MATRIX-RELAXATION METHODS IN THE SOLUTION OF BOUNDARY-VALUE PROBLEMS

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Preface

At the inception of Project No. 21 of the Research Foundation at Oklahoma A. and M. College, Dr. Alvin C. Sugar began the preparation of an atlas of inverse matrices which could be used to solve the Dirichlet problem. This paper embodies the results of that investigation, which may be extended to examine allied problems that are suggested.

The fundamental theory for the simple cases has been worked out in great detail with the hope that some clue may become apparent for generalizing more complicated cases.

The bibliography, deliberately inextensive, is basic. References to it throughout the text are indicated by bracketed numerals and page numbers.

The paper does not represent all that has been and is being done on the project; as a matter of fact, an approach to the problem is being made at present through the use of integral equations and variation principles.

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1 Introduction

The principal result of this paper is the development of certain numerical methods of solving the Dirichlet problem for a long rectangular domain. Less general results are obtained for squares and other rectangular boundaries, and there is included a sketch of how the methods may be extended to equations other than Laplace's.

The problem is formulated as follows:
The set of lines $x = h$, $x = 2h$, ... $x = nh$ intersects the set $y = k$, $y = 2k$, ... $y = mk$ in mn <u>interior points</u> on a cartesian coordinate system for all integral m and n. Beginning at $(x,y) = (h,mk)$, number these points P_i serially from left to right in each row and count off the rows consecutively from top to bottom so that (x,y) will be designated $P_{(m-y/k)n+x/h}$. The pairs of <u>boundary lines</u> $y = (m+1)k$, $y = 0$, and $x = 0$, $x = (n+1)h$, which constitute the <u>boundary</u>, intersect this configuration in $2(m+n)$ <u>boundary points</u> $(x,(m+1)k)$, $(x,0)$ $[x = h, 2h, \ldots nh]$ and $(0,y)$, $((n+1)h,y)$ $[y = mk, (m-1)k, \ldots k]$, which are named, respectively, $P^1_{x/h}$, $P^2_{x/h}$, and $P^3_{m+1-y/k}$, $P^4_{m+1-y/k}$. Now suppose a function $u = u(x,y)$, defined at every point of this net, to have the value u_i at P_i and u^j_i at P^j_i $[i = 1, 2, \ldots mn$; and the superscript $j = 1, 2, 3, 4]$. The Dirichlet problem seeks the u_i when (1) the u^j_i are known and (2) $u(x,y)$ satisfies Laplace's equation in two dimensions

$$(1) \qquad u_{xx}+u_{yy} = 0 \,.$$

It is possible, in a manner to be described presently, to replace (1) by the linear algebraic system of difference equations

$$(2) \qquad Mu = u^* \,,$$

where M is a nonsingular square matrix of constants, and u and u* are column matrices containing u_i and linear combinations of u_i^j, respectively [$i = 1, 2, \ldots mn$; $j = 1, 2, 3, 4$]. Then for a given rectangular boundary, as m and n increase without bound, h and k approach zero and the solutions of (2) converge to those of (1), subject to the given boundary conditions. A proof of this is given in [1]. The solution

$$(3) \qquad u = M^{-1}u^*$$

of the matrix equation (2) requires (1) the inversion of M and (2) the multiplication of M^{-1} by the column matrix u^*. Once M^{-1} is known, each u_i can be computed by adding mn pairs of products of numbers. Thus it appears that tabulation of inverses of M for areas divided into a large number of rectangles would facilitate the complete solution of many numerically difficult engineering problems.

2 The $1 \times n$ rectangle

Equation(1) is reduced in [7, p. 163] and [5, p. 20] to the difference equation

$$(4) \quad u(x+h,y)+u(x,y+h)+u(x-h,y)+u(x,y-h)-4u(x,y) = 0 \ ,$$

which is a simplification of

$$(5) \quad \frac{u(x+h,y)+u(x-h,y)-2u(x,y)}{h^2} + \frac{u(x,y+k)+u(x,y-k)-2u(x,y)}{k^2} = 0$$

in the case where the rectangle is subdivided into squares so that $h = k$. Now suppose $m = 1$; then in (2) M has 4's in the principal diagonal, -1's in the immediately adjacent diagonals, and 0's elsewhere, while u has u_1, u_2, $\ldots$ u_n, and u^* has $u_1^1+u_1^2+u_1^3$, $u_2^1+u_2^2$, $\ldots$ $u_{n-1}^1+u_{n-1}^2$, $u_n^1+u_n^2+u_n^4$, reading downward in both cases. The square matrix, which in this instance ($m = 1$) will be called M_n, is not only symmetric but also has the <u>reversibility</u> property that the elements of the i th row [j th column] read forward [downward] are the same as those of the (n+1-i)th row [(n+1-j)th column] read backward [upward]. More precisely, if J is a square matrix with 1's in the secondary diagonal and 0's elsewhere, and if M = JMJ, then M is called a <u>reversible matrix</u>. The theorems below follow from the

<u>Lemma</u>: $J^2 = I$, where I is the unit matrix of the same order as J; for if the elements of J are d_{ij} [i, j = 1, 2, ... n], then $d_{ij} = \partial_{i,n+1-j} = \partial_{n+1-i,j}$, where

$$\partial_{ij} = \begin{matrix} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{matrix}$$

is used throughout for the Kronecker delta. Now if $J^2 = C$ and the elements of C are c_{ij}, then $c_{ij} = \Sigma\, d_{ik}d_{ij}$, where the summation runs from k = 1 to n. The only nonzero elements are those of the form $d_{i,n+1-i}d_{n+1-j,j}$; hence n+1-i = n+1-j, or i = j. Therefore, $c_{ij} = \partial_{ij}$, so that C = I.

<u>Theorem 1</u>: $J^{-1} = J$.

<u>Theorem 2</u>: A reversible matrix is not necessarily symmetric; e.g., if $b \neq d$,

$$\begin{matrix} a & b & c \\ d & e & d \\ c & b & a \end{matrix} .$$

<u>Theorem 3</u>: A symmetric matrix is not necessarily reversible; e.g., if $a \neq d$,

$$\begin{matrix} a & b \\ b & d \end{matrix} .$$

<u>Theorem 4</u>: The inverse of a reversible matrix is reversible; for suppose M = JMJ, then $M^{-1} = J^{-1}M^{-1}J^{-1} = JM^{-1}J$.

<u>Theorem 5</u>: If M is reversible, then JM = MJ; for suppose M = JMJ, then JM = JJMJ = IMJ = MJ.

Theorem 6: The sum or difference of two reversible matrices is reversible; for suppose $M = JMJ$ and $N = JNJ$, then $M \pm N = JMJ \pm JNJ = (JM \pm JN)J = J(M \pm N)J$.

Theorem 7: The product of two reversible matrices is reversible; for suppose $M = JMJ$ and $N = JNJ$, then $MN = JMJJNJ = JMINJ = JMNJ$.

Definition: A symmetric, reversible matrix is called a symverse.

Since M_n is a symverse, so also is M_n^{-1}; thus it is necessary to compute only those elements in the fundamental triangle, which is that part of M_n^{-1} to the left of and including the left halves of the principal and secondary diagonals. These inverses are calculated exactly and to five decimal places in Table 1 up to $n = 7$. It is possible to determine any element in M_n^{-1} once the element a_n in the first row and first column is known. Moreover, a_{n+1} can be expressed in terms of a_n. Furthermore, it will be shown that each of the elements approaches a limiting value as n increases without bound.

Let a symmetric matrix be partitioned

$$M = \begin{matrix} a & c' \\ c & d \end{matrix},$$

where a and d are square, symmetric, and nonsingular, but c and hence its transpose c' may be rectangular. Then its inverse

$$M^{-1} = \begin{matrix} A & C' \\ C & D \end{matrix},$$

whose submatrices are of the same order as similarly placed quantities in M, is worked out according to the following steps, which are adapted from [4, p. 112, ff.]:

1) Compute a^{-1}

(6) 2) Premultiply 1) by c: ca^{-1}

3) Postmultiply 2) by c': $ca^{-1}c'$

(6)

4) Subtract 3) from d: $d-ca^{-1}c'$

5) Invert 4): $D = (d-ca^{-1}c')^{-1}$

6) Premultiply 2) by the negative of 5): $C = -Dca^{-1}$

7) Premultiply 6) by c': $c'C$

8) Subtract 7) from the unit matrix I: $I-c'C$

9) Premultiply 8) by 1): $A = a^{-1}(I-c'C)$

The submatrices A, C, D are thus determined from 9), 6), 5).

Suppose M_n^{-1} is known and that M_{n+1} is partitioned

$$M_{n+1} = \begin{matrix} M_n & K' \\ K & 4 \end{matrix} ,$$

where K is a row matrix with n-1 0's and having -1 as the rightmost element. Then, since the element in the n th row and the n the column of M_n^{-1} is equal to a_n (because M_n is reversible), it follows after applying the first five steps of (6) that

$$a_{n+1} = 1/(4-a_n) . \quad (7)$$

Since $a_1 = 1/4$, subsequent a_n are rational fractions; thus if $a_n = N_n/D_n$, where N_n and d_n are relatively prime integers, (7) is equivalent to

$$N_{n+1} = D_n, \quad D_{n+1} = 4D_n-N_n .$$

Combination of these yields the relations

$$\begin{matrix} N_1 = 1 , & N_2 = 4 , & N_{n+1} = 4N_n-N_{n-1} \\ D_1 = 4 , & D_2 = 15, & D_{n+1} = 4D_n-D_{n-1} \end{matrix} \quad [n = 3, 4, \ldots] . \quad (8)$$

The values of N_n and ten-place approximations of a_n are entered in Table 2. An enlargement of this table gives the leading element in M_n^{-1} for any n.

Now, in order to determine the other elements of the fundamental triangle, designate the elements of M_n^{-1} by a_{ij} and those of M_n by b_{ij}; then, since $M_n^{-1}M_n = M_nM_n^{-1} = I_n$ (the unit matrix of order n), the n^2 equations

$$(9) \qquad \begin{aligned} \Sigma\, a_{ik}b_{kj} &= \partial_{ij} \\ \Sigma\, b_{ik}a_{kj} &= \partial_{ij} \end{aligned} \qquad [i,\ j = 1,\ 2,\ \ldots\ n]\ ,$$

hold, the summation running from $k = 1$ to n. From the definition of M,

$$\begin{aligned} b_{ii} &= 4 & [i = 1,\ 2,\ \ldots\ n] \\ b_{i,i+1} &= -1 & [i = 1,\ 2,\ \ldots\ n-1]; \\ b_{i,i-1} &= -1 & [i = 2,\ 3,\ \ldots\ n] \end{aligned}$$

thus for $i,\ j = 1,\ 2,\ \ldots\ n$, (9) may be written

$$(10) \qquad \begin{aligned} -a_{i,j-1}+4a_{ij}-a_{i,j+1} &= \partial_{ij} \\ -a_{i-1,j}+4a_{ij}-a_{i+1,j} &= \partial_{ij}\ , \end{aligned}$$

provided that

$$(11) \qquad a_{i0} = a_{i,n+1} = a_{0j} = a_{n+1,j} = 0\ .$$

Elements of the first column are determined from the equations

$$(12) \qquad a_{11} = a_n\ , \quad a_{i+1,1} = 4a_{i1}-a_{i-1,1}-\partial_{i1} \quad [i = 1,\ 2,\ \ldots\ n-1].$$

Now since a_{i1} can be written $a_{i1} = N^{(i)}/D_n$, where $N^{(i)}$ is an integer, it follows that $N^{(i)}$ satisfies a difference equation

$$N^{(i+1)} = 4N^{(i)}-N^{(i-1)}\ , \quad N^{(1)} = N_n$$

similar to that for N_n; however, the sequence $N^{(i)}$ decreases from N_n to 1 while assuming the same values as N_i in reverse order.

To determine the remaining elements, first eliminate a_{ij} from (10):

$$(13) \qquad a_{i,j+1} = a_{i-1,j}+a_{i+1,j}-a_{i,j-1}\ .$$

An induction shows that

$$(14) \qquad a_{ij} = \Sigma_{i-(j-1)}^{i+(j-1)}\ a_{r1}\ ,$$

the summation running over either odd or even integers r, not both, since

$$\Sigma_{i-j}^{i+j} = \Sigma_{i-j}^{i+j-2} + \Sigma_{i-j+2}^{i+j} - \Sigma_{i-j+2}^{i+j-2}$$

and (14) holds for $j = 1,\ 2$. After applying (7) and (12), it is most expeditious to calculate first the elements along the principal diagonal, then those along the diagonal just below, etc.; work toward the center of the matrix. This solves completely the problem of inverting the matrix for the $1 \times n$ rectangle.

It will now be shown that as $n \to \infty$, corresponding elements of M_n^{-1} have limiting values; the infinite matrix M_∞^{-1} with these limiting elements is accordingly called the <u>limit</u> <u>matrix</u> of M_n^{-1}. This is accomplished by proving (1) all the a_{ij} are bounded for any n and (2) corresponding a_{ij} form a monotone sequence. Suppose in (3) that u^* has 1 in the j th row and 0's elsewhere, then if $M = [a_{ij}]$, $u_i = a_{ij}$ $[i,j = 1, 2, \dots n]$; thus any a_{ij} is actually a solution of (2) for a particular set of boundary values; viz., 1 at some point and 0 at all other boundary points. Since by (5) $u(x,y)$ is the average of values of u at the four neighboring points $(x+h,y)$, $(x-h,y)$, $(x,y+h)$, $(x,y-h)$, it follows as shown in [3, p. 735] that u attains its maximum and minimum on the boundary; hence, all the a_{ij} lie between 0 and 1 for any n. In (7) therefore $4-a_n > 0$ so that the difference

(15) $$a_{n+1}-a_n = (a_n-a_{n-1})/(4-a_n)(4-a_{n-1})$$

is positive if $a_n-a_{n-1} > 0$. Since $a_2-a_1 = 4/15-1/4 = 1/60 > 0$, it follows by induction from (15) that the a_n form a monotone nondecreasing sequence with 1 as an upper bound and have a limit s which is the smaller root of

(16) $$s = 1/(4-s)\ , \quad s^2 = 4s-1\ , \quad \text{or } s = 2-\sqrt{3}\ .$$

Similar arguments using (12) and (14) demonstrate the convergence of remaining a_{ij}. To calculate these limits for $j = 1$, first take the limits as $n \to \infty$ of the first terms in (12) for $i = 1$:

$$\lim a_{21} = 4s-1\ ;$$

comparison of this with (16) makes

$$\lim a_{21} = s^2 .$$

Furthermore,

$$(17) \qquad \begin{aligned} \lim a_{i1} &= s^i \\ \lim a_{ij} &= \Sigma_{i-(j-1)}^{i+(j-1)} s^r , \end{aligned}$$

where Σ is defined as after (14). Elements of M_∞^{-1} calculated along successive diagonals are displayed in Table 3. In the inverses computed in Table 1 it appears that if the elements in the fundamental triangle of M_n^{-1} are replaced by those of M_∞^{-1}, the following may be said about differences between corresponding elements: (1) they decrease as n increases, (2) they increase rather rapidly away from the principal diagonal, (3) they increase rather slowly along the principal diagonal towards the center of the matrix; therefore, for large n the matrix built up by symmetry and reversibility in this manner from the appropriate fundamental triangle is a good approximation to M_n^{-1} and can be improved by using the formula

$$M_n^{-1} = M_\infty^{-1}(2I - M_n M_\infty^{-1}) ,$$

which is the first step of an iterative procedure described in [4, p. 120]. As a numerical example consider a 1×8 rectangle with boundary values

$$u_i^1 = 2i , \quad u_1^4 = 9 , \quad \text{all other } u_i^j = 0 .$$

The results obtained by using elements of M_∞^{-1} differ from the exact values $u_i = i$ in the fifth decimal place, as attested in Table 4. The exact solution is $u = xy$.

3 The $m \times n$ rectangle

In applying (5) to the case of m rows of points [m = 1, 2, ...] the matrix M in (2) is of order mn and is composed of m th order submatrices: m M_n's in the principal diagonal, $-I_n$'s in the immediately adjacent diagonals, and O_n's elsewhere, where I_n and O_n are n th order unit and zero matrices, respectively; u* is a column matrix of n th order column submatrices $U_1, U_2, \ldots U_m$, where

$$\begin{aligned} &U_1 \text{ has } u_1^1+u_1^3,\ u_2^1,\ \ldots\ u_{n-1}^1,\ u_n^1+u_1^4\ , \\ &U_i \text{ has } u_i,\ 0,\ \ldots\ 0,\ u_i^4 \qquad [i = 2, 3, \ldots m-1]\ , \\ &U_m \text{ has } u^2+u_m^3,\ u_2^2,\ \ldots\ u_{n-1}^2,\ u_n^2+u_m^4\ , \end{aligned}$$

all reading downwards. The square matrix, which in this instance will be called M_{mn} [n = 1, 2, ...], is a symverse, and so also is its inverse; M_{22}^{-1}, M_{23}^{-1}, M_{24}^{-1}, and M_{33}^{-1} are exhibited in Table 5.

Results analogous to those in the preceding section are now presented. Suppose M_{mn} is known and that $M_{m+1,n}$ is partitioned

$$M_{m+1,n} = \begin{matrix} M_{mn} & K_n' \\ K_n & M_n \end{matrix}\ ,$$

where K_n is a row matrix of submatrices: m-1 O_n's and $-I_n$ at the extreme right. Let the submatric elements of M_{mn}^{-1} be A_{ij} [i, j = 1, 2, ...m], each of n th order. Also let $A_{mm} = A_m$, then application of the first five steps of (6) gives

$$(18) \qquad A_1 = M_n^{-1}, \quad A_{m+1} = (M_n - A_m)^{-1}\ ,$$

whence it follows by Theorems 4 and 6 and mathematical induction that A_{m+1} is a symverse and therefore $A_{11} = A_{mm} = A_m$ for all m. Furthermore, if the submatric elements of the last row of $M_{m+1,n}$ are designated $A_{m+1,j}^{(1)}$ [j = 1,2, ... m+1], they may be calculated according to step 6 of (6) by the formula

$$(19) \qquad A_{m+1,m+2-j}^{(1)} = A_{m+1}A_{m,m+1-j}\ ,$$

because $A_{m+1,m+1}^{(1)} = A_{m+1}$. However, since M_{mn}^{-1} is a symverse,

$$(20) \qquad A_{ij} = A_{ji} = A_{m+1-i,m+1-j} = A_{m+1-j,m+1-i}\ ,$$

so that (19) becomes

(21) $$A_{11} = A_{m+1}\ , \quad A_{i1}^{(1)} = A_{m+1}A_{i1} \qquad [i = 2, 3, \ldots]\ ,$$

where, in accord with convention, i replaces j to designate the row.

To determine the other A_{ij} of M_{mn}^{-1}, note that the equations for finding the ij th and ji th submatric elements in the product $M_{mn}M_{mn}^{-1}$ can be written

(22) $$-A_{i-1,j}+M_nA_{ij}-A_{i+1,j} = \partial_{ij}$$

(23) $$-A_{j-1,i}+M_nA_{ji}-A_{j+1,i} = \partial_{ji}\ ,$$

respectively, provided $A_{i0} = A_{i,m+1} = A_{0j} = A_{m+1,j} = 0$ and

$$\partial_{ij} = \begin{matrix} I_n \text{ if } i = j \\ 0_n \text{ if } i \neq j \end{matrix}\ .$$

Substitution of (20) into (23) yields

(24) $$-A_{i,j-1}+M_nA_{ij}-A_{i,j+1} = \partial_{ij}\ ;$$

elimination of M_nA_{ij} from (22) and (24) gives the analog of (13); finally, induction is applied to this and the analog of (12)

(25) $$A_{i+1,1} = M_nA_{i1}-A_{i-1,1}-\partial_{i1} \qquad [i = 1, 2, \ldots m]$$

to produce

(26) $$A_{ij} = \Sigma_{i-(j-1)}^{i+(j-1)}A_{r1}\ ,$$

the summation running over r as described after (14). In summary, to calculate M_{mn}^{-1}:

(27)
1) Use the methods of Section 2 to calculate M_n^{-1}
2) Use (18) to find the appropriate A_m, which is the leading submatric element in M_{mn}^{-1}
3) Use (21) to find the remaining A_{i1}
4) Use (26) to determine all the other A_{ij} in the fundamental triangle of symmetric elements

Several theorems are now proved:

<u>Theorem 8</u>: $M_nA_{ij} = A_{ij}M_n \qquad [i, j = 1, 2, \ldots m; n = 1, 2, \ldots]\ .$

Proof: The product $M_{mn}^{-1}M_{mn}$ has the ij th submatric element given by

$$(28) \qquad -A_{i,j-1}+A_{ij}M_n-A_{i,j+1} = \partial_{ij} ,$$

with the same notation as in (23). The theorem follows immediately on comparison of (24) and (28).

<u>Theorem</u> <u>9</u>: If (1) A and B are symmetric matrices and (2) AB = BA, then AB is symmetric.

Proof: Let A', B', (AB)' be the respective transposes of A, B, (AB); then (AB)' = B'A' = BA = AB.

<u>Theorem</u> <u>10</u>: A_{ij} is a symverse by application of Theorems 8, 9, 7, 6 and equations (18), (25), (26). Consequently, it is necessary to compute only the elements in the fundamental triangle of each A_{ij}.

<u>Theorem</u> <u>11</u>: $A_mA_{m+1} = A_{m+1}A_m$ $\qquad [m = 1, 2, ...]$.

Proof: Pre- and postmultiplication of the second equation in (18) by (M_n-A_m) yield

$$M_nA_{m+1}-A_mA_{m+1} = I_n$$
$$A_{m+1}M_n-A_{m+1}A_m = I_n ,$$

respectively. By the definition of A_{m+1} and Theorem 8 the first terms in each of these equations are identical; hence, the theorem.

<u>Theorem</u> <u>12</u>: The corresponding elements in A_m increase monotonically with m.

Proof: The difference between successive terms in the sequence (17) may be written

$$\begin{aligned} A_{m+1}-A_m &= A_{m+1}(A_mA_m^{-1})-(A_mA_{m+1})A_{m+1}^{-1} \\ &= (A_{m+1}A_m)A_m^{-1}-(A_{m+1}A_m)A_{m+1}^{-1} \\ &= A_{m+1}A_m(A_m^{-1}-A_{m+1}^{-1}) \end{aligned}$$

$$= A_{m+1}A_m[(M_n-A_{m-1})-(M_n-A_m)]$$
$$= A_{m+1}A_m(A_m-A_{m-1}) \ .$$

According to the statement preceding (15) A_m has only positive elements; hence the matrix $A_{m+1}-A_m$ contains all positive elements if and only if A_m-A_{m-1} does also. To complete the induction note that

$$A_2-A_1 = A_2(I_n-A_2^{-1}A_1) = A_2[I_n-(M_n-A_1)A_1] = A_2[I_n-(A_1^{-1}-A_1)A_1] = A_2A_1^2$$

has positive elements.

Since each element in A_m is never more than 1 for any m, it follows by Theorem 12 that corresponding elements have limits and that the limit matrix is a solution of

$$S = (M_n-S)^{-1} \ .$$

which upon postmultiplication by (M_n-S) and rearrangement becomes

$$S^2-SM_n = -I_n \ .$$

The left side could be written as a perfect square thus:

$$S^2-SM_n/2-M_nS/2+M_n^2/4 = M_n^2/4-I_n \ ,$$
$$(29) \qquad S(S-M_n/2)-(M_n/2)(S-M_n/2) = (M_n^2-4I_n)/4 \ ,$$
$$(S-M_n/2)^2 = (M_n^2-4I_n)/4 \ ,$$

provided $SM_n = M_nS$, but this is true by Theorem 8 and the convergence of A_m to S; consequently, from (29),

$$S = (M_n-Q_n^{1/2})/2 \ ,$$

where

$$Q_n = M_n^2-4I_n$$

is an n th order matrix having, for $n > 3$, 14's in the principal diagonal with the exception of the two corner elements which are 13's, -8's in the two immediately adjacent diagonals, 1's in the two next diagonals, and 0's elsewhere. The steps for the determination of $Q_n^{1/2}$ by using Sylvester's theorem are sketched below (for the application of the theorem to fractional exponents see [4, p. 81]:

1) Calculate the characteristic roots x_r of Q_n $[r = 1, 2, \ldots n]$
2) For each x_r form the product F_r of all matrices $x_j I_n - Q_n$ $[j \neq r]$. This can be expanded into a matric polynomial of degree n-1 in Q_n; thus it is necessary to calculate powers of Q_n

(30) 3) Also form the product Δ_r of all numbers $x_j - x_r$ $[j \neq r]$

4) Form the matrices $Z_r = F_r / \Delta_r$ $\qquad [r = 1, 2, \ldots n]$
5) Then $Q_n^{1/2} = \Sigma \pm x_r^{1/2} Z_r$, the summation running from 1 to n; the sign $\pm$ before each term must be determined in such a way that S has elements all less than 1.

The appropriate square root of Q_2 and the limit matrix $M_{\infty 2}$ are given in Table 6. In general submatric elements of $M_{\infty n}$ are expressible as sums of powers of S in the same manner that the a_{ij} are calculated in (17); consequently, once S is known, a good approximation to M_{mn}^{-1} for large m can be obtained by multiplication and addition of n th order matrices.

4 The relaxation method

Recall that the elements in the first column of M^{-1} are the solutions of (2) when u* has 1 for its first element and 0's elsewhere, so that the problem of solving (2) is equivalent to solving the Laplace boundary value problem where the sum of the values of u at the boundary points near the upper left corner interior point of the rectangle is 1 and all other boundary values are 0. The <u>relaxation</u> procedure, which is used to solve this problem, is described in [2] and [5]; an interesting geometric interpretation is given in [6]. First guess a set of values u_p $[p = 1, 2, \ldots mn]$, numbered as in

Section 1, and substitute them into the left side of (5); generally instead of being 0 this will equal some <u>residual</u> R_p. Equations for the interior points P_p and the surrounding normal neighbors P_{p-1}, P_{p+1}, P_{p-n}, P_{p+n}, some of which may be boundary points, are

(31) $$R_p = 4u_p - u_{p-1} - u_{p+1} - u_{p-n} - u_{p+n}$$

(32) $$R_{p-1} = 4u_{p-1} - u_{p-2} - u_p - u_{p-1-n} - u_{p-1+n}$$

and three others for u_{p+1}, u_{p-n}, u_{p+n}. In case $R_p \neq 0$ it is possible to reduce it to 0 by adding $-R_p$ to both sides of (31); this can be accomplished by adding $-R_p/4$ to u_p, not only in (31) but also in (32) and the three other equations; then to balance (32), etc., $-R_p/4$ must be subtracted from R_{p-1}, R_{p+1}, R_{p-n}, R_{p+n}. Usually it is best not to reduce R_p to 0 because the surrounding residuals are thereby increased in absolute value. Therefore, an arbitrary positive or negative number q is added which reduces the left side of (31) to almost zero, so that (31) and (32) etc. become

$$R_p + q = 4(u_p + q/4) - u_{p-1} - u_{p+1} - u_{p-n} - u_{p+n}$$
$$R_{p-1} - q/4 = 4u_{p-1} - u_{p-2} - (u_p + q/4) - u_{p-1-n} - u_{p-1+n}$$

and three other equations. The procedure is outlined as follows:

1) Guess a set of values u_p
2) Calculate R_p from (31) for $p = 1$ take $u_{p-1} + u_{p-n} = 1$
   for $p = kn$ [$k = 1, 2, \ldots m$] take $u_{p+1} = 0$
   for $p = kn + 1$ [$k = 1, 2, \ldots m-1$] take $u_{p-1} = 0$
(33) for $p < 1$, $p > n$ take $u_p = 0$
3) At a point where R_p is largest: add q to R_p, add $q/4$ to u_p, diminish R_{p-1}, R_{p+1}, R_{p-n}, R_{p+n} each by $q/4$ (boundary values are not to be used)
4) Continue repeating (3) until every R_p is less than a prescribed value

This method is now used to determine the first columns of inverse matrices for certain square boundaries up to 15×15. The u_p in these squares are symmetric with respect to the diagonal of the square which runs from the upper left to the lower right corner; i.e., $u_{ij} = u_{(i-1)n+j}$ $[i, j = 1, 2, \ldots n]$. Consequently, the residual at a noncorner diagonal point is

$$R_{(i-1)n+i} = 2u_{(i-1)n+i}+2u_{(i-1)n+i+n}-4u_{(i-1)n+i} ,$$

while, for the corner points,

$$R_1 = 1+2u_{n+1}-4u_1 , \qquad R_{nn} = 2u_{nn+1}-4u_{nn} \qquad [nn = n^2] .$$

In applying step 3 of (33) to any u_{in+i} just below the diagonal, the residuals $R_{(i-1)n+i}$, R_{in+i+1} at the diagonal points must be diminished by $2q/4$. In guessing take values somewhat greater than those at corresponding points in smaller squares for which the problem has already been solved and fill the remaining rows with quantities so that the u_p decreases in any column toward the bottom. These values, written out only to three decimal places, are relaxed so that the absolute value of R_p never exceeds 2 in the third place; then a fourth figure is guessed and the above repeated, etc. Various stages of this process for the 10×10 rectangle are shown in Table 7, and Table 8 contains solutions for several $n \times n$ squares $[n = 1, 2, 3, 4, 7, 10, 15]$. Finally, (26) is used to calculate the other elements for the 15×15 case and the fundamental triangles of the first submetric column in the corresponding 225×225 matrix are exhibited in Table 9. If this approximating matrix is called $M^*_{15\ 15}$, then

$$M_{15\ 15} = M^{*-1}_{15\ 15}\ (2I-M_{15\ 15}M^{*-1}_{15\ 15})$$

is an improvement.

5 Extensions

A few types of equations to which the above method may be applied are now described. The solution of Poisson's equation

$$u_{xx}+u_{yy} = \phi(x,y) \tag{34}$$

is equivalent to solving Laplace's equation with altered boundary conditions, since the difference equation corresponding to (34) for a network of squares $(h = k)$ is

$$4u(x,y)-u(x+h,y)-u(x,y+h)-u(x-h,y)-u(x,y-h) = \phi(x,y)h^2,$$

which differs from (2) only in having the i th element of u^* increased by $\phi_i h^2$, where ϕ_i is the value of $\phi(x,y)$ at P_i $[i = 1, 2, \ldots mn]$.

The matrix for the biharmonic equation

$$u_{xxxx}+2u_{xxyy}+u_{yyyy} = 0$$

in the case of an $m \times n$ rectangle is of mn th order and contains n th order submatric elements L_n in the principal diagonal, F_n in the immediately adjacent diagonals, I_n in the next two diagonals, and O_n elsewhere, where L_n has 20's in the principal diagonal, -8's in the adjacent diagonals, 1's in the next diagonals, and 0's elsewhere, while F_n has -8's in the principal diagonal, 2's in the adjacent diagonals, and 0's elsewhere.

Table 1

Inverses for $1 \times n$ rectangles

n	M_n		M_n^{-1}	M_n^{-1} (5 decimals)
1	4	$\frac{1}{4}$	1	25000
2	4 -1	$\frac{1}{15}$	4 1	26667 6667
	-1 4		1 4	6667 26667
3	4 -1 0		15 4 1	26786 7143 1786
	-1 4 -1	$\frac{1}{56}$	4 16 4	7143 28571 7143
	0 -1 4		1 4 15	1786 7143 26786
4	4 -1 0 0		56 15 4 1	26794 7177 1914 478
	-1 4 -1 0	$\frac{1}{209}$	15 60 16 4	7177 28708 7656 1914
	0 -1 4 -1		4 16 60 15	1914 7656 28708 7177
	0 0 -1 4		1 4 15 56	478 1914 7177 26794
5	4		209	26795
	-1 4		56 224	7179 28718
	0 -1 4	$\frac{1}{780}$	15 60 225	1923 7692 28846
	0 0		4 16	513 2051
	0		1	128
6	4		780	26795
	-1 4		209 836	7180 28719
	0 -1 4	$\frac{1}{2911}$	56 224 840	1924 7695 28856
	0 0 -1		15 60 225	515 2061 7729
	0 0		4 16	137 550
	0		1	34
7	4		2911	26795
	-1 4		780 3120	7180 28719
	0 -1 4		209 836 3135	1924 7695 28857
	0 0 -1 4	$\frac{1}{10864}$	56 224 840 3136	515 2062 7732 28866
	0 0 0		15 60 225	138 552 2071
	0 0		4 16	37 147
	0		1	9

For $n > 4$ only fundamental triangles are shown

Decimal points and nonsignificant zeros are omitted from last columns

Table 2

Leading elements and numerators for $1 \times n$ rectangles

n	N_n	a_n
1	1	25000 00000
2	4	26666 66667
3	15	26785 71428
4	56	26794 25837
5	209	26794 87179
6	780	26794 91584
7	2911	26794 91900
8	10864	26794 91922
9	40545	26794 91924
10	1 51316	26794 91924
11	5 64719	
12	21 07560	
13	78 65521	
14	293 54524	
15	1095 52575	
16	4088 55776	
17	15258 70529	
18	56946 26340	
19	2 12526 34831	
20	7 93159 12984	

Table 3

Some elements of the limit matrix M_{∞}

Row	*	Column 1	Column 2	Column 3	Column 4	Column 5
1	0	26794 91924				
2	1	71796 76971	28718 70789			
3	1	19237 88646	76951 54585	28856 82970		
4	2	51547 76140	20619 10456	77321 64213	28866 74640	
5	2	13812 18104	55248 72416	20718 27156	77348 21385	28867 45839
6	3	37009 62755	14803 85102	55514 44133	20725 39143	77350 12161
7	3	99166 99813	39666 79926	14875 04972	55533 51896	20725 90261
8	4	26571 71706	10628 68683	39857 57561	14880 16156	55534 88867
9	5	71198 70127	28479 48051	10679 80519	39871 27274	14880 52884
10	5	19077 63451		28616 45176	10683 47801	39872 25687
11	6	51118 36760			28626 29303	10683 74170
12	6	13697 12532				28626 99960
13	7	36728 13160				
14	8	98412 73201				
15	8	26369 61206				
16	9	70657 16255				
17	9	18932 52964				
18	10	50729 56027				

Row	*	Column 6	Column 7	Column 8	Column 9
6	0	28867 50591			
7	1	77350 25858	28867 51318		
8	1	20725 93934	77350 26842	28867 51344	
9	2	55534 98708	20725 94198	77350 26912	28867 51346
10	2	14880 55521	55534 99415	20725 94217	77350 26918
11	3	39872 32753	14880 55710	55534 99466	
12	3	10683 76064	39872 33260		
13	4	28627 05033			

*Each figure in second column indicates number of zeros between decimal point and first digit of elements read along appropriate diagonal

Table 4

Approximation to M_8^{-1} by M_∞^{-1}

M^{-1}								u*	u
26794	91924	7179	67697	1923	78864	515	47761	2	1.00005 69579
7179	67697	28718	70789	7695	15458	2061	91046	4	2.00004 98388
1923	78864	7695	15458	28856	82970	7732	16421	6	3.00004 27189
515	47761	2061	91046	7732	16421	28866	74640	8	4.00003 55985
138	12181	552	48724	2071	82716	7734	82138	10	5.00002 84779
37	00963	148	03851	555	14441	2071	82716	12	6.00002 13566
9	91670	39	66680	148	03851	552	48724	14	7.00001 42385
2	65717	9	91670	37	00963	138	12181	25	8.00000 71176

Only the first four columns of M^{-1} are shown here

Decimal points and nonsignificant zeros are omitted from M^{-1}

Table 5

Inverses for m $\times$ n rectangles

m n	M_{mn}							M^{-1}_{mn} (exact)				M^{-1}_{mn} (5 decimals)			
	4	-1	-1	0				56	16	16	8	29333	8333		
	-1	4	0	-1				16	56	8	16	8333	29333		
2 2	-1	0	4	-1			$\frac{1}{192}$	16	8	56	16	8333	4167		
	0	-1	-1	4				8	16	16	56	4167	8333		
	4	-1						712	208			29482	8613		
	-1	4						208	712			8613	29482		
3 2	-1	0	4	-1			$\frac{1}{2415}$	225	120	780	255	9317	4969	32298	10559
	0	-1						120	225	255	780	4969	9317	10559	32298
	0	0						68	47			2816	1946		
	0	0						47	68			1946	2816		
	4	-1						8948	2623			29256	8655		
	-1	4						2623	8948			8655	29256		
	-1	0	4	-1				2864	1544	9912	3312	9451	5095	32707	10929
4 2	0	-1	-1	4			$\frac{1}{30305}$	1544	2864	3312	9912	5095	9451	10929	32707
	0	0	-1	0				964	689	3167	1792	3181	2274	10450	5913
	0	0	0	-1				689	964	1792	3167	2274	3181	5913	10450
	0	0						303	248			1000	818		
	0	0						248	303			818	1000		
	4	-1	0					67	55			29910	33036		
	-1	4	-1					22	74			9821			
	0	-1	4					7				3125			
	-1	0	0	4	-1	0	$\frac{1}{224}$	22		74		9821		33036	
3 3	0	-1	0	-1	4	-1		14	28	28	84	6250	12500	12500	37500
	0	0	-1	0	-1	4		6		10		2679		4464	
	0	0	0					7				3125			
	0	0	0					6	10			2679	4464		
	0	0	0					3				1339			

Decimal points and nonsignificant zeros are omitted from last columns

Table 6

The limit matrix $M_{\infty 2}$

$$Q_2 = \begin{matrix} 13 & -8 \\ -8 & 13 \end{matrix}$$

$$X_1 = 21 \qquad X_2 = 5$$

$$\Delta_1 = -16 \qquad \Delta_2 = 16$$

$$F_1 = \begin{matrix} -8 & 8 \\ 8 & -8 \end{matrix} \qquad F_2 = \begin{matrix} 8 & 8 \\ 8 & 8 \end{matrix}$$

$$Z_1 = \begin{matrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{matrix} \qquad Z_2 = \begin{matrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{matrix}$$

$$Q_2^{1/2} = X_1 Z_1 + X_2 Z_2$$

$$S = \begin{matrix} .29533\ 9082 & .08662\ 6929 \\ .08662\ 6929 & .29533\ 9082 \end{matrix}$$

Table 7

Stages in approximating first column elements for 10 x 10 square

1	2	3	4	5	6	7	8	9	10
303									
105	75								
43	46	39							
21	30	29	19						
10	17	20	15	10					
4	8	11	7	6	3				
2	5	6	6	5	2	1			
1	3	4	5	3	1	0	0		
0	1	1	2	2	1	0	0	0	
0	0	1	1	1	0	0	0	0	0
3025									
1050	750								
425	449	354							
198	276	255	222						
102	160	178	187	140					
51	94	112	109	94	68				
31	56	69	69	60	44	25			
18	32	41	43	36	26	16	14		
8	14	22	23	20	13	3	4	1	
3	6	10	10	9	6	2	1	0	0
30230									
10461	7419								
4193	4377	3355							
1934	2539	2333	1864						
1003	1511	1573	1395	1134					
568	928	1051	1006	872	707				
339	582	698	705	641	541	427			
205	363	452	474	446	387	312	232		
118	213	271	291	280	248	203	153	102	
54	99	127	138	135	121	100	76	51	26

Decimal points and nonsignificant zeros are omitted

Table 8

Relaxation approximations to first columns for n x n squares

n										
1	25000									
2	29333									
	8333	4167								
3	29910									
	9821	6250								
	3125	2679	1339							
4	30105									
	10211	6938								
	3802	3665	2392							
	1333	1530	1120	560						
7	30216									
	10432	7362								
	4150	4292	3231							
	1875	2425	2169	1656						
	925	1363	1366	1142	842					
	464	735	790	702	542	360				
	197	323	361	332	263	178	89			
10	30233									
	10468	7433								
	4202	4395	3381							
	1944	2561	2363	1898						
	1015	1534	1604	1429	1170					
	579	948	1079	1039	911	741				
	347	597	718	730	670	570	451			
	209	370	464	490	467	410	333	250		
	120	214	275	300	294	265	218	164	109	
	53	99	129	142	143	129	108	81	56	27

Decimal points and nonsignificant zeros are omitted

Table 9

First column of submatric elements for 15 x 15 square

30240								
10480	34467							
4227	12461	35561						
1981	5321	13075	35945					
1094	2595	5705	13331	36121				
614	1478	2851	5881	13468	36227			
384	870	1654	2988	5987	13548	36283		
256	560	1007	1760	3068	6043	13596	36303	A_{11}
176	393	666	1087	1816	3116	6063		
137	282	473	722	1135	1836			
106	217	338	521	742				
80	162	265	358					
56	128	182						
48	76							
20								

10480								
7455	14909							
4429	10098	16497						
2643	6017	11102	17157					
1588	3647	6677	11566	17503				
1004	2248	4111	7023	11831	17709			
660	1468	2594	4376	7229	11988	17820		
464	1006	1733	2800	4533	7340	12065	17860	A_{21}
346	729	1212	1890	2911	4610	7380		
265	552	886	1323	1967	2951			
206	422	663	963	1363				
157	317	499	703					
111	234	357						
77	151							
40								

4227								
4429	7618							
3391	6836	9274						
2407	5047	7986	10105					
1656	3557	5878	8608	10575				
1150	2487	4179	6348	8976	10870			
831	1772	2957	4547	6643	9210	11043		
622	1301	2140	3252	4781	6816	9331	11103	A_{31}
470	990	1596	2374	3425	4902	6876		
368	765	1224	1769	2495	3485			
295	602	938	1345	1829				
234	468	723	998					
173	356	528						
121	233							
60								

Table 9

(continued)

1981								
2643	4388							
2407	4604	5901						
1961	3920	5759	6787					
1513	3116	4806	6446	7329				
1155	2399	3803	5348	6887	7700			
886	1842	2941	4244	5719	7196	7928		
687	1428	2283	3312	4553	5947	7370	8006	A_{41}
542	1128	1799	2592	3540	4727	6025		
441	913	1437	2027	2766	3714			
371	750	1141	1611	2105				
309	599	924	1219					
228	483	677						
174	306							
78								
1094								
1588	2750							
1656	3101	4034						
1513	2940	4154	4890					
1284	2566	3796	4850	5462				
1053	2140	3262	4368	5327	5899			
856	1749	2712	3739	4805	5729	6157		
696	1428	2226	3149	4141	5063	5882	6235	A_{51}
572	1173	1865	2628	3407	4294	5141		
477	1009	1575	2123	2781	3485			
437	879	1267	1728	2201				
402	695	1032	1345					
258	555	773						
153	336							
78								
614								
1004	1764							
1150	2159	2817						
1155	2203	3080	3605					
1053	2076	2991	3747	4170				
921	1841	2743	3556	4229	4581			
788	1588	2406	3225	3967	4570	4828		
667	1353	2070	2817	3566	4214	4727	4905	A_{61}
565	1149	1764	2411	3064	3723	4291		
482	976	1490	2011	2568	3141			
411	823	1223	1647	2088				
341	658	980	1300					
247	498	735						
157	324							
77								

Table 9

(continued)

384								
660	1215							
831	1546	2071						
886	1687	2334	2774					
856	1674	2390	2952	3311				
788	1559	2292	2927	3420	3702			
703	1406	2096	2760	3318	3734	3934		
618	1240	1874	2487	3074	3550	3885	4008	A_{71}
537	1086	1631	2188	2719	3225	3624		
468	928	1400	1863	2339	2793			
391	782	1160	1551	1937				
314	623	933	1234					
232	465	697						
151	306							
74								
256								
464	878							
622	1151	1574						
687	1318	1818	2192					
696	1354	1936	2381	2692				
667	1314	1917	2436	2843	3062			
618	1230	1814	2379	2806	3134	3277		
563	1118	1692	2184	2670	3021	3275	3347	A_{81}
500	1025	1488	1983	2399	2811	3091		
462	870	1316	1703	2124	2469			
370	753	1085	1457	1773				
291	585	894	1155					
215	432	655						
141	285							
70								
176								
346	646							
470	888	1218						
542	1042	1453	1755					
572	1107	1579	1953	2221				
565	1109	1607	2045	2357	2556			
537	1065	1575	2011	2380	2622	2755		
500	1003	1469	1910	2276	2579	2751	2819	A_{91}
466	904	1338	1734	2109	2405	2643		
404	801	1169	1537	1863	2173			
335	669	1000	1298	1601				
265	534	798	1064					
199	394	598						
129	263							
64								

Table 9

(continued)

137								
265	505							
368	706	982						
441	845	1188	1450					
477	923	1313	1650	1854				
482	945	1385	1717	1994	2141			
468	944	1349	1729	2004	2226	2317		
462	872	1288	1636	1961	2180	2344	2377	$A_{10,1}$
404	806	1159	1520	1812	2079	2240		
344	691	1038	1335	1638	1872			
287	576	867	1156	1395				
232	463	694	927					
176	350	523						
118	236							
60								
106								
206	401							
295	571	838						
371	732	988	1229					
437	782	1123	1358	1564				
411	828	1152	1458	1645	1801			
391	781	1163	1439	1695	1847	1956		
370	726	1068	1400	1641	1850	1953	2010	$A_{11,1}$
335	657	963	1270	1555	1747	1904		
287	572	859	1118	1376	1609			
237	489	727	965	1172				
202	392	595	781					
155	308	446						
106	209							
54								
80								
157	314							
234	466	716						
309	636	807	1030					
402	650	950	1098	1295				
341	716	941	1215	1330	1497			
314	632	981	1173	1417	1500	1630		
291	579	864	1183	1343	1550	1591	1678	$A_{12,1}$
265	523	781	1034	1316	1434	1598		
232	467	693	914	1125	1364			
202	402	600	784	952				
170	335	493	648					
133	261	383						
91	181							
48								

Table 9

(continued)

56								
173	339	487						
228	431	586	719					
258	475	663	801	918				
247	490	690	862	977	1073			
232	462	689	866	1017	1110	1181		
215	431	638	844	999	1125	1186	1218	$A_{13,1}$
199	391	586	771	952	1075	1162		
176	354	524	694	847	989			
155	309	462	600	731				
133	263	385	499					
108	209	300						
76	145							
37								

48								
77	169							
121	251	322						
174	274	408	473					
153	331	425	549	602				
157	304	472	554	667	708			
151	298	433	590	660	758	784		
141	280	416	539	681	736	815	812	$A_{14,1}$
129	259	386	507	615	738	764		
118	235	350	462	564	643			
106	209	311	407	490				
91	182	266	339					
76	148	210						
57	104							
28								

20								
40	80							
60	118	158						
78	138	195	232					
78	155	212	265	296				
77	152	225	276	325	350			
74	147	216	285	330	373	387		
70	138	207	270	333	367	401	402	$A_{15,1}$
64	130	192	255	307	361	382		
60	118	178	229	283	322			
54	108	155	206	244				
48	91	136	170					
37	76	106						
28	52							
15								

Bibliography

[1] Courant, R.; Friedrichs, K.; Lewy, H. ''Ueber die partiellen Differenzengleichungen der mathematischen Physik.'' Mathematische Annalen, 100 (1928), 22-74.

[2] Emmons, H. W. ''The Numerical Solution of Partial Differential Equations.'' Quarterly of Applied Mathematics, 2 (1944), 173-195.

[3] Frank, P.; von Mises, R. Die Differential- und Integralgleichungen [8. Auflage von Riemann-Webers Partiellen Differentialgleichungen der mathematischen Physik]. Braunschweig: Friedr. Vieweg, 1930.

[4] Frazer, R. A.; Duncan, W. J.; Collar, A. R. Elementary Matrices and Some Applications to Dynamics and Differential Equations. New York: The Macmillan Company, 1946.

[5] Southwell, R. V. Relaxation Methods in Theoretical Physics. Oxford: The Clarendon Press, 1946.

[6] Synge, J. L. ''A Geometric Interpretation of the Relaxation Method.'' Quarterly of Applied Mathematics, 2 (1944), 87-89.

[7] Tamarkin, J. D.; Feller, W. Partial Differential Equations. Providence: Brown University, 1941.

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