# A GENERALIZATION OF MENGER'S RESULT ON THE STRUCTURE OF LOGICAL FORMULAS

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# Preface

The problem of generalizing Menger's[1] result was first raised by S. Hoberman, and suggested to the author of this paper by Dr. J. C. C. McKinsey. The formulation of the principal theorem here proved is due to Miss Helen Dayton. The proof, the first given for this theorem, is original.

In Menger's paper there is a proof for the special case n = 2. The proof here given is valid for all n > 1, and thus includes his result. The theorem provides a decision method, that is, a method for determining whether an arbitrary expression is a formula.

[1] Karl Menger, Eine elemtare Bemarkung über die Structure logischer Formeln.

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## Section I. Definitions and Lemmas.

All numbers used in the following are integers, for which the results of arithmetic are assumed.

Notation: The symbol $p_i$ is a sentential variable for any integer i, and will be referred to as a 'variable.' The symbol $R_j^n = R$ is an n-ary connective for any j and $n > 1$, and will be called simply a connective.

**Definitions:**

(1). An expression is a sequence $s_1 \ldots s_k$ such that $s_i$ for $i = 1, \ldots, k$ is a variable or connective.

(2). An initial segment of an expression is an expression $s_1 \ldots s_i$, where $i < k$.

(3). A terminal segment of an expression is an expression $s_t \ldots s_k$, where $t > 1$.

(4). For each $n > 1$, a formula is an expression contained in every set K such that:

(a) Every variable is in K.

(b) If $x_1, \ldots, x_n$ are in K, $Rx_1 \ldots x_n$ is in K.

From (4) we have immediately the lemmas:

(5). If all variables have a property, and if when $x_1, \ldots, x_n$ have the property, $Rx_1 \ldots x_n$ has the property, then all formulas have the property.

(6). Every variable is a formula. If $x_1, \ldots, x_n$ are formulas, then $Rx_1 \ldots x_n$ is a formula.

## Section II. Principal Theorem.

Necessary and sufficient conditions that an expression $x = s_1 \ldots s_k$ be a formula are:

(C1) $v_i < (n - 1)\, c_i + 1$

(C2) $v_k = (n - 1)\, c_k + 1$

where $x_i$ = an initial segment of x for i = 1,...,k-1

$v_i$ = the number of variables in $x_i$

$c_i$ = the number of connectives in $x_i$

$v_k$ = the number of variables in x

$c_k$ = the number of connectives in x.

The conditions are necessary:

Every variable satisfies (1) vacuously and (2), since $c_k = 0$.

To show by (5) that (C1) and (C2) hold for all formulas, we assume they hold for $z_1,\ldots z_n$ and consider $x = Rz_1\ldots z_n$. Let y be an initial segment of x. Then one of the following is true:

(7) y is an initial segment of $Rz_1$.

(8) $y = Rz_1\ldots z_h$ for some $h < n$.

(9) $Rz_1\ldots z_h$ is an initial segment of y for some $h < n$.

If (7) holds, x obviously satisfies (C1).

If (8) holds, let

$u_i$ = number of variables in $z_i$ for i = 1,...,h

$q_i$ = number of connectives in $z_i$ for i = 1,...,h

$u_y$ = number of variables in y

$q_y$ = number of connectives in y.

Then we have:

(10) $q_y = q_1 + \ldots + q_h + 1$

$u_y = u_1 + \ldots + u_h$

$= (n-1)(q_1 + \ldots + q_h) + h$ by (2) for each $z_i$

$= (n-1)(q_y - 1) + h$ by (10)

$= (n-1)q_y - n + h + 1$

But $h - n < 0$, since $h < n$, so that $u_y < (n-1)q_y + 1$.

If (9) holds, consideration of (8) and $z_{h+1}$ leads to the desired result; thus (C1) holds for all formulas.

The proof that (C2) holds for x is the same as the proof for case (8) above, with $u_y = v_k$, $q_y = c_k$, and $h = n$. Thus (C2) holds for all formulas.

The conditions are sufficient:

This is proved by an induction on the length of the expression. If x is an expression of length one, this one symbol by (C2) must be a variable. This is a formula by (6). Suppose then that all expressions of length $< k$ satisfying (C1) and (C2) are formulas, and that $x = s_1 \ldots s_k$ satisfies (C1) and (C2), where $k > 1$.

If $s_1$ is a variable, by (C1) we have $1 < 1$. Hence $s_1$ is a connective. In any terminal segment, if $v_t$ is the number of variables, and $c_t$ the number of connectives, we have for $t = 1 + i$ for some $i$,

$$
\begin{aligned}
c_t &= c_k - c_i \\
v_t &= v_k - v_i \\
&> (n-1)c_k + 1 - (n-1)c_i - 1 \quad \text{by (C1) and (C2)} \\
&= (n-1)(c_k - c_i) \\
&= (n-1)c_t
\end{aligned}
$$

or (11) $v_t > (n-1)c_t$.

Write $x = Rx' = Rs_2 \ldots s_k$. If $s_2 = x_1$ is a variable, it is a formula by (6). If $s_2$ is a connective, this initial segment of $x'$ satisfies (C1). Let $x_1$ be the shortest segment which does not satisfy (C1), i.e. such that $v_1 = (n-1)c_1 + 1$, where $v_1$ and $c_1$ are defined for $x_1$ as usual. There is such a segment by (11) for $x'$. Thus $x_1$ satisfies (C1) and (C2), and is a formula by the induction hypothesis. We write $x = Rx_1x''$, and construct in the same manner formulas $x_2, \ldots, x_m$ so that $x = Rx_1 \ldots x_m$. It is possible to exhaust the symbols of x in this manner, since k is an integer, and each $x_i$ contains at least one symbol.

As in the proof of (C1) for case (8), we have

$$v_k = (n-1)c_k - n + 1 + m$$

but $$v_k = (n-1)c_k + 1 \text{ by (C2) for } x.$$

Hence $m - n = 0$ or $m = n$. We conclude by (6) that x is a formula.

This completes the proof of the theorem.

BIBLIOGRAPHY

Menger, Karl, Eine elementare Bemarkung über die Structure logischer Formeln, Ergebnisse eines mathematischen Kolloquiums, Heft 3 (pp. 22-23).

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