}, \boldsymbol{\alpha}) = 0$. The Lagrangian is formed as usual, its derivatives taken with respect to the decision variables and the Lagrange multipliers, and each derivative set equal to zero. If the necessary second-order conditions hold, the Implicit Function Theorem allows these n first-order conditions to be solved simultaneously for the optimum values of the variables $\boldsymbol{x}^*$ as functions of the parameters $\boldsymbol{\alpha}$. When these values are substituted into the objective function, the result is the indirect objective function, or value function, $\phi(\boldsymbol{\alpha}) = f(\boldsymbol{x}^*(\boldsymbol{\alpha}), \boldsymbol{\alpha})$.

Now form a new function, $z = F(\boldsymbol{x}, \boldsymbol{\alpha}) = f(\boldsymbol{x}, \boldsymbol{\alpha}) - \phi(\boldsymbol{\alpha})$. Silberberg calls this the "Primal-Dual objective function," to be maximized subject to $g(\boldsymbol{x}, \boldsymbol{\alpha}) = 0$. Since $\phi(\boldsymbol{\alpha})$ is the maximum of $f(\boldsymbol{x}, \boldsymbol{\alpha})$ for a given set of parameter values $\boldsymbol{\alpha}$, and subject to the constraint, F (which is a function of $m + n$ independent variables) is non-positive, and has a maximum of zero when $\boldsymbol{x} = \boldsymbol{x}^*$. The first-order conditions of this function with respect to $\boldsymbol{x}$ are the first-order conditions of the original objective function $f(\boldsymbol{x}, \boldsymbol{\alpha})$ (subject to

[29] As the optimization of the difference between the objective function and its dual or "value function," that is, the function which represents the optimized value of the objective function as a function of its parameters.
[30] Silberberg, 1990, first edition 1978.

constraint), and the first-order conditions of this function with respect to $\boldsymbol{\alpha}$ are the envelope results (also subject to constraint).

The second-order conditions for maximization of F are that the matrix of second partials of *its* Lagrangean be negative (semi) definite subject to constraint. This matrix can be written as the partitioned matrix

$$\boldsymbol{H} = \left[\begin{array}{c|c} \mathcal{L}^*_{xx} & \mathcal{L}^*_{x\alpha} \\ \hline \mathcal{L}^*_{\alpha x} & \mathcal{L}^*_{\alpha\alpha} \end{array}\right]$$

It is easy to show from the definition of a negative (semi) definite matrix (as a product of the matrix with a vector satisfying the constraint) that if the entire matrix above is negative semi-definite subject to constraint, the matrix in the lower-right quadrant, consisting of partials with respect to the parameters only, is negative semi-definite as well. Another way of expressing this condition is that the border-preserving principal minors of order k of the following bordered Hessian determinant have sign $(-1)^k$ or 0:

$$|\bar{\boldsymbol{H}}_{\alpha\alpha}| = \begin{vmatrix} \mathcal{L}^*_{\alpha\alpha} & g_\alpha \\ g_\alpha & 0 \end{vmatrix}$$

What do the terms in $\mathcal{L}^*_{\alpha\alpha}$ look like? By differentiating the first-order envelope results with respect to $\boldsymbol{\alpha}$, it can be shown that for values of $\boldsymbol{x}$ that maximize F and satisfy the constraint, each element of $\mathcal{L}^*_{\alpha\alpha}$ is an expression containing all the partial derivatives of $\boldsymbol{x}^*$ and λ^* with respect to α_j, along with the cross partials of the original objective function and the constraint function with respect to the parameters and the decision variables. But these are all the relations we have involving the CS derivatives (the partial derivatives of $\boldsymbol{x}_i$ with respect to α_j). Therefore it can be asserted that the negative (semi) definiteness of the above Hessian matrix captures *all* the refutable hypotheses implied by the maximization hypothesis.

Note that this process does not necessarily generate expressions for the CS derivatives themselves, as does the conventional method using Cramer's Rule. In general, it does not. What it does do is to generate inequalities involving those derivatives, in fact, all such relations that follow from the maximization hypothesis alone, i.e., without any assumptions regarding the nature of the functions involved.

Of course, this includes not only such simple relations as the lemmas of Shephard and Hotelling, but many complex relations which may be impossible to test empirically. Practical CS results emerge only when the form of the particular objective function under study (or special assumptions made with regard to it) produces simple, tractable expressions for *some* of the border-preserving principal minors of $|\overline{\boldsymbol{H}}|$. Fortunately, this is the case for many important economic models.

At first glance it might seem as if the primal-dual approach (PD) is but a long and scenic route to a familiar destination. But this is not the case. Just as we do not have to go through the (perhaps interesting) process of inventing the digital computer each time we wish to type a paper, neither do we have to go through the derivation of PD each time we need to derive a CS result: The work has already been done for us, and PD stands as a tool, ready to use. Silberberg summarizes the advantages of the PD approach thus:[31]

[31] Silberberg, 1974 (1), 171.

1. The derivation of comparative statics results is drastically simplified one merely needs to calculate mechanically a certain matrix and various determinants associated with it.
2. The common basis of the comparative statics results obtained in differing economic models is more clearly exhibited than in the older analysis.
3. Through the effects of 1 and 2, the discovery of new theorems in comparative statics should be easier and the limits of the qualitative results that are obtainable in models involving a maximization format should be more easily delineated. It is simply a better way to do comparative statics analysis in economics.

Current Research

The theory of comparative statics is more or less fully developed at the present time. A recent search of the EconLit database found 828 items since 1969, with some duplications (sequences of working papers, journal articles later published in collective volumes, etc.). Searches of ABI/Inform Global and Archive turned up only a half dozen items not found in the EconLit search. Excluding titles with phrases such as "comparative statics of . . .," articles in which comparative statics was an aspect of the analysis of a particular model, general textbooks, etc., at most 30 could be considered contributions to basic theory. About one-third of these dealt with dynamic or stochastic models. About one-fourth dealt with "monotone" comparative statics, most of them building on a 1994 article by Milgrom and Shannon which established conditions for the solution set of an optimization problem to be monotonic in the parameters, primarily in game theoretic and programming contexts. No other single sub-topic generated more than four references, even obliquely; the remainder of the papers published dealt either with historic material or very narrow topics.

Searches for keywords "primal" and "dual" found only 36 items on EconLit, but 297 on ABI/Inform Global (since 1971). Virtually all of the papers in both cases dealt explicitly with mathematical programming problems, primarily linear. Only five papers mentioned Silberberg's or related methods. One of these was Currier's 2002 paper, "Long-run equilibrium in a monopolistically competitive industry."[32] The other four were all by Michael Caputo, a student of Silberberg's and one of the few who have used or referred to the Primal-Dual method in published works. Indeed, one of the four papers proves the superiority of the Primal-Dual method to another method put forth by Hatta,[33] and another playing on the title of Silberberg (1974 (1)) but referring to "comparative dynamics."[34] Both of the other papers employ primal-dual methodology in dynamic contexts, one theoretical, one in Caputo's specialty of resource economics.[35]

Plan of the Work

What is simple in principle often turns out to be less so in practice. Even apparently simple models can generate quite complex expressions, many of which are difficult or impossible to interpret. Furthermore, models of apparently similar form can generate very different results. And sometimes a model which is susceptible to conventional analysis generates a bordered Hessian all of the border-preserving principal

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[32] Currier, 2002.
[33] Caputo, 1999.
[34] Caputo, 1990 (1).
[35] Caputo, 1990 (2); Caputo, 1990 (3)

minor (BPPM) determinants of which are zero, meeting the second-order necessary condition for an optimum but not the sufficient condition, and providing few testable results.

The analysis of the familiar utility-maximization model by the PD method in Chapter II was undertaken as an attempt to understand the structure of a model with this characteristic. The Safety Rule model of Lichtenberg and Zilberman is similar in structure to the utility-maximization model, in that it has no parameters in the objective function and four in the constraint, one of which is a level parameter.[36] The fact that the latter is fully susceptible to PD analysis, generating all the standard results plus some novel ones, demonstrates that models with similar structures in terms of the arrangement of parameters in the objective function and the constraint (an important consideration in PD analysis) may generate entirely different CS results, depending on the particular form of the model and of its derivatives.

The simple banking model analyzed in Chapter III, while not markedly more complex than the utility-maximization model, has two parameters in the objective function and two in the constraint, one of which is a level. It generates a number of interesting and plausible testable hypotheses, although many of the results depend on an assumption that the bank incurs monotonically increasing costs in the servicing of deposits. The results are compared with those from conventional CS analysis.

The limit pricing model in Chapter IV cannot be analyzed by the usual Lagrangian method in the terms one would ordinarily expect for a monopoly situation. It

[36] Lichtenberg and Zilberman, 1988.

requires manipulation as mentioned above (p. 7) in order for its objective function to have an optimum without the use of an inequality constraint.

Specifically, with the constraint that the potential entrant's profit equals zero, the incumbent's output has no minimum (the monopolist having ordinarily an incentive to restrict output): It is always possible in principle for the entrant to operate inefficiently and earn zero profit no matter how high the price becomes as a result of the incumbent's restricted output. There is, however, a *maximum* output for the incumbent at which satisfies the constraint, and graphical analysis shows that it is indeed this level of output that is desired. Any lower output would allow the entrant positive profit, while any higher output would lead to negative profit for the entrant under all circumstances, and to unnecessary reduction in the incumbent's monopoly profits. This inversion of the usual situation leads to an unexpected formulation of the dual problem as well.

This model, like utility maximization, has no parameters in the objective function and two (one a level) in the constraint. Its analysis, however, is at least as complex as that for utility maximization, and its results resemble those of the latter model only in a very broad sense. In both this model and the Safety Rule model, the pristine symmetry of the utility-maximization model is absent, and both contain functions of the variables of unspecified form (including probabilistic functions in the Safety Rule model) which are absent from the utility-maximization model. It is not surprising in retrospect that these differences prevail completely over the similarities in terms of the arrangement of the parameters. While it is true as Silberberg maintains that the arrangement of the parameters has profound consequences for duality, these consequences are no less profound than those proceeding from the forms of the functions themselves.

Chapter V considers the conclusions that can be drawn from the analysis of these models, and possibilities for further work.

CHAPTER II

THE FAMILIAR UTILITY MAXIMIZATION MODEL AND AN EXTENSION

The elementary utility maximization model with two goods and a budget constraint illustrates the differences between the results of conventional CS analysis and those of PD analysis. In the standard treatment the first-order conditions (FOC) provide the result that at the point of optimality the ratio of the marginal utility of a good to its price is the same for both goods (for all goods in the extended model). The FOC are totally differentiated and the differentials of the quantities of the goods and of the Lagrange multiplier are found by Cramer's Rule. These are then divided by the differentials of prices and income to generate the standard CS derivatives, which are the partials of the quantity of each good purchased at the point of optimality with respect to its own price, to the price of the other good, and to income.

As such, these derivatives are of little use, because their arguments include the unobservable second partials of utility, (although assumptions regarding the signs of those partials, corresponding to specific economic assumptions, can specify the signs of some of the CS derivatives). But most important for consumer theory are the various relations among these derivatives and those arising from the dual problem (expenditure minimization given a fixed level of utility), which relations are themselves consequences of duality, along with certain characteristics of the functions involved.

Primal-dual (PD) analysis, on the other hand, generates those familiar relations (and often others) more or less directly, by taking duality as its starting point, without generating the standard CS derivatives at all. The standard derivatives are taken with respect to the parameters, but they typically have among their arguments derivatives of the objective function with respect to the decision variables. It is *those* derivatives which are absent from PD analysis, and hence from any expressions for the standard CS derivatives which may emerge when the fundamental primal-dual relations happen to simplify in particular ways.

The PD method is developed in the context of the Envelope Theorem, which has to do with the (partial) derivative of an objective function with respect to a parameter ("parameterized optimization"). Envelope relations in this sense do not exist for parameters which appear in a constraint, because such relations rest upon fixing the decision variables at their optimum values for some given values of the parameters, then varying the parameters. But for parameters which enter the constraint this is not possible: When a constraint parameter is varied, the previously determined values for the decision variables are then in violation of the constraint, a problem which does not arise for parameters which do *not* appear in a constraint.

On the other hand, the PD formulation of this problem reveals directly certain characteristics of the functions involved which must be proved *ad hoc* in the conventional development. If we maximize $U(x_1, x_2)$ s.t. $p_1x_1 + p_2x_2 = M$, the result – the value function – is the indirect utility function, $v(p_1, p_2, M)$. If we then set up the PD problem,

$$\max\ U(x_1, x_2) - v(p_1, p_2, M) \text{ s.t. } p_1x_1 + p_2x_2 = M$$

by forming the PD Lagrangian,

$$\mathcal{L}^* = U(x_1, x_2) - v(p_1, p_2, M) + \lambda(M - p_1x_1 - p_2x_2),$$

we see that all the derivatives of $U(x_1, x_2)$ with respect to the parameters, which are the only derivatives of concern in this method, are zero. Thus the problem is in effect to *minimize* the *indirect* utility function, subject to the constraint. Therefore, if the assumptions are met that guarantee the existence of a maximum of the original objective function, the indirect function must be convex in the parameters (subject to the constraint). The linearity of the constraint in the parameters then means that the indirect utility function is *quasi-convex* in the parameters.[1] This familiar element of consumer theory is thus shown to emerge directly from the mathematical structure of the problem.[2]

Differentiating the PD Lagrangian yields the first-order conditions (FOC) with respect to the parameters of the form $\mathcal{L}^*_\alpha = 0$:

$$\mathcal{L}^*_{p1} = -\lambda x_1 - v_{p1} = 0 \Rightarrow v_{p1} \equiv -\lambda^* x_1^*$$

$$\mathcal{L}^*_{p2} = -\lambda x_1 - v_{p2} = 0 \Rightarrow v_{p2} \equiv -\lambda^* x_2^*$$

$$\mathcal{L}^*_M = \lambda - v_M = 0 \Rightarrow v_M \equiv \lambda^*$$

In an unconstrained model with parameters in the primal objective function, these would be the envelope relations between the derivatives of the direct and indirect functions with respect to the parameters. In a model with constraints which also include those parameters, the relations would be between the derivatives of the indirect function and those of the *Lagrangian* of the direct function. In this model, with parameters in the constraint *only*, these relations are between the derivatives of the indirect function and those of the *constraint* (including the multiplier): They do not involve the (direct)

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[1] Silberberg, *Structure*, p. 202; cf. Chiang, pp. 351, 396-399.
[2] See, for example, Varian, p. 102.

objective function at all, even in the limit, and thus are not envelope relations in the usual sense. Nevertheless they play an important part in the analysis, and will be referred to herein as "pseudo-envelope" relations.

In the third of these relations, the interpretation of the Lagrange multiplier as the "Marginal Utility of Money" (MUM) is direct and obvious compared to the standard derivation. The first two relations correspond directly to the FOC of the primal problem, which say that the change in optimum utility in response to a change in the quantity chosen of a good is equal to the MUM times the price of the good. These say that the change in optimum utility in response to a change in the *price* of a good is equal to the *negative* of the MUM times the quantity chosen of the good. (More of the good $\rightarrow$ more utility; higher price $\rightarrow$ less utility.)

To obtain the second partials $\mathcal{L}^*_{\alpha\alpha}$ of the PD Lagrangian, we note that in general (i.e., with the $\mathcal{L}^*_{\alpha}$ not restricted to equal zero) the only argument of the $\mathcal{L}^*_{\alpha}$ that depends on the parameters is v. Then by inspection, we see that the second partials $\mathcal{L}^*_{\alpha\alpha}$ are simply the negatives of $v_{\alpha\alpha}$. To calculate these derivatives, we differentiate the "pseudo-envelope" identities above:

$$\mathcal{L}^*_{p_1p_1} = -v_{p_1p_1} = \lambda^*(\partial x_1^*/\partial p_1) + x_1^*(\partial\lambda^*/\partial p_1)$$

$$\mathcal{L}^*_{p_1p_2} = -v_{p_1p_2} = \lambda^*(\partial x_1^*/\partial p_2) + x_1^*(\partial\lambda^*/\partial p_2)$$

$$\mathcal{L}^*_{p_1M} = -v_{p_1M} = \lambda^*(\partial x_1^*/\partial M) + x_1^*(\partial\lambda^*/\partial M)$$

$$\mathcal{L}^*_{p_2p_1} = -v_{p_2p_1} = \lambda^*(\partial x_2^*/\partial p_1) + x_2^*(\partial\lambda^*/\partial p_1)$$

$$\mathcal{L}^*_{p_2p_2} = -v_{p_2p_2} = \lambda^*(\partial x_2^*/\partial p_2) + x_2^*(\partial\lambda^*/\partial p_2)$$

$$\mathcal{L}^*_{p_2M} = -v_{p_1M} = \lambda^*(\partial x_2^*/\partial M) + x_2^*(\partial\lambda^*/\partial M)$$

$$\mathcal{L}^*_{Mp_1} = -v_{Mp_1} = -\partial\lambda^*/\partial p_1$$

$$\mathcal{L}^*_{Mp_2} = -v_{Mp_2} = -\partial\lambda^*/\partial p_2$$

$$\mathcal{L}^*_{MM} = -v_{MM} = -\partial\lambda^*/\partial M$$

Three relations follow from the symmetry of the cross partials:

$$\partial\lambda^*/\partial p_i = -\lambda^*(\partial x_i^*/\partial M) - x_i^*(\partial\lambda^*/\partial M)$$

$$-\lambda^*(\partial x_1^*/\partial p_2) - x_1^*(\partial\lambda^*/\partial p_2) = -\lambda^*(\partial x_2^*/\partial p_1) - x_2^*(\partial\lambda^*/\partial p_1)$$

The first pair of relations say that the response of MUM to a change in the price of one good (with income and all other prices held constant) is equal and opposite to the response of the *product* of the quantity chosen of that good with MUM to a change in income. This is of some interest: When the price of a good increases with income held constant, MUM must decrease because an additional dollar then purchases less utility than before. On the other side of the equation, the amount of a normal good purchased increases with an increase in income, while MUM decreases if marginal utility is diminishing. This relation (actually a set of relations, one for each choice variable) is "almost patentable:" It is novel, and it is non-obvious. Unfortunately, since it involves not only the MUM itself but its derivatives, it is not clear that it is useful.

The last relation is of the general form derived by Silberberg for reciprocity relations.[3] It says that the response of the product of MUM and the quantity chosen of one good in response to a change in the price of the *other* good is equal to that for the opposite combination. This, too, is non-obvious, though intuitively attractive because of its symmetry. The problem again is that it is not clear how one might apply it given the prominence of λ, which is unobservable, and its derivatives. One way around this problem is to combine the relations in such a way that the λ's cancel out, and indeed the

[3] Silberberg, *Structure*, p. 214, equation (7-44).

"cross-price relation" follows immediately from the symmetrical reciprocity relation, or from the quotient of the first pair, if λ^* is assumed to be constant.

While these relations may simply reveal the structure underlying the "obvious" $(\partial x_1^*/\partial p_2) = (\partial x_2^*/\partial p_1)$, they seem to suggest something a little more profound and unexpected. If each of the "money pair" is divided by $-\partial\lambda^*/\partial p_i$, the λ's are not eliminated, but we get the tantalizing relation

$$\lambda^*\left(\frac{\partial x_1^*/\partial M}{\partial\lambda^*/\partial p_1}\right)+x_1^*\left(\frac{\partial\lambda^*/\partial M}{\partial\lambda^*/\partial p_1}\right)=\lambda^*\left(\frac{\partial x_2^*/\partial M}{\partial\lambda^*/\partial p_2}\right)+x_2^*\left(\frac{\partial\lambda^*/\partial M}{\partial\lambda^*/\partial p_2}\right)=-1$$

If one of the "money pair" is subtracted from the other, we get

$$-(\partial\lambda^*/\partial p_2-\partial\lambda^*/\partial p_2)=\lambda^*(\partial x_1^*/\partial M-\partial x_2^*/\partial M)+(x_1^*-x_2^*)(\partial\lambda^*/\partial M)$$

While each of these looks like something that should have a name, both are still heavily dependent on the unobservable λ and its derivatives. Furthermore, the expression $x_1^* - x_2^*$ literally represents "apples minus oranges," so the significance (if any) of these relations is unclear, and they are certainly not testable as they stand.

Next we form the bordered Hessian matrix $\mathfrak{L}^*_{\alpha\alpha}$ of the second partials of the PD function with respect to the parameters, whose elements are the $\mathfrak{L}^*_{\alpha\alpha}$ above:

$$\mathfrak{L}^*_{\alpha\alpha}=\begin{bmatrix}\lambda^*\left(\frac{\partial x_1^*}{\partial p_1}\right)+x_1^*\left(\frac{\partial\lambda^*}{\partial p_1}\right) & \lambda^*\left(\frac{\partial x_1^*}{\partial p_2}\right)+x_1^*\left(\frac{\partial\lambda^*}{\partial p_2}\right) & \lambda^*\left(\frac{\partial x_1^*}{\partial M}\right)+x_1^*\left(\frac{\partial\lambda^*}{\partial M}\right) & -x_1^* \\ \lambda^*\left(\frac{\partial x_2^*}{\partial p_1}\right)+x_2^*\left(\frac{\partial\lambda^*}{\partial p_1}\right) & \lambda^*\left(\frac{\partial x_2^*}{\partial p_2}\right)+x_2^*\left(\frac{\partial\lambda^*}{\partial p_2}\right) & \lambda^*\left(\frac{\partial x_2^*}{\partial M}\right)+x_2^*\left(\frac{\partial\lambda^*}{\partial M}\right) & -x_2^* \\ \frac{\partial\lambda^*}{\partial p_1} & \frac{\partial\lambda^*}{\partial p_2} & \frac{\partial\lambda^*}{\partial M} & 1 \\ -x_1^* & -x_2^* & 1 & 0\end{bmatrix}$$

The sufficient second-order condition for (x_1^*, x_2^*) to be a maximum is that this matrix be negative definite, in which case the principal minors of order k have sign $(-1)^k$. Assuming this condition is met, eliminating the third row and column yields

$$\begin{vmatrix} \lambda^*(\partial x_1^*/\partial p_1) + x_1^*(\partial\lambda^*/\partial p_1) & \lambda^*(\partial x_1^*/\partial p_2) + x_1^*(\partial\lambda^*/\partial p_2) & -x_1^* \\ \lambda^*(\partial x_2^*/\partial p_1) + x_2^*(\partial\lambda^*/\partial p_1) & \lambda^*(\partial x_2^*/\partial p_2) + x_2^*(\partial\lambda^*/\partial p_2) & -x_2^* \\ -x_1^* & -x_2^* & 0 \end{vmatrix} > 0$$

$$\begin{aligned} &x_1^*x_2^*[\lambda^*(\partial x_1^*/\partial p_2) + x_1^*(\partial\lambda^*/\partial p_2)] - x_1^{*2}[\lambda^*(\partial x_2^*/\partial p_2) + x_2^*(\partial\lambda^*/\partial p_2)] \\ &\quad - x_2^{*2}[\lambda^*(\partial x_1^*/\partial p_1) + x_1^*(\partial\lambda^*/\partial p_1)] + x_1^*x_2^*[\lambda^*(\partial x_2^*/\partial p_1) + x_2^*(\partial\lambda^*/\partial p_1)] > 0 \end{aligned}$$

$$\lambda^*\{x_1^*x_2^*[(\partial x_1^*/\partial p_2) + (\partial x_2^*/\partial p_1)] - x_1^{*2}(\partial x_2^*/\partial p_2) - x_2^{*2}(\partial x_1^*/\partial p_1)\} > 0$$

And since $\lambda^* > 0$,

$$x_1^*x_2^*[(\partial x_1^*/\partial p_2) + (\partial x_2^*/\partial p_1)] - x_1^{*2}(\partial x_2^*/\partial p_2) - x_2^{*2}(\partial x_1^*/\partial p_1) > 0$$

$$x_1^*x_2^*[(\partial x_1^*/\partial p_2) + (\partial x_2^*/\partial p_1)] > x_1^{*2}(\partial x_2^*/\partial p_2) + x_2^{*2}(\partial x_1^*/\partial p_1)$$

This relation says that the sum of the cross-price effects, multiplied by the product of the chosen quantities of the two goods, is greater than the sum of the own-price effects (the slopes of the Marshallian demand curves), *each* multiplied by the *square* of the chosen quantity of the *other* good. Since this relation involves no unobservable quantities, it is in principle a testable hypothesis and apparently new. It is plausible but not obviously true, and it does not arise mechanically from the standard analysis.[4] That its significance also is not obvious does not necessarily mean that it is not significant, or that the comparable relation in a model with similar structure might not be. In any case, it demonstrates the power of the PD method to reveal relationships of potential significance which conventional comparative statics analysis may overlook.

[4] For example, Varian, Chapters 7 and 8; Henderson and Quandt, Chapter 2.

Eliminating the second row and column yields

$$\begin{vmatrix} \lambda^*(\partial x_1^*/\partial p_1) + x_1^*(\partial\lambda^*/\partial p_1) & \lambda^*(\partial x_1^*/\partial M) + x_1^*(\partial\lambda^*/\partial M) & -x_1^* \\ -\partial\lambda^*/\partial p_1 & -\partial\lambda^*/\partial M & 1 \\ -x_1^* & 1 & 0 \end{vmatrix} > 0$$

$$x_1^*(\partial\lambda^*/\partial p_1) - \lambda^*(\partial x_1^*/\partial p_1) - x_1^*(\partial\lambda^*/\partial p_1)$$
$$+ x_1^*[\, x_1^*(\partial\lambda^*/\partial M) - \lambda^*(\partial x_1^*/\partial M) - x_1^*(\partial\lambda^*/\partial M)] > 0$$

$$-\lambda^*(\partial x_1^*/\partial p_1) - \lambda^* x_1(\partial x_1^*/\partial M) > 0$$

$$\lambda^*[\partial x_1^*/\partial p_1 + x_1(\partial x_1^*/\partial M)] < 0$$

The final order-2 determinant is similar in x_2, p_2, yielding

$$\lambda^*[\partial x_2^*/\partial p_2 + x_2(\partial x_2^*/\partial M)] < 0$$

MUM $= \lambda^* > 0$, so we have the result

$$\partial x_i^*/\partial p_i + x_i(\partial x_i^*/\partial M) < 0$$

But this is just the substitution effect, because the left side is the slope of the compensated demand function, which is always negative. So PD analysis provides no new information in this case, but could provide a route to a useful result for other models with a similar structure.

The order-3 principal minor expands to a very complex expression which cannot be simplified, and which is not likely to have any economic interpretation. No "conjugate pairs" relations are generated, because they arise from the envelope relations. Therefore, the above constitute all the results obtainable from this model by PD analysis, which Silberberg claims to be all the refutable hypotheses that follow from the maximization hypothesis alone.

Some additional results emerge, however, when PD method is applied to the dual problem of minimizing expenditure given a fixed level of utility:

$$e(p_1, p_2, \bar{U}) = \min\,(p_1x_1 + p_2x_2) \text{ s.t. } U(x_1,x_2) = \bar{U}$$

The PD formulation is

$$\min\ p_1x_1 + p_2x_2 - e(p_1, p_2, \bar{U}) \text{ s.t. } U(x_1,x_2) = \bar{U}$$

The PD Lagrangian is

$$\mathcal{L}^* = p_1x_1 + p_2x_2 - e(p_1, p_2, \bar{U}) + \mu[\bar{U} - U(x_1,x_2)]$$

The FOC with respect to the parameters are:

$$\mathcal{L}^*_{p1} = x_1 - e_{p1} = 0 \Rightarrow e_{p1} \equiv x_1^*$$

$$\mathcal{L}^*_{p2} = x_2 - e_{p2} = 0 \Rightarrow e_{p2} \equiv x_2^*$$

$$\mathcal{L}^*_U = \mu - e_U = 0 \Rightarrow e_U \equiv \mu^*$$

In these relations, the x_i^* are the compensated (Hicksian) demands, henceforth designated h_i^* (although the compensated demands are usually assumed to be optimized).

The first two relations on the right are true envelope relations, because the parameters p_1 and p_2 appear in the objective function, and h_1^* and h_2^* are the derivatives of the objective function with respect to those parameters, evaluated at the point of optimality. We still cannot, however, derive meaningful conjugate pairs relations from these envelope identities, because as derivatives of the objective function h_1^* and h_2^* are degenerate, functions only of themselves and thus of the parameters directly. The conjugate pairs relations depend on the first derivatives of the objective function with respect to a parameter in the function being expressible as functions of all the arguments of the objective function, which are then expressed as functions of the parameters at the point of optimality. These first two relations say that the change in the minimum necessary expenditure in response to a change in the price of a good is equal to the quantity of the good chosen at the point of optimality. This is a novel result, which should be easy to derive by conventional methods, but which is not found in standard texts.

The third relation says that the change in expenditure in response to a change in the given level of utility is equal to the value of the (dual) Lagrange multiplier at the optimum, which is as usual the reciprocal of that in the primal problem. (Perhaps it should be called the "marginal money of utility," the additional increment of expenditure necessary to obtain an additional unit of utility.)

In order to form the SOC we need the second derivatives of the Lagrangian:

$$\mathcal{L}^*_{p_i p_i} = -e_{p_i p_i}^* = -\partial h_i^*/\partial p_i$$

$$\mathcal{L}^*_{p_i p_j} = -e_{p_i p_j}^* = -\partial h_i^*/\partial p_j$$

$$\mathcal{L}^*_{p_i U} = -e_{p_i U}^* = -\partial h_i^*/\partial \bar{U}$$

$$\mathcal{L}^*_{UU} = -e_{UU}^* = -\mu_U^*$$

$$\mathcal{L}^*_{U p_i} = -e_{U p_i}^* = -\mu_{p_i}^*$$

The symmetry of the cross partials yields the unsurprising cross-price relation,

$$\mathcal{L}^*_{p_i p_j} = -\partial h_i^*/\partial p_j = -\partial h_j^*/\partial p_i = \mathcal{L}^*_{p_j p_i}$$

But symmetry also yields the following pair, which appear to be new:

$$\mathcal{L}^*_{U p_i} = -\mu_{p_i}^* = -\partial \mu^*/\partial p_i = -\partial h_i^*/\partial \bar{U} = \mathcal{L}^*_{p_i U}$$

These correspond roughly to the pair of novel asymmetric relations for the primal problem, and are also reciprocity relations.[5] They can be interpreted as meaning that the response of the quantity purchased of good *i* to a change in the specified level of utility is equal to the response of the "marginal money of utility" to a change in the price of good *i*. Such relations are not of practical use, as the only observable quantity is the price. But

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[5] See Currier's Theorem 5-3B, p. 87.

they are of theoretical interest insofar as they provide an additional and unusual window into the structure of this familiar model.

The bordered Hessian matrix $\mathcal{L}^*_{\alpha\alpha}$ will be positive definite (the sufficient condition), since this is a minimization problem, which means that all the border-preserving principal minors will be negative:

$$\mathcal{L}^*_{\alpha\alpha} = \begin{bmatrix} -\partial h_1^*/\partial p_1 & -\partial h_1^*/\partial p_2 & -\partial h_1^*/\partial \bar{U} & 0 \\ -\partial h_2^*/\partial p_1 & -\partial h_2^*/\partial p_2 & -\partial h_2^*/\partial \bar{U} & 0 \\ -\partial \mu^*/\partial p_1 & -\partial \mu^*/\partial p_2 & -\partial \mu^*/\partial \bar{U} & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

We have 2 minors of order 2:

$$\begin{vmatrix} -\partial h_i^*/\partial p_i & -\partial h_i^*/\partial \bar{U} & 0 \\ -\partial \mu^*/\partial p_i & -\partial \mu^*/\partial \bar{U} & 1 \\ 0 & 1 & 0 \end{vmatrix} = \partial h_i^*/\partial p_i < 0$$

That is simply to say that the compensated demand is always downward-sloping.

The determinant of the entire matrix, expanded by the fourth row, is

$$-\begin{vmatrix} -\partial h_1^*/\partial p_1 & -\partial h_1^*/\partial p_2 & 0 \\ -\partial h_2^*/\partial p_1 & -\partial h_2^*/\partial p_2 & 0 \\ -\partial \mu^*/\partial p_1 & -\partial \mu^*/\partial p_2 & 1 \end{vmatrix} < 0$$

This in turn expands to

$$-\left[(\partial h_1^*/\partial p_1)(\partial h_2^*/\partial p_2) - (\partial h_2^*/\partial p_1)(\partial h_1^*/\partial p_2)\right] < 0$$

$$(\partial h_1^*/\partial p_1)(\partial h_2^*/\partial p_2) > (\partial h_2^*/\partial p_1)(\partial h_1^*/\partial p_2) = (\partial h_2^*/\partial p_1)^2 = (\partial h_1^*/\partial p_2)^2$$

This relation says that the product of the compensated own-price effects is greater than the product of the compensated cross-price effects (which is equal to the square of either one, since they were shown above to be equal). While not testable due to the unobservability of the compensated demands, this relation does not emerge readily from conventional analysis and does not appear to have been noted previously.

We would have serious reservations about a novel analytical method if it failed to lead to universally accepted general consequences of a well-understood problem. For this problem, Roy's identity and the Slutsky equation should follow directly from the above results, and they do. Both are consequences of the FOC of the PD Lagrangian.

From the primal problem we have (p. 20, above)

$$\mathcal{L}^*_{p_i} = -\lambda x_i - v_{p_i} = 0 \Rightarrow v_{p_i} \equiv -\lambda^* x_i^*$$

$$\mathcal{L}^*_M = \lambda - v_M = 0 \Rightarrow v_M \equiv \lambda^*$$

Dividing the first relation by the second yields Roy's identity,

$$(\partial v/\partial p_i)/(\partial v/\partial M) = -x_i^*.$$

The Slutsky equation follows as always from the identity of the optimizing bundle for the primal and dual problems,

$$h_i^*(p_1, p_2, \bar{U}) \equiv x_i^*(p_1, p_2, e(p_1, p_2, \bar{U}))$$

Differentiating with respect to p_j,

$$\partial h_i^*/\partial p_j = \partial x_i^*/\partial p_j + (\partial x_i^*/\partial M)(\partial e/\partial p_j).$$

But from the FOC for the dual problem (p. 26, above), $\partial e/\partial p_j \equiv h_j^*$, so rearranging the preceding relation and substituting yields the Slutsky equation,

$$\partial x_i^*/\partial p_j = \partial h_i^*/\partial p_j - x_j^*(\partial x_i^*/\partial M).$$

Analyzing the primal and dual problems simultaneously *should* make general conclusions stand out, while tending to suppress intermediate results. Specifying the form or changing the arguments of the objective function can be expected to produce different results, or at least results that differ in appearance. For example, a quasi-linear utility function is sometimes specified in this problem just because the demand functions in that case have particularly striking characteristics. PD analysis does generate some novel relations in that case, but they can be interpreted only in light of the conventional results, which do *not* emerge from the PD formulation. Such relations as are generated seem less significant than those from the conventional analysis, which are derived *ad hoc*.

Suppose on the other hand that prices enter the utility function. This situation is sometimes dismissed as a strictly academic exercise. Persons not fully convinced by Friedman's "Positive Economics" may consider such a situation to be unrealistic despite anecdotal suggestions. For example, a man was heard to crow over having bought a new $50,000 Cadillac (which he could well afford but had not intended to buy) because he was able to save over $10,000 in financing costs due to incentives offered by auto manufacturers in the fall of 2001. But since utility is not observable, we cannot say whether he actually valued the car more highly for this reason, or whether he simply moved down his demand curve in response to the lower effective price to the point at which his demand for Cadillacs rose to a quantity of 1.

What about people whose budget constraint is relatively severe? Is the satisfaction they derive from a given quantity of a particular good determined solely by the quantities of the various goods they consume, or do they actually get more satisfaction (or possibly less) from consuming a good obtained at a low price than they would from consuming the same quantity of the same good if the price were higher? We usually assume that their preference would be to consume the cheaper good, *so that they might then be able to consume more of some other good, given the same budget.* But again, the premise is that we cannot tell by observing their behavior. On the other hand, the present writer is a self-proclaimed cheapskate, and unless we would assert that one's subjective report regarding one's own utility function is also irrelevant, we are forced to grant that some people might actually derive a higher level of utility from the *same bundle of goods*, if purchased at lower prices with a lower overall budget: More *stuff* is better, money is only a veil, but some people may yet have preferences regarding the characteristics of the veil!

Everybody loves a bargain. The conventional wisdom is that whatever changes might result in the demand function as a result of prices in the utility function can be modeled simply as a change in elasticity, indistinguishable from that due to other causes. The matter should not be taken so lightly, however, if prices in the utility function lead to different behavioral predictions. It turns out that they do, and as usual they emerge easily and succinctly from PD analysis. Consider the PD formulation,

$$\max\ U(x_1, x_2, p_1, p_2) - v(p_1, p_2, M) \text{ s.t. } p_1x_1 + p_2x_2 = M.$$

The PD Lagrangian is

$$\mathcal{L}^* = U(x_1, x_2, p_1, p_2) - v(p_1, p_2, M) + \lambda(M - p_1x_1 - p_2x_2).$$

The FOC are

$$\mathcal{L}^*_{p_1} = U_{p_1} - v_{p_1} - \lambda x_1 = 0 \Rightarrow v_{p_1} \equiv U^*_{p_1} - \lambda^* x_1^*$$

$$\mathcal{L}^*_{p_2} = U_{p_2} - v_{p_2} - \lambda x_1 = 0 \Rightarrow v_{p_2} \equiv U^*_{p_2} - \lambda^* x_2^*$$

$$\mathcal{L}^*_M = U_M - v_M + \lambda = 0 \Rightarrow v_M \equiv U^*_M + \lambda^*$$

Note that while it is still true that $v = U^*$, their *derivatives* are no longer equal because they are taken with respect to parameters which appear in *both* the objective function *and the constraint*. These are the envelope relations for a problem of that sort.[6] Nevertheless, when these identities are differentiated in order to obtain the second

[6] See Silberberg, *Structure*, p. 200.

partials of v, the interpretation still can only be the relevant second partials of U, evaluated at the optimum.

When the first partials of $\mathfrak{L}^*$ (not equated to zero) are differentiated a second time, the resulting second partials of U which appear in the second-order conditions must also be evaluated at the optimum. These second partials of $\mathfrak{L}^*$ are then simplified by substituting the second partials of v obtained by differentiating the first-order identities. The resulting expressions for the second partials of $\mathfrak{L}^*$ with respect to the parameters are the same as those for the model without prices in the utility function. The second partials of U disappear in the present case because they occur both in their own right, as derivatives of the direct objective function, and as a term in the expression for the derivatives of the indirect function, a consequence of the primal-dual formulation. They do not occur in the other case because U is not a function of the parameters. For example,

$$\mathfrak{L}^*_{p_1p_1} = U^*_{p_1p_1} - v_{p_1p_1}$$

But since $v_{p_1p_1} = U_{p_1p_1} - \partial(\lambda^* x_1^*)/\partial p_1 = U_{p_1p_1} - \lambda^*(\partial x_1^*/\partial p_1) - x_1^*(\partial\lambda^*/\partial p_1)$, we have

$$\mathfrak{L}^*_{p_1p_1} = U_{p_1p_1} - U_{p_1p_1} - \lambda^*(\partial x_1^*/\partial p_1) - x_1^*(\partial\lambda^*/\partial p_1)$$

$$= - \lambda^*(\partial x_1^*/\partial p_1) - x_1^*(\partial\lambda^*/\partial p_1),$$

just as for the case without prices in the utility function. The same is true for all the other second derivatives.

But dividing either of the first two FOC by the third to derive Roy's identity yields

$$(v_{p_i} - U_{p_i})/(v_m - U_M) = -x_i.$$

This relation, while clearly analogous to Roy's identity, is distinguished by the presence of the *differences* between the derivatives of U and v at the optimum (instead of those of v only), as it is based upon the envelope relations which, as mentioned above, differ from those for parameters which do not appear in the objective function.

The PD formulation of the dual problem is

$$\min p_1x_1 + p_2x_2 - e(p_1, p_2, \bar{U}) \text{ s.t. } U(x_1, x_2, p_1, p_2) = \bar{U}$$

The PD Lagrangian is

$$\mathcal{L}^* = p_1x_1 + p_2x_2 - e(p_1, p_2, \bar{U}) + \mu[\bar{U} - U(x_1, x_2, p_1, p_2)].$$

The FOC are

$$\mathcal{L}^*_{p_1} = x_1 - e_{p_1} - \mu U_{p_1} = 0 \Rightarrow e_{p_1} \equiv h_1^* - \mu^* U_{p_1}$$

$$\mathcal{L}^*_{p_2} = x_2 - e_{p_2} - \mu U_{p_2} = 0 \Rightarrow e_{p_2} \equiv h_2^* - \mu^* U_{p_2}$$

$$\mathcal{L}^*_{\mu} = -e_{\mu} + \mu = 0 \Rightarrow e_{\mu} \equiv -\mu^*.$$

So differentiating the identity $h_i^*(p_1, p_2, \bar{U}) \equiv x_i^*(p_1, p_2, e(p_1, p_2, \bar{U}))$ with respect to p_j and substituting for e_{p_j} from the FOC above, the Slutsky relation for this problem is

$$\partial x_i^*/\partial p_j = \partial h_i^*/\partial p_j - (\partial x_i^*/\partial M)(x_j^* - \mu^*(\partial U/\partial p_j))$$

$$= \partial h_i^*/\partial p_j - x_j^*(\partial x_i^*/\partial M) + \mu^*(\partial x_i^*/\partial M)(\partial U/\partial p_j)$$

$$\partial x_i^*/\partial p_j + x_j^*(\partial x_i^*/\partial M) = \partial h_i^*/\partial p_j + \mu^*(\partial x_i^*/\partial M)(\partial U/\partial p_j)$$

Kalman and Intriligator first recognized the existence of generalized Slutsky relations of this sort for a very general class of models in 1973,[7] and they noted the application to consumer theory for the case in which prices enter the utility function. It is possible without using their results to derive the above relation by conventional methods, but PD analysis does the job succinctly and efficiently. It is easy to see that the matrix of generalized substitution effects (the right hand side of the last equation above) is

[7] Kalman and Intriligator, 1973. The last equation above is their equation 3.3, p. 482

symmetric. Kalman and Intriligator prove this is true for the general case, and also that this matrix is negative semidefinite under fairly general conditions.

CHAPTER III

A SIMPLE BANKING MODEL

A simplified banking model can illustrate further how the results of PD analysis depend on whether parameters appear in the objective function or in the constraint, and also the benefits and limitations of the method. Consider a profit-maximizing bank that earns rate r_L on loans L and pays rate r_D on demand deposits D. Assume that the bank is subject to a reserve requirement $q = R/D$, $0 < q < 1$. Assume that the bank also incurs costs related to both the loans it services and the deposits it administers expressed as $C(L,D)$. (We will consider the shape of the cost function later.) The bank thus maximizes the function

$$\pi = r_L L - r_D D - C(L, D).$$

The bank's only assets are the loans and the reserves, and the demand deposits are its only liabilities. Therefore its balance sheet is represented by the constraint that its "net worth" (actually, owners' equity) $w = L + R - D$, but since $R = qD$, we rewrite this constraint as

$$\begin{aligned} w &= L + qD - D \\ &= L - (1 - q)D. \end{aligned}$$

In one version of this problem it is assumed that w is identically zero, because otherwise profit can be increased in direct proportion to an infusion of capital. But this assumption obscures certain aspects of the model while simplifying the analysis only slightly, so it is not made here.

If the original objective function is maximized over the decision variables L and D, subject to the constraint, by the Lagrangian technique, the result is the maximum profit as a function of the parameters, $\pi^*(r_L, r_D, w, q)$. This process also generates expressions for the value of the multiplier at the point of optimality,

$$\lambda^* = r_L - C_L = (r_D + C_D)/(1-q) > 0,$$

assuming that costs are increasing with an increase in deposits ($C_D > 0$).[1]

Next we form the primal-dual Lagrangian,

$$\mathcal{L}^* = r_L L - r_D D - C(L, D) - \pi^*(r_L, r_D, w, q) + \lambda(w - L + (1-q)D),$$

and set its derivatives with respect to the parameters equal to zero for the FOC:

$$\mathcal{L}^*_{r_L} = L - \pi^*_{r_L} = 0 \Rightarrow \pi^*_{r_L} \equiv L^* > 0 \text{ (because } L > 0 \text{ by definition)}$$

$$\mathcal{L}^*_{r_D} = -D - \pi^*_{r_D} = 0 \Rightarrow \pi^*_{r_D} \equiv -D^* < 0 \text{ (because } D > 0 \text{ by definition)}$$

$$\mathcal{L}^*_q = -\lambda D - \pi^*_q = 0 \Rightarrow \pi^*_q \equiv -\lambda^* D^* < 0 \text{ (if } C_D > 0 \Rightarrow \lambda^* > 0)$$

$$\mathcal{L}^*_w = \lambda - \pi^*_w = 0 \Rightarrow \pi^*_w \equiv \lambda^* > 0 \text{ (if } \lambda^* > 0).$$

Now we have an interesting situation. The first two FOC are true envelope relations, involving derivatives (of the direct and indirect objective functions) with respect to parameters which enter the objective function only. The latter two are "pseudo-envelope" relations involving derivatives of the indirect objective function only with respect to parameters that enter the constraint only. In the utility-maximization problem,

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[1] The existence of a maximum for the objective function depends on the characteristics of the cost function, in particular upon the second partials with respect to L and D. More on this later. The Lagrange multiplier, however, is interpreted here as the "shadow price" or value of an additional unit of net worth, and is therefore definitely positive as an increase in w increases profit directly.

it is from these latter relations that Roy's identity is derived. Here we can derive a closely analogous relation,

$$(\partial\pi^*/\partial q)/(\partial\pi^*/\partial w) = -D^*.$$

The ratio of the response of the maximum possible profit to a change in the reserve ratio to that to a change in the net worth of the bank is negative (as might be expected) because $D > 0$ by definition, and it is equal in absolute value to the level of deposits, which is surprising and potentially useful.

On the other hand, the ratio of the derivatives with respect to the parameters appearing in the objective function is simply the negative of the ratio of the corresponding choice variables. This results from the fact that the objective function is linear in those parameters, and from the fact that they are absent from the constraint.

Of the various ratios of one of the former class of relations to one of the latter, perhaps the most elegant are

$$(\partial\pi^*/\partial q)/(\partial\pi^*/\partial r_D) = \lambda^*$$

$$(\partial\pi^*/\partial w)/(\partial\pi^*/\partial r_L) = \lambda^*/L^*.$$

The interpretation of these relations, and of the last 2 of the FOC, is enhanced by knowledge of the sign of λ^*, which is positive if $C_D > 0$, but not necessarily otherwise. The value of all Roy-like relations as testable hypotheses is limited by the fact that they involve not only the optimum value of the objective function (which may plausibly be taken as the actual value if that is the assumption being tested), but also the Lagrange multiplier, which is not directly observable (but which can sometimes be calculated).

Taking the second partials of $\mathcal{L}^*$ and substituting the derivatives of the FOC gives the second partials,

$$\mathcal{L}^*_{r_L r_L} = -\pi^*_{r_L r_L} = -\partial L^*/\partial r_L$$

$$\mathcal{L}^*_{r_L r_D} = -\pi^*_{r_L r_D} = -\partial L^*/\partial r_D$$

$$\mathcal{L}^*_{r_L q} = -\pi^*_{r_L q} = -\partial L^*/\partial q$$

$$\mathcal{L}^*_{r_L w} = -\pi^*_{r_L w} = -\partial L^*/\partial w$$

$$\mathcal{L}^*_{r_D r_L} = -\pi^*_{r_D r_L} = \partial D^*/\partial r_L$$

$$\mathcal{L}^*_{r_D r_D} = -\pi^*_{r_D r_D} = \partial D^*/\partial r_D$$

$$\mathcal{L}^*_{r_D q} = -\pi^*_{r_D q} = \partial D^*/\partial q$$

$$\mathcal{L}^*_{r_D w} = -\pi^*_{r_D w} = \partial D^*/\partial w$$

$$\mathcal{L}^*_{q r_L} = -\pi^*_{q r_L} = \partial(\lambda^* D^*)/\partial r_L = \lambda^*(\partial D^*/\partial r_L) + D^*(\partial \lambda^*/\partial r_L)$$

$$\mathcal{L}^*_{q r_D} = -\pi^*_{q r_D} = \partial(\lambda^* D^*)/\partial r_D = \lambda^*(\partial D^*/\partial r_D) + D^*(\partial \lambda^*/\partial r_D)$$

$$\mathcal{L}^*_{qq} = -\pi^*_{qq} = \partial(\lambda^* D^*)/\partial q = \lambda^*(\partial D^*/\partial q) + D^*(\partial \lambda^*/\partial q)$$

$$\mathcal{L}^*_{qw} = -\pi^*_{qw} = \partial(\lambda^* D^*)/\partial w = \lambda^*(\partial D^*/\partial w) + D^*(\partial \lambda^*/\partial w)$$

$$\mathcal{L}^*_{w r_L} = -\pi^*_{w r_L} = -\partial \lambda^*/\partial r_L$$

$$\mathcal{L}^*_{w r_D} = -\pi^*_{w r_D} = -\partial \lambda^*/\partial r_D$$

$$\mathcal{L}^*_{wq} = -\pi^*_{wq} = -\partial \lambda^*/\partial q$$

$$\mathcal{L}^*_{ww} = -\pi^*_{ww} = -\partial \lambda^*/\partial w.$$

Consider first the symmetry relations, of which there are 6:

$$\mathcal{L}^*_{r_L r_D} = -\partial L^*/\partial r_D = \partial D^*/\partial r_L = \mathcal{L}^*_{r_D r_L}$$

$$\mathcal{L}^*_{r_L q} = -\partial L^*/\partial q = \partial(\lambda^* D^*)/\partial r_L = \mathcal{L}^*_{q r_L}$$

$$\mathcal{L}^*_{r_L w} = -\partial L^*/\partial w = -\partial \lambda^*/\partial r_L = \mathcal{L}^*_{w r_L}$$

$$\mathcal{L}^*_{r_D q} = \partial D^*/\partial q = \partial(\lambda^* D^*)/\partial r_D = \mathcal{L}^*_{q r_D}$$

$$\mathcal{L}^*_{r_D w} = \partial D^*/\partial w = -\,\partial\lambda^*/\partial r_D = \mathcal{L}^*_{w r_D}$$

$$\mathcal{L}^*_{qw} = \partial(\lambda^* D^*)/\partial w = -\,\partial\lambda^*/\partial q = \mathcal{L}^*_{wq}$$

All of these relations except the first are limited in their usefulness by the presence of derivatives of the multiplier (three of them derivatives of the product $\lambda^* D^*$). With that reservation, each constitutes a testable hypothesis of whether the firm is maximizing profit on the basis of the model. In practice, the first relation, involving only the quantities of loans and deposits and the rates on both would probably be the most feasible to test empirically.

Now consider the Hessian matrix of the primal-dual Lagrangian:

$$\mathcal{L}^*_{\alpha\alpha} = \begin{bmatrix} -\,\partial L^*/\partial r_L & -\,\partial L^*/\partial r_D & -\,\partial L^*/\partial q & -\,\partial L^*/\partial w & 0 \\ \partial D^*/\partial r_L & \partial D^*/\partial r_D & \partial D^*/\partial q & \partial D^*/\partial w & 0 \\ \partial(\lambda^* D^*)/\partial r_L & \partial(\lambda^* D^*)/\partial r_D & \partial(\lambda^* D^*)/\partial q & \partial(\lambda^* D^*)/\partial w & -D^* \\ -\,\partial\lambda^*/\partial r_L & -\,\partial\lambda^*/\partial r_D & -\,\partial\lambda^*/\partial q & -\,\partial\lambda^*/\partial w & 1 \\ 0 & 0 & -D^* & 1 & 0 \end{bmatrix}$$

The necessary second-order condition for a maximum is that this matrix be negative semi-definite, implying that its border-preserving principal minors of order k have sign $(-1)^k$ or zero. Among the order-2 minors, that including columns 1 and 2 is equal to zero because of the bordering zeroes. Those including columns 1 and 3 and columns 2 and 3 happen to be standard CS derivatives, whose signs are thus determined, and those including columns 1 and 4, and 2 and 4, produce the same results:

$$\partial L^*/\partial r_L \geq 0$$

$$\partial L^*/\partial r_D \leq 0.$$

The symmetry result applied to the second of these produces another,

$$\partial D^*/\partial r_L \geq 0.$$

And a basic conjugate pairs result[2] yields

$$\partial D^*/\partial r_D \leq 0.$$

This result and the first one above also follow from the negativity of the diagonal elements of the matrix. Similarly, we can also state that

$$\partial \lambda^*/\partial w \geq 0,$$

and

$$\partial(\lambda^* D^*)/\partial q \leq 0$$

$$\lambda^*(\partial D^*/\partial q) + D^*(\partial \lambda^*/\partial q) \leq 0.$$

Since $D^* > 0$ by definition, then if $\lambda^* > 0$, this expressions says that either $\partial D^*/\partial q$ or $\partial \lambda^*/\partial q$ or both must be non-positive, and it sets a condition on the relative magnitudes of these responses if they are of opposite sign. While this relation may be difficult to apply in practice, the next one may be less so and its interpretation is similar.

The minor that includes columns 3 and 4 produces a complex expression that simplifies to a potentially useful result:

$$\mathrm{D}_{13} = +D^*(D^*(\partial L^*/\partial r_L)) = D^{*2}(\partial L^*/\partial r_L) \geq 0 \Rightarrow \partial L^*/\partial r_L \geq 0$$

$$\mathrm{D}_{23} = +D^*(-D^*(\partial D^*/\partial r_D)) = -\mathrm{D}^{*2}(\partial D^*/\partial r_D) \geq 0 \Rightarrow \partial D^*/\partial r_D \leq 0$$

$$\mathrm{D}_{34} = -D^*[\partial(\lambda^* D^*)/\partial w - D^*(\partial \lambda^*/\partial w)] - [\partial(\lambda^* D^*)/\partial q - D^*(\partial \lambda^*/\partial q)] \geq 0$$

$$D^*[\lambda^*(\partial D^*/\partial w) + D^*(\partial \lambda^*/\partial w) - D^*(\partial \lambda^*/\partial w)]$$

[2] See Currier, Theorem 5-1B, p. 81.

$$+ [\lambda^*(\partial D^*/\partial q) + D^*(\partial\lambda^*/\partial q) - D^*(\partial\lambda^*/\partial q)] \leq 0$$

$$\lambda^*[D^*(\partial D^*/\partial w) + \partial D^*/\partial q] \leq 0$$

But if $\lambda^* > 0$, then $D^*(\partial D^*/\partial w) + \partial D^*/\partial q \leq 0$. Since $D^* > 0$ by definition, this says that deposits must decrease in response to an increase either in w or in q or in both, and it sets a condition on the relative sizes of the responses if they are of opposite sign. We might expect to find $\partial D^*/\partial q \leq 0$, but this is not assured by any of the foregoing. (In fact, the sign of $\partial D^*/\partial q$ cannot be determined by conventional analysis, either.)

Two of the order-3 minors and the entire determinant yield complex expressions which, because none of the terms vanishes or is equal to unity, do not simplify. The order-3 minors that include columns 1 and 2 and *either* column 3 or column 4 produce the same expression:

$$D_{124} = -[(\partial L^*/\partial r_D)(\partial D^*/\partial r_L) - (\partial L^*/\partial r_L)(D^*/\partial r_D)] \leq 0$$

$$= (\partial L^*/\partial r_L)(D^*/\partial r_D) - (\partial L^*/\partial r_D)(\partial D^*/\partial r_L) \leq 0$$

$$(\partial L^*/\partial r_L)(D^*/\partial r_D) \leq (\partial L^*/\partial r_D)(\partial D^*/\partial r_L).$$

This says that the product of the "own-rate effects" is less than or equal to the product of the "cross-rate effects." But we know from the first two order-2 minors that the former are opposite in sign, and from the first symmetry relation that the latter are also opposite in sign. Therefore, what this relation actually says is that the product of the *magnitudes* of the "own-rate effects" is *greater* than or equal to that of the "cross-rate effects."

$$|(\partial L^*/\partial r_L)|\ |(D^*/\partial r_D)| \geq |(\partial L^*/\partial r_D)|\ |(\partial D^*/\partial r_L)|$$

Next consider the dual problem,

minimize $w = L - (1 - q)D$, subject to $r_L L - r_D D - C(L, D) = \overline{\pi}$.

Now we see why it helps to include w as a parameter in the primal problem. If the constraint in the primal problem is written simply as $L = (1 - q)D$, or even if the expression $L - (1 - q)D$ is assigned a specific numerical value (e.g., zero), careful attention is needed to recognize what it is that must be minimized in the dual problem. Indeed, the nature of the duality itself is obscured. Including w as a parameter makes it clear that the duality is between w and π (or w^* and $\overline{\pi}$), and that if $\overline{\pi}$ is the level of profit that results when $w = 0$, the minimum level of w that can produce a profit of $\overline{\pi}$ is $w = 0$.

The PD Lagrangian (of the dual problem) is

$$\mathcal{L}^* = L - (1 - q)D + \mu(\overline{\pi} - r_L L + r_D D + C(L, D)) - w^*(r_L, r_D, q, \overline{\pi}).$$

The FOC with respect to the parameters are

$$\mathcal{L}^*_{r_L} = -\mu L - w^*_{r_L} = 0 \Rightarrow w^*_{r_L} \equiv -\mu^* L^*$$

$$\mathcal{L}^*_{r_D} = \mu D - w^*_{r_D} = 0 \Rightarrow w^*_{r_D} \equiv \mu^* D^*$$

$$\mathcal{L}^*_{q} = D - w^*_{q} = 0 \Rightarrow w^*_{q} \equiv D^*$$

$$\mathcal{L}^*_{\pi} = \mu - w^*_{\pi} = 0 \Rightarrow w^*_{\pi} \equiv \mu^*.$$

Because of the way the respective Lagrangians are set up, $\mu^* = 1/\lambda^*$, which can be verified from the FOC (with respect to the decision variables) for the ordinary dual problem. Now, the optimized values of the variables in the dual problem, L^* and D^*, are "compensated" variables, for which the bank's net worth w is adjusted for changes in the parameters r_L, r_D, and q in order to maintain a constant level of profit $\overline{\pi}$. Therefore, rename these variables L^c and D^c. There are 6 "Roy-like" relations for the dual problem, which are ratios of the implied identities above (12 relations counting reciprocals). They are not especially enlightening, but examples are included here for completeness and for comparison with those for the primal problem:

$$(\partial w^*/\partial q)/(\partial w^*/\partial \bar{\pi}) = D^c/\mu^* = \lambda^* D^c$$

$$(\partial w^*/\partial r_L)/(\partial w^*/\partial r_D) = -L^c/D^c$$

$$(\partial w^*/\partial r_L)/(\partial w^*/\partial \bar{\pi}) = -L^c$$

$$(\partial w^*/\partial r_D)/(\partial w^*/\partial q) = \mu^*.$$

The signs of the second and third expressions are negative, from the definitions of L and D, and those of the first and fourth are positive if $\lambda^* > 0$.

We can also derive generalized "Slutsky-type" relations for this model. As in the utility-maximization model, at the point of optimality the choice variables have the same values as their counterparts in the primal problem, L^* and D^*. In other words,

$$L^c(r_L, r_D, q, \bar{\pi}) \equiv L^*(r_L, r_D, q, w^*(r_L, r_D, q, \bar{\pi}))$$

$$D^c(r_L, r_D, q, \bar{\pi}) \equiv D^*(r_L, r_D, q, w^*(r_L, r_D, q, \bar{\pi})).$$

Differentiating these relations with respect to r_L and r_D,

$$\partial L^c/\partial r_L = \partial L^*/\partial r_L + (\partial L^*/\partial w^*)(\partial w^*/\partial r_L)$$

$$\partial L^c/\partial r_D = \partial L^*/\partial r_D + (\partial L^*/\partial w^*)(\partial w^*/\partial r_D)$$

$$\partial D^c/\partial r_L = \partial D^*/\partial r_L + (\partial D^*/\partial w^*)(\partial w^*/\partial r_L)$$

$$\partial D^c/\partial r_D = \partial D^*/\partial r_D + (\partial D^*/\partial w^*)(\partial w^*/\partial r_D).$$

Rearranging and substituting for the derivatives of w^* (from the FOC) gives the generalized Slutsky relations,

$$\partial L^*/\partial r_L = \partial L^c/\partial r_L + \mu^* L^*(\partial L^*/\partial w^*)$$

$$\partial L^*/\partial r_D = \partial L^c/\partial r_D - \mu^* D^*(\partial L^*/\partial w^*)$$

$$\partial D^*/\partial r_L = \partial D^c/\partial r_L + \mu^* L^*(\partial D^*/\partial w^*)$$

$$\partial D^*/\partial r_D = \partial D^c/\partial r_D - \mu^* D^*(\partial D^*/\partial w^*).$$

Differentiating with respect to q adds another pair:

$$\partial L^*/\partial q = \partial L^c/\partial q - D^*(\partial L^*/\partial w^*)$$

$$\partial D^*/\partial q = \partial D^c/\partial q - D^*(\partial D^*/\partial w^*).$$

Except for the presence of the Lagrange multiplier in these relations, all but the last pair are identical in form to the classical Slutsky relation for the utility maximization model. On the other hand, the generalized Slutsky relations for the utility maximization model with prices in the objective function have an additional term, which does contain the multiplier. The reason for this difference is the arrangement of the parameters. In this model (the primal problem) two parameters (r_L and r_D) appear in the objective function only, while the others (w and q) appear in the constraint only. Thus both the objective function and the constraint are special cases of the general form considered by Kalman and Intriligator, just as in the standard utility maximization model, in which all the parameters are in the constraint. In the extended utility maximization model, prices appear in *both* the objective function *and* the constraint. As a result, derivatives of the Lagrangian with respect to those parameters in the generalized Slutsky relation for that model contain *two terms*, instead of one as in these models in which the parameters are separated.[3]

As noted earlier, the value of the Lagrange multiplier in this model depends on known parameters, and on the derivative of the cost function with respect either to loans or to deposits. Thus it is not unobservable in the same sense as in the utility maximization model. But the first four Slutsky relations can only be evaluated numerically to the extent

[3] For further details on the way in which the form of the generalized Slutsky relation depends on the forms of the objective function and the constraint, see Kalman and Intriligator (1973), especially pp. 474, 478-479, and 482-483.

that the value of C_L or C_D can be determined. If one of these values *can* be determined, then those relations could conceivably open up an analysis of the way in which a bank would adjust its loans and deposits in response to a change in the rate on either one if its net worth were automatically adjusted to keep profit at the same level, with implications and applications comparable to those of compensated demand and supply. In any case, all of these relations provide a new way of looking at certain aspects of banking.

Next, differentiating and substituting again gives the second partials of $\mathcal{L}^*$:

$$\mathcal{L}^*_{r_L r_L} = -w^*_{r_L r_L} = \partial(\mu^* L^c)/\partial r_L = \mu^*(\partial L^c/\partial r_L) + L^c(\partial\mu^*/\partial r_L)$$

$$\mathcal{L}^*_{r_L r_D} = -w^*_{r_L r_D} = \partial(\mu^* L^c)/\partial r_D = \mu^*(\partial L^c/\partial r_D) + L^c(\partial\mu^*/\partial r_D)$$

$$\mathcal{L}^*_{r_L q} = -w^*_{r_L q} = \partial(\mu^* L^c)/\partial q = \mu^*(\partial L^c/\partial q) + L^c(\partial\mu^*/\partial q)$$

$$\mathcal{L}^*_{r_L \pi} = -w^*_{r_L \pi} = \partial(\mu^* L^c)/\partial\overline{\pi} = \mu^*(\partial L^c/\partial\overline{\pi}) + L^c(\partial\mu^*/\partial\overline{\pi})$$

$$\mathcal{L}^*_{r_D r_L} = -w^*_{r_D r_L} = -\partial(\mu^* D^c)/\partial r_L = -\mu^*(\partial D^c/\partial r_L) - D^c(\partial\mu^*/\partial r_L)$$

$$\mathcal{L}^*_{r_D r_D} = -w^*_{r_D r_D} = -\partial(\mu^* D^c)/\partial r_D = -\mu^*(\partial D^c/\partial r_D) - D^c(\partial\mu^*/\partial r_D)$$

$$\mathcal{L}^*_{r_D q} = -w^*_{r_D q} = -\partial(\mu^* D^c)/\partial q = -\mu^*(\partial D^c/\partial q) - D^c(\partial\mu^*/\partial q)$$

$$\mathcal{L}^*_{r_D \pi} = -w^*_{r_D \pi} = -\partial(\mu^* D^c)/\partial\overline{\pi} = -\mu^*(\partial D^c/\partial\overline{\pi}) - D^c(\partial\mu^*/\partial\overline{\pi})$$

$$£^*_{q r_L} = -w^*_{q r_L} = -\partial D^c/\partial r_L$$

$$\mathcal{L}^*_{q r_D} = -w^*_{q r_D} = -\partial D^c/\partial r_D$$

$$\mathcal{L}^*_{qq} = -w^*_{qq} = -\partial D^c/\partial q$$

$$\mathcal{L}^*_{q\pi} = -w^*_{q\pi} = -\partial D^c/\partial\overline{\pi}$$

$$\mathcal{L}^*_{\pi r_L} = -w^*_{\pi r_L} = -\partial\mu^*/\partial r_L$$

$$\mathcal{L}^*_{\pi r_D} = -w^*_{\pi r_D} = -\partial\mu^*/\partial r_D$$

$$\mathcal{L}^*_{\pi q} = -w^*_{\pi q} = -\partial\mu^*/\partial q$$

$$\mathfrak{L}^*_{\pi\pi} = -\,w^*_{\pi\pi} = -\partial\mu^*/\partial\overline{\pi}.$$

For the dual problem we have the following symmetry relations:

$$\mathfrak{L}^*_{r_L r_D} = \partial(\mu^* L^c)/\partial r_D = -\partial(\mu^* D^c)/\partial r_L = \mathfrak{L}^*_{r_D r_L}$$

$$\mathfrak{L}^*_{r_L q} = \partial(\mu^* L^c)/\partial q = -\partial D^c/\partial r_L = \mathfrak{L}^*_{q r_L}$$

$$\mathfrak{L}^*_{r_L \pi} = \partial(\mu^* L^c)/\partial\overline{\pi} = -\partial\mu^*/\partial r_L = \mathfrak{L}^*_{\pi\, r_L}$$

$$\mathfrak{L}^*_{r_D q} = -\partial(\mu^* D^c)/\partial q = -\partial D^c/\partial r_D = \mathfrak{L}^*_{q r_D}$$

$$\mathfrak{L}^*_{r_D \pi} = -\partial(\mu^* D^c)/\partial\overline{\pi} = -\partial\mu^*/\partial r_D = \mathfrak{L}^*_{\pi\, r_D}$$

$$\mathfrak{L}^*_{q\pi} = -\partial D^c/\partial\overline{\pi} = -\partial\mu^*/\partial q = \mathfrak{L}^*_{\pi\, q}$$

These relations are subject to essentially the same remarks as those made with regard to the symmetry relations for the primal problem.

The Hessian matrix for this problem provides very few useful results. Two of its rows rather than one contain derivatives of products, which doubles the number of terms in the expansions of minors that include those rows, while the borders contain two elements (rather than one, as in the primal problem) which are neither 0 nor 1. Only three of the order-2 minors produce expressions which are not too complex to be of use, and they all reduce to the same expression. Since the dual problem involves minimization, the bordered Hessian is *positive* semidefinite, so that with a single constraint, all of its principal minors are non-positive.

$$\begin{vmatrix} -\partial D^c/\partial q & -\partial D^c/\partial\overline{\pi} & 0 \\ -\partial\mu^*/\partial q & -\partial\mu^*/\partial\overline{\pi} & 1 \\ 0 & 1 & 0 \end{vmatrix} = \partial D^c/\partial q \leq 0$$

In the primal problem, the sign of the corresponding derivative is impossible to determine: Even though $D^* = D^c$ at the point of optimality, $\partial D^*/\partial q$ is not necessarily equal to $\partial D^c/\partial q$. It is possible that this information could be used in conjunction with the various relations among those derivatives that have been developed above to determine the signs of other individual CS derivatives in addition to those determined earlier for the primal problem (p. 39, above), but attempts to do so have yet to succeed.

Silberberg emphasizes repeatedly that the results generated by the PD method consist of all the consequences that follow from the maximization hypothesis alone, and no others. Since it does not involve any derivatives with respect to the decision variables (the starting point for conventional analysis), it does not take account of anything that may be known or assumed about those derivatives. In the standard utility-maximization model, the signs of the standard CS derivatives are determined by the signs of the second partials of the utility function, and hence by the assumptions made regarding its shape.

The same is true for this model with regard to the cost function. It has been noted that the signs of a number of the relations here can be determined if the sign of λ^* is known, and that $\lambda^* > 0$ if $C_D > 0$. Since the cost function is exogenous to the model, there is no way to determine the sign of its derivative within the model. But if the reasonable assumption is made that the bank incurs costs to administer both loans and deposits ($C_L > 0$, $C_D > 0$), and the only slightly less plausible assumption that diminishing returns to scale result in $C_{LL} > 0$, $C_{DD} > 0$, with $C_{LD} = 0$, is also made, it is possible to determine the signs of many of the standard CS derivatives by conventional analysis, which results are stated below. These assumptions in themselves do not allow us to determine directly the signs of any expressions in the PD analysis other than those dependent on the sign of λ^*:

For the primal problem,

$$\partial L^*/\partial r_L > 0,\ \partial L^*/\partial r_D < 0,\ \partial L^*/\partial q < 0,\ \partial L^*/\partial w > 0$$

$$\partial D^*/\partial r_L > 0,\ \partial D^*/\partial r_D < 0,\ \partial D^*/\partial q\ ?,\ \partial D^*/\partial w < 0$$

$$\partial \lambda^*/\partial r_L > 0,\ \partial \lambda^*/\partial r_D > 0,\ \partial \lambda^*/\partial q > 0,\ \partial \lambda^*/\partial w < 0.$$

For the dual problem,

$$\partial L^c/\partial r_L\ ?,\ \partial L^c/\partial r_D\ ?,\ \partial L^c/\partial q < 0,\ \partial L^c/\partial \bar{\pi} > 0$$

$$\partial D^c/\partial r_L > 0,\ \partial D^c/\partial r_D < 0,\ \partial D^c/\partial q < 0,\ \partial D^c/\partial \bar{\pi} < 0$$

$$\partial \mu^*/\partial r_L < 0,\ \partial \mu^*/\partial r_D\ ?,\ \partial \mu^*/\partial q < 0,\ \partial \mu^*/\partial \bar{\pi} > 0.$$

These relations are consistent with those from the PD analysis in which these derivatives appear, all of which are more general, being derived with no assumptions on the signs of the derivatives of C except where noted. The combination of the two methods still leaves indeterminate the signs of the derivatives which are indeterminate under conventional analysis alone.

Which set of conditions is more useful is open to question. On the one hand, it is usually simpler empirically to test the sign of a single derivative than to test any more complex relation, but this is not always the case. The deeper relations may require significant insight and manipulation to develop from the conventional analysis. All of the conventional results above rely on the assumptions that the second partials of cost with respect to both loans and deposits are positive and that the cross partials are zero, without which the second-order conditions for a maximum either fail or are not assured. These assumptions are essentially equivalent to the common assumption of a U-shaped LRAC curve, which is relatively benign, but which is unnecessary for the PD analysis.

One might argue that the question of "usefulness" is academic on the premise that this model does not sufficiently resemble actual banking operations to make refutable hypotheses derived from it of any use in the "real world." Of course, many textbook models share this limitation in varying degrees, but two features of this model stand out as being particularly questionable.

As this model is constructed loans and hence profit can be pumped up without limit (except for that imposed by the convex cost function) simply by pouring in more financial capital. Strictly speaking, this does not affect the bank's net worth, because the injected assets are offset by an increase in the equity of the investors. Neither the asset category nor the equity is separately accounted for in the model; the parameter w called "net worth" is actually just the difference between loans plus required reserves and deposits. But this is not necessarily unrealistic: If a real bank is "loaned up" (to the limit of its reserve requirement), the investors can always in principle issue more stock and buy it themselves, thus increasing the bank's excess reserves (assumed to be zero in the model), which can then be loaned out to produce additional profit. Presumably the investors would not consider such an action unless they were certain that the market for additional loans was favorable. As we have shown, including this possibility does nothing to make the model unworkable. The essential factor that is ignored in the model is not net worth but excess reserves.

A more complete model would recognize that profits contribute to net worth by way of retained earnings. In such a model it might then make sense to maximize net worth, rather than minimize it. Such a model would then have to recognize other types of constraints based on market conditions and perhaps accounting realities. It would also almost have to be a multi-period model, as many realistic models must be, regardless of philosophical arguments regarding the importance of dynamics in economics.[4]

The other factor omitted from this model is the multiplier effect. The constraint that $L = (1 - q)D$ says that deposits in excess of required reserves are loaned out, and that these funds then just disappear or are stashed under a mattress, as there is no mechanism in the model for them to be redeposited. But there is less here than meets the eye. If a redeposit mechanism were incorporated, the theory of the simple deposit multiplier would predict that the total amount of money existing after an initial deposit d would be d/q. Since all of this money except the initial deposit exists in the form of loans, and since the relation holds for all deposits, the only change in the result would be that the coefficient of D would be $1/q$ instead of $1 - q$. But the essential relation would still be strict proportionality between L and D.

Therefore it appears that this model, simple as it is, may not be significantly more "unrealistic" than many other simple models which form the basis of empirical research in economics. Perhaps it would be worthwhile to test some of the "refutable hypotheses" implied by this model to see whether they can in fact be refuted.

One aspect of banking that has generated a great deal of interest for 40 years with no sign of slacking off is that of expense-preference behavior. As early as 1957, Becker showed how non-pecuniary considerations could have economic content and could therefore provide managers to pursue objectives other than or in addition to maximizing profits for the firm. Stigler (1956) had recognized a year earlier that a significant portion of monopoly rents could be hidden in the form of compensation for executives and others

[4] It is worth noting that Samuelson viewed dynamics as the second "Foundation" of economics in addition to comparative statics, and in a sense more fundamental. See Samuelson (1983), p. 5.

and thus not reported as profits. Baumol (1959) soon made the same observation in a different context, as did others.

Williamson (1963) may have been the first to attempt to operationalize the idea into a specific model and examine its performance against data, and also the first to recognize that the opportunity for such managerial discretion depends on weak oversight of managers by owners (which may be as prevalent in family-owned firms as in large corporations). Alchian and Kessel (1962) argued that regulation, like monopoly, provides a climate in which managers can divert revenue to uses other than compensating the owners of a firm, according to their own preferences and with little to hinder them.

Edwards (1964, 1977) saw the banking industry as especially open to this sort of exploitation by managers, being highly regulated, often highly concentrated, and increasingly managed by hired technocrats while owned by a large number of relatively small shareholders. A majority of the listings in both the EconLit and ABI/Inform Global databases are for articles on banking or other sectors of the financial services industry, including studies done in Europe and Latin America. Other industries which have received considerable study are public utilities, especially electric power, and common carriers such as the trucking industry.

Many of the studies of the financial services industry focus on the savings and loan industry. One reason is that some thrifts are organized as mutual companies, in which the depositors are the legal owners but exercise negligible oversight (often signing away their voting rights when they join), while others are stock corporations. It is believed by many investigators that the former are more vulnerable to expense-preference behavior on the part of managers than are the latter. The evidence on this point is mixed, and may depend on the particular model used or on the type of econometric tests employed.[5]

Another reason for interest in this sector is the fact that it underwent significant deregulation in the 1980s, and it is thought that this change of industry structure should reduce the prevalence of expense-preference behavior. Indeed, studies seem to support this hypothesis, although it does not appear to be as widely tested as that regarding the form of organization.[6]

Perhaps the most notable fact about the sample of papers reviewed for this study is that very few attempted to do any comparative-statics analysis of the model employed. One reason for this is that virtually all of them, and the vast majority of those listed in the databases for which only the abstract was examined, are econometric studies. Very few are theoretical, and the econometric models tend to be simple linear or log-linear types, although some incorporate several parameters of a theoretical model into a single regression parameter. Quite a few of the savings and loan studies in fact employ an extended version of the model used in this chapter.

But as has been amply argued by authorities already cited, it is ultimately comparative statics that generates testable hypotheses. Indeed, the signs of regression coefficients constitute a rudimentary kind of comparative statics, but only when the derivatives with respect to the corresponding parameters are obvious, as they are for the kinds of models generally used. It would appear that this area is ripe for theoretical work,

[5] For example, Akella and Greenbaum (1988), Mester (1989), Keating and Keating (1992), and Gropper and Hudson (2003). The latter includes an extensive and up-to-date bibliography.
[6] Gropper and Hudson (2003). Gropper has written related articles with other collaborators within the past decade, and this one also contains references to other works on the effects of banking deregulation, as well as a list of survey articles on methodology.

including analysis of models such as that used by Akella and Greenbaum and their critics and supporters which incorporate several model parameters into a single regression parameter. Considering the effects of the model parameters separately could provide more precise and specific results. And Primal-Dual analysis would generating a much wider range of testable hypotheses than just the CS derivatives themselves, providing material for another generation of econometricians.

CHAPTER IV

A LIMIT-PRICING MODEL

It has been suggested that a monopolist facing a threat of competition from a potential entrant might increase output above the monopoly level, thereby lowering its price, in an effort to make entry unattractive, a strategy known as "limit pricing." In addition, several strategies for raising the fixed cost of entry are widely known. A simple limit-pricing model demonstrates how different the analysis of models with very similar form can be.

While the origin of the theory of limit pricing by a monopolist to deter entry into the industry by a potential competitor is generally ascribed to Bain (1949), the idea was at least suggested by Kaldor (1935). Interestingly, Kaldor mentions the reduction of prices by incumbents in an oligopolistic industry not as a strategic behavior the incumbents might employ to deter entry, but as a theoretically possible way to prevent loss of total surplus due to excess investment into a declining-cost industry which the incumbents' self-interest would not lead them to take. Investment in excess capacity is one of the strategic options that has emerged as a more plausible alternative to limit prices in one group of models that have grown out of Bain's original formulation.

Bain actually cites no fewer than seven possible reasons a monopolist might depart from a profit-maximizing pricing strategy. He also recognizes at least implicitly that such a situation is inherently dynamic (involving possible discontinuities and path dependence) and strategic (and thus – today – implicitly game-theoretic). Bain's insights were further developed by Sylos-Labini (1962) and Modigliani (1958), and by numerous others in the 1970s (see Salop, 1979, for further references).

Dixit (1980), noting the inevitable strategic interactions between incumbent and potential entrant, pointed out that most early efforts to analyze the situation adopted the "Bain-Sylos postulate" that the entrant would believe that the incumbent would maintain the price-limiting level of output even in the event that entry occurred, thus giving it a Stackelberg leadership role. But as Dixit also pointed out, the incumbent actually has two contradictory incentives: Faced with "an irrevocable fact of entry," a rational agent might be expected to make the best of the situation by reducing output to the profit-maximizing level given the level of output chosen by the entrant. On the other hand, he would like to present a credible threat that in the event of entry he would actually *increase* output in a predatory move to drive down the price still further, since the potential entrant is aware of the incentive to do otherwise. Dixit refers to Sherer (1970, ch. 8) for a detailed exposition of the Bain-Sylos-Modigliani model and its critique.

Building on the work of Schelling (1960) and Spence (1977), Dixit (1979, 1980) considered prior commitment, primarily in the form of investment in capacity, as the pre-entry signal to prospective entrants regarding potential profit should they choose to enter. Employing only the simplest game-theoretic models he found that outcomes depend in sensitive ways upon numerous factors necessarily omitted from any one model. For example, he concluded that in the absence of "agreement about the rules of the post-entry game" entry may result in a temporary state of disequilibrium, and that in that case, "Financial positions of the firms may then acquire an important role."[1] Emphasizing the distinction between the rules assumed to govern the game and the *initial conditions* with which the game begins, Dixit observed that ". . . the role of an irrevocable commitment of investment in pre-entry deterrence is to alter the initial conditions of the post-entry game to the advantage of the established firm, *for any fixed rule under which the game is to be played*" (emphasis added).[2] He ended that paper with a question, whether one firm can *change* the rules to its own advantage.

Salop (1979) also quoted Schelling regarding "the paradox that the power to constrain an adversary may depend on the power to bind oneself."[3] Salop further asserted that some strategies (such as advertising) may rationally be expected to be abandoned if entry occurs, a claim that is certainly open to question.[4] But he noted as well Spence's observation that some strategies, such as cartels, function best as deterrents to entry if they are designed to ensure their self-destruction if entry occurs, because competition would decrease the gains from entry and hence the incentive to enter.

J. Friedman (1979) was one of the first to employ a more modern and sophisticated game-theoretic approach to entry deterrence, and he made a different observation which cast doubt on the plausibility of limit pricing as a workable strategy to deter entry. In a complete-information game-theoretic context, pre-entry prices have no effect on post-entry costs or demand, and hence none on post-entry profit potential. And since the entrant is fully informed regarding that potential, the incumbent's pre-entry

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[1] Dixit (1980), p. 95.
[2] Dixit (1980), p. 106.
[3] Salop (1979), p. 335.
[4] *ibid.* p. 336.

price behavior would have no effect on the decision to enter, and limit pricing therefore would not occur in equilibrium. Scheffman and Spiller (1992), however, found limit pricing to be a likely equilibrium even *with* full information.

Recognizing that in the Bain-Sylos framework the entrant is assumed to be using the pre-entry price as a signal regarding the incumbent's costs, and hence on the price and market shares that can be expected after entry, Milgrom and Roberts (1982) devised a game-theoretic model in which the incumbent and potential entrant in a monopolistic industry do *not* have complete information regarding costs and hence *do* use pre-entry price as a signal regarding costs. They found multiple equilibria (a common feature in games of this type), and concluded that while limit pricing can exist in equilibrium under these assumptions, "The probability that entry actually occurs . . . can be lower, the same, or even higher than in a regime of complete information . . . ."[5]

The Milgrom and Roberts model has been extended and modified in many ways, serving as the basis of a great many of the very large number of articles on entry deeterrence published in the past two decades. Bagwell and Ramey (1991) extended the basic model to a 2-firm oligopoly and found what they termed "robust, no-distortion equilibria" in which incumbents played exactly as if they possessed complete information or there were no threat of entry, with entry occurring exactly when it would be profitable (the latter as under the Milgrom-Roberts assumptions). They concluded that the basic conclusions of Milgrom and Roberts were correct, but that the latter were incorrect in

[5] Milgrom, Paul, and J. Roberts (1982), p. 443.

associating the process by which these outcomes emerged with distortions in pre-entry prices.

Bagwell and Ramey also drew a distinction which does not seem accurate between models based on commitment and those based on signaling. Many if not most game theoretic models involve signaling of some sort. The questions has to do with both the nature and content of the signal, that is, whether the signal being read is price or quantity or something else, and whether it is taken as an indicator of costs, of output capacity, of financial strength, or of something else. The different parameters that might be assumed to be the content of the signal are Dixit's "initial conditions;" the parameter observed and the meaning assumed to be imputed to it are his "rules of the game."

Most theoretical studies since Milgrom and Roberts have employed some sort of sequential game using modern techniques. Most have found at least qualified support for the existence of equilibria in which limit pricing occurs, which may or may not deter entry depending on the assumptions and the circumstances. Sorenson (2004) extends the basic results of Milgrom and Roberts to multiple-period games. Chowdhury (2002) finds multiple Nash Equilibria when one firm has an absolute cost advantage, but only two perfect equilibria, which converge to the limit pricing outcome when the range of allowable prices becomes small. But with symmetric costs he finds only two equilibria which converge to Baumol's contestable-markets outcome, which has otherwise found little support.[6] Scheffman and Spiller (1992) consider a variety of game types under a

[6] Baumol, 1982, and references therein. As a partial justification for the "contestable markets" approach, Baumol observes (p. 2) that, ". . . in the standard analysis (including that of many of our fellow rebels), the properties of oligopoly models are heavily dependent on the assumed expectations and reaction patterns characterizing the firms that are involved. When there is a change in the assumed nature of these expectations or reactions, the implied behavior of the oligopolistic industry may change drastically."

wide variety of assumptions in markets for intermediate goods in which *buyers'* ability to make a "credible but costly commitment to switch suppliers."[7] They find that such commitments, *along with sellers' sunk costs* (contrary to some other studies of commitment), may significantly limit sellers' market power. They find robust limit-pricing equilibria even with a finite horizon, "unlike many infinite-horizon games."[8]

While the search for this review focused on articles dealing specifically with limit pricing, the literature on the theory of entry deterrence is vast. Much less vast is the empirical literature in this area. Bergman and Rudholm (2003) studied the effects of actual and potential competition in the Swedish pharmaceutical industry and found that prices tended to decrease for certain drugs as the date of patent expiration neared, which could be construed as a form of limit pricing. Significantly, a law in Sweden requires drug manufacturers, if the price of a drug if it has ever been reduced, to present hard evidence of increased costs before it can ever be raised again. This helps to make the Bain-Sylos assumption a credible threat.

Siegfried and Evans (1994) surveyed over 70 empirical studies of entry-deterrence behavior in at least 11 countries. They found little support for economies of scale, or for strategic use of excess capacity or limit pricing, as entry barriers. Neither was high research and development cost an impediment. In fact, innovation may attract entry by firms seeking a protected niche. The effects of product differentiation and advertising were found to be ambiguous. High absolute cost may be a barrier, depending on the specific source, as may multiplant operations. Highly concentrated industries usually experience less entry, but the direction of causation is unclear. The authors find

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[7] Scheffman and Spiller, 1992, p. 418.
[8] Scheffman and Spiller, 1992, p. 427.

these results surprising in light of the many game-theoretic models in which the importance of both structural barriers and limit pricing depend on the specific form of oligopoly behavior. They found few if any empirical studies which capture this dynamic interaction between market structure and firm conduct. They also found few if any studies on textbook structural barriers such as patents and control of essential resources. They speculate that the weak observed effects of structural barriers may be the result of compensating behavior not captured in the empirical studies.

Given all of the above, especially the sensitivity to the assumptions of the model of both the existence of limit pricing and its ability to deter entry, and especially Dixit's observation regarding the potential importance of the financial positions of the firms in the temporary disequilibrium state that must ensue once a new firm has entered an industry, it would appear that the Bain-Sylos postulate is not as lacking in credibility as some writers have assumed. Indeed, even Laffont (1991), who dismisses the credibility of the postulate based on Friedman (1979), notes that in extensions of the basic Milgrom-Roberts model, ". . . the interpretation of that signal is highly sensitive to model specification and only extremely precise knowledge of the industry under study can lead to correct interpretation, a consideration which applies to all models with incomplete information."[9]

Dixit's remark quoted earlier regarding the importance of the respective financial positions of the firms involved seems especially pertinent. From local craft markets to global markets for goods such as automobiles, a financially strong producer is always cognizant of the potential for using that financial strength to prevent or eliminate

[9] Laffont, 1991, p. pp. 167-168.

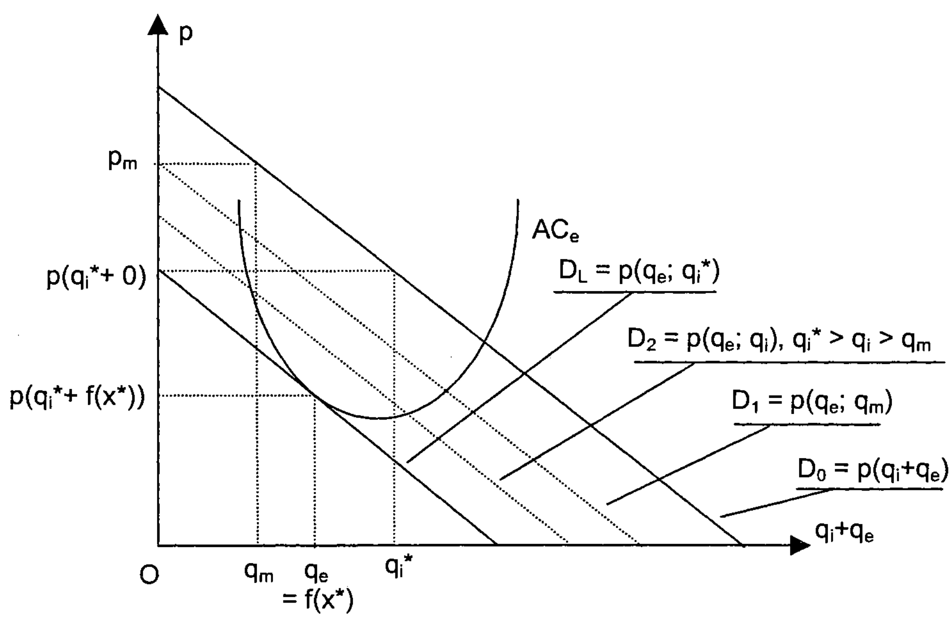
competitors lacking such resources by pricing below cost for long enough to drive them out of the market, even if the potential competitors are not. The perils facing any startup business, in any industry at any scale, are sufficiently daunting that a non-naïve entrant would be foolish not to take seriously the threat of an immediate price war to decide who is able to sell below cost for the longer time. Therefore, let us proceed to examine the comparative statics of a simple limit-pricing model.

The model is a static, one-period model, the limitations of which are acknowledged. Such a model, however, is nonetheless the basis for whatever may occur in subsequent periods of any strategic game, although many other strategic considerations obviously affect even the first period, Dixit's "initial conditions."

Let an incumbent monopolist facing demand $D = p(q)$ produce output q_i. Let a potential entrant threaten to produce output q_e, so that $q = q_i + q_e$, with the entrant facing residual demand $D_e = p(q - q_i) = p(q_e)$. In order to parameterize the model, let the entrant face a fixed cost of entry F, and let $q_e = f(x)$, with x representing an input with unit cost w. Then the entrant will make profit $\pi_e = p(q_i + f(x))f(x) - wx - F$.

The incumbent will attempt to optimize his output q_i in order to maximize his own profit. In the absence of a threat of entry (and assuming demand for the output is elastic at the point of optimality), the incumbent will maximize his profit by producing monopoly output $q_m < q_c$, where q_c is the output that would be produced under perfect competition. The resulting increase in price above the competitive level p_c provides the incentive for a competitor to try to enter the market and share the monopoly profits. In the simplest limit-pricing models the incumbent increases output above the monopoly level, reducing the price if possible to a point at which no feasible output q_e results in positive profit for the entrant given the fixed cost of entry.

In reality, the goal of the incumbent is to produce the *minimum* output consistent with non-positive profit for the entrant. But because lower levels of q_i (closer to the monopoly level) can result in positive profit for the entrant, to implement this condition mathematically requires an inequality constraint. A different approach, however, allows the use of an equality constraint, which is simpler and consistent with the PD formulation. Assume that the potential entrant faces a typical U-shaped average cost curve, and demand as stated above.



Here, the market demand is $D_0 = p(q_i + q_e)$, and the residual demand facing the potential entrant when the incumbent produces optimum output q_i^* is $D_L = p(q_e; q_i^*)$. If the incumbent were to producing the monopoly output q_m, the residual demand facing the entrant would be $D_1 = p(q_e; q_m)$, and at level of output intermediate between q_m and q_i^* the entrant would face residual demand $D_2 = p(q_e; q_i)$. Observe that the residual demand curve for the potential entrant shifts to the left as the incumbent *increases* output from zero to q_m and ultimately to q_i*.

If the incumbent, facing the entire market demand curve D_0, produces output q_m, the resulting price is p_m. If the incumbent successfully prevents entry ($q_e = 0$) by producing output q_i*, the resulting price is $p(q_i*)$. If the "Bain-Sylos postulate" that the incumbent will maintain the pre-entry (limit-pricing) level is credible, the entrant will indeed produce no output because his average cost will exceed the price at every level of output except that at which $p = AC$, yielding no incentive to enter the market.

If the incumbent produces a level of output between q_m and q_i*, however, there will be two levels of output for the potential entrant for which his average cost $AC = p$. Between those two levels, $p > AC$ for the potential entrant, who then is likely to enter and reap positive profit.

The incumbent, wishing to restrict output to reap monopoly profits, would actually prefer to minimize output subject to $p \leq AC_e$. The inequality constraint is necessary because at any level of q_i less than q_i* (including q_m), $p > AC_e$, in which case the potential entrant can make positive profit and thus has an incentive actually to enter the market. But if the condition for monopoly profit exists in the first place ($\varepsilon < -1$), $p = AC_e$ for at least one value of q_e for all q_i such that $q_i* \geq q_i > q_m$. Therefore it is possible to consider q_i* as the *maximum* level of q_i for which $p(q_i) = AC_e$ with $q_e = 0$ ($p(q_e; q_i) < p(0; q_i) \forall\ q_i, q_e > 0$). Since $q_e = f(x)$, it is straightforward to show that the condition $p(q_i + q_e) = AC_e = (wx + F)/q_e$ is equivalent to the condition $\pi_e = p(q_i + f(x))f(x) - wx - F = 0$, and the first-order conditions assure that the solution is restricted to the point of tangency in the diagram above.

The incumbent's optimum output $q_i^* = q_i^*(w, F)$, then, represents the industry supply as long as entry is prevented. Notice that q is not a function of p, or even of the elasticity of the demand function as is the case for a monopolist not facing a threat of entry. In this model the quantity supplied is fixed, determined only by factor cost and the fixed cost of entry facing a potential competitor. The monopolist's whole attention has been captured by the necessity of dealing with the potential competitor, and he considers no other factors.

The problem for the incumbent, then, is to maximize output q_i, subject to the constraint, $p(q_i + f(x))f(x) - wx - F = 0$. Let the Lagrangian be

$$\mathcal{L} = q_i + \lambda[p(q_i + f(x))f(x) - wx - F].$$

Then at the point of optimality,

$$\mathcal{L}_q = 1 + \lambda f(x)(\partial p/\partial q_i) = 0$$

$$\mathcal{L}_x = \lambda[p(q_i + f(x))f'(x) + f(x)(\partial p/\partial q_i)f'(x) - w] = 0$$

$$= \lambda f'(x)[p(q_i + f(x)) + f(x)(\partial p/\partial q_i)] - \lambda w = 0$$

From the first condition,

$$\lambda^* = -1/[f(x)(\partial p/\partial q_i)]$$

Since $f(x) > 0$ and $\partial p/\partial q_i < 0$, $\lambda^* > 0$. The second condition yields an expression for λ^* of indeterminate sign. Although this expression for the multiplier is more complex than those in most simple models, it can still be interpreted as the shadow price of F in a sense that will be explained below.

It is then straightforward to show that

1. The indirect objective function is decreasing in both parameters.
2. The indirect objective function is quasi-convex in both parameters.

3. The incumbent's profit is increasing in F.
4. An expression can be derived for the maximum value of F needed to make entry unprofitable at any level of output.
5. The response of the incumbent's profit to a change in factor price w is in general indeterminate unless the cost function $C(w, q_i)$ is known.

The P-D Lagrangian involves the same objective function with the same constraint, minus the value function of the original objective function,

$$\mathcal{L}^* = q_i - q_i^* + \lambda[p(q_i + f(x))f(x) - wx - F].$$

Then

$$\mathcal{L}^*_w = -\partial q_i^*/\partial w - \lambda x = 0 \Rightarrow \partial q_i^*/\partial w \equiv -\lambda^* x^* < 0$$

$$\mathcal{L}^*_F = -\partial q_i^*/\partial F - \lambda = 0 \Rightarrow \partial q_i^*/\partial F \equiv -\lambda^* < 0$$

Thus the indirect objective function is indeed decreasing in both parameters, and the quasi-convexity of q_i in w and F follows from the linearity of the constraint in these parameters. These are pseudo-envelope relations, derivatives of the indirect objective function with respect to parameters which occur only in the constraint. Indeed, not only are there no parameters in the objective function, but the objective function is itself one of the choice variables, somewhat unusual but entirely acceptable.

These relations also confirm the nature of the multiplier λ^* as the "shadow price of F." In the derivation from the primal Lagrangian, the dimensions of λ^* are the negative reciprocal of the quantity, "units of output times price divided by units of output," or $-1/p$. But the dimensions of p are dollars per unit of output, so those of λ^* are the negative of units of output per dollar. In the first pseudo-envelope relation, the dimensions of λ^* are units of *incumbent's* output divided by the product of dollars per unit of input with units of *entrant's* input (a cost to the entrant). In the second relation above, they are simply the negative of units of incumbent's output per dollar of fixed entry cost. That is, λ^* represents the value to the incumbent of a \$1 increase in the fixed cost of entry, in terms of the number of units that the incumbent can *reduce* output *below* the limit-pricing level, toward the monopoly level.

The second derivatives of the Lagrangian w/r/t the parameters, where $g(w, F)$ is the constraint function $p(q_i + f(x))f(x) - wx - F$ (which is equal to zero at the optimum), are

$$\mathcal{L}^*_{ww} = \partial(-q_i{}^*{}_w - \lambda x)/\partial w = -\partial^2 q_i{}^*/\partial w^2 = \lambda^* x^*{}_w + \lambda^*{}_w x^*$$

$$\mathcal{L}^*_{wF} = \partial(-q_i{}^*{}_w - \lambda x)/\partial F = -\partial^2 q_i{}^*/\partial w \partial F = \lambda^* x^*{}_F + \lambda^*{}_F x^*$$

$$\mathcal{L}^*_{Fw} = \partial(-q_i{}^*{}_F - \lambda)/\partial w = -\partial^2 q_i{}^*/\partial F \partial w = \lambda^*{}_w$$

$$\mathcal{L}^*_{FF} = \partial(-q_i{}^*{}_F - \lambda)/\partial F = -\partial^2 q_i{}^*/\partial F^2 = \lambda^*{}_F$$

$$g_w = -x^*$$

$$g_F = -1,$$

leading to the P-D Hessian:

$$\bar{\boldsymbol{H}} = \begin{bmatrix} \lambda^* x^*{}_w + \lambda^*{}_w x^* & \lambda^* x^*{}_F + \lambda^*{}_F x^* & -x^* \\ \lambda^*{}_w & \lambda^*{}_F & -1 \\ -x^* & -1 & 0 \end{bmatrix}$$

As always, $\bar{\boldsymbol{H}}$ must be negative semi-definite at the optimum (maximum value of $q_i = q_i{}^*$ subject to constraint). The only border-preserving principal minor of the determinant of this matrix is the determinant of the entire matrix, which is of order 2 and therefore non-negative:

$$-x^*[-(\lambda^* x^*{}_F + \lambda^*{}_F x^*) + \lambda^*{}_F x^*] + [-(\lambda^* x^*{}_w + \lambda^*{}_w x^*) + \lambda^*{}_w x^*] \geq 0.$$

The second and third terms in each bracket cancel, leaving only

$$\lambda^* x^* x^*_F - \lambda^* x^*_w \geq 0$$

$$x^* x^*_F \geq x^*_w,$$

since $\lambda^* > 0$.

This is an important result in the sense that it makes specific predictions about the ways that the potential entrant's optimum use of input would change when either of the cost parameters w and F change. If the production function is assumed to be monotonic, changes in output are of the same sign as changes in input. The signs of these derivatives are of particular interest in the analysis of the dual problem, and in the generalized Slutzky relation.

Unfortunately, the sign of x^*_w appears in general to be indeterminate, strongly dependent on the shapes of the demand and AC curves. The shape of the AC curve in turn depends on that of the production function, which determines the shape of $MC(y)$, given $C = wx - F$. It can be shown that $x^*_F > 0$ if the demand function is linear, but in general the sign of x^*_F is also indeterminate. Since $x^* > 0$, if $x^*_F < 0$ then $x^*_w < 0$ as well, but this appears to be the only general statement that can be made about these derivatives. It does appear from graphical analysis that the changes in the potential entrant's output in response to changes in w or F are relatively small.

If $x^* = 0$, then $x^*_w = 0$, but the fact that entry has been thwarted does not in general result in $x^* = 0$: Instead, x^* is the non-zero level of output at which the entrant would earn exactly zero profit when the incumbent is practicing limit pricing, as is clear from the diagram.

It is also the case that the principal diagonal elements of the bordered Hessian must be non-positive:

$$\lambda^* x^*_w + \lambda^*_w x^* \leq 0$$

$$\lambda^*_F \leq 0$$

From the first of these we get

$$\lambda^*_w \leq -\lambda^* x^*_w / x^*$$

Since $\lambda^* > 0$ and $x^* > 0$, this relation places a condition on the sign and/or value of λ^*_w, depending on the sign of x^*_w.

Also, from the symmetry of the cross partials,

$$\lambda^*_w = \lambda^* x^*_F + \lambda^*_F x^*$$

$$(\lambda^*_w - \lambda^* x^*_F)/x^* = \lambda^*_F \leq 0$$

And since $x^* > 0$,

$$\lambda^*_w - \lambda^* x^*_F \leq 0$$

$$x^*_F \geq \lambda^*_w / \lambda^*.$$

Since λ^*_w has an upper bound as shown above, this places a lower bound on x^*_F. But since the sign of λ^*_w is unknown and x^*_F is known to be positive at least under some conditions, this relation is of little use.

It can be shown that the response of the incumbent's profit to a change in fixed cost of entry $\pi^*_F > 0$ as follows. As long as entry is prevented so that q_i is the entire output and $q_i = D(p)$,

$$\pi^* = p(q_i^*)q_i^* - C(w, q_i^*), \text{ where } q_i^* = q_i^*(w, F).$$

Then,

$$\pi^*_F = q_i^*(dp/dq)(\partial q_i^*/\partial F) + p(\partial q_i^*/\partial F) - (\partial C/\partial q_i)(\partial q_i^*/\partial F)$$

$$= [q_i^*(dp/dq) + p - (\partial C/\partial q_i)]\cdot(\partial q_i^*/\partial F)$$

Since $\partial q_i^*/\partial F = -\lambda^* < 0$ (from the FOC of the P-D Lagrangian), then

$$\pi^*_F = [q_i^*(dp/dq) + p - (\partial C/\partial q_i)]\cdot(\partial q_i^*/\partial F) > 0, \text{ if and only if}$$

$$[q_i^*(dp/dq) + p - (\partial C/\partial q_i)] < 0$$

$$p < -\, q_i^*(dp/dq) + (\partial C/\partial q_i)$$

$$1 < -(q_i^*/p)(dp/dq) + (1/p)(\partial C/\partial q_i)$$

$$1 < -1/\varepsilon + (1/p)(\partial C/\partial q_i)$$

$$1 - (1/p)(\partial C/\partial q_i) < -1/\varepsilon$$

$$(1/p)(\partial C/\partial q_i) > 1 + 1/\varepsilon,$$

where ε is the price elasticity of demand for the output, and $p = p(q_i^*(w, F))$.

For monopoly profits to be possible, $\varepsilon < -1$, so

$$-1 < 1/\varepsilon < 0 \text{ which means that}$$

$$0 < 1 + 1/\varepsilon.$$

Substituting into the above, the condition for $\pi^*_F > 0$ then becomes

$$(1/p)(\partial C/\partial q_i) > 0.$$

Therefore it is always the case that $\pi^*_F > 0$ as long as $\partial C/\partial q_i = MC(q_i) > 0$.

Substituting MC for $\partial C/\partial q_i$ in the previous inequality and rearranging yields

$$(p - MC)/p < -1/\varepsilon.$$

The left side of this inequality is the price-cost margin, and also the Lerner Index of market power. The profit-maximizing condition for a monopoly is that the price-cost margin be *equal* to $-1/\varepsilon$. Thus the inequality means that if $\pi^*_F > 0$ the monopolist is *not* maximizing profit but is pricing below the profit-maximizing level, i.e., limit pricing.

The previous inequality can also be rearranged to read

$$p < (\partial C/\partial q_i)[1/(1 + 1/\varepsilon)].$$

This means that if the incumbent is receiving any monopoly profit,

$$MC < p < MC\cdot[1/(1 + 1/\varepsilon)],$$

where the multiplier in brackets is greater than one, approaching unity as demand approaches unitary elasticity. If demand is only slightly elastic at the optimum output $q_i^*(w, F)$, then at that level of output $p \sim MC$ (monopoly profit becomes negligible).

Determining the level of F which will make entry unprofitable at any level is tantamount to asking how high F must be to cause p and q_i to have the same values as when fending off entry is not a consideration. In other words, the value of F that absolutely prevents entry is equal to the maximum profit (not considering F) the potential entrant could obtain by any choice of the level of input x when the incumbent is producing the monopoly output q_m which is exogenous and therefore parametric to the present model.

The potential entrant's profit is

$$\pi_e = p(q_m + f(x))\cdot f(x) - wx$$

$$\partial\pi_e/\partial x = p(q_m + f(x))\cdot f'(x) + f(x)(\partial p/\partial x) - w$$

$$= p(q_m + f(x))\cdot f'(x) + f(x)(dp/dq)f'(x) - w$$

$$= [p(q_m + f(x)) + f(x)(dp/dq)]f'(x) - w.$$

Since $f'(x) > 0$, and $dp/dq < 0$, in order for $\partial\pi_e/\partial x$ to equal zero (interior maximum) it must be the case that

$$p(q^*(w) + f(x)) + f(x)(dp/dq) > 0$$

$$dp/dq > -[p(q_m + f(x))]/f(x)$$

$$-\, dp/dq < [p(q_m + f(x))]/f(x)$$

$$-\, dq/dp > f(x)/[p(q_m)+ f(x))]$$

$$-(p/q)(dq/dp) > pf(x)/q[p(q_m + f(x))]$$

$$-\varepsilon > f(x)/q_m$$

$$-\varepsilon > f(x)/(q_m + f(x)),$$

which is true for all $\varepsilon < -1$.

Therefore, the potential entrant's profit is maximized for the x^* that solves

$$\partial\pi_e/\partial x = p(q_m + f(x))\cdot f'(x) + f(x)(\partial p/\partial x) - w = 0$$

$$p(q_m + f(x^*))\cdot f'(x^*) + f(x^*)(\partial p/\partial x) = -\, w,$$

and the value of F that absolutely prevents entry is

$$F = \pi_e{}^* = p(q_m + f(x^*))\cdot f(x^*) - wx^*.$$

The fact that the value of F is even a consideration for the monopolist is evidence that he is not practicing what some writers call "naïve profit maximization," that is, setting output at the level at which $MR = MC$. In fact, the preceding analysis shows that the monopolist faced with a threat of entry has an incentive to take action (at a certain cost to himself) to increase the value of F, as well as to adjust his output in response to a change in F, if possible to the point at which a potential competitor cannot earn profit above the rental rate of capital. Salop (1979) reports a personal communication in which Stiglitz observed that "all deterrence instruments create intertemporal relationships in the profit function."[10] Salop goes on to interpret this as meaning that all deterrence instruments, including limit pricing, function as capital, thus generalizing the concepts of

[10] Salop, 1979, p. 337.

specific kinds of investment (or "binding commitment") as entry deterrents discussed earlier.

Regarding the response of the incumbent's profit to a change in factor price w, consider the monopolist's profit function,

$$\pi^* = p(q(i^*))q - wi^*,$$

where $i^* = i^*(w)$, the level of input that maximizes the incumbent's monopoly profits in the absence of a threat of entry. If this function is differentiated with respect to w and the first-order condition with respect to i substituted, the result is the same as for a similar competitive firm, $\pi^*_w = -i^*(w) < 0$. But the incumbent's input i is not a variable in the limit-pricing model. As before, we must differentiate the function

$$\pi^* = p(q_i^*(w, F))\cdot q_i^* - C(w, q_i^*)$$

$$\pi^*_w = q_i^*(\partial p/\partial w) + p(\partial q_i^*/\partial w) - [C_w + (\partial C/\partial q_i)(\partial q_i^*/\partial w)]$$

$$= q_i^*(dp/dq)(\partial q_i^*/\partial w) + p(\partial q_i^*/\partial w) - C_w - (\partial C/\partial q_i)(\partial q_i^*/\partial w)$$

$$= [q_i^*(dp/dq) + p - (\partial C/\partial q_i)](\partial q_i^*/\partial w) - C_w.$$

It was shown earlier (when evaluating π^*_F) that the expression in brackets is negative if and only if $(1/p)(\partial C/\partial q_i) > 1 + 1/\varepsilon$, which is the case if the incumbent is operating in the elastic part of the demand curve and is producing all the output. It was also shown from the FOC (envelope conditions) that $\partial q_i^*/\partial w = \mu^* x^* \leq 0$. If the incumbent is the sole producer, then the potential entrant's actual output $f(x) = 0$, so the corresponding input $x = 0$. But x^* is the non-zero input which would produce zero profit at the given levels of w and F. Thus the first term in the above expression is positive.

But it is also the case that $C_w > 0$, and so the sign of π^*_w depends on the relative magnitudes of these two quantities. Unless the cost function $C(w, q_i)$ is known, nothing further can be said of the sign of π^*_w. If, however, empirical observation were to discover that the incumbent's profits increased upon an increase in factor prices, this would be evidence that the incumbent was not maximizing the ordinary (monopoly) profit function, but some other, possibly our limit-pricing function.

In principle, we could analyze this situation in terms of the cost function $C(w, q_i^*) = wi^*$, where $i^*(w, F)$ is the input needed by the incumbent to produce output q_i^*. If this form is adopted, $C_w = i^* + w(\partial i^*/\partial w)$, and

$$\begin{aligned}\pi^*_w &= q_i^*(\partial p/\partial w) + p(\partial q_i^*/\partial w) - i^* + w(\partial i^*/\partial w)\\ &= q_i^*(dp/dq)(\partial q_i^*/\partial w) + p(\partial q_i^*/\partial w) - i^* + w(\partial i^*/\partial w)\\ &= [q_i^*(dp/dq) + p](\partial q_i^*/\partial w) - i^* + w(\partial i^*/\partial w).\end{aligned}$$

Inclusion of a fixed cost not dependent upon w would not alter this relation. It was shown earlier that $\partial q_i^*/\partial w < 0$, and if output q is monotonic in input i, then $\partial i^*/\partial w < 0$ as well. Therefore, the sign of π^*_w depends on the relative magnitudes of $p > 0$ and $q_i^*(dp/dq) < 0$. If $p > |q_i^*(dp/dq)|$, then the factor in brackets is positive and the first term is negative, in which case $\pi^*_w < 0$, as usual. But if $|q_i^*(dp/dq)| > p$, the factor in brackets is negative, the first term is positive, and the sign of π^*_w is again indeterminate. This would leave open again the empirical possibility that $\pi^*_w > 0$ might be observed, indicating that the incumbent was not maximizing the ordinary profit function and might be limit pricing.

In the expression above, $\partial i^*/\partial w$ is a conventional CS derivative for the primal limit-pricing objective function. But that function is not expressed in terms of i but of q, so to obtain an expression for $\partial i^*/\partial w$ would require re-framing the original problem in terms of i. Such an exercise might also make it possible to obtain more useful results from the P-D Hessian.

There does exist a "Roy-like" relation, the ratio of the two pseudo-envelope relations,

$$(\partial q_i^*/\partial w)/(\partial q_i^*/\partial F) = (-\lambda^* x^*)/(-\lambda^*) = x^*,$$

precisely analogous to Roy's Identity in the Utility-maximization model.

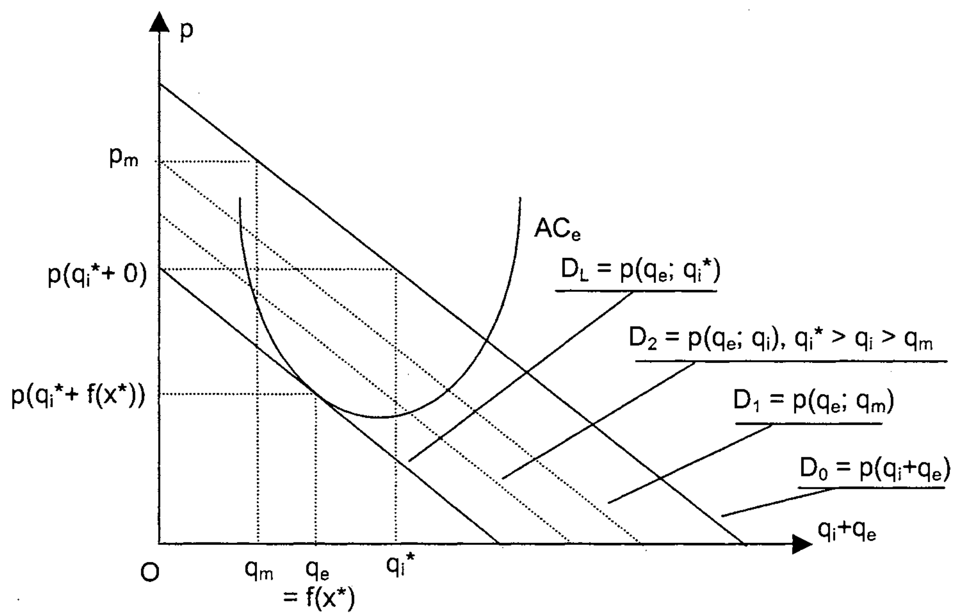
This relation provides no testable hypothesis, because this "optimum" level of entrant's input, the ratio of these two derivatives of the incumbent's output with respect to the parameters, is not chosen on any consideration of the producer who might actually use that input. It does, however, lend some insight into the workings of the model. This level of the entrant's input is in no sense an optimum to the entrant himself, but is the result of a choice by the incumbent to produce a level of output which makes it impossible for the entrant to earn positive profit. Given that his *maximum* potential profit is zero, the potential entrant either does not enter (and consumes *no* input, $x \neq x^*$), or else engages in some strategic behavior intended to call the incumbent's bluff and force a change in *his* behavior.

Analysis of the dual problem provides some further insight. The original (primal) problem is

"Choose q_i and x to maximize q_i, subject to $p(q_i + f(x))\cdot f(x) - w(x) - F = 0$."

We would therefore expect the dual problem to be

"Choose x to minimize $\pi_e = p(q_i + f(x))\cdot f(x) - w(x) - F$, subject to $q_i = \overline{q_i}$," where $\bar{q_i}$ is the optimum value of q_i found in the primal problem. But the considerations that forced us to maximize the primal objective function force us to maximize the dual function as well. To see why, return to the diagram.



As the incumbent increases production from zero (D_0, the entire market demand curve) first to q_m, the monopoly output, and eventually to q_i*, the limit-pricing output, the residual demand curve left to the potential entrant shifts steadily to the left (D_1, D_2, D_L). As in the primal problem, for any given levels of w and F, which fix the potential entrant's AC curve, and given $q_i = \bar{q_i} = q_i^*$ from the primal problem, if the entrant chooses to use $x^d = x^*$ and produce output $f(x^*)$, the price at the resulting level of industry output is just equal to his average cost and entry is unprofitable. For *any other* choice of input and hence of output, the entrant's average cost exceeds the resulting price and his potential profit is negative.

In other words, *the dual function has no minimum, but it does have a maximum.*

The function given above, however, is unduly complicated. It contains a constant, F, which does not affect the optimum value. All that is necessary is to maximize the difference between the potential entrant's revenue $p(q_i + f(x)) \cdot f(x)$ and his *variable* cost wx for any given level of q_i, recognizing that at $q_i^*(w, F)$ (and the level of F that generated it) the *maximum* value of this difference and hence of π_e is zero.

The dual problem, then, will be

"Choose x (and q_i) to maximize $I_e = p(q_i + f(x)) \cdot f(x) - w(x)$, subject to $q_i = \bar{q_i}$,"

where I_e can be considered as the income of the potential entrant net of *variable* costs.

Note that q_i is not really a choice variable in this case, and it would be possible to incorporate the constraint into the objective function. We will consider it separately, however, paralleling the analysis of the primal problem.

The Lagrangian for this problem, using the same format as for the primal problem, is

$$\mathcal{L}^d = p(q_i + f(x)) \cdot f(x) - w(x) + \mu(q_i - \bar{q_i})$$

The first-order conditions are

$$\mathcal{L}^d_x = f(x)(\partial p/\partial x) f'(x) + p(q_i + f(x) \cdot f'(x) - w$$

$$= f'(x)[f(x)(\partial p/\partial x) + p(q_i + f(x)] - w = 0$$

$$\mathcal{L}^d_x = (\partial p/\partial x) f(x) + \mu = 0 \Rightarrow \mu^d = -f(x)(\partial p/\partial x) > 0$$

Let $\phi(w, \bar{q_i})$ be the value function of the above. Then the Primal-Dual Lagrangian is

$$\Phi = p(q_i + f(x)) \cdot f(x) - w(x) - \phi(w, \bar{q_i}) + \mu(q_i - \bar{q_i})$$

The first-order conditions with respect to the parameters are

$$\Phi_w = -x - \phi_w = 0 \Rightarrow \phi_w \equiv -x^d(w, \bar{q_i}) < 0$$

$$\Phi_q = -\phi_q - \mu = 0 \Rightarrow \phi_q \equiv -\mu^d(w, \bar{q_i}) < 0$$

Interpretation: The optimum (maximum) value of I_e decreases when either w or $\bar{q_i}$ increases. The former is clear: An increase in w raises the potential entrant's AC across the board, reducing the difference between his revenue and his variable cost. An increase in $\bar{q_i}$ (above q_m and even above q_i*) does the same thing by shifting his residual demand curve to the left relative to his fixed AC curve.

The second derivatives of the Primal-Dual Lagrangian are

$$\Phi_{ww} = -\phi_{ww} = -\partial(-x^d)/\partial w = \partial x^d/\partial w$$

$$\Phi_{wq} = \partial(\mathrm{I}^d{}_w)/\partial\bar{q_i} = \partial(-\phi_w)/\partial\bar{q_i} = -\ \phi_{wq} = \partial x^d/\partial\bar{q_i}$$

$$\Phi_{qw} = \partial(\mathrm{I}^d{}_q)/\partial w = \partial(-\phi_q)/\partial w = -\ \phi_{qw} = \partial\mu^d/\partial w$$

$$\Phi_{qq} = -\phi_{qq} = -\partial(-\mu^d)/\partial\bar{q_i} = \partial\mu^d/\partial\bar{q_i}$$

The second-order necessary condition is that the Hessian matrix

$$\boldsymbol{H} = \begin{bmatrix} \partial x^d/\partial w & \partial x^d/\partial\bar{q_i} & 0 \\ \partial\mu^d/\partial w & \partial\mu^d/\partial\bar{q_i} & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

must be negative semi-definite. The determinant of this matrix has no BPPM other than itself, which is readily seen to be

$$(\partial x^d/\partial w)(\partial\mu^d/\partial\bar{q_i}) - (\partial x^d/\partial\bar{q_i})(\partial\mu^d/\partial w) \geq 0$$

Making use of the symmetry of the cross partials,

$$\partial x^d/\partial\bar{q_i} = \partial\mu^d/\partial w,$$

the inequality above can be rewritten as

$$(\partial x^d/\partial w)(\partial\mu^d/\partial\bar{q_i}) \geq (\partial x^d/\partial\bar{q_i})(\partial\mu^d/\partial w) = (\partial x^d/\partial\bar{q_i})^2 = (\partial\mu^d/\partial w)^2$$

Furthermore, the principal diagonal elements of $\boldsymbol{H}$ are non-positive:

$$\partial x^d/\partial w \leq 0;\ \partial \mu^d/\partial \bar{q}_i \leq 0.$$

The former is to be expected. The latter is difficult to interpret in light of $\mu^d = -f(x)(\partial p/\partial x)$. But note that the optimum value of the multiplier in this dual problem is the reciprocal of that in the primal problem, as it should be.

By Currier's Theorem 5-3B[11] there is a reciprocity relation involving the other two derivatives:

$$I^d{}_{xw}(\partial x^d/\partial \bar{q}_i) = I^d{}_{\mu q}(\partial \mu^d/\partial w)$$

But returning to the first-order conditions for the Primal-Dual Lagrangian, above, both of the derivatives of I^d are -1, so this is simply the symmetry condition above.

A generalized Slutsky relation can be obtained by setting the optimum value of x^d in the dual problem equal to that of x^* in the primal problem and differentiating with respect to w:

$$x^d(w, \bar{q}_i) \equiv x^*(w, \Phi(w, \bar{q}_i))$$

$$\partial x^d/\partial w = \partial x^*/\partial w + (\partial x^*/\partial \phi)(\partial \phi/\partial w)$$

$$= \partial x^*/\partial w + (\partial x^*/\partial \phi)(-x^d)$$

But at the optimum, $x^d = x^*$. Furthermore, as the optimum value of $p(q_i + f(x))\cdot f(x) - w(x)$ subject to the constraint, $\phi(w, \bar{q}_i)$ is equal to the value of F that resulted in x^*, because F is the amount by which $\phi(w, \bar{q}_i)$ is reduced to yield $\pi_e = 0$ at the optimum. The Slutsky relation thus becomes

[11] Currier, p. 87.

$$\partial x^d/\partial w = \partial x^*/\partial w - x^*(\partial x^*/\partial F), \text{ or}$$

$$\partial x^*/\partial w = \partial x^d/\partial w + x^*(\partial x^*/\partial F).$$

This is identical to the ordinary Slutsky equation in the utility-maximization model except for the sign of the compensating term. In the utility-maximization model an increase in M leads to an increase in x^*, meaning that the slope of the compensated demand curve is always less negative than that of the uncompensated demand. In this limit-pricing model, however, $\partial x^*/\partial F$ can be either positive or negative, which means that the slope of the "compensated factor demand," in effect adjusting the fixed cost of entry to return the entrant's factor demand to the value it would have if the incumbent's output q_i were a choice variable, can be greater or less than the latter value.

Since $x^*(\partial x^*/\partial F) \geq \partial x^*/\partial w$ (p. 67), and $\partial x^d/\partial w \leq 0$ (p. 78), the Slutsky relation is consistent, but since the sign of $\partial x^*/\partial w$ is unknown, little more can be said.

CHAPTER V

CONCLUSIONS

The Primal-Dual analysis of the utility-maximization model accomplishes three things:

- It confirms the validity of the method by generating all results of conventional analysis (Roy's identity and the Slutsky equation) not dependent upon assumptions regarding the signs of the derivatives of utility.
- It confirms that despite the important role of the arrangement of the parameters in the objective function and the constraint function (which itself is evident from PD theory but relatively obscure in conventional analysis[1]), it is the particular form of those functions that determines the nature of the CS relations.[2] This is not especially surprising, but one could be led from Silberberg's discussion to expect that the arrangement or distribution of the parameters plays a larger role.

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[1] See Silberberg, 1990, 210-216.

[2] A protracted effort was made to analyze a model of decision making for environmental regulation using PD methodology. Like the utility-maximization model, that model has no parameters in the objective function, and three in the constraint, including one representing a level of the constraint. It was determined that all of the BPPM of the Hessian determinant for that model are equal to zero, which means that while the second-order necessary condition for a maximum (negative *semi*-definiteness of the Hessian matrix) is met, the second-order sufficient condition (negative definiteness) is not. Roughly speaking, the constrained objective function is "flat" in the vicinity of any optimum that may exist, and no conditions emerge from the PD analysis except the FOC and the symmetry of the second partials. This is in strong contrast to the richness of the results for the utility-maximization model, which has a constraint of much simpler form.

- It provides some results which, if not entirely new, are at least not evident in any of the standard textbook analyses of the model. In particular, the relation involving the cross-price effect and the own-price effects (above, p. 24) is important, and the symmetry relations from which it is derived, as well as the other relations which follow therefrom, are important in principle despite their dependence on the value of the Lagrange multiplier (the Marginal Utility of Money). The similar relations in the dual problem (expenditure minimization), $\mu_{p_i}{*} = \partial\mu{*}/\partial p_i = \partial h_i{*}/\partial\bar{U}$, and the relation between the *compensated* own-price and cross-price effects (p. 28, quite different from that for the uncompensated effects) are also of at least theoretical importance. The version of Roy's identity for the case with prices in the utility function appears to be new as well, although it can be derived less directly using conventional methods.

These results demonstrate the ability of the PD method to bring forth all of the relations which follow from the maximization hypothesis, including those which may be overlooked by conventional analysis.

The banking model probably should also be viewed more as an exercise than as a contribution to banking theory. One common criterion for judging the usefulness of a model is whether it captures significant features of the phenomenon modeled. Certain features of the nature of banking are clearly inherent in the model while others are not. On the other hand, this is true for most elementary textbook models, and perhaps more so for the banking industry than for some others. The model predicts that the profit-maximizing value of loans increases and that of deposits decreases when the rates on those variables increase, in agreement with intuition at least. Similarly, the result corresponding to Roy's Identity says that increases in the bank's net worth and in the reserve ratio have opposite effects on its bottom line.

The other Roy-like relations (p. 37) provide testable hypotheses which accord with intuition if the Lagrange multiplier is positive, which is "almost certainly" the case.[3] The symmetry relations, 6 in number, make firm predictions about the behavior of the quantities of loans and deposits in response to changes in the various parameters, but all but one involve *derivatives* of the multiplier with respect to the parameters. While PD analysis has nothing to say about the signs of these CS derivatives, they can be shown by conventional analysis to depend on the shape of the cost function, in particular upon its second derivatives with respect to loans and deposits. If these derivatives are determined empirically or by assumption, all 6 of these relations become testable hypotheses.

The primal problem also generates two other testable hypotheses. One states that deposits must decrease in response to an increase in either net worth or reserve ratio or both, which is not trivial since conventional analysis cannot determine the sign of $\partial D^*/\partial q$. The other states that the products of the magnitudes of the "own-rate effects" (the response of loans to an increase in the rate on loans, and the same for deposits) is greater than or equal to the product of the magnitudes of the "cross-rate effects." Both of these relations involve only observable quantities and so are definitely testable.

The generalized Slutsky relations are very interesting. Assuming that the value of the response of the bank's costs to a change in either loans or deposits can be determined, they can be computed numerically. They would then make firm predictions about

[3] See discussion on p. 36, above.

"compensated" values of loans and deposits, the way that the bank would adjust the values of these variables if its net worth were adjusted to keep profits at the same level. This is of at least theoretical interest.

Finally, while it is impossible to determine the sign of $\partial D^*/\partial q$, PD analysis shows (and conventional analysis confirms) that $\partial D^c/\partial q \leq 0$. While compensated values of loans and deposits would be difficult to observe (less so, perhaps, than compensated demand), this prediction too is significant if the concept is considered meaningful at all.

Remarkably, it appears that the comparative-statics properties of this rudimentary limit pricing model have not previously been explored. While early writers (Bain, Modigliani, Sylos-Labini) were concerned with the broader questions of barriers to entry in general, and about the circumstances in which various possible strategic actions might be plausible, most writers after Dixit, and especially after Milgrom and Roberts, were concerned primarily with the strategic aspects and outcomes of the game itself.

Given the familiarity of the principle involved, the main results proved for this model are unsurprising. But the remaining limited results are almost certainly incomplete, and it is anticipated that further work, especially with regard to the dual problem and the Slutsky relation, will uncover additional testable and meaningful predictions. The entire area of limit pricing remains an open question, and the fact that more complex behavior is possible and almost certainly does occur does not rule out the possibility of simpler behavior of the very sort envisioned by Bain over 50 years ago.

Comparative statics appears to many upon first encountering it to be a complex and arcane procedure. A little experience with standard models soon dispels this impression and makes it appear simple, routine, cut-and-dried. The former impression may actually be more accurate when considering the more complex models encountered in modern applications. While many linear models with two or three choice variables and a similar number of parameters still lend important insights, modern research models commonly include uncertainty, multiple periods, and game-theoretic aspects. If such effects are difficult to model, the models they generate can be extremely daunting to analyze.

It is with these more complex models that Primal-Dual method provides significant advantages. In essence, it pre-calculates the general forms of the testable relations that are available from an optimization model, independently from any assumptions upon the character of the functions involved; and it provides an organized system for deriving their specific forms for a particular model. In contrast, the conventional method reinvents the wheel not just for each problem but for each combination of variable and parameter in a given problem, incurring the expense of much tedious, repetitive, and error-prone calculation. Careful use of PD method also assures that all the fundamental results will be found, and it separates results that depend on assumptions regarding the shape of the functions involved or their derivatives from those that are not thus dependent

The price paid for these advantages is that effective use of PD method, even in relatively simple problems, requires a thorough understanding of concepts of duality and of concavity/convexity, fluent mastery of the subtleties of multivariate calculus, and of course, practice. This is a considerably greater investment in the techniques of analytical mathematics than is required by conventional comparative-statics methods, which normally require nothing more than partial differentials and Cramer's Rule. Furthermore, despite the power inherent in the PD method, skill, insight, and perseverance are still necessary in order to derive results which are meaningful and useful.

The emphasis in economic education today is on statistical methods, themselves arcane and specialized, and on iterative techniques such as mathematical programming and numerical approximation which, like statistical analysis, are necessarily implemented by computer. Such tools are essential to address the increasing variety of questions which economists are called upon to answer. Programming methods, for example, were developed specifically to determine the values of the independent variables in real-world optimization problems too complex for practical solution by traditional methods of the calculus. Statistical methods, on the other hand, can be seen as complementary to comparative statics (as well as to comparative dynamics, an increasingly important field of investigation), in that it is the latter that is the source of the hypotheses which statistical studies are called upon to test. Despite the additional educational burden in the specialized mathematical methods involved, Primal-Dual method appears to be the appropriate tool, if not the only practical one, for generate testable hypotheses for the more complex models encountered today

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#2

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