A PRESENTATION OF SEVERAL METHODS OF BEAM ANALYSIS AND A COMPARISON OF THE FACILITY WITH WHICH THEY MAY BE APPLIED

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By

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PART I

DEFINITION OF THE SCOPE OF THE REPORT

The primary purpose of this paper is to present, in comparatively compact form, a demonstration of the use of several of the more common methods of beam analysis. It is intended that these examples serve as an aid to others who, though somewhat unfamiliar with the techniques involved, might wish to extend one (or more) of these methods to apply to a particular problem not covered herein.

It would be undesirable and practically impossible to attempt, within the scope of a single report, analyses demonstrating every possible combination of support and loading conditions. For this reason, only a few of the more common classes of loading will be shown. The types of beams to be covered include: (1) the simple beam, (2) the cantilever beam, (3) beams fixed at one end and supported at the other, (4) fixed-ended beams, and (5) continuous beams.

For the beams which are statically determinate, the object of the analysis will be to determine the maximum deflection Δ. For statically indeterminate beams, the reactions and moments at the supports (and, in some cases, the deflections) will be found. Our only concern will be the magnitude of the deflection, since the direction usually can be determined easily by inspection.

The methods of analysis to be used will include: (A) double integration, (B) area-moment, (C) conjugate beam, (D) column analogy, (E) slope deflection, (F) virtual work, (G) real work, (H) least work, (J) theorem of three moments, and (K) moment distribution. In order to facilitate understanding of these methods, the basic theories upon which they are founded will be pointed out. For certain of the beams discussed, some of the above methods of analysis will be either inapplicable or so cumbersome and impractical as to be of little value. In such cases, no attempt will be made to include them as a part of the report.

As a result of these calculations, it should be possible to point out, in conclusion, which of these methods offer the most facile means of inquiry for each type of beam studied.

PART II

AN OUTLINE OF THE ANALYTICAL METHODS TO BE USED

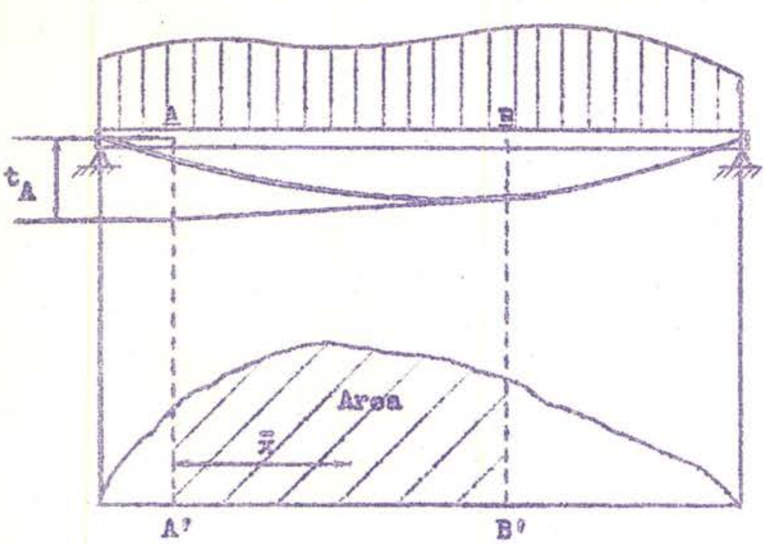
As a preliminary step a brief outline will be presented of the theory supporting each method of investigation.

(A) Method of Double Integration: The expression for the radius of curvature, ρ, of any curve is $\rho = \dfrac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$. Since the curvature of most (initially) straight beams is quite small when subjected to stresses below the elastic limit, the second order differential $\left(\frac{dy}{dx}\right)^2$ is very small and may be neglected with no appreciable error. Hence, for our purposes, $\rho = 1/\frac{d^2y}{dx^2}$. It can be shown further that $\rho = EI/M$. By equating the two expressions for ρ we arrive at the basic relationship, $EI\,\frac{d^2y}{dx^2} = M$, which is the general equation for the elastic curve of a beam. M is the bending moment, expressed in terms of x, at a distance x from the origin and y is the deflection of the beam at the same point.

(B) Area-Moment Method: Proof of the two theorems used in this method may be found in most strength of materials textbooks. The theorems may be stated as follows:

Theorem I - The change in the slope between two points on the elastic curve of a straight beam subjected to bending is represented in magnitude by the area under the M/EI diagram between the two points.

Theorem II - When a straight beam is subjected to bending, the distance of any point on the elastic curve, measured normal to the original position of the bending axis, from a tangent drawn at any other point on the elastic curve, is represented in magnitude by the moment of the area under the M/EI diagram between the two points about an ordinate through the first point.



Load Diagram

$\theta_B - \theta_A = \text{Area}$

$t_A = \text{Area}\ (\bar{\bar{x}})$

M/EI Diagram

Fig. 1

The two theorems above may be expressed mathematically by the equations: $\Delta\theta = \int_A^B \frac{Mdx}{EI}$ and $t_A = \int_A^B \frac{Mxdx}{EI}$, where M is expressed in terms of the distance, x, measured from the point, A.

(C) Conjugate Beam Method: From the similarity of the relationships $\frac{d^2M}{dx^2} = w$ and $\frac{d^2y}{dx^2} = \frac{M}{EI}$, it may be seen that the load bears the same relationship to the moment that the M/EI bears to the deflection. Thus if the real beam is replaced by a conjugate beam (which, in some cases differs from the real beam in type of support) and this conjugate beam then loaded with the M/EI diagram, the deflection of the real beam at a given point will be equal in magnitude to the moment in the conjugate beam at the same point.

The slope in the real beam at a given point, incidentally, will be equal in magnitude to the shear in the conjugate beam.

(D) Column Analogy: This is the method devised by Professor Hardy Cross for determining the moment at any point of a statically indeterminate structure which forms a continuous ring (the earth is assumed to be a part of this ring) without any members crossing or intersecting at a joint. The method applies only to single spans.

In the use of Column Analogy, the member is treated as a short column of width L and thickness 1/EI, with its axis symmetrical to the axis of the structure, and loaded with the angle change (from any cause). If, as in this paper, the angle change is due to moment produced by loading the structure, the structure is altered in some way (so as to be statically determinate) and the moment curve of this simple structure applied as a load on the analogous column. The stress ($P/A \pm Mc/I$) in the column is then computed and that stress is equal to the indeterminate moment at the point under consideration. Then the actual moment at that point is the statical moment (for the assumed statically determinate condition) minus the indeterminate moment.

This method may be applied to single bents as well as to single span beams.

(E) Slope Deflection: The moment at the end of a rigid beam may be influenced by four factors: (1) the fixed end moment due to loads on the beam, (2) the angle through which that end rotates, (3) the angle through which the far end rotates, and (4) the relative deflection of the two ends. If M_{ab} represents the moment at end, a, of span ab, then the general equation which takes these four factors into consideration is:

$$M_{ab} = M_{Fab} + \frac{EI}{L}\left(4\theta_a + 2\theta_b - 6\frac{\Delta}{L}\right)$$

and for the moment at end, b

$$M_{ba} = M_{Fba} + \frac{EI}{L}\left(2\theta_a + 4\theta_b - 6\frac{\Delta}{L}\right)$$

If these equations are applied, together with the statical equations of equilibrium which may be written, there will generally be a sufficient number of equations to determine the unknown values for moment and angle change in the structure. In the above equations the numerical values of the fixed end moments should be used with their proper signs: positive when the resisting moment acts clockwise and negative when it acts counterclockwise. The unknown angles θ should be assumed positive or clockwise. The sign of the fixed end moments due to Δ must be consistent with the direction of Δ. Sometimes the ratio Δ/L is represented by the letter R.

(F) <u>Virtual Work:</u> The expression for the elastic deflection of a beam due to moment is $\Delta = \int \frac{Mmdx}{EI}$, where M is the bending moment at x distance from the origin due to the applied loads and m is the bending moment at the same point due to a unit load applied at the point where the deflection is to be found. This equation may be derived as follows:

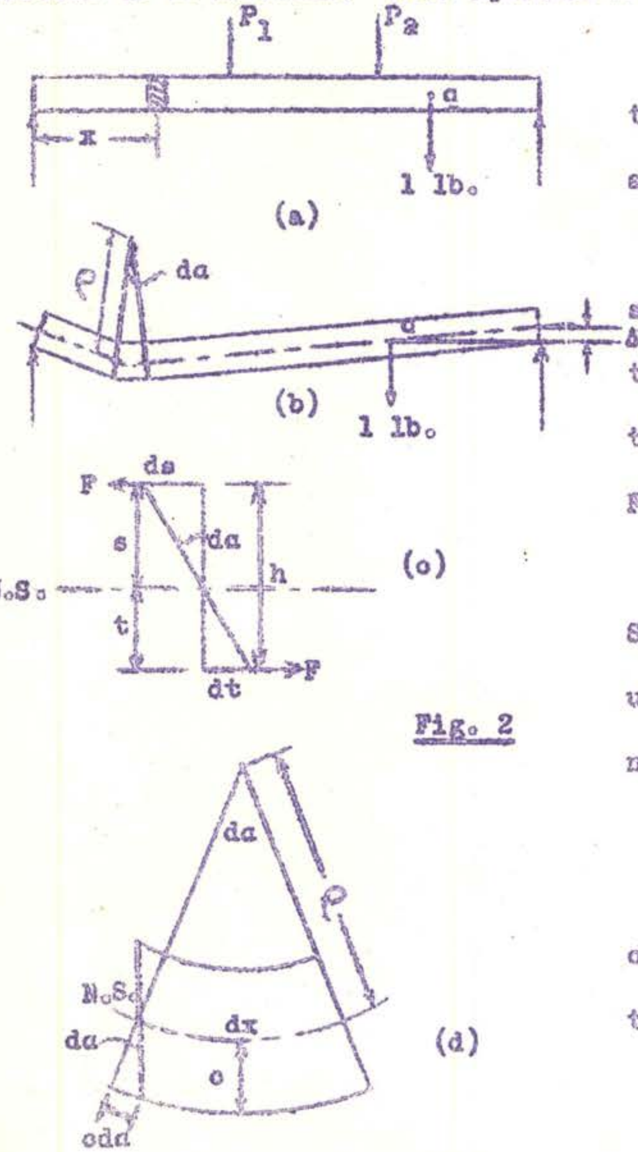


Fig. 2

Assume that all portions of the beam, except the dx portion, are infinitely stiff.

Let F (fig. 2c) be the fiber stress due to a unit load. Then the work done on the dx portion of the beam by the unit load is Fds + Fdt. But Fds + Fdt =

Fsdα + Ftdα = Fdα(s+t) or Fhdα

Since Fh = m, this may be written u = mdα. And, equating the external and internal work we get:

$$1 \cdot \Delta = m d\alpha \quad \text{(Eq. 1)}$$

Examining the deformed portion of the beam (fig. 2d) it is seen

that $\varepsilon = \frac{cd\alpha}{dx} = \frac{S}{E} = \frac{Mc}{EI}$

$$\text{or} \quad d\alpha = \frac{Mdx}{EI}$$

Then from Eq. 1 we get $1 \cdot \Delta = \frac{Mmdx}{EI}$ or $\Delta = \int\frac{Mmdx}{EI}$ (Eq. 2) if the beam is deformed along its entire length.

The equations expressing the deflection due to torque and shear may be derived similarly. They are $\Delta_T = \int\frac{Ttdx}{E_sJ}$ and $\Delta_s = \int\frac{Vvdx}{E_sA}$.

(G) Real Work: This method is somewhat limited in application since it can be used to find a deflection only where there is a load, only in the direction of the load, and only when there are no other loads. It, of course, will not apply to distributed loads.

The expression for the elastic energy which will be absorbed by a given volume of material when uniformly stressed is $U = 1/2\ S^2/E \times Vol.$ In the case of beams, where the material is not uniformly stressed, this fact may be used in setting up a differential expression for the energy. For the internal energy or work,

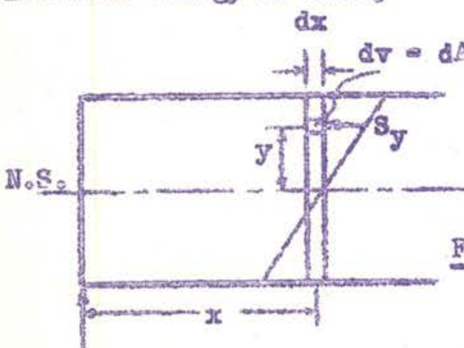


Fig. 3

$$dU = \frac{1}{2}\frac{S_y^2}{E}\,dV = \frac{S_y^2}{2E}\,dAdx$$

but $S_y = \frac{My}{I}$

so, $dU = \frac{M^2y^2}{2EI^2}\,dAdx$

$$U = \int\frac{M^2dx}{2EI^2}\int y^2dA = \int\frac{M^2dx}{2EI}$$

Since the external work must equal the internal work, $1/2\ P\Delta = \int\frac{M^2dx}{2EI}$, or $\Delta = \frac{1}{PEI}\int M^2dx$.

(H) Least Work: Castigliano's Theorem, which states that the derivative of the work with respect to a given load $\frac{dU}{dP_1}$ is equal to the distance through which the load moves (y_1), provides a tool by means of which deflections may be determined. A fairly brief, semi-geometric proof[1] of Castigliano's Theorem is as follows:

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1 Irving P. Church, Mechanics of Internal Work, pp. 126-127. Proof due to Prof. E. W. Rettger, Cornell University.

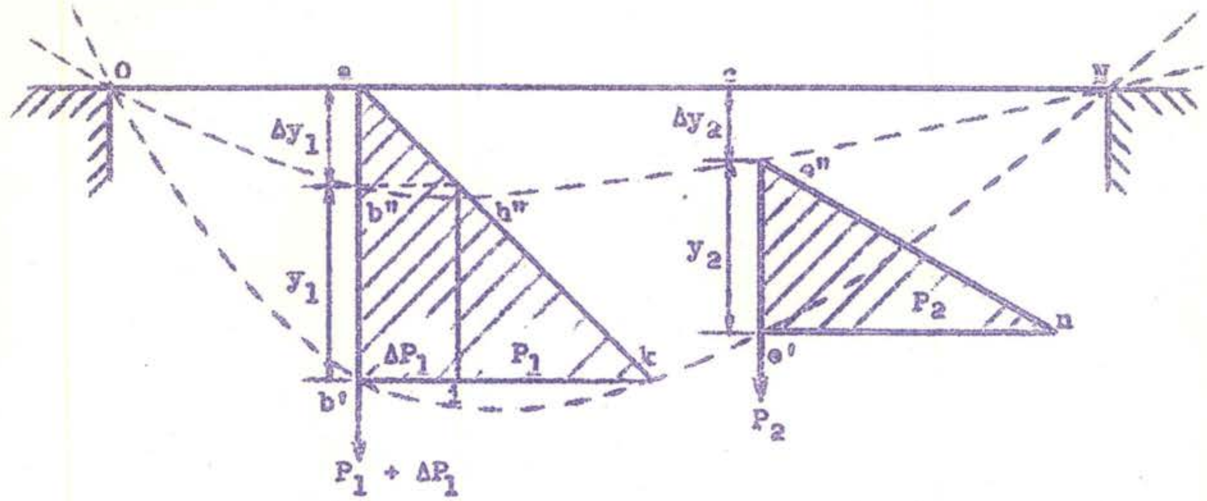


Fig. 4

Loads P_1 and P_2 are applied at points a and c, respectively, of the span shown above. In investigating the effect on the internal work, U, of applying an increment, ΔP_1, to the load P_1, let us utilize the fact that it is immaterial whether the loads be applied simultaneously, or successively in any order, the final result being the same. Then conceive the increment ΔP_1 of the variable load P_1 to be applied first of all, even before P_1 and P_2 are placed on the structure.

If ΔP_1 is applied gradually, the total external work done so far is $\frac{1}{2}\Delta P_1 \Delta y_1$, which is represented by the area of the triangle ab"h", where b"h" equals ΔP_1 (to some scale).

Now, let the loads P_1 and P_2 be applied gradually and simultaneously until the final positions b' and e' of points a and c are reached. The external work done by P_1 is $\frac{1}{2} P_1 y_1$, represented by the area h"ik; and that done by P_2 is $\frac{1}{2} P_2 y_2$, represented by the area e"e'n. The additional external work done by ΔP_1, which acts with constant force through the distance y_1, is equal to $\Delta P_1 y_1$. Thus the total external work (which is equal to the total internal work, U') is represented by the entire shaded area in figure 4.

Next, let the structure be entirely unloaded and the two loads, P_1 and P_2, be applied gradually and simultaneously. The total external work (which is equal to the total internal work, U) is equal to $\frac{1}{2} P_1 y_1 + \frac{1}{2} P_2 y_2$. These two terms are represented, respectively, by the areas of the two triangles, h"ik and e"e'n, which form a part of the shaded areas shown in figure 4. Therefore, it is evident that the difference U' - U, or ΔU, is represented by the sum of the areas of the triangle, ab"h", and the rectangle, b"b'ih". Therefore

$$\Delta U = \frac{1}{2} \Delta P_1 \Delta y_1 + \Delta P_1 y_1$$

$$\text{or} \quad \frac{\Delta U}{\Delta P_1} = \frac{1}{2} \Delta y_1 + y_1$$

$$\text{as} \quad \Delta P_1 \rightarrow 0 \, , \; \Delta y_1 \rightarrow 0$$

$$\text{and} \quad \lim_{\Delta P_1 \rightarrow 0} \text{ of } \frac{\Delta U}{\Delta P_1} = y_1$$

$$\text{That is,} \quad \frac{dU}{dP_1} = y_1$$

Since any of the loads may be considered as the variable, the preceding relationship is generally expressed as a partial derivative:

$$\frac{\partial U}{\partial P_1} = y_1$$

The internal energy due to bending has been shown (on page 7) to be

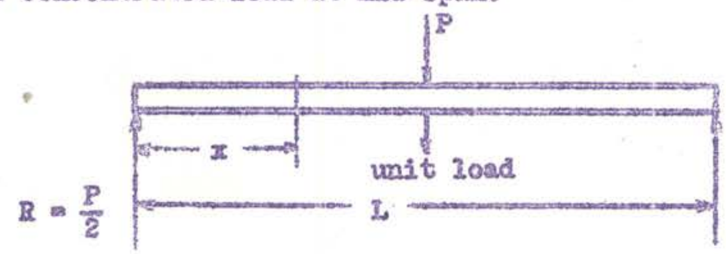
$$U = \int \frac{M^2 dx}{2EI}$$

$$\text{Then} \quad \frac{\partial U}{\partial P} = \int \frac{M \frac{\partial M}{\partial P} dx}{EI}$$

$$\text{or} \quad \Delta = \int \frac{M \frac{\partial M}{\partial P} dx}{EI}$$

where Δ represents the deflection of the beam at the point of application of the load P.

The similarity between this equation and the virtual work equation (shown on page 7) should be pointed out at this time. It will be noted that, where in the virtual work equation there is an m denoting the moment due to a dummy unit load, here we have $\frac{\partial M}{\partial P}$. The two equations lead to the same result since m and $\frac{\partial M}{\partial P}$ are identical. For example, consider a simple beam with a concentrated load at mid-span:

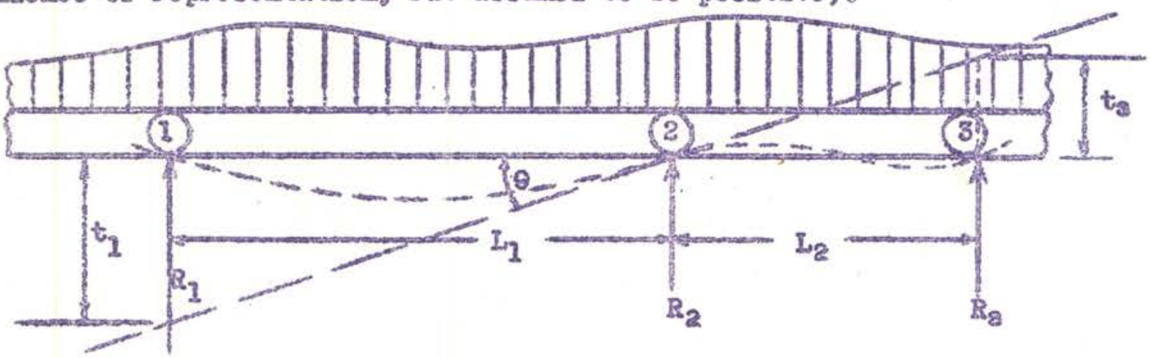


$M_x = \frac{P}{2} x$ and $m_x = \frac{1}{2} x$

$$\frac{\partial M_x}{\partial P} = \frac{\partial \left(\frac{P}{2} x\right)}{\partial P} = \frac{1}{2} x$$, which is the same as m_x.

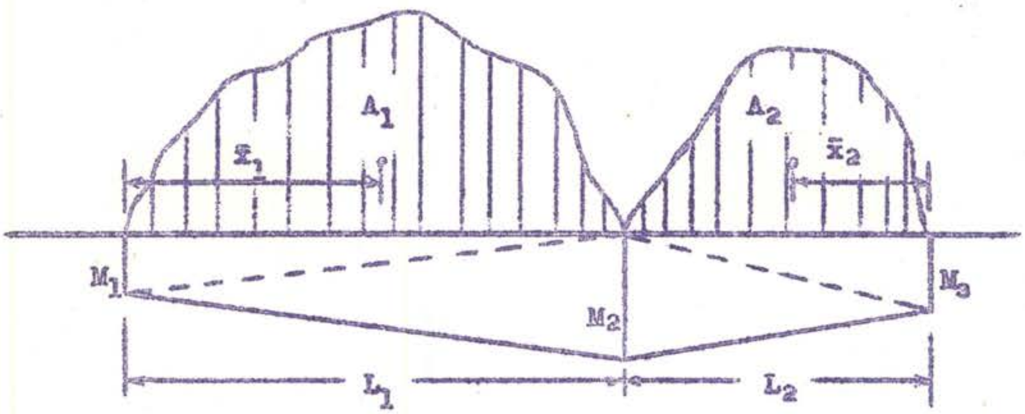
If it is desired to find a deflection, by the method of least work, at some point other than where a load is acting it will be necessary to apply a dummy load of zero magnitude at the point under consideration. The value of zero for P_1 (the dummy load) may be substituted at any time after the partial derivative has been extracted.

(J) <u>Theorem of Three Moments</u>: If a beam is continuous over three or more supports, a certain relationship exists between the moments at any three consecutive supports and the loads on the two included spans. A convenient derivation of this relationship may be made by use of the method of area-moments--as previously outlined under (B). In fig. 5 let: θ be the slope of the tangent drawn to the elastic curve at the center support; t_1 and t_3, the tangential deviations of points 1 and 3, respectively, from the tangent drawn at the center support; A_1 and A_2, the areas under the simple moment diagrams for the two spans; $\bar{x}_1$, the distance from the left support to the centroid of the A_1 area; $\bar{x}_2$, the distance from the right support to the centroid of the A_2 area; and M_1, M_2, and M_3, the moments at the left, center, and right supports, respectively (plotted below the base line for convenience of representation, but assumed to be positive).



$$\frac{t_1}{L_1} = \frac{-t_3}{L_2} \quad \text{(Eq. 1)}$$

Fig. 5



$$t_1 = \left[A_1\bar{x}_1 + \frac{1}{2} M_1 L_1 \cdot \frac{L_1}{3} + \frac{1}{2} M_2 L_1 \cdot \frac{2L_1}{3}\right] \frac{1}{EI_1}$$

$$t_1 = \frac{1}{6EI}\left[6A_1\bar{x}_1 + M_1 L_1^2 + 2M_2 L_1^2\right]$$

$$t_2 = \left[A_2\bar{x}_2 + \frac{1}{2} M_3 L_2 \cdot \frac{L_2}{3} + \frac{1}{2} M_2 L_2 \cdot \frac{2L_2}{3}\right] \frac{1}{EI_2}$$

$$t_2 = \frac{1}{6EI_2}\left[6A_2\bar{x}_2 + M_3 L_2^2 + 2M_2 L_2^2\right]$$

then, from Eq. 1

$$\frac{1}{6EI_1L_1}\left[6A_1\bar{x}_1 + M_1 L_1^2 + 2M_2 L_1^2\right] = \frac{-1}{6EI_2L_2}\left[6A_2\bar{x}_2 + M_3 L_2^2 + 2M_2 L_2^2\right]$$

and if $I_1 = I_2$, then

$$\frac{6A_1\bar{x}_1}{L_1} + M_1L_1 + 2M_2L_1 = -\frac{6A_2\bar{x}_2}{L_2} - M_3L_2 - 2M_2L_2$$

$$\text{or} \quad M_1L_1 + 2M_2 (L_1 + L_2) + M_3L_2 = -\frac{6A_1\bar{x}_1}{L_1} - \frac{6A_2\bar{x}_2}{L_2}$$

which is the general Theorem of Three Moments equation for continuous beams of constant cross section and made of one material, where points 1, 2, and 3 remain on the same straight line.

(K) Moment Distribution:[2] The moment distribution method of analyzing statically indeterminate structures was developed by Professor Hardy Cross while at the University of Illinois, and originally published by the American Concrete Institute in 1929.

If the B end of an unloaded structural member AB is fully restrained against both translation and rotation, and the A end restrained against translation but caused to rotate through an angle θ by a moment M applied at that end, there will be a moment M' produced at the B end which bears a fixed relationship to the moment M such that the ratio M'/M is a constant. This constant will be termed the "carry-over factor". If the member is of constant cross-section, it may be demonstrated by the method of area-moments that $M_A = 4EI\Delta\theta/L$ and that $M_B = -\frac{1}{2} M_A$, if tension in the lower fibers is taken as a positive moment and tension in the upper fibers, as a negative moment. That is, the carry-over factor is equal to a negative one-half. For a given angle change $\Delta\theta$, the moments are proportional to the I/L ratio for the member. This ratio will be called the "stiffness factor" and will be denoted by K. The effect of translation of a joint will not be considered in this report.

To make use of the relations shown above, consider all joints of a structure to be locked in position against translation and rotation.

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[2] Fred L. Plummer, Fundamentals of Indeterminate Structures. pp. 135-139

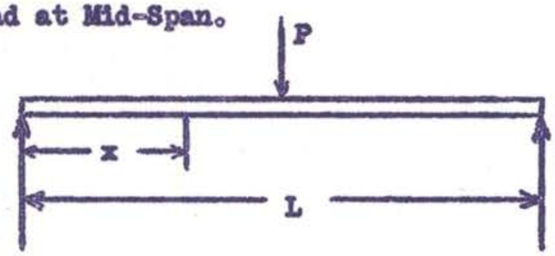
Compute the fixed-end moments due to the loads on the members. In general, there will now exist an unbalanced moment at each joint. Now unlock one joint for rotation only while keeping all other joints locked. The joint will rotate sufficiently to balance the moments at that joint. The change in moment in each of the members coming into the joint will be proportional to the stiffness factor K of the member, and the total of these corrections will equal the original unbalanced moment. At the same time there will be produced at the far end of each member a carry-over moment of the opposite algebraic sign and one-half the magnitude of the change in moment at the near end. Now if this joint is again considered locked in position against translation and rotation and each other joint, in turn, unlocked for rotation, there will result a first set of corrected values much nearer to the actual values of the moments than were the original fixed-end moments. Due to the carry-over moments, the moments about each joint will still be unbalanced, and the process must be repeated until the carry-over moments are negligible as compared to the actual moment in the member. Usually no more than five repetitions of the process are required to produce values very close to the true moments in the structure.

PART III

APPLICATION OF THE VARIOUS METHODS OF ANALYSIS TO SEVERAL SPECIFIC CASES

THE SIMPLY SUPPORTED BEAM

1. Concentrated Load at Mid-Span.



$R_1 = P/2$ $R_2 = P/2$

(A) By Double Integration:

$$EI\,d^2y/dx^2 = M_x = Px/2$$

$$EI\,\frac{dy}{dx} = \frac{Px^2}{4} + C_1$$

when $x = L/2$, $dy/dx = 0$, so $C_1 = -\frac{PL^2}{16}$

$$EI\,\frac{dy}{dx} = \frac{Px^2}{4} - \frac{PL^2}{16}$$

$$EIy = \frac{Px^3}{12} - \frac{PL^2x}{16} + C_2$$

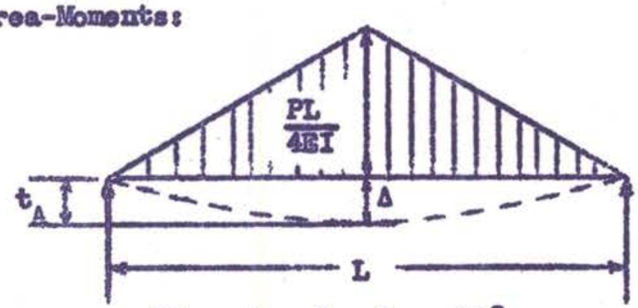
when $x = 0$, $y = 0$, so $C_2 = 0$

and when $x = L/2$, $y = \Delta$, then

$$EI\Delta = \frac{PL^3}{96} - \frac{PL^3}{32} = -\frac{PL^3}{48}$$

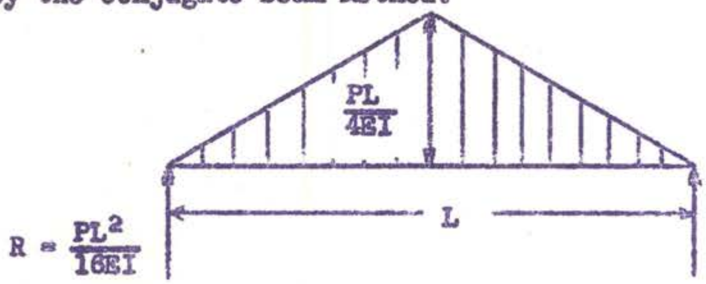
$$\Delta = \frac{PL^3}{48EI}$$

(B) By Area-Moments:



$$t_A = \Delta = \frac{PL}{4EI} \cdot \frac{L}{2} \cdot \frac{1}{2} \cdot \frac{L}{3} = \frac{PL^3}{48EI}$$

(C) By the Conjugate Beam Method:



The maximum moment in the conjugate beam will occur at mid-span and will be equal to the maximum deflection Δ of the real beam

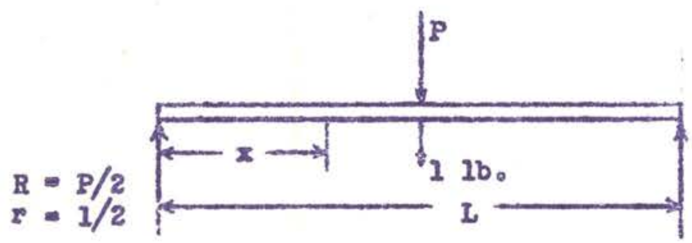
$$\Delta = \frac{PL^2}{16EI} \cdot \frac{L}{2} - \frac{PL}{4EI} \cdot \frac{L}{2} \cdot \frac{1}{2} \cdot \frac{L}{6}$$

$$\Delta = \frac{PL^3}{32EI} - \frac{PL^3}{96EI} = \frac{PL^3}{48EI}$$

(D) Column Analogy does not apply.

(E) Slope Deflection may be used, but is uncommon for this purpose.

(F) By the Method of Virtual Work:



$$M_x = Px/2 \text{ , } m_x = x/2$$

$$\Delta = \frac{1}{EI} \int_0^L M_x m_x dx$$

and, due to symmetry of loading and support

$$\Delta = \frac{2}{EI} \int_0^{L/2} M_x m_x dx$$

$$\Delta = \frac{2}{EI}\int_0^{L/2} \frac{Px}{2} \cdot \frac{x}{2} dx = \frac{P}{2EI}\int_0^{L/2} x^2 dx$$

$$\Delta = \frac{P}{6EI} (L/2)^3 = \frac{PL^3}{48EI}$$

(G) By the Method of Real Work:

$$\Delta = \frac{1}{PEI} \int_0^L M^2 dx$$

$$M_x = Px/2$$

$$\Delta = \frac{2}{PEI}\int_0^{L/2} \left(\frac{Px}{2}\right)^2 dx = \frac{P}{2EI}\int_0^{L/2} x^2 dx$$

$$\Delta = \frac{P}{6EI}\,(L/2)^3 = \frac{PL^3}{48EI}$$

(H) By the Method of Least Work:

$$U = \frac{1}{2EI}\int_0^L M^2 dx$$

$$M_x = Px/2$$

$$U = \frac{2}{2EI}\int_0^{L/2}\left(\frac{Px}{2}\right)^2 dx = \frac{1}{4EI}\int_0^{L/2} P^2x^2 dx$$

$$\Delta = \frac{\partial U}{\partial P} = \frac{1}{4EI}\int_0^{L/2} 2Px^2 dx$$

$$\Delta = \frac{P}{2EI}\,\frac{(L/2)^3}{3} = \frac{PL^3}{48EI}$$

(J) In the derivation of the Theorem of Three Moments (Part II, J) it was assumed that points 1, 2, and 3 were originally on the same straight line, and that they retained these positions after the application of the loads. The three points chosen are not necessarily reaction points, nor need they retain their original positions when the loads are applied. A formula[3] which takes into account the deflection d of point 2 with respect to a straight line through points 1 and 3 is:

$$\frac{M_1L_1}{6} + \frac{M_2(L_1 + L_2)}{3} + \frac{M_3L_2}{6} + \frac{A_1\bar{x}_1}{L_1} + \frac{A_2\bar{x}_2}{L_2} = EId\left(\frac{1}{L_1} + \frac{1}{L_2}\right)$$

This method is not commonly applied to statically determinate beams, though it does provide rather an easy means for determining the deflection at any desired point. For this particular case: $L_1 = L_2 = L/2$, M_1 and M_3 are equal to zero, $M_2 = PL/4$, and both $A_1\bar{x}_1$ and $A_2\bar{x}_2$ are equal to zero since the only load acts at point 2. Therefore

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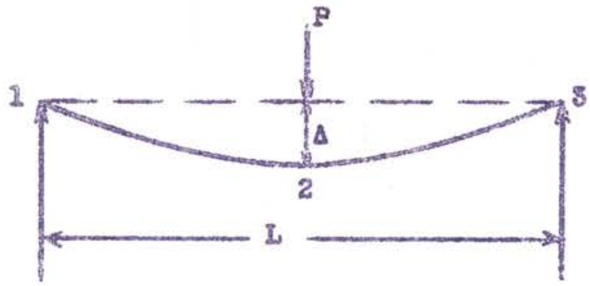
3 George, Rettger, and Howell, Mechanics of Materials, 2d edition, Chapter IX.



$$0 + \frac{PL}{4}\,\frac{\frac{L}{2} + \frac{L}{2}}{3} + 0 + 0 + 0 = EI\Delta\left(\frac{1}{L/2} + \frac{1}{L/2}\right)$$

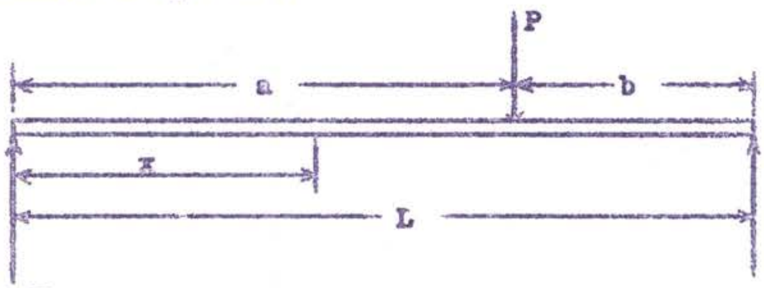
$$\frac{PL^2}{12} = EI\Delta\left(\frac{4}{L}\right)$$

$$\Delta = \frac{PL^3}{48EI}$$

(K) Moment Distribution does not apply.

2. Concentrated Load at Any Point.

(A) By Double Integration:



$a \gg \frac{L}{2}$

$R = \frac{Pb}{L}$

For portion of beam to left of load:

(1) $EI\,\frac{d^2y}{dx^2} = \frac{Pb}{L}\,x$

(2) $\frac{EI}{P}\,\frac{dy}{dx} = \frac{bx^2}{2L} + C_1$

For portion of beam to right of load:

$EI\,\frac{d^2y}{dx^2} = \frac{Pb}{L}\,x - P(x - a)$

$\frac{EI}{P}\,\frac{dy}{dx} = \frac{bx^2}{2L} - \frac{(x - a)^2}{2} + C_1'$

when $x = a$

$$\frac{dy}{dx} = \frac{dy}{dx}$$

$$\therefore C_1 = C_1'$$

$$(3)\quad \frac{EI}{P}y = \frac{bx^3}{6L} + C_1x + C_2 \qquad\Bigg|\qquad \frac{EI}{P}y = \frac{bx^3}{6L} - \frac{(x-a)^3}{6} + C_1x + C_2'$$

when $x = 0$, $y = 0$, $\therefore C_2 = 0$, when $x = a$

$$y = y$$

$$\frac{ba^3}{6L} + C_1a = \frac{ba^3}{6L} - 0 + C_1a + C_2'$$

$$0 = C_2'$$

when $x = L$, $y = 0$

$$\therefore C_1 = \frac{(L-a)^3}{6L} - \frac{bL^3}{6L^2}$$

Now, to determine the distance x to the point of maximum deflection Δ, it will be assumed that the maximum deflection occurs to the left of the load. Then at this point the slope (dy/dx) of the elastic curve will be equal to zero. If, in the equation (2) for the slope for the portion of the beam to the left of the load, zero is substituted for dy/dx and the proper value for C_1 inserted, the resulting equation may be solved for x, the distance to the point of maximum deflection.

$$\frac{bx^2}{2L} + \frac{(L-a)^3}{6L} - \frac{bL^3}{6L^2} = 0$$

Since $L - a = b$, this may be written

$$bx^2 + \frac{b^3}{3} - \frac{bL^2}{3} = 0 \text{ , whence}$$

$$x^2 = \frac{L^2 - b^2}{3}$$

$$x = \frac{1}{3}\sqrt{3(L^2 - b^2)}$$

This expression for x may now be used in equation (3), with the constants properly evaluated, to determine the maximum deflection Δ.

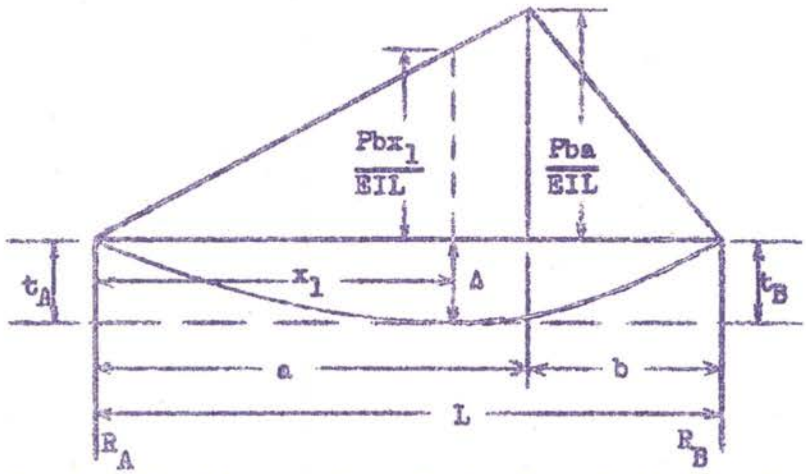
$$\frac{EI}{P}\Delta = \frac{b\left(\frac{1}{3}\sqrt{3(L^2 - b^2)}\right)^3}{6L} + \frac{b^3L - bL^3}{6L^2} \cdot \frac{1}{3}\sqrt{3(L^2 - b^2)}$$

$$\frac{EI}{P}\Delta = -\frac{b}{27L}(L^2 - b^2)\sqrt{3(L^2 - b^2)}$$

$$\Delta = \frac{Pb(L^2 - b^2)\sqrt{3(L^2 - b^2)}}{27EIL}$$

The deflection y of the beam at any point to the left of the load may be found from equation (3) by substituting therein the value for x measured from the left end.

(B) By Area-Moments:



Let x be the distance from R_A to the point of maximum deflection Δ.

(1) $t_A = \frac{Pbx_1}{EIL} \cdot \frac{x_1}{2} \cdot \frac{2x_1}{3} = \frac{Pbx_1^3}{3EIL}$

(2) $t_B = \frac{Pba}{EIL} \cdot \frac{a}{2}\left(b + \frac{a}{3}\right) + \frac{Pba}{EIL} \cdot \frac{b}{2} \cdot \frac{2b}{3} - \frac{Pbx_1}{EIL} \cdot \frac{x_1}{2}\left(L - \frac{2x_1}{3}\right)$

$= \frac{Pba}{2EIL}\left(ab + \frac{a^2}{3} + \frac{2b^2}{3}\right) - \frac{Pbx_1^2}{2EI} + \frac{Pbx_1^3}{3EIL}$

and, since $t_A = t_B$

$$\frac{Pbx_1^3}{3EIL} = \frac{Pba}{2EIL}\left(ab + \frac{a^2}{3} + \frac{2b^2}{3}\right) - \frac{Pbx_1^2}{2EI} + \frac{Pbx_1^3}{3EIL}$$

from which $x_1^2 = \frac{a(a + b)}{3}$

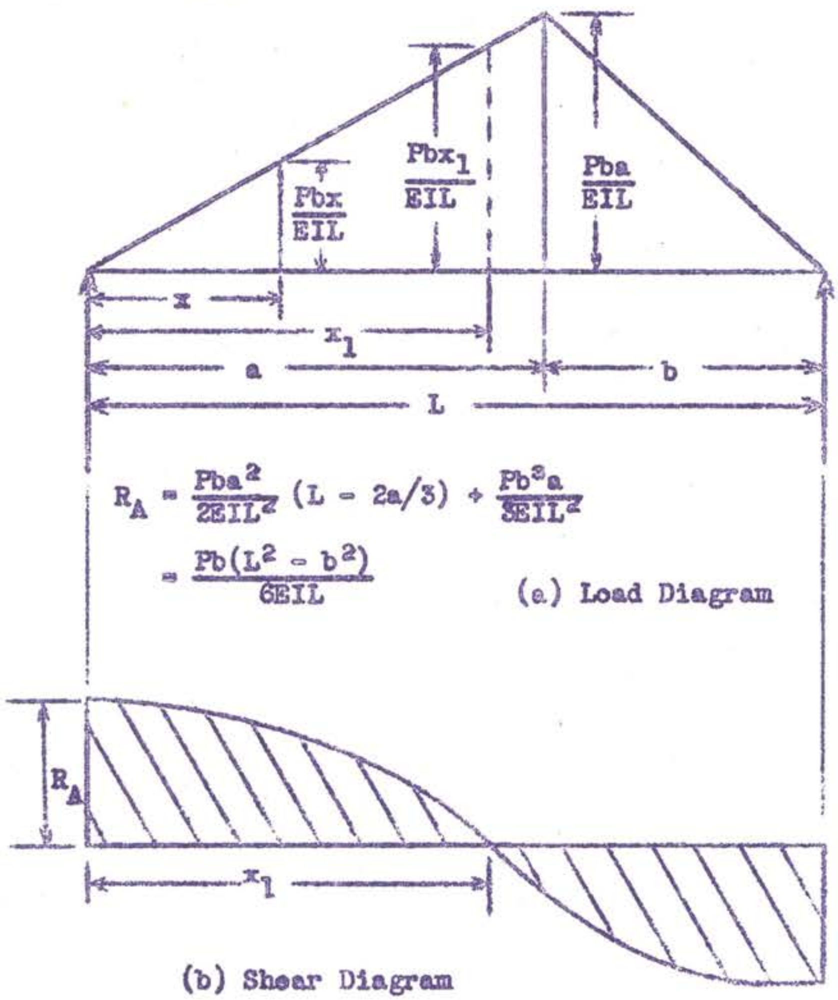
If $L - b$ is substituted for a in the preceding expression, it is found that

$$x_1 = \frac{1}{3}\sqrt{3(L^2 - b^2)}$$

By substituting this value for x_1 into equation (1), it is seen that

$$t_A = \Delta = \frac{Pb(L^2 - b^2)\sqrt{3(L^2 - b^2)}}{27EIL}$$

(C) By the Conjugate Beam Method:



(a) Load Diagram

(b) Shear Diagram

The shear equation for the conjugate beam for values of x between zero and a is:

$$V_x = \frac{Pb(L^2 - b^2)}{6EIL} - \frac{Pbx}{EIL} \cdot \frac{x}{2}$$

Where the shear in the conjugate beam is equal to zero, the moment in the conjugate beam will be a maximum; and the maximum deflection in the real beam will occur at this point. If the above shear equation is set equal to zero, the value for x found therefrom will be equal to x_1.

$$\frac{Pb(L^2 - b^2)}{6EIL} - \frac{Pbx^2}{2EIL} = 0$$

$$x^2 = \frac{L^2 - b^2}{3}$$

$$x = \frac{1}{3}\sqrt{3(L^2 - b^2)} = x_1$$

Now the maximum moment in the conjugate beam (Δ for the real beam) will be

$$M\ (\text{max.}) = \Delta = R_A \cdot x_1 - \frac{Pbx_1}{EIL} \cdot \frac{x_1}{2} \cdot \frac{x_1}{3}$$

$$\Delta = \frac{Pb(L^2 - b^2)}{6EIL} \cdot \frac{1}{3}\sqrt{3(L^2 - b^2)} - \frac{Pb}{6EIL}\left(\frac{1}{3}\sqrt{3(L^2 - b^2)}\right)^3$$

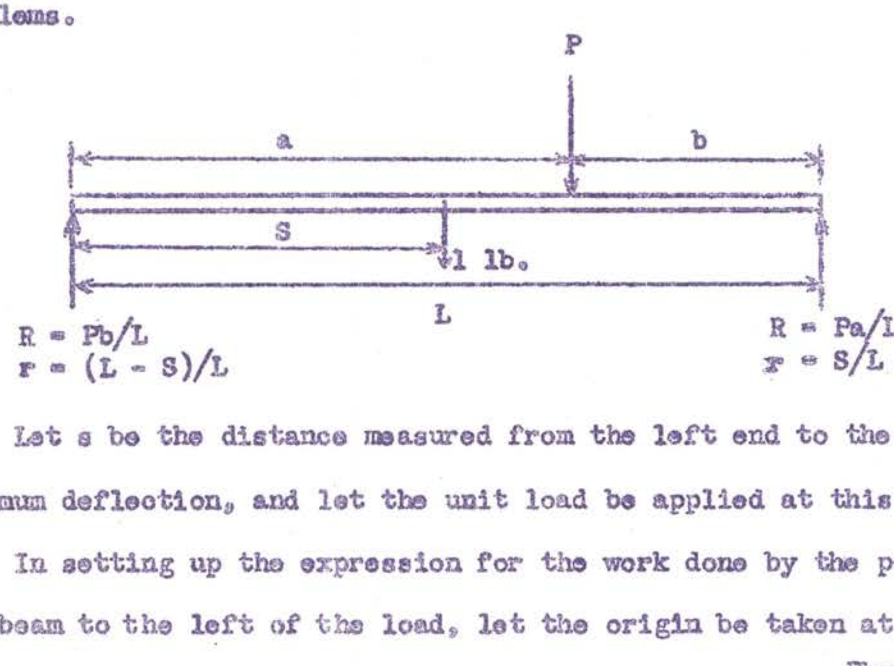
$$\Delta = \frac{Pb(L^2 - b^2)\sqrt{3(L^2 - b^2)}}{27EIL}$$

(D) Column Analogy does not apply.

(E) Slope Deflection is not commonly used.

(F) By the Method of Virtual Work:

The solution of this problem by virtual work is quite tedious and the likelihood of making mechanical errors (in algebra, etc.) is great. It is shown here principally to indicate the method of attack for such problems.



Let s be the distance measured from the left end to the point of maximum deflection, and let the unit load be applied at this point.

In setting up the expression for the work done by the portion of the beam to the left of the load, let the origin be taken at the left reaction. Then for values of x between zero and s, $M_x = \frac{Pbx}{L}$, and

$m_x = \frac{(L - s)}{L} x$. For the interval between $x = s$ and $x = a$, $M_x = \frac{Pbx}{L}$, and $m_x = \frac{(L - s)}{L} x - (x - s)$.

For the part of the beam to the right of the load, let the origin be taken at the right reaction. For values of x between zero and b, $M_x = \frac{Pax}{L}$, and $m_x = \frac{sx}{L}$. Then

$$(1) \quad EI\Delta = \int_0^s \frac{Pb(L - s)x^2dx}{L^2} + \int_s^a \frac{Pbx}{L}\left[\frac{(L - s)x}{L} - (x - s)\right] dx$$

$$+ \int_0^b \frac{Pasx^2dx}{L^2}$$

If the indicated operations are performed, the preceding expression reduces to

$$(2) \quad EI\Delta = \frac{Pba^3}{3L^2}(L - s) - \frac{Pba^3}{3L} + \frac{Pbs^3}{3L} + \frac{Pba^2s}{2L} - \frac{Pbs^3}{2L} + \frac{Pasb^3}{3L^2}$$

$$\text{or, } \frac{6EIL^2\Delta}{Pb} = - 2a^3s + 3La^2s - Ls^2 + 2ab^2s$$

If $\underline{L - b}$ is substituted for $\underline{a}$, it is found that

$$(3) \quad \frac{6EIL\Delta}{Pb} = (L^2 - b^2)s - s^3$$

In order to find the value of s for which the deflection Δ is a maximum, the first derivative of Δ with respect to s may be set equal to zero, and the resulting equation solved for s.

$$\frac{6EIL}{Pb}\frac{d\Delta}{ds} = (L^2 - b^2) - 3s^2 = 0$$

$$s = \frac{1}{3}\sqrt{3(L2 - b^2)}$$

Now, placing this value of s in equation (3)

$$\frac{6EIL\Delta}{Pb} = (L^2 - b^2) \cdot \frac{1}{3}\sqrt{3(L^2 - b^2)} - \left(\frac{1}{3}\sqrt{3(L^2 - b^2)}\right)^3$$

$$\text{or} \quad \Delta = \frac{Pb(L^2 - b^2)}{27EIL} \cdot \sqrt{3(L^2 - b^2)}$$

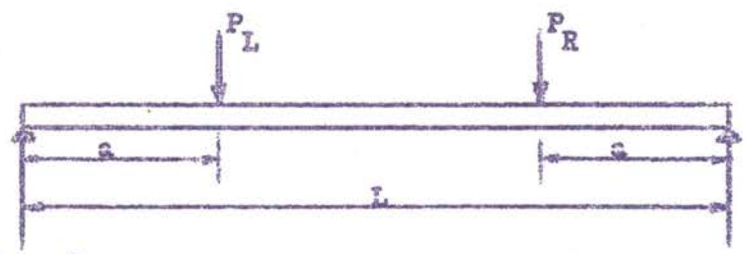
(G) The Method of Real Work does not apply.

(H) The Method of Least Work may be used in very much the same manner as was the method of virtual work. No solution will be shown.

(J) The Theorem of Three Moments is not often used for this type beam.

(K) Moment distribution does not apply.

3. Two Equal Concentrated Loads Symmetrically Placed.



$R_1 = P$

(A) By Double Integration:

For portion of beam between R_1 and P_L:

$$M_x = Px$$

$$\frac{EI}{P}\frac{d^2y}{dx^2} = x$$

$$\frac{EI}{P}\frac{dy}{dx} = \frac{x^2}{2} + C_1$$

For portion of beam between P_L and P_R:

$$M_x = Pa$$

$$\frac{EI}{P}\frac{d^2y}{dx^2} = a$$

$$\frac{EI}{P}\frac{dy}{dx} = ax + C_1'$$

when $x = \frac{L}{2}$, $\frac{dy}{dx} = 0$

$$\therefore C_1' = -\frac{aL}{2}$$

when $x = a$

$$\frac{a^2}{2} + C_1 = a^2 + C_1' = a^2 - \frac{aL}{2}$$

$$C_1 = \frac{a}{2}(a - L)$$

$$\frac{EI}{P}y = \frac{x^3}{6} + \frac{ax}{2}(a - L) + C_2$$

when $x = 0$, $y = 0$

$$\therefore C_2 = 0$$

$$\frac{EI}{P}y = \frac{ax^2}{2} - \frac{aLx}{2} + C_2'$$

when $x = a$

$$\frac{a^3}{6} + \frac{a^2}{2}(a - L) = \frac{a^3}{2} - \frac{a^2L}{2} + C_2'$$

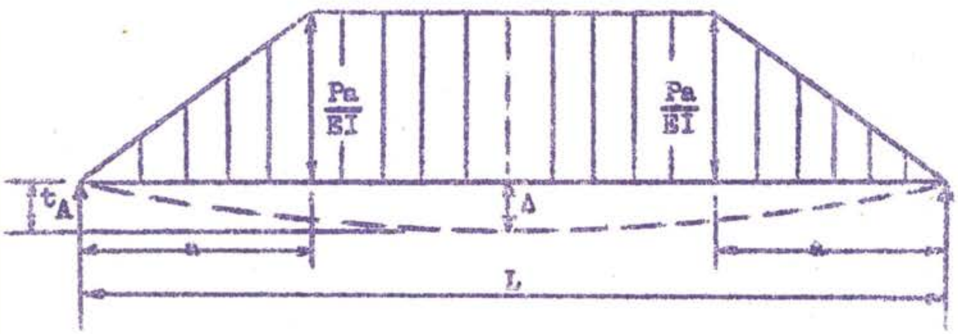
$$\frac{a^3}{6} = C_2'$$

when $x = \frac{L}{2}$, $y = \Delta$

$$\frac{EI}{P}\,\Delta = \frac{aL^2}{8} - \frac{aL^2}{4} + \frac{a^3}{6}$$

$$\Delta = \frac{Pa}{24EI}\left(3L^2 - 4a^2\right)$$

(B) By Area-Moments:



$$\Delta = t_A = \frac{Pa}{EI} \cdot \frac{a}{2} \cdot \frac{2a}{3} + \frac{Pa}{EI}\left(\frac{L}{2} - a\right)\left[a + \frac{1}{2}\left(\frac{L}{2} - a\right)\right]$$

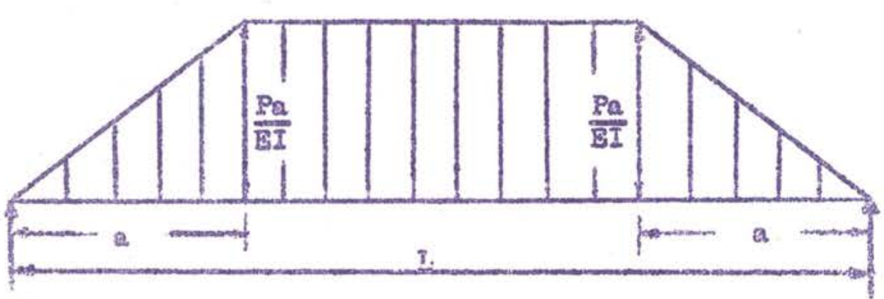
$$\Delta = \frac{Pa^3}{3EI} + \frac{Pa}{EI}\left(\frac{L}{2} - a\right) \cdot \frac{1}{2}\left(\frac{L}{2} + a\right)$$

$$= \frac{Pa^3}{3EI} + \frac{Pa}{2EI}\left(\frac{L^2}{4} - a^2\right)$$

$$= \frac{Pa^3}{3EI} + \frac{PaL^2}{8EI} - \frac{Pa^3}{2EI} = \frac{PaL^2}{8EI} - \frac{Pa^3}{6EI}$$

$$\Delta = \frac{Pa}{24EI}\left(3L^2 - 4a^2\right)$$

(C) By the Conjugate Beam Method:



$$R = \frac{Pa}{2EI}\left(L - a\right)$$

$$\Delta = M\ (\text{max.}) = \frac{Pa}{2EI}(L - a) \cdot \frac{L}{2} - \frac{Pa}{EI} \cdot \frac{a}{2} \cdot \left(\frac{L}{2} - \frac{2a}{3}\right)$$
$$- \frac{Pa}{EI}\left(\frac{L}{2} - a\right) \cdot \frac{1}{2}\left(\frac{L}{2} - a\right)$$

$$\frac{EI}{P}\Delta = \frac{aL^2}{4} - \frac{a^2L}{4} - \frac{a^2L}{4} + \frac{a^3}{3} - \frac{aL^2}{8} + \frac{a^2L}{2} - \frac{a^3}{2}$$

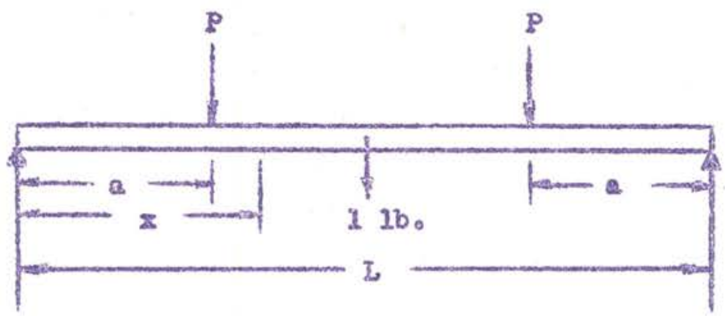
$$\frac{EI}{P}\Delta = \frac{a}{24}(3L^2 - 4a^2)$$

$$\Delta = \frac{Pa}{24EI}(3L^2 - 4a^2)$$

(D) Column Analogy does not apply.

(E) Slope Deflection is not commonly used.

(F) By the Method of Virtual Work:



$R = P$
$r = 1/2$

For the values of x between zero and a

$$M_x = Px\ ,\ m_x = x/2$$

For the values of x between a and $(L - a)$

$$M_x = Pa\ ,\ m_x = x/2$$

Since the beam is symmetrically loaded and supported, the total internal work of the beam will be equal to twice the work of one-half the beam. Then

$$\Delta = 2\int_0^a \frac{Px}{EI}\ \frac{x}{2} \cdot dx + 2\int_a^{L/2} \frac{Pa}{EI} \cdot \frac{x}{2}\,dx$$

$$\frac{EI\Delta}{P} = \int_0^a x^2dx + \int_a^{L/2} axdx$$

$$\frac{EI}{P}\Delta = \frac{a^3}{3} + \frac{aL^2}{8} - \frac{a^3}{2}$$

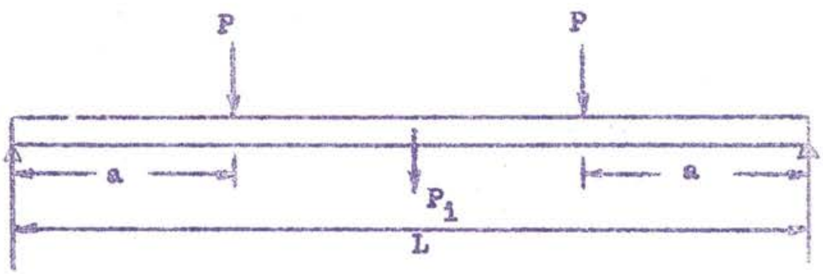
$$\frac{EI}{P}\Delta = \frac{a}{24}(3L^2 - 4a^2)$$

$$\Delta = \frac{Pa}{24EI}(3L^2 - 4a^2)$$

(G) The method of real work does not apply.

(H) By the Method of Least Work:

Since there is no load acting at the center of the beam where the deflection is a maximum, it will be necessary to apply a dummy load of zero magnitude at mid-span.



$R = P + P_1/2$

$$U = \frac{1}{2EI}\int_0^L M^2 dx$$

$$EIU = \int_0^a \left(Px + \frac{P_1 x}{2}\right)^2 dx + \int_a^{L/2}\left(Pa + \frac{P_1 x}{2}\right)^2 dx$$

$$EI\Delta = EI\frac{\partial U}{\partial P_1} = \int_0^a 2\left(Px + \frac{P_1 x}{2}\right)\cdot\frac{x}{2}\,dx + \int_a^{L/2} 2\left(Pa + \frac{P_1 x}{2}\right)\cdot\frac{x}{2}\,dx$$

and since $P_1 = 0$

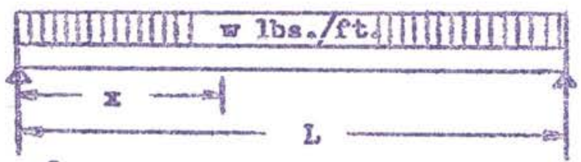
$$EI\Delta = \int_0^a Px^2 dx + \int_a^{L/2} Pax\,dx \text{ , from which}$$

$$\Delta = \frac{Pa}{24EI}(3L^2 - 4a^2)$$

(J) The Theorem of Three Moments is not generally used.

(K) Moment Distribution does not apply.

4. Load Uniformly Distributed Over the Entire Length of the Beam.



$$R = \frac{wL}{2}$$

(A) By Double Integration:

$$EI\,\frac{d^2y}{dx^2} = M_x = \frac{wLx}{2} - \frac{wx^2}{2}$$

$$EI\,\frac{dy}{dx} = \frac{wLx^2}{4} - \frac{wx^3}{6} + C_1$$

$$\text{when } x = L/2\,,\ \frac{dy}{dx} = 0\,,\ \therefore C_1 = -\frac{wL^3}{24}$$

$$EIy = \frac{wLx^3}{12} - \frac{wx^4}{24} - \frac{wL^3x}{24} + C_2$$

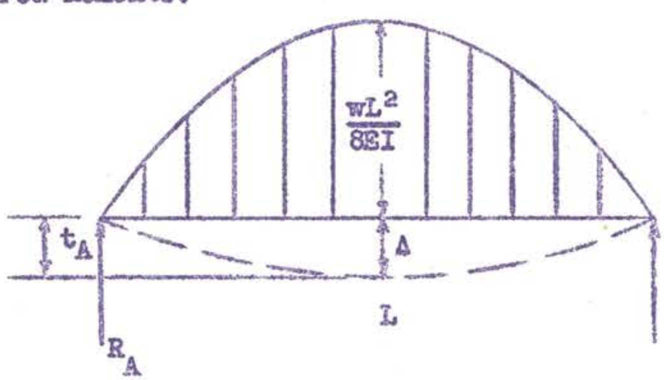
$$\text{when } x = 0\,,\ y = 0\,,\ \therefore C_2 = 0$$

$$\text{when } x = L/2\,,\ y = \Delta$$

$$EI\Delta = \frac{wL^4}{96} - \frac{wL^4}{384} - \frac{wL^4}{48} = \frac{5wL^4}{384}$$

$$\Delta = \frac{5}{384} \cdot \frac{wL^4}{EI}$$

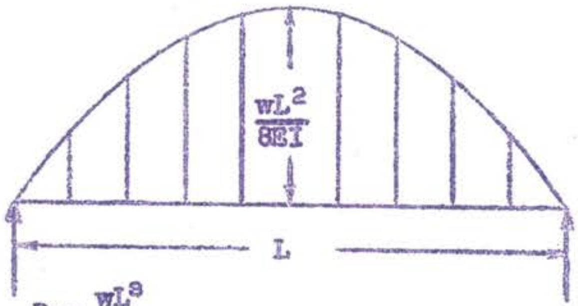
(B) By Area-Moments:



$$\Delta = t_A = \frac{2}{3} \cdot \frac{L}{2} \cdot \frac{wL^2}{8EI} \cdot \frac{5}{8} \cdot \frac{L}{2}$$

$$\Delta = \frac{5}{384}\ \frac{wL^4}{EI}$$

(C) By the Conjugate Beam Method:



$R = \frac{wL^3}{24EI}$

$$\Delta = M\ (\text{max.}) = \frac{wL^3}{24EI} \cdot \frac{L}{2} - \frac{2}{3} \cdot \frac{L}{2} \cdot \frac{wL^2}{8EI} \cdot \frac{3}{8} \cdot \frac{L}{2}$$

$$\Delta = \frac{wL^4}{48EI} - \frac{wL^4}{128EI} = \frac{5}{384} \cdot \frac{wL^4}{EI}$$

(D) Column Analogy does not apply.

(E) Slope Deflection is not commonly used.

(F) By the Method of Virtual Work:

$R = wL/2$
$r = 1/2$

Since the beam is symmetrically loaded and supported, the internal work done by the beam is equal to twice the work of one-half the beam. Then

$$\Delta = \frac{2}{EI} \int_0^{L/2} Mm\,dx$$

For values of x between zero and $L/2$

$$M_x = \frac{wLx}{2} - \frac{wx^2}{2}$$

$$m_x = x/2$$

$$\Delta = \frac{2}{EI} \int_0^{L/2} \left(\frac{wLx^2}{4} - \frac{wx^3}{4}\right) dx$$

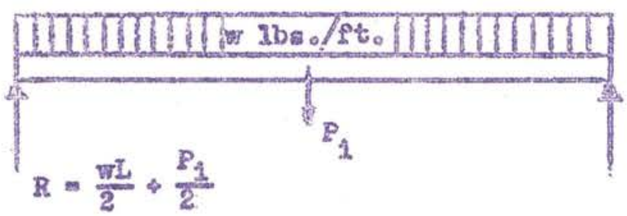
$$EI\Delta = \frac{wL^4}{48} - \frac{wL^4}{128} = \frac{5wL^4}{384}$$

$$\Delta = \frac{5}{384} \cdot \frac{wL^4}{EI}$$

(G) The method of real work does not apply.

(H) By the Method of Least Work:

Again, since there is no concentrated load acting at mid-span, the point of maximum deflection, it will be necessary to apply a dummy load of zero magnitude at this point.



$$U = \frac{2}{2EI}\int_0^{L/2} M^2 dx$$

$$M_x = \frac{wLx}{2} + \frac{P_1 x}{2} - \frac{wx^2}{2}$$

$$EIU = \int_0^{L/2}\left(\frac{wLx}{2} + \frac{P_1 x}{2} - \frac{wx^2}{2}\right)^2 dx$$

$$EI\Delta = EI\frac{\partial U}{\partial P_1} = \int_0^{L/2} 2\left(\frac{wLx}{2} + \frac{P_1 x}{2} - \frac{wx^2}{2}\right)\frac{x}{2}\,dx$$

and since $P_1 = 0$

$$EI\Delta = \int_0^{L/2}\left(\frac{wLx^2}{2} - \frac{wx^3}{2}\right)dx$$

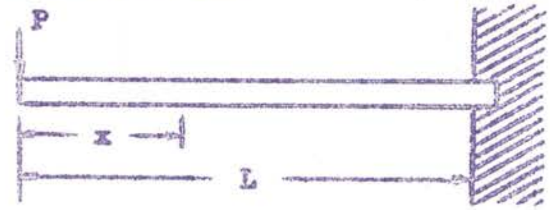
$$\Delta = \frac{5}{384} \cdot \frac{wL^4}{EI}$$

(J) The Theorem of Three Moments is not generally used.

(K) Moment Distribution does not apply.

THE CANTILEVER BEAM

1. Concentrated Load at the Free End.



(A) By Double Integration:

$$EI \frac{d^2y}{dx^2} = M_x = -Px$$

$$EI \frac{dy}{dx} = -\frac{Px^2}{2} + C_1$$

when $x = L$, $\frac{dy}{dx} = 0$, $\therefore C_1 = \frac{PL^2}{2}$

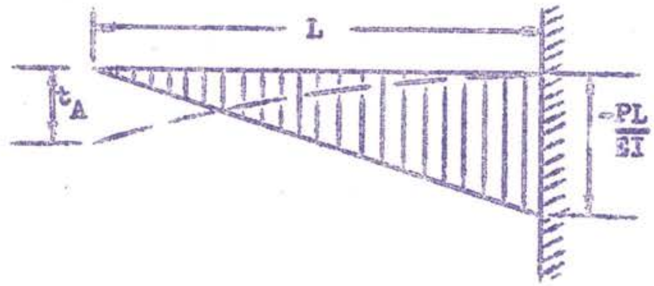
$$EIy = -\frac{Px^3}{6} + \frac{PL^2x}{2} + C_2$$

when $x = L$, $y = 0$, $\therefore C_2 = -\frac{PL^3}{3}$

when $x = 0$, $y = \Delta$

$$\Delta = \frac{PL^3}{3EI}$$

(B) By Area-Moments:

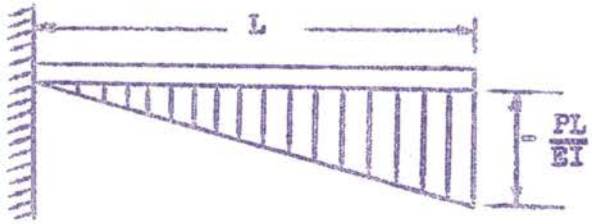


$$t_A = \Delta = -\frac{PL}{EI} \cdot \frac{L}{2} \cdot \frac{2L}{3}$$

$$\Delta = \frac{PL^3}{3EI}$$

(C) By the Conjugate Beam Method:

The end of the beam which is free, in the real beam, becomes the fixed end of the conjugate beam, and the fixed end of the real beam becomes the free end of the conjugate beam.



$$\Delta = M\ (max.) = -\frac{PL}{EI} \cdot \frac{L}{2} \cdot \frac{2L}{3}$$

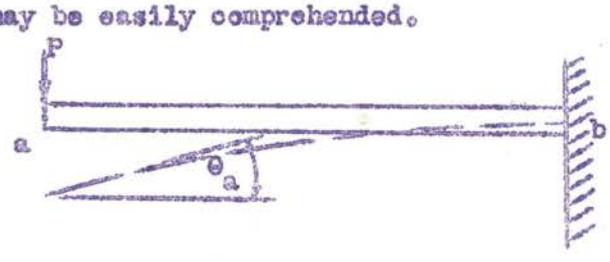
$$\Delta = \frac{PL^3}{3EI}$$

It should be noted that, for the cantilever beam, this method is exactly like the method of Area-Moments, except for the somewhat different approach which is noticeable only in the difference of the two identification sketches.

(D) Column Analogy does not apply.

(E) By the Slope Deflection Method:

The reader is referred to page 5, part II, of this report for a discussion of this method and for the basic equations of slope deflection. The method will apply directly for the cantilever beam whereas it applies only artificially to the simple beam. In this (first) application of the method, the general basic equations will be shown as well as the specific equations for this case, so that the transformation may be easily comprehended.



(1) $M_{ab} = M_{Fab} + \frac{EI}{L}\left(4\theta_a + 2\theta_b - 6\frac{\Delta}{L}\right)$

(2) $M_{ba} = M_{Fba} + \frac{EI}{L}\left(2\theta_a + 4\theta_b - 6\frac{\Delta}{L}\right)$

Then, from equation (1)

(3) $0 = 0 + \frac{EI}{L}\left(4\theta_a + 0 - 6\frac{\Delta}{L}\right)$

and from equation (2)

(4) $-PL = 0 + \frac{EI}{L}\left(2\theta_a + 0 - 6\frac{\Delta}{L}\right)$

From equation (3)

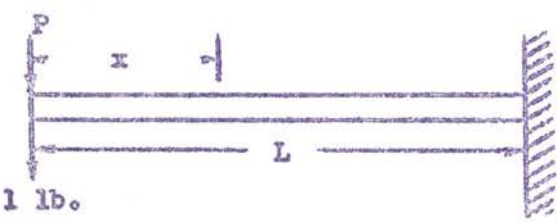
$$\theta = \frac{3\Delta}{2L}$$

and if this is substituted for θ_a in equation (4)

(5) $-PL = 0 + \frac{EI}{L}\left(\frac{3\Delta}{L} - \frac{6\Delta}{L}\right) = \frac{EI}{L^2}(-3\Delta)$

$$\Delta = \frac{PL^3}{3EI}$$

(F) By the Method of Virtual Work:



$$M_x = -Px$$

$$m_x = -x$$

$$EI\Delta = \int_0^L -Px(-x)dx = \int_0^L Px^2dx$$

$$\Delta = \frac{PL^3}{3EI}$$

(G) By the Method of Real Work:

$$\Delta = \frac{M^2dx}{PEI}$$

$$PEI\ \Delta = \int_0^L P^2x^2dx = \frac{P^2L^3}{3}$$

$$\Delta = \frac{PL^3}{3EI}$$

(H) By the Method of Least Work:

$$U = \int \frac{M^2 dx}{2EI} = \int_0^L \frac{P^2 x^2 dx}{2EI}$$

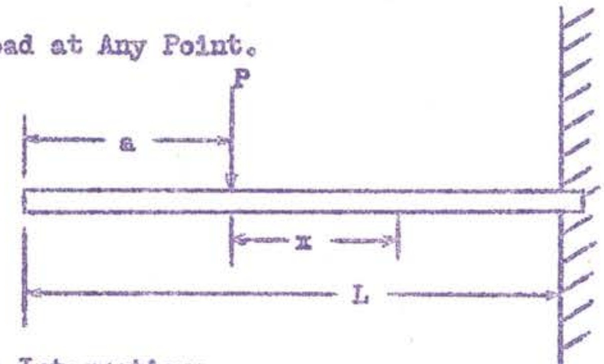
$$\Delta = \frac{\partial U}{\partial P} = \int_0^L \frac{2Px^2 dx}{2EI}$$

$$\Delta = \frac{PL^3}{3EI}$$

(J) The Theorem of Three Moments does not apply.

(K) Moment Distribution does not apply.

2. Concentrated Load at Any Point.



(A) By Double Integration:

Since the beam to the left of the load P is not subjected to bending--i.e. the radius of curvature is infinitely large--it will be convenient to select the origin at the load point and to determine the deflection y_P under the load. The deflection Δ of the end of the beam may be found by adding to y_P the additional deflection $(\theta_P \cdot a)$ due to the change in slope at the load point (see sketch below).

To determine y_P, consider a beam having a length of (L - a).

Then

$$y_P = \frac{P(L - a)^3}{3EI}$$

$$EI \frac{d^2y}{dx^2} = M_x = -Px$$

$$EI \frac{dy}{dx} = \frac{Px^2}{2} + C_1$$

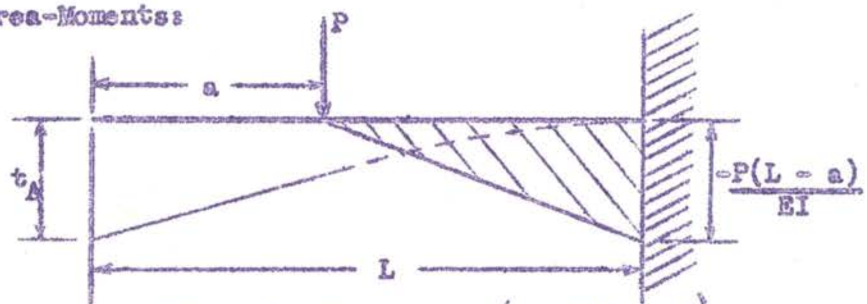
when $x = L - a$, $\frac{dy}{dx} = 0$, $\therefore C_1 = \frac{P(L-a)^2}{2}$

when $x = 0$, $\frac{dy}{dx} = \Theta_P = \frac{P(L-a)^2}{2EI}$

$$\Delta = y_P + \Theta_P \cdot a = \frac{P(L-a)^3}{3EI} + \frac{Pa(L-a)^2}{2EI}$$

$$\Delta = \frac{P(L-a)^2(2L+a)}{6EI}$$

(B) By Area-Moments:

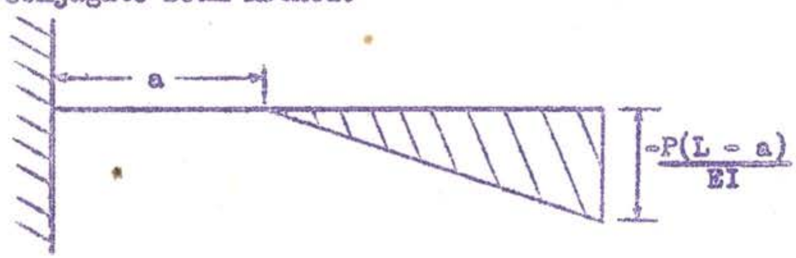


$$\Delta = t_A = \frac{P(L-a)}{EI} \cdot \frac{1}{2}(L-a)\left(a + \frac{2(L-a)}{3}\right)$$

$$\Delta = \frac{P(L-a)^2}{2EI} \quad \frac{1}{3}(3a + 2L - 2a)$$

$$\Delta = \frac{P(L-a)^2}{6EI}(2L + a)$$

(C) By the Conjugate Beam Method:



$$\Delta = M\ (\text{max.}) = \frac{P(L-a)}{EI} \cdot \frac{1}{2}(L-a)\left(a + \frac{2(L-a)}{3}\right)$$

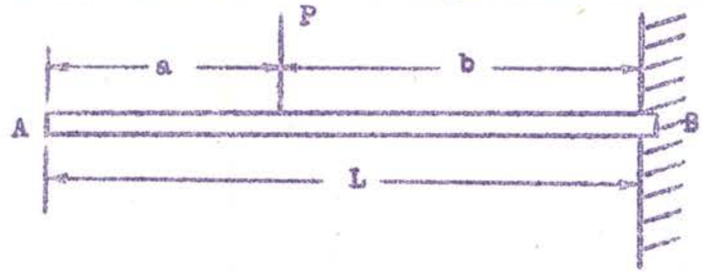
$$\Delta = \frac{P(L-a)^2}{2EI} \quad \frac{1}{3}(3a + 2L - 2a)$$

$$\Delta = \frac{P(L-a)^2}{6EI}(2L + a)$$

(D) Column Analogy does not apply.

(E) By the Method of Slope Deflection:

This method is, in general, rather cumbersome when applied to statically determinate beams. Since the method entails the use of the fixed-end moments, which themselves are sometimes difficult of solution (being statically indeterminate), the slope deflection theories are seen to apply more readily to indeterminate cases. However, a solution will be shown here to demonstrate the use of the method, and at the same time, to illustrate its comparative awkwardness.



The two basic equations in this case are:

$$(1)\quad 0 = -\frac{Pb^2a}{L^2} + \frac{EI}{L}\left(4\theta_A + 0 - \frac{6\Delta}{L}\right)$$

$$(2)\quad Pb = \frac{Pba^2}{L^2} + \frac{EI}{L}\left(2\theta_A + 0 - \frac{6\Delta}{L}\right)$$

The solution of equation (1) for θ_A gives

$$\theta_A = \frac{Pb^2a}{4EIL} + \frac{3\Delta}{2L}$$

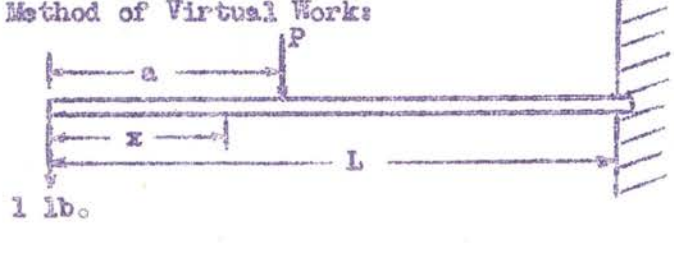
Now, if this value for θ_A is placed into equation (2) and the resulting equation solved for Δ, it is found that

$$\Delta = \frac{Pb}{6EI}\left(2a^2 + ab - 2L^2\right)$$

and since $b = L - a$

$$\Delta = \frac{P(L-a)^2}{6EI}\left(2L + a\right)$$

(F) By the Method of Virtual Work:



For values of x between zero and a, M_x is zero and the product Mm is zero, therefore the work equation will cover only the portion of the beam to the right of the load, where

$$M_x = -P(x-a)$$

$$\text{and } m_x = -x$$

$$EI\Delta = \int_a^L P(x-a)x\,dx = \int_a^L Px^2dx - \int_a^L Pax\,dx$$

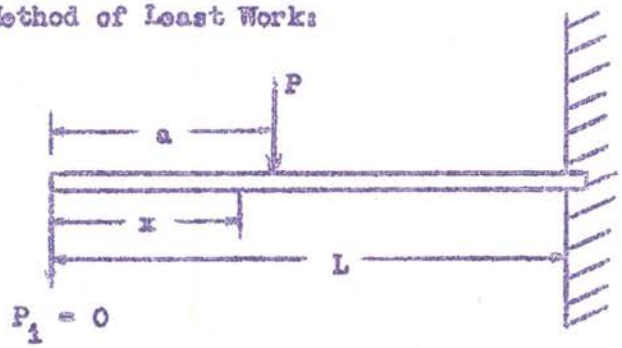
$$\frac{EI\Delta}{P} = \frac{L^3}{3} - \frac{a^3}{3} - \frac{aL^2}{2} + \frac{a^3}{2}$$

$$\frac{6EI\Delta}{P} = 2L^3 - 2a^3 - 3aL^2 + 3a^3 = 2L^3 - 3aL^2 + a^3$$

$$\Delta = \frac{P(L-a)^2(2L+a)}{6EI}$$

(G) The Method of Real Work does not apply.

(H) By the Method of Least Work:



For the values of x between zero and a, $M_x = -P_1x$; and for values of x between a and L, $M_x = -P_1x - P(x-a)$. Then

$$U = \frac{1}{2EI}\int_0^a (-P_1x)^2dx + \frac{1}{2EI}\int_a^L (-P_1x - Px + Pa)^2dx$$

$$2EI\Delta = 2EI\frac{\partial U}{\partial P_1} = \int_0^a 2P_1x^2 + \int_a^L 2(-P_1x - Px + Pa)\,(-x)dx$$

$$\text{since } P_1 = 0$$

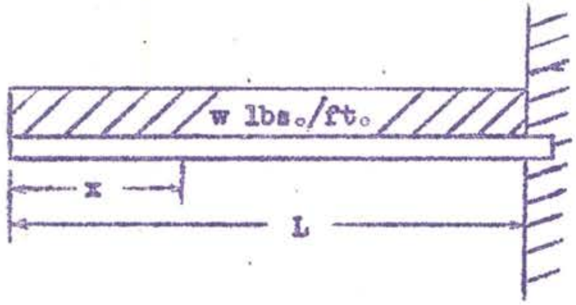
$$EI\Delta = \frac{PL^3}{3} - \frac{Pa^3}{3} - \frac{PaL^2}{2} + \frac{Pa^3}{3}\text{, from which}$$

$$\Delta = \frac{P(L-a)^2}{6EI}(2L+a)$$

(J) The Theorem of Three Moments does not apply.

(K) Moment Distribution does not apply.

3. Load Uniformly Distributed Over the Entire Length of the Beam.



(A) By Double Integration:

$$EI \frac{d^2y}{dx^2} = M_x = -\frac{wx^2}{2}$$

$$EI \frac{dy}{dx} = -\frac{wx^3}{6} + C_1$$

when $x = L$, $\frac{dy}{dx} = 0$, $\therefore C_1 = \frac{wL^3}{6}$

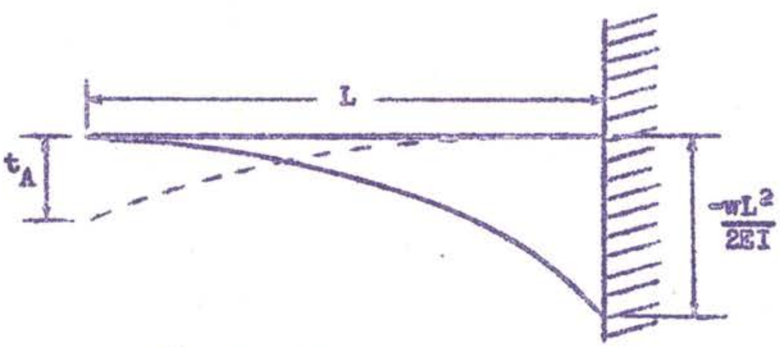
$$EIy = -\frac{wx^4}{24} + \frac{wL^3x}{6} + C_2$$

when $x = L$, $y = 0$, $\therefore C_2 = -\frac{wL^4}{8}$

when $x = 0$, $y = \Delta$

$$\Delta = \frac{wL^4}{8EI}$$

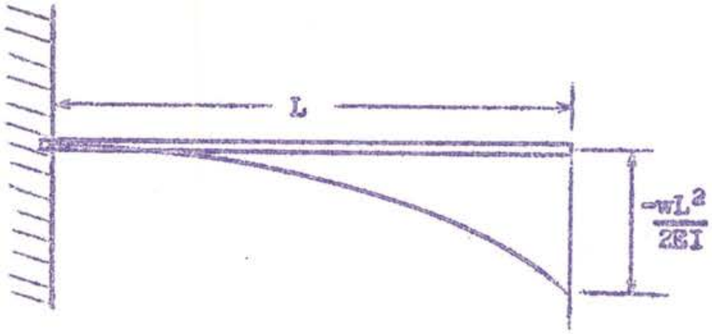
(B) By Area-Moments:



$$\Delta = t_A = \frac{wL^2}{2EI} \cdot \frac{L}{3} \cdot \frac{3L}{4}$$

$$\Delta = \frac{wL^4}{8EI}$$

(C) By the Conjugate Beam:



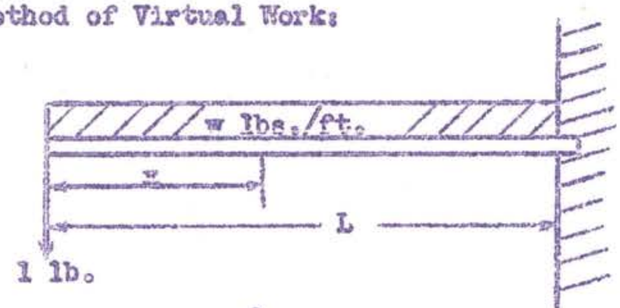
$$\Delta = M\ (\text{max.}) = -\frac{wL^2}{2EI} \cdot \frac{L}{3} \cdot \frac{3L}{4}$$

$$\Delta = \frac{wL^4}{8EI}$$

(D) Column Analogy does not apply.

(E) The method of Slope Deflection may be applied; but, for reasons previously stated, is too unwieldy to warrant consideration. The same general method as that shown under 2(E), page 35, applies.

(F) By the Method of Virtual Work:



$$M_x = -\frac{wx^2}{2}$$

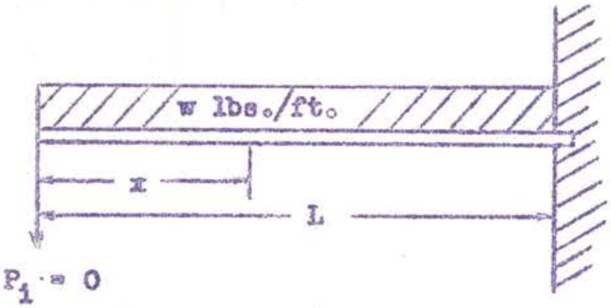
$$m_x = -x$$

$$EI\Delta = \int_0^L -\frac{wx^2}{2}\,(-x)\,dx = \frac{wL^4}{8}$$

$$\Delta = \frac{wL^4}{8EI}$$

(G) The Method of Real Work does not apply.

(H) By the Method of Least Work:



$P_1 = 0$

$$M_x = -\frac{wx^2}{2} - P_1 x$$

$$U = \int_0^L \frac{\left(-\frac{wx^2}{2} - P_1 x\right)^2 dx}{2EI}$$

$$\Delta = \frac{\partial U}{\partial P_1} = \int_0^L \frac{2\left(-\frac{wx^2}{2} - P_1 x\right)(-x)\,dx}{2EI}$$

since $P_1 = 0$

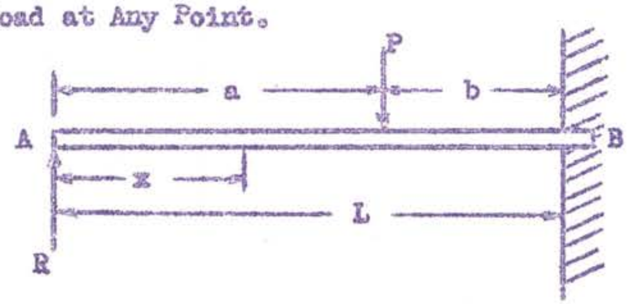
$$\Delta = \frac{wL^4}{8EI}$$

(J) The Theorem of Three Moments does not apply.

(K) Moment Distribution does not apply.

BEAMS FIXED AT ONE END AND SUPPORTED AT THE OTHER

1. Concentrated Load at Any Point.



(A) By Double Integration:

For values of x between o and a:

$$EI\frac{d^2y}{dx^2} = M_x = Rx$$

$$EI\frac{dy}{dx} = \frac{Rx^2}{2} + C_1$$

For values of x between a and L:

$$EI\frac{d^2y}{dx^2} = Rx - P(x-a)$$

$$EI\frac{dy}{dx} = \frac{Rx^2}{2} - \frac{P}{2}(x-a)^2 + C_1'$$

when $x = L$, $\frac{dy}{dx} = 0$

$$\therefore C_1' = \frac{Pb^2}{2} - \frac{RL^2}{2}$$

when $x = a$

$$\frac{Ra^2}{2} + C_1 = \frac{Ra^2}{2} + C_1'$$

$$C_1 = C_1' = \frac{Pb^2}{2} - \frac{RL^2}{2}$$

$$EIy = \frac{Rx^3}{6} + \frac{Pb^2x}{2} - \frac{RL^2x}{2} + C_2 \qquad \Big| \qquad EIy = \frac{Rx^3}{6} - \frac{P}{6}(x - a)^3 + \frac{Pb^2x}{2} - \frac{RL^2x}{2} + C_2'$$

when $x = 0$, $y = 0$, $\therefore C_2 = 0$

when $x = a$

$$\frac{Ra^3}{6} + \frac{Pb^2a}{2} - \frac{RL^2a}{2} = \frac{Ra^3}{6} + \frac{Pb^2a}{2} - \frac{RL^2a}{2} + C_2'$$

$$0 = C_2'$$

when $x = L$, $y = 0$

$$\therefore R = \frac{Pb^2}{2L^3}(3L - b)$$

The moment at B may then be determined statically.

$$M_B = RL - Pb = \frac{Pb^2}{2L^3}(3L - b)L - Pb$$

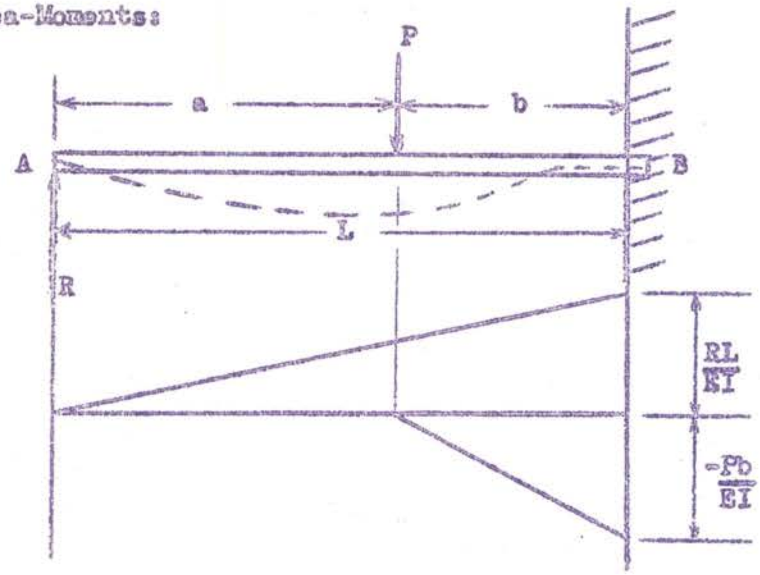
$$M_B = -\frac{Pab}{2L^2}(L + a)$$

The deflection at any point may be determined from the equations of the elastic curve, shown above. For any problem involving numerical values the cumputations would be quite simple.

If $a = b = \frac{L}{2}$ the above expressions for the reaction R and the moment at B become $R = \frac{5P}{16}$ and $M_B = \frac{3PL}{16}$

If more than one load acts on the span, the resulting reaction and end moment may be found by algebraically adding the reaction and moment due to each load acting separately.

(B) By Area-Moments:



The tangential deviation of point A with respect to the tangent drawn at point B is equal to zero.

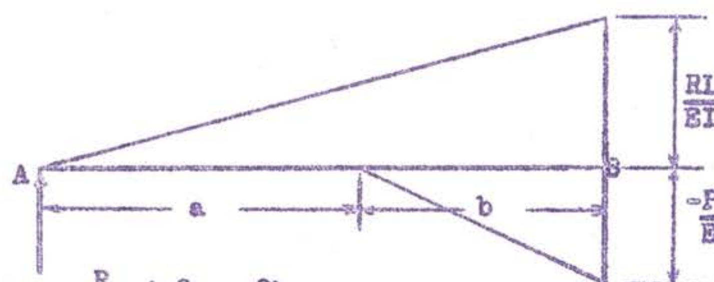
$$t_A = 0 = \frac{RL}{EI} \cdot \frac{L}{2} \cdot \frac{2L}{3} - \frac{Pb}{EI} \cdot \frac{b}{2}\left(a + \frac{2b}{3}\right)$$

$$\frac{RL^3}{3} = \frac{Pb^2a}{2} + \frac{Pb^3}{3} = \frac{Pb^2}{6}\ (3a + 2b)$$

$$R = \frac{Pb^2}{2L^3}\,(3a + 2b) = \frac{Pb^2}{2L^3}\,(3L - b)$$

The moment at B may be determined by statics as shown under 1 (A).

(C) By the Conjugate Beam Method:



$$R = \frac{R}{2EI}\,(L^2 - b^2)$$

The deflection of the real beam at both A and B is equal to zero. Therefore, the moment in the conjugate at either A or B may be set equal to zero and the resulting equation solved for R. It will be most convenient to take moments about A in order to eliminate the moment of the conjugate beam reaction.

$$A_B = M_B = 0 = \frac{RL}{EI} \cdot \frac{L}{2} \cdot \frac{2L}{3} - \frac{Pb}{EI} \cdot \frac{b}{2}\left(a + \frac{2b}{3}\right)$$

Solving this equation for R gives

$$R = \frac{Pb^2}{2L^2}(3L - b)$$

(D) By Column Analogy:

In the use of the Column Analogy, the analogous column is loaded with the moment diagram for the structure altered in any way so as to make it statically determinate. However, it is important to choose the most convenient curve of determinate moments. Furthermore, the statically determinate condition should not be achieved by imposing a condition of restraint where none already exists. For example, if the structure under consideration in this problem were to be made statically determinate by freeing the fixed end and fixing the hinge supported end the column analogy could be applied to determine correctly the moment at the fixed end B. But the method fails when it is attempted to show that the moment at the hinge A is zero.

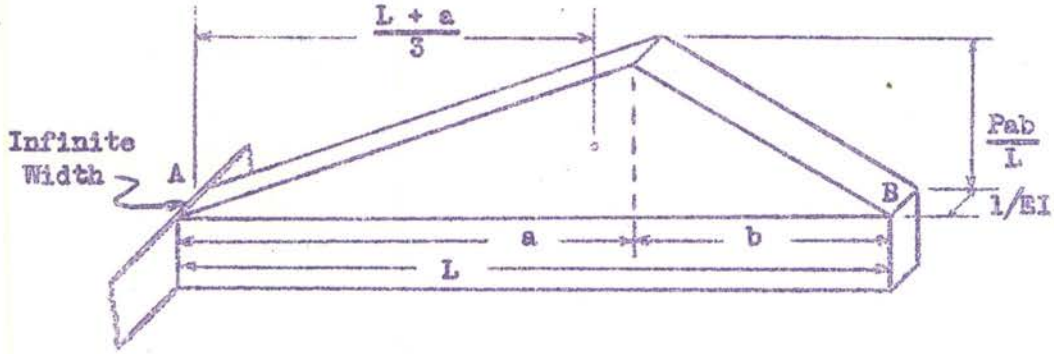
It must be noted that, since it offers no resistance to rotation due to moment, a hinge is considered to have an infinite elastic area. Thus, the area of the analogous column is infinitely great, and the P/A term (of $f = P/A + Mc/I$) is infinitely small and need not be considered. Both the centroid and the kern point of the infinite column section lie at the hinge.

In this case let the analogous column be loaded with moment curve for a simple span.(see sketch on page 43).

$$f_B = M_{i_B} = 0 + \frac{\frac{Pab}{L} \cdot \frac{L}{2} \cdot \frac{1}{EI} \cdot \frac{L + a}{3} \cdot L}{\frac{1}{EI} \cdot \frac{L^3}{3}} = \frac{Pab}{2L^2}(L + a)$$

$$M_B = M_{s_B} - M_{i_B} = 0 - \frac{Pab}{2L^2}(L + a)$$

$$M_B = -\frac{Pab}{2L^2}(L + a)$$



(E) By the Method of Slope Deflection:

This method, of course, implies a knowledge of the fixed end moments for a given span. Hence, the use of slope-deflection in the solution of this problem might be considered somewhat questionable. However, the solution will be shown to illustrate the method.

$$M_{FAB} = -\frac{Pb^2a}{L^2}, \quad M_{FBA} = +\frac{Pa^2b}{L^2}$$

$$M_{AB} = 0, \ \theta_B = 0, \ \Delta = 0$$

(1) $M_{AB} = -\frac{Pb^2a}{L^2} + \frac{EI}{L}(4\theta_A + 0 - 0) = 0$

(2) $M_{BA} = \frac{Pa^2b}{L^2} + \frac{EI}{L}(2\theta_A + 0 - 0)$

From equation (1)

$$\theta_A = \frac{Pb^2a}{4EIL}, \quad \text{then}$$

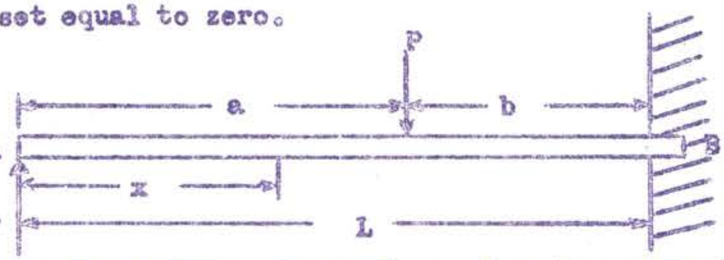
$$M_{BA} = \frac{Pa^2b}{L^2} + \frac{EI}{L}\left(\frac{2Pb^2a}{4EIL}\right) = \frac{Pa^2b}{L^2} + \frac{Pb^2a}{2L^2}$$

$$M_{BA} = \frac{Pab}{2L^2}(L + a)$$

It should be noted that the sign of M_{BA} is positive because the resisting moment at B acts in a clockwise direction, which is in accordance with the assumptions made in developing the slope deflection equations.

(F) By the Method of Virtual Work:

Let the unit load, in this case, be applied at the left reaction. Since there is no deflection of the elastic curve at this point the work done by the unit load is zero. Therefore, the internal work equation may be set equal to zero.



For values of x between zero and a, $M_x = Rx$ and $m_x = x$. For values of x between a and L, $M_x = Rx - P(x - a)$ and $m_x = x$. Then

$$EIy_A = 0 = \int_0^a Rx^2dx + \int_a^L Rx^2dx - \int_a^L P(x - a)xdx$$

$$0 = \frac{Ra^3}{3} + \frac{RL^3}{3} - \frac{Ra^3}{3} - \frac{PL^3}{3} + \frac{Pa^3}{3} + \frac{PaL^2}{2} - \frac{Pa^3}{2}$$

$$2RL^3 = 2PL^3 - 3PaL^2 + Pa^3$$

and since $a = L - b$

$$R = \frac{Pb^2}{2L^3}(3L - b)$$

The moment at B may now be found by statics since $M_B = RL - Pb$.

(G) The Method of Real Work does not apply.

(H) By the Method of Least Work:

As has been pointed out, this method is substantially the same as the method of virtual work. Here, there is no deflection of the elastic curve at the left reaction. Then for values of x between zero and a, $M_x = Rx$; and for values of x between a and L, $M_x = Rx - P(x - a)$.

$$U = \frac{M^2dx}{2EI}$$

$$2EIU = \int_0^a (Rx)^2dx + \int_a^L (Rx - Px + Pa)^2dx$$

$$EI\frac{\partial U}{\partial R} = 0 = \int_0^a (Rx)xdx + \int_a^L (Rx - Px + Pa)xdx$$

This may be rewritten

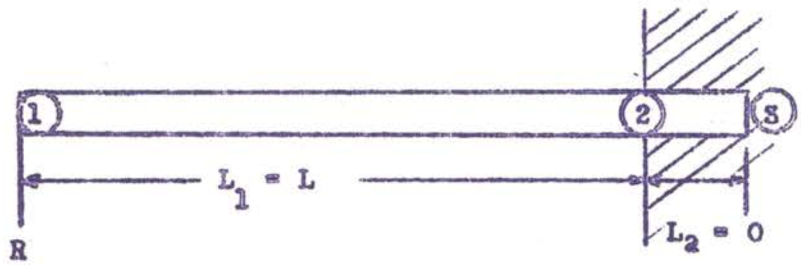
$$0 = \int_0^a Rx^2dx + \int_a^L Rx^2dx - \int_a^L P(x - a)xdx$$

which is exactly the same form as the virtual work equation. Therefore

$$R = \frac{Pb^2}{2L^3}(3L - b)$$

(J) By the Theorem of Three Moments:

In order to apply this theorem, an additional span of zero length must be assumed to extend beyond point 2 into the wall.



The simple moment diagram for span L_1 is shown below.

$$M_1L_1 + 2M_2(L_1 + L_2) + M_3L_2 = -\frac{6A_1\bar{x}_1}{L_1} - \frac{6A_2\bar{x}_2}{L_2}$$

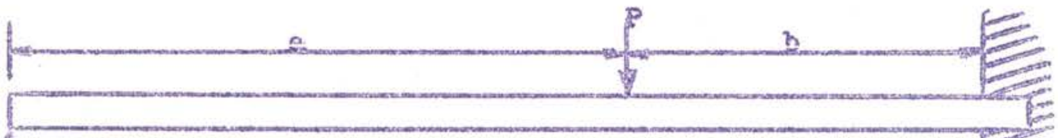
$$0 + 2M_2(L + 0) + 0 = -6\frac{Pab}{L} \cdot \frac{L}{2} \cdot \frac{L + a}{3} \cdot \frac{1}{L}$$

$$2LM_2 = -\frac{Pab}{L}(L + a)$$

$$M_2 = \frac{-Pab}{2L^2}(L + a)$$

(E) By Moment Distribution:

This method, just as did the method of slope deflection, must proceed on the assumption that the fixed-end moments for the span are already known, since the fixed-end moments are merely redistributed as the joints are allowed to rotate in turn.



$-\frac{Pb^2a}{L^2}$ F. E. Moments $-\frac{Pa^2b}{L^2}$

$+\frac{Pb^2a}{L^2}$ Balancing Moment

Carry-over Moment $-\frac{Pb^2a}{2L^2}$

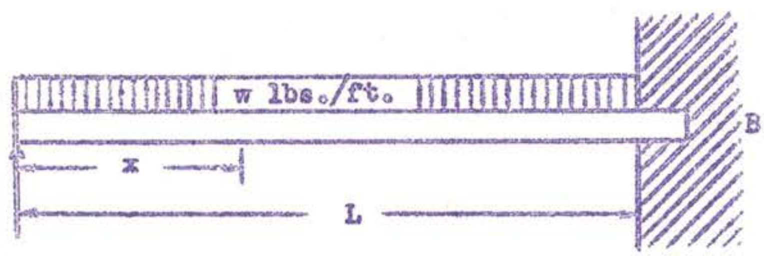
0

$-\frac{Pab}{2L^2}(2a+b)$

$= -\frac{Pab}{2L^2}(L+a)$

The fixed-end moments are assumed to be negative if the top fibers of the beam are in tension. The carry-over factor is $-\frac{1}{2}$.

2. Load Uniformly Distributed the Entire Length of the Beam.



(A) By Double Integration:

$$EI\frac{d^2y}{dx^2} = M_x = Rx - \frac{wx^2}{2}$$

$$EI\frac{dy}{dx} = \frac{Rx^2}{2} - \frac{wx^3}{6} + C_1$$

$$\text{when } x = L,\ \frac{dy}{dx} = 0,\ \therefore C_1 = \frac{wL^3}{6} - \frac{RL^2}{2}$$

$$EIy = \frac{Rx^3}{6} - \frac{wx^4}{24} + \frac{wL^3x}{6} - \frac{RL^2x}{2} + C_2$$

when $x = 0$, $y = 0$, $\therefore C_2 = 0$

when $x = L$, $y = 0$, so

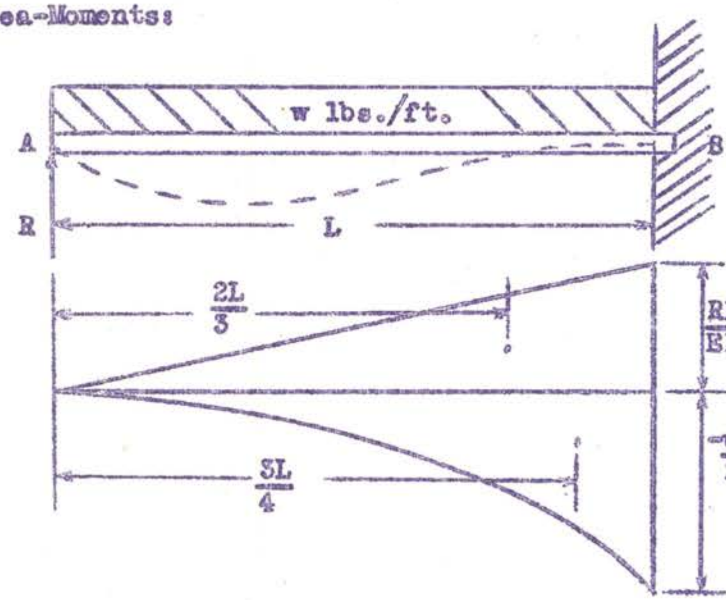
$$\frac{RL^3}{6} - \frac{wL^4}{24} + \frac{wL^4}{6} - \frac{RL^3}{2} = 0$$

$$R = \frac{3}{8} wL$$

$$M_B = RL - \frac{wL^2}{2} = \frac{3}{8} wL^2 - \frac{wL^2}{2} = -\frac{wL^2}{8}$$

The deflection at any point may be found by the use of the equation for the elastic curve, shown above. If the point of maximum deflection is desired, the equation for the slope of the elastic curve may be set equal to zero and solved for x.

(B) By Area-Moments:



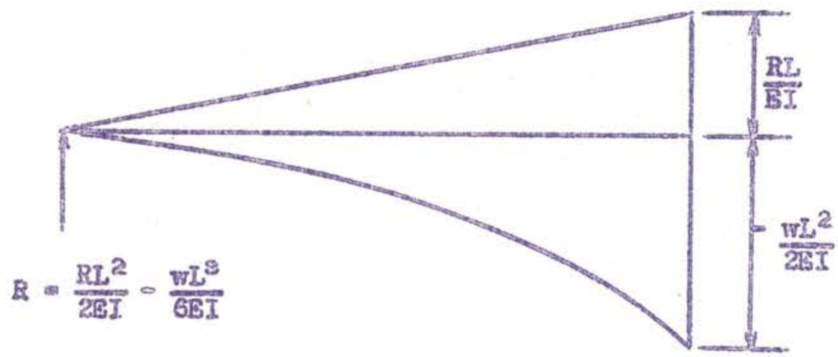
$$t_A = 0 = \frac{RL}{EI} \cdot \frac{L}{2} \cdot \frac{2L}{3} - \frac{wL^2}{2EI} \cdot \frac{L}{3} \cdot \frac{3L}{4}, \text{ from which}$$

$$R = \frac{3}{8} wL$$

$$M_B = \frac{3}{8} wL \cdot L - \frac{wL^2}{2} = -\frac{wL^2}{8}$$

To determine the deflection of any point on the elastic curve it is necessary only to take the moment of the area under the M/EI diagram between that point and B, since the tangent at B is parallel to the original position of the beam.

(C) By the Conjugate Beam Method:



The moment in the conjugate beam at R is equal to the deflection in the real beam at R. It is known, however, that this deflection is zero.

$$M_R = 0 = \frac{RL}{EI} \cdot \frac{L}{2} \cdot \frac{2L}{3} - \frac{wL^2}{2EI} \cdot \frac{L}{3} \cdot \frac{3L}{4}$$

$$R = \frac{3}{8} wL$$

If the shear equation for the conjugate beam is set equal to zero, the distance x_1 to the point of maximum deflection may be found. The maximum deflection then will be equal to the moment in the conjugate beam at this point.

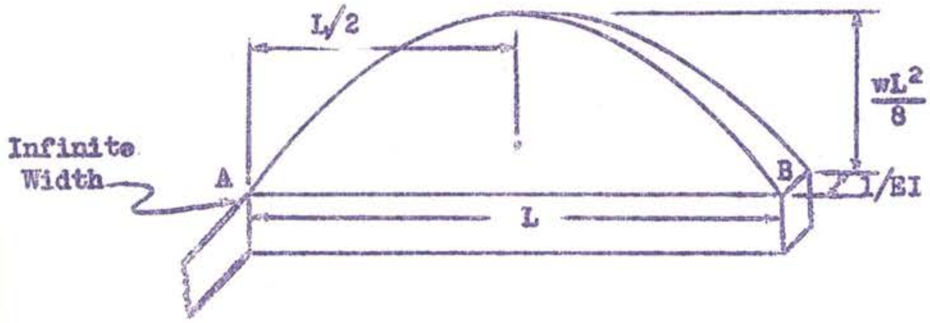
(D) By Column Analogy:

Let the analogous column be loaded with the moment diagram for a simple span, remembering that the centroid of the infinite column area lies at the hinge (see sketch on page 49).

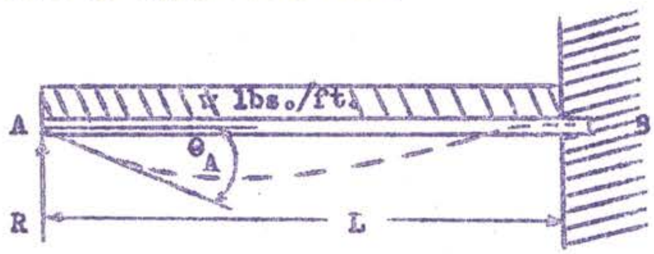
$$M_{1_B} = f_B = \frac{P'}{A} + \frac{Mc}{I_o} = 0 + \frac{\frac{wL^2}{8} \cdot \frac{2L}{3} \cdot \frac{1}{EI} \cdot \frac{L}{2} \cdot L}{\frac{1}{EI} \cdot \frac{L^3}{3}}$$

$$M_{1_B} = \frac{wL^2}{8}$$

$$M_B = M_{s_B} - M_{i_B} = 0 - \frac{wL^2}{8} = -\frac{wL^2}{8}\,; \quad \text{By statics, } R = \frac{3}{8}wL$$



(E) By the Method of Slope Deflection:



$$M_A = 0\,, \ \theta_B = 0\,, \ -M_{F_{AB}} = M_{F_{BA}} = \frac{wL^2}{12}$$

(1) $M_{AB} = -\frac{wL^2}{12} + \frac{EI}{L}(4\theta_A + 0 - 0) = 0$

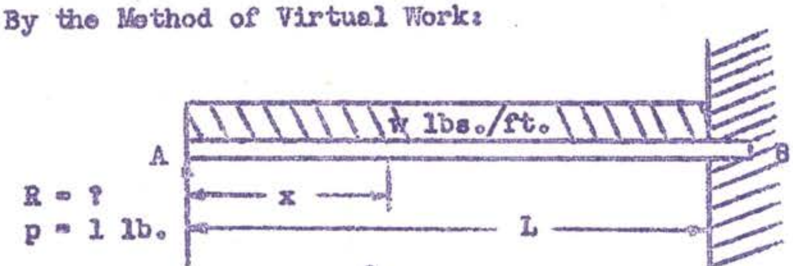
(2) $M_{BA} = \frac{wL^2}{12} + \frac{EI}{L}(2\theta_A + 0 - 0)$

From equation (1) $\theta_A = \frac{wL^3}{48EI}$, then

$$M_{BA} = \frac{wL^2}{12} + \frac{EI}{L} \cdot \frac{2wL^3}{48EI} = \frac{wL^2}{8}$$

The positive sign indicates that the resisting moment at B acts in a clockwise direction - i.e., there is tension in the top fibers.

(F) By the Method of Virtual Work:



$$M_x = Rx - \frac{wx^2}{2}\,, \ m_x = x$$

$$EIy_A = 0 = \int_0^L Rx^2dx - \int_0^L \frac{wx^3}{2}\,dx$$

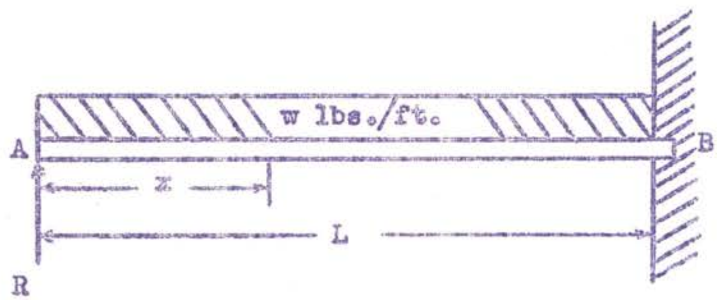
$$0 = \frac{RL^3}{3} - \frac{wL^4}{8}$$

$$R = \frac{3}{8} wL$$

$$M_B = \frac{3}{8} wL \cdot L - \frac{wL^2}{2} = - \frac{wL^2}{8}$$

(G) The Method of Real Work does not apply.

(H) By the Method of Least Work:



$$M_x = Rx - \frac{wx^2}{2}$$

$$2EIU = \int_0^L (Rx - wx^2)^2 dx$$

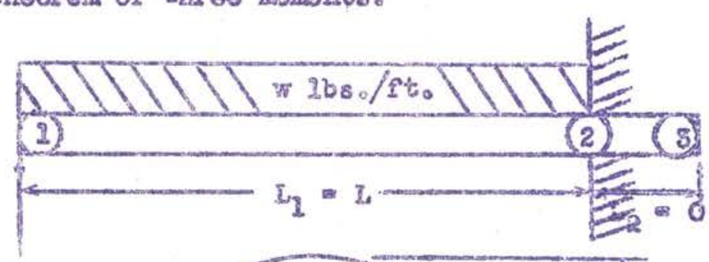
$$2EIy_A = 0 = 2EI \frac{\partial U}{\partial R} = \int_0^L 2\left(Rx - \frac{wx^2}{2}\right) x dx$$

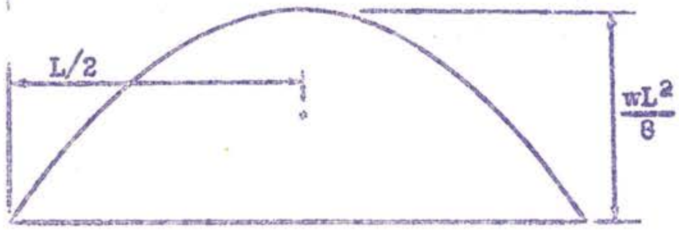
which may be written

$$0 = \int_0^L Rx^2 dx - \int_0^L \frac{wx^3}{2} dx$$

from which, $R = \frac{3}{8} wL$

(J) By the Theorem of Three Moments:





Simple Moment Diagram

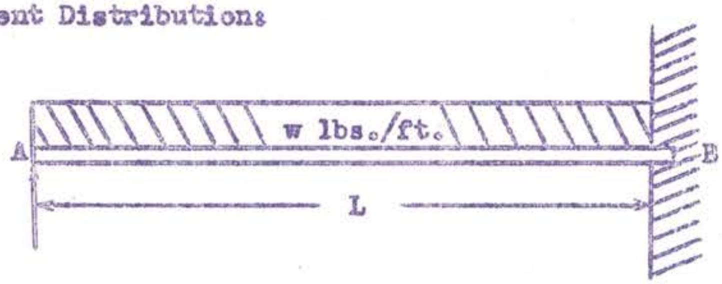
$$M_1L_1 + 2M_2(L_1 + L_2) + M_3L_2 = -\frac{6A_1\bar{x}_1}{L_1} - \frac{6A_2\bar{x}_2}{L_2}$$

$$0 + 2M_2(L + 0) + 0 = -\frac{6wL^2}{8} \cdot \frac{2L}{3} \cdot \frac{L}{2} \cdot \frac{1}{L} - 0$$

$$2M_2L = -\frac{wL^3}{4}$$

$$M_2 = -wL^2/8$$

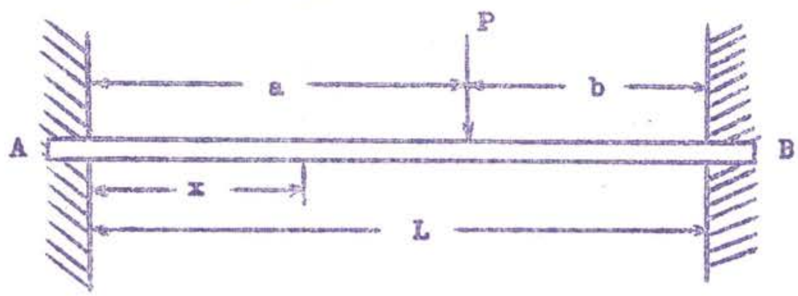
(K) By Moment Distribution:



$-\frac{1}{12}wL^2$	F.E.M.		$-\frac{1}{12}wL^2$
$+\frac{1}{12}wL^2$	Bal. Mom.	C.O.M.	$-\frac{1}{24}wL^2$
0	Final Mom.		$-\frac{1}{8}wL^2$

BEAMS FIXED AT BOTH ENDS

1. Concentrated Load at Any Point



(A) By Double Integration:

For values of x between o and a:

$$EI\frac{d^2y}{dx^2} = M_x = M_A + V_Ax$$

$$EI\frac{dy}{dx} = M_Ax + \frac{V_Ax^2}{2} + C_1$$

For values of x between a and L:

$$EI\frac{d^2y}{dx^2} = M_x = M_A + V_Ax - P(x - a)$$

$$EI\frac{dy}{dx} = M_Ax + \frac{V_Ax^2}{2} - \frac{P}{2}(x - a)^2 + C_1'$$

when $x = 0$, $\frac{dy}{dx} = 0$, $\therefore C_1 = 0$

when $x = a$

$$M_A a + \frac{V_A a^2}{2} = M_A a + \frac{V_A a^2}{2} - 0 + C_1'$$

$$0 = C_1'$$

$$EIy = \frac{M_A x^2}{2} + \frac{V_A x^3}{6} + C_2$$

$$EIy = \frac{M_A x^2}{2} + \frac{V_A x^3}{6} - \frac{P}{6}(x - a)^3 + C_2'$$

when $x = 0$, $y = 0$, $\therefore C_2 = 0$

when $x = a$

$$\frac{M_A a^2}{2} + \frac{V_A a^3}{6} = \frac{M_A a^2}{2} + \frac{V_A a^3}{6} - 0 + C_2'$$

$$0 = C_2'$$

when $x = L$, $y = 0$, and $\frac{dy}{dx} = 0$

Therefore

$$M_A L + \frac{V_A L^2}{2} - \frac{P}{2}(L - a)^2 = 0 \qquad (1)$$

$$\frac{M_A L^2}{2} + \frac{V_A L^3}{6} - \frac{P}{6}(L - a)^3 = 0 \qquad (2)$$

Multiplying eq. (1) by $L/3$

$$\frac{M_A L^2}{3} + \frac{V_A L^3}{6} - \frac{PL}{6}(L - a)^2 = 0 \qquad (3)$$

Subtracting eq. (3) from eq. (2) (note that $L - a = b$)

$$\frac{M_A L^2}{6} + 0 + \frac{Pb^2}{6}(L - b) = 0 \qquad (4)$$

$$M_A = -\frac{Pb^2 a}{L^2}$$

and from eq. (1) , $V_A = \frac{Pb^2(3a + b)}{L^3}$

The other two unknown reactions, M_B and V_B, may be found by statics.

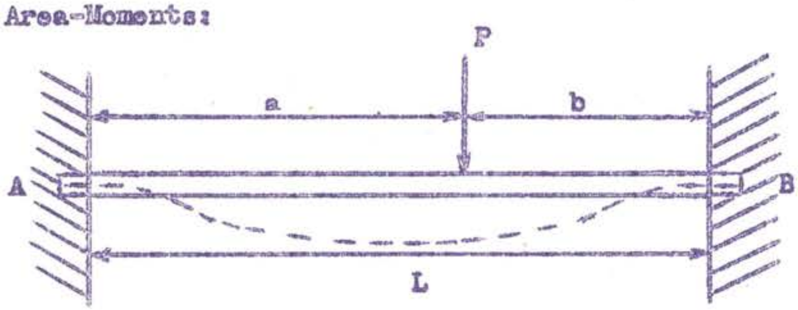
$$M_B = -\frac{Pa^2 b}{L^2}$$

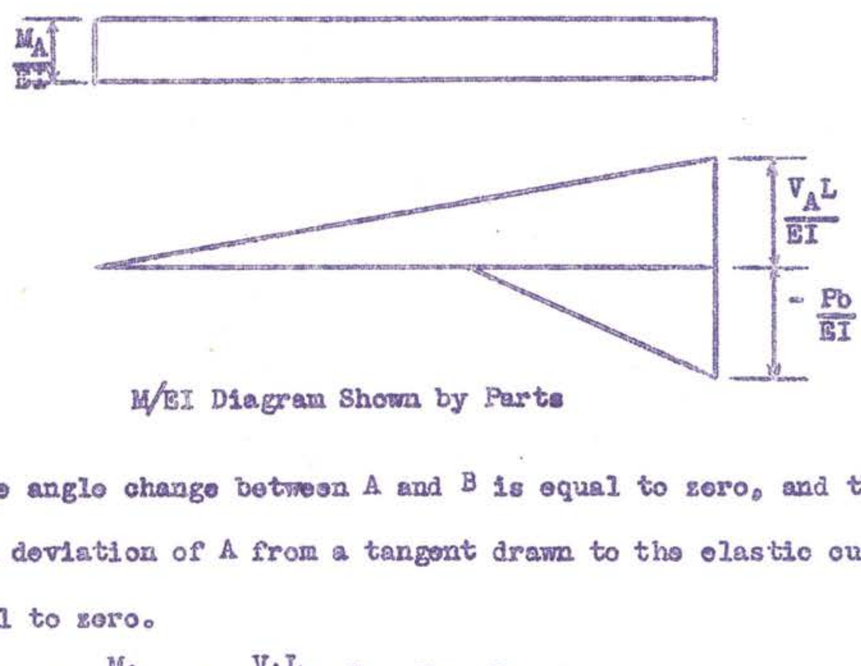
$$V_B = \frac{Pa^2(a + 3b)}{L^3}$$

It should be observed that, if the two end moments for any span be known, the end shears may be determined by adding to the simple shear at the end of the greater moment the algebraic difference of the end moments divided by the span length. Therefore, if a is greater than b, $V_A = \frac{Pb}{L} - \frac{M_B - M_A}{L}$, and $V_B = \frac{Pa}{L} + \frac{M_B - M_A}{L}$.

In this problem if $a = b = L/2$, it is found that $M_A = M_B = -PL/8$ and that $V_A = V_B = P/2$.

(B) By Area-Moments:





M/EI Diagram Shown by Parts

The angle change between A and B is equal to zero, and the tangential deviation of A from a tangent drawn to the elastic curve at B is equal to zero.

$$\Delta\theta = 0 = \frac{M_A}{EI} \cdot L + \frac{V_A L}{EI} \cdot \frac{L}{2} - \frac{Pb}{EI} \cdot \frac{b}{2}$$

$$0 = M_A L + \frac{V_A L^2}{2} - \frac{Pb^2}{2} \qquad (1)$$

$$t_A = 0 = \frac{M_A}{EI} \cdot L \cdot \frac{L}{2} + \frac{V_A L}{EI} \cdot \frac{L}{2} \cdot \frac{2L}{3} - \frac{Pb}{EI} \cdot \frac{b}{2} \left(a + \frac{2b}{3}\right)$$

$$0 = \frac{M_A L^2}{2} + \frac{V_A L^3}{3} - \frac{Pb^2}{6}(3a + 2b) \qquad (2)$$

Multiplying eq. (1) by $\frac{2L}{3}$

$$0 = \frac{2M_A L^2}{3} + \frac{V_A L^3}{3} - \frac{Pb^2 L}{3} \qquad (3)$$

Subtract eq. (3) from eq. (2)

$$\frac{-M_A L^2}{6} + 0 - \frac{Pb^2}{6}(3a + 2b - 2L) = 0 \qquad (4)$$

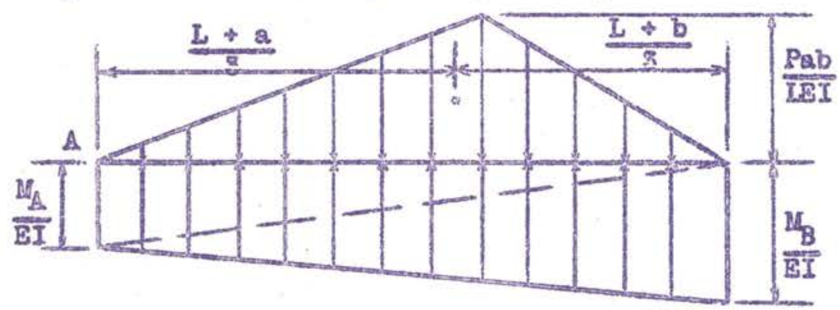
$$M_A = -\frac{Pb^2 a}{L^2}$$

$$V_A = \frac{Pb^2(3a + b)}{L^3} \quad \text{from eq. (1)}$$

By statics it is found that $M_B = -\frac{Pa^2 b}{L^2}$ and that $V_B = \frac{Pa^2(a + 3b)}{L^3}$.

(C) By the Conjugate Beam Method:

The ends which are fixed in the real beam become free ends in the conjugate beam, which is assumed to be held in equilibrium in space by the pseudo-pressures exerted by the M/EI diagram.



$$EI\Delta_A = 0 = M_A\ (\text{conj.}) = \frac{Pab}{L} \cdot \frac{L}{2} \cdot \frac{L + a}{3} + M_A \cdot \frac{L}{2} \cdot \frac{L}{3} + M_B \cdot \frac{L}{2} \cdot \frac{2L}{3}$$

$$0 = Pab(L + a) + M_A L^2 + 2M_B L^2 \qquad (1)$$

$$EI\Delta_B = 0 = M_B\ (\text{conj.}) = \frac{Pab}{L} \cdot \frac{L}{2} \cdot \frac{L + b}{3} + M_A \cdot \frac{L}{2} \cdot \frac{2L}{3} + M_B \cdot \frac{L}{2} \cdot \frac{L}{3}$$

$$0 = Pab(L + b) + 2M_A L^2 + M_B L^2 \qquad (2)$$

Multiplying eq. (1) by 2

$$0 = 2Pab(L + a) + 2M_A L^2 + 4M_B L^2 \qquad (3)$$

subtracting eq. (3) from eq. (2)

$$0 = Pab(L + b - 2L - 2a) - 0 - 3M_B L^2 \qquad (4)$$

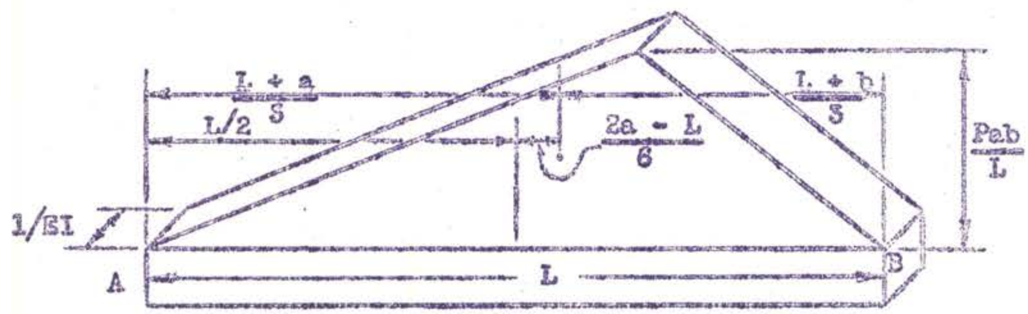
$$3M_B L^2 = Pab(-3a)$$

$$M_B = -\frac{Pa^2b}{L^2}$$

$$M_A = -\frac{Pb^2a}{L^2} \quad \text{from eq. (1)}$$

(D) By Column Analogy:

Let the analogous column be loaded with the moment diagram of a simply supported beam.



$$M_{i_A} = f_A = \frac{P'}{A} + \frac{Mc}{I} = \frac{\frac{Pab}{L} \cdot \frac{L}{2} \cdot \frac{1}{EI}}{L \cdot \frac{1}{EI}} + \frac{\frac{Pab}{L} \cdot \frac{L}{2} \cdot \frac{1}{EI} \cdot \frac{(2a - L)}{6} \cdot \left(\frac{-L}{2}\right)}{\frac{1}{12} \cdot \frac{1}{EI} \cdot L^3}$$

$$M_{i_A} = \frac{Pab}{2L} - \frac{Pa^2b}{L^2} + \frac{Pab}{2L} = \frac{Pab^2}{L^2}$$

$$M_A = M_{s_A} - M_{i_A} = 0 - \frac{Pab^2}{L^2} = -\frac{Pab^2}{L^2}$$

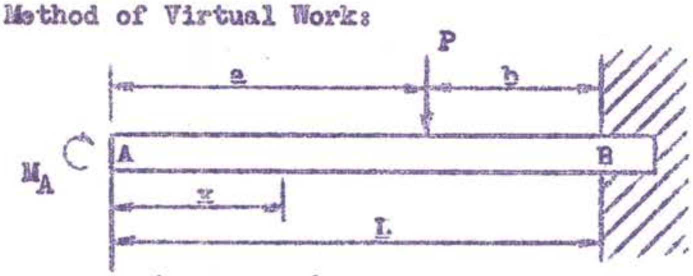
$$M_{i_B} = f_B = \frac{\frac{Pab}{L} \cdot \frac{L}{2} \cdot \frac{1}{EI}}{L \cdot \frac{1}{EI}} + \frac{\frac{Pab}{L} \cdot \frac{L}{2} \cdot \frac{1}{EI} \cdot \frac{(2a - L)}{6} \cdot \frac{L}{2}}{\frac{1}{12} \cdot \frac{1}{EI} \cdot L^3}$$

$$M_{i_B} = \frac{Pab}{2L} + \frac{Pa^2b}{L^2} - \frac{Pab}{2L} = \frac{Pa^2b}{L^2}$$

$$M_B = M_{s_B} - M_{i_B} = 0 - \frac{Pa^2b}{L^2} = -\frac{Pa^2b}{L^2}$$

(E) Since the basic equations of the slope deflection method contain the terms M_{Fab} and M_{Fba}, and since $M_A = M_{Fab}$ and $M_B = M_{Fba}$, it is not possible to use the method in the solution of this problem.

(F) By the Method of Virtual Work:



V_A + (p = 1 lb.)

First, let a unit load be applied at the left end of the beam, which is taken as the origin. For values of x between zero and a, $M_x = M_A + V_A x$ and $m_x = x$; and for values between a and L, $M_x = M_A + V_A x - P(x - a)$ and $m_x = x$. Then

$$EI p\Delta = 0 = \int_0^a (M_A x + V_A x^2)dx + \int_a^L (M_A x + V_A x^2 - Px^2 + Pax)dx$$

from which it is found that

$$3M_A L^2 + 2V_A L^3 - 2PL^3 + 3PaL^2 - Pa^3 = 0 \qquad (1)$$

Next, let the right end of the beam be taken as the origin and the unit load applied at the right end.

For values of x between zero and b, $M_x = M_B + V_B x$ and $m_x = x$; and for values between b and L, $M_x = M_B + V_B x - P(x - b)$ and $m_x = x$. Then

$$EI p\Delta = 0 = \int_0^b (M_B x + V_B x^2)dx + \int_b^L (M_B x + V_B x^2 - Px^2 + Pbx)dx$$

from which it is found that

$$3M_B L^2 + 2V_B L^2 - 2PL^3 + 3PbL^2 - Pb^3 = 0 \qquad (2)$$

Since $M_B = M_A + V_A L - Pb$, and $V_B = P - V_A$, equation (2) may be rewritten

$$3M_A L^2 + V_A L^3 - Pb^3 = 0 \qquad (3)$$

Now, multiply equation (3) by two.

$$6M_A L^2 + 2V_A L^3 - 2Pb^3 = 0 \qquad (4)$$

If equation (4) is subtracted from equation (1) it is found that

$$3M_AL^2 = -3Pab^2$$

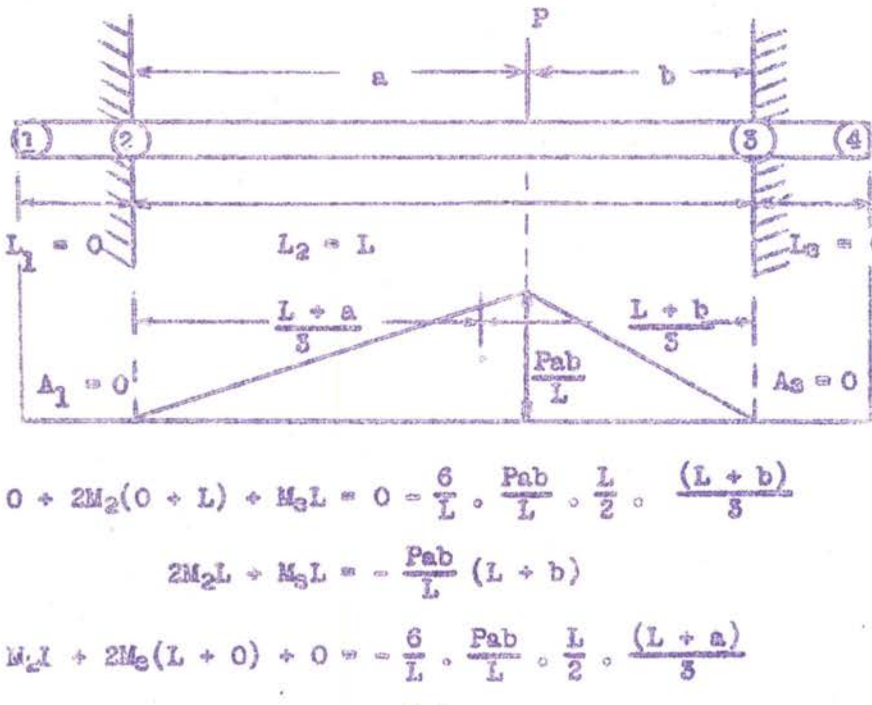
$$M_A = -\frac{Pab^2}{L^2}$$

V_A may now be determined from equation (1) and M_B and V_B found by statics.

(G) The Method of Real Work does not apply.

(H) The Method of Least Work may be applied in much the same way as was the Method of Virtual Work. Instead of applying a unit load at each end in turn, the partial derivative of U with respect to V_A is set equal to zero.

(J) By the Theorem of Three Moments:



$$0 + 2M_2(0 + L) + M_3L = 0 - \frac{6}{L} \cdot \frac{Pab}{L} \cdot \frac{L}{2} \cdot \frac{(L + b)}{3}$$

$$2M_2L + M_3L = -\frac{Pab}{L}(L + b) \qquad (1)$$

$$M_2L + 2M_3(L + 0) + 0 = -\frac{6}{L} \cdot \frac{Pab}{L} \cdot \frac{L}{2} \cdot \frac{(L + a)}{3}$$

$$M_2L + 2M_3L = -\frac{Pab}{L}(L + a) \qquad (2)$$

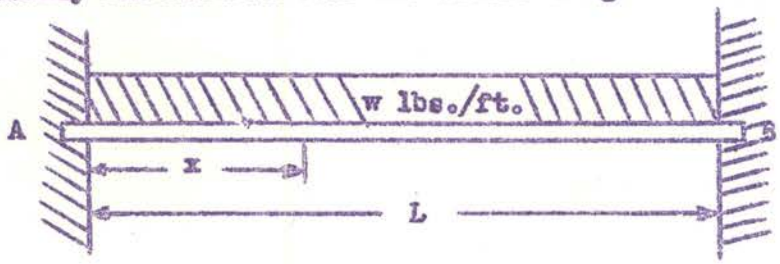
The simultaneous solution of equations (1) and (2) produces the values

$$M_2 = -Pab^2/L^2$$

$$M_3 = -Pa^2b/L^2$$

(K) The Method of Moment Distribution does not apply since there is no rotation of the joints when the load is applied.

2. Load Uniformly Distributed Over the Entire Length of the Beam.



(A) By Double Integration:

$$EI \frac{d^2y}{dx^2} = M_x = M_A + V_A x - \frac{wx^2}{2}$$

$$EI \frac{dy}{dx} = M_A x + \frac{V_A x^2}{2} - \frac{wx^3}{6} + C_1$$

when $x = 0$, $\frac{dy}{dx} = 0$, $\therefore C_1 = 0$

$$EIy = \frac{M_A x^2}{2} + \frac{V_A x^3}{6} - \frac{wx^4}{24} + C_2$$

when $x = 0$, $y = 0$, $\therefore C_2 = 0$

when $x = L$, $\frac{dy}{dx} = 0$, and $y = 0$, therefore

$$6M_A + 3V_A L - wL^2 = 0 \qquad (1).$$

$$12M_A + 4V_A L - wL^2 = 0 \qquad (2)$$

If equations (1) and (2) are solved simultaneously it will be found that

$$V_A = wL/2$$

$$M_A = -wL^2/12$$

In cases of this sort, where there is symmetry of loading and support, some time may be gained by observing that $V_A = V_B = \frac{wL}{2}$, thus removing the necessity for the simultaneous solution of two equations. By symmetry, also, $M_A = M_B$.

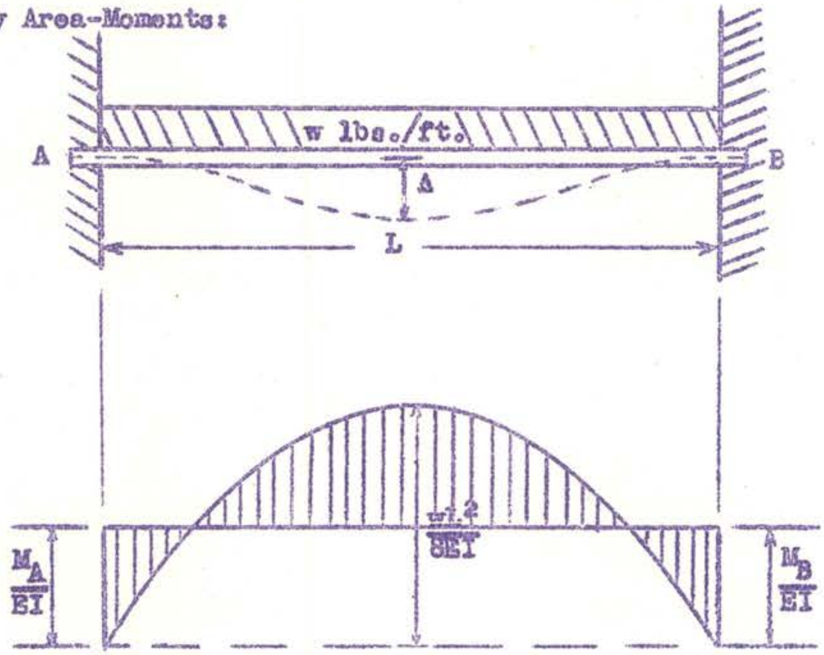
The maximum deflection occurs at mid-span so when $x = \frac{L}{2}$, $y = \Delta$ in the equation of the elastic curve.

$$EI\Delta = \frac{M_A L^2}{8} + \frac{V_A L^3}{48} - \frac{wL^4}{384}$$

$$EI\Delta = -\frac{wL^4}{96} + \frac{wL^4}{96} - \frac{wL^4}{384} = \frac{wL^4}{384}(-4 + 4 - 1)$$

$$\Delta = \frac{wL^4}{384EI}$$

(B) By Area-Moments:



The deviation of point A from a tangent drawn at point B is equal to zero.

$$t_A = 0 = \frac{2}{3} \cdot \frac{wL^2}{8EI} \cdot L \cdot \frac{L}{2} + \frac{M_A}{EI} \cdot L \cdot \frac{L}{2}$$

$$M_A = \frac{-1}{12} wL^2$$

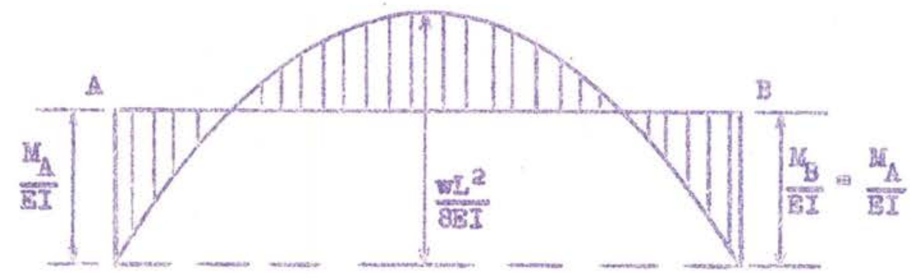
All other unknown reactions may be found by statics and by taking advantage of the symmetry.

With respect to a tangent drawn to the elastic curve at the mid-point

$$t_A = \Delta = \frac{2}{3} \cdot \frac{wL^2}{8EI} \cdot \frac{L}{2} \cdot \frac{5L}{16} + \frac{M_A}{EI} \cdot \frac{L}{2} \cdot \frac{L}{4}$$

$$\Delta = \frac{5wL^4}{384EI} - \frac{wL^4}{96EI} = \frac{wL^4}{384EI}$$

(C) By the Conjugate Beam Method:



The deflection of the real beam at point A is equal to zero. Therefore the moment in the conjugate beam at A is equal to zero.

$$\frac{2}{3} \cdot \frac{wL^2}{8EI} \cdot L \cdot \frac{L}{2} + \frac{M_A}{EI} \cdot L \cdot \frac{L}{2} = 0$$

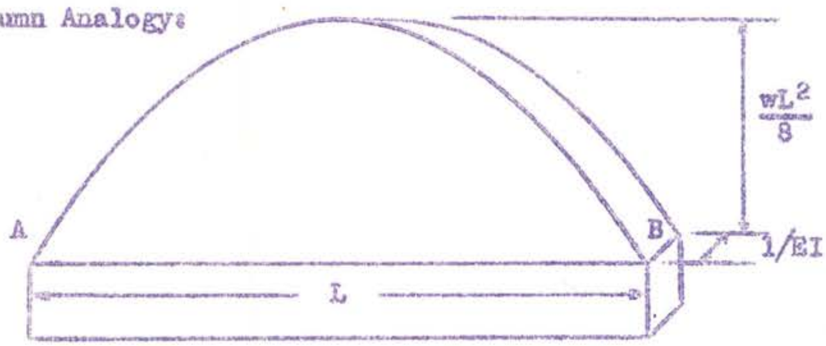
$$M_A = - \frac{wL^2}{12EI}$$

The maximum moment in the conjugate beam is at its center and is equal to the maximum deflection in the real beam.

$$\Delta = M\ (\text{max.}) = \frac{wL^2}{8EI} \cdot \frac{2}{3} \cdot \frac{L}{2} \cdot \frac{3}{8} \cdot \frac{L}{2} + \frac{M_A}{EI} \cdot \frac{L}{2} \cdot \frac{L}{4}$$

$$\Delta = \frac{wL^4}{128EI} - \frac{wL^4}{96EI} = - \frac{wL^4}{384EI}$$

(D) By Column Analogy:



The statically determinate condition chosen will be that of a simple span, which provides an analogous column loaded as shown above. Note that the load is a concentric load and that Mc/I will therefore be equal to zero.

$$M_{i_A} = f_A = \frac{P'}{a} \quad \frac{Mc}{I}$$

$$M_{i_A} = \frac{\frac{wL^2}{8} \cdot \frac{2L}{3} \cdot \frac{1}{EI}}{L \cdot \frac{1}{EI}} + 0 = \frac{wL^2}{12}$$

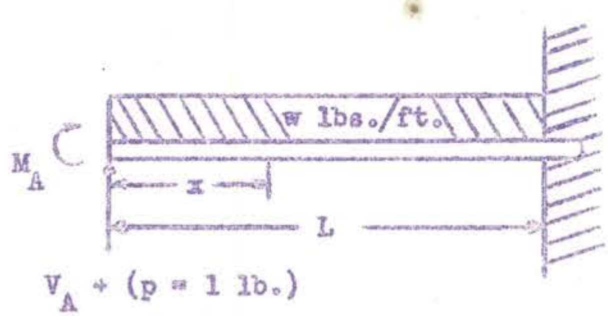
$$M_A = M_{s_A} - M_{i_A} = 0 - \frac{wL^2}{12} = -\frac{wL^2}{12}$$

It may be observed that, since the load is concentric, the indeterminate moment at any point in the beam is equal to the P'/a stress in the analogous column, i.e. $\frac{wL^2}{12}$

(E) The Method of Slope Deflection does not apply.

(F) By the Method of Virtual Work:

Let the origin be taken at A and the unit load applied at A so that $p\Delta = 0$.



$$M_x = M_A + V_A x - \frac{wx^2}{2}, \; m_x = x$$

$$EIp\Delta = 0 = \int_0^L \left(M_A + V_A x - \frac{wx^2}{2}\right) x\,dx \,, \quad \text{from which}$$

$$12M_A + 8V_A L - 3wL^2 = 0$$

Now, since $V_A = \frac{wL}{2}$

$$12M_A + 4wL^2 - 3wL^2 = 0$$

$$M_A = -\frac{wL^2}{12}$$

(G) The Method of Real Work does not apply.

(H) By the Method of Least Work:

The procedure closely parallels that shown for virtual work. Again, the partial derivative of U with respect to V_A is equal to zero. Then $M_x = M_A + V_A x - \frac{wx^2}{2}$.

$$2EIU = \int_0^L \left(M_A + V_A x - \frac{wx^2}{2}\right)^2 dx$$

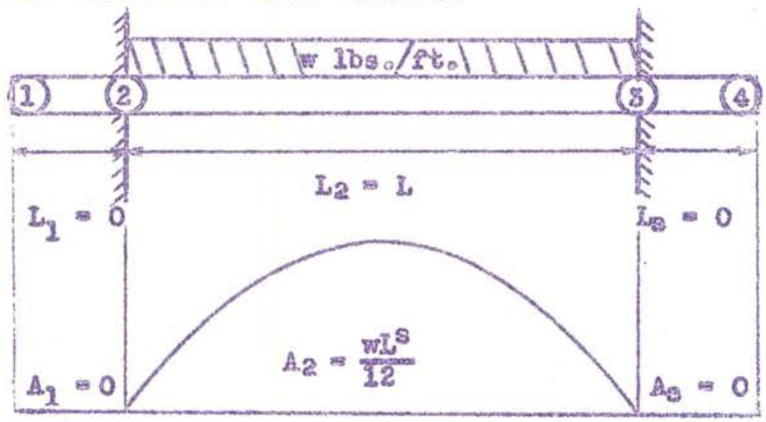
$$2EI\Delta = 0 = 2EI\frac{\partial U}{\partial V_A} = \int_0^L 2\left(M_A + V_A x - \frac{wx^2}{2}\right)x\,dx$$

Note that this equation is the same as that for virtual work.

Therefore

$$M_A = -\frac{wL^2}{12}$$

(J) By the Theorem of Three Moments:



$$0 + 2M_1(0 + L) + M_2L = 0 - \frac{6}{L} \cdot \frac{wL^3}{12} \cdot \frac{L}{2}$$

$$2M_1L + M_2L = -\frac{wL^3}{4}$$

By symmetry $M_1 = M_2$, therefore

$$3M_1L = -\frac{wL^3}{4}$$

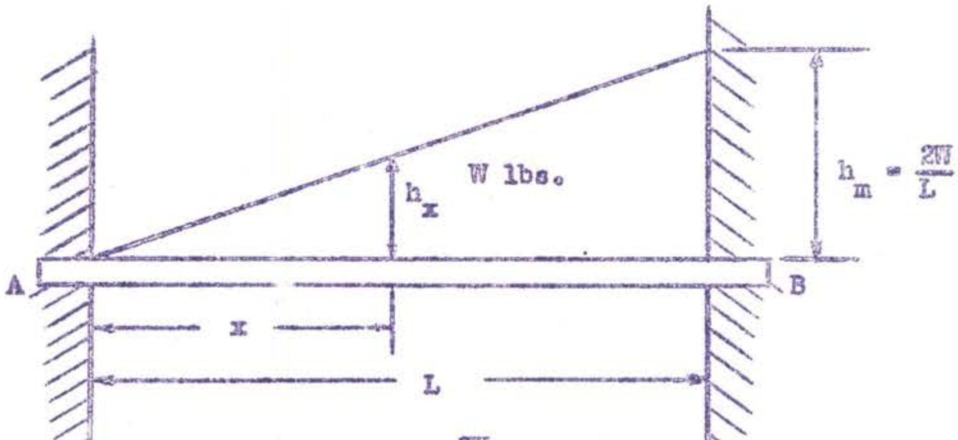
$$M_1 = -\frac{wL^2}{12}$$

(K) The Method of Moment Distribution does not apply.

3. Load Varying Uniformly from Zero at One End to a Maximum at the Other.

This problem is included only to point out a special method which may be applied to the solution of problems involving distributed loads. The analysis will first be made by means of double integration and then by the special method so that there will exist some basis for comparison.

(A) By the Method of Double Integration:



$$h_x = h_m \cdot \frac{x}{L} = \frac{2Wx}{L^2}$$

$$EI\frac{d^2y}{dx^2} = M_x = M_A + V_A x - \frac{2Wx}{L^2} \cdot \frac{x}{2} \cdot \frac{x}{3}$$

$$EI\frac{dy}{dx} = M_A x + \frac{V_A x^2}{2} - \frac{Wx^4}{12L^2} + (C_1 = 0)$$

$$EIy = \frac{M_A x^2}{2} + \frac{V_A x^3}{6} - \frac{Wx^5}{60L^2} + (C_2 = 0)$$

when $x = L$, $\frac{dy}{dx} = 0$, and $y = 0$. When $\frac{dy}{dx} = 0$

$$M_A L + \frac{V_A L^2}{2} - \frac{WL^2}{12} = 0 \text{, or}$$

$$12M_A + 6V_A L - WL = 0 \qquad (1)$$

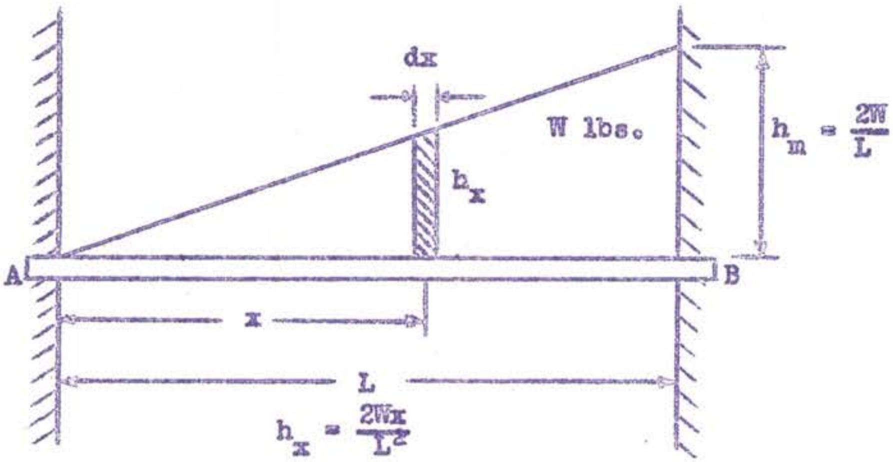
when $y = 0$

$$\frac{M_A L^2}{2} + \frac{V_A L^3}{6} - \frac{WL^3}{60} = 0 \text{, or}$$

$$30M_A + 10V_A L - WL = 0 \qquad (2)$$

If equations (1) and (2) are solved simultaneously it will be found that $M_A = -\frac{WL}{15}$ and $V_A = \frac{3W}{10}$. The moment and shear at B are found by statics to be $M_B = -\frac{WL}{10}$ and $V_B = \frac{7W}{10}$.

(B) By a Special Method of Integration:



As has been shown previously, the moment at end A of a fixed-ended beam due to a concentrated load P applied at distance a from end A is $M_A = -\frac{Pab^2}{L^2}$. Now, assume the distributed load to be made up of an infinite number of concentrated loads of magnitude dP acting at the variable distance x from end A. Then the differential moment at A due to the load dP will be $dM_A = -\frac{dP \cdot x(L-x)^2}{L^2}$. Note now that $dP = h_x dx = \frac{2Wxdx}{L^2}$. Then

$$dM_A = -\frac{2Wxdx}{L^4} \cdot x(L-x)^2 = -\frac{2W}{L^4}(x^2L^2 - 2x^3L + x^4)dx$$

$$M_A = -\frac{2W}{L^4}\int_0^L (x^2L^2 - 2x^3L + x^4)dx$$

$$M_A = -\frac{2W}{L^4}\left[\frac{L^5}{3} - \frac{L^5}{2} + \frac{L^5}{5}\right]$$

$$M_A = -\frac{WL}{15}$$

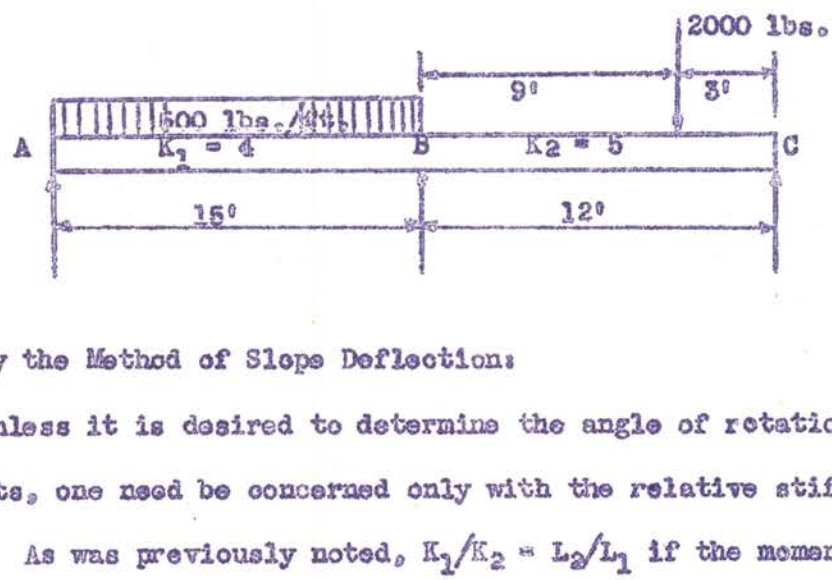
The moment at B may be found similarly.

CONTINUOUS BEAMS

Of the ten methods of analysis studied previously in this report, all but three prove to be generally impractical or laborious in the analysis of continuous beams. Therefore, this section will be limited to the study of only these three methods--(1) slope deflection, (2) the theorem of three moments, and (3) moment distribution. The beams chosen will be of constant cross-section, although all of these procedures apply as well to beams of variable cross-section. The stiffness ratio I/L, for which K is the common notation, will vary then inversely with the length of the span.

It is believed advisable, in the interest of clarity, to use numerical values for the span lengths and the loads. Since the moment of inertia of the cross-section is to remain constant, it will not be necessary to employ a numerical value in its stead. In general, the basic equations required will not be restated each time they are used. It falls upon the reader to refer to Part II of this report to acquire familiarity with such equations. Where the fixed-end moments are needed in the solution they will not be here derived, but will be taken from the portion of this report dealing with fixed-ended beams.

1. Beam Continuous Over Two Spans; All Supports Simple.



(A) By the Method of Slope Deflection:

Unless it is desired to determine the angle of rotation at the supports, one need be concerned only with the relative stiffness of the spans. As was previously noted, $K_1/K_2 = L_2/L_1$ if the moment of inertia is constant (since K is proportional to I/L). Then $K_1 = K_2L_2/L_1$ in which any convenient value may be chosen for K_2. In this problem let $K_2 = 5$; then $K_1 = 5\,\frac{12}{15} = 4$. The fixed-end moments at A and B in the first span are numerically equal to $wL^2/12$, the moment at A being taken as negative and that at B positive. In the second span the fixed-end moment at B is equal to $-Pab^2/L^2$ and that at C equal to $-Pa^2b/L^2$.

$$M_{Fab} = -M_{Fba} = -\frac{500(15)^2}{12} = -9370 \text{ ft.-lbs.}$$

$$M_{Fbc} = -\frac{2000(3)^2 9}{(12)^2} = -1125 \text{ ft.-lbs.}$$

$$M_{Fcb} = \frac{2000(9)^2 3}{(12)^2} = 3375 \text{ ft.-lbs.}$$

The following relationships may be further noted:

(a) $M_{ab} = 0$; (b) $M_{cb} = 0$; (c) $M_{ba} + M_{bc} = 0$

Now, using the basic slope-deflection equations:

$$M_{ab} = -9370 + 4E(4\theta_a + 2\theta_b) = 0 \text{ , or}$$

$$16\theta_a + 8\theta_b = 9370/E \qquad (1)$$

$$M_{ba} = 9370 + 8E\theta_a + 16E\theta_b \qquad (2)$$

$$M_{bc} = -1125 + 5E(4\theta_b + 2\theta_c) \text{ , or}$$

$$M_{bc} = -1125 + 20E\theta_b + 10E\theta_c \qquad (3)$$

$$M_{cb} = 3375 + 10E\theta_b + 20E\theta_c = 0 \text{ , or}$$

$$10\theta_b + 20\theta_c = -3375/E \qquad (4)$$

Now, from equations (c), (2), and (3)

$$8\theta_a + 36\theta_b + 10\theta_c = -8245/E \qquad (5)$$

From equation (4)

$$\theta_c = -\theta_b/2 - 169/E \qquad (6)$$

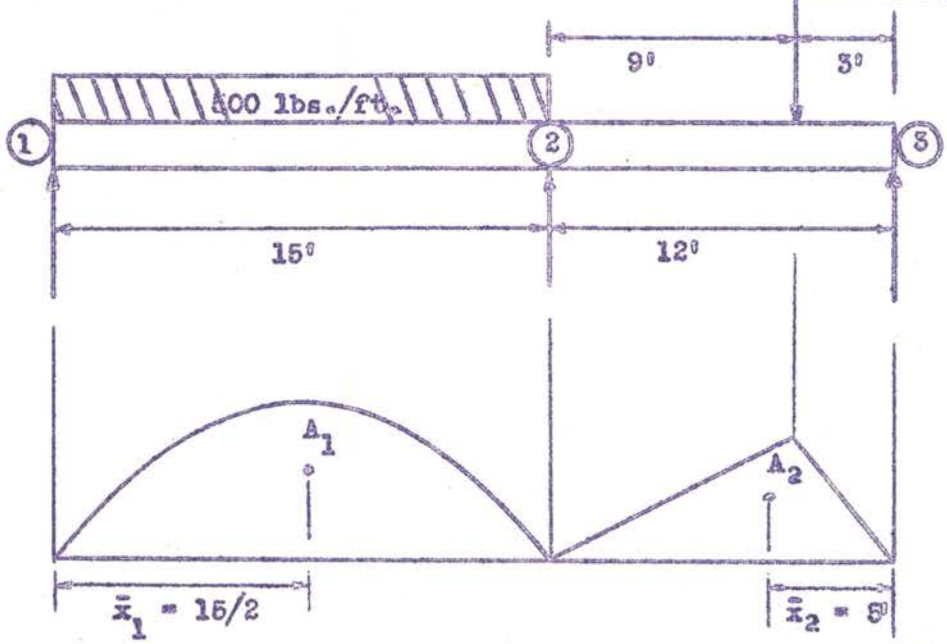
If equations (5) and (6) are combined to eliminate θ_c , it is found that

$$8\theta_a + 31\theta_b = -6555/E \qquad (7)$$

The simultaneous solution of equations (1) and (7) show that $\theta_a = 793/E$, $\theta_b = -416/E$, and $\theta_c = 39/E$. Then equations (2) and (3) may be used to show that $M_{ba} = 9060$ ft.-lbs. and $M_{bc} = -9055$ ft.-

lbs, which is a close check. The solution will be assumed to be correct.

(B) By the Theorem of Three Moments:



The general equation is $M_1L_1 + 2M_2(L_1 + L_2) + M_3L_2 = -\frac{6}{L_1}A_1\bar{x}_1 - \frac{6}{I_2}A_2\bar{x}_2$. In this problem $M_1 = M_3 = 0$.

$$0 + 2M_2(15 + 12) + 0 = -\frac{6}{15}\ \frac{2}{3}(15)\ \frac{1}{8}(500)(15)^2\ \ \frac{15}{2}$$

$$-\frac{6}{12}\ \frac{1}{2}(12)\ \frac{2000(9)(3)}{12}\ \ (6)$$

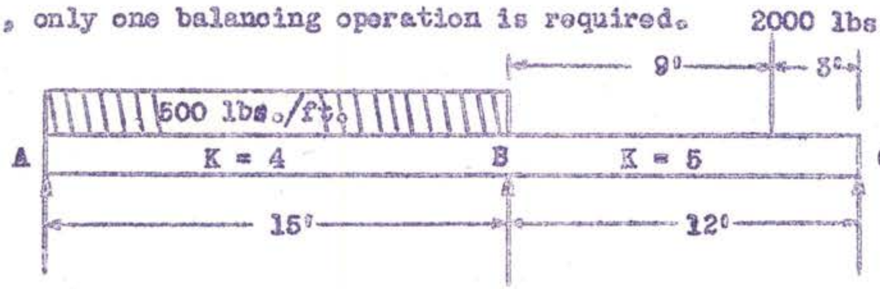
$$54M_2 = -422{,}000 - 67{,}500 = -489{,}500$$

$$M_2 = -\frac{489{,}500}{54} = -9060 \text{ ft.-lbs.}$$

(C) By Moment Distribution:

In order to determine what proportion of the unbalanced moment at a joint is taken by each member as the joint is unlocked, it is necessary to determine the distribution factors for the members. The distribution factor for member 1 at a joint where n members meet is $\frac{K_1}{K_1 + K_2 + K_3 \dots + K_n}$ or $\frac{K_1}{\Sigma K}$. Thus in this problem, the distribution factor for member AB at joint B is 4/9 or 0.445, and that for BC

5/9 or 0.555. Joints A and C will be unlocked in succession, balanced, and the moment carried over to B. Then joint B will be unlocked and balanced. It will be unnecessary to carry over the balancing moments from B to joints A and C, since anything carried over to the simple end joints will be reflected back to B in the same proportion and, hence, will not affect the final moment at B. To facilitate checking the tabulations a line should be drawn beneath each balancing moment. Here, only one balancing operation is required.

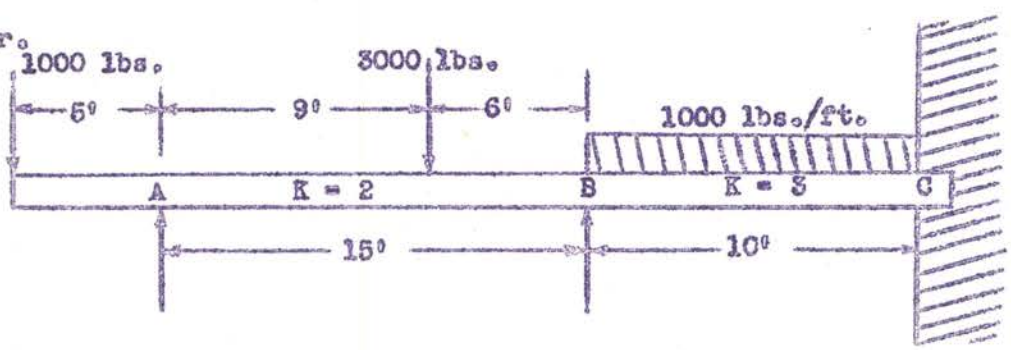


		.445	.555		
- 9370		- 9370	- 1125		- 3375
+ 9370	C. O. M.	- 4686	- 1687	C. O. M	+ 3375
	Bal. M.	+ 5000	- 6243	Bal. M.	
0		- 9055	- 9055		0

Final Moments in Ft.-Lbs.

It may be seen that there is practically no difference in the accuracy of the three methods, the results differing by about 0.055 per cent in this case.

2. Beam Continuous Over Two Spans; Fixed at One End and Overhanging the Other.



(A) By the Method of Slope Deflection:

$$M_{Fab} = -\frac{3000(9)(6)^2}{(15)^2} = -4320 \text{ ft.-lbs.}$$

$$M_{Fba} = \frac{3000(9)^2 6}{(15)^2} = 6480 \text{ ft.-lbs.}$$

$$M_{Fbc} = -M_{Fcb} = -\frac{1000(10)^2}{12} = -8330 \text{ ft.-lbs.}$$

Known relationships:

(a) $M_{ab} = -5000$ ft.-lbs., (b) $M_{ba} + M_{bc} = 0$, (c) $\theta_c = 0$

Slope-deflection equations:

$$M_{ab} = -5000 = -4320 + 2E(4\theta_a + 2\theta_b) \text{, or}$$

$$8\theta_a + 4\theta_b = -680/E \qquad (1)$$

$$M_{ba} = 6480 + 2E(2\theta_a + 4\theta_b) \text{, or}$$

$$M_{ba} = 4E\theta_a + 8E\theta_b + 6480 \qquad (2)$$

$$M_{bc} = -8330 + 3E(4\theta_b + 0) \text{, or}$$

$$M_{bc} = 12E\theta_b - 8330 \qquad (3)$$

$$M_{cb} = 8330 + 3E(0 + 2\theta_b) \text{, or}$$

$$M_{cb} = 6E\theta_b + 8330 \qquad (4)$$

From equations (b), (2), and (3)

$$4\theta_a + 20\theta_b = 1850/E \qquad (5)$$

Now, if equations (1) and (5) are solved simultaneously, it will be found that $\theta_b = 122/E$ and that $\theta_a = -146/E$. Then from equation (2)

$$M_{ba} = 4(-146) + 8(122) + 6480 = 6872 \text{ ft.-lbs.}$$

and from equation (3)

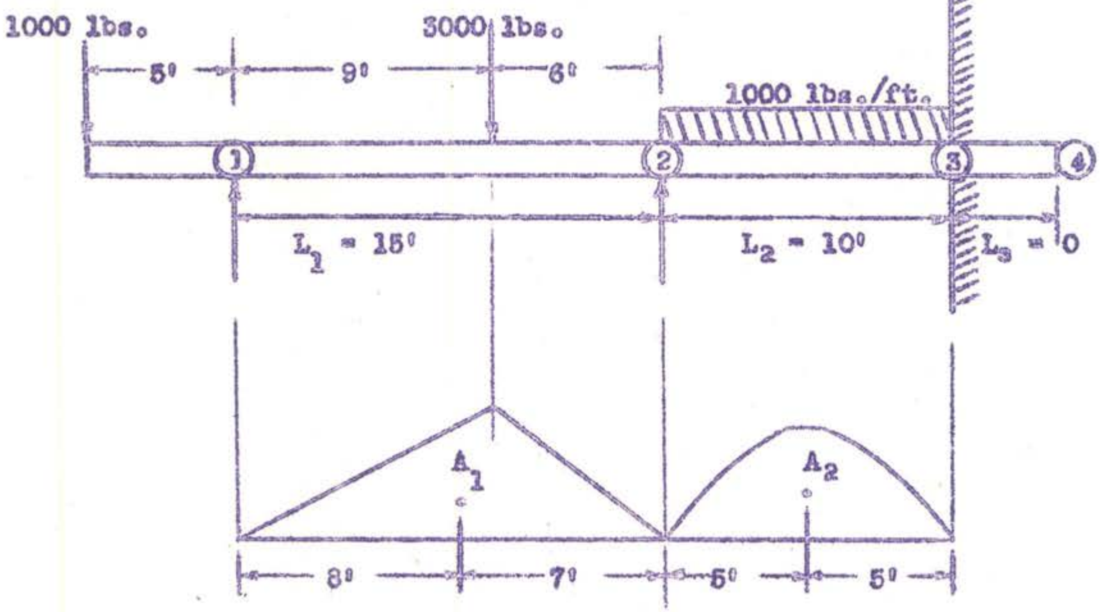
$$M_{bc} = 12(122) - 8330 = -6866 \text{ ft.-lbs.}$$

and from equation (4)

$M_{cb} = 6(122) + 8330 = 9062$ ft.-lbs.

The six foot-pound difference between M_{ba} and M_{bc} indicate a small error in the simultaneous solution of the equations, but the error is not large enough to cause any worry. To get an exact check, one should solve the equations on a calculator and remove the errors by successive approximation.

(B) By the Theorem of Three Moments:



By inspection, $M_1 = -5000$ ft.-lbs. Then for spans 1 - 2 and 2 - 3:

$$-5000(15) + 2M_2(15 + 10) + M_3(10) = -\frac{6}{15}\ \frac{15}{2}\cdot\frac{3000(9)(6)}{15}\ (8)$$

$$-\frac{6}{10}\ \frac{2}{3}\ 10\ \frac{1000(10)^2}{8}\ (5)$$

$$-75{,}000 + 50M_2 + 10M_3 = -259{,}200 - 250{,}000\ , \quad \text{or}$$

$$50M_2 + 10M_3 = -434{,}200 \qquad (1)$$

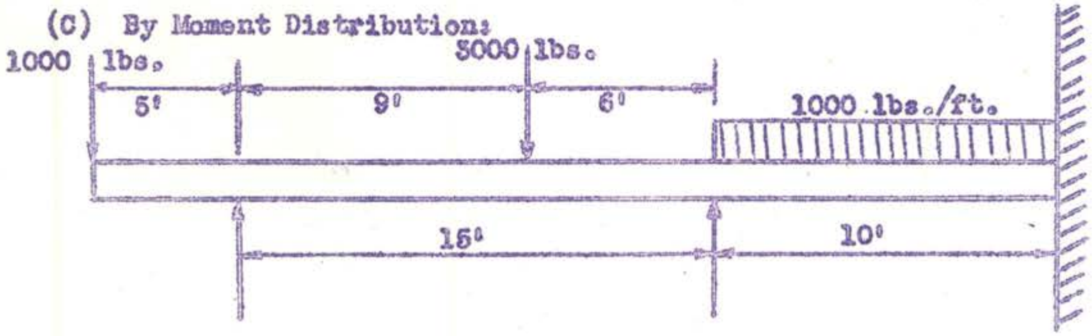
Now, for spans 2 - 3 and 3 - 4:

$$10M_2 + 2M_3(10 + 0) + 0 = -250{,}000\ , \quad \text{or}$$

$$10M_2 + 20M_3 = -250{,}000 \qquad (2)$$

The simultaneous solution of equations (1) and (2) provides the values $M_2 = -6870$ ft.-lbs. and $M_3 = -9065$ ft.-lbs., which closely check the values obtained by slope deflection.

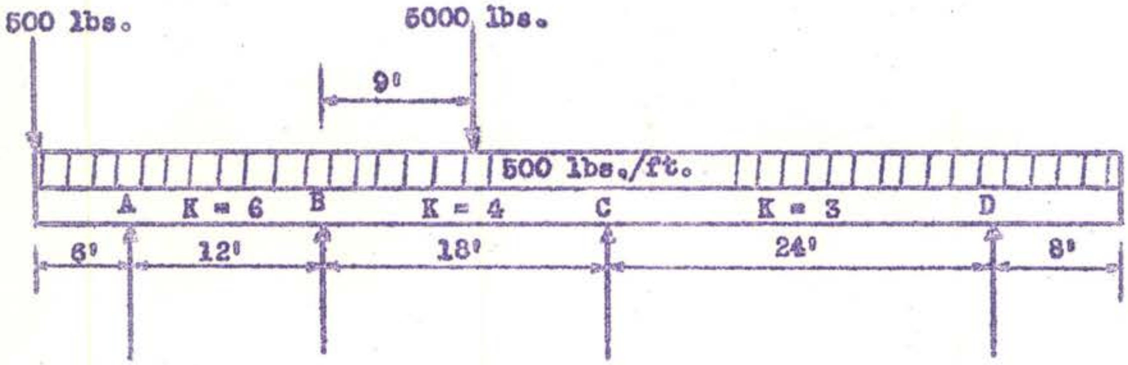
(C) By Moment Distribution:



0	1	.4	.6	1
- 5000	- 4320	- 3480	- 8330	- 8330
	- 680	+ 340		
	+ 438	- 876	+ 1314	- 657
	- 438	+ 219		
	+ 44	- 88	+ 131	- 65
	- 44	+ 22		
	+ 4	- 9	+ 13	- 7
	- 4	+ 2		
	0	- 1	+ 1	- 1
- 5000	- 5000	- 6871	- 6871	- 9060

Final Moments are in ft.-lbs.

3. Beam Continuous Over Three Spans; Both Ends Overhanging.



(A) By the Method of Slope Deflection:

$$M_{Fab} = -M_{Fba} = -\frac{500(12)^2}{12} = -6000 \text{ ft.-lbs.}$$

$$M_{Fbc} = -M_{Fcb} = -\frac{500(18)^2}{12} - \frac{5000(18)}{8} = -24{,}750 \text{ ft.-lbs.}$$

$$M_{Fcd} = -M_{Fdc} = -\frac{500(24)^2}{12} = -24{,}000 \text{ ft.-lbs.}$$

Known relationships:

(a) $M_{ab} = -12{,}000$ ft.-lbs. (b) $M_{dc} = 16{,}000$ ft.-lbs.

(c) $M_{ba} + M_{bc} = 0$ (d) $M_{cb} + M_{cd} = 0$

Slope-Deflection Equations:

$$M_{ab} = -12{,}000 = -6{,}000 + 6E(4\theta_a + 2\theta_b)$$

$$24\theta_a + 12\theta_b = -6{,}000/E \qquad (1)$$

$$M_{ba} = 6{,}000 + 6E(2\theta_a + 4\theta_b)$$

$$M_{ba} = 12E\theta_a + 24E\theta_b + 6{,}000 \qquad (2)$$

$$M_{bc} = -24{,}750 + 4E(4\theta_b + 2\theta_c)$$

$$M_{bc} = 16E\theta_b + 8E\theta_c - 24{,}750 \qquad (3)$$

$$M_{cb} = 24{,}750 + 4E(2\theta_b + 4\theta_c)$$

$$M_{cb} = 8E\theta_b + 16E\theta_c + 24{,}750 \qquad (4)$$

$$M_{cd} = -24{,}000 + 3E(4\theta_c + 2\theta_d)$$

$$M_{cd} = 12E\theta_c + 6E\theta_d - 24{,}000 \qquad (5)$$

$$M_{dc} = 16{,}000 = 24{,}000 + 3E(2\theta_c + 4\theta_d)$$

$$6\theta_c + 12\theta_d = -8{,}000/E \qquad (6)$$

From equations (c), (2), and (3)

$$12E\theta_a + 40E\theta_b + 8E\theta_c - 18{,}750 = 0$$

$$12\theta_a + 40\theta_b + 8\theta_c = 18{,}750/E \qquad (7)$$

From equations (d), (4), and (5)

$$8E\theta_b + 28E\theta_c + 6E\theta_d + 750 = 0$$

$$8\theta_b + 28\theta_c + 6\theta_d = -750/E \qquad (8)$$

TABLE I

Operation	Eq. No.	θ_a	θ_b	θ_c	θ_d	Constant Term	Check Term
	1	+ 24	+ 12			- 6000/E	- 5964
	2			+ 6	+ 12	- 8000/E	- 7982
	3	+ 12	+ 40	+ 8		+18750/E	+18810
	4		+ 8	+ 28	+ 6	- 750/E	- 708
1 ÷ 24	1'	+ 1	+0.5			- 250/E	- 248.5
2	2'			+ 6	+ 12	- 8000/E	- 7982
3 ÷ 12	3'	+ 1	+3.333	+0.667		+ 1562/E	+ 1567
4	4'		+ 8	+ 28	+ 6	- 750/E	- 708
2'	5			+ 6	+ 12	- 8000/E	- 7982
1' - 3'	6		-2.833	-0.667		- 1812/E	- 1815.5
4'	7		+ 8	+ 28	+ 6	- 750/E	- 708
5	5'			+ 6	+ 12	- 8000/E	- 7982
6 ÷ -2.833	6'		+ 1	+0.236		+ 640/E	+641.236
7 ÷ 8	7'		+ 1	+ 3.5	+ 0.75	-93.75/E	- 88.5
5'	8			+ 6	+ 12	- 8000/E	- 7982
6' - 7'	9			-3.264	- 0.75	+733.75/E	+729.736
8 ÷ 6	8'			+ 1	+ 2	- 1333/E	- 1330
9 ÷ -3.264	9'			+ 1	0.23	- 224.5/E	- 223.27
8' - 9'	10				+ 1.77	-1108.5/E	-1106.73

(from 10) $\theta_d = -\ 626/E$

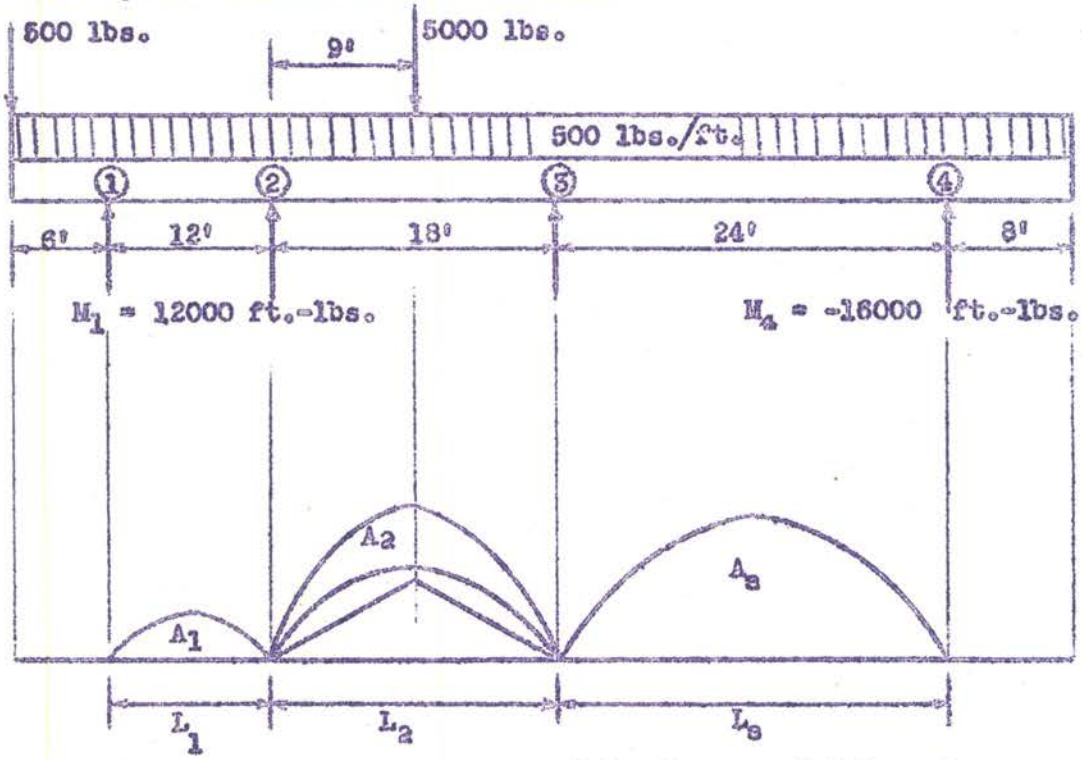
(from 9') $\theta_c = 1252/E - 1333/E = -\ 81/E$

(from 7') $\theta_b = \dfrac{+283.5 + 469.5 - 93.75}{E} = +\dfrac{659.25}{E}$

(from 1') $\theta_a = -329.63/E - 250/E = -\dfrac{579.63}{E}$

The four equations - (1), (6), (7), and (8) - contain the four unknown slopes and may be solved simultaneously. Where there are more than three equations to solve simultaneously it is generally most convenient to set up the equations in tabular form and systematically reduce the number of unknown quantities until a solution is had. Table I shows the solution of this problem in the aforementioned manner. Using the values shown in Table I for θ_a, θ_b, θ_c, and θ_d, it is possible to determine the moments at the supports. Thus, from equation (2), M_{ba} = 14,850 ft.-lbs. From equation (3), M_{bc} = - 14,850 ft.-lbs. From equation (4), M_{cb} = 28,680 ft.-lbs. From equation (5), M_{cd} = - 28,720 ft.-lbs. Note that there is a very slight discrepancy between M_{cb} and M_{cd}. Such discrepancies are typical of slide rule solutions.

(B) By the Theorem of Three Moments:



The composite area A_2 is equal to the sum of triangular area of the moment diagram due to the concentrated load and the parabolic area of the moment diagram due to the distributed load.

$$12M_1 + 2M_2(12 + 18) + 18M_3 = -\frac{6}{12}\frac{2}{3} \quad \frac{500(12)^2}{8}(12)\ (6)$$

$$-\frac{6}{18}\frac{2}{3} \cdot \frac{500(18)^2}{8}\ (18)\ (9) - \frac{6}{18}\frac{1}{2} \cdot \frac{5000(18)}{4}\ (18)\ (9)$$

$$-144,000 + 60M_2 + 18M_3 = -216,000 - 729,000 - 607,000 = -1,552,000$$

$$60M_2 + 18M_3 = -1,408,000 \qquad (1)$$

$$18M_2 + 2M_3(18 + 24) + 24M_4 = -729,000 - 607,000$$

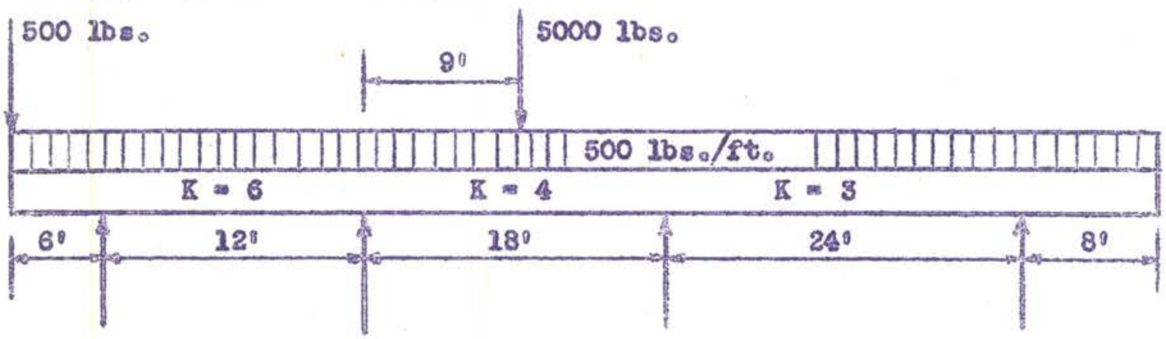
$$-\frac{6}{24}\frac{2}{3} \cdot \frac{500(24)^2}{8}\ (24)\ (12)$$

$$18M_2 + 84M_3 - 384,000 = -729,000 - 607,000 - 1,728,000$$

$$18M_2 + 84M_3 = -2,680,000 \qquad (2)$$

The simultaneous solution of equations (1) and (2) shows that $M_2 = -14,850$ ft.-lbs. and that $M_3 = -28,700$ ft.-lbs., which check the values found by slope-deflection.

(C) By Moment Distribution:



0	1	.6	.4	.572	.428	1	0
-12000	-6000	-6000	-24750	-24750	-24000	-24000	-16000
	-6000	+3000					
	+6525	-15050	+8700	-4350			
	-6525	+3262	-1460	+2920	-2180	+1090	
	+1416	-2832	+1890	-945	-3455	+6910	
	-1416	+708	+717	-1435	+1075	-537	
	-2	+5	-4	+2	-268	+537	
	+2	-1	+77	-154	+116	-58	
	-23	+47	-31	+15	-29	+58	
	+23	-11	+12	-25	+19	-9	
	-7	+14	-9	+4	-4	+9	
	+7	-3	+2	-5	+3	-1	
		+3	-2			+1	
-12000	-12000	-14858	-14858	-28723	-28723	-16000	-16000

Final Moments

PART IV

CONCLUSIONS

Statically Determinate Beams: Of the methods available for determining deflections of statically determinate beams, the methods of double integration and of area-moments are more universally understood and employed than any of the others. This doubtless is a result of the trend, in elementary mechanics textbooks, toward the presentation of these two methods to the exclusion of others. However, this does not seem to be an undesirable bias since the area-moment method usually provides a solution more readily than any other method.

In cases where the beam is symmetrically loaded and supported and where it is desired to find the maximum deflection, the use of double integration is perhaps almost as easy as the area-moment method. But lack of symmetry, in a simple beam, complicates the solution by double integration much more than is the case with some of the other methods. In general the difficulty of making a solution by double integration is proportional to the difficulty encountered in evaluating the constants of integration, which are peculiar to this method.

There is actually very little difference between the area-moment and conjugate beam methods. The only difference lies in the dissimilar frames of mind, or philosophies, with which the attack is begun. For some students the conjugate beam method may be more easily remembered because it is the duplication of a common, everyday operation—that of finding the moment at a given section of a beam. Care must be exercised, however, in choosing the proper type of support for the conjugate beam. Very little difficulty is caused by this if a few simple rules are learned.

The methods of work very often provide an expedient means for finding the deflections of statically determinate beams. Frequently they may be employed with as much facility as is possible by the area-moment method. The method, however, becomes increasingly complicated as the number of loads on the structure increases. Separate moment equations must be set up, integrated, and evaluated between certain limits corresponding to adjacent, abrupt changes in the shear diagram.

Other methods, such as the Theorem of Three Moments and the slope-deflection method, are rather easily applied but involve more or less artificial approaches. Further, they require the use of basic formulas which are easily forgotten or misapplied if not frequently used.

In short, it would seem that the area-moment method is the one most readily used in the greatest variety of conditions. In addition it is easily comprehended and easily recalled, so that it provides an excellent tool for occasional use as well as one for routine use.

Statically Indeterminate Beams: For single-span indeterminate beam analysis, it is rather hard to choose between the area-moment method and the Column Analogy. The Column Analogy seems somewhat easier to apply, but its advantage over the area-moment method is hardly great enough to justify its use for occasional problems of the type covered in this report. For single-span bents, for curved beams, or for arches it is a singularly useful method.

The Theorem of Three Moments should not be discounted for the solution of single-span beams. In some cases (particularly for single-span indeterminates which overhang one hinged support) this method provides a solution more readily than does any other.

The methods of work are generally cumbersome to apply and would certainly not seem desirable for sporadic use.

For multi-span indeterminate beams the method of moment distribution is to be preferred by far for general use. Where there is only one redundant reaction the Theorem of Three Moments provides a satisfactory method of solution. If the number of redundants exceeds two, necessitating the solution of three or more equations simultaneously, the Theorem of Three Moments solution becomes unwieldly as compared to that by moment distribution.

The method of slope-deflection, which usually involves the solution of three or more equations simultaneously, is fraught with the possibility for error. The solution can almost never be accomplished the first time without mistake unless the equations are set up and solved in a systematic manner and a constant check maintained as the solution progresses. The method shown previously in Table I provides one way of reducing the possibility of error.

It should be repeated that the method of moment distribution far excels any other for determining the moments in multi-span beams. The method, furthermore, is easily used for determining the moments in the members of multi-story continuous structures. It is regrettable that so many engineering students are graduated with very little or no knowledge of this important method of analysis.

The applications shown in this report are by no means the only uses for many of the methods described. Some of them, for example the methods of work, furnish useful means for determining the deflections of trusses. There are many other applications for the methods, and it is to be hoped that the examples shown in this report may serve to introduce some of the methods and indicate the manner in which they may be applied.

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