THE UNIVERSITY OF OKLAHOMA

GRADUATE COLLEGE

PRESSURE TRANSIENT ANALYSIS OF WELLS WITH PARTIAL-FLUID-INJECTION IN AN ANISOTROPIC PETROLEUM RESERVOIR

A DISSERTATION

SUBMITTED TO THE GRADUATE FACULTY

in partial fulfillment of the requirements for the

degree of

DOCTOR OF PHILOSOPHY

By

NAGMEDDIN ABDALLA ARIFI

Norman, Oklahoma

1978

PRESSURE TRANSIENT ANALYSIS OF WELLS WITH PARTIAL-

FLUID-INJECTION IN AN ANISTROPIC

PETROLEUM RESERVOIR

APPROVED BY

D E Menzie

Henry Crichlow

A. W. McCray

Arthur Bernhardt

F M Townsend

DISSERTATION COMMITTEE

ACKNOWLEDGEMENTS

The author is entirely grateful to Dr. D. E. Menzie, Professor of Petroleum Engineering, for his careful review, helpful guidance, continuous encouragement, and for generously giving his knowledge and time throughout this work as chairman of the author's Graduate Committee.

The author wishes to express his sincere appreciation to Dr. H. B. Crichlow, Associate Professor and Chairman of the Department of Petroleum Engineering; Dr. A. W. McCray, Professor of Petroleum Engineering, for their review of this work, for their guidance and encouragements.

Particular thanks are also extended to Dr. A. F. Bernhart, Professor of Mathematics; Dr. M. Townsend, Professor of Chemical Engineering, for their cooperation and for serving as members of the author's Graduate Committee.

The author gratefully acknowledges The University of Alfateh, Tripoli, Libya for their financial support.

The author wishes to express his love to his wife, Naima, his daughter Na'hla and his son, Usama for their patience, understanding, and devotion.

The author is deeply indebted to his mother and father for their inspiration, encouragement, and the many sacrifices during the course of study.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF ILLUSTRATIONS

PRESSURE TRANSIENT ANALYSIS OF WELLS WITH PARTIAL-FLUID-INJECTION IN AN ANISOTROPIC PETROLEUM RESERVOIR

CHAPTER I

INTRODUCTION

Petroleum reservoirs are those rock formations or portions of rock formations within the earth wherein hydrocarbons have been accumulated locally. Three physical properties are inherent to any petroleum reservoir. First, it must possess porosity. Second, the reservoir must possess permeability. Third, the reservoir must be a fluid trap of some sort.

Because a reservoir has permeability, the space occupied by the reservoir fluids is an interconnected network. The fluids within this continuous pore space are therefore a common composite mixture and they will possess some common character. Another consequence of the continuity of the fluid containing pore space is that there will be a continuity of hydraulic conditions within the reservoir.

The recognition of such characteristics of petroleum reservoirs has led to an understanding that the fluid behavior in one portion of a reservoir cannot be considered separately from the fluid behavior in another part of the reservoir. A reduction of pressure at one point is felt in all portions of the reservoir. The speed of transmission of such effects is, of course, dependent upon the individual characteristics of the reservoir and reservoir fluids.

The interdependence between reservoir behavior and well behavior is quite broad. The type fluids injected into or produced from a well and the rate at which they will be injected or produced depend upon reservoir characteristics such as pressure, temperature, permeability properties, and the type of displacement mechanism in action, as well as, upon the number of wells which penetrate the reservoir and the manner in which thewells are completed. Conversely, the operation of wells can control the mechanism by which oil is produced from a given reservoir. Whether a reservoir is under control of a water drive or solution gas drive, for example, may depend upon the manner in which wells are completed and operated.

In considering well behavior one ordinarily thinks of a producing well. Present technology often requires for many purposes the use of injection wells. Engineering calculations for such wells may appear to require new concepts or different approaches. So far as the mechanism of reservoir fluid flow is considered, it is of no consequence whether fluid may be flowing into or from the well. Therefore, the engineering techniques which are applicable to producing wells ought also to be applicable to injection wells.

One of the commonly used measures of performance on a producing well is the productivity index. The corresponding measure on the injection well has been termed injectivity index. Either can be defined as the rate of liquid transfer through the well in barrels per day per pound pressure drop between the flowing bottom-hole pressure and the formation pressure. Another useful test on producing wells has been the bottom-hole pressure buildup curve. This is observed by shutting in production and recording as a function of time the pressure readings opposite the production formation. An analogous test for injection wells is the bottom-hole-pressure fall-off test. During flow, the injection well is a formation pressure source. When the injection stream is cut off, the pressure opposite the formation in the well tends to fall to some equilibrium value. This pressure history can be recorded in the same manner as in the case of pressure buildup. Information of this nature can be analyzed to yield the same type reservoir quantities as bottom-hole-pressure buildup data.

Concluding from the above analogies that the individual factors which apply for producing wells may differ from those which apply to injection wells, but one can nevertheless apply the same general engineering techniques to both situations.

It is of considerable interest and importance to be able to determine the characteristics of the reservoir in an area surrounding an injection well. These include the relative amount of damage or improvement existing in the formation immediately at the wellbore. One of the factors controlling injectivity of water, for example, in a water flooding recovery program is the resistance of the formation in the first few feet around the wellbore. A comparison of tests before and after workovers will indicate the degree of success accomplished by the well treatment. Similarily, determination of the effective permeability of the formation around the injection well in conjunction with skin effect, the injection rate, and the injection pressure will allow a better understanding of the well's performance.

Since pressure analysis is a well developed technique in the industry, it was desired to find methods that apply to injection wells which would utilize previous knowledge and data of this sort.

In using pressure buildup equations in the analysis of producing wells, we have dealt exclusively with the case of wells that completely penetrate an isotropic formation. Two-dimensional radial symmetry was achieved, thereby, and the problem was greatly simplified.

This ideal situation is seldom encountered in practice. Therefore, we must ask how different would the results be of such cases when wells penetrate an anisotropic reservoir and fluids are effectively injected through only a portion of the formation in question.

Thus, the purpose of this study is to utilize information obtained from injection wells to:

1. Investigate the effect of partial fluid injection on pressure fall-off curves in the presence of permeability anisotropy
2. Estimate the vertical and horizontal permeabilities.
3. Estimate the injectivity impairment due to partial injection and permeability anisotropy.

CHAPTER II

LITERATURE SURVEY

In many petroleum reservoirs, it is a common occurrence that a well is completed as partially penetrating, that is, only a portion of the zone is open to flow. It is often desirable and necessary to complete a well with restricted fluid entry. This may be done for many reasons; one would be to deliberately produce from only a portion of a formation when bottom waters underlie the oil zone and the difficulties of premature water breakthrough would ensue from high well penetrations. Another common reason is to prevent undesirable zones or channels from thieving injection water in water flooding programs.

Another important fact that is normally encountered in petroleum reservoirs is the existence of vertical permeability. Vertical flow is an important mechanism in many petroleum reservoirs. The knowledge of the average vertical permeability of a formation is sometimes necessary to anticipate properly the production or injection performance of a reservoir.

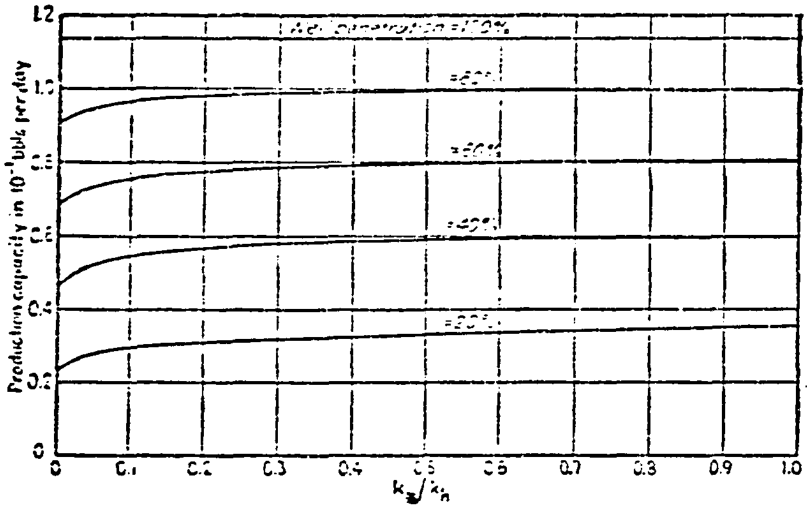
In this chapter a close look is taken at the information concerning the effects of restricted fluid entry and the existence of vertical permeability on pressure buildup analysis. Methods of vertical permeability determinations are also discussed.

Partial Penetration Effects

The problem of fluid flow to wells with partial penetration or restricted entry has received much attention in the last few years. A number of steady state[21, 24, 25] and unsteady state[2, 3, 4, 22, 30] solutions have appeared in the literature, and methods for analyzing pressure data have been proposed by several authors [2, 5, 26, 27].

Although many different techniques have been used for solving the partial penetration problem (namely, the point source solution[3, 5, 9, 10, 22], Fourier[24] and Laplace transforms [13, 30], Hankel and Laplace transforms[8], Green's functions[10], finite differences[2, 17]). The analytical expressions and the numerical results obtained for reservoir pressures by the different methods were identical.

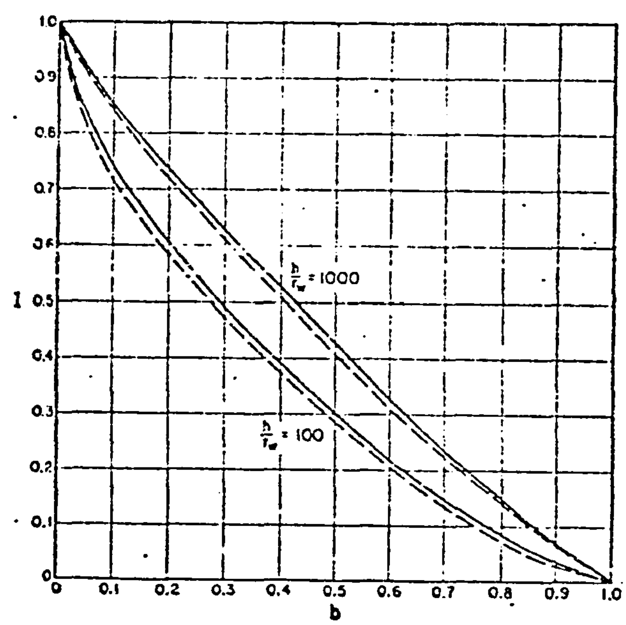
The partially-penetrating well problem was first studied by Muskat[21] for steady-state conditions. He calculated pressure distributions and productive capacities for an anisotropic system in which the horizontal and vertical permeabilities are not equal and concluded that the productivity depended slightly on the permeability ratio ($K_z/K_r > 0.1$), Fig. 1. He also showed that the steady state problem is identical between a well in an isotropic and an



Permeability Ratio

Fig. 1. Steady state homogeneous fluid production capacities of partially penetrating wells as a function of the permeability ratio K_z/K_r, reference 21.

Fractional Loss of Productivity



Penetration Ratio

Fig. 2. Comparison between fractional loss of productivity calculated by Brons et al[3] (solid lines) and Muskat[21] (dashed lines).

anisotropic medium, with the only change that the radius of the well and the external radius have to be multipled by (K_θ/K_r).

Nisle[22] studied the effect of partial penetration on pressure buildup curves. He discussed mainly the effect of partial penetration, assuming that the formation would be open from its top to some depth within the formation. He used the instantaneous point source solution to the diffusivity equation to solve the constant flux, isotropic problem. He constructed synthetic pressure buildup curves for various penetration ratios, and found that the extrapolated buildup curves show two semilog straight line portions for the case of partial penetration. Generally an early slope and a late slope may be recognized. The early slope is always steeper than the late slope. It follows that if one uses the early slope to calculate the transmissibility, one will find a lower transmissibility when using the late slope. This means that the early slope reflects the transmissibility near the wellbore only, where the late slope represents the transmissibility of the total formation. Nisle suggested that it was theoretically possible to calculate the penetration ratio from the ratio of the slope of the late portion to that of the slope of the early portion of the buildup curve.

Brons and Marting[4] supplemented the findings of Nisle and concluded that partial penetration gave rise to a pseudo damage, which reflected itself as a skin factor in pressure buildup calculations. Using the pseudo skin, they calculated the impairment of productivity of the production wells, which agreed well with the results of Muskat, Fig. 2.

Odeh[24] used a finite cosine transform to arrive at a solution for the steady state flow problem where the open interval was located anywhere within the producing formation. Odeh[25] also used the finite cosine transform to derive an analytic solution for pseudo steady-state flow of an oil well with limited entry and with an altered zone. He found that the effect of the limited entry and altered zone can be accounted for by a skin factor. The component of the skin factor due to the altered zone can be calculated by a simple equation. This component is mainly a function of the permeabilities of the formation and the altered zone, the total thickness of the productive sand, the length and location of the open interval, the radius of the well, the radius of the altered zone, and the ratio of the horizontal to the vertical permeability.

Hantush[13] solved the transient, anisotropic, partial-penetration problem by the successive use of LaPlace and Fourier transforms for the infinite reservoir case. In his work he assumed the wellbore radius was vanishing and the flow rate was uniform for each point along the vertical sand section open to flow. Seth[30] extended

Hantush's work to bounded reservoirs. Kazemi and Seth[17] investigated the combined effects of anisotropy, stratification with crossflow, and restricted flow entry on pressure transient analysis of flow and buildup tests. They solved the partial-penetration problem by using a numerical finite difference model. They reproduced Hantush's solution for the case of an infinite conductivity inner boundary condition. They found that the productivity impairment chart of Brons and Marting can be extended to include the effect of anisotropy. They also found that the conventional interference test analysis, which are based on total well penetration, are also applicable to reservoirs with restricted flow entry. Gringarten and Ramey[11] solved the infinite conductivity partial penetration problem analytically by use of Green's functions. The method used was to superimpose a number of discrete flux segments of different strengths such that the pressure at the center of each segment was the same. This approach was first suggested by Muskat, and again later by Burns in a discussion of the paper by Kazemi and Seth [17]. Clegg and Mills[8] applied Laplace and Hankel transforms to arrive at an approximate solution to the problem of the pressure distributions arising from the production of a compressible liquid in a partially penetrating well. They concluded that the effects of partial penetration, as reflected in the pressure observations at the wells, are such that the unperforated zone will appear to behave as a zone with permeability-thickness product $K_z h$, and this $K_z h$ characterizes the unperforated section of the formation.

Finally, despite the number of studies of the partial penetration problem, there is only one recent study[2] that has combined the effects of a finite-radius well, wellbore storage, wellbore damage, and the infinite conductivity source condition. Bilhartz and Ramey[2] used a two-dimensional, single-phase finite difference numerical model to generate dimensionless pressure vs. dimensionless time graphs for a closed, anisotropic (r-z) circular model. The inner boundary condition consisted of a partially-penetrating, infinite conductivity (hollow source) well influenced by a flux dependent infinitesimal skin effect, and wellbore storage. They concluded that the flux-dependent infinitesimal skin effect changes the zero skin curve by an additional amount equal to the dimensionless skin factor for all times. And wellbore storage causes two periods of flow prior to joining the zero storage curve, Fig. 3. The first period is the storage control period. For this period, the pressure response for partial and complete penetration is the same. After the initial period, a transition from wellbore storage control to the usual zero storage curve takes place.

Vertical Permeability Determination

The formation vertical permeability is often a dominant influence in reservoir recovery processes with

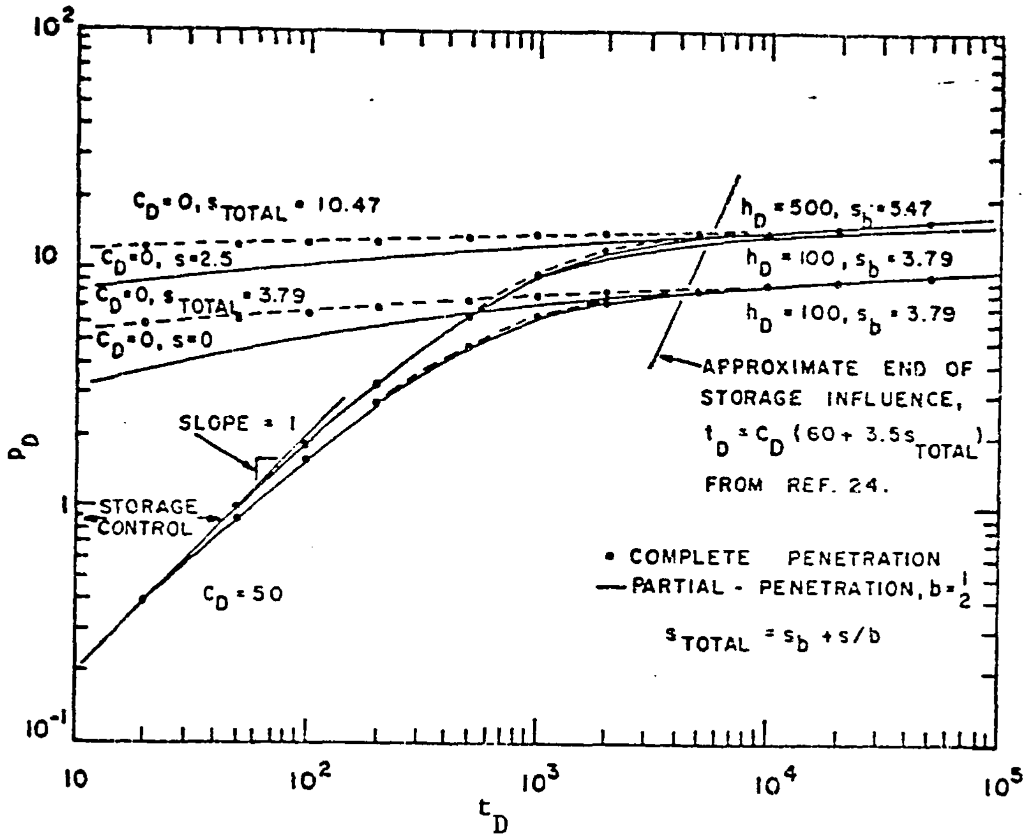


Fig. 3. Log-log type curve of P_D vs t_D (after Bilhartz et al[2]).

h_D = dimensionless wellbore radius, $\frac{h_w}{r_w} \sqrt{\frac{K_r}{K_z}}$

C_D = dimensionless wellbore constant,

s = skin factor,

P_D = dimensionless pressure,

t_D = dimensionless time.

vertical fluid flow such as water or gas coning, gravity drainage of high relief reservoirs, and displacement by water or gas in a heterogeneous formation. How reliably numerical reservoir simulators can predict the recovery performance of these processes depends upon how accurately the significant reservoir parameters are estimated. Core measurements of vertical permeability do not necessarily represent effective in-situ values. Cores may be representative of only localized pockets of high or low permeability, and total reliance on cores might cause one to overlook the presence of vertical fractures altogether.

Transient pressure techniques for estimating in-situ vertical permeability have been introduced by Burns[5] and by Prats [26]. Both techniques require injection or production at a constant rate from a short perforated interval and measurement of the pressure response at another perforated interval that is isolated from the first by a packer. The interpretation technique of Burns required a computer-generated type curve or a single-phase numerical reservoir simulator. This type-curve approach is applicable for an anisotropic, homogeneous, infinite reservoir model, and the numerical simulator with a regression analysis program is needed for finite or layered reservoir models. The technique presented by Prats did not require a computer program because the result of the analysis was presented on a single graph. The horizontal and vertical permeabilities could be estimated from the slope and the intercept of the pressure response and the appropriate value from the graph. The method of Prats was based on an infinite, anisotropic homogeneous reservoir model.

Both the Burns and Prats methods require two-point well testing. That is, they require injecting or producing at a constant rate through one set of perforations while measuring the pressure response opposite another set of perforations isolated from the first by a packer. Perhaps because of this elaborate procedure, two-point vertical permeability tests are seldom performed.

Raghavan and Clark[27] examined the applicability of the spherical flow equations to a well producing from a limited section of a thick anisotropic formation. They concluded that the vertical permeability may be obtained using the spherical flow approximation from the slope of the pressure drawdown and/or buildup graphs if the horizontal permeability is known. This calculation is unaffected by the magnitude of the Van Everdingen and Hurst skin effect. However, time checks are essential to insure that the proper straight line was chosen. And the horizontal permeability may be obtained using the spherical flow approximation if the skin effect is negligible, or if the skin effect can be determined by an independent study.

Finally, Bilhartz and Ramey's study suggests that vertical permeability can be evaluated from a drawdown test.

They emphasize, however, that the procedure depends upon early time drawdown data for small wellbore storage effects. Thus the method may be difficult or impossible to apply if wellbore storage is significant.

TRANSIENT PRESSURE ANALYSIS

CHAPTER III

Pressure testing of wells was generally limited to the determination of producing and static mean formation pressures. In this so-called "static" measurements, a pressure-measuring device was lowered to the bottom of a well which had been closed for a period of time. These static measurements sufficed to indicate the pressure in permeable, high productivity reservoirs. However, it was soon recognized that in most formations the static pressure measurements were very much functions of closed-in time. The lower the permeability, the longer the time required for the pressure in a well to equalize at the prevailing reservoir pressure. Thus, it was realized very early that the rapidity with which pressure buildup or falloff occurred when a well was closed-in was a reflection of the permeability of the reservoir rock around that well. This qualitative observation was an important step in developing an understanding of well pressure behavior. This understanding led to the other basic type of measurement, called transient pressure testing. In this type of measurement, the pressure variation with time is recorded after the flow rate of the well is changed. It is this type of measurement which is mainly our concern.

In this chapter a presentation of basic principles and theoretical equations used in the analysis of well tests will be made.

Basic to this discussion is understanding of the theory of fluid flow in porous media. Review of the fundamental concepts is contained within this chapter, largely in summary form.

The Differential Equation for Fluid Flow in Porous Media

Flow in porous media is a very complex phenomenon and cannot be described as explicitly as flow through pipes and conduits.

A mathematical description of fluid flow in a porous medium can be obtained by a combination of the following physical principles[18]:

(1) The law of conservation of mass;

(2) Darcy's law which states that the rate of flow of a homogenous fluid per unit cross sectional area at any point in a uniform porous medium is: $u_r = \frac{-K\rho g}{\mu} \frac{\partial \Phi}{\partial r}$, where u_r = velocity in positive r; K = permeability; Φ = potential; ρ = density; g = gravitational acceleration.

(3) Equation(s) of state.

$$\frac{\partial}{\partial x} \left(\rho \frac{K_x}{\mu} \frac{\partial \Phi}{\partial x}\right) + \frac{\partial}{\partial y} \left(\rho \frac{K_y}{\mu} \frac{\partial \Phi}{\partial y}\right) + \frac{\partial}{\partial z} \left(\rho \frac{K_z}{\mu} \frac{\partial \Phi}{\partial z}\right) = \phi \frac{\partial P}{\partial t} \qquad (3.1)$$

This is known as the diffusivity equation, and in this general form[7], describes the flow potential generated by the movement of a fluid through a porous medium subject only to the restrictions cited for the continuity equations and Darcy's law. However, in this general form, the equation is nearly impossible to solve for the potential. Various simplifying assumptions and boundary conditions are used depending upon the type of reservoir problem to be studied. Solution to all reservoir engineering problems which involve the flow of fluids in the reservoir are based on a solution to some form of the diffusivity equation.

The Ideal Reservoir Model

Several simplifying assumptions about the well and reservoir to be modeled must be made before development, analysis, and design techniques of pressure testing. These assumptions are needed to combine the above mentioned physical principles in order to obtain simple and useful solutions to the equations describing our situation.

The assumptions made in the development of pressure analysis theory are summarized as follows:

1. Radial flow into a well opened over entire thickness of formation;
2. Homogeneous and isotropic porous media;
3. Porosity and permeability constant (independent of pressure);

4. Fluid of small and constant compressibility;

5. Constant fluid viscosity;

6. Small pressure gradients;

7. Negligible gravity forces; and

8. Rock and fluid properties are independent of pressure.

The details of this work are only outlined here but can be found in reference 18. Emphasis will be only on the results of this analysis.

Now consider radial flow toward a well in a circular reservoir. If we combine the law of conservation of mass and Darcy's law for the isothermal flow of fluids of small and constant compressibility--a highly satisfactory model for single phase flow--we obtain a partial differential equation which simplifies to:

$$\frac{\partial^2 P}{\partial r^2} + \frac{1}{r}\frac{\partial P}{\partial r} = \frac{\phi \mu c}{.000264K}\frac{\partial P}{\partial t} \tag{3.2}$$

This equation, named the diffusivity equation, is written in terms of field units, the term $\frac{.000264K}{\phi \mu c}$ is called the hydraulic diffusivity, which is in ft^2/hr. Pressure, P, is in psi; distance, r, is in ft.; porosity, ϕ, is a fraction; viscosity, μ, is in cp; compressibility, c, is in psi^{-1}, permeability, K, is in md; and time, t, is in hours.

Solutions to Diffusivity Equation

There are three solutions to equation (3.2) which are particularly useful in well pressure testing (see Fig. 4).

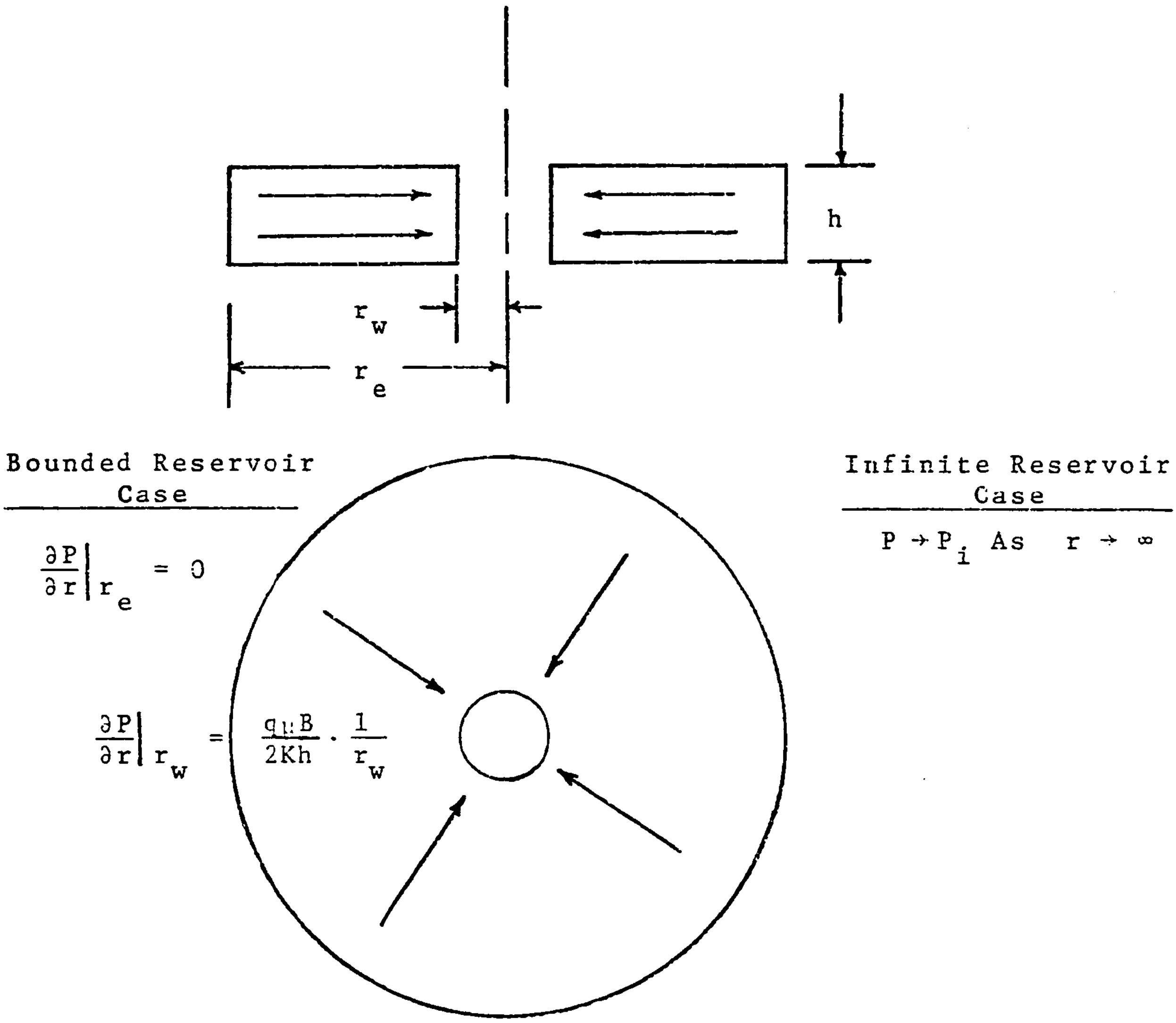


Fig. 4. Schematic drawing of geometry and boundary condition for radial flow constant rate case, Matthews[18].

(1) The solution for a bounded cylindrical reservoir with a well assumed to be located in the center of a cylindrical reservoir with no flow across the exterior boundary, and

(2) The solution for an infinite reservoir with a well considered to be a "line source" with zero wellbore radius, and

(3) The semi-steady-state solution.

Bounded circular reservoir. The analytical solution of equation (3.2) requires the existence of at least two boundary conditions and an initial condition. A realistic and practically useful solution is obtained if the following assumptions are made:

1. A well is producing at constant rate, qB, into the wellbore, $\left.\frac{\partial P}{\partial r}\right|_{r_w} = \frac{q\mu B}{2 \Pi Kh} \cdot \frac{1}{r_w}$;
2. The well, with wellbore radius r_w, is centered in a cylindrical reservoir of radius r_e, and that there is no flow across the outer boundary; $\left.\frac{\partial P}{\partial r}\right|_{r_e} = 0$: and finally
3. The reservoir is at uniform pressure, P_i, at time = 0.

The most useful form of the desired solution relates flowing pressure, P_{wf}, at the sandface, to time and to reservoir rock and fluid properties is given by Russell and Mathews[18]:

$$P_{wf} = P_i - \frac{141.2\ q\mu B}{Kh}\left(\frac{2\ t_{Dw}}{r_{e_D}^2} + \ln r_{e_D} - \frac{3}{4} + 2\sum_{n=1}^{\infty}\frac{\exp\left(-\alpha_n^2\ t_{Dw}\right) J_1^2\left(\alpha_n\ r_{e_D}\right)}{\alpha_n^2\left[J_1^2(\alpha_n\ r_{e_D}) - J_1^2(\alpha_n)\right]}\right) \tag{3.3}$$

where,

$$r_{e_D} = r_e/r_w$$

$$t_{Dw} = 0.000264\ Kt/\phi\mu\ cr_w^2$$

and where the α_n are the roots of

$$J_1(\alpha_n\ r_{e_D})\ Y_1\ (\alpha_n) - J_1\ (\alpha_n)\ Y_1\ (\alpha_n\ r_{e_D}) = 0$$

and where J_1 and Y_1 are Bessel functions.

The most important fact about equation (3.3) is to note that it is an exact solution to equation (3.2) along with the assumption made.

It is not necessary to use equation (3.3) in its complete form to calculate the numerical values of P_{wf}; instead, limiting forms of the solution are used for computations.

Infinite Reservoir With Line Source Well

The solution to this case is an approximate solution to the exact solution above. The simplest system one can visualize is a uniform, homogeneous, infinitely large reservoir, i.e., $P \to P_i$ as $r \to \infty$ penetrated by a single well of radius r_w. In most cases the wellbore radius is very small compared to both the dimensions of the reservoir as well as the distance between wells. Thus the wellbore may be considered to have a vanishing small radius. This leads to the concept of a "line-source" well.

Assume that we produce a well at a constant rate qB; that before production begins, the reservoir is at uniform pressure, P_i. Under these conditions, the solution to equation (3.2) is:

$$P = P_i + 70.6 \frac{q\mu B}{Kh} E_i \left(\frac{-r^2}{4\zeta t}\right) \tag{3.4}$$

where,

P, the pressure at distance r from the well at time t, ζ, hydraulic diffusivity $\left(\frac{0.000264K}{\phi\mu c}\right)$; and $-E_i(-x)$ is a mathematical function known as the exponential integral and is given by

$$-E_i(-x) = \int_x^\infty \frac{e^{-u}}{u}\, du \tag{3.5}$$

A tabulation of this function is given in Table 1. If the argument of the exponential integral is small enough ($x < 0.5$), then the exponential integral may be replaced by the following relation:

$$-E_i(-x) \simeq \ln(x) + .80907 \tag{3.6}$$

where ln (x) is the natural logarithm of x.

In terms of dimensionless variables Eq. (3.4) may be written as[17]:

$$P_D(r_D, t_D) = -\frac{1}{2} E_i\left(\frac{-\ r_D^2}{4\ t_D}\right) \simeq \frac{1}{2}\left[\ln \frac{t_D}{r_D^2} + 0.80907\right]; \quad \frac{t_D}{r_D^2} > 0 \tag{3.7}$$

where, the dimensionless pressure is

$$p_D(r_D,\ t_D) = \frac{Kh}{141.2\ q\mu B}\left(P_i - P(r,\ t)\right) \qquad (3.8)$$

the dimensionless time is

$$t_D = \frac{0.000264Kt}{\phi\mu c\ r_w^{\ 2}} \qquad (3.9)$$

The dimensionless radius is

$$r_D = r/r_w \qquad (3.10)$$

In Eq. (3.9): r_w is the well radius (feet), and t is the time (hrs.). Equations (3.8) and (3.9) are in field units (psi, hours, md, STB/day, cp, RB/STB, psi^{-1}). The definition of dimensionless pressures p_D, and dimensionless time, t_D, in equations (3.8) and (3.9) are commonly referred to as the Van Everdingen and Hurst definitions. Note that all dimensionless groups are directly proportional to the named real variable.

A table of dimensionless pressure and dimensionless time for a given problem is an easy way of providing a solution to the problem of interest without going into complex mathematical details. If real time data behave in a manner analogous to the p_D vs t_D data selected, then the model formulation is a correct one.

The line source solution[28] is shown graphically in Fig. 5. Though the above solution corresponds to a vanishing small radius, Mueller and Witherspoon[20] have shown that this solution is also applicable for a well of finite radius. They concluded that the line source solution is an adequate approximation for locations $r_D \geq 20$ for all t_D/r_D^2 and for all r_D for $t_D/r_D^2 \geq 25$. Their solution is shown in Fig. 6.

Semi-Steady-State Solution

The semi-steady-state solution is simply a limiting form of equation (3.3), which describes pressure behavior with time for a well centered in a cylindrical reservoir of radius r_e.

The α_n values in equation (3.3) take on monotonically increasing values as n increases; i.e., $\alpha_1<\alpha_2<\alpha_3\ldots.$ Thus, for a given value of t_{Dw}, the exponentials decrease monotonically $(e^{-\alpha_1^2 t_{Dw}} > e^{-\alpha_2^2 t_{Dw}} > \ldots.)$. Also, the Bessel function portion of the terms of the series becomes less as n increases. Thus, as t_{Dw} becomes large, the terms for large n become progressively small. For t_{Dw} sufficiently large, all the terms of the summation become negligibly small and equation (3.3) becomes:

$$P_{wf} = P_i - 141.2 \frac{q\mu B}{Kh} \left(\frac{2\, t_{Dw}}{r_{e_D}^2} + \ln r_{e_D} - \frac{3}{4}\right) \qquad (3.11)$$

Note that, during this time period (which turns out to be, $\left(t > \frac{\phi\mu c r_e^2}{(4)(0.000264)K}\right)$, the rate of pressure decline at the well during semi-steady flow can be found by differentiating equation (3.11) with respect to time.

$$\frac{\partial P_{wf}}{\partial t} = \frac{\partial P_i}{\partial t} - \frac{\partial}{\partial t}\left[141.2 \frac{q\mu B}{Kh}\left(\frac{0.000528Kt}{\phi\mu c\, r_e^2} + \ln \frac{r_e}{r_w} - \frac{3}{4}\right)\right]$$

$$\frac{\partial P_{wf}}{\partial t} = - \frac{.0745\, q\, B}{\phi c\, h\, r_e^2} \qquad (3.12)$$

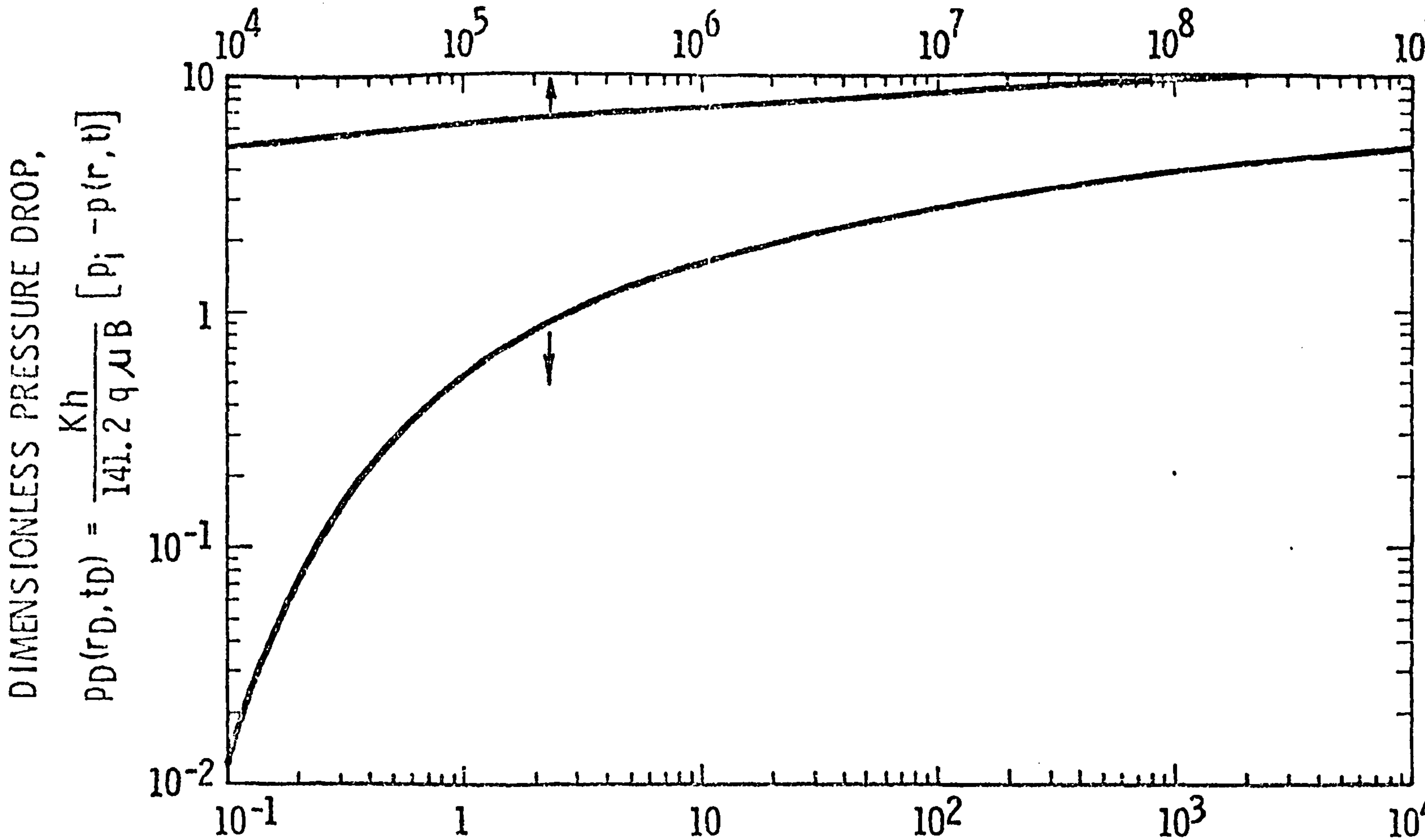


Fig. 5. RATIO OF DIMENSIONLESS TIME TO DIMENSIONLESS DISTANCE, SQUARED t_D/r_D^2

THE LINE SOURCE SOLUTION (after Raghavan[28])

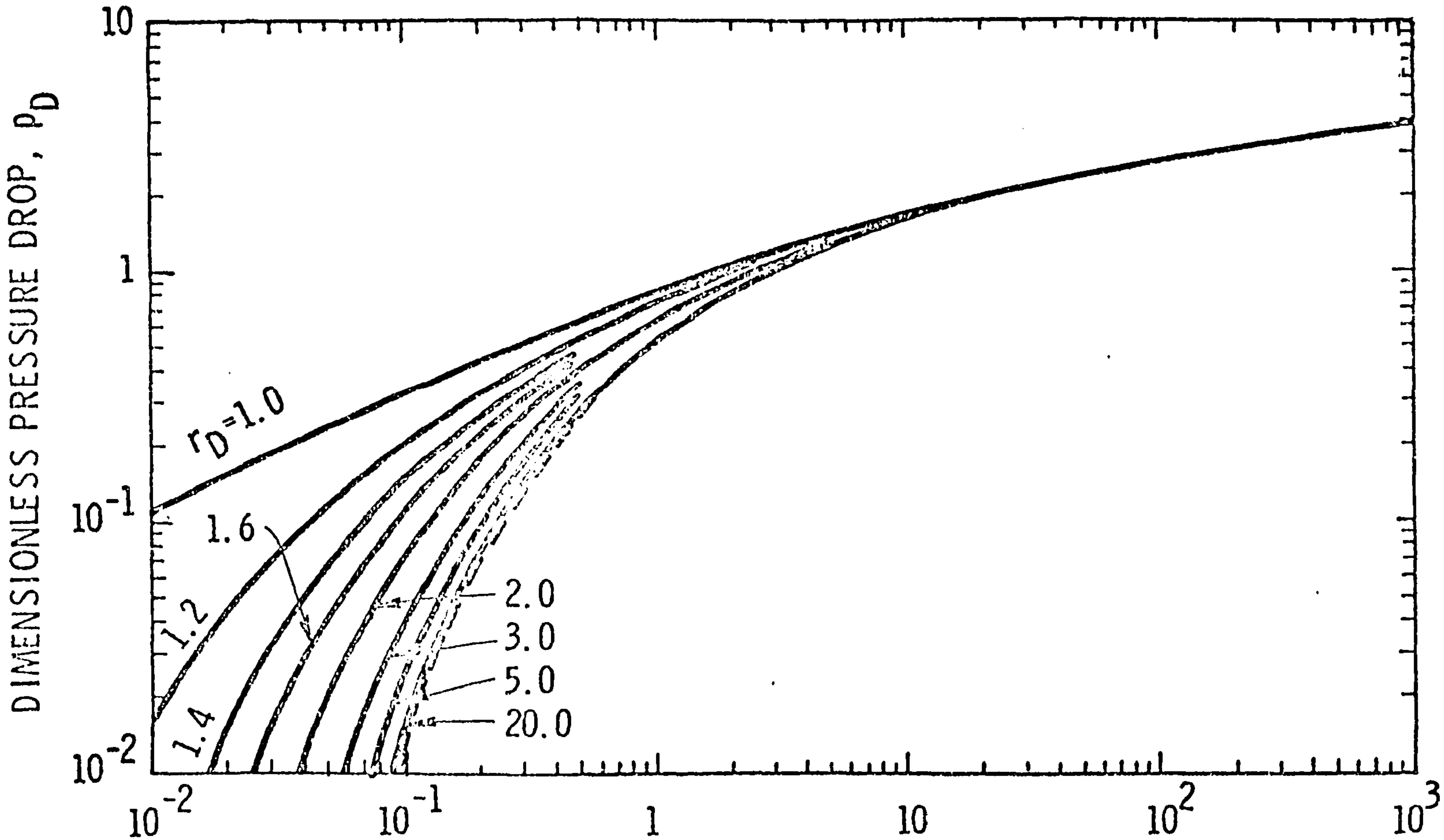


Fig. 6. RATIO OF DIMENSIONLESS TIME TO DIMENSIONLESS DISTANCE, SQUARED t_D/r_D^2
THE FINITE WELL BORE RADIUS SOLUTION (after Mueller[20, 28])

Since the fluid-filled pore volume of the reservoir, V_p, is

$$V_p = \Pi \phi h r_e^2$$

then

$$\frac{\partial P_{wf}}{\partial t} = -0.234 qB/cV_p \tag{3.13}$$

Thus, the rate of pressure decline is inversely proportional to the fluid-filled pore volume. This fact is the basis for so-called "reservoir limit tests," which seeks to determine reservoir size from the rate of pressure decline in a wellbore with time.

Another interesting fact is that the difference between the average reservoir pressure and the flowing wellbore pressure is constant during semi-steady-state flow. The volumetric average pressure within the drainage volume of the well is

$$\bar{P} = P_i - \frac{5.615\ q\mu B\ t/24}{\Pi \phi c r_e^2\ h} \tag{3.14}$$

Substitution of equation (3.14) into equation (3.11) gives:

$$\bar{P} - P_{wf} = \frac{141.2\ q\mu B}{Kh} \left(\ln \frac{r_e}{r_w} - \frac{3}{4} \right) \tag{3.15}$$

Since the productivity index of the well is defined as

$$J = \frac{q}{\bar{P} - P_{wf}} \tag{3.16}$$

Equation (3.16) implies that during semi-steady state flow the productivity index is constant.

Skin Effect

In practice, we find that most wells are either damaged or stimulated. Equation (3.4) fails to model such wells properly; explicit in its derivation was the assumption of uniform permeability throughout the drainage area of the well up to the wellbore. In many cases, it has been found that the permeability of the formation near the wellbore is reduced as a result of drilling and completion practices. Invasion by drilling fluids, dispersion of clays, presence of a mud cake and of cement, and plugging of perforations are some of the factors responsible for this reduction in permeability.

Thus the sand-face pressure of damaged or stimulated wells can be modelled using the logarithmic approximation for the E_i-function in equation (3.4) and by introducing an additional pressure drop ΔP_s, reflecting either damage or stimulation.

$$P_{wf} = P_i + 70.6 \frac{q\mu B}{Kh} \ln \left(\frac{1.78 r_w^2}{4\zeta t}\right) - \Delta P_s \qquad (3.17)$$

This additionally pressure drop is proportional to the rate, qB; and a skin factor, s, can be quantitatively defined, after Van Everdingen[31], as a constant which relates the pressure drop in the skin to the dimensionless rate of flow:

$$\Delta P_s = 141.2 \frac{q\mu B}{Kh} (s) \qquad (3.18)$$

Thus equation 3.12 can be written as

$$P_{wf} = P_i + 70.6 \frac{q\mu B}{Kh} \left(\ln \frac{0.445\ r_w^2}{\zeta t} - 2s \right) \qquad (3.19)$$

If the damaged or stimulated region near a wellbore is considered to be an "altered zone" of uniform permeability K_s and radial extent r_s, the skin factor, s, is related to these altered zone properties by[14]:

$$s = \left(\frac{K}{K_s} - 1\right) \ln \frac{r_s}{r_w} \qquad (3.20)$$

Some physical significance of the sign of the skin factor may be obtained from equation (3.20): If a well is damaged ($K_s < K$), s will be positive. It should be noted that there is no upper limit for s; some newly drilled wells may not flow at all prior to stimulation, for example; for these wells $K_s \to 0$, and $s \to \infty$. If a well is neither damaged nor stimulated, that is the permeabilities are equal ($K = K_s$), $s = 0$. Finally, if the permeability in the skin is greater than that in the formation ($K_s > K$), as from stimulation, s will be negative.

It is very important to point out that existence of an altered zone near a particular well affects only the pressure near that well, i.e., the pressure in the unaltered formation away from the well is not affected by the existence of the altered zone.

Effective Wellbore Radius

Noting that it is not possible to obtain both the radius of the skin and its permeability from equation

(3.20) even if K, s and r_w are known, an effective wellbore radius can be defined in order to alleviate this difficulty. This radius, r_w', would be such that it makes the calculated pressure drop in an ideal reservoir equal to that in an actual reservoir with skin. Thus,

$$\ln \frac{r_e}{r_w'} = \ln \frac{r_e}{r_w} + s$$

or

$$r_w' = r_w \, e^{-s} \tag{3.21}$$

If s is positive, the effective wellbore radius r_w' is smaller than r_w. If s is negative, the effective wellbore radius is larger than r_w.

Radius of Investigation

The radius of investigation is simply the distance that a pressure transient has moved into a formation following a rate change in a well. To show the significance of this radius, consider the following example:

A volume of liquid is injected instantaneously into a well. From the solution to the diffusivity equation for an instantaneous line source in an infinite reservoir(6, 22).

$$\Delta P \text{ (line)} = P_{wf} - P_i = \frac{-Q(\phi \mu c)}{4 \Pi \, Kt} \, e^{\frac{\phi \mu c r^2}{4 \, Kt}} \tag{3.22}$$

Where Q is a constant related to the strength of the instantaneous source.

The distance, r, at which the pressure disturbance is a maximum at time t can be obtained by differentiating the above equation with respect to time and setting the derivative equal to zero:

$$\frac{\partial P}{\partial t} = + \frac{Q\ \phi\mu c}{4\ \Pi K t^2}\ e^{\frac{-\phi\mu c r^2}{4Kt}} - \frac{Q\ (\phi\mu c)^2 r^2}{(4K)^2\ \Pi t^3}\ e^{\frac{-\phi\mu c r^2}{4Kt}} = 0$$

$$r^2 = 4\ \frac{Kt}{\phi\mu c}$$

$$r = \sqrt{\frac{4\ Kt}{\phi\mu c}}\ ;$$

and in field units it is; $r = \sqrt{4\zeta\ t}$ (3.23)

Equation (3.23) can also be modified to estimate the length of time required to achieve the maximum pressure disturbance at radius r for an instantaneous source

$$t = r^2/4\zeta \qquad (3.24)$$

From equation (3.24), the time required for a pressure transient to reach the boundaries of a reservoir where stabilized conditions prevail can also be determined. For example, if a well has a drainage radius r_e, then if $r = r_e$ and time $t = t_b$, the time required to reach the boundary, then

$$t_b = r_e^{\ 2}/4\zeta \qquad (3.25)$$

t_b can be described to be the time at which semi-steady-state flow begins. It must be pointed out now that equation (3.22) is correct only for a homogeneous and isotropic cylindrical reservoir.

Principle of Superposition

The solution to the diffusivity equation presented earlier was applicable only for describing the pressure distribution in a reservoir for the case of a constant volumetric rate of flow at flowing bottom hole conditions beginning at time zero. In general, however, a well will not have produced or injected into at constant rate throughout its life. For this situation, a very powerful mathematical technique commonly referred to as the principle of superposition is used.

To develop the principle of superposition and gain an understanding of its use, consider the simple case of a well which flows at three different flow rates as shown in Fig. 7. In this case, the well produces at rate q_1, from time 0 to time t_1; at t_1, the rate is changed to q_2; and, at time t_2, the rate is changed to q_3.

The first pressure drop in reservoir pressure is contributed by producing the well at rate q_1, starting at $t = 0$. Assuming that the well has skin, then from equation (3.19):

$$\Delta P_1 = (P_i - P_{wf})_1 = -70.6 \frac{q_1 \mu B}{Kh} \left(\ln \frac{0.445 r_w^2}{\zeta t} - 2s \right) \tag{3.26}$$

Stating at time t_1, the new total rate is q_2. In order that the total rate after t_1 is the required q_2, a well 2 located at the same position in the reservoir will be turned on to flow at rate $(q_2 - q_1)$ starting at t_1.

Thus the pressure drop in the reservoir due to well 2 is:

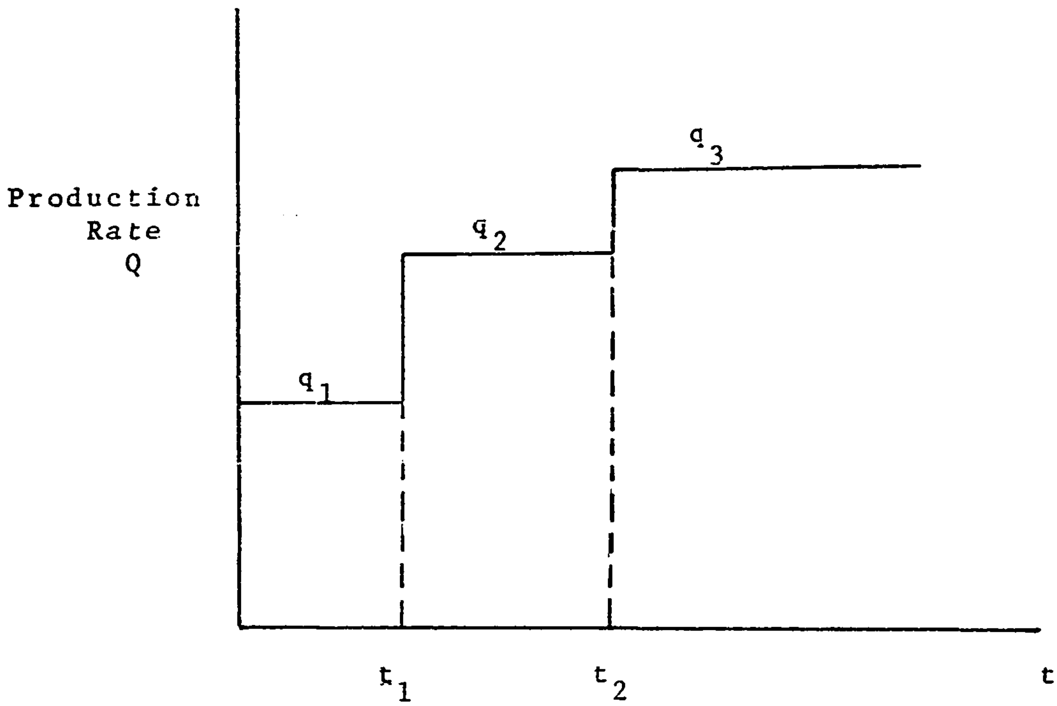


Fig. 7. Production history of a well with stepwise increasing flow rate.

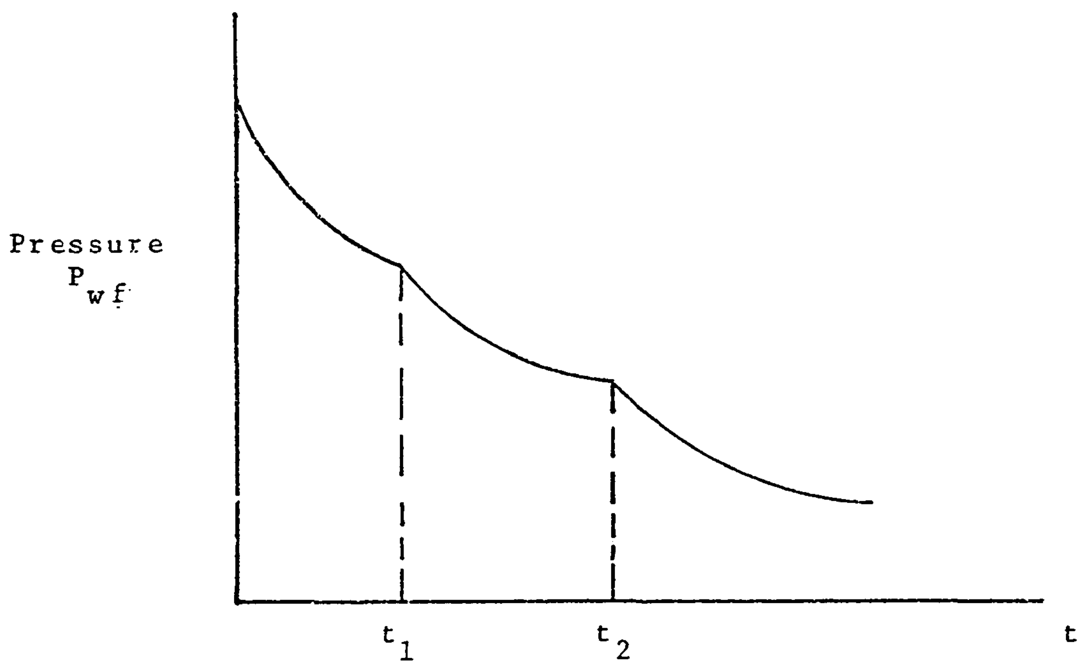


Fig. 8. Pressure history of a well with stepwise increasing flow rate.

$$(\Delta P)_2 = (P_i - P_{wf})_2 = \frac{-70.6\ (q_2-q_1)\mu B}{Kh}\left(\ln \frac{.445\ r_w^2}{\zeta (t-t_1)} - 2s\right) \quad (3.27)$$

Note that the total time since well 2 started flowing at q_2 is $(t-t_1)$.

Similarly the pressure drop in the reservoir due to a third well is:

$$(\Delta P)_3 = (P_i - P_{wf})_3 = \frac{-70.6(q_3-q_2)\mu B}{Kh}\left(\ln \frac{.445 r_w^2}{\zeta(t-t_2)} - 2s\right) \quad (3.28)$$

Therefore, the total pressure drop in the reservoir due to a well that flows at three different flow rates is:

$$(P_i - P_{wf})_{total} = \Delta P_1 + \Delta P_2 + \Delta P_3$$

$$= \frac{-70.6\ q_1 \mu B}{Kh}\left(\ln \frac{.445 r_w^2}{\zeta t} - 2s\right) - \frac{70.6\ (q_2-q_1)\mu B}{Kh}$$

$$\left(\ln \frac{.445 r_w^2}{\zeta(t-t_1)} - 2s\right) - \frac{70.6\ (q_3-q_2)\mu B}{Kh}$$

$$\left(\ln \frac{.445 r_w^2}{\zeta(t-t_2)} - 2s\right) \quad (3.29)$$

Concluding from the above example, the principle of superposition can be stated, for practical purposes, in the following manner: "The total pressure drop at any point in a reservoir is the sum of the pressure drops at the point caused by flow in each of the wells in the reservoir as if each well were alone in the reservoir."

CHAPTER IV

PRESSURE FALL-OFF ANALYSIS OF WATER INJECTION WELLS

Unsteady-state pressure analyses of buildup tests have long been recognized as a valuable and reliable tool for determining reservoir flow characteristics. This one topic has received much essential research for the last 20 years. As similar information about water injection is equally useful in operating water-flooding and pressure maintenance projects, several investigations[12, 15, 16, 23] studied the application of the fluid flow equations which have been developed for producing wells to injection wells. These authors have confirmed that these same equations are also applicable to injection wells providing suitable modifications are made for the differences in the direction of flow and in the physical characteristics of the fluids.

Nowak and Lester[23] reported that, theoretically, in an injection well the flow of water into the formation will continue as long as the sand-face pressure is greater than the reservoir pressure. They also observed that, in practice, upon stopping injection and opening the wellhead to atmospheric pressure, some wells backflow water for time intervals ranging from several seconds to several hours although the sand-face pressure, as calculated from the hydrostatic head, is considerably greater than the static reservoir pressure. Further, field observations indicated that upon shutting in a water injection well, the wellhead pressure does not drop off immediately, but a pressure decline occurs lasting from several seconds to several days. These observations make it evident that when an injection well is shut in, a residual pressure lingers in the vicinity of the wellbore which is greater than the average reservoir pressure. This residual pressure declines first rapidly, then more slowly to the static reservoir pressure; thus it is called pressure fall-off.

Quantitative Analysis of Pressure Fall-Off Tests

The mathematical analysis of pressure fall-off curves involves consideration of two distinct processes:

1. Pressure-rise during injection, and
2. Pressure fall-off during subsequent shutin.

In the first, the pressure will rise similar to the lowering of the pressure in oil producing wells. In the latter, the pressure fall-off is similar to pressure build-up in an oil producing well.

Pressure Rise During Injection

For a homogeneous infinite reservoir, the pressure distribution satisfying the basic governing differential equation can be obtained by integrating a unit impulse point source with respect to time and space.

The basic governing differential equation is:

$$K_x \frac{\partial^2 P}{\partial x^2} + K_y \frac{\partial^2 P}{\partial y^2} + K_z \frac{\partial^2 P}{\partial z^2} = \phi \mu c \frac{\partial P}{\partial t} \qquad (4.1)$$

The solution to equation (4.1) is given in the Kelvin instantaneous point source[6,15];

$$\Delta P(x,y,z,t) = \frac{-Q\ (\phi\mu c)^{3/2}}{8(\Pi^3 t^3 K_x K_y K_z)^{1/2}} \left[\exp \frac{-\phi\mu c}{4t} \left(\frac{(x-x')^2}{K_x} + \frac{(y-y')^2}{K_y} + \frac{(z-z')^2}{K_z} \right) \right] \qquad (4.2)$$

Where Q is a constant related to the strength of the instantaneous source.

The solution to the partial differential equation represented by equation (4.2) is called the pressure due to an instantaneous point source of strength Q at (x', y', z') at t = 0. Inspecting equation (4.2), it is found that as $t \rightarrow 0$, the expression tends to zero at all points except at (x', y', z') where it becomes infinite.

In Appendix (C), it is shown the solution (4.2) satisfies the partial differential equation (4.1).

In order to obtain the continuous infinite line source, we must integrate equation (4.2) with respect to the space variable, thus:

$$\Delta P(\text{line}) = -\frac{Q\ (\phi\mu c)}{4\ \phi c\ (\Pi^2 t^2 K_x K_y)^{1/2}} \exp\left[\left(\frac{(x-x')^2}{K_x} + \frac{(y-y')^2}{K_y}\right) - \left(\frac{\phi\mu c}{4t}\right)\right]$$

$$\frac{1}{\sqrt{\Pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{(z-z')^2 \phi\mu c}{4\ K_z\ t}\right) \frac{dz}{\left(\frac{4\ K_z t}{\phi\mu c}\right)^{1/2}} \qquad (4.3)$$

we have,

$$\text{erf } x = \frac{2}{\sqrt{\Pi}} \int_0^x \overline{e}^{\xi^2}\, d\xi \qquad , \text{ and}$$

$$\text{erf } \infty = \frac{2}{\sqrt{\Pi}} \int_0^\infty \overline{e}^{\xi^2}\, d\xi = 1$$

A tabulation of this function is given in Table (2)

Let $\xi = \dfrac{z-z'}{\left(\frac{4\ K_z t}{\phi\mu c}\right)^{1/2}}$

then, $d\xi = \dfrac{dz}{\left(\frac{4K_z t}{\phi\mu c}\right)^{1/2}}$

Therefore,

$$\frac{1}{2} * \frac{2}{\sqrt{\Pi}} \int_{-\infty}^{\infty} \exp\left(\frac{-(z-z')^2}{\frac{4K_z t}{\phi\mu c}}\right) \frac{dz}{\left(\frac{4K_z t}{\phi\mu c}\right)^{1/2}} = \frac{1}{2} * \frac{2}{\sqrt{\Pi}} \int_{-\infty}^{\infty} \overline{e}^{\xi^2}\, d\xi$$

$$= \frac{1}{2}\left[\frac{2}{\sqrt{\Pi}} \int_{-\infty}^{0} \overline{e}^{\xi^2}\, d\xi + \frac{2}{\sqrt{\Pi}} \int_0^\infty \overline{e}^{\xi^2}\, d\xi\right] \qquad (4.4)$$

Since the error function is symmetric, equation (4.4) is unity; and equation (4.3) becomes:

$$\Delta P(\text{line}) = -\frac{Q\mu}{4\ (\Pi^2 t^2 K_x K_y)^{1/2}} \exp\left[\left(\frac{(x-x')^2}{K_x} + \frac{(y-y')^2}{K_y}\right)\left(-\frac{\phi\mu c}{4t}\right)\right] \tag{4.5}$$

Which is the solution for the instantaneous infinite line source.

If the liquid injection rate is q_i, the total injection, Q, in the time interval, dt, is

$$Q = \int q_i dt$$

let $r^2 = (x-x')^2 + (y-y')^2$

The reservoir is homogeneous by definition, then integrating equation(4.5) with respect to time

$$\Delta P = \frac{-q_i \mu}{4\Pi K}\int_0^T \exp\left(-\frac{\phi\mu c r^2}{4Kt}\right)\frac{dt}{t} \tag{4.6}$$

let $t = T - t'$

$dt = -dt'$;

As $t' \to 0,\ t \to T$

$t' \to T,\ t \to 0$;

$$\Delta P = +\frac{q_i \mu}{4\Pi K}\int_T^0 \exp \frac{-\phi\mu c r^2}{4K(T-t')}\ \frac{dt'}{T-t'}$$

$$= -\frac{q_i \mu}{4\Pi K}\int_0^T \exp \frac{-\phi\mu c r^2}{4K(T-t')}\ \frac{dt'}{T-t'} \tag{4.7}$$

let $\nu = \dfrac{\phi\mu c r^2}{4K(T-t')}$;

$$d\nu = \frac{\phi\mu c r^2}{4K(T-t')^2}\ dt'$$

$$\frac{d\nu}{\nu} = \frac{dt'}{T-t'}$$

As $t' \to 0, \ \nu \to \frac{\phi\mu c r^2}{4KT}$

$t' \to T, \ \nu \to \infty$

Equation (4.7) becomes

$$\Delta P = \frac{-q_i \mu}{4\pi K} \int_{\frac{\phi\mu c r^2}{4Kt}}^{\infty} e^{-\nu} \frac{d\nu}{\nu} \tag{4.8}$$

The integral in equation (4.8) is by definition the exponential function E_i, where

$$-E_i(-x) = \int_x^{\infty} \frac{e^{-u}}{u} du$$

In equation (4.8), q_i is the injection rate per unit length of the infinite line source, hence

$$q_i = -q/h,$$

where $-q$ is the total injection over a portion, h, of the infinite line source. Equation (4.8) can be written

$$\Delta P = \frac{q \mu}{4\pi Kh} \left(- E_i \left(- \frac{\phi\mu c r^2}{4Kt}\right)\right) \tag{4.9}$$

and in oil field units

$$\Delta P = \frac{70.6 \ q\mu B}{Kh} \left(- E_i \left(- \frac{r^2}{4\zeta t}\right)\right) \tag{4.10}$$

Therefore the theoretical rise in pressure P at radial distance r from an injection well caused by injecting at rate $-q$ into an infinite reservoir of uniform thickness and permeability can be calculated by equation (4.10).

Equation (4.10) may then be approximated as shown earlier if values of $\frac{r^2}{4\zeta t}$ are relatively small, e.g., when r is equal to the wellbore radius r_w.

Thus:

$$P_{wf}(t)-P_i = -70.6 \frac{q\mu B}{Kh} \ln\left(\frac{.445\ r_w^2}{\zeta t}\right) \qquad (4.11)$$

$$P_{wf}(t)-P_i = 162.6 \frac{q\mu B}{Kh} \left[\log t - \log\left(\frac{.445\ r_w^2}{\zeta}\right)\right] \qquad (4.12)$$

$$P_{wf}(t)-P_i = 162.6 \frac{q\mu B}{Kh} \left[\log t - \log\left(1.68*10^3\ \frac{\phi\mu c r_w^2}{K}\right)\right] \qquad (4.13)$$

In equation (4.13) the slope of the straight line plot is represented by the quantity on the outside of the brackets. Usually the unknown factors occurring in the slope of the line are K and h.

Injective Capacity

Estimation of the injective capacity of the reservoir, the product Kh, is possible by plotting the observed injection pressure P_{wf} against log t and evaluating the slope of the resulting straight line (equation 4.13) to calculate Kh.

Pressure Fall-Off During Shut-in

Consider the case of a well that is charging our infinite acting reservoir, that the formation and fluids have uniform properties so that the E_i function solution applies. It is assumed that the well is inside a zone of altered permeability and that the well has been injecting at a constant flow rate, -q, during a considerable time, t, prior to shut in. If the pressure decline upon closing in is recorded as a function of the closed in time, Δt, and only those pressure declines are used after the effects of storage have died down. By superposing a positive rate, +q, so that the rate of injection to the formation becomes zero, see Figure 9, then:

$$P_{ws} - P_i = -70.6 \frac{-q\mu B}{Kh}\left(\ln \frac{.445\, r_w^2}{\zeta(t+\Delta t)} - 2s\right)$$

$$-70.6 \frac{+q\mu B}{Kh}\left(\ln \frac{.445\, r_w^2}{\zeta \cdot \Delta t} - 2s\right) \qquad (4.14)$$

$$P_{ws} - P_i = 70.6 \frac{-q\mu B}{Kh}\left(\ln \frac{t+\Delta t}{\Delta t}\right)$$

$$P_{ws} - P_i = 162.6 \frac{-q\mu B}{Kh}\left(\log \frac{t+\Delta t}{\Delta t}\right) \qquad (4.15)$$

Because the injection rate is not held constant over the injection history up to the time of shut-in, use of Horner's approximation will facilitates this problem. The stabilized injection rate q_s is substituted for q and the pseudo-injection time, t_p, for t in equation(4.14).

$$t_p \text{ (hr)} = \frac{\text{Cumulative volume of water injected STB} * \frac{24\text{hr}}{\text{D}}}{\text{Stabilized injection rate } q_s \frac{\text{STB}}{\text{D}}}$$

Therefore, equation (4.15) becomes

$$P_{ws} - P_i = 162.6 \frac{-q_s \mu B}{Kh} \log \frac{t_p + \Delta t}{\Delta t} \qquad (4.16)$$

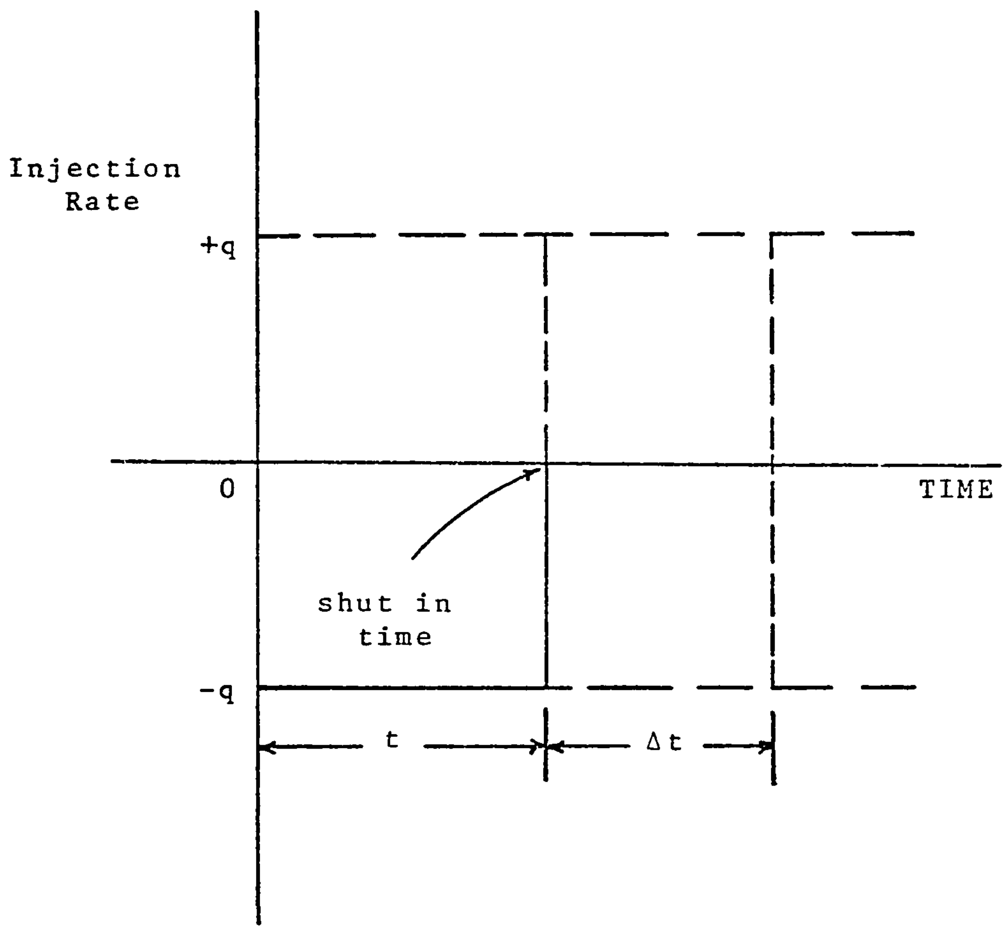


Fig. 9. Superposition of flowrate for shutting in a well.

From equation (4.16), a plot of shut-in bottom hole pressure, P_{ws}, versus the logarithm of time after a well is shut in yields a curve with a straight line section, Figure 10. Further, the absolute value of the slope, m, of this straight line should be:

$$m = 162.6 \frac{q\mu B}{Kh} \tag{4.17}$$

The slope of the straight line can be determined by measurement. As values for q, B, m, and h are known or can be estimated, the permeability may be calculated from equation (4.17). The calculated effective permeabilities when compared with permeability values obtained by other means, as from core analyses and injectivity indices, have been useful in evaluating well completions. Effective permeability values calculated from pressure fall-off curves are particularly useful because they represent the actual effective permeability to the flowing liquid of the formation away from the wellbore to the boundary of the reservoir.

If, for example, a permeability value indicated by injectivity index for a given well is lower than that indicated by an analysis of a pressure fall-off curve, the well evidently has been damaged by completion or injection methods.

Calculation of Skin Factor

The skin factor, s, can be determined from the idealized pressure fall off test theory. At shut-in time, the

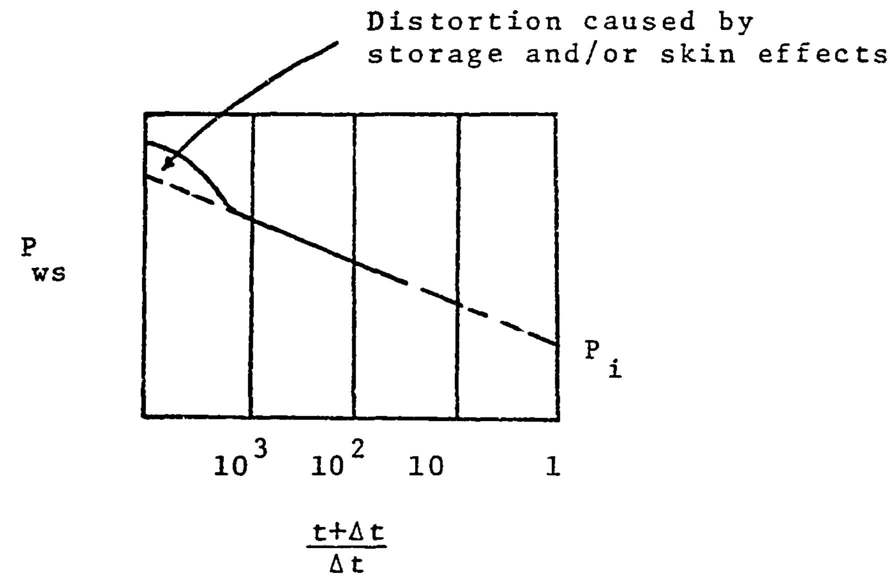


Fig. 10. Horner Fall-Off Type Plot

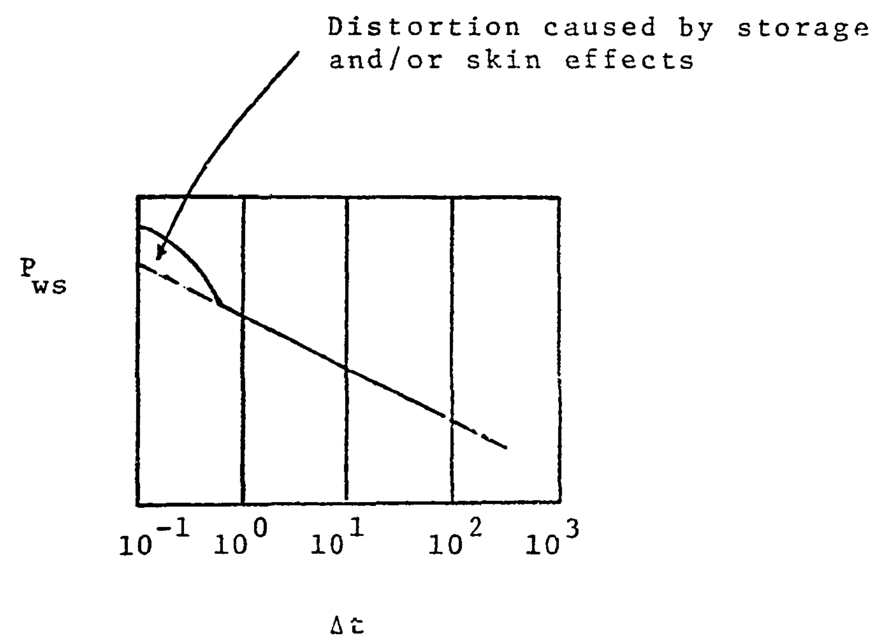


Fig. 11. Miller-Dyes-Hutchinson Fall-Off Type Plot

flowing bottom hole pressure, $P_{wf,s}$ is

$$P_{wf,s} = P_i - 70.6 \frac{q\mu B}{Kh}\left(\ln \frac{.445 r_w^2}{\zeta t} - 2s\right)$$

or

$$P_{wf,s} = P_i - 162.6 \frac{q\mu B}{Kh}\left(\log \frac{.445 r_w^2}{\zeta t} - .87s\right) \qquad (4.18)$$

From equation (4.15)

$$P_{ws} = P_i + 162.6 \frac{q\mu B}{Kh} \log \frac{t+\Delta t}{\Delta t}$$

Combining (4.15) and (4.18)

$$P_{wf,s} - P_{ws} = - m \log \frac{.445 r_w^2}{\zeta t} + .87s - m \log \frac{t+\Delta t}{\Delta t}$$

$$s = 1.151 \frac{P_{wf,s} - P_{ws}}{m} + 1.151 \log \frac{.445 r_w^2}{\zeta t} + 1.151 \log \frac{t+\Delta t}{\Delta t}$$

$$s = 1.151 \frac{P_{wf,s} - P_{ws}}{m} + 1.151 \log \frac{.445 r_w^2}{\zeta \Delta t} + 1.151 \log \frac{t+\Delta t}{t} \qquad (4.19)$$

It is conventional practice to choose a fixed shut-in time, Δt, of one hour and the corresponding shut in pressure, P_{1hr}, to use in equation (4.19). Using this simplification, $\log \frac{t+\Delta t}{t}$ is negligible. Thus,

$$s = 1.151 \frac{P_{wf,s} - P_{1hr}}{m} - \log \frac{K}{\phi \mu c r_w^2} + 3.23 \qquad (4.20)$$

The additional pressure drop, $(\Delta P)_s$, across the altered zone can be obtained using the slope, m:

since $(\Delta P)_s = 141.2 \frac{q\mu B}{Kh}(s)$

Thus, in terms of m

$$(\Delta P)_s = .87 \text{ ms} \qquad (4.21)$$

In addition to determination of formation permeability and skin factor, the straight line portion of the fall-off curve may be extrapolated to infinite shut in time, i.e., $\frac{t+\Delta t}{\Delta t} = 1$ to obtain the original formation pressure, P_i.

Conventional practice in the industry is the use of either the Horner or Miller-Dyes-Hutchinson (MDH) semi-log graphs. For buildup or fall-off the Horner method involves plotting the shut in pressure, P_{ws} versus $(t+\Delta t)/\Delta t$ where t is the "producing or injecting time" and Δt is the shut in time (Figure 10). In the MDH method the shut in pressure is graphed as a function of shut in time, Δt, Figure 11.

Qualitative Analysis of Pressure Fall-Off Tests

In order that the foregoing pressure fall-off equations apply, the following well-fluid-reservoir conditions were imposed:

1. A single well of infinitesmal diameter penetrates the entire thickness of the reservoir,
2. The well is charging the reservoir at a constant rate during injection;
3. The fluid is a single phase of constant viscosity and of constant small compressibility; and
4. The reservoir is homogeneous of infinite radial extent.

The restrictions cited above have varying degrees of importance and effect in actual practice. Some of these limitations can be handled by incorporating suitable modifications in the mathematical analysis, or by choosing suitable portions of the data which predominantly reflect the behavior in the region of the reservoir under investigation.

A study of the literature assures that the size of the wellbore diameter can be considered negligible. The effect of the variation in injection rate and of other wells in the reservoir can be considered by use of the superposition principle.

In water injection operations, the restriction of a single phase fluid is flowing offers fewer uncertainties. If the fluid mobility and compressibility are considerably different ahead of the flood front from those in the watered out region, a change of the pressure fall-off curve should be observed in the latter part of the shut in time. If liquid and gas are flowing simultaneously, equations describing two phase flow must be used in the analysis. They have been discussed by Miller, Dyes, and Hutchinson [19]. A method which would apply when there is a gas saturation was developed for pressure behavior in water injection wells by Hazebrock, Rainbow, and Matthews [15].

The restriction on the size of the reservoir can be overcome since if the data are obtained during early injection history, the reservoir will behave as if it were infinite in size.

Another important consideration that is necessary for the pressure fall-off curve to reflect the permeability of the formation to water, enough water must have been injected before the test to insure that the pressure drop as reflected in the measurements will occur entirely in the water phase. Moreover, the liquid saturation of the formation should be high enough to preclude the possibility of a continuous gas phase. Hence, this technique applies to liquid filled reservoirs with mobility of the injected fluid essentially equal to the mobility of the in-situ fluid. If the unit mobility ratio condition is not satisfied, results of this technique may not be valid. However, if the radius of investigation is not beyond the water (injected fluid) bank, valid analysis can be made for permeability and skin, but not for static reservoir pressure.

Aside from the skin effect, another factor causing a deviation between observed and theoretical pressure fall-off behavior. This factor is a function of fluid compressibility as well as the storage capacity of the well for the flowing fluid. Because the conditions giving rise to the factors occurs in the wellbore region, they affect the shape of the pressure fall-off curve in the early portion of the shut in time.

In fall-off tests on water injection wells, the storage frequently changes from compressive wellbore storage to falling fluid level. One of the best methods for eliminating this problem is to go to a two-rate fall-off test. A two-rate fall-off test is run by injecting at a relatively high rate and then decreasing the injection rate while observing the pressure decrease as a result of the rate decrease. If rates are chosen correctly, surface pressure is maintained and changing wellbore storage effects are eliminated.

CHAPTER V

MATHEMATICAL MODEL

The assumption of an isotropic porous medium has been in most of the applications of the diffusivity equation. This was because solutions are much easier to obtain if the permeability, viscosity, and density terms can be removed from the left side of equation (3.1).

Since the viscosity can be considered constant, as in the case of Newtonian fluids, the density can be replaced with potential or pressure in many systems of interest, where the fluid is slightly compressible, the diffusivity equation for an isotropic porous media will be chosen such that the permeability will be included on the left side of the equation. Thus, for a homogeneous medium with small and constant compressibility, and unsteady state fluid flow:

$$\frac{\partial}{\partial x}\left(K_x \frac{\partial P}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_y \frac{\partial P}{\partial y}\right) + \frac{\partial}{\partial z}\left(K_z \frac{\partial P}{\partial z}\right) = \phi \mu c \frac{\partial P}{\partial t} \quad (5.1)$$

The form of equation (5.1) can be simplified using Muskat's[21] change of variables. The change of variables will amount to the transformation of this equation to a new coordinate system in which the axes are still orthogonal but are functions of the old axes and the proper permeability term. This can be done by defining a new Cartesian coordinate system $(\bar{x}, \bar{y}, \bar{z})$ such that:

$$\bar{x} = \frac{x}{\sqrt{K_x}} \; ;$$

$$\bar{y} = \frac{y}{\sqrt{K_y}} \; ; \text{ and}$$

$$\bar{z} = \frac{z}{\sqrt{K_z}} \; ;$$

With first partial derivatives:

$$\frac{\partial \bar{x}}{\partial x} = \frac{1}{\sqrt{K_x}} \; ; \quad \frac{\partial \bar{x}}{\partial y} = \frac{\partial \bar{x}}{\partial z} = 0$$

$$\frac{\partial \bar{y}}{\partial y} = \frac{1}{\sqrt{K_y}} \; ; \quad \frac{\partial \bar{v}}{\partial x} = \frac{\partial \bar{y}}{\partial z} = 0$$

$$\frac{\partial \bar{z}}{\partial z} = \frac{1}{\sqrt{K_z}} \; ; \quad \frac{\partial \bar{z}}{\partial x} = \frac{\partial \bar{z}}{\partial y} = 0$$

Because the medium is homogeneous by definition, K_x, K_y and K_z will be considered constants (although presumably unequal). Therefore, they can be moved outside the parentheses in each term to obtain a slightly different form of equation (5.1):

$$K_x \frac{\partial}{\partial x} \left(\frac{\partial P}{\partial x}\right) + K_y \frac{\partial}{\partial y} \left(\frac{\partial}{\partial y}\right) + K_z \frac{\partial}{\partial z} \left(\frac{\partial P}{\partial z}\right) = \phi \mu c \frac{\partial P}{\partial t} \qquad (5.2)$$

Using the chain rule for differentiation and substituting for the partial derivatives with respect to x, y and z such as:

$$K_x \frac{\partial}{\partial x}\left(\frac{\partial P}{\partial \bar{x}} \frac{\partial \bar{x}}{\partial x}\right) = K_x \frac{\partial}{\partial \bar{x}}\left(\frac{\partial P}{\partial \bar{x}} \frac{\partial \bar{x}}{\partial x}\right) \frac{\partial \bar{x}}{\partial x}$$

$$= K_x \frac{\partial}{\partial \bar{x}}\left(\frac{\partial P}{\partial \bar{x}} \frac{1}{\sqrt{K_x}}\right) \frac{1}{\sqrt{K_x}}$$

$$= K_x \frac{\partial}{\partial \bar{x}}\left(\frac{\partial P}{\partial \bar{x}}\right) \frac{1}{K_x}$$

$$= \frac{\partial}{\partial \bar{x}}\left(\frac{\partial P}{\partial \bar{x}}\right) = \frac{\partial^2 P}{\partial \bar{x}^2}$$

it follows at once:

$$\frac{\partial^2 P}{\partial \bar{x}^2} + \frac{\partial^2 P}{\partial \bar{y}^2} + \frac{\partial^2 P}{\partial \bar{z}^2} = \phi \mu c \frac{\partial P}{\partial t} \qquad (5.3)$$

This equation now looks exactly like the equation for an isotropic porous medium for solution purposes. It is a second order, linear partial differential equation.

It thus appears that the effect of anisotropy in the permeability can be replaced by an equivalent shrinking or expansion of the coordinates.

The general solution would give P as a function of $\bar{x}$, $\bar{y}$ and $\bar{z}$. It would be necessary to change the boundary conditions using the transform equations before the constants of integration could be evaluated once obtained as a complete solution, however, the transform equations may be again used to place the solution in terms of the axes of the anisotropic porous medium.

In equation(5.2), the gravity effect is ignored. The geometry of the problem is shown in Figure 12.

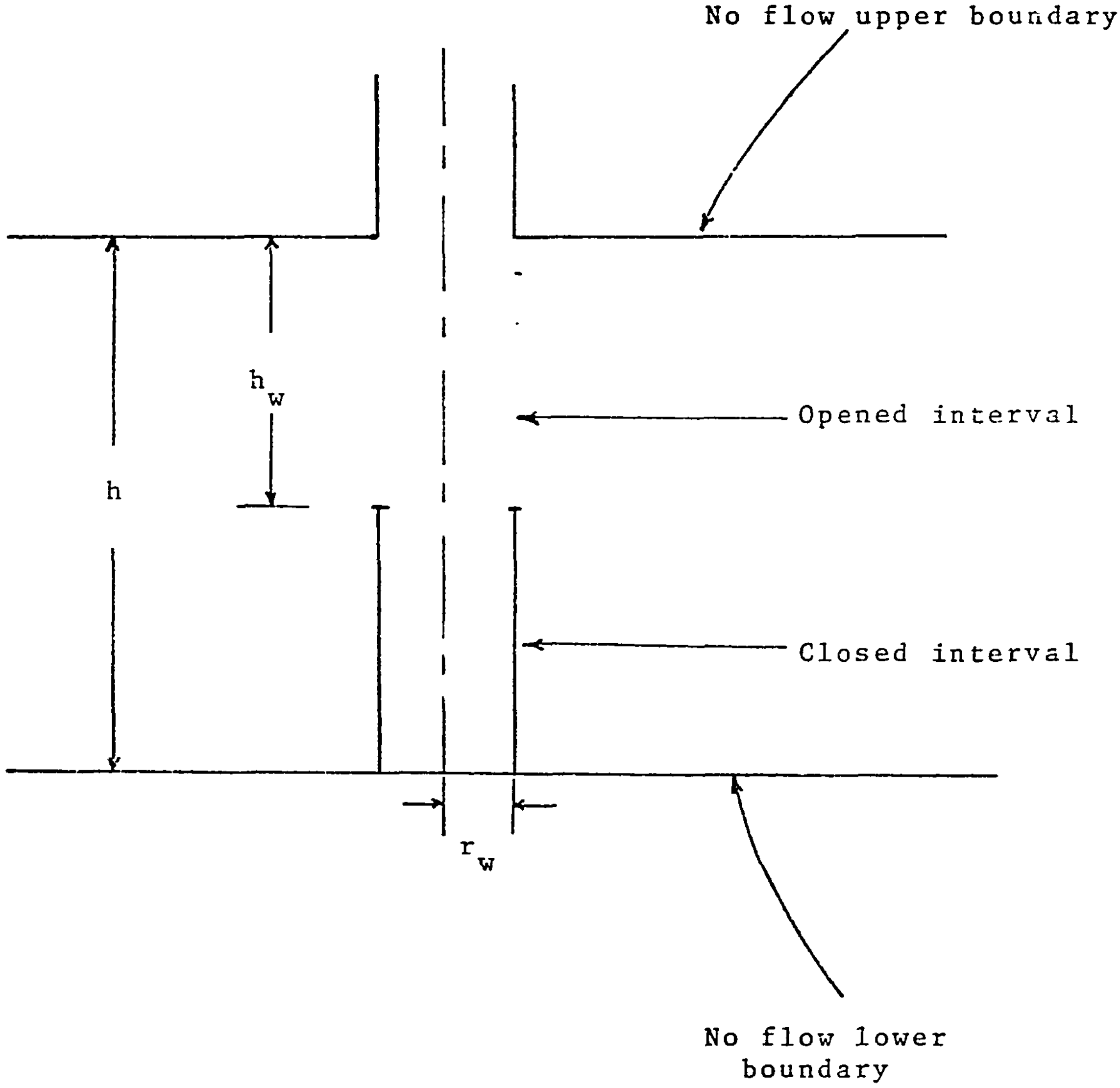


Fig. 12. Diagram of the geometry of the problem.

The initial and boundary conditions are as follows:

1. The initial reservoir pressure is constant

$$P\ (x,\ y,\ z,\ 0) = P_i \tag{5.4}$$

2. The reservoir is infinite, the pressure remains unchanged at infinity.

$$\frac{\partial P}{\partial t}\ (\infty,\ \infty,\ z,\ t) = P_i \tag{5.5}$$

3. No flow of fluids across the upper or lower boundaries.

$$\frac{\partial P}{\partial z}\ (x,\ y,\ 0,\ t) = \frac{\partial P}{\partial z}\ (x,\ y,\ h,\ t) = 0 \tag{5.6}$$

4. The flow into the open interval is constant.

$$r\ \frac{\partial P}{\partial r}\bigg|_{r=r_w} = \begin{cases} \dfrac{(-q\mu)}{2\,\Pi\ K\quad h_w}\ ; & 0 < z < h_w \\ 0\ ; & h_w < z < h \end{cases} \tag{5.7}$$

For injection, q, is positive (+).

From Chapter IV, the instantaneous point source solution to the partial differational equation 5.2 is:

$$\Delta P\ (x,y,z,t) = \frac{-Q\ \ (\phi\mu c\,)^{3/2}}{8(\Pi^3 t^3 K_x K_y K_z)^{1/2}}\ \exp\left[\frac{-\phi\mu c}{4t}\left(\frac{(x-x')^2}{K_x} + \frac{(y-y')^2}{K_y} + \frac{(z-z')^2}{K_z}\right)\right] \tag{5.8}$$

In order to obtain the instantaneous line source of length h_w, we must integrate equation (5.8) along this line. The boundary condition to be satisfied is $\frac{\partial P}{\partial z} = 0$ in the boundary plane, taken to be the x-y plane. Introduction of the image† Q of length h_w and located as shown in Figure 13 accomplishes this result, thus integrating along this line source from $-h_w$ to $+h_w$:

$$\Delta P = -\frac{Q\ (\phi\mu c)^{3/2}}{4\ \phi c \sqrt{\Pi^2 K_x K_y}\, t^2} \exp\left[-\frac{\phi\mu c}{4t}\left(\frac{(x-x')^2}{K_x} + \frac{(y-y')^2}{K_y}\right)\right] *$$

$$\frac{1}{\sqrt{\Pi}} \int_{-h_w}^{+h_w} \exp\left(\frac{-\phi\mu c}{4t}\,\frac{(z-z')^2}{K_z}\right) \frac{dz}{\sqrt{4K_z t}} \qquad (5.9)$$

$$\text{Let} \quad \xi = (z - z')\sqrt{\frac{\phi\mu c}{4K_z t}}$$

$$d\xi = -dz'\sqrt{\frac{\phi\mu c}{4K_z t}}$$

$$\text{Thus, as} \quad z' \rightarrow h_w, \quad \xi \rightarrow (z-h_w)\sqrt{\frac{\phi\mu c}{4K_z t}}$$

$$z' \rightarrow -h_w, \quad \xi \rightarrow (z+h_w)\sqrt{\frac{\phi\mu c}{4K_z t}}$$

Therefore, equation 5.9 becomes:

$$\Delta P = +\frac{Q\mu}{4\sqrt{\Pi^2}\, t^2 K_x K_y}\left[\exp\ \frac{-\phi\mu c}{4t}\left(\frac{(x-x')^2}{K_x} + \frac{(y-y')^2}{K_y}\right) *\right.$$

---

†The method of images which yields closed form solutions by deductive reasoning requires that we define an image singularity or point source across the boundary in question from the fundamental singularity. The image singularity must have just the strength, location, and character to successfully oppose the original one at the boundary[29].

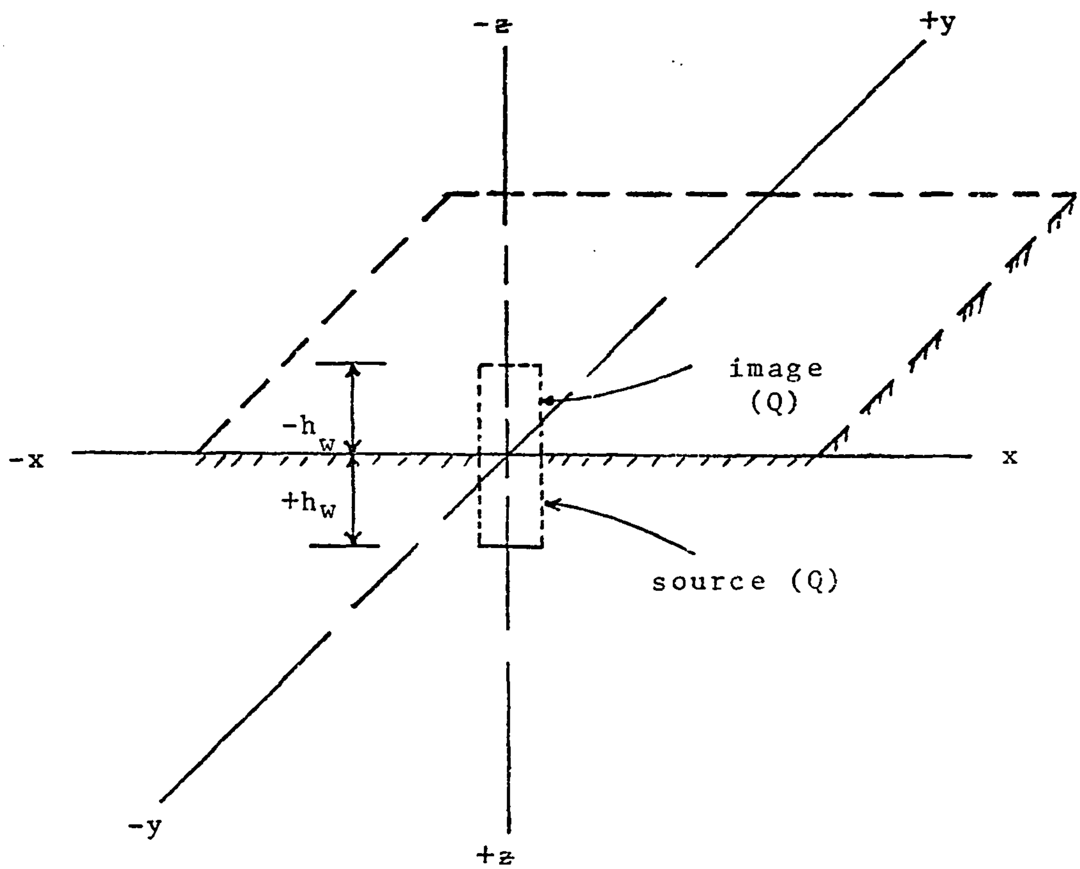


Fig. 13. Partially injecting line source well in a reservoir infinite in radial extent but semi-infinite in the vertical direction.

$$\frac{1}{\sqrt{\Pi}} \int_{(z+h_w)\sqrt{\frac{\phi\mu c}{4K_z t}}}^{(z-h_w)\sqrt{\frac{\phi\mu c}{4K_z t}}} \exp\ (-\xi^2)\ d\xi \qquad (5.10)$$

$$\Delta P = + \frac{Q\mu}{4\Pi t\sqrt{K_x K_y}} \exp\left[\frac{-\phi\mu c}{4t}\left(\frac{(x-x')^2}{K_x} + \frac{(y-y')^2}{K_y}\right)\right] *$$

$$\frac{1}{\sqrt{\Pi}} \int_{o}^{(z-h_w)\sqrt{\frac{\phi\mu c}{4K_z t}}} \exp\ (-\xi^2) d\xi - \int_{o}^{(z+h_w)\sqrt{\frac{\phi\mu c}{4K_z t}}} \exp\ (-\xi^2) d\xi$$

$$\Delta P = - \frac{Q\mu}{8\Pi t\sqrt{K_x K_y}} \exp\left[\frac{-\phi\mu c}{4t}\left(\frac{(x-x')^2}{K_x} + \frac{(y-y')^2}{K_y}\right)\right]$$

$$\left(\operatorname{erf} \frac{z+h_w}{\sqrt{\frac{4K_z t}{\phi\mu c}}} - \operatorname{erf} \frac{z-h_w}{\sqrt{\frac{4K_z t}{\phi\mu c}}}\right) \qquad (5.11)$$

Partially Injecting Source in a Thick Formation of Infinite Radial Direction

The no-flow boundary conditions specified by equation (5.5) may be satisfied by adding an infinite series of image sources[22] to the solution of equation (5.11), as illustrated in Figure 14. Each of these images may be represented by an integral of the type used in equation (5.9). Each of these integrals may be transformed as in equation (5.10) and the results expressed as equation (5.11).

Assume that the permeability is uniform and homogeneous

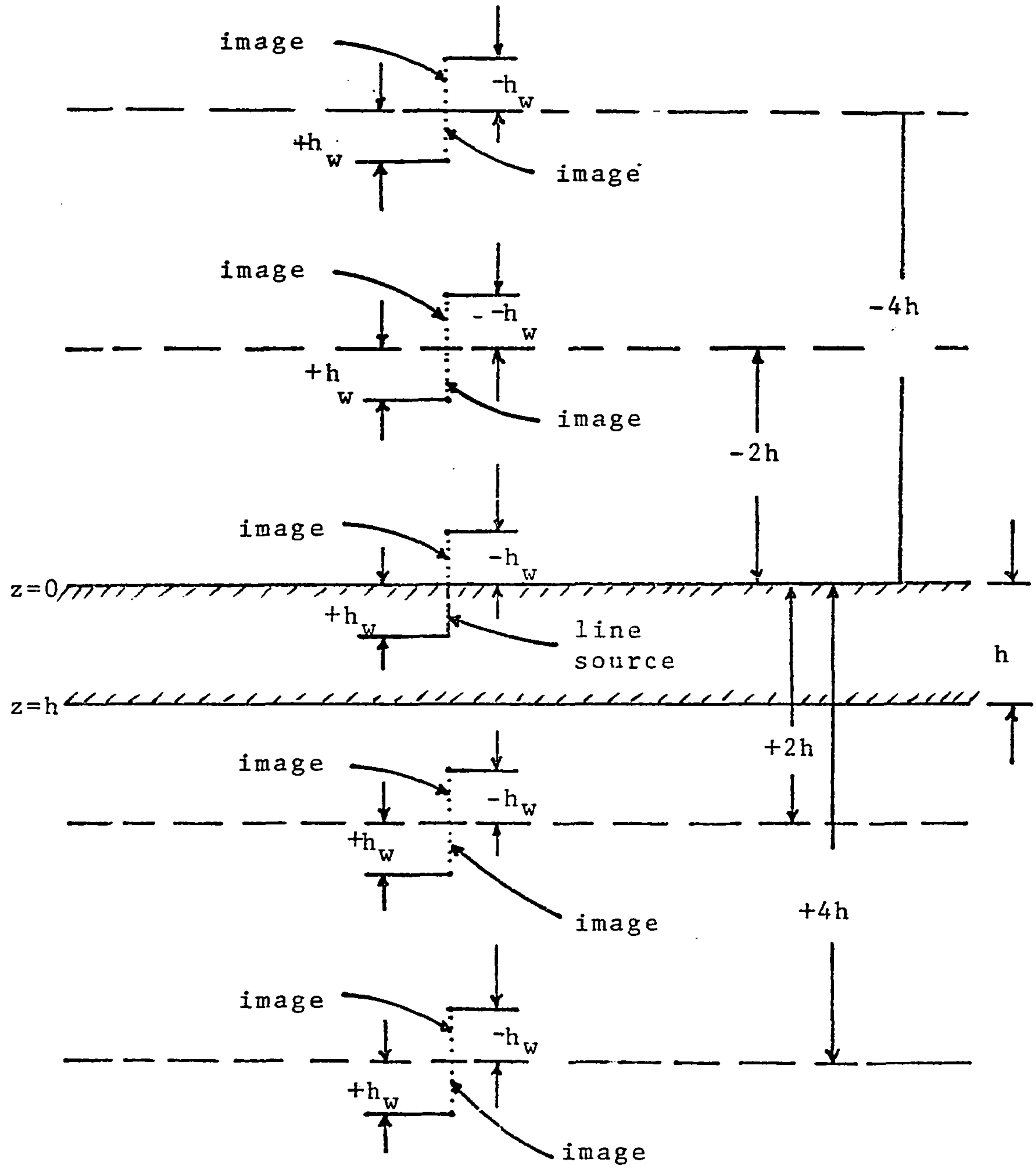


Fig. 14. Images used to satisfy no-flow boundary conditions for a partially injecting well in an infinite thick formation.

in the radial direction, i.e., $K_x = K_y = K_r$:

$$K_z = Kv;$$

$$Kv \neq K_r; \text{ and}$$

$$r^2 = (x-x')^2 + (y-y')^2$$

Then the resulting expression for the partially injecting case is:

$$\Delta P = -\frac{Q\mu}{8\pi K_r}\, e^{\left(-\frac{\phi\mu c r^2}{4K_r t}\right)} \Bigg[\left(\operatorname{erf} \frac{z+h_w}{\sqrt{\frac{4Kvt}{\phi\mu c}}} - \operatorname{erf} \frac{z-h_w}{\sqrt{\frac{4Kvt}{\phi\mu c}}} \right) +$$

$$\sum_{n=1}^{\infty} \left(\operatorname{erf} \frac{z+(2nh-h_w)}{\sqrt{\frac{4Kvt}{\phi\mu c}}} - \operatorname{erf} \frac{z-(2nh+h_w)}{\sqrt{\frac{4Kvt}{\phi\mu c}}} + \right.$$

$$\left. \operatorname{erf} \frac{z+(2nh+h_w)}{\sqrt{\frac{4Kvt}{\phi\mu c}}} - \operatorname{erf} \frac{z+(2nh-h_w)}{\sqrt{\frac{4Kvt}{\phi\mu c}}} \right) \Bigg] \tag{5.12}$$

It is noted that in equation (5.12) when $h_w = h$, the integration extends from $-\infty$ to $+\infty$. This case reduces in the limit to a continuous infinite line source, as shown in Chapter IV. This situation assures the convergence of the summation because the error function is known to converge for the limits $-\infty$ and $+\infty$.

Finally the solution for the continuous partially injecting infinite line source can be obtained by integrating the latter solution (5.12) over the interval 0 to T. For convenience the following substitutions are made:

$$b = h_w/h$$

$$z_D = z/h$$

$$\Delta P = \frac{q\mu B}{8\pi K_r h} \int_o^T e^{\left(\frac{-\phi\mu c r^2}{4K_r t}\right)} \left[\left\{ \text{erf} \left(\frac{(z_D+b)^2 h^2}{\frac{4Kv}{\phi\mu c}} \right)^{1/2} - \text{erf} \left(\frac{(z_D-b)^2 h^2}{\frac{4Kv}{\phi\mu c}} \right)^{1/2} \right\} \right.$$

$$+ \sum_{n=1}^{\infty} \left\{ \text{erf} \left(\frac{(2n+b-z_D)^2 h^2}{\frac{4Kv}{\phi\mu c}} \right)^{1/2} - \text{erf} \left(\frac{(2n-b-z_D)^2 h^2}{\frac{4Kv}{\phi\mu c}} \right)^{1/2} + \right.$$

$$\left. \left. \text{erf} \left(\frac{(2n+b+z_D)^2 h^2}{\frac{4Kv}{\phi\mu c}} \right)^{1/2} - \text{erf} \left(\frac{(2n-b+z_D)^2 h^2}{\frac{4Kv}{\phi\mu c}} \right)^{1/2} \right\} \right] \frac{dt}{t}$$

(5.13)

or

$$\Delta P = \frac{q\mu B}{4\pi K_r h_w} \int_o^T U(t)\, dt \qquad (5.14)$$

where,

$$U(t) = \frac{1}{2t} e^{\left(-\frac{\phi\mu c r^2}{4K_r t}\right)} \left[\left(\text{erf} \frac{(z_D+b)h}{\sqrt{\frac{4Kvt}{\phi\mu c}}} - \text{erf} \frac{(z_D-b)h}{\sqrt{\frac{4Kvt}{\phi\mu c}}} \right) \right.$$

$$+ \sum_{n=1}^{\infty} \left(\text{erf} \frac{(2n+b-z_D)h}{\sqrt{\frac{4Kvt}{\phi\mu c}}} - \text{erf} \frac{(2n-b-z_D)h}{\sqrt{\frac{4Kvt}{\phi\mu c}}} + \right.$$

$$\left. \left. \text{erf} \frac{(2n+b+z_D)h}{\sqrt{\frac{4Kvt}{\phi\mu c}}} - \text{erf} \frac{(2n-b+z_D)h}{\sqrt{\frac{4Kvt}{\phi\mu c}}} \right) \right] \qquad (5.15)$$

In order to make the results of this study generally useful, the problem will be converted to dimensionless variables. The variables are:

Dimensionless time:

$$t_D = \frac{K_r t}{948\ \phi\mu c r_w^2} \qquad (5.16)$$

Dimensionless Press;

$$P_D = \frac{K_r h \ \Delta P}{141.2 \ q\mu B} \tag{5.17}$$

Dimensionless radius;

$$r_D = \frac{r}{r_w} \tag{5.18}$$

Dimensionless thickness:

$$h_D = \frac{h}{r_w} \sqrt{\frac{K_r}{Kv}} \tag{5.20}$$

and,

$$\begin{aligned} M_1(n) &= 2n + b - z_D \\ M_2(n) &= 2n - b - z_D \\ M_3(n) &= 2n + b + z_D \\ M_4(n) &= 2n - b + z_D \end{aligned} \tag{5.21}$$

The quantity $\dfrac{h}{\left(\dfrac{Kvt}{948 \ \phi\mu c}\right)^{1/2}}$ can now be rearranged:

$$\frac{h}{\left(\dfrac{Kvt}{948 \ \phi\mu c}\right)^{1/2}} = \frac{h}{\left(\dfrac{Kvt}{948} \dfrac{K_r}{K_r} \dfrac{r_w 2}{\phi\mu c r_w^{\ 2}}\right)} \ 1/2$$

$$= \frac{h}{r_w} \sqrt{\frac{K_r}{Kv}} \ \frac{1}{\sqrt{t_D}}$$

$$= h_D \ / \sqrt{t_D}$$

Also,

$$dt = \frac{948 \ \phi\mu c \ r_w^2}{K_r} \ dt_D$$

Equation 5.15 becomes:

$$U(t_D) = \frac{1}{2} * \frac{1}{t_D} * \frac{K_r}{948 \phi\mu c r_w^{\ 2}} * e^{\left(\frac{-r_D^{\ 2}}{t_D}\right)} *$$

$$\left[\left(\operatorname{erf}\frac{(z_D+b)h_D}{\sqrt{t_D}} - \operatorname{erf}\frac{(z_D-b)h_D}{\sqrt{t_D}}\right) + \sum_{n=1}^{\infty}\left(\operatorname{erf}\frac{M_1 h_D}{\sqrt{t_D}} - \operatorname{erf}\frac{M_2 h_D}{\sqrt{t_D}} + \operatorname{erf}\frac{M_3 h_D}{\sqrt{t_D}} - \operatorname{erf}\frac{M_4 h_D}{\sqrt{t_D}}\right)\right] \tag{5.22}$$

and equation 5.13 becomes:

$$P_D = \frac{1}{2b}\int_o^{T_D} e^{\left(\frac{-r_D^2}{t_D}\right)}\left[\left(\operatorname{erf}\frac{(z_D+b)h_D}{\sqrt{t_D}} - \operatorname{erf}\frac{(z_D-b)h_D}{\sqrt{t_D}}\right) + \sum_{n=1}^{\infty}\left(\operatorname{erf}\frac{M_1 h_D}{\sqrt{t_D}} - \operatorname{erf}\frac{M_2 h_D}{\sqrt{t_D}} + \operatorname{erf}\frac{M_3 h_D}{\sqrt{t_D}} - \operatorname{erf}\frac{M_4 h_D}{\sqrt{t_D}}\right)\right]\frac{dt_D}{2t_D} \tag{5.23}$$

and equation (5.14) becomes:

$$P_D = \frac{1}{2b}\int_o^{T_D} U\ (t_D)\ dt_D \tag{5.24}$$

It is therefore apparent that Nisle's study becomes a special case of this formulation, i.e., if we have complete isotropy ($K_x = K_y = K_z$), we obtain Nisle's results.

CHAPTER VI

NUMERICAL CALCULATIONS

In order to investigate how partial injection and the existence of vertical permeability affect pressure transient curves, numerical calculations must be performed over a suitable range of values of these parameters. More realistic conditions would also, consider wellbore storage and skin effect[2]. The purpose of this study is to develop a method of determining horizontal to vertical permeability ratio and skin regardless of the existence of wellbore storage. The line source well was considered because a study by Agarwal et al[1] showed that wellbore storage causes time delays sufficient that the line-source assumption is usually valid.

It is now important to note that equation (5.24) is valid only in an approximate way. The dimensionless pressure P_D is constant throughout the borehole, i.e., in reality we expect the injection pressures to be independent of z and they would differ only by virtue of the potential head of flowing fluid and friction over the injection interval, assuming the wellbore is open. However, the flowlines (see Fig. 15) diverge away from the

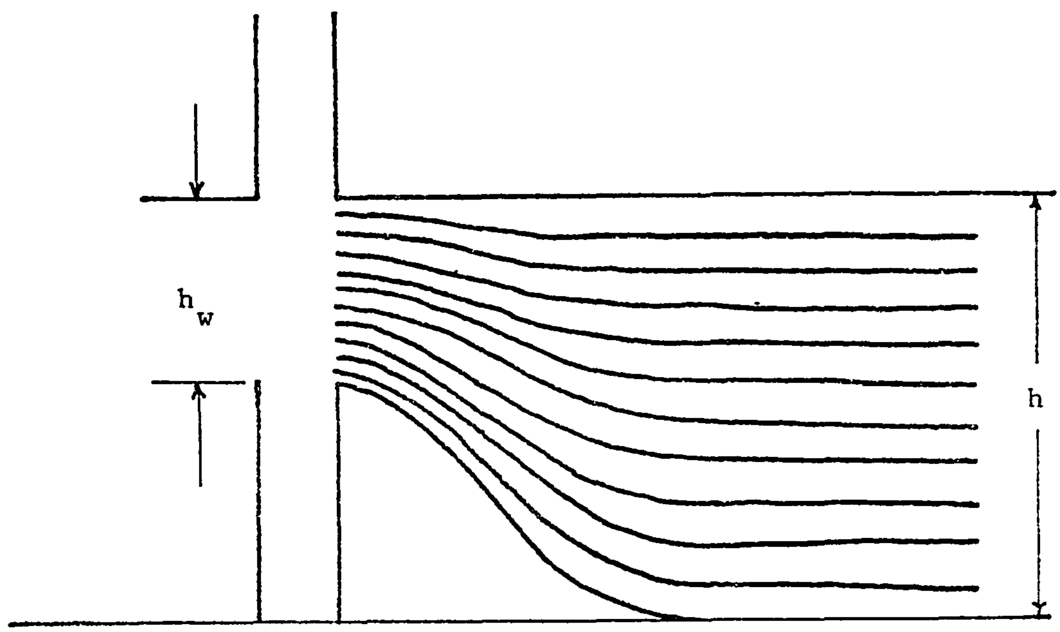


Fig. 15. Flow lines of a partially injecting well in a homogeneous medium.

penetrated part of the formation and consequently the flow rate per unit thickness, q/h, changes with depth. The integral therefore must be a function of the vertical position of the point of reference, which is expressed by $z_D = z/h$.

To express P_D as a function of the total (constant) rate of flow, which is the integrated value of the varying rates per unit thickness, an average value of the integral must be used. If the assumption that fluid enters the open interval at a constant flux is made, an unreal pressure distribution along the interval is caused. The problem faced, therefore, is to decide what pressure to take as the correct borehole pressure in the uniform flux solution. Muskat[21] was the first to point out this problem. He made a comparison between uniform flux and infinite conductivity line-source solutions at steady state. Infinite conductivity implies that the pressure in the wellbore is uniform and the integral of the flux over the open interval is equal to the constant specified rate. Muskat determined the exact infinite conductivity solution by breaking the well into discrete elements and adjusting their strength by trial and error until the potential at the well surface was found to be uniform, to as high an accuracy as desired. He concluded that the exact steady-state wellbore pressure for a partially penetrating well (infinite conductivity), could be obtained within 0.5 percent, assuming a uniform flux distribution at the wellbore, by computing the pressure at a location three-fourths of the distance from the top of the open interval to the bottom of the well. In another study, Gringarten and Ramey[11] computed an approximate infinite conductivity line-source unsteady-state solution and compared the results with the usual uniform flux line-source solution. Their results showed that the simpler uniform flux unsteady-state solution of the problem of a partially penetrating well yields the value of the transient wellbore pressure corresponding to the theoretical case of uniform wellbore potential, when computed at an effective average pressure point within the wellbore. This point is located about 70 percent of the well length from the reservoir impermeable boundary when the open interval is at the top or bottom of the formation, and about 70 percent of the well half-length from the center of the open interval if the well is open at the center of the formation.

It was decided to use the selected value of the integral at $z_D = .75b$ because a recent result for transient flow, obtained by Bilhartz and Ramey[2] agreed with Muskat's.

The calculation of the function U (t_D) and its subsequent integration was programmed for the IBM 370. Fig. 16 is an illustration of $U(t_D)$ vs t_D and Fig. (17) is a flow chart of the calculation procedure.

The basic reservoir data used to generate the dimensionless pressure vs dimensionless time Tables (4-8) are shown in Table 3. Details of the integration program are included in Appendix (D).

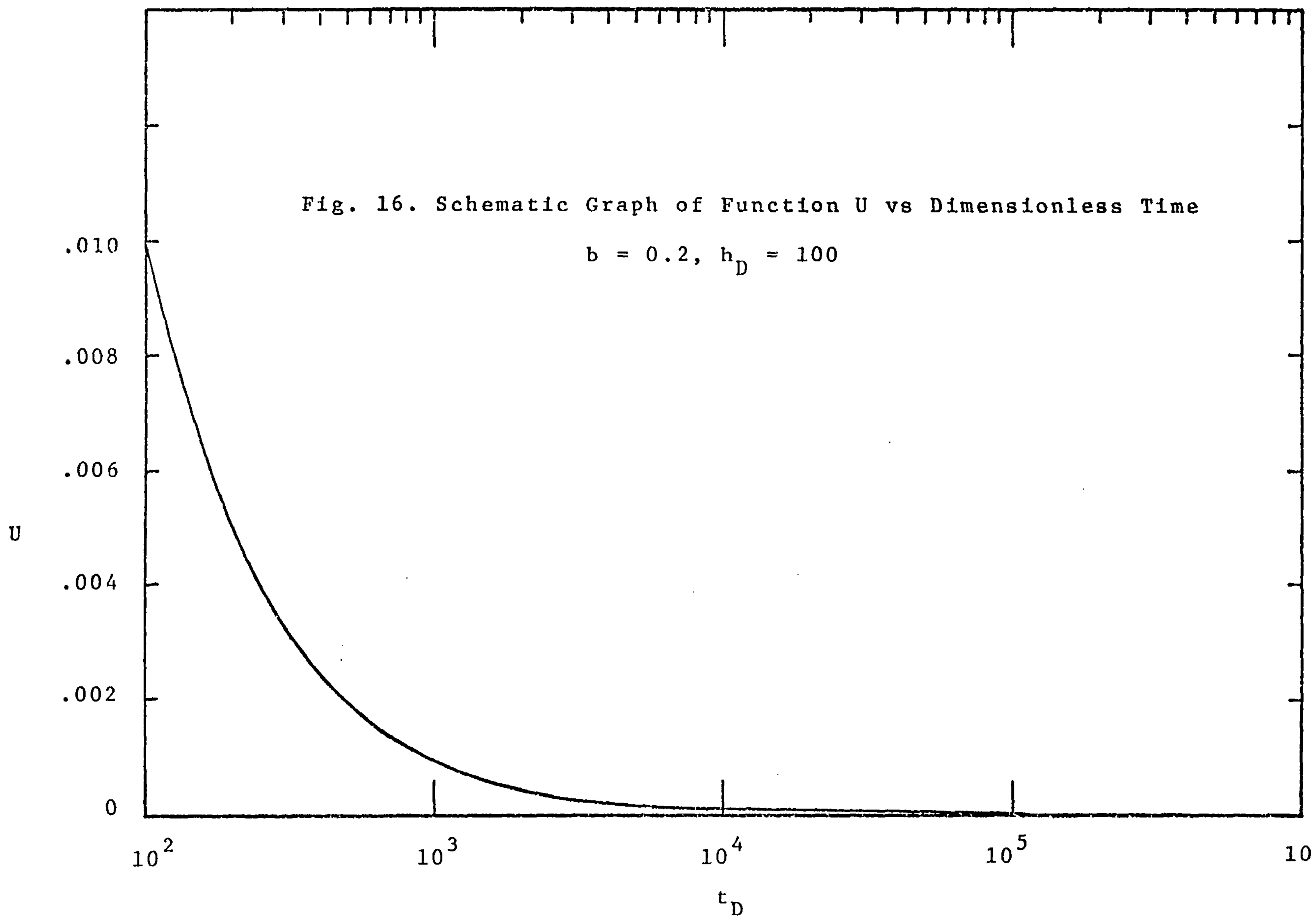


Fig. 16. Schematic Graph of Function U vs Dimensionless Time

$b = 0.2,\ h_D = 100$

TABLE 3

BASIC PARAMETERS USED TO GENERATE DIMENSIONLESS PRESSURE VS DIMENSIONLESS TIME FOR A PARTIALLY INJECTING WELL

K_r = 5, 20, 80, 500, 2000, 200000 md

Kv = 20 md

r_w = 0.25 ft.

h = 25 ft.

$\frac{h}{r_w}$ = 100

K_r/Kv	$(K_r/Kv)^{1/2}$	$h_D = \frac{h}{r_w} \left(\frac{K_r}{Kv}\right)^{1/2}$
0.25	0.5	50
1	1	100
4	2	200
25	5	500
100	10	1000
10000	100	10000

TABLE 4

DIMENSIONLESS PRESS VS DIMENSIONLESS TIME
FOR A PARTIALLY INJECTING WELL

	P_D (t_D, h_D, b)			h_D = 50
t_D	b = .2	b = .4	b = .6	b = .8
1E02	13.5356	6.7680	4.5119	3.3839
2	14.8718	7.4751	4.9837	3.7378
4	16.2238	8.1996	5.4675	4.1006
8	17.5850	8.9376	5.9607	4.4705
1E03	18.0243	9.1775	6.1212	4.5909
2	19.3910	9.9290	6.6244	4.9683
4	20.7585	10.6884	7.1336	5.3502
8	20.8249	10.7013	7.6479	5.4861
1E04	20.8742	10.9260	7.8583	5.6110
2	20.9108	11.0699	8.0275	6.0012
4	21.2574	11.2726	8.1187	6.3941
8	21.6039	11.4167	8.1957	6.7799
1E05	21.7155	11.5280	8.2741	6.8915
2	22.0621	11.8746	8.3207	7.2388
4	22.4087	12.2212	8.8582	7.5846
8	22.7553	12.5678	9.0275	7.9312
1E06	22.8668	12.6793	9.1143	8.0428
2	22.2134	13.0259	9.4719	8.3894
4	23.5599	13.3725	9.8185	8.7359
8	24.9065	13.7190	10.1551	9.0825
1E07	24.0181	13.8306	10.2657	9.1941

PRESS = PRESSURE

TABLE 5

DIMENSIONLESS PRESS VS DIMENSIONLESS TIME
FOR A PARTIALLY INJECTING
WELL

	P_D (t_D, h_D, b)			h_D = 100
t_D	b = .2	b = .4	b = .6	b = .8
1E02	13.5356	6.7678	4.5119	3.3839
2	14.9512	7.4756	4.9837	3.7378
4	16.3992	8.2012	5.4675	4.1006
8	17.8752	8.9411	5.9607	4.4705
1E03	18.3551	9.1818	6.1212	4.5909
2	19.8581	9.6366	6.6244	4.9683
4	20.3765	9.7005	7.1336	5.2503
8	20.9071	10.1108	7.3349	5.5061
1E04	21.4010	10.1221	7.8147	5.6310
2	21.9453	10.9024	7.8409	5.9112
4	22.2919	11.5883	7.8588	5.4041
8	22.6385	11.9237	8.2632	6.8899
1E05	22.7501	12.0353	8.3748	6.9035
2	23.0966	12.3819	8.7214	7.1481
4	23.4432	12.6285	9.0679	7.6146
8	23.7898	13.0751	9.4145	7.9412
1E06	23.9014	13.1867	9.5261	8.0548
2	24.2479	13.5332	9.8726	8.3994
4	24.5945	13.9807	10.2192	8.7459
8	24.9411	14.2253	10.5658	9.1025
1E07	25.0526	14.3379	10.6774	9.2104

TABLE 6

DIMENSIONLESS PRESS VS DIMENSIONLESS TIME
FOR A PARTIALLY INJECTING WELL

	P_D $(t_D,\ h_D,\ b)$		h_D = 200	
t_D	b = .2	b = .4	b = .6	6 = .8
1E02	13.5356	6.7680	4.5119	3.3839
2	14.9512	7.4756	4.9837	3.7378
4	16.4024	8.2012	5.4675	4.1006
8	17.8821	8.9411	5.9607	4.4705
1E03	18.3636	9.1818	6.1212	4.5909
2	17.8731	9.9366	6.6244	4.9683
4	21.4008	10.7005	7.1337	5.3503
8	22.9439	11.4722	7.6481	5.6261
1E04	23.4436	11.7221	7.8147	5.7410
2	23.4624	12.3998	8.4282	6.1312
4	23.5535	12.4910	8.5668	6.5241
8	23.6974	12.6349	8.7107	6.9099
1E05	23.8089	12.7464	8.8699	7.0235
2	24.1555	13.0930	9.1688	7.3681
4	24.5021	13.4396	9.5154	7.7146
8	24.8486	13.7862	9.9620	8.0612
1E06	24.9603	13.8977	9.8736	8.1748
2	25.3068	14.2443	10.3201	8.5194
4	25.6534	14.5908	10.6667	8.8659
8	25.999	14.9375	11.0133	9.2225
1E07	26.1115	15.0490	11.1282	9.3301

TABLE 7

DIMENSIONLESS PRESS VS DIMENSIONLESS TIME FOR A PARTIALLY INJECTING WELL

t_D	$P_D(t_D, h_D, b)$			$h_D = 500$
	$b = .2$	$b = .4$	$b = .6$	$b = .8$
1E02	13.5356	6.7680	4.5119	3.3839
2	14.9512	7.4756	4.9837	3.7378
4	16.4024	8.2012	5.4675	4.1006
8	17.8821	8.9411	5.9607	4.4705
1E03	18.3636	9.1818	6.1213	4.5909
2	19.8731	9.9366	6.6244	4.9683
4	21.4008	10.7005	7.1337	5.3503
8	22.9439	11.4722	7.6481	5.7361
1E04	23.4436	11.7221	7.8147	5.8610
2	24.0042	11.9024	8.3349	6.2512
4	25.5760	12.6883	8.8588	6.6441
8	25.9226	13.1774	9.1106	7.0299
1E05	26.0341	13.2889	9.2223	7.1415
2	26.3807	13.6355	9.5688	7.4881
4	26.7273	13.9821	9.9154	7.8346
8	27.0738	14.3286	10.2620	8.1812
1E06	27.1854	14.4402	10.3735	8.2928
2	27.5320	14.7868	10.7201	8.6294
4	27.8786	15.1334	11.0667	8.9859
8	28.2251	15.4799	11.4133	9.3325
1E07	28.3367	15.5915	11.5248	9.4441

TABLE 8

DIMENSIONLESS PRESS VS DIMENSIONLESS TIME
FOR A PARTIALLY INJECTING WELL

t_D	$P_D\ (t_D,\ h_D,\ b)$			$h_D = 1000$
	$b = .2$	$b = .4$	$b = .6$	$b = .8$
1E02	13.5356	6.7678	4.5119	3.3839
2	14.95117	7.4756	4.9837	3.7378
4	16.4024	8.2012	5.4675	4.1006
8	17.8821	8.9411	5.9607	4.4705
IE03	18.3636	9.1818	6.1212	4.5909
2	14.8731	9.9366	6.6244	4.9683
4	21.4008	10.7005	7.1337	5.3503
8	22.9439	11.4722	7.6481	5.7361
1E04	23.4436	11.7221	7.8147	5.8610
2	25.0042	12.5024	8.3349	6.2512
4	26.5760	13.2883	8.8588	6.6441
8	27.3191	13.9931	9.3560	7.0395
1E05	27.4307	14.1047	9.4626	7.1511
2	27.7772	14.4513	9.8091	7.4977
4	28.1238	14.7978	10.1557	7.8442
6	28.4704	15.1444	10.5023	8.1908
1E06	28.5819	15.2560	10.6138	8.3024
2	28.9285	15.6026	10.9604	8.6489
4	29.2751	15.94913	11.3070	8.9955
8	29.6217	16.2957	11.6536	9.3421
1E07	29.7333	16.4073	11.7651	9.4537

CHAPTER VII

DISCUSSION AND RESULTS

In Chapter VI, a mathematical model was devised to study the effect of permeability anisotropy and of limited entry to flow on the pressure distribution resulting from injecting a compressible liquid at a constant rate. The mathematical analysis was simplified by considering the case of a well located in an infinite reservoir of constant thickness, and it was assumed that this well is open over an interval adjacent to the upper (or lower) impereable boundary of the formation. A well injecting into an arbitrary interval within the bulk of the formation could be treated in a similar way.

The effect of taking permeability anisotropy into account was dealt with by a simple transformation of the coordinates. The equation describing the pressure distribution for a well in a porous medium of uniform but unequal horizontal and vertical permeabilities was reduced to the equation for an isotropic medium making the introduction of anisotropy equivalent to a contraction or expansion of the coordinates. Obtaining an analytical solution to the problem of a well partially injecting into an anisotropic medium was therefore greatly simplified.

It was not apparent how the combination of anisotropy and limited entry affected a pressure transient curve from an examination of equation (5.23), hence, numerical calculations were employed over a suitable range of values. The integral in equation (5.23) was calculated numerically. Synthetic dimensionless pressure - dimensionless time data were obtained from these calculations. Graphs 18 and 19 are semi-logarithmic presentation of P_D vs t_D for a few particular cases. The symbol in these graphs refers to a dimensionless injection pressure.

Limited Entry Effects

In Figure 18, a dimensionless pressure quantity based on the injection interval, h_w, is plotted vs dimensionless time. The abcissa is the product of b and P_D. It is noted that all of the curves of different injection ratios have two features in common:

(1) At early times the slope of the first semi-log straight line approaches that for 100 percent injection, and

(2) At late times the slope of the second straight line becomes very nearly b times the slope for 100 percent injection.

However, by plotting dimensionless pressure P_D vs dimensionless time t_D for a certain value of h_D and

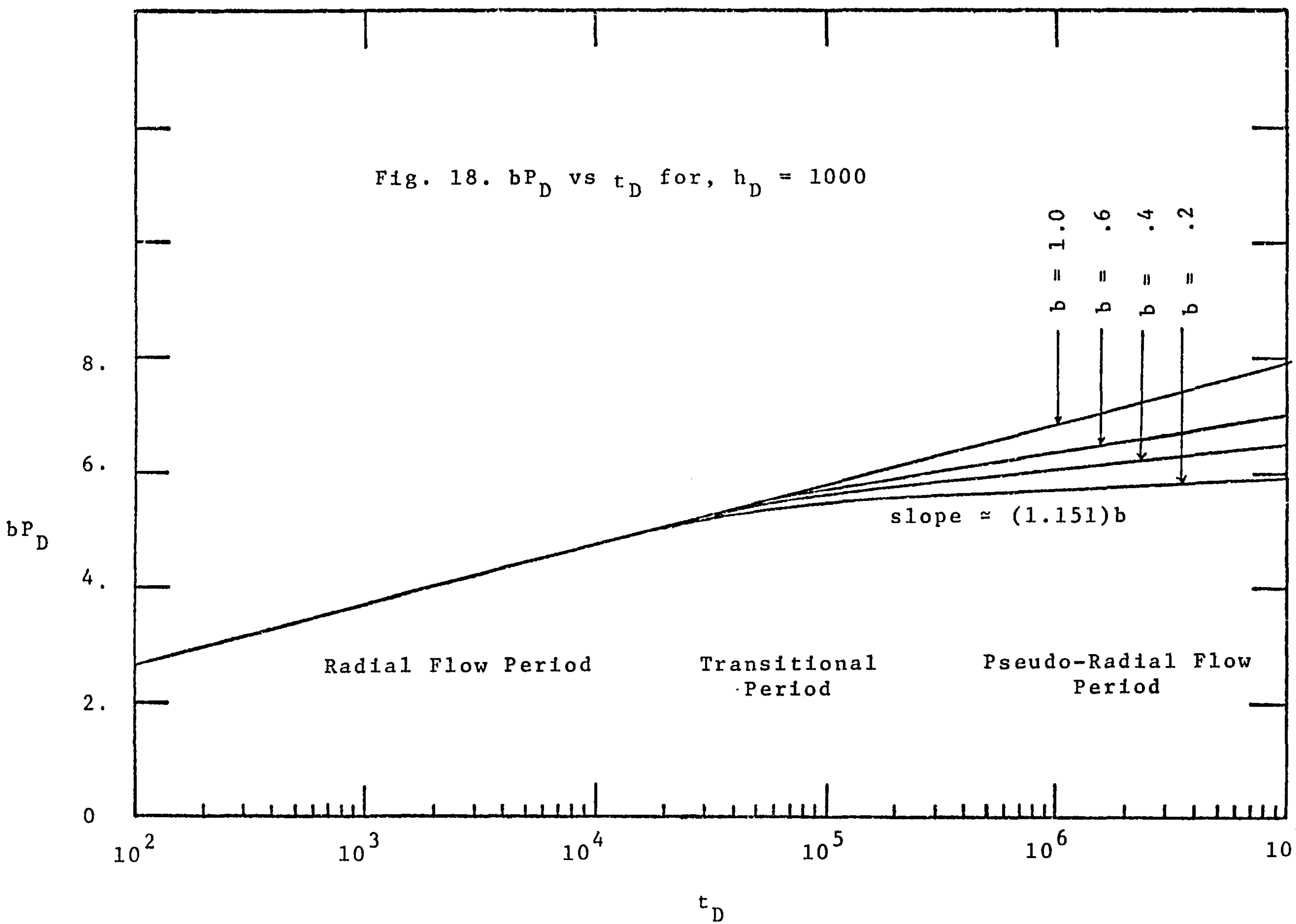


Fig. 18. bP_D vs t_D for, $h_D = 1000$

different injection ratios, Figure 19 illustrates that at early times the slope of the first semi-log straight line differ from the slope of the line obtained when the total formation thickness is effective to flow by the reciprocal of the injection ratio and at late times the slopes of the second semi-log straight line segments are equal to the slopes of total injection, but the curves for each injection ratio are parallel to the curve of total injection, i.e., each of the curves obtained for different injection ratios is displaced from the curve of total injection by a constant amount at late times. This constant pressure drop, pseudo-skin, S_b, may be caused by divergence of flow out of the open interval as seen in Fig. 15 where near the wellbore a smaller thickness of the formation is effective to flow and a lower transmissibility exists than further out in the reservoir. This also follows since the effect of both limited entry and an anisotropic formation would tend to create a hinderance to the flow of fluids into the formation. The pseudo-skin factor may be found by subtracting the usual semi-logarithmic form of the radial-flow dimensionless pressure at large dimensionless time from values in Tables 4 through 8.

In Table (9), the pseudo-skin factors for various values of partial injection ratios and dimensionless thickness are given. These values are shown graphically as Figures (20) and (21).

Fig. 19. P_D vs t_D for, h_D = 1000

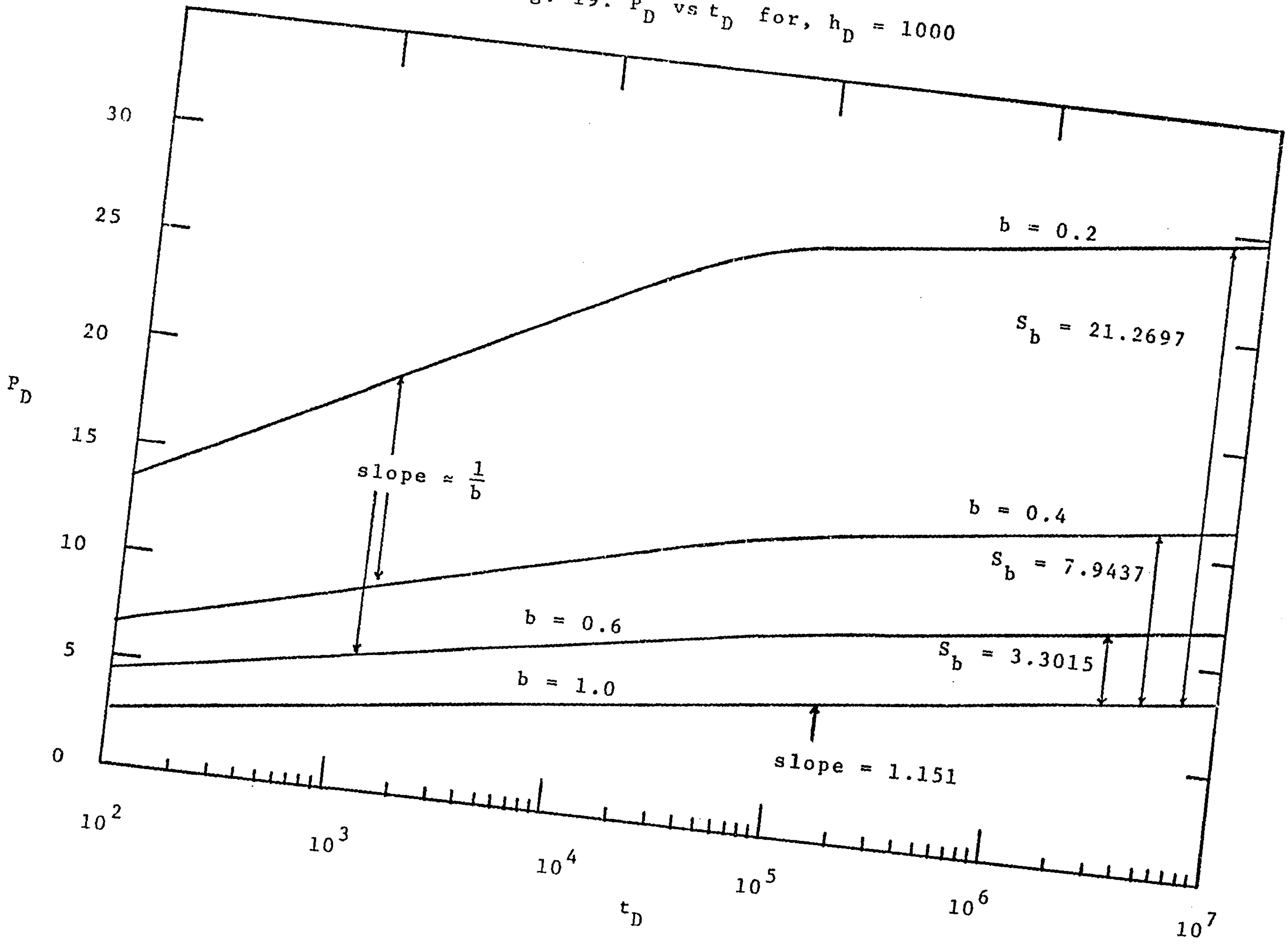


TABLE 9

PSEUDO-SKIN FACTOR FOR TABULATED b, h_D VALUES

h_D \ b	0.2	0.4	0.6	0.8
50	15.5545	5.3670	1.8021	.7305
100	16.5890	5.8743	2.2138	.7468
200	17.6479	6.5854	2.6646	0.8665
500	19.8731	7.1279	3.0612	0.9805
1000	21.2697	7.9437	3.3015	0.9901
5000	25.9166	9.5166	3.9785	1.1777
10,000	26.2208	9.6257	4.0112	1.1837

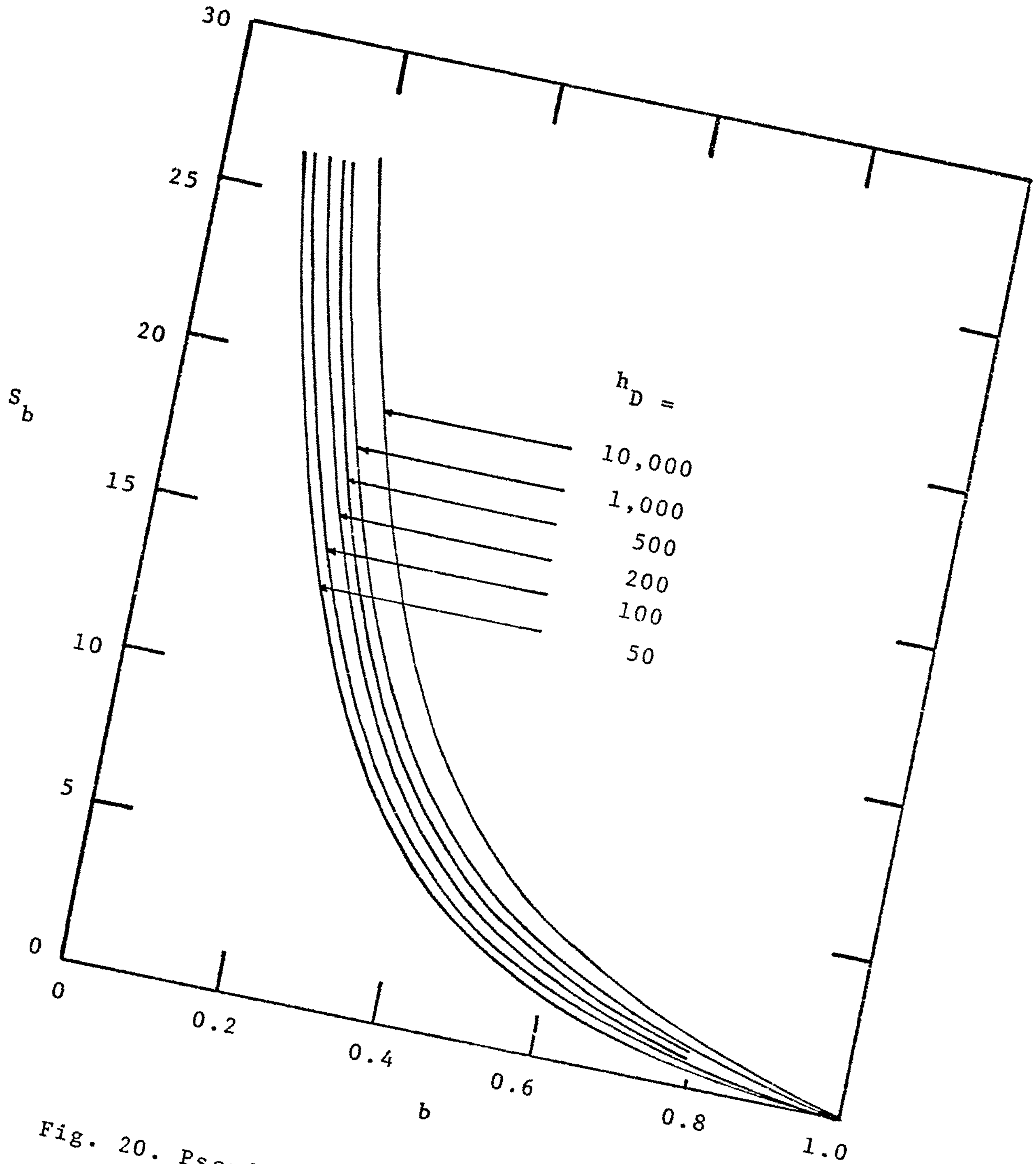
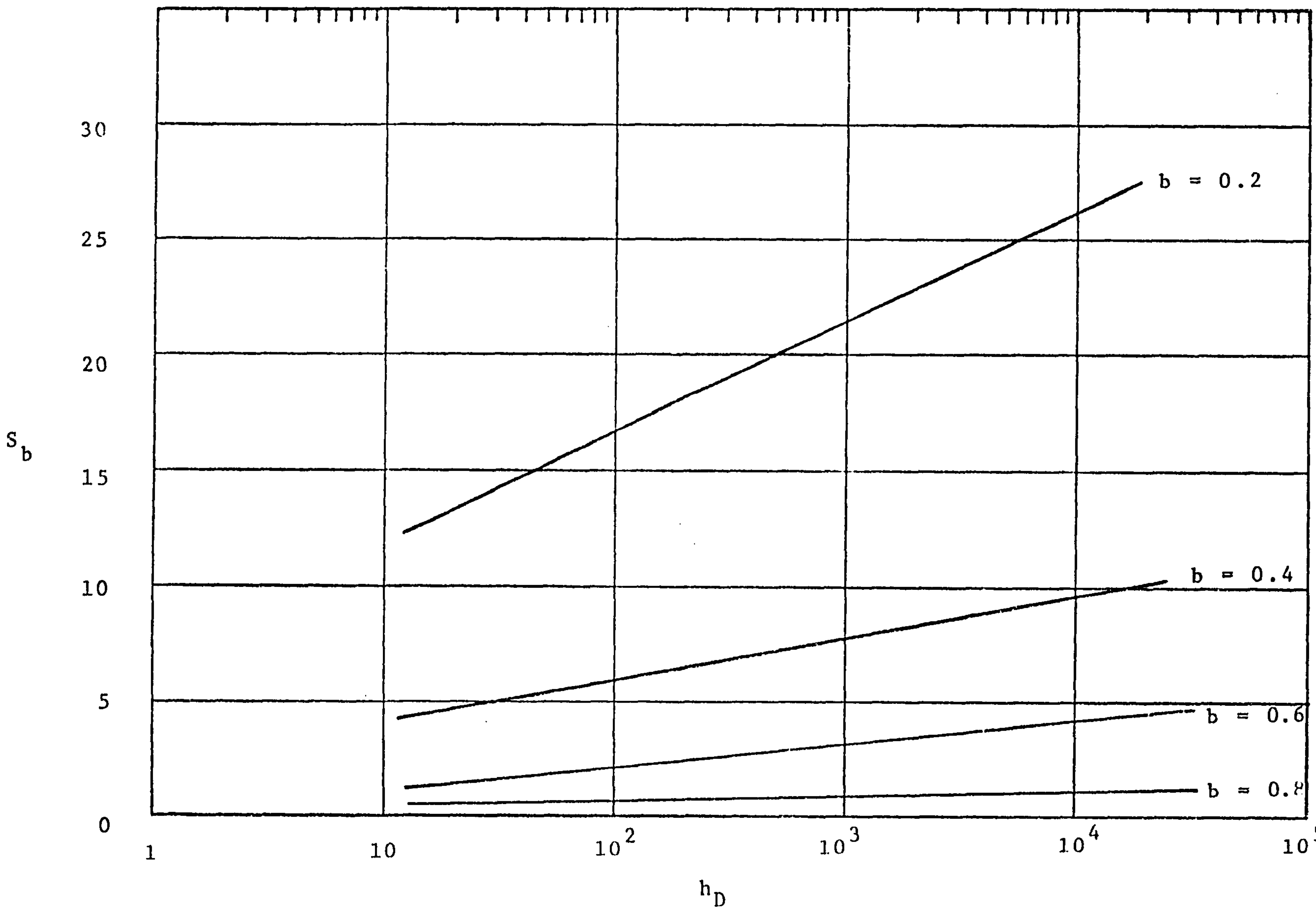


Fig. 20. Pseudo-Skin Factor vs Partial Injection Ratio.

Fig. 21. Pseudo-Skin Factor vs Dimensionless Thickness



Thus, it can be concluded that flow into the wellbore when only a fraction of the formation is open to flow produces a characteristic shape of pressure-time data. There exists two distinct flow periods: a radial flow period occurs at early time and after a period of transition, there is a pseudo-radial flow period. These flow periods are evident in Figure 18 where the curves for the different injection ratios merge because the apparent early radial flow period is the same for all injection ratios. The second semi-log straight line is reached when pseudo-radial flow is fully developed. It must also be pointed out that the time the second straight line begins decreases as the injection ratio increases.

In conclusion, limited entry to flow affects pressure transient curves such that at early times the slope of the first straight line, radial flow period, is inversely proportional to the injection interval thickness and formation horizontal permeability, $K_r h_w$. At the late times the slope of the second straight line, pseudo-radial flow period, is inversely proportional to the total formation thickness and the formation horizontal permeability, $K_r h$.

Effect of Permeability Anisotropy

The assumption of complete isotropy of petroleum reservoir strata is as much an idealization as that of strict permeability uniformity. While the consensus that permeability may vary very slightly in different directions in the planes of bedding, core analyses show that exact equality between the permeability in the bedding plane and normal to it is the exception rather than the rule.

When the flow is two-dimensional in the bedding plane, as in the case of completely injecting wells, the permeability normal to that plane does not enter the problem, whether or not the medium is isotropic as shown in chapter IV. On the other hand this permeability must be taken into account when there is a flow component normal to the bedding planes. The values of $K_r/K_z = 1$ and $K_z = 0$ correspond, respectively, to the case of an isotropic formation, and to the case of strict radial flow confined to the part of the formation actually penetrated by the well where limited entry effects would vanish and the well would behave as though the full formation thickness were h_w.

Thus the dimensionless thickness parameters, h_D, equation (5.20), represents an anisotropic effect. The limiting condition of $h_D \to \infty$ would correspond to zero vertical permeability. This dimensionless thickness influences the pressure-time response such that it determines the time at which the first radial flow period of the limited entry curve diverges from the total injection curve. It also determines the time at which the pseudo-radial flow period begins. Fig. (22) shows this observation. It is also observed in Fig. (22) that as h_D increases, i.e., vertical permeability decreases, the

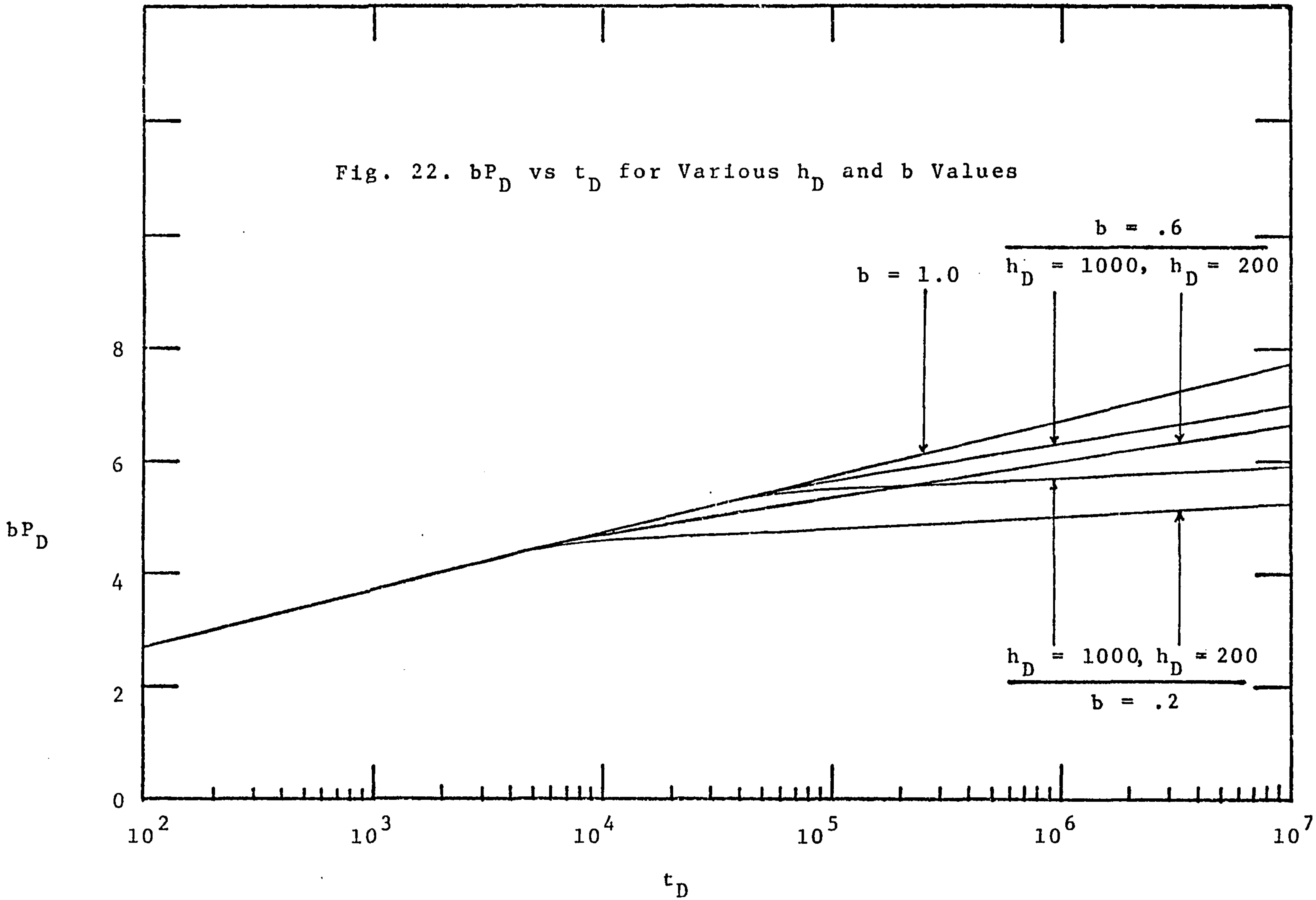


Fig. 22. bP_D vs t_D for Various h_D and b Values

transition between the two straight lines is delayed in time. It must also be noted that pseudo-skin effect, S_b, is also a function of h_D. These results verify Bilhartz's findings.

Determination of Skin Effects in a Well of Limited Entry

In chapter III, it was found that the presence of a low permeability skin around the wellbore results in a loss of injectivity. Since limited entry also contributes to the loss of injectivity, therefore, if pressure fall-off data obtained on a well with limited entry are used to establish the presence or absence of skin (i.e., formation damage), and a correction is not made for this loss of injectivity, the calculations would result in an erroneous skin value. They might indicate the presence of formation damage when there is none, or they might indicate a value larger than the true value. This could lead to an erroneous decision for planning remedial work.

From the above analysis of dimensionless pressure vs dimensionless time data for an injecting well the pressure response for the early time period is given by equation (4.11) where the slope is proportional to 1/b,

$$P_{wf}(t) - P_i = -70.6 \frac{q\mu B}{K_r h b} \ln\left(\frac{.445 r_w^2}{\zeta t}\right) \qquad (7.1)$$

where $P_{wf}(t)$ and t correspond to any point on the initial semi-log straight line.

If wellbore damage exists, the pressure drop due to skin will be related to the injection thickness, h_w.

$$(\Delta P)_s = 141.2 \frac{q\mu B}{K_r h_w}(s) \tag{7.2}$$

$$(\Delta \check{P})_s = 141.2 \frac{q\mu B}{K_r hb}(s) \tag{7.3}$$

Equation (7.1) becomes:

$$P_{wf} - P_i = -70.6 \frac{q\mu B}{K_r h b}\left(\ln \frac{.445 r_w^2}{\zeta t} - 2s\right) \tag{7.4}$$

$$P_{wf} - P_i = 162.6 \frac{q\mu B}{K_r h b}\left(\log \frac{K_r t}{\phi\mu c r_w^2} + .87s - 3.23\right) \tag{7.5}$$

The slope of the first semi-log straight line is:

$$m_1 = 162.6 \frac{q\mu B}{k_r hb} \tag{7.6}$$

If the formation thickness, h, is unknown, then

$$m_1 = 162.6 \frac{q\mu B}{k_r h_w} \tag{7.7}$$

The radial permeability can be calculated from:

$$k_r = 162.6 \frac{q\mu B}{m_1 h_w} \tag{7.8}$$

The skin effect can be calculated by rearranging equation (7.5),

$$s = 1.151\left(\frac{P_{wf}(t) - P_i}{m_1} - \log \frac{K_r t}{\phi\mu c r_w^2} + 3.23\right) \tag{7.9}$$

From a practical standpoint, the early slope will not always be observed since it occurs in most cases during the very first seconds, or possibly, minutes of a test. Further, the early slope is readily obscured by effects such as afterflow or physical wellbore damage. The late slope is normally used to calculate the total formation transmissibility.

As it was previously noted the slope of the late times second semi-log straight line, pseudo-radial flow period, is equal to the slope of a totally injecting well, but the curve is displaced by a constant amount. This constant pressure drop due to pseudo-skin effect, S_b, is:

$$(\Delta P)_{S_b} = 141.2 \frac{q\mu B}{K_r h} \left(S_b\right) \qquad (7.10)$$

Thus, the pressure response for the late time period is:

$$P_{wf}(t) - P_i = -70.6 \frac{q\mu B}{K_r h}\left(\ln \frac{.445 r_w^2}{\zeta t} - 2\ S_b\right) \qquad (7.11)$$

where P_{wf} (t) and t correspond to any point on the second semi-log straight line.

If wellbore damage exists, the pressure drop due to skin will still be related to the injection thickness h_w.

Equation (7.11) becomes:

$$P_{wf}(t) - P_i = 162.6 \frac{q\mu B}{K_r h}\left(\log \frac{K_r t}{\phi\mu c r_w^2} + .87 S_b + .87 \frac{s}{b} - 3.23\right) \qquad (7.12)$$

The slope of the second semi-log straight line is:

$$m_2 = 162.6 \frac{q\mu B}{k_r h} \qquad (7.13)$$

The radial permeability can be calculated from:

$$k_r = 162.6 \frac{q\mu B}{m_2 h} \qquad (7.14)$$

If both of the semi-log straight lines are present, then the partial injection ratio, b, can be obtained by combining equations (7.7) and (7.13).

$$\frac{m_2}{m_1} = \frac{h_w}{h} = b \qquad (7.15)$$

The skin effect can be calculated by rearranging equation (7.12):

$$s = 1.151(b)\left(\frac{P_{wf}(t) - P_i}{m_2} - \log \frac{K_r t}{\phi \mu c r_w^2} - .87S_b + 3.23\right) \qquad (7.16)$$

However, the skin effect is now a function of the pseudo-skin, S_b, which is a function of the injection ratio, b, and dimensionless radius h_D.

Determination of Vertical Permeability

At the stabilized state (often described as the semi-steady-state) the equation for the ideal flow into a well is:

$$q_i = \left(7.07*10^{-3}\right) \frac{K_r h \left(P_{wf} - P_i\right)}{\mu B \left(\ln \frac{r_e}{r_w} - .75\right)} \qquad (7.17)$$

The injectivity index is:

$$II = \frac{q_i \mu B}{P_{wf} - P_i} = \frac{7.07*10^{-3} K_r h}{\ln \frac{r_e}{r_w} - .75} \qquad (7.18)$$

Thus the impairment of injectivity due to partial injection effects and due to the difference in the vertical and horizontal permeabilities may be given by:

$$I = \frac{q}{q_i} = \frac{\ln \frac{r_e}{r_w} - .75}{\ln \frac{r_e}{r_w} - .75 + S_b} \qquad (7.19)$$

Using the dimensionless variables of Table (3), a generated set of tables and/or plots will give the ability to determine the dimensionless thickness easily and accurately.

The impairment ratio, I, can be calculated as a function of h_D and b for different drainage radii. The results are shown in Tables (10-12), and graphically in Figures (23-25).

METHOD OF CALCULATING VERTICAL PERMEABILITY AND SKIN FACTORS OF PARTIAL-FLUID-INJECTION WELLS

Before proposition of this method, it must be pointed out that limited entry to flow decreases well injectivity as does permeability anisotropy. Decrease in vertical permeability lowers I. As a limit when the vertical permeability is zero ($h_D \to \infty$), I becomes equal to b, where b is the fraction of thickness open to flow. Since some degree of anisotropy is the usual case, injectivities calculated on the basis of complete isotropy usually will be higher than actual injectivities. This is true since $K_z < K_r$ in most practical cases.

The terms needed in this simple approach are easily obtained. They are the wellbore radius, the radius of drainage, the formation thickness, the thickness of the interval open to flow, the actual injection rate and the ideal injection rate. In order to calculate the vertical permeability, the injection ratio can be calculated from the ratio of the slopes of the early and late straight

TABLE 10

IMPAIRMENT RATIO I AS A FUNCTION OF b, h_D, $\frac{r_e}{r_w}$ = 1320

h_D	b	S_b	$\frac{r_e}{r_w}$	$\ln \frac{r_e}{r_w}$	$\ln \frac{r_e}{r_w} - .75$	$\ln \frac{r_e}{r_w} - .75 + S_b$	I
50	.2	15.5545	1320	7.1854	6.4354	21.9899	.2927
	.4	5.3670				11.8024	.5453
	.6	1.8021				8.2375	.7812
	.8	.7305				7.1659	.8981
100	.2	16.5890				23.0244	.2795
	.4	5.8743				12.3097	.5228
	.6	2.2138				8.6492	.7440
	.8	.7468				7.1822	.8960
200	.2	17.6479				24.0833	.2672
	.4	6.5854				13.0208	.4942
	.6	2.6646				9.100	.7072
	.8	.8665				7.3019	.8813
500	.2	19.8731				26.3085	.2446
	.4	7.1279				13.5633	.4745
	.6	3.0612				9.4966	.6777
	.8	.9805				7.4159	.8678
1000	.2	21.2697				27.7051	.2323
	.4	7.9437				14.3791	.4476
	.6	3.3015				9.7369	.6609
	.8	.9901				7.4255	.8667
10000	.2	26.2208				32.6562	.1971
	.4	9.6257				16.0611	.4007
	.6	4.0112				10.4466	.6160
	.8	1.1837				7.6191	.8446

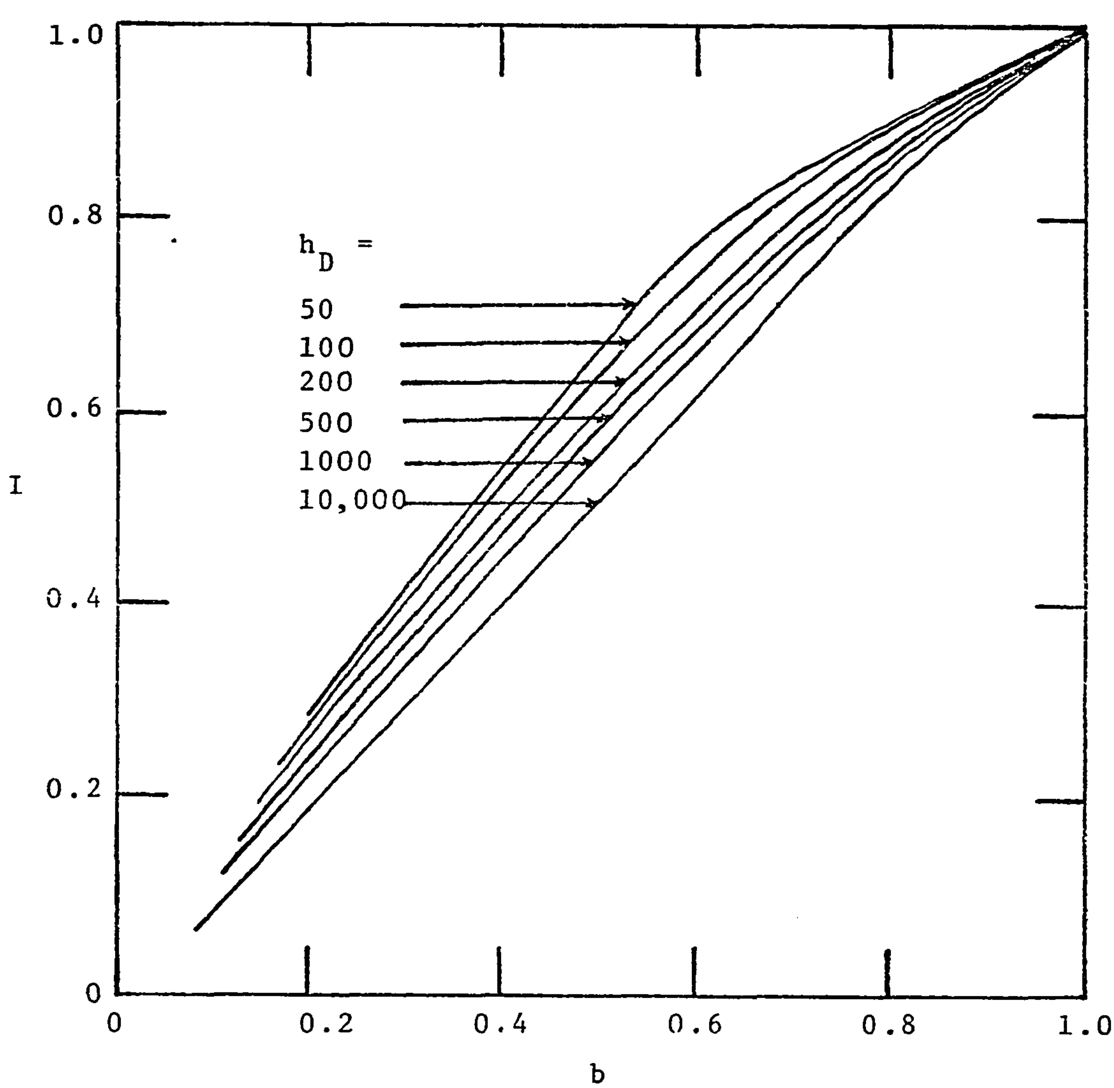


Fig. 23. Impairment Ratio I as a Function of b, h_D, $\frac{r_e}{r_w} = 1320$

TABLE 11

IMPAIRMENT RATIO I AS A FUNCTION OF b, h_D, $\frac{r_e}{r_w} = 2640$

h_D	b	S_b	$\frac{r_e}{r_w}$	$\ln\frac{r_e}{r_w}$	$\ln\frac{r_e}{r_w} - .75$	$\ln\frac{r_e}{r_w} - .75 + S_b$	I
50	.2	15.5545	2640	7.8785	7.1285	22.6830	.3143
	.4	5.3670				12.4953	.5705
	.6	1.8021				8.9306	.7982
	.8	.7305				7.8590	.9070
100	.2	16.5890				23.7175	.3006
	.4	5.8743				13.0028	.5482
	.6	2.2138				9.3423	.7630
	.8	.7468				7.8753	.9052
200	.2	17.6479				24.7764	.2877
	.4	6.5854				13.7139	.5198
	.6	2.6646				9.7931	.7279
	.8	.8665				7.9950	.8916
500	.2	19.8731				27.0016	.2640
	.4	7.1279				14.2564	.5000
	.6	3.0612				10.1897	.6996
	.8	.9805				8.1090	.8791
1000	.2	21.2697				28.3982	.2510
	.4	7.9437				15.0722	.4730
	.6	3.3015				10.4301	.6835
	.8	.9901				8.1186	.8780
10000	.2	26.2208				33.3493	.2138
	.4	9.6251				16.7542	.4255
	.6	4.0112				11.1397	.6399
	.8	1.1837				8.3122	.8576

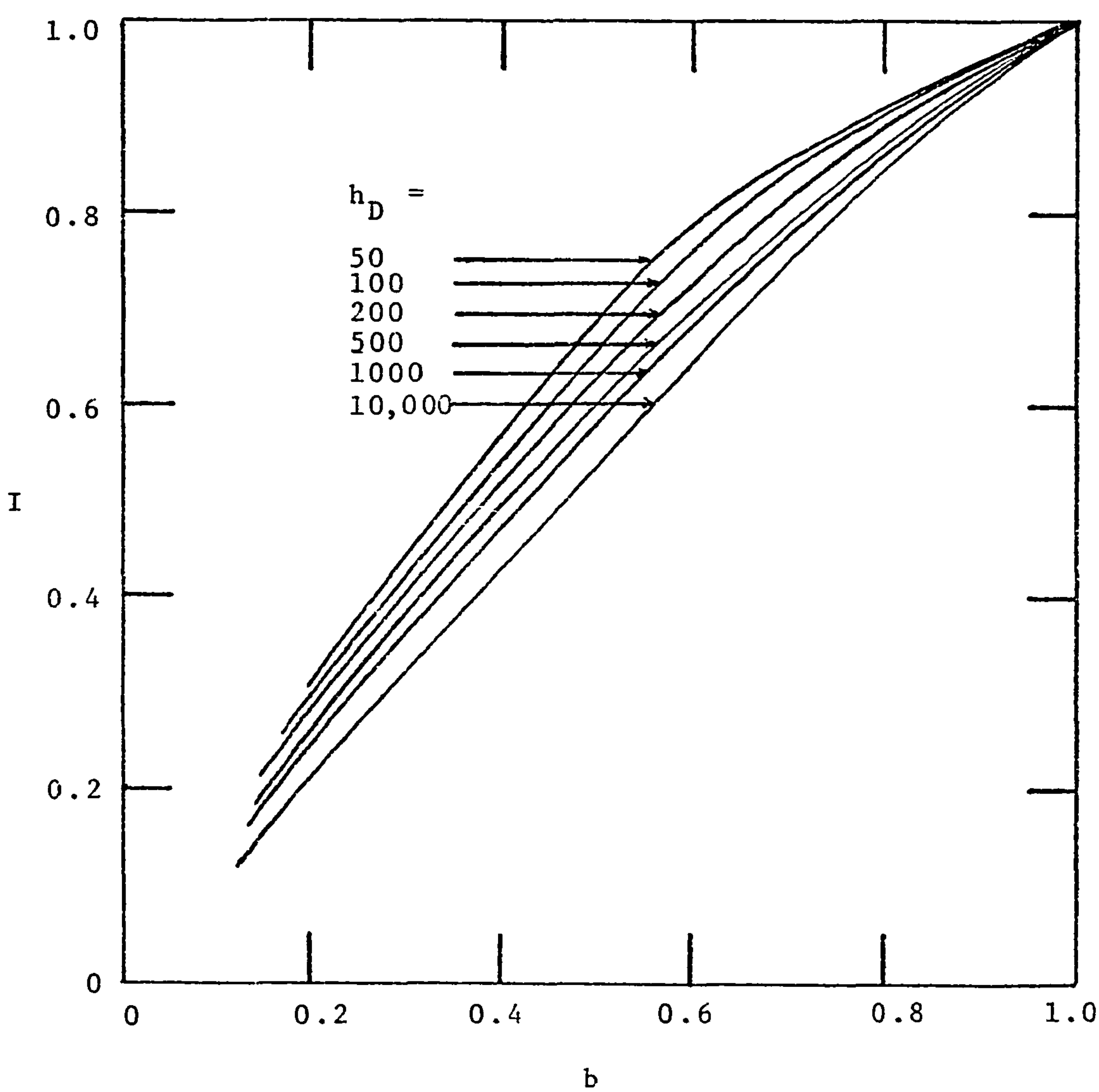


Fig. 24. Impairment Ratio I as a Function of b, h_D, $\frac{r_e}{r_w}$ = 2640

TABLE 12

IMPAIRMENT RATIO I AS A FUNCTION OF b,

h_D, $\frac{r_e}{r_w} = 8000$

h_D	b	S_b	$\frac{r_e}{r_w}$	$\ln\frac{r_e}{r_w}$	$\ln\frac{r_e}{r_w}$ -.75	$\ln\frac{r_e}{r_w}$ .75+S_b	I
50	.2	15.5545	8000	8.9872	8.2372	23.7917	.3462
	.4	5.3670				13.6042	.6055
	.6	1.8021				10.0393	.8205
	.8	.7305				8.9677	.9185
100	.2	16.5890				24.8262	.3318
	.4	5.8743				14.1115	.5837
	.6	2.2138				10.4510	.7882
	.8	.7468				8.9840	.9169
200	.2	17.6479				25.8851	.3182
	.4	6.5854				14.8226	.5557
	.6	2.6646				10.9018	.7556
	.8	.8665				9.1037	.9048
500	.2	19.8731				28.1103	.2930
	.4	7.1279				15.3651	.5361
	.6	3.0612				11.2984	.7291
	.8	.9805				9.2177	.8936
1000	.2	21.2697				29.5069	.2792
	.4	7.9437				16.1809	.5091
	.6	3.3015				11.5387	.7139
	.8	.9901				9.2273	.8927
10000	.2	26.2708				34.4580	.2391
	.4	9.6257				17.8629	.4611
	.6	4.0112				12.2484	.6725
	.8	1.1837				9.4209	.8744

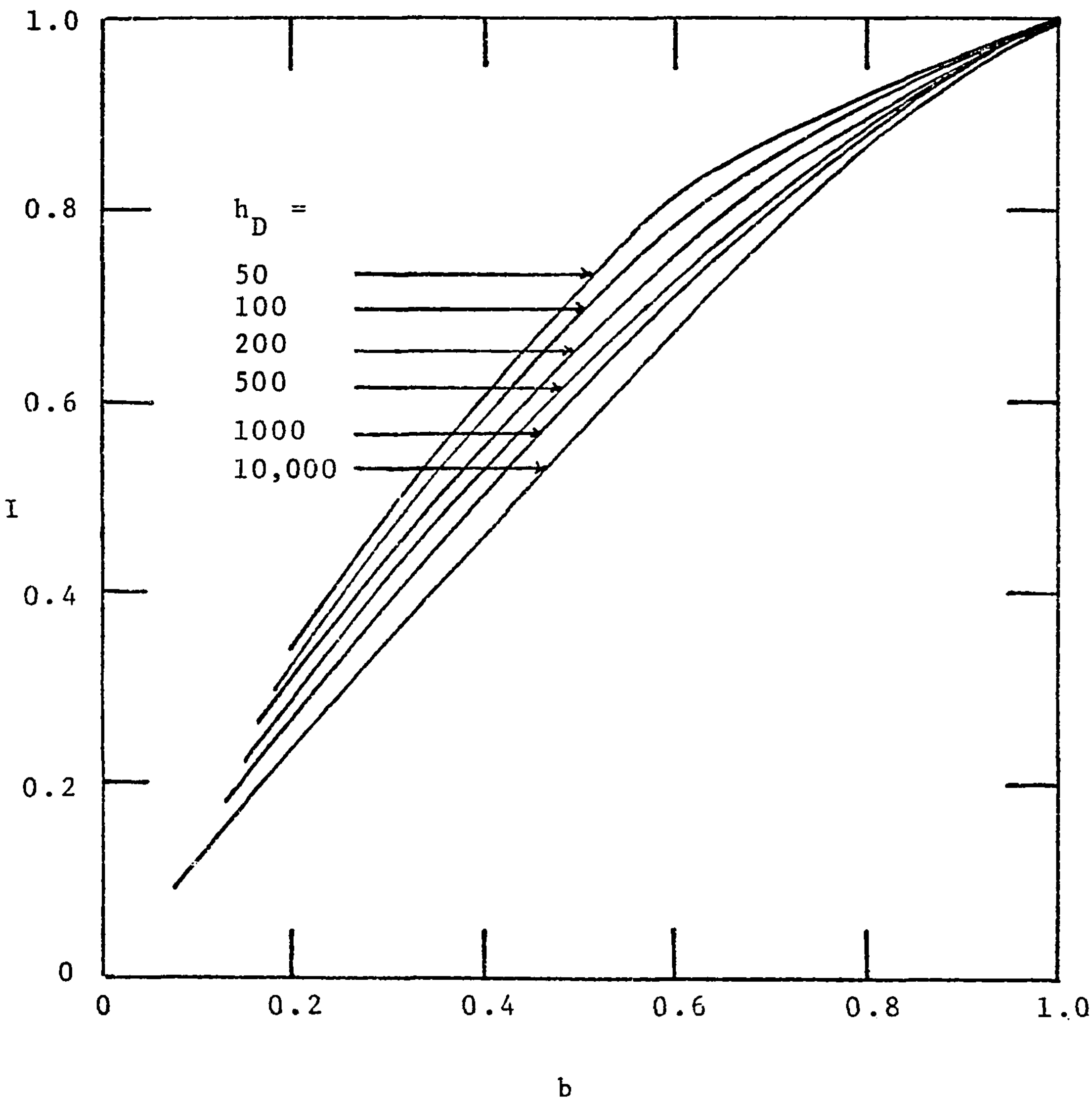


Fig. 25. Impairment Ratio I as a Function of b, h_D, $\frac{r_e}{r_w} = 8000$

lines if both are present or estimated as shown below. The radial permeability can be calculated from either straight line portions.

Fig. 27 is a flow chart outlining the proposed method. Based on this outline an example calculation will be used to illustrate the use of this method.

Partial Injection Ratio Calculation

In studying the effect of partial penetration, the mathematical derivation was based on the assumption that the part of the open hole is near the top of the formation, such as in the case of partial injection with open hole completion. However, by making use of symmetry consideration, the methods and curves can also be used if the open interval is somewhere else in the well[4].

1. Figure 26a represents the case of a well which penetrates a fraction h_w = 20 ft. of a formation of thickness h = 100 ft.

$$b = \frac{h_w}{h} = \frac{20}{100} = 0.2$$

2. In Figure 26b, the wellbore is open to the formation over an interval of 20 feet in the middle of the formation. Using symmetry consideration, it is seen that the same theory applies if the effective h is half the total formation thickness and consequently,

$$b = \frac{10}{50} = 0.2$$

3. In Figure 26c, the wellbore is open to the

formation in five intervals, totaling 20 feet. Symmetry considerations again show that the same theory applies if the effective h is half the total sand thickness divided by the number of intervals, and consequently,

$$b = \frac{2}{10} = 0.2$$

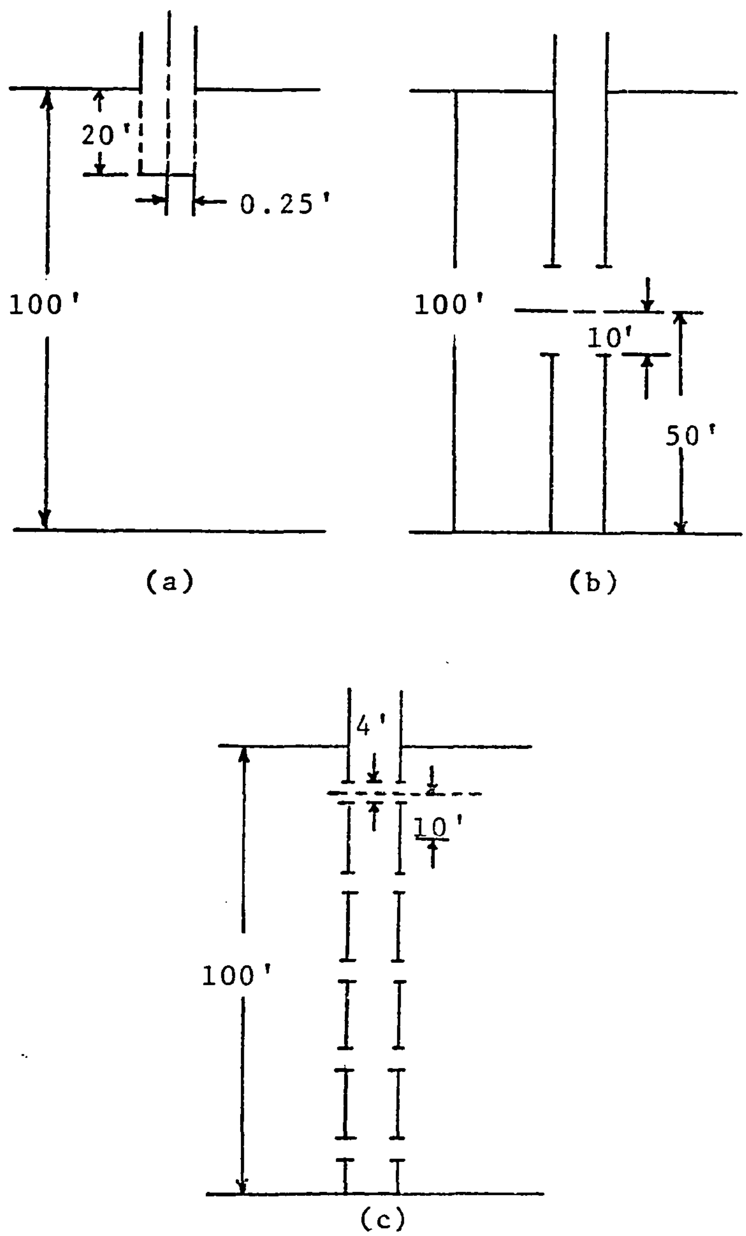


Fig. 26 (a). Partial injection in top of formation.

(b). Partial injection in center of formation.

(c). Partial injection at various intervals in the formation.

Calculate radial permeability, K_r, from the semi-log straight line portion of the fall-off curve using equation (7.13)

↓

Calculate the impairment Ratio, I.

↓

Determine the value of partial injection Ratio, b.

↓

Determine the value of the dimensionless thickness, h_D, from Figures 23-25.

↓

Calculate the value of vertical permeability, Kv from equation (5.20)

↓

Determine the value of the pseudo-skin factor, S_b, from Figures 20-21.

↓

Calculate the skin factor, s, from equation (7.16)

Fig. 27. Flow Chart Outlining the Method of Calculating Vertical Permeability and Skin Factor of Partial-Fluid-Injection Wells.

EXAMPLE CALCULATION

A water injection well has been injecting at constant rate; q = 103 BPD, then it was shut-in for pressure fall-off analysis.

Data

$P_{wf,s}$ = 2200 psi, P_i = 1200 psi, μ = 1.0 cp

B = 1.05, $c = 3.*10^{-6}\ psi^{-1}$, h_w=20', h=40'

r_w = 0.25', r_e = 660', ϕ = .18, q = 103 BPD

Procedure

1. From the fall-off curve, the absolute value of the slope of the straight line segment,

$$m_2 = 84.5$$

2. Using equation(7.14):

$$K_r = 162.6\ \frac{q\mu B}{m_2 h}$$

$$K_r = \frac{(162.6)(103)(1.)(1.05)}{(84.5)(40)} = 4.55 \text{ md}$$

3. Using equation(7.17):

$$q_i = \frac{(7.07)(10^{-3})(4.55)(40)(2200-1200)}{(1.05)(1.)(\ln 2640-.75)}$$

$$= 172 \text{ BBL/D}$$

4. $$I = \frac{q}{q_i} = \frac{103}{172} = .6$$

From Fig. 24, we obtain h_D = 500

5. The vertical permeability can be calculated using equation(5.20):

$$h_D = \frac{h}{r_w} \sqrt{\frac{K_r}{Kv}}$$

$$500 = \frac{40}{.25} \sqrt{\frac{4.55}{Kv}}$$

$$Kv = 0.47 \text{ md}$$

6. From Fig. 20 we obtain the amount of pseudo-skin

$$S_b = 4$$

7. Using the fall-off curve, at shut-in time of (10) hours P_{wf} from the straight line portion is 2000 psi. Thus we can now calculate the skin from equation 7.16.

$$s = (1.151)(.5)\left(\frac{2000-1200}{84.5} - \log \frac{(4.55)(10)}{(.18)(1.)(3*10^{-6})(.25)^2} - (.87)(4) + 3.23\right)$$

$$s = +0.0505$$

8. If we increase the ratio from 0.5 to 0.75, without further damage to the wellbore, therefore, from Fig. 24:

$$I = .79$$

and $q = (.79)\ q_i = (.79)(172) = 135.88$ BBL/D

Thus, the injection rate can be increased by

$$\frac{135.88 - 103}{103} * 100 = 32\%$$

CONCLUSIONS

The work represented here may be summarized as follows:

1. From the mathematical analysis of the pressure behavior of an injection well in a liquid-filled unit-mob- ility- ratio reservoir, it has been demonstrated that the reservoir pressure, the potential injection capacity of the formation and the degree of formation damage in the immediate vicinity of the wellbore can be estimated.

2. A mathematical model was derived and numerical calculations were used to study the effect of anisotropy on the pressure behavior of wells with partial-fluid-injection.

3. From the theoretical results of this study, partial injection produces a characteristic shape of the pressure transient curve which differs from the classic form, and this difference may be utilized to estimate the partial injection ratio. However, it does not appear feasible to estimate the amount of partial injection from actual fall-off curves due to skin and storage effects.

4. The effect of partial injection is similar to that of wellbore damage. This additional pressure drop

may be estimated by procedure outlined in this study.

5. The amount of wellbore damage may also be approximated for both isotropic and anisotropic media by procedures outlined in this study.

6. Injectivity impairment chart similar to productivity impairment chart of Brons and Marting was extended to include the effect of anisotropy.

7. A simple method of determining vertical permeability in partially injecting wells is proposed. This method utilized information from pressure transient testing and injectivity impairment charts.

NOMENCLATURE

B = formation volume factor, rub/STB

b = partial injection ratio

c = isothermal compressibility, psi^{-1} or atm^{-1}

h = formation thickness, ft or cm

h_w = length of line source or perforated thickness, ft or cm

h_D = dimensionless thickness

II = injectivity index, BPD/psi or $\frac{cm^3/sec}{atm}$

I = impairment ratio

K = permeability, md or μm^2

K_r = radial permeability, md or μm^2

K_x = permeability in the x-direction, md or μm^2

K_y = permeability in the y-direction, md or μm^2

K_z = vertical permeability, md or μm^2

Kv = vertical permeability, md or μm^2

$\ln$ = natural logarithm, base e

m_1 = slope of early-time semi-log straight line

m_2 = slope of late-time semi-log straight line

P = pressure, psi or atm

P_D = dimensionless pressure

q = surface flow rate, cm^3/sec or STB/D

r = radial direction, ft or cm

r_w = wellbore radius, ft or cm

r_e = external boundary radius, ft or cm

r_D = dimensionless radius

s = true skin effect

S_b = pseudo skin effect

t = time, sec or hr or days

t_D = dimensionless time

x,y,z = Cartesian coordinates

z = vertical direction

ϕ = fractional porosity

μ = viscosity, cp

ζ = hydraulic diffusivity, $\frac{0.000264K}{\phi\mu c}$ in $\frac{ft^2}{hr.}$

ξ = a variable of integration

Π = 3.14159

e = 2.71828

E_i = exponential function, equation (3.5), Table 1

erf = error function, Table 2

* = multiplication sign

BIBLIOGRAPHY

1. Agarwal, R. G.; Al-Hussainy, R.; and Ramey, H. J., Jr. "An Investigation of Wellbore Storage and Skin Effect in Transient Liquid Flow--I. Analytical Treatment." SPEJ. September 1970, p. 279.

2. Bilhartz, H. L., Jr., and Ramey, H. J., Jr. "The Combined Effects of Storage, Skin, and Partial Penetration on Well Test Analysis." Preprint of Paper SPE 6753 presented at 52nd Annual Fall Meeting of SPE, (Oct. 9-12, 1977) in Denver.

3. Brons, F., and Marting, V. E. "The Effect of Restricted Fluid Entry on Well Productivity." Preprint of Paper 1322-G presented at 34th Annual Fall Meeting of SPE (Oct. 4-7, 1959) in Dallas.

4. Brons, F., and Marting, V. E. "The Effect of Restricted Fluid Entry on Well Productivity." JPT, Feb. 1961, pp. 172-174.

5. Burns, W. A., Jr. "New Single-Well Test for Determining Vertical Permeability." JPT, June 1969, pp. 743-752; Trans. AIME, v. 246.

6. Carslaw, H. S., and Jaeger, J. C. Conduction of Heat in Solids. London: Oxford Univ. Press, 1947.

7. Candle, B. H. Fundamentals of Reservoir Engineering. SPE, AIME, 1967.

8. Clegg, M. W., and Mills, M. "A Study of the Behavior of Partially Penetrating Wells." SPEJ. June 1969, pp. 189-203; Trans. AIME, v. 246.

9. Gringarten, A. C., and Ramey, H. J., Jr. "Unsteady State Pressure Distributions Created by a Well With a Single Horizontal Fracture, Partial Penetration, or Restricted Entry." SPEJ. August 1974, pp. 413-426; Trans. AIME, v. 257.

10. Gringarten, A. C., and Ramey, H. J., Jr. "The Use of Source and Green's Functions in the Solution of Unsteady Flow Problems in Reservoirs." SPEJ. October 1973, pp. 285-296.

11. Gringarten, A. C., and Ramey, H. J., Jr. "An Approximate Infinite Conductivity Solution for a Partially Penetrating Line-Source Well." SPEJ, April 1975, p. 140.

12. Groeneman, A. R., and Wright, F. F. "Analysis of Fluid Input Wells by Shut-In Pressures." JPT, July 1956, pp. 21-24.

13. Hantush, M. Advances in Hydroscience, Ven Te Chow, ed., New York and London: Academic Press (1964), p. 307.

14. Hawkins, M. F., Jr. "A Note on the Skin Effect." Trans. AIME (1956), v. 207, pp. 356-357.

15. Hazebroek, P.; Rainbow, H.; and Matthews, C. S. "Pressure Fall Off in Water Injection Wells." Trans. AIME (1958), v. 213, pp. 250-260.

16. Joers, J. C, and Smith, R. V. "Determination of Effective Formation Permeabilities and Operation Efficiencies of Water-Input Wells." Producers Monthly, October 1954, pp. 21-28.

17. Kazemi, H., and Seth, M. S. "Effect of Anisotropy and Stratification Pressure Transient Analysis of Wells with Restricted Flow Entry." JPT, May 1969, pp. 639-647.

18. Matthews, C.S., and Russell, D. G. Pressure Buildup and Flow Tests in Wells. Dallas:SPE Monograph Series, (1967), 1.

19. Miller, O. C.; Dyes, A. B.; and Hutchinson, C. A., Jr. "The Estimation of Permeability and Reservoir Pressure from Bottom-Hole Pressure Buildup Characteristics." JPT, April 1950, v. 2, No. 4, pp. 91-104.

20. Mueller, T. D., and Witherspoon, P. A. "Pressure Interference Effects Within Reservoirs and Aquifers." Trans. AIME (1965), v. 234, p. 471.

21. Muskat, M. Physical Principles of Oil Production. New York: McGraw-Hill Book Co., Inc. (1948).

22. Nisle, R. G. "The Effect of Partial Penetration on Pressure Buildup in Oil Wells." Trans. AIME, v. 213 (1958), pp. 85-90.

23. Nowak, T. J., and Lester, G. W. "Analysis of Pressure Fall-Off Curves Obtained in Water Injection Wells to Determine Injective Capacity and Formation Damage." Trans. AIME, v. 204, (1955), pp. 96-102.

24. Odeh, A. A. "Steady-State Flow Capacity of Wells with Limited Entry to Flow." SPEJ. March 1968, pp. 43-51.

25. Odeh, A. A. "Pseudo Steady-State Flow Capacity of Oil Wells with Limited Entry and with an Altered Zone Around the Wellbore." Prepring of Paper 6132 presented at 51st Annual Fall Meeting of SPE (Oct. 3-6, 1976) in New Orleans.

26. Prats, M. "A Method for Determining the Net Vertical Permeability Neon a Well from In-Situ Measurements." JPT, May 1970. pp. 637-643; Trans. AIME, v. 249.

27. Raghavan, R., and Clark, K. K. "Vertical Permeability from Limited Entry Flow Tests in Thick Formation." SPEJ. February 1975, pp. 65-73.

28. Raghavan, R. Modern Well Test Analysis - The Log-Log Type Curve Approach. The Mid-Continent Section, Well Pressure Test Analysis. SPE-AIME, Continuing Education Course No. 9, 1975.

29. Street, R. L. Analysis and Solution of Partial Differential Equations. Monterey, California: Brooks/Cole Publishing Co. (1973).

30. Seth, M. S. "Unsteady State Pressure Distribution in a Finite Reservoir with Partial Wellbore Opening." JCPT, v. 7, October-December 1968, pp. 153-163.

31. Van Everdingen, A. F. "The Skin Effect and Its Influence on the Productive Capacity of a Well." JPT, v. 5, no. 6, June 1953, pp. 171-176.

APPENDIX A

PRACTICAL UNITS USED IN THIS STUDY

Variable	Units
c	vol/vol/psi
ϕ	fraction
h	ft
K	md
μ	cp
P	psia
q	STB/day
t	days

APPENDIX B

TABLE 1

The exponential integral $-E_i(-x)$ vs x (after Raghavan[28])

u	$\times 10^{-13}$	$\times 10^{-12}$	$\times 10^{-11}$	$\times 10^{-10}$	$\times 10^{-9}$	$\times 10^{-8}$	$\times 10^{-7}$	$\times 10^{-6}$	$\times 10^{-5}$	$\times 10^{-4}$	$\times 10^{-3}$	$\times 10^{-2}$	$\times 10^{-1}$	$\times 10^{-0}$
1.0	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
1.1	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
1.2	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
1.3	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
1.4	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
1.5	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
1.6	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
1.7	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
1.8	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
1.9	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
2.0	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
2.1	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
2.2	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
2.3	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
2.4	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
2.5	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
2.6	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
2.7	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
2.8	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
2.9	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
3.0	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
3.1	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
3.2	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
3.3	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
3.4	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
3.5	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
3.6	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
3.7	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
3.8	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
3.9	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
4.0	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
4.1	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
4.2	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
4.3	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
4.4	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
4.5	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
4.6	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
4.7	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
4.8	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
4.9	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
5.0	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
5.1	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
5.2	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
5.3	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
5.4	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
5.5	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
5.6	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
5.7	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
5.8	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
5.9	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
6.0	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
6.1	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
6.2	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
6.3	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
6.4	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
6.5	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
6.6	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
6.7	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
6.8	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
6.9	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
7.0	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
7.1	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
7.2	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
7.3	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
7.4	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
7.5	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
7.6	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
7.7	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
7.8	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
7.9	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
8.0	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
8.1	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
8.2	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
8.3	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
8.4	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
8.5	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
8.6	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
8.7	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
8.8	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
8.9	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
9.0	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
9.1	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
9.2	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
9.3	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
9.4	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
9.5	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
9.6	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
9.7	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
9.8	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
9.9	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

TABLE 2

The probability integral, erf (x) vs x, (After Pierce[+])

$$\left(\frac{2}{\sqrt{\pi}}\int_0^x e^{-x^2}dx.\right)$$

x	0	1	2	3	4	5	6	7	8	9
0.00	0.00000	00113	00226	00339	00451	00564	00677	00790	00903	01016
0.01	0.01128	01241	01354	01467	01580	01692	01805	01918	02031	02144
0.02	0.02256	02369	02482	02595	02708	02820	02933	03046	03159	03271
0.03	0.03384	03497	03610	03722	03835	03948	04060	04173	04286	04398
0.04	0.04511	04624	04736	04849	04962	05074	05187	05299	05412	05525
0.05	0.05637	05750	05862	05975	06087	06200	06312	06425	06537	06650
0.06	0.06762	06875	06987	07099	07212	07324	07437	07549	07661	07773
0.07	0.07886	07998	08110	08223	08335	08447	08559	08671	08784	08896
0.08	0.09008	09120	09232	09344	09456	09568	09680	09792	09904	10016
0.09	0.10128	10240	10352	10464	10576	10687	10799	10911	11023	11135
0.10	0.11246	11358	11470	11581	11693	11805	11916	12028	12139	12251
0.11	0.12362	12474	12585	12697	12808	12919	13031	13142	13253	13365
0.12	0.13476	13587	13698	13809	13921	14032	14143	14254	14365	14476
0.13	0.14587	14698	14809	14919	15030	15141	15252	15363	15473	15584
0.14	0.15695	15805	15916	16027	16137	16248	16358	16468	16579	16689
0.15	0.16800	16910	17020	17130	17241	17351	17461	17571	17681	17791
0.16	0.17901	18011	18121	18231	18341	18451	18560	18670	18780	18890
0.17	0.18999	19109	19218	19328	19437	19547	19656	19766	19875	19984
0.18	0.20094	20203	20312	20421	20530	20639	20748	20857	20966	21075
0.19	0.21184	21293	21402	21510	21619	21728	21836	21945	22053	22162
0.20	0.22270	22379	22487	22595	22704	22812	22920	23028	23136	23244
0.21	0.23352	23460	23568	23676	23784	23891	23999	24107	24214	24322
0.22	0.24430	24537	24645	24752	24859	24967	25074	25181	25288	25395
0.23	0.25502	25609	25716	25823	25930	26037	26144	26250	26357	26463
0.24	0.26570	26677	26783	26889	26996	27102	27208	27314	27421	27527
0.25	0.27633	27739	27845	27950	28056	28162	28268	28373	28479	28584
0.26	0.28690	28795	28901	29006	29111	29217	29322	29427	29532	29637
0.27	0.29742	29847	29952	30056	30161	30266	30370	30475	30579	30684
0.28	0.30788	30892	30997	31101	31205	31309	31413	31517	31621	31725
0.29	0.31828	31932	32036	32139	32243	32346	32450	32553	32656	32760
0.30	0.32863	32966	33069	33172	33275	33378	33480	33583	33686	33788
0.31	0.33891	33993	34096	34198	34300	34403	34505	34607	34709	34811
0.32	0.34913	35014	35116	35218	35319	35421	35523	35624	35725	35827
0.33	0.35928	36029	36130	36231	36332	36433	36534	36635	36735	36836
0.34	0.36936	37037	37137	37238	37338	37438	37538	37638	37738	37838
0.35	0.37938	38038	38138	38237	38337	38436	38536	38635	38735	38834
0.36	0.38933	39032	39131	39230	39329	39428	39526	39625	39724	39822
0.37	0.39921	40019	40117	40215	40314	40412	40510	40608	40705	40803
0.38	0.40901	40999	41096	41194	41291	41388	41486	41583	41680	41777
0.39	0.41874	41971	42068	42164	42261	42358	42454	42550	42647	42743
0.40	0.42839	42935	43031	43127	43223	43319	43415	43510	43606	43701
0.41	0.43797	43892	43988	44083	44178	44273	44368	44463	44557	44652
0.42	0.44747	44841	44936	45030	45124	45219	45313	45407	45501	45595
0.43	0.45689	45782	45876	45970	46063	46157	46250	46343	46436	46529
0.44	0.46623	46715	46808	46901	46994	47086	47179	47271	47364	47456
0.45	0.47548	47640	47732	47824	47916	48008	48100	48191	48283	48374
0.46	0.48466	48557	48648	48739	48830	48921	49012	49103	49193	49284
0.47	0.49375	49465	49555	49646	49736	49826	49916	50006	50096	50185
0.48	0.50275	50365	50454	50543	50633	50722	50811	50900	50989	51078
0.49	0.51167	51256	51344	51433	51521	51609	51698	51786	51874	51962

[+] Pierce, B. O. A Short Table of Integrals, 3rd ed. Boston: Ginn and Co. (1929), p. 116.

TABLE 2-Continued

$$\left(\frac{2}{\sqrt{\pi}}\int_0^x e^{-x^2}\,dx.\right)$$

x	0	1	2	3	4	5	6	7	8	9
0.50	0.52050	52138	52226	52313	52401	52488	52576	52663	52750	52837
0.51	0.52924	53011	53098	53185	53272	53358	53445	53531	53617	53704
0.52	0.53790	53876	53962	54048	54134	54219	54305	54390	54476	54561
0.53	0.54646	54732	54817	54902	54987	55071	55156	55241	55325	55410
0.54	0.55494	55578	55662	55746	55830	55914	55998	56082	56165	56249
0.55	0.56332	56416	56499	56582	56665	56748	56831	56914	56996	57079
0.56	0.57162	57244	57326	57409	57491	57573	57655	57737	57818	57900
0.57	0.57982	58063	58144	58226	58307	58388	58469	58550	58631	58712
0.58	0.58792	58873	58953	59034	59114	59194	59274	59354	59434	59514
0.59	0.59594	59673	59753	59832	59912	59991	60070	60149	60228	60307
0.60	0.60386	60464	60543	60621	60700	60778	60856	60934	61012	61090
0.61	0.61168	61246	61323	61401	61478	61556	61633	61710	61787	61864
0.62	0.61941	62018	62095	62171	62248	62324	62400	62477	62553	62629
0.63	0.62705	62780	62856	62932	63007	63083	63158	63233	63309	63384
0.64	0.63459	63533	63608	63683	63757	63832	63906	63981	64055	64129
0.65	0.64203	64277	64351	64424	64498	64572	64645	64718	64791	64865
0.66	0.64938	65011	65083	65156	65229	65301	65374	65446	65519	65591
0.67	0.65663	65735	65807	65878	65950	66022	66093	66165	66236	66307
0.68	0.66378	66449	66520	66591	66662	66732	66803	66873	66944	67014
0.69	0.67084	67154	67224	67294	67364	67433	67503	67572	67642	67711
0.70	0.67780	67849	67918	67987	68056	68125	68193	68262	68330	68398
0.71	0.68467	68535	68603	68671	68738	68806	68874	68941	69009	69076
0.72	0.69143	69210	69278	69344	69411	69478	69545	69611	69678	69744
0.73	0.69810	69877	69943	70009	70075	70140	70206	70272	70337	70403
0.74	0.70468	70533	70598	70663	70728	70793	70858	70922	70987	71051
0.75	0.71116	71180	71244	71308	71372	71436	71500	71563	71627	71690
0.76	0.71754	71817	71880	71943	72006	72069	72132	72195	72257	72320
0.77	0.72382	72444	72507	72569	72631	72693	72755	72816	72878	72940
0.78	0.73001	73062	73124	73185	73246	73307	73368	73429	73489	73550
0.79	0.73610	73671	73731	73791	73851	73911	73971	74031	74091	74151
0.80	0.74210	74270	74329	74388	74447	74506	74565	74624	74683	74742
0.81	0.74800	74859	74917	74976	75034	75092	75150	75208	75266	75323
0.82	0.75381	75439	75496	75553	75611	75668	75725	75782	75839	75896
0.83	0.75952	76009	76066	76122	76178	76234	76291	76347	76403	76459
0.84	0.76514	76570	76626	76681	76736	76792	76847	76902	76957	77012
0.85	0.77067	77122	77176	77231	77285	77340	77394	77448	77502	77556
0.86	0.77610	77664	77718	77771	77825	77878	77932	77985	78038	78091
0.87	0.78144	78197	78250	78302	78355	78408	78460	78512	78565	78617
0.88	0.78669	78721	78773	78824	78876	78928	78979	79031	79082	79133
0.89	0.79184	79235	79286	79337	79388	79439	79489	79540	79590	79641
0.90	0.79691	79741	79791	79841	79891	79941	79990	80040	80090	80139
0.91	0.80188	80238	80287	80336	80385	80434	80482	80531	80580	80628
0.92	0.80677	80725	80773	80822	80870	80918	80966	81013	81061	81109
0.93	0.81156	81204	81251	81299	81346	81393	81440	81487	81534	81580
0.94	0.81627	81674	81720	81767	81813	81859	81905	81951	81997	82043
0.95	0.82089	82135	82180	82226	82271	82317	82362	82407	82452	82497
0.96	0.82542	82587	82632	82677	82721	82766	82810	82855	82899	82943
0.97	0.82987	83031	83075	83119	83162	83206	83250	83293	83337	83380
0.98	0.83423	83466	83509	83552	83595	83638	83681	83723	83766	83808
0.99	0.83851	83893	83935	83977	84020	84061	84103	84145	84187	84229

TABLE 2-Continued

$$\left(\frac{2}{\sqrt{\pi}}\int_0^x e^{-x^2}dx.\right)$$

x	0	1	2	3	4	5	6	7	8	9
1.00	0.84270	84312	84353	84394	84435	84477	84518	84559	84600	84640
1.01	0.84681	84722	84762	84803	84843	84883	84924	84964	85004	85044
1.02	0.85084	85124	85163	85203	85243	85282	85322	85361	85400	85439
1.03	0.85478	85517	85556	85595	85634	85673	85711	85750	85788	85827
1.04	0.85865	85903	85941	85979	86017	86055	86093	86131	86169	86206
1.05	0.86244	86281	86318	86356	86393	86430	86467	86504	86541	86578
1.06	0.86614	86651	86688	86724	86760	86797	86833	86869	86905	86941
1.07	0.86977	87013	87049	87085	87120	87156	87191	87227	87262	87297
1.08	0.87333	87368	87403	87438	87473	87507	87542	87577	87611	87646
1.09	0.87680	87715	87749	87783	87817	87851	87885	87919	87953	87987
1.10	0.88021	88054	88088	88121	88155	88188	88221	88254	88287	88320
1.11	0.88353	88386	88419	88452	88484	88517	88549	88582	88614	88647
1.12	0.88679	88711	88743	88775	88807	88839	88871	88902	88934	88966
1.13	0.88997	89029	89060	89091	89122	89154	89185	89216	89247	89277
1.14	0.89308	89339	89370	89400	89431	89461	89492	89522	89552	89582
1.15	0.89612	89642	89672	89702	89732	89762	89792	89821	89851	89880
1.16	0.89910	89939	89968	89997	90027	90056	90085	90114	90142	90171
1.17	0.90200	90229	90257	90286	90314	90343	90371	90399	90428	90456
1.18	0.90484	90512	90540	90568	90595	90623	90651	90678	90706	90733
1.19	0.90761	90788	90815	90843	90870	90897	90924	90951	90978	91005
1.20	0.91031	91058	91085	91111	91138	91164	91191	91217	91243	91269
1.21	0.91296	91322	91348	91374	91399	91425	91451	91477	91502	91528
1.22	0.91553	91579	91604	91630	91655	91680	91705	91730	91755	91780
1.23	0.91805	91830	91855	91879	91904	91929	91953	91978	92002	92026
1.24	0.92051	92075	92099	92123	92147	92171	92195	92219	92243	92266
1.25	0.92290	92314	92337	92361	92384	92408	92431	92454	92477	92500
1.26	0.92524	92547	92570	92593	92615	92638	92661	92684	92706	92729
1.27	0.92751	92774	92796	92819	92841	92863	92885	92907	92929	92951
1.28	0.92973	92995	93017	93039	93061	93082	93104	93126	93147	93168
1.29	0.93190	93211	93232	93254	93275	93296	93317	93338	93359	93380
1.30	0.93401	93422	93442	93463	93484	93504	93525	93545	93566	93586
1.31	0.93606	93627	93647	93667	93687	93707	93727	93747	93767	93787
1.32	0.93807	93826	93846	93866	93885	93905	93924	93944	93963	93982
1.33	0.94002	94021	94040	94059	94078	94097	94116	94135	94154	94173
1.34	0.94191	94210	94229	94247	94266	94284	94303	94321	94340	94358
1.35	0.94376	94394	94413	94431	94449	94467	94485	94503	94521	94538
1.36	0.94556	94574	94592	94609	94627	94644	94662	94679	94697	94714
1.37	0.94731	94748	94766	94783	94800	94817	94834	94851	94868	94885
1.38	0.94902	94918	94935	94952	94968	94985	95002	95018	95035	95051
1.39	0.95067	95084	95100	95116	95132	95148	95165	95181	95197	95213
1.40	0.95229	95244	95260	95276	95292	95307	95323	95339	95354	95370
1.41	0.95385	95401	95416	95431	95447	95462	95477	95492	95507	95523
1.42	0.95538	95553	95568	95582	95597	95612	95627	95642	95656	95671
1.43	0.95686	95700	95715	95729	95744	95758	95773	95787	95801	95815
1.44	0.95830	95844	95858	95872	95886	95900	95914	95928	95942	95956
1.45	0.95970	95983	95997	96011	96024	96038	96051	96065	96078	96092
1.46	0.96105	96119	96132	96145	96159	96172	96185	96198	96211	96224
1.47	0.96237	96250	96263	96276	96289	96302	96315	96327	96340	96353
1.48	0.96365	96378	96391	96403	96416	96428	96440	96453	96465	96478
1.49	0.96490	96502	96514	96526	96539	96551	96563	96575	96587	96599

TABLE 2-Continued

$$\left(\frac{2}{\sqrt{\pi}}\int_0^x e^{-x^2}dx\right)$$

x	0	2	4	6	8	x	0	2	4	6	8
1.50	0.96611	96634	96658	96681	96705	2.00	0.99532	99536	99540	99544	99548
1.51	0.96728	96751	96774	96796	96819	2.01	0.99552	99556	99560	99564	99568
1.52	0.96841	96864	96886	96908	96930	2.02	0.99572	99576	99580	99583	99587
1.53	0.96952	96973	96995	97016	97037	2.03	0.99591	99594	99598	99601	99605
1.54	0.97059	97080	97100	97121	97142	2.04	0.99609	99612	99616	99619	99622
1.55	0.97162	97183	97203	97223	97243	2.05	0.99626	99629	99633	99636	99639
1.56	0.97263	97283	97302	97322	97341	2.06	0.99642	99646	99649	99652	99655
1.57	0.97360	97379	97398	97417	97436	2.07	0.99658	99661	99664	99667	99670
1.58	0.97455	97473	97492	97510	97528	2.08	0.99673	99676	99679	99682	99685
1.59	0.97546	97564	97582	97600	97617	2.09	0.99688	99691	99694	99697	99699
1.60	0.97635	97652	97670	97687	97704	2.10	0.99702	99705	99707	99710	99713
1.61	0.97721	97738	97754	97771	97787	2.11	0.99715	99718	99721	99723	99726
1.62	0.97804	97820	97836	97852	97868	2.12	0.99728	99731	99733	99736	99738
1.63	0.97884	97900	97916	97931	97947	2.13	0.99741	99743	99745	99748	99750
1.64	0.97962	97977	97993	98008	98023	2.14	0.99753	99755	99757	99759	99762
1.65	0.98038	98052	98067	98082	98096	2.15	0.99764	99766	99768	99770	99773
1.66	0.98110	98125	98139	98153	98167	2.16	0.99775	99777	99779	99781	99783
1.67	0.98181	98195	98209	98222	98236	2.17	0.99785	99787	99789	99791	99793
1.68	0.98249	98263	98276	98289	98302	2.18	0.99795	99797	99799	99801	99803
1.69	0.98315	98328	98341	98354	98366	2.19	0.99805	99806	99808	99810	99812
1.70	0.98379	98392	98404	98416	98429	2.20	0.99814	99815	99817	99819	99821
1.71	0.98441	98453	98465	98477	98489	2.21	0.99822	99824	99826	99827	99829
1.72	0.98500	98512	98524	98535	98546	2.22	0.99831	99832	99834	99836	99837
1.73	0.98558	98569	98580	98591	98602	2.23	0.99839	99840	99842	99843	99845
1.74	0.98613	98624	98635	98646	98657	2.24	0.99846	99848	99849	99851	99852
1.75	0.98667	98678	98688	98699	98709	2.25	0.99854	99855	99857	99858	99859
1.76	0.98719	98729	98739	98749	98759	2.26	0.99861	99862	99863	99865	99866
1.77	0.98769	98779	98789	98798	98808	2.27	0.99867	99869	99870	99871	99873
1.78	0.98817	98827	98836	98846	98855	2.28	0.99874	99875	99876	99877	99879
1.79	0.98864	98873	98882	98891	98900	2.29	0.99880	99881	99882	99883	99885
1.80	0.98909	98918	98927	98935	98944	2.30	0.99886	99887	99888	99889	99890
1.81	0.98952	98961	98969	98978	98986	2.31	0.99891	99892	99893	99894	99896
1.82	0.98994	99003	99011	99019	99027	2.32	0.99897	99898	99899	99900	99901
1.83	0.99035	99043	99050	99058	99066	2.33	0.99902	99903	99904	99905	99906
1.84	0.99074	99081	99089	99096	99104	2.34	0.99906	99907	99908	99909	99910
1.85	0.99111	99118	99126	99133	99140	2.35	0.99911	99912	99913	99914	99915
1.86	0.99147	99154	99161	99168	99175	2.36	0.99915	99916	99917	99918	99919
1.87	0.99182	99189	99196	99202	99209	2.37	0.99920	99920	99921	99922	99923
1.88	0.99216	99222	99229	99235	99242	2.38	0.99924	99924	99925	99926	99927
1.89	0.99248	99254	99261	99267	99273	2.39	0.99928	99928	99929	99930	99930
1.90	0.99279	99285	99291	99297	99303	2.40	0.99931	99932	99933	99933	99934
1.91	0.99309	99315	99321	99326	99332	2.41	0.99935	99935	99936	99937	99937
1.92	0.99338	99343	99349	99355	99360	2.42	0.99938	99939	99939	99940	99940
1.93	0.99366	99371	99376	99382	99387	2.43	0.99941	99942	99942	99943	99943
1.94	0.99392	99397	99403	99408	99413	2.44	0.99944	99945	99945	99946	99946
1.95	0.99418	99423	99428	99433	99438	2.45	0.99947	99947	99948	99949	99949
1.96	0.99443	99447	99452	99457	99462	2.46	0.99950	99950	99951	99951	99952
1.97	0.99466	99471	99476	99480	99485	2.47	0.99952	99953	99953	99954	99954
1.98	0.99489	99494	99498	99502	99507	2.48	0.99955	99955	99956	99956	99957
1.99	0.99511	99515	99520	99524	99528	2.49	0.99957	99958	99958	99958	99959
2.00	0.99532	99536	99540	99544	99548	2.50	0.99959	99960	99960	99961	99961

APPENDIX C

MATHEMATICAL DERIVATIONS SHOWING THAT EQUATION (4.2) IS A SOLUTION TO THE PARTIAL DIFFERENTIAL EQUATION (4.1)

A solution of a partial differential equation in some region R of the space of the independent variables is a function which has all the partial derivatives appearing in the equation is some domain containing R, and satisfies the equation everywhere in R. (Often one merely requires that the function is continuous at the boundary of R, has those derivatives in the interior of R, and satisfied the equation in the interior of R.)

Applying this concept to the problem at hand; we have:

The differential equation is:

$$Kx \frac{\partial^2 P}{\partial x^2} + Ky \frac{\partial^2 P}{\partial y^2} + Kz \frac{\partial^2 P}{\partial z^2} = \phi \mu c \frac{\partial P}{\partial t} \qquad \text{(C1)}$$

The solution is:

$$\Delta P\ (x,y,z,t) = - \frac{Q\ (\phi \mu c)}{8\ (\Pi^3 t^3 K_x K_y K_z)^{1/2}} \exp \left[\frac{-\phi \mu c}{4t} \left(\frac{(x-x')^2}{K_x} + \frac{(y-v')^2}{K_y} + \frac{(z-z)^2}{K_z} \right) \right] \qquad \text{(C2)}$$

Differentiate (C2) twice with respect to x, y, and z:

$$\frac{\partial P}{\partial x} = \frac{-Q(\phi\mu c)^{3/2}}{8(\Pi^3 t^3 K_x K_y K_z)^{1/2}} \cdot \frac{-2(x-x')\phi\mu c}{4(K_x)t} e^A$$

where, $A = \frac{(x-x')^2}{K_x} + \frac{(y-y')^2}{K_y} + \frac{(z-z')^2}{K_z}$

$$\frac{\partial^2 P}{\partial x^2} = \frac{-2(\phi\mu c)}{4K_x t} \cdot \frac{Q(\phi\mu c)^{3/2}}{8(\Pi^3 t^3 K_x k_y k_z)^{1/2}} e^A + \frac{-2(x-x')\phi\mu c}{4K_x t} \cdot \frac{-2(x-x'\phi\mu c}{4K_x t} \cdot \frac{Q(\phi\mu c)^{3/2}}{8(\Pi^3 t^3 K_x K_y K_z)^{1/2}} e^A$$

Therefore,

$$K_x \frac{\partial^2 P}{\partial x^2} = \frac{-1}{16} \cdot \frac{Q(\phi\mu c)^{5/2}}{(\Pi^3 t^5 K_x K_y K_z)^{1/2}} e^A + \frac{1}{32} \cdot \frac{Q(\phi\mu c)^{7/2}}{(\Pi^3 t^7 K_x K_y K_z)^{1/2}} \cdot \frac{(x-x')^2}{K_x} e^A \tag{C3}$$

Similarily,

$$K_y \frac{\partial^2 P}{\partial y^2} = \frac{-1}{16} \frac{Q(\phi\mu c)^{5/2}}{(\Pi^3 t^5 K_x K_y K_z)^{1/2}} e^A + \frac{1}{32} \frac{Q(\phi\mu c)^{7/2}}{(\Pi^3 t^7 K_x K_y K_z)^{1/2}} \cdot \frac{(y-y')^2}{K_y} e^A \tag{C4}$$

And,

$$K_z \frac{\partial^2 P}{\partial z^2} = \frac{-1}{16} \frac{Q(\phi\mu c)^{5/2}}{(\Pi^3 t^5 K_x K_y K_z)^{1/2}} e^A + \frac{1}{32} \frac{Q(\phi\mu c)^{7/2}}{(\Pi^3 t^7 K_x K_y K_z)^{1/2}} \cdot \frac{(z-z')^2}{K_z} e^A \tag{C5}$$

Therefore, the left hand side of equation (C1) is:

$$-\frac{3}{16} \frac{Q(\phi\mu c)^{5/2}}{(\Pi^3 t^5 K_x K_y K_z)^{1/2}} e^A + \frac{Q}{32} \frac{(\phi\mu c)^{7/2}}{(\Pi^3 t^7 K_x K_y K_z)^{1/2}} e^A \left(\frac{(x-x')^2}{K_x} + \frac{(y-y')^2}{K_y} + \frac{(z-z')^2}{K_z}\right) \tag{C6}$$

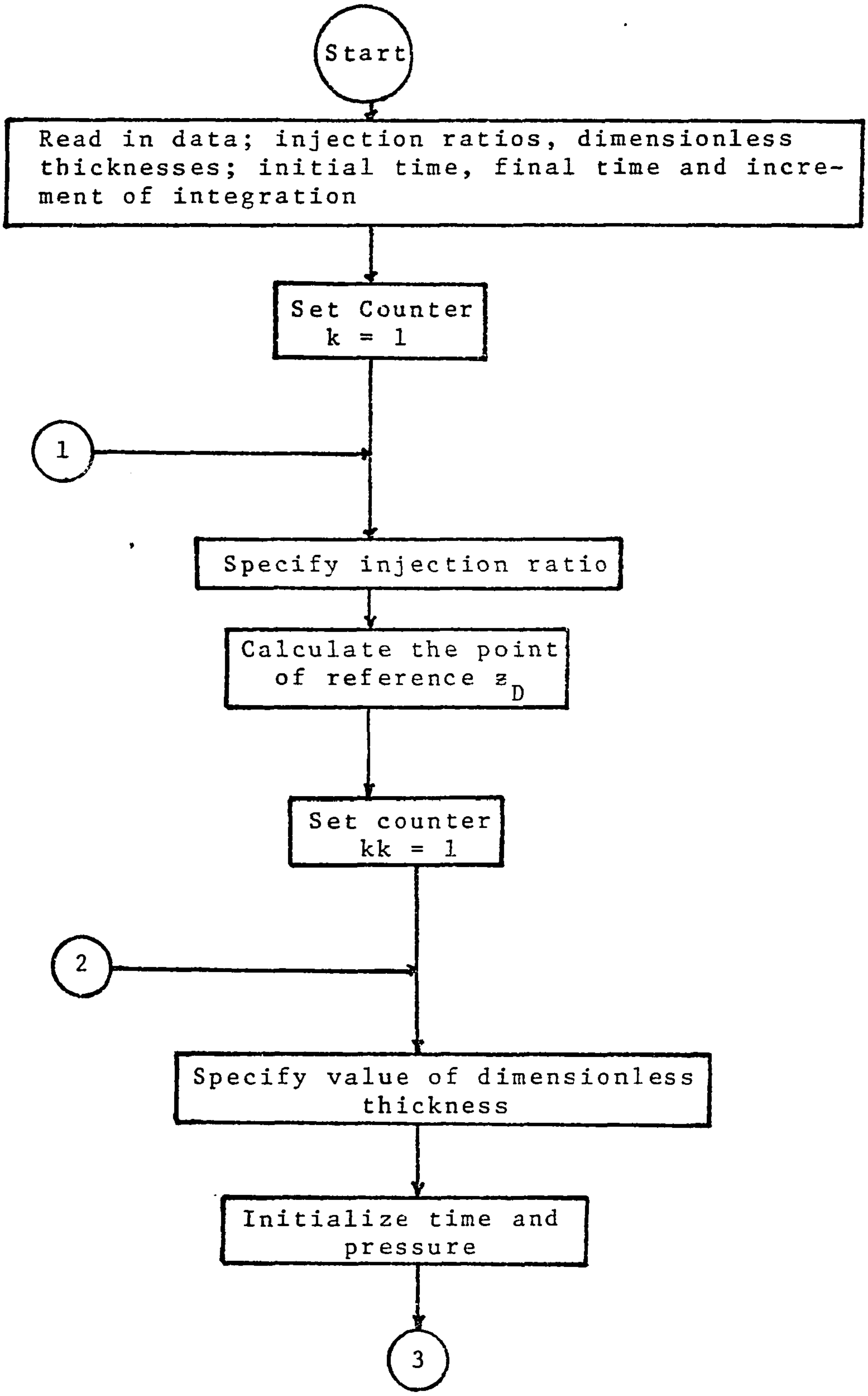
Now differentiating equation (C2) with respect to t, we obtain the right hand side of equation (C1):

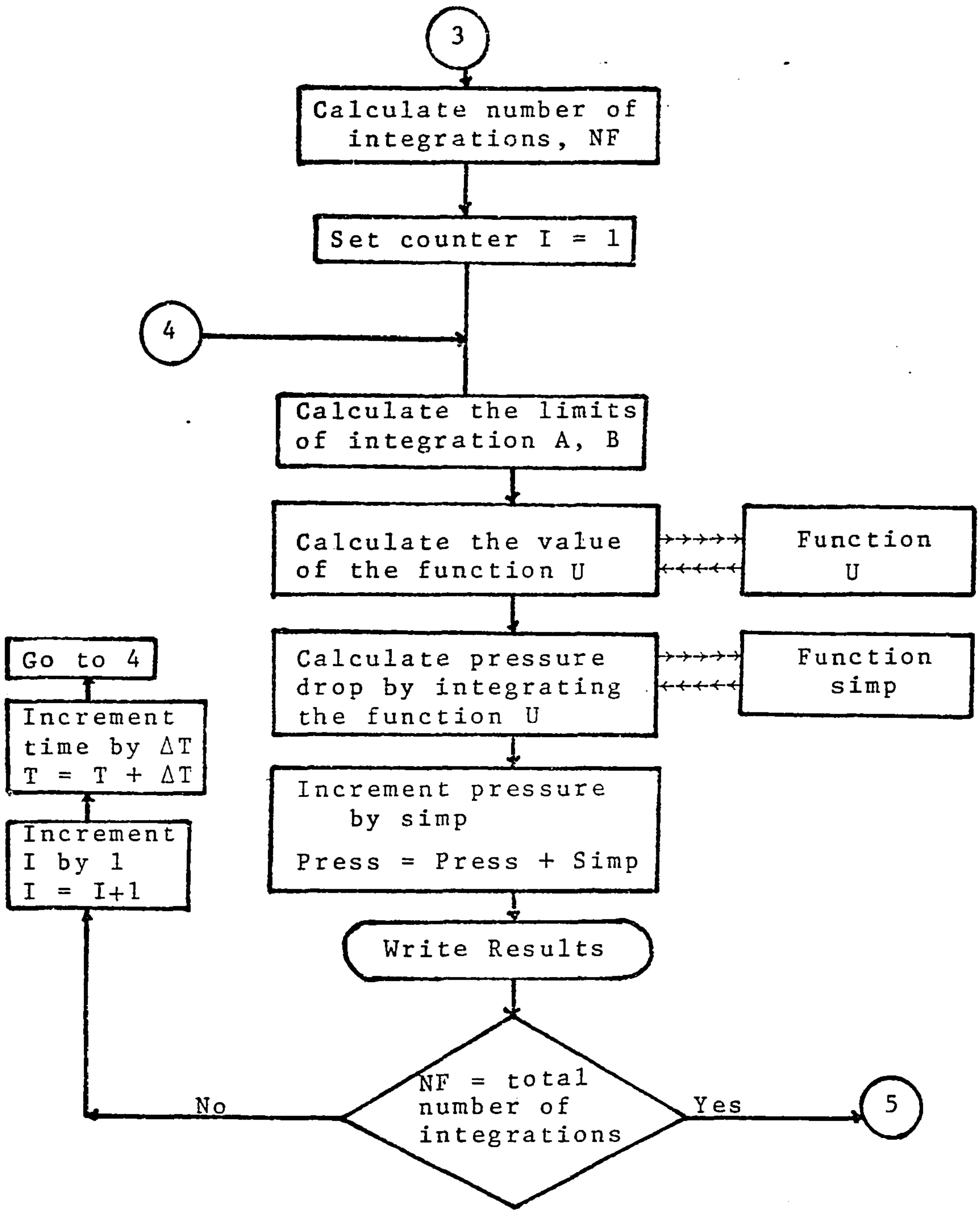
$$\frac{\partial P}{\partial t} = \frac{-3}{16}\,\frac{Q\,(\phi\mu c)^{5/2}}{(\Pi^3 t^5 K_x K_y K_z)^{1/2}}\,e^A + \frac{Q}{32}\,\frac{(\phi\mu c)^{7/2}\,e^A}{(\Pi^3 t^7 K_x K_y K_z)^{1/2}} \left(\frac{(x-x')^2}{K_x} + \frac{(y-y')^2}{K_y} + \frac{(z-z')^2}{K_z}\right) \qquad \text{(C7)}$$

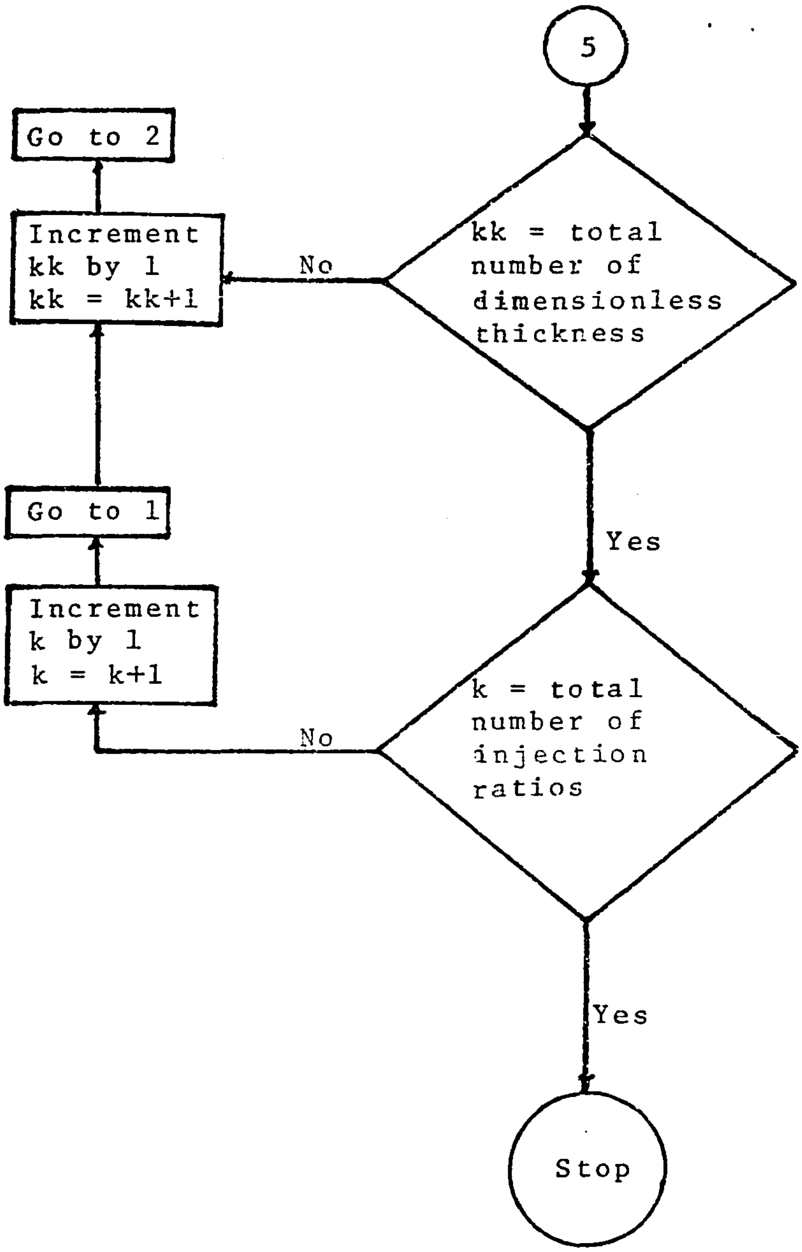
From equations (C6) and (C7), the left hand side of equation (C1) is equal to the right hand side, i.e., equation (C2) is a solution to equation (C1).

APPENDIX D

Fig. 17. Flow chart of computer program.




3
Calculate number of integrations, NF
Set counter I = 1
4
Calculate the limits of integration A, B
Calculate the value of the function U
Function U
Calculate pressure drop by integrating the function U
Function simp
Increment pressure by simp
Press = Press + Simp
Write Results
NF = total number of integrations
Yes
5
No
Increment I by 1
I = I+1
Increment time by ΔT
T = T + ΔT
Go to 4


5
Go to 2
Increment
kk by 1
kk = kk+1
No
kk = total
number of
dimensionless
thickness
Yes
Go to 1
Increment
k by 1
k = k+1
No
k = total
number of
injection
ratios
Yes
Stop

FORTRAN IV G LEVEL 21 MAIN DATE = 78337 11/32/24

```
      DIMENSION BB(6),BETA( 6),PRESS( 5) ,D(6),TF(5),DT(5)
      EXTERNAL U
C     BB IS PENETRATION RATIO,BETA IS DIMENSIONLESS THICKNESS
      READ(5,1)(BB(K),K=1,II)
    1 FORMAT(4F10.2)
      READ(5,2)(BETA(KK),KK=1,JJ)
    2 FORMAT(2F10.2)
C     TI,TF,AND DT ARE THE INITIAL TIME, THE FINAL TIME AND INCREMENT
C     OF INTEGRATION RESPECTIVELY.
C     THE PROCESS OF INTEGRATION HAS BEEN DIVIDED INTO FOUR PARTS.
      READ(5,20)(DT(J),TF(J),J=2,5)
   20 FORMAT(8E7.0)
      READ(5,33)M,N,TI,II,JJ,EPS
   33 FORMAT(I2,I2,F5.0,2I2,F10.7)
      DO 3  K=1,II
C     D IS ZD=Z/H
      D(K)=0.75*BB(K)
      DO 3  KK=1,JJ
      A=ALOG(TI)
      J=1
C     SINCE THE INTEGRATION CANNOT BE STARTED AT TIME ZERO , THE INSTANT
C     -ANEOUS INFINITE LINE SOURCE EI FUNCTION APPROXIMATION IS USED
C     THE AREA UNDER THE CURVE OF THE FUNCTION U(TD) IS EQUAL TO THE
C     DIMENSIONLESS PRSEEURS DROP.
      PRESS(J)=(ALOG( TI  )+.80907)/(BB(K)*2.0)
      DO 3 J=2,4
      WRITE(6,200)
  200 FORMAT('1',4X,'    BB     ',10X,' BETA',18X,'DIMENSIONLESS PRESS',
     &10X,'DIMENSEONLESS TIME' ,//)
      PRESS(J)=PRESS(J-1)
C     DETERMINE THE NUMBER OF INTEGRATIONS,NF
      NF=(TF(J)-TI)/DT(J)
      DO 30 I=1,NF
      TD=FLOAT(I)*DT(J)
      B=ALOG(TD)
      F=Q(D,BB,BETA,M,K,KK,TD)
C     FUNCTION Q IS USED TO DETERMINE THE VALUE OF  FUNCTION U .
      IF(F.LE.EPS) GO TO 297
C     THE VALUE OF F DETERMINES THE TIME AT WHICH THE INTEGRAL IS NO
C     LONGER A FUNCTION OF THE PENETRATION RATIO AND/OR THE ANISOTROPY.
      GO TO 298
  297 P=PRESS(J)+0.5*(B-A)
      WRITE(6,300) BB(K),BETA(KK),P,TD
      GO TO 30
  298 PRESS(J)=PRESS(J)+SIMP(A,B,N,U,D,BB,BETA,M,K,KK)/(BB(K)*2.0)
  299 WRITE(6,300) BB(K),BETA(KK),PRESS(J),TD
  300 FORMAT(1X,F10.2,8X,F10.2,8X,F25.5,8X,F15.0)
      A=B
   30 CONTINUE
    3 CONTINUE
      STOP
      END
```

```
      FUNCTION SIMP(A,B,N,U,D,BB,BETA,M,K,KK)
      DIMENSION D(6),BB(6),BETA( 6)
C     THE FUNCTION SIMP USES N APPLICATIONS OF SIMPSON'S RULE TO
C     CALCULATE NUMERICALLY THE INTEGRAL OF U(T)*DT BETWEEN INTEGRATION
C     LIMITS A AND B .  SUMEND IS THE SUM OF ALL U(T(I)) FOR EVEN I
C     (EXCEPT FOR U(T(2*N)) WHILE SUMMID IS THE SUM OF ALL U(T(I)) FOR
C     I ODD.   H IS THE STEPSIZE BETWEEN ADJACENT T(I) AND TWOH IS THE
C     LENGTH OF THE INTERVAL OF INTEGRATION FOR EACH INDIVIDUAL
C     APPLICATION OF SIMPSON'S RULE. LL IS THE ITERATION COUNTER.
C     .....INITIALIZE PARAMETERS ....
      TWOH=(B-A)/N
      H=TWOH/2.0
      SUMEND=0.0
      SUMMID=0.0
C     .... EVALUATE SUMEND AND SUMMID ....
      DO 200 LL=1,N
      T=A+FLOAT(LL-1)*TWOH
      TH=T+H
      SUMEND=SUMEND+U( D,BB,BETA,M,K,KK,T)
  200 SUMMID=SUMMID+U(D,BB,BETA,M,K,KK,TH)
C     .....RETURN ESTIMATED VALUE OF THE INTEGRAL .......
      AAA=U(D,BB,BETA,M,K,KK,A)
      BBB=U(D,BB,BETA,M,K,KK,B)
      SIMP=(2.0*SUMEND+4.0*SUMMID-AAA+BBB) *H/3.0
      RETURN
      END
```

```
FORTRAN IV G LEVEL  21                    U                   DATE = 77351          11/31/41

              FUNCTION U(D,BB,BETA,M,K,KK,TD)
              DIMENSION BB(6),BETA( 6),D(6),EM1(20),EM2(20),EM3(20),EM4(20)
C              U IS A FUNCTION OF PENETRATION RATIO , DIMENSIONLESS WELLBORE
C             THICKNESS , AND DIMENSIONLESS TIME.
C             FUNCTION U IS USED TO BE INTEGRATED WHERE THE LIMITS OF INTEGRAT-
C             ION ARE IN THE FORM OF NATURAL LOGS.
C             TD IS DIMENSIONLESS TIME.M IS THE NUMBER OF SOURCE IMAGES USED.
              TDS=SQRT(TD)
              BTD=BETA(KK)/TDS
              SUM=0.0
              DO 100 J=1,M
              Y=FLOAT(J)
              EM1(J)=2.0*Y+BB(K)-D(K)
              EM2(J)=2.0*Y-BB(K)-D(K)
              EM3(J)=2.0*Y+BB(K)+D(K)
              EM4(J)=2.0*Y-BB(K)+D(K)
          100 SUM=SUM+ERF(EM1(J)*BTD)-ERF(EM2(J)*BTD)+ERF(EM3(J)*BTD)-ERF(
             & EM4(J)*BTD)
              E=EXP(-1.0/TD)
              G=ERF((D(K)+BB(K))*BTD)
              H=ERF((D(K)-BB(K))*BTD)
              U=(E*((G-H)+SUM))/(2.0)
              RETURN
              END
```

FORTRAN IV G LEVEL 21 Q DATE = 77351 11/31/41

```
      FUNCTION Q(D.BB.BETA.M.K.KK.TD)
      DIMENSION BB(6).BETA( 6).D(6).EM1(20).EM2(20).EM3(20).EM4(20)
      TDS=SQRT(TD)
      BTD=BETA(KK)/TDS
      SUM=0.0
      DO 100 J=1.M
      Y=FLOAT(J)
      EM1(J)=2.0*Y+BB(K)-D(K)
      EM2(J)=2.0*Y-BB(K)-D(K)
      EM3(J)=2.0*Y+BB(K)+D(K)
      EM4(J)=2.0*Y-BB(K)+D(K)
  100 SUM=SUM+ERF(EM1(J)*BTD)-ERF(EM2(J)*BTD)+ERF(EM3(J)*BTD)-ERF(
     & EM4(J)*BTD)
      E=EXP(-1.)/TD)
      G=ERF((D(K)+BB(K))*BTD)
      H=ERF((D(K)-BB(K))*BTD)
      Q=(E*((G-H)+SUM))/(2.0*TD)
      RETURN
      END
```