UNIVERSITY OF OKLAHOMA

GRADUATE COLLEGE

DEFORMATIONS OF SIMPLE REPRESENTATIONS OF TWO-GENERATOR HNN EXTENSIONS

A Dissertation

SUBMITTED TO THE GRADUATE FACULTY

in partial fulfillment of the requirements for the

degree of

Doctor of Philosophy

By

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Norman, Oklahoma

2002

UMI Number: 3042506

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A Dissertation APPROVED FOR THE
DEPARTMENT OF MATHEMATICS

BY

Acknowledgements

I would like to thank my advisor, Dr. Magid, for his immeasurable help, advice and support in writing this thesis. I will always hold his guidance in the highest regard and will always be grateful to him.

Secondly, I want to thank my wife, Linda, for all of her support. She has always been my biggest fan and supporter and I am so grateful to have her. She always talks about how I make getting my work done look effortless. Well, she makes being my wife look effortless and I think that says a lot about her. I'll always be her biggest fan, too.

I would also like to thank my parents for all of their support and advice. They have always been behind me in pursuit of my Ph.D., and I am proud to have accomplished all that they ever hoped I would. My future is due in great part to them.

Finally, I would like to thank my committee members for their time, good advice, and patience with me.

April 16, 2002

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Abstract

DEFORMATIONS OF SIMPLE REPRESENTATIONS OF TWO-GENERATOR HNN EXTENSIONS

This thesis demonstrates the existence of simple representations in any dimension of a specific collection of Baumslag-Solitar groups, $BS(m, n)$, and investigates their deformations. Specifically, the dimension of the tangent space of the representation scheme of this collection of groups, at these specific representations, is computed. Moreover, we then deduce the dimension of the representation scheme itself. We also investigate the geometry of the character variety of the group in a neighborhood of these special representations.

Chapter 1 – Introduction

The goal of this thesis is to investigate the computation of dimensions of representation varieties of certain finitely presented groups. With the rapid advancement of readily available computing power, it was believed that many more examples than in the past could now be computed. This investigation began with the attempt to compute the dimension of the representation varieties of some finitely presented groups in low-dimensional cases. It was discovered rather quickly that a group's presentation has a great effect on the success or failure of these computations. The fifth chapter of the thesis is devoted to presenting a sampling of the attempted computations, both successful and failed.

It should be mentioned that all computations in this thesis were attempted in the *CoCoA* computer algebra system (CAS). The CAS *Macaulay 2* was also used to compute many of the given examples, mainly to determine if one of the two CAS's was more capable than the other. The collections of successful and unsuccessful examples were the same for both systems.

It was determined, after attempting computations on groups with somewhat complicated presentations, i.e. many generators or many relators, that an appropriate class of groups with which to begin is the class of two-generator, one-relator groups. In particular, this work focuses on the *Baumslag-Solitar* groups, named after G. Baumslag and D. Solitar. These groups are indexed by pairs of positive integers, and are given by the finite presentation

$BS(m, n) = \langle a, t : ta^m = a^n t \rangle$. The second chapter of the thesis describes some of the characteristics of these groups as well as places them in the geometric group theory collection of *HNN extensions*. Many of the dimensions computed with the Baumslag-Solitar groups formed the basis for the results of this thesis. The machine computations *suggested* some results, however the results obtained are purely theoretical and are valid in all dimensions.

The third chapter of the thesis gives all necessary background regarding representation varieties and schemes and their deformations. The set of k-dimensional complex representations of a finitely presented (or generated) group, Γ, forms an affine algebraic variety which we denote by $R_k(\Gamma)$. We call this structure a *representation variety of* Γ. $R_k(\Gamma)$ is indeed an affine variety because we can associate a representation with a tuple of matrices with complex entries satisfying all of the group relators. As a result, we have a collection of polynomials, the solution set of which forms the aforementioned variety.

There also exists the related notion of a *representation scheme* of Γ. We refer to the functor, $\mathcal{R}_k(\Gamma)$, of k-dimensional representations of Γ which is representable by an affine algebra, making it an affine scheme. $\mathcal{R}_k(\Gamma)$ is the functor from commutative $\mathbb{C}$-algebras to sets defined by $\mathcal{R}_k(\Gamma)(\boldsymbol{A}) = \mathrm{Hom}(\Gamma, GL_k(\boldsymbol{A}))$. Due to a result of A. Lubotzky and A. Magid in [LM], this functor's coordinate ring, $\mathcal{A}_k(\Gamma)$, is easily constructible (on paper!) and represents the functor. We will see that $\mathrm{Hom}(\Gamma, GL_k(*)) \simeq \mathrm{Hom}(\mathcal{A}_k(\Gamma), *)$.

Again, the complication that comes associated with the construction of $\mathcal{A}_k(\Gamma)$ is the fact that if we have a presentation for Γ,

$$\Gamma = \langle \gamma_1, \ldots, \gamma_d : r_q, \; q \in Q \rangle \, ,$$

then $\mathcal{A}_k(\Gamma)$ will consist of the quotient of the polynomial ring in $d(k^2+1)$ indeterminates with coefficients in $\mathbb{C}$ modulo an ideal generated by a certain collection of $d + k^2 | \, Q \, |$ polynomials. (This is assuming a finite presentation. i.e. $| \, Q \, | < \infty$.) These polynomials are obtained from the relators given in the presentation. One sees very quickly that the more "interesting" the group, the more complicated $\mathcal{A}_k(\Gamma)$ is.

We also study various algebraic operations on the representation schemes of Γ. In particular, we will discuss the algebraic action of GL_k on representations (by conjugation). We also see that new representation schemes can be created from old via certain scheme morphisms. As a sample, we display the morphism

$$\mu : \mathcal{R}_k(\Gamma) \times \mathcal{R}_l(\Gamma) \longrightarrow \mathcal{R}_{k+l}(\Gamma) : (\rho_1, \rho_2) \mapsto (\rho_1 \oplus \rho_2).$$

There will be discussion of $R_k^s(\Gamma)$, the collection of *simple* representations of the group Γ, as well as $\mathcal{R}_k^s(\Gamma)$, a subfunctor of $\mathcal{R}_k(\Gamma)$ which is also an open subscheme $\mathcal{R}_k(\Gamma)$. We are able to define $S_k(\Gamma)$ and $SS_k(\Gamma)$, the schemes of (conjugacy) classes of simple and semi-simple representations of Γ. We then describe many of the topological properties of these objects, as well as their relationships to one another. For example, we note that $SS_k(\Gamma)$ is a categorical quotient of $R_k(\Gamma)$ while $S_k(\Gamma)$ is a geometric quotient of $R_k^s(\Gamma)$.

Given the overall goal of computing the dimension of representation varieties and schemes, some time is spent in describing the known facts regarding the dimensions of the aforementioned objects, as well as the *tangent spaces* to those objects at a specific representation. For example, we use the notation $T_\rho(\mathcal{R}_k(\Gamma))$ to denote the tangent space to the representation scheme of Γ at ρ. As one might expect, we define a representation to be *scheme non-singular* if $\dim_{\mathbb{C}}\Big(T_\rho(\mathcal{R}_k(\Gamma))\Big) = \dim_\rho(\mathcal{R}_k(\Gamma))$. We make a similar definition to describe a representation that is *non-singular on the variety*.

Moreover, we note the following inequalities hold regarding the dimensions of the objects of interest:

$$\dim_{\mathbb{C}}\Big(T_\rho(\mathcal{O}(\rho))\Big) \leq \dim_\rho\Big(R_k(\Gamma)\Big) = \dim_\rho\Big(\mathcal{R}_k(\Gamma)\Big) \leq \dim_{\mathbb{C}}\Big(T_\rho(\mathcal{R}_k(\Gamma))\Big),$$

where $\mathcal{O}(\rho)$ denotes the *orbit* of ρ, or the collection of representations conjugate to ρ.

The remainder of the exposition on representation varieties and schemes focuses on the computational relationship between $T_\rho(\mathcal{R}_k(\Gamma))$ and group cohomology using *Fox Calculus* in [CF, Chap. 7]. We see that many of the computations we wish to perform can be translated into the setting of group cohomology. As a result, the computation of $\dim\Big(T_\rho(\mathcal{R}_k(\Gamma))\Big)$ can be accomplished using a good deal of linear algebra. It is precisely this work that leads to the major results of this thesis. The discussion of representation varieties and schemes is completed with a discussion of the *deformations* of simple representations. It is this notion that allows us eventually to compute the dimension of the actual representation varieties and schemes, instead of merely computing the dimensions of their tangent spaces at particular representations.

The particular groups studied in this thesis are the Baumslag-Solitar groups. These make up the collection of two-generator, one-relator groups given as follows:

$$\boldsymbol{\mathcal{BS}} := \{BS(m,n) : m, n \in \mathbb{N}\}$$

where $BS(m,n) = \langle a, t : ta^m = a^n t\rangle$.

The second chapter of this dissertation is devoted to some of the properties of these groups. We will see that each Baumslag-Solitar group is an example of an algebraic construct known as an *HNN extension*. HNN extensions are similar in nature to the free product with amalgamation of two groups. We also give a constructive proof of the existence of a subcollection of $\boldsymbol{\mathcal{BS}}$ consisting of *non-Hopfian* groups.

The fourth chapter contains the major results of the thesis. It is shown that if $\gcd(m,n) = 1$, then $BS(m,n)$ has a simple k-dimensional representation, for any $k \in \mathbb{N}$. The proof of this result, while long, is constructive and algebraic in nature. We then use the interconnections between the tangent space to the representation scheme, group cohomology, and the Fox Calculus to obtain the result:

$$\dim\Big(T_\rho\big(\mathcal{R}_k(BS(m,n))\big)\Big) = k^2,$$

where ρ is one of the constructed simple representations of $BS(m,n)$ with m and n relatively prime. We may then use the cohomological result of Lubotzky and Magid that states if $\rho \in R_k(\Gamma)$, $Ad \circ \rho$ is the action of ρ on $M_k(\mathbb{C})$ via conjugation, and $\mathcal{O}(\rho)$ is the orbit of ρ, then

$$\begin{aligned} Z^1(\Gamma, Ad \circ \rho) &\simeq T_\rho(\mathcal{R}_k(\Gamma)) \,. \\ B^1(\Gamma, Ad \circ \rho) &\simeq T_\rho(\mathcal{O}(\rho)) \end{aligned}$$

This presents us with the corollary that $\dim\Big(H^1(BS(m,n), Ad \circ \rho)\Big) = 1$, thus providing us with more of a geometric notion of what the representation scheme "looks like" near the representation ρ.

A discussion of the deformations of the simple representations in question leads to the computation of the dimension of the representation variety at our specific representation. We obtain $\dim_\rho\Big(R_k(BS(m,n))\Big) = k^2$ and $\dim_{[\rho]}\Big(S_k(BS(m,n))\Big) = 1$ and can thus conclude that our specific representation ρ is scheme non-singular.

The final chapter of the thesis is given to presenting several examples of the types of computations attempted in the *CoCoA* computer algebra system.

Chapter 2 — Baumslag-Solitar Groups

The groups of interest in this thesis were devised by mathematicians Gilbert Baumslag and Donald Solitar through their work in [BS]. Their intent in that article was to show that there existed a finitely generated group with one defining relator which was isomorphic to a proper factor of itself. Put another way, they searched for surjective group endomorphisms with non-trivial kernel.

The groups they investigated formed the collection

$$\mathcal{BS} := \{BS(m,n) : m,n \in \mathbb{N}\}$$

where $BS(m,n) = \langle a,t : ta^m = a^n t\rangle$, for each $m,n \in \mathbb{N}$.

One of the more interesting properties of these groups is that they are all examples of *HNN Extensions*. HNN extensions, named for G. Higman, B. H. Neumann and H. Neumann, are a group-theoretic construct similar to the *free product with amalgamation* of two groups. In this chapter, we will summarize the findings on $\mathcal{BS}$ with regards to Baumslag and Solitar's original investigation.

Free Products With Amalgamation and HNN Extensions

In this section, we describe the construction of HNN extensions and show how all of the groups in $\mathcal{BS}$ are examples of such. Few proofs will be given in this chapter, since the main goal is the description of HNN extensions and placing the Baumslag-Solitar groups in that class of groups.

Definition 2.1 *Given groups A, B and C and given homomorphisms $i : C \longrightarrow A$ and $j : C \longrightarrow B$, we call a triple (G, f, g), consisting of a group and two homomorphisms, a* solution *if it creates a commutative diagram as given below:*

$$\begin{tikzcd}
C \arrow[r, "i"] \arrow[d, "j"'] \arrow[dr, phantom, "\circ"] & A \arrow[d, "f"] \\
B \arrow[r, "g"'] & G
\end{tikzcd}$$

We note that a solution will always exist under these circumstances, using any group G, and by letting f and g be the zero homomorphisms. More important, though, is the existence of a solution satisfying a universal property:

Definition 2.2 *A* pushout *of the data $A \xleftarrow{i} C \xrightarrow{j} B$ is a solution (P, ϕ_1, ϕ_2) having the property that if (G, f, g) is any solution of the data, then there exists a unique homomorphism $h : P \longrightarrow G$ such that $f = h \circ \phi_1$ and $g = h \circ \phi_2$.*

Ket to the previous definition is that given any set of data $A \xleftarrow{i} C \xrightarrow{j} B$, pushouts always exist. The proof is constructive in that the pushout, P, will be:

$$P = (A * B)/\langle\{i(c)j(c)^{-1} : c \in C\}\rangle$$

while ϕ_1 and ϕ_2 are inclusion maps.

For the remainder of this section, we will assume our groups have the following presentations: $A = \langle X_A : R_A \rangle$, $B = \langle X_B : R_B \rangle$ and group C is generated by the set X_C. The group P thus becomes:

$$P = \langle X_A \amalg X_B : R_A \cup R_B \cup \{i(c)j(c)^{-1} : c \in X_C\} \rangle.$$

As for some examples of the above construction, it is easy to see that if $C = \{1\}$ then $P = A * B = \langle X_A \amalg X_B : R_A \cup R_B \rangle$, the usual free product of A and B, since $i(c)j(c)^{-1} = 1$, for each $c \in C$. If we assume $B = \{1\}$, then we have $P = (A * \{1\})/\langle i(c) : c \in X_C \rangle = A/\langle i(c) : c \in X_C \rangle$. A similar result is obtained if we assume $A = \{1\}$.

A special case of a pushout occurs when the maps i and j are injective:

Definition 2.3 *Given the data* $A \xleftarrow{i} C \xrightarrow{j} B$ *with both* i *and* j *group monomorphisms, the pushout of this data is called the* free product of A and B amalgamated over the subgroup C *and is denoted* $A *_C B$.

An *HNN extension* is a special type of free product with amalgamation. If we assume $A = B$ and that the two monomorphisms are $i, j : C \longrightarrow A$, then the construction is denoted $A *_C$ and the maps i and j are typically understood in context. This is summarized in the following definition:

Definition 2.4 *Suppose* $i, j : C \longrightarrow A$ *are group monomorphisms. An* HNN extension *of* A *is the pushout of the data* $A \xleftarrow{i} C \xrightarrow{j} A$ *and is denoted* $A *_C$.

Similar to the explicit construction of the free product of A and B amalgamated over C, an HNN extension of A can be described with an explicit presentation:

Theorem 2.5 *The HNN extension resulting from the data* $A \xleftarrow{i} C \xrightarrow{j} A$ *has the algebraic structure:*

$$A *_C = \langle X_A, t : R_A, \{ti(c) = j(c)t : c \in X_C\} \rangle$$

Admittedly, the notation $A *_C$ is a bit vague, but the maps i and j are generally understood in the appropriate context. A proof of Theorem 2.5 can be found in [Ha].

As a simple first example, if we set $A = C$ and let $i = j = \mathbb{I}_A$, the identity on A, then we see that

$$\begin{aligned} A *_A &= \langle X_A, t : R_A, \{ta = at : a \in X_A\} \rangle . \\ &= A \times \mathbb{Z} \end{aligned}$$

Specific to our discussion, we see that all of our Baumslag-Solitar groups are HNN extensions:

$$BS(m, n) = \langle a, t : ta^m = a^n t \rangle = \mathbb{Z} *_{\mathbb{Z}}$$

where the two monomorphisms involved are given by the assignments $i : \mathbb{Z} \longrightarrow \mathbb{Z} : \mathbf{1} \longmapsto \boldsymbol{m}$ and $j : \mathbb{Z} \longrightarrow \mathbb{Z} : \mathbf{1} \longmapsto \boldsymbol{n}$. In fact, since all $\phi \in \mathrm{Hom}(\mathbb{Z}, \mathbb{Z})$ are of the form $\phi : 1 \longmapsto \boldsymbol{m}$, where $m \in \mathbb{Z}$, the collection $\boldsymbol{BS}$ can be parametrized as follows:

$$\boldsymbol{BS} \longleftrightarrow \Big\{ (\phi_1, \phi_2) \in \mathrm{Hom}(\mathbb{Z}, \mathbb{Z})^{(2)} : \phi_i : \mathbf{1} \longmapsto \boldsymbol{m_i},\ m_i \in \mathbb{N}, i = 1, 2 \Big\}.$$

Hopficity and Baumslag and Solitar's Original Investigation

As mentioned earlier in this chapter, Baumslag and Solitar's original goal was to investigate groups having surjective endomorphisms with non-trivial kernel. We first define the appropriate terminology for the above description:

Definition 2.6 *A group, G, is called* Hopfian *if every surjective endomorphism of G is an automorphism. i.e.,*

$$G/N \simeq G \Rightarrow N = \{1\}$$

What we will see first is, for example, $BS(2,3)$ is non-Hopfian:

Example 2.7 *The group $\Gamma = BS(2,3)$ is non-Hopfian.*

Details We first define the map $\eta : \Gamma \longrightarrow \Gamma : \begin{cases} t \mapsto t \\ a \mapsto a^2 \end{cases}$. We now define an element of Γ which is non-trivial and yet lies in the kernel of η. However, we first see that η is indeed a surjective endomorphism. It is easy to see that η is a homomorphism: Given any reduced word in Γ, $w_0(a,t)$, we have $\eta(w_0(a,t)) = w_0(a^2,t) = w_0(\eta(a),\eta(t))$. Clearly $t \in \eta(\Gamma)$. Now note $\eta(tat^{-1}a^{-1}) = ta^2t^{-1}a^{-2} = (ta^2t^{-1})a^{-2} = (a^3)a^{-2} = a$ and thus $a \in \eta(\Gamma)$. Therefore, η is surjective.

Now, define the element $\gamma_w = [t,a]^2a^{-1}$. It is non-trivial to see that γ_w is non-trivial in Γ, but one can see this by looking at the group from the geometric point-of-view. One can see from the universal cover of the presentation 2-complex of Γ that the path created by γ_w does not form a loop. Equivalently, one can compute that the normal form of γ_w is not trivial. A standard treatment of the topic of normal forms can be found in [PS].

We now know that $\gamma_w \neq 1_\Gamma$. We now compute $\eta(\gamma_w)$ and see that $\eta(\gamma_w) = 1_\Gamma$:

$$\begin{aligned}\eta(\gamma_w) &= [t, a^2]^2 a^{-2} \\ &= ta^2t^{-1}a^{-2}ta^2t^{-1}a^{-2}a^{-2} \\ &= a^3tt^{-1}a^{-2}a^3tt^{-1}a^{-4} \\ &= a^3a^{-2}a^3a^{-4} \\ &= 1_\Gamma\end{aligned}$$

Thus, $N := \ker(\eta)$ is non-trivial, and so $\Gamma \simeq \Gamma/N$. □

This example refutes the claim by G. Higman in [Hi] that every finitely generated one-relator group is Hopfian. Clearly the computation above could be replicated for many other $BS(m, n)$, for example where $n - m = 1$.

For such a $BS(m, n)$, we would define the map $\eta : \Gamma \longrightarrow \Gamma : \begin{cases} t \longmapsto t \\ a \longmapsto a^m \end{cases}$ and consider the element $\gamma_w = [t, a]^m a^{-1}$. Again, we can verify that η is indeed a surjective endomorphism of $BS(m, n)$ and that $\gamma_w \neq 1_\Gamma$, as described above. We then obtain

$$\begin{aligned}\eta(\gamma_w) &= [t, a^m]^m a^{-m} \\ &= \left(ta^m t^{-1} a^{-m}\right)^m a^{-m} \\ &= \left(a^n a^{-m}\right)^m a^{-m} \\ &= \left(a^{n-m}\right)^m a^{-m} \\ &= \left(a^1\right)^m a^{-m} \\ &= a^{m-m} = a^0 = 1_\Gamma\end{aligned}$$

so that the collection $\{BS(m, n) : n - m = 1\} \subset \boldsymbol{\mathcal{BS}}$ consists only of non-Hopfian groups.

In fact, the work done in the 1970's on characterizing the Hopficity of the Baumslag-Solitar groups by, among others, D. Collins and S. Meskin, can be summed up in Theorem 2.8 (from [Co] or [Me]) below.

Theorem 2.8 *If $m = 1$ or $n = 1$ or $m = n$, then $BS(m, n)$ is residually finite and hence Hopfian. Otherwise, $BS(m, n)$ is Hopfian if and only if $\pi(m) = \pi(n)$ (where $\pi(m), \pi(n)$ denote the sets of prime divisors of m and n, respectively.)*

Note that a group G is *residually finite* if, for each non-identity element $g \in G$, there is a finite index normal subgroup $N_g \subset G$ such that $g \notin N_g$. It was proven by Mal'cev in [Ma 1] that residual finiteness of a group implies its Hopficity.

Chapter 3 — Representation Varieties and Schemes and Deformations

In this chapter, we will define the appropriate terminology for this thesis and recall many of the known results which are relevent to the discussion in this thesis. Much of the exposition from this chapter will be a discussion of the relevant information from [LM], [AM], and [Ma].

We shall see that the k-dimensional representations of a finitely presented group, Γ, form an algebraic variety, $R_k(\Gamma)$, called a *representation variety* while there exists the related notion of a functor $\mathcal{R}_k(\Gamma)$ of k-dimensional representations of Γ which is representable by an affine algebra, thus making it an affine scheme. We term $\mathcal{R}_k(\Gamma)$ a *representation scheme*. We shall investigate many of the algebraic and geometric properties of these objects, including the notion of the algebraic action of GL_k on the representation scheme.

It should be noted now that the results from this section can be given in the context of an algebraically closed field, K, of characteristic zero. However, the major results of this thesis have been obtained for $K = \mathbb{C}$.

We make some assumptions and conventions on the objects in this chapter: Γ is a finitely generated group, all $\mathbb{C}$-algebras are considered commutative, and the affine $\mathbb{C}$-algebras are finitely generated as algebras.

Definition 3.1 *A* complex k-dimensional representation *of a group* Γ *is a homomorphism* $\rho : \Gamma \longrightarrow GL_k(\mathbb{C})$. *The representation* ρ *is said to be* simple *if* $\mathbb{C}^{(k)}$ *has no proper* ρ*-invariant subspaces. Moreover a representation,* ρ*, is* faithful *if* $\ker(\rho) = \{1_\Gamma\}$.

Definition 3.2 [LM, Def. 1.1] $\mathcal{R}_k(\Gamma)$ *denotes the functor from commutative* $\mathbb{C}$*-algebras to sets defined by* $\mathcal{R}_k(\Gamma)(A) = \mathrm{Hom}(\Gamma, GL_k(A))$. *If* $f : A \longrightarrow B$ *is a* $\mathbb{C}$*-algebra homomorphism, then* $f_* : \mathcal{R}_k(\Gamma)(A) \longrightarrow \mathcal{R}_k(\Gamma)(B)$ *denotes the function sending the representation* $\rho : \Gamma \longrightarrow GL_k(A)$ *into the composite* $\Gamma \longrightarrow GL_k(A) \longrightarrow GL_k(B)$.

The first thing we note from Definition 3.2 is that $\mathcal{R}_k(\Gamma)(A)$ consists of representations of Γ in $GL_k(A)$. Thus if $\Gamma = \langle a_1, \ldots, a_d : s_q,\ q \in Q \rangle$, then we see we have the following bijection:

$$\mathcal{R}_k(\Gamma)(A) \longleftrightarrow \left\{ \boldsymbol{C} = (C_1, \ldots, C_d) \in GL_k(A)^{(d)} : s_q(\boldsymbol{C}) = I_k \right\}$$

Given the correspondence above, for general A, we obtain that $\mathcal{R}_k(\Gamma)$ is representable by an affine $\mathbb{C}$-scheme:

Proposition 3.3 [LM, Prop. 1.2] *There is an affine* $\mathbb{C}$*-algebra,* $\mathcal{A}_k(\Gamma)$*, and a representation,* $\rho_0 : \Gamma \longrightarrow GL_k(\mathcal{A}_k(\Gamma))$*, such that for any commutative* $\mathbb{C}$*-algebra* A*, and representation,* $\rho : \Gamma \longrightarrow GL_k(A)$*, there is a unique* $\mathbb{C}$*-algebra homomorphism* $f : \mathcal{A}_k(\Gamma) \longrightarrow A$*, such that* $\rho = f_*(\rho_0)$.

Proof Let $\Gamma = \langle \gamma_1, \ldots, \gamma_d : r_q,\ q \in Q \rangle$ be a presentation for Γ. We define the set of indeterminates $\{x_{ij}^{(p)} : 1 \leq i, j \leq k,\ 1 \leq p \leq d\}$ and create a collection of $k \times k$ matrices, each defined by $X^{(p)} := [x_{ij}^{(p)}]$, $p = 1, \ldots, d$.

We now consider the polynomial ring $B = \mathbb{C}[x_{ij}^{(p)} : 1 \leq i, j \leq k,\ 1 \leq p \leq d]$. Moreover, we invert $z_p := \det(X^{(p)})$ by creating additional indeterminates y_p, $p = 1, \ldots, d$, defining the ideal $I = \langle y_p z_p - 1 : 1 \leq p \leq d \rangle$, and considering the $\mathbb{C}$-algebra $D := B/I$.

Now, for each $q \in Q$, we consider the matrix $r_q(X^{(1)}, \ldots, X^{(d)}) \in GL_k(D)$ and denote its $(i,j)^{th}$ entry by $(r_q)_{ij}$. We let $J = \Big\langle \big((r_q)_{ij} - \delta_{ij}\big) : q \in Q,\ 1 \leq i, j \leq k \Big\rangle$ and define $\mathcal{A}_k(\Gamma) := D/J$ and let $\overline{x}_{ij}^{(p)}$ and $\overline{X}^{(p)}$ denote the images of $x_{ij}^{(p)}$ and $X^{(p)}$ in $\mathcal{A}_k(\Gamma)$.

As a result of the fact that the matrices $\overline{X}^{(p)}$ were constructed to satisfy the relations $\{r_q : q \in Q\}$, we know there must exist a representation $\rho_u : \Gamma \longrightarrow GL_k(\mathcal{A}_k(\Gamma))$ such that $\rho_u(\gamma_p) = \overline{X}^{(p)}$, $p = 1, \ldots, d$. Now if ρ is any representation of Γ into some $GL_k(A)$ with $\rho(\gamma_p) = [a_{ij}^{(p)}]$, we obtain a $\mathbb{C}$-algebra homomorphism $f_0 : D \longrightarrow A$ defined by $f_0(x_{ij}^{(p)}) = a_{ij}^{(p)}$ and $f_0\Big(\det(X^{(p)})^{-1}\Big) = \det([a_{ij}^{(p)}])^{-1}$. Since $r_q(\rho(\gamma_1), \ldots, \rho(\gamma_d)) = I_k$, for each $q \in Q$, we have $f_0(J) = \{0\}$ and thus f_0 determines a $\mathbb{C}$-algebra homomorphism $f : \mathcal{A}_k(\Gamma) \longrightarrow A$.

By construction, we have $f_*(\rho_u) = \rho$. Moreover, f is uniquely determined by this equation since it implies

$$f(\overline{x}_{ij}^{(p)}) = a_{ij}^{(p)} \quad \text{and} \quad f\Big(\det(\overline{X}^{(p)})\Big) = \det(\rho(\gamma_p)),\ p = 1, \ldots, d$$

as desired. $\square$

As a result of the above, we see that the algebra $\mathcal{A}_k(\Gamma)$ represents the functor $\mathcal{R}_k(\Gamma)$. Moreover, $\mathcal{A}_k(\Gamma)$ is the coordinate ring of the scheme $\mathcal{R}_k(\Gamma)$. This universal property can be applied to a representation $\rho \in \mathcal{R}_k(\Gamma)(\mathbb{C})$: There exists a unique homomorphism $f_\rho : \mathcal{A}_k(\Gamma) \longrightarrow \mathbb{C}$ such that $\rho = GL_k(f_\rho) \circ \rho_0$, the kernel of which, M_ρ, is a maximal ideal.

One good thing is that the proof of Prop. 3.3 is constructive. It provides a method, given a presentation of Γ, to analyze the algebra and the functor it represents. One merely creates a quotient of a polynomial ring. However, it is, in general, a major difficulty to compute a wide array of examples using Prop. 3.3, due to the number of indeterminates required per group generator (k^2), not to mention the number of polynomials generating the ideal in the quotient. Despite the incredible increases in computing power in the last decade, computer algebra systems such as *Mathematica*©, *CoCoA*, and *Macaulay 2* cannot compute examples one might expect to be computable. In Chapter 5, we will see examples of successful and unsuccessful computations coming from Proposition 3.3.

Example 3.4 *Compute $\mathcal{A}_2(\Gamma)$, for $\Gamma = \mathbb{Z}_3 * \mathbb{Z}$.*

Solution We note that $\mathbb{Z}_3 * \mathbb{Z}$ is given by the presentation

$$\mathbb{Z}_3 * \mathbb{Z} = \langle x, y : x^3 \rangle$$

We use the following matrices of indeterminates, using a simpler notation:

$$X^{(1)} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad X^{(2)} = \begin{pmatrix} e & f \\ g & h \end{pmatrix}.$$

Then the determinants of the matrices are $\det(X^{(1)}) = ad - bc$ while $\det(X^{(2)}) = eh - fg$. We introduce new indeterminates y_1 and y_2 and define the polynomial ring

$$B = \mathbb{C}[a, b, c, d, e, f, g, h, y_1, y_2]$$

and ideal defined by

$$I := \langle z_1, z_2 \rangle = \langle y_1(ad - bc) - 1, \; y_2(eh - fg) - 1 \rangle$$

Thus, we denote $D := B/I$ as our $\mathbb{C}$-algebra with determinants inverted.

We also compute

$$r_1(X^{(1)}, X^{(2)}) = X^{(1)^3} = \begin{pmatrix} a(a^2 + bc) + c(ab + bd) & b(a^2 + bc) + d(ab + bd) \\ a(ac + cd) + c(bc + d^2) & b(ac + cd) + d(bc + d^2) \end{pmatrix}$$

and denote this matrix $[p_{ij}]$, where the p_{ij} denote the homogeneous degree three polynomials above. Thus, our $\mathbb{C}$-algebra is given by:

$$\begin{aligned} \mathcal{A}_2(\Gamma) &= D/\langle p_{11} - 1, p_{12}, p_{21}, p_{22} - 1 \rangle \\ &= \mathbb{C}[a, b, c, d, e, f, g, h, y_1, y_2]/\langle z_1, z_2, p_{11} - 1, p_{12}, p_{21}, p_{22} - 1 \rangle \end{aligned}$$

□

Notation 3.5 *The affine $\mathbb{C}$-scheme* $\mathrm{Spec}(\mathcal{A}_k(\Gamma))$ *is denoted by $\mathcal{R}_k(\Gamma)$ and is referred to as the k^{th}* representation scheme *of Γ. When referring to its functor of points, it will be called the k^{th}* representation functor *of Γ.*

Given the above definition, we consider various operations on the representation schemes of Γ. They will be seen as natural transformations of representable functors.

Proposition 3.6 [LM, Prop. 1.5] *Each of the following operations on representations is a scheme morphism:*

(a) $\mathcal{R}_k(\Gamma) \to \mathcal{R}_k(\Gamma)$ by $\rho \mapsto \rho^* : \gamma \mapsto (\rho(\gamma)^{-1})^T$

(b) $\mathcal{R}_k(\Gamma) \times \mathcal{R}_l(\Gamma) \to \mathcal{R}_{k+l}(\Gamma)$ by $(\rho_1, \rho_2) \mapsto (\rho_1 \oplus \rho_2)$

(c) $\mathcal{R}_k(\Gamma) \times \mathcal{R}_l(\Gamma) \to \mathcal{R}_{kl}(\Gamma)$ by $(\rho_1, \rho_2) \mapsto (\rho_1 \otimes \rho_2)$

We also see that GL_k operates on representations via conjugation:

Proposition 3.7 [LM, Prop. 1.6] *The map $\alpha : GL_k \times \mathcal{R}_k(\Gamma) \longrightarrow \mathcal{R}_k(\Gamma)$, defined by $\alpha(T, \rho) = T \cdot \rho$, where $T \cdot \rho(\gamma) = T\rho(\gamma)T^{-1}$, is a morphism of schemes. Moreover, α is a group scheme action in the sense that:*

(a) $I \cdot \rho = \rho$, for all $\rho \in \mathcal{R}_k(\Gamma)$

(b) $T_1 \cdot (T_2 \cdot \rho) = (T_1 T_2) \cdot \rho$, for all T_1, T_2, ρ .

A representation of a group Γ into $GL_k(A)$ turns the free module $A^{(k)}$ into a module over the group algebra $A[\Gamma]$. We give a convenient notation for this module:

Definition 3.8 **[LM, Def. 1.8]** *Let A be a commutative $\mathbb{C}$-algebra and $\rho \in \mathcal{R}_k(\Gamma)(A)$. Then we denote by $V(\rho)$ the free module $A^{(k)}$ with $A[\Gamma]$-structure given by $(\sum a_i\gamma_i)v = \sum a_i\rho(\gamma_i)(v)$, for $a_i \in A$ and $\gamma_i \in \Gamma$.*

Using the notation from this definition, we see there exists, for $\rho \in \mathcal{R}_k(\Gamma)(A)$, an A-algebra homomorphism

$$A[\rho] : A[\Gamma] \longrightarrow \mathrm{End}_A(V(\rho))$$

In particular, if A is a commutative $\mathbb{C}$-algebra and $\rho \in \mathcal{R}_k(\Gamma)(A)$, then we may define ρ to be a *simple* representation of Γ if $A[\rho]$ is surjective. This characterization of simplicity is more understandable by considering an equivalent definition of a simple representation: ρ is simple if and only if its image spans $M_k(\mathbb{C})$. We also denote the set of all simple representations of Γ by $\mathcal{R}_k^s(\Gamma)(A) \subset \mathcal{R}_k(\Gamma)(A)$.

Proposition 3.9 **[LM, Prop. 1.10]** *$\mathcal{R}_k^s(\Gamma)$ is a subfunctor of the k^{th} representation functor, is stable under the action of GL_k, and is an open subscheme of the representation scheme.*

As we will see below, all of the orbits of the action of GL_k on $\mathcal{R}_k^s(\Gamma)$ are closed subsets. Thus, by [MF, 1.3, p.30] we see that we have a *geometric quotient* by GL_k on each of these affine subsets, where the quotient is in the sense of that in [MF, Def. 0.6, p.4]. We now give an official account of these results:

Definition 3.10 [LM, Def. 1.11] *Let $\mathcal{B}_k(\Gamma) = \mathcal{A}_k(\Gamma)^{GL_k(K)}$ denote the ring of invariants of $\mathcal{A}_k(\Gamma)$ under the action of GL_k as in Prop. 3.7. We also let $\mathcal{SS}_k(\Gamma)$ denote* $\text{Spec}(\mathcal{B}_k(\Gamma))$ *and let $\pi : \mathcal{R}_k(\Gamma) \longrightarrow \mathcal{SS}_k(\Gamma)$ the induced morphism of schemes. Moreover, we let $\mathcal{S}_k(\Gamma)$ denote the image $\pi(\mathcal{R}_k^s(\Gamma))$.*

We call $\mathcal{S}_k(\Gamma)$ and $\mathcal{SS}_k(\Gamma)$ the schemes of classes of simple and semi-simple representations, respectively. We now investigate the geometric and topological properties of these objects:

Proposition 3.11 [LM, Prop. 1.12] *$\mathcal{SS}_k(\Gamma)$ is an affine scheme and π is a universal categorical quotient of $\mathcal{R}_k(\Gamma)$ by GL_k. $\mathcal{S}_k(\Gamma)$ is an open subscheme of $\mathcal{SS}_k(\Gamma)$ and π restricted to $\mathcal{R}_k^s(\Gamma)$ is a geometric quotient of $\mathcal{R}_k^s(\Gamma)$ by GL_k.*

Definition 3.12 [LM, Def. 1.13] *For $\rho \in \mathcal{R}_k(\Gamma)(A)$ there exists a map $\psi_\rho : GL_k \times \text{Spec}(A) \longrightarrow \mathcal{R}_k(\Gamma) \times \text{Spec}(A)$ defined for $(T, f) \in GL_k(B) \times \text{Spec}(A)(B)$ by $\psi_\rho(T, f) = (T \cdot f_* \rho, f)$. The image of ψ_ρ is denoted $\mathcal{O}(\rho)$ and is called the* orbit *of ρ.*

Proposition 3.13 [LM, Cor. 1.17] *Let A be a finitely-generated $\mathbb{C}$-algebra and $\rho \in \mathcal{R}_k^s(\Gamma)(A)$. Then, $\mathcal{O}(\rho)(\mathbb{C})$, the collection of $\mathbb{C}$-points of the $GL_k(\mathbb{C})$ orbit of ρ, is closed in $\mathcal{R}_k(\Gamma)(\mathbb{C}) \times \text{Spec}(A)(\mathbb{C})$. In particular, if $\rho \in \mathcal{R}_k^s(\Gamma)(\mathbb{C})$, then the $GL_k(\mathbb{C})$ orbit of ρ is closed in $\mathcal{R}_k(\Gamma)(\mathbb{C})$.*

We now re-emphasize that the results of this thesis are those regarding *complex* representations of Baumslag-Solitar groups. Thus, much of our attention has been focused on the action of $GL_k(\mathbb{C})$ and on the objects $\mathcal{R}_k(\Gamma)(\mathbb{C})$ and $\mathcal{R}_k^s(\Gamma)(\mathbb{C})$. We now establish some special notation for this situation:

Definition 3.14 [LM, Def. 1.18] *$R_k(\Gamma)$ denotes the set $\mathcal{R}_k(\Gamma)(\mathbb{C})$ of $\mathbb{C}$-points of the scheme $\mathcal{R}_k(\Gamma)$. We call $R_k(\Gamma)$ the k^{th}* representation variety *of Γ. It is affine and we use $A_k(\Gamma)$ to denote its coordinate ring. In particular, $R_k^s(\Gamma)$ will denote $\mathcal{R}_k^s(\Gamma)(\mathbb{C})$.*

From the above definition, we see we have $R_k(\Gamma) = \mathcal{R}_k(\Gamma)(\mathbb{C})$. As a result, $R_k(\Gamma)$ is the space of *maximal* ideals of the coordinate ring $\mathcal{A}_k(\Gamma)$. Thus, $A_k(\Gamma)$ is the quotient of $\mathcal{A}_k(\Gamma)$ by its nilpotent radical. For example, if $\Gamma = \mathbb{Z}^2 \times S_3$ then $\mathcal{R}_2(\Gamma)$ is not reduced. i.e. $\mathcal{A}_2(\Gamma)$ had nilpotent elements. Refer to [LM, Ex. 2.10.4] for the non-trivial details for this example.

More important, though, is the fact that the elements of $R_k(\Gamma)$ can be viewed as both representations of Γ as well as geometric points. This is seen via a universal representation and representing algebra, as in Proposition 3.3:

Proposition 3.15 [LM, Prop. 1.19] *There is a representation $\rho_u : \Gamma \longrightarrow GL_k(A_k(\Gamma))$ such that for any affine algebraic set, X, and any morphism $\phi : X \longrightarrow R_k(\Gamma)$, there is a unique $\mathbb{C}$-algebra homomorphism $f : A_k(\Gamma) \longrightarrow \mathbb{C}[X]$ such that $\phi(x)$ is the representation obtained by evaluating $f_*(\rho_u) : \Gamma \longrightarrow GL_k(\mathbb{C}[X])$ at the point x. In particular, (for the case where $X = \{\rho\}$), a representation $\rho \in R_k(\Gamma)$ is given by evaluating the entries of ρ_u at ρ.*

Many of the same results hold in our new situation as before: The group $GL_k(\mathbb{C})$ acts on $R_k(\Gamma)$, and this is indeed an algebraic group action. The actions and operations from Propositions 3.6 and 3.7 are valid morphisms of representation varieties. We also have that $R_k^s(\Gamma)$ is an open algebraic subset of $R_k(\Gamma)$ and is stable under the $GL_k(\mathbb{C})$ action. In this respect, we can investigate the categorical (universal) quotient of $R_k(\Gamma)$ (or $R_k^s(\Gamma)$) by $GL_k(\mathbb{C})$. We set this investigation up using similar notation as earlier:

Definition 3.16 [LM, Def. 1.20] *Let $B_k(\Gamma) = A_k(\Gamma)^{GL_k(\mathbb{C})}$, the ring of invariants of $A_k(\Gamma)$ under the $GL_k(\mathbb{C})$ action. Let $SS_k(\Gamma)$ denote the variety of $\mathbb{C}$-points of $\mathcal{SS}_k(\Gamma)$, $p : R_k(\Gamma) \longrightarrow SS_k(\Gamma)$ the variety morphism on $\mathbb{C}$-points obtained from π and $S_k(\Gamma)$ the $\mathbb{C}$-points of the scheme $\mathcal{S}_k(\Gamma)$. We call $SS_k(\Gamma)$ ($S_k(\Gamma)$) the variety of semi-simple (simple) representations.*

Proposition 3.17 [LM, Prop. 1.21] *$p : R_k(\Gamma) \longrightarrow SS_k(\Gamma)$ is a categorical quotient; that is to say that the coordinate ring of $SS_k(\Gamma)$ is $\mathbb{C}[SS_k(\Gamma)] = B_k(\Gamma)$, and $p : R_k^s(\Gamma) \longrightarrow S_k(\Gamma)$ is a geometric quotient; that is to say that $p^{-1}p(\rho) = \mathcal{O}(\rho)$, the orbit of ρ. Moreover, the map $p : S_k(\Gamma) \longrightarrow SS_k(\Gamma)$ is an open inclusion.*

Tangent Spaces and First Cohomology

The goal of this section of exposition is to set up the discussion of computing the dimension of representation schemes and varieties. However, one realizes that that computational goal is more easily stated than achieved. As a result, we rely on the close relationship between tangent spaces to representation varieties and first cohomology, $H^1(\Gamma, Ad \circ \rho)$.

We first, however, assume $\rho \in R_k(\Gamma)$ and consider the tangent spaces at ρ to $\mathcal{O}(\rho)$, $R_k(\Gamma)$, and $\mathcal{R}_k(\Gamma)$. We always have the inclusions

$$T_\rho(\mathcal{O}(\rho)) \subset T_\rho(R_k(\Gamma)) \subset T_\rho(\mathcal{R}_k(\Gamma)).$$

Moreover, if ρ is simple, then we have the equalities

$$T_\rho(R_k(\Gamma))/T_\rho(\mathcal{O}(\rho)) = T_{[\rho]}(S_k(\Gamma)) = T_{[\rho]}(SS_k(\Gamma))$$

What we note now is the tangent space to $R_k(\Gamma)$ at the representation ρ, denoted $T_\rho(R_k(\Gamma))$, can be identified with a subspace of the space $Z^1(\Gamma, Ad \circ \rho)$, of one cocycles of Γ with coefficients in the representation $Ad \circ \rho$. $Ad \circ \rho$ denotes the action of Γ on $M_n(\mathbb{C})$ via $A \longmapsto \rho(\gamma)A\rho(\gamma)^{-1}$. We will, in fact, see that the tangent space to the representation *scheme* is equal to the cocycle space. It also turns out that the tangent space to the orbit of ρ, $T_\rho(\mathcal{O}(\rho))$, can be identified with the space of coboundaries $B^1(\Gamma, Ad \circ \rho)$.

We summarize the preceding discussion in the following proposition:

Proposition 3.18 *Let $\rho \in R_k(\Gamma)$. Then there exist $\mathbb{C}$-linear isomorphisms*

$$Z^1(\Gamma, Ad \circ \rho) \simeq T_\rho(\mathcal{R}_k(\Gamma))$$
$$B^1(\Gamma, Ad \circ \rho) \simeq T_\rho(\mathcal{O}(\rho))$$

As a result, we have the inclusions

$$B^1(\Gamma, Ad \circ \rho) \subset T_\rho\big(R_k(\Gamma)\big) \subset Z^1(\Gamma, Ad \circ \rho),$$

where the first inclusion can be explained a bit further:

Proposition 3.19 [LM, Cor. 2.24] *Let $\rho \in R_k(\Gamma)$ and let $\mathcal{O}(\rho)$ denote its orbit in $R_k(\Gamma)$. Then $T_\rho(\mathcal{O}(\rho)) \longrightarrow T_\rho(\mathcal{R}_k(\Gamma))$ is injective. In terms of the isomorphism from Prop. 3.18, this map corresponds to the inclusion $B^1(\Gamma, Ad \circ \rho) \longrightarrow Z^1(\Gamma, Ad \circ \rho)$.*

Thus, we have the following inequalities regarding the dimensions of these objects:

$$\dim_{\mathbb{C}}\Big(T_\rho(\mathcal{O}(\rho))\Big) \le \dim_\rho\Big(R_k(\Gamma)\Big) = \dim_\rho\Big(\mathcal{R}_k(\Gamma)\Big) \le \dim_{\mathbb{C}}\Big(T_\rho(\mathcal{R}_k(\Gamma))\Big)$$

Note the middle equality is due to the facts that Krull dimension is not affected by the presence of nilpotents in the coordinate ring and that the coordinate ring of $R_k(\Gamma)$ is that of $\mathcal{R}_k(\Gamma)$ modulo its nilradical.

By this time, one might conceive of a notion of a representation ρ being *scheme non-singular* as well as being *non-singular on the variety*:

Definition 3.20 *Let $\rho \in R_k(\Gamma)$. Then we say that ρ is* scheme non-singular *if*

$$\dim_\rho\Big(\mathcal{R}_k(\Gamma)\Big) = \dim_{\mathbb{C}}\Big(T_\rho(\mathcal{R}_k(\Gamma))\Big)$$

Moreover, we say that ρ is non-singular on the variety $R_k(\Gamma)$ *if*

$$\dim_\rho\Big(R_k(\Gamma)\Big) = \dim_{\mathbb{C}}\Big(T_\rho(R_k(\Gamma))\Big)$$

One should refer to [LM, Example 2.10] for examples of groups which have representations which are non-singular on the variety, but are not scheme non-singular.

Now we further consider the dimensional consequences of the (categorical) quotient map $p : R_k(\Gamma) \longrightarrow SS_k(\Gamma)$ referred to in Definition 3.16. There is another known result regarding (dimensions of) tangents spaces at simple representations:

Proposition 3.21 [LM, Thm. 2.13] *Let $\rho \in R_k^s(\Gamma)$ be simple. Then there exists an exact sequence of tangent spaces:*

$$0 \longrightarrow T_\rho(\mathcal{O}(\rho)) \longrightarrow T_\rho(R_k^s(\Gamma)) \longrightarrow T_{p(\rho)}(S_k(\Gamma)) \longrightarrow 0$$

In particular,

$$\dim_{[\rho]}\Big(S_k(\Gamma)\Big) \leq \dim_{\mathbb{C}}\Big(T_{p(\rho)}(S_k(\Gamma))\Big) \leq \dim_{\mathbb{C}}\Big(H^1(\Gamma, Ad \circ \rho)\Big)$$

What we will see in the next chapter, which includes the main results of this thesis, is that the preceding inequalities of dimensions will aid us in computing the fact that many of the Baumslag-Solitar groups are indeed scheme non-singular. In order to make those computations, though, we will need an explicit connection between the finitely presented groups, Γ, where our interests lie, their representation varieties, and the ability to actually compute $Z^1(\Gamma, Ad \circ \rho)$. In particular, we will see that the computation of one cocycles of a group Γ involves using *Fox Calculus*. Furthermore, these calculations amount to arriving at the Jacobian matrix of a specific morphism.

The morphism alluded to at the end of the previous section is defined as follows: Let Γ be given by the finite presentation

$$\Gamma = \langle a_1, \ldots, a_d : s_q, \, q = 1, \ldots, m \rangle$$

and we consider the map

$$f : GL_k(\mathbb{C})^{(d)} \longrightarrow GL_k(\mathbb{C})^{(m)} : \boldsymbol{A} = (\boldsymbol{A}_1, \ldots, \boldsymbol{A}_d) \mapsto \left(\boldsymbol{s}_q(\boldsymbol{A})\right)_{q=1}^{m}.$$

One sees easily that the preimage under f of the m-tuple of identity matrices is, in fact, $R_k(\Gamma)$. Moreover, f is indeed a morphism of varieties and $R_k(\Gamma)$ is a fibre of f. Since every fibre of f has dimension bounded below by the difference in the dimensions of its range and domain, we obtain the fact that every irreducible component of $R_k(\Gamma) = f^{-1}(I_k, \ldots I_k)$ has dimension at least $k^2(d-m)$. One can refer to [Mu, Thm 2, p. 92] for more details on this note.

As an aside, we note that the number $(d-m)$ is called the *presentation deficiency* for Γ. Since this value can be at most $\mathrm{Rank}(\Gamma^{ab})$, we see that the presentation deficiencies have a maximum, called the *deficiency* of Γ and is denoted $\mathrm{Def}(\Gamma)$. Using this definition, we obtain several results:

Proposition 3.22 [LM, Prop. 3.4] *If* $\mathrm{Def}(\Gamma) = \mathrm{Rank}(\Gamma^{ab})$, *then the trivial representation,* ρ_0, *of* Γ *in* $GL_k(\mathbb{C})$ *is scheme non-singular and the dimension of the unique irreducible component of* $R_k(\Gamma)$ *through* ρ_0 *is* $\mathrm{Rank}(\Gamma^{ab})k^2$. *If*

Def(Γ) $=$ Rank(Γ^{ab}) $= 1$, *then this unique irreducible component consists of all representations factoring through* Γ^{ab} *modulo torsion.*

Examples of groups satisfying the hypotheses of the above Proposition would be one-relator groups where the relator is not contained in the commutator subgroup.

What we shall do now, using the presentation $\Gamma = \langle x_1, \dots, x_d : s_q, q = 1, \dots, m \rangle$, is to calculate the one cohomology $Z^1(\Gamma, \rho)$ using the Fox Calculus. Let $\Lambda = \langle x_1, \dots, x_d \rangle$ be a free group on d generators and let $F = \mathbb{Z}[\Lambda]$ be the integral group algebra considered as a left Λ-module. Now if V is any Λ-module, then we have the following isomorphism:

$$Z^1(\Lambda, V) \longrightarrow V^{(d)} : \boldsymbol{\alpha} \mapsto \big(\boldsymbol{\alpha}(\boldsymbol{x}_1), \dots, \boldsymbol{\alpha}(\boldsymbol{x}_d)\big)$$

It is important to note here that the Fox Calculus is concerned with explicitly constructing an inverse of the above map. This is done as follows: Let e_i, $i = 1, \dots, d$ be the standard basis for $F^{(d)}$ and define $D : \Lambda \longrightarrow F^{(d)}$ to be the cocycle with $D(x_i) = e_i$. If $\boldsymbol{a} = (a_1, \dots, a_d) \in V^{(d)}$, let $h_a : F^{(d)} \longrightarrow V$ be defined by $h_a(f_1, \dots, f_d) = f_1 a_1 + \cdots + f_d a_d$. Then $(h_a \circ D) : \Lambda \longrightarrow V$ is a cocycle with the image of x_i being a_i, for all i, so that $(h_a \circ D)$ is the cocycle corresponding to the d-tuple $\boldsymbol{a}$. Moreover, D itself is a d-tuple of cocycles: $D(x) = \big(D_1(x), \dots, D_d(x)\big)$, with each $D_i : \Lambda \longrightarrow F$. The standard notation of Fox, then, is $\partial/\partial x_i := D_i$, and the above becomes the following:

Formulae 3.23 – *The Fox Derivative Formulae*

(1) $\quad \frac{\partial}{\partial x_i}$ *is a cocycle, and so for* $x, y \in \Lambda$ *we have:*

(a)	$\frac{\partial xy}{\partial x_i} = \frac{\partial x}{\partial x_i} + x\frac{\partial y}{\partial x_i}$
(b)	$\frac{\partial x^{-1}}{\partial x_i} = -x^{-1}\frac{\partial x}{\partial x_i}$

(2) $\frac{\partial x_i}{\partial x_j} = \delta_{ij}$

(3) *If* $\alpha \in Z^1(\Lambda, V)$ *and* $x \in \Lambda$, *then* $\alpha(x) = \sum \frac{\partial x}{\partial x_i}\alpha(x_i)$.

Now, if $g : \Lambda \longrightarrow \Gamma$ and ρ is a representation of Γ, then we can explicitly describe the space of one cocycles: $Z^1(\Gamma, \rho) = \{\alpha \in Z^1(\Lambda, \rho \circ g) : \alpha(s_q) = 0, q = 1, \dots m\}$. Using the third part of the preceding set of formulae, this becomes the following:

Proposition 3.24 [LM, Prop. 3.5] *Let* $\Gamma = \langle x_1, \dots, x_d : s_q,\ q = 1, \dots, m\rangle$ *and let* $\rho \in R_k(\Gamma)$. *Let* $v_1, \dots, v_d \in V(\rho)$, *where* $V(\rho)$ *is any* Λ*-module. Then there is a cocycle* $\alpha \in Z^1(\Gamma, \rho)$ *with* $\alpha(x_i) = v_i$ *for* $i = 1, \dots, d$ *if and only if* $\sum \frac{\partial s_q}{\partial x_i} v_i = 0$, *for* $q \in Q$.

As a result of this proposition, we have the following matrix associated to the set of relations $\{s_1, \dots, s_m\}$:

$$\frac{\partial s}{\partial x} := \begin{bmatrix} \frac{\partial s_1}{\partial x_1} & \cdots\cdots & \frac{\partial s_1}{\partial x_d} \\ \vdots & & \vdots \\ \frac{\partial s_m}{\partial x_1} & \cdots\cdots & \frac{\partial s_m}{\partial x_d} \end{bmatrix}$$

If we consider $\frac{\partial s}{\partial x} : V(\rho)^{(d)} \longrightarrow V(\rho)^{(m)}$, then $\ker(\frac{\partial s}{\partial x}) = Z^1(\Gamma, \rho)$ by using 3.24 above.

If we now assume that ρ is replaced by $Ad \circ \rho$, then $V(Ad \circ \rho)$ can be identified with $M_k(\Gamma)$, with Γ acting via conjugation through ρ, which is the tangent space of $GL_k(\mathbb{C})$. Thus, the matrix above is the Jacobian of the map f at the d-tuple $\big(\rho(x_1), \ldots, \rho(x_d)\big)$.

Proposition 3.25 [LM, Prop. 3.7] *Let $r = x_{i_1}^{e_{i_1}} \cdots x_{i_s}^{e_{i_s}}$, where $e_{i_j} = \pm 1$, be a word in the free group $\Lambda = \langle x_1, \ldots, x_d \rangle$ and let $f : GL_k(\mathbb{C})^{(d)} \longrightarrow GL_k(\mathbb{C}) : \boldsymbol{T} = (\boldsymbol{T}_1, \ldots, \boldsymbol{T}_d) \mapsto r(\boldsymbol{T})$. Let $\boldsymbol{A} = (A_1, \ldots, A_d) \in GL_k(\mathbb{C})^{(d)}$ and let $f(\boldsymbol{A}) = Y$. Then there is a commutative diagram*

$$\begin{tikzcd}
M_n(\mathbb{C})^{(d)} \arrow[r] \arrow[d, "D"] & T_{\boldsymbol{A}}\big(GL_k(\mathbb{C})^{(d)}\big) \arrow[d, "(Df)_{\boldsymbol{A}}"] \\
M_k(\mathbb{C}) \arrow[r] & T_Y\big(GL_k(\mathbb{C})\big)
\end{tikzcd}$$

where the horizontal maps are isomorphisms and $D(B_1, \ldots, B_d) = \sum \frac{\partial s}{\partial x_i} B_i$. Note that Λ acts on $M_k(\mathbb{C})$ with x_i being conjugation by A_i.

What we see now is that Proposition 3.24, along with the $m = 1$ case of the above proposition and considering the projections on each factor give the following result:

Corollary 3.26 *Let $\rho \in R_k(\Gamma)$. Then $T_\rho\big(\mathcal{R}_k(\Gamma)\big)$ is isomorphic to $Z^1(\Gamma, Ad \circ \rho)$.*

This result is easily applied to our Baumslag-Solitar groups. We will see in the next chapter that the Fox Calculus will allow us to compute the dimension of the representation varieties of $BS(m, n)$, when m and n are relatively prime, at its simple representations.

Deformations of Representations

The last section of this chapter will introduce the concept of a *deformable* representation. The representations we discuss in this section need not be simple. However, in the context of the results of this thesis we may assume, without loss of generality, that all representations mentioned in this section are simple unless otherwise noted.

It should be noted here that most of the discussion in this section is based loosely around the exposition found in [AM] and [Ma]. We begin with a definition:

Definition 3.27 *Let $\rho \in R_k(\Gamma)$. Then we say that ρ is* deformable *if ρ belongs to a one-parameter family of non-isomorphic representations. Equivalently, there exists a connected curve, $C \subset R_k(\Gamma)$ on which ρ lies.*

By considering the irreducible component of $[\rho]$, the projection map $p : R_k(\Gamma) \longrightarrow S_k(\Gamma)$, and the aforementioned curve C, the definition above leads to the following result.

Proposition 3.28 *Let $\rho \in R_k(\Gamma)$ be a deformable representation. Then:*

1. *There is an irreducible curve in $R_k(\Gamma)$ which contains a simple representation isomorphic to ρ and a simple representation not isomorphic to ρ*
2. *There is an irreducible curve in $S_k(\Gamma)$ through $[\rho]$.*

The case where we can explicitly construct the curve going through ρ is called a geometric deformation of ρ:

Definition 3.29 *Let $\rho \in R_k^s(\Gamma)$. A* geometric deformation *of ρ is an embedding $\alpha \mapsto \rho_\alpha$ of an irreducible affine curve D with base point $\alpha = 0$ such that $\rho_{\alpha=0} = \rho$. The deformation is* non-trivial *if the map $\alpha \mapsto \chi(\rho_\alpha)$ is non-constant.*

Obviously, a representation which is deformable has a geometric deformation. The converse is also true:

Proposition 3.30 *A simple representation is deformable if and only if it has a non-trivial geometric deformation.*

One is referred to [AM] for the proof of the above result.

We note that there is yet one more concept of deformability. We define a formal deformation of ρ:

Defintion 3.31 *Let $\rho \in R^s_k(\Gamma)$. A* formal deformation *of ρ is a representation over formal power series, $\rho_x : \Gamma \longrightarrow GL_k(\mathbb{C}[[x]])$, such that the representation $\rho_0 := \rho_x\big|_{x=0}$ coincides with ρ. The deformation is* non-trivial *if $\chi(\rho_x)$ is non-constant. The* non-triviality degree *of ρ_x is the smallest $m \in \mathbb{N}$ such that there is $\gamma \in \Gamma$ so that the coefficient of x^m in $\chi(\rho_x)(\gamma)$ is non-zero.*

A trivial deformation of ρ would be one in which $\chi(\rho_x)$ is constant. There always exists a trivial deformation of ρ, found by setting $\rho_x = \rho$. More important now is the fact that we can connect the three notions of deformability, geometric deformability, and formal deformability of ρ:

Theorem 3.32 *A simple representation has a non-trivial formal deformation if and only if it has a geometric déformation.*

As a result, we may speak of a representation as merely "deformable" and know that the term unambiguously refers to the existence of both a geometric and non-trivial formal deformation for ρ.

We finish off this section by including a brief discussion of the family of representations a formal deformation, $\rho_x : \Gamma \longrightarrow GL_k(\mathbb{C}[[x]])$, can produce. We can clearly construct the family of representations

$$\rho_i : \Gamma \longrightarrow GL_k(\mathbb{C}[[x]]/\langle x^{i+1}\rangle),\ i = 1, \ldots$$

by taking advantage of the surjections $\mathbb{C}[[x]] \longrightarrow \mathbb{C}[x]/\langle x^{i+1}\rangle$. Each of these representations reduces to the original ρ by setting $x = 0$. Moreover, when $i > j$, we see ρ_i reduces to ρ_j modulo x^{j+1}. A notation for that concept is to say

$\rho_i \equiv \rho_j \pmod{x^{j+1}}$. We say that the collection $\{\rho_i : i \in \mathbb{N}\}$ is a *consistent* family of representations.

Conversely, such a family produces a formal deformation of ρ by taking its inverse limit.

Definition 3.33 *Let $\rho \in R_k^s(\Gamma)$. A* lifting of ρ to level i *is a representation $\sigma : \Gamma \longrightarrow GL_k(\mathbb{C}[x]/\langle x^{i+1}\rangle)$ such that the residual representation $\sigma_0 = \sigma\big|_{x=0}$ coincides with ρ. The lifting is* non-trivial *if the character $\chi(\sigma)$ is non-constant. The* non-triviality degree *of σ is the smallest positive integer m such that there is $\gamma \in \Gamma$ so that the coefficient of x^m in $\chi(\sigma)(\gamma)$ is non-zero.*

Thus, a lifting is trivial if its character is constant. Now suppose that ρ_a is a lifting of ρ to level $a \geq 1$. We say that ρ_a *extends to level* $a+1$ if there is a lifting ρ_{a+1} of ρ where $\rho_{a+1} \equiv \rho_a \pmod{x^{a+1}}$. If there is no such lifting, then we say that ρ_a is *obstructed*.

What one could see is that extensions and obstructions can have interpretations in group cohomology. This is a topic beyond the scope of this thesis, but one can find an excellent handling of the topic in [AM].

Chapter 4 — Main Results

It was found, in general, that computing dimensions of representation varieties of arbitrary groups using *CoCoA* (or any computer algebra system), was usually not feasible. The number of indeterminates, and hence polynomials, involved in performing the computations proved to require more memory than the typical computer has. Chapter 5 contains the code and output from several examples attempted in *CoCoA*, some of which were successful and some of which were not.

Due to the computational issues in finding such dimensions, two-generator, one-relator groups form an attractive collection of groups on which to attempt these computations. Moreover, the Baumslag-Solitar groups are a well-known class of such groups and thus form an excellent starting point for an investigation. We recall from Chapter 2 our notation for the collection of Baumslag-Solitar groups:

$$\mathcal{BS} := \{BS(m,n) : m,n \in \mathbb{N}\}$$

The first observation about the Baumslag-Solitar groups is that we are guaranteed a simple arbitrary-dimensional representation for a subcollection of $\mathcal{BS}$:

Theorem 4.1 *If $m,n \in \mathbb{N}$, $n > m$, and $\gcd(m,n) = 1$, then there exists a k-dimensional simple representation, ρ, of $BS(m,n)$ for any $k \in \mathbb{N}$ where $\rho(a)$ is a diagonal matrix with distinct non-zero complex entries and $\rho(t)$ is a matrix which cyclically permutes those diagonal elements by conjugation.*

Proof We recall that our presentation for $BS(m,n)$ is:

$$BS(m,n) = \langle a, t : ta^m = a^n t \rangle$$

We first fix k, m, and n and prove the existence of a representation $\rho : BS(m,n) \longrightarrow GL_k(\mathbb{C})$ of the form where $\rho(a)$ is diagonal on the distinct non-zero complex numbers $\lambda_1, \lambda_2, \dots, \lambda_k$ and $\rho(t)$ is a cyclic permutation of those elements. The explicit matrices for such a representation are given below.

Representations of this type are simple provided that the λ_i are distinct. This is easy to see, for the eigenvectors of matrix $\rho(a)$, the diagonal matrix with distinct entries, are merely the standard basis vectors. However, none of the standard basis vectors are eigenvectors for $\rho(t)$. Thus, ρ is indeed simple.

We note for all of the choices of $\rho(t)$ as some cyclic permutation of the diagonal elements, the systems of equations we obtain are of similar forms. Without loss of generality, we consider the representation below where $\rho(t)$ is the cyclic permutation $(1\,2\,3 \cdots k)$.

$$\rho(a) = A_0 = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & \lambda_k \end{pmatrix}, \quad \rho(t) = T_0 = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 & \vdots \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

As a result, the relation $\rho(ta^m) = \rho(a^n t)$ requires $T_0 A_0^m = A_0^n T_0$. i.e.,

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & \lambda_k^m - \lambda_1^n \\ \lambda_1^m - \lambda_2^n & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2^m - \lambda_3^n & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 & \vdots \\ 0 & 0 & 0 & \lambda_{k-1}^m - \lambda_k^n & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

We need a solution, $(\lambda_1, \lambda_2, \dots, \lambda_k) \in (\mathbb{C}^*)^{(k)}$, with distinct entries, of the system

$$\begin{aligned} \lambda_1^m &= \lambda_2^n \\ \lambda_2^m &= \lambda_3^n \\ &\vdots \\ \lambda_{k-1}^m &= \lambda_k^n \\ \lambda_k^m &= \lambda_1^n \end{aligned}$$

We will find a solution to the system using the primitive $\mu = 2(n^k - m^k)^{th}$ root of unity $\theta = e^{2\pi i/\mu}$. As such, we will then prove a solution of the desired form exists where $\lambda_i = \theta^{\alpha_i}$ for each i, and such that the α_i are distinct.

We note that finding a solution to the above system is equivalent to solving the following system of congruences:

$$\begin{aligned} m\alpha_1 &\equiv n\alpha_2 \pmod{\mu} \\ m\alpha_2 &\equiv n\alpha_3 \pmod{\mu} \\ &\vdots \\ m\alpha_{k-1} &\equiv n\alpha_k \pmod{\mu} \\ m\alpha_k &\equiv n\alpha_1 \pmod{\mu} \end{aligned}$$

In order to proceed, we require a lemma

Lemma *If $m, n \in \mathbb{N}$ with $n > m$ and $\gcd(m, n) = 1$, then either $\gcd(m, \mu) = 1$ or $\gcd(n, \mu) = 1$. Specifically, this holds for whichever of m or n is odd.*

Proof Assuming the hypotheses on m and n, we see that m and n cannot simultaneously be even. Thus, at least one of m or n is odd. Without loss of generality, suppose n is odd. If $\gcd(n, \mu) \neq 1$, then there would exist an odd prime p dividing both n and μ. Note $p \mid (n^k - m^k)$, since p is odd. Since $p \mid n$, we would then have $p \mid \left[n^k - (n^k - m^k)\right] = m^k$. Therefore, $p \mid m$. This, however, is impossible due to the assumption of $\gcd(m, n) = 1$. The argument is identical if we assume m is odd. This completes the proof of the Lemma.

For the remainder of the proof of the theorem, we will assume that n is odd.

The system is solvable now because the lemma guarantees the existence of $n^{-1} \pmod{\mu}$. So if we (without loss of generality) assume $\alpha_1 = 2$, then we solve the first congruence to get $\alpha_2 \equiv n^{-1}m\alpha_1 \equiv 2n^{-1}m \pmod{\mu}$. We then obtain $\alpha_3 \equiv n^{-1}m\alpha_2 \equiv 2(n^{-1}m)^2 \pmod{\mu}$. Continuing in this manner, we see $\alpha_i \equiv n^{-1}m\alpha_{i-1} \equiv 2(n^{-1}m)^{i-1} \pmod{\mu}$, for $i = 2, \ldots, k$.

The only remaining detail is to whether the system is consistent. i.e., does this method yield the same/correct $\alpha_1 = 2$ in the final step as we assumed in the first congruence? In order to investigate this notion, we employ back-substitution. We begin with the next-to-last congruence, $m\alpha_{k-1} \equiv n\alpha_k \pmod{\mu}$, solve for α_k, and try to obtain an expression for $m\alpha_k$ based upon the preceding congruences.

We express $m\alpha_k$ in terms of α_1 by way of the following sequences of congruences:

$$\begin{aligned}
m\alpha_k &\equiv m\left(n^{-1}m\alpha_{k-1}\right) && \pmod{\mu} \\
&\equiv m\left((n^{-1}m)^2\alpha_{k-2}\right) && \pmod{\mu} \\
&\equiv m\left((n^{-1}m)^3\alpha_{k-3}\right) && \pmod{\mu} \\
&\ \ \vdots \\
&\equiv m\left((n^{-1}m)^{k-1}\alpha_1\right) && \pmod{\mu} \\
&\equiv \left(n^{k-1}\right)^{-1}m^k\alpha_1 && \pmod{\mu}
\end{aligned}$$

We realize that for this system to be consistent, the last expression above must be congruent to $n\alpha_1 \pmod{\mu}$. In other words, we must prove the following congruence holds:

$$\left(n^{k-1}\right)^{-1} m^k \alpha_1 \equiv n\alpha_1 \pmod{\mu}$$

This is seen as follows:

$$\begin{aligned} m^k \alpha_1 &\equiv n^k \alpha_1 && \pmod{\mu} \\ m^k \alpha_1 &\equiv n \cdot n^{k-1} \alpha_1 && \pmod{\mu} \\ m^k \cdot \left(n^{k-1}\right)^{-1} \alpha_1 &\equiv n\alpha_1 && \pmod{\mu} \\ \left(n^{k-1}\right)^{-1} m^k \alpha_1 &\equiv n\alpha_1 && \pmod{\mu} \end{aligned}$$

As a result, we have obtained a k-tuple $(\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathbb{N}^k$ which solves the system of congruences, and thus we obtained our k-tuple solution $(\lambda_1, \lambda_2, \ldots, \lambda_k) \in (\mathbb{C}^*)^{(k)}$.

We now must show, in addition, this k-tuple solution has distinct entries. If there exist $k \geq j > i \geq 1$ such that $\alpha_i = \alpha_j$, then we can arrive at a contradiction by combining the congruences in our system.

$$\begin{aligned} \alpha_j &\equiv n^{-1} m \alpha_{j-1} && \pmod{\mu} \\ &\equiv \left(n^{-1} m\right) n^{-1} m \alpha_{j-2} && \pmod{\mu} \\ &\equiv \left(n^{-1} m\right)^2 \alpha_{j-2} && \pmod{\mu} \\ &\equiv \left(n^{-1} m\right)^3 \alpha_{j-3} && \pmod{\mu} \\ &\vdots \\ &\equiv \left(n^{-1} m\right)^{j-i} \alpha_i && \pmod{\mu} \\ &\equiv \left(n^{-1} m\right)^{j-i} \alpha_j && \pmod{\mu} \end{aligned}$$

and thus, as a result of our formula for α_i, namely $\alpha_i \equiv 2(n^{-1}m)^{i-1} \pmod{\mu}$, we see

$$2(n^{-1}m)^{i-1} \equiv 2(n^{-1}m)^{i-1} \cdot (n^{-1}m)^{j-i} (\operatorname{mod} \mu)$$

and this implies

$$(n^{-1}m)^{i-1} \equiv (n^{-1}m)^{i-1} \cdot (n^{-1}m)^{j-i} (\operatorname{mod} \mu/2)$$

Recalling $\mu = 2(n^k - m^k)$, we note that m is invertible modulo $\mu/2$ and thus has an inverse modulo $\mu/2$. This is seen by observing if a prime p divides both m and $\mu/2$, then p would divide n. Clearly, this violates the assumption that $\gcd(m, n) = 1$. As a result of the equivalence relation above, we have

$$\begin{aligned} 1 &\equiv (n^{-1}m)^{j-i} \quad (\operatorname{mod} \mu/2) \\ n^{j-i} &\equiv m^{j-i} \qquad\quad (\operatorname{mod} \mu/2) \end{aligned}$$

Consequently, we would say that $(n^k - m^k) \mid (n^{j-i} - m^{j-i})$, which is possible if and only if $k \leq j - i$. (One can find this result in many number theory books, such as [Ko, Chap. I, §4]). This cannot be the case since $k \geq j > i \geq 1$. Thus, all the elements of $(\alpha_1, \alpha_2, \dots, \alpha_k)$, and hence of $(\lambda_1, \lambda_2, \dots, \lambda_k)$, are distinct. $\square$

We use the group $BS(2, 3) = \langle a, t : ta^2 = a^3 t \rangle = \langle a, t : ta^2 t^{-1} a^{-3} = 1 \rangle$ as an ongoing example in the case $k = 3$. In this case, we see the matrix equation to be solved is

$$\begin{pmatrix} 0 & 0 & \lambda_3^2 - \lambda_1^3 \\ \lambda_1^2 - \lambda_2^3 & 0 & 0 \\ 0 & \lambda_2^2 - \lambda_3^3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We let θ be the primitve $2(3^3 - 2^3)^{th} = 38^{th}$ root of unity $\theta = e^{\pi i/19}$ and we find a triple $(\lambda_1, \lambda_2, \lambda_3) \in (\mathbb{C}^*)^{(3)}$ that solves the above matrix equation where $\lambda_i = \theta^{\alpha_i}, i = 1, 2, 3$ and each α_i is an integer between 1 and 38 (inclusive). In other words, we solve the system of congruences:

$$\begin{aligned} 4 \equiv 2\alpha_1 \equiv 3\alpha_2 & \quad \pmod{38} \\ 2\alpha_2 \equiv 3\alpha_3 & \quad \pmod{38} \\ 2\alpha_3 \equiv 3\alpha_1 = 6 & \quad \pmod{38} \end{aligned}$$

We set $\alpha_1 = 2$ as a first step. We then use the fact that $3^{-1} \equiv 13 \pmod{38}$ and obtain $\alpha_2 = 3^{-1} \cdot 4 \equiv 13 \cdot 4 \equiv 14 \pmod{38}$ and $\alpha_3 = 3^{-1} \cdot 28 \equiv 13 \cdot 28 \equiv 22 \pmod{38}$. Since $\theta = e^{\pi i/19}$, our solution is the triple

$$(\lambda_1, \lambda_2, \lambda_3) = (e^{2\pi i/19}, e^{14\pi i/19}, e^{22\pi i/19}) \in (\mathbb{C}^*)^{(3)}.$$

Therefore, the simple representation of $BS(2,3)$ we have just computed is the homomorphism $\rho : BS(2,3) \longrightarrow GL_3(\mathbb{C})$ where

$$\rho(a) = \begin{pmatrix} e^{2\pi i/19} & 0 & 0 \\ 0 & e^{14\pi i/19} & 0 \\ 0 & 0 & e^{22\pi i/19} \end{pmatrix}, \quad \rho(t) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Now, we shall see how to compute the dimension of (the tangent space to) the representation scheme at any of these special types of simple representations of $BS(m,n)$:

Theorem 4.2 *Suppose $k, n, m \in \mathbb{N}$, $n > m$, $\gcd(m,n) = 1$, and let ρ be a k-dimensional simple representation of $BS(m,n)$ where $\rho(a)$ is diagonal with non-zero complex entries $\lambda_1, \lambda_2, \ldots, \lambda_k$, and $\rho(t)$ cyclically permutes those diagonal elements by conjugation. Also, suppose $i \neq j$ implies $\lambda_i^n \neq \lambda_j^n$. Then*

$$\dim_{\mathbb{C}}\Big(T_\rho(\mathcal{R}_k(BS(m,n)))\Big) = \dim_{\mathbb{C}}\Big(Z^1(BS(m,n), Ad \circ \rho)\Big) = k^2.$$

Proof This proof is achieved using the Fox calculus, as described in Chapter 3. We fix k and recall our group is given by the presentation

$$BS(m,n) = \langle a, t : ta^m = a^n t \rangle = \langle a, t : ta^m t^{-1} a^{-n} = 1 \rangle$$

and our representation, ρ, is given by

$$\rho(a) = A_\rho = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & \lambda_k \end{pmatrix}, \ \rho(t) = T_\rho = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 & \vdots \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

The matrix associated to the relator $r = ta^m t^{-1} a^{-n}$ is a 1-by-2 matrix whose columns corresponding to the generators a and t:

$$\frac{\partial R}{\partial(a,t)} := \begin{bmatrix} \frac{\partial r}{\partial a} & \frac{\partial r}{\partial t} \end{bmatrix}.$$

We obtain the entries of $\frac{\partial R}{\partial(a,t)}$ using the Fox calculus derivative formulas as given in Formulae 3.23. Thus,

$$\begin{aligned}
\frac{\partial r}{\partial t} &= \frac{\partial(ta^m t^{-1} a^{-n})}{\partial t} \\
&= \frac{\partial t}{\partial t} + t\frac{\partial(a^m t^{-1} a^{-n})}{\partial t} \\
&= 1 + t\left[\frac{\partial(a^m)}{\partial t} + a^m \frac{\partial(t^{-1}a^{-n})}{\partial t}\right] \\
&= 1 + t\left[0 + a^m \frac{\partial(t^{-1}a^{-n})}{\partial t}\right] \\
&= 1 + ta^m\left[\frac{\partial(t^{-1})}{\partial t} + t^{-1}\frac{\partial(a^{-n})}{\partial t}\right] \\
&= 1 + ta^m\left[-t^{-1} + t^{-1}(0)\right] \\
&= 1 + ta^m\left[-t^{-1}\right] \\
&= 1 - ta^m t^{-1}
\end{aligned}$$

And,

$$\begin{aligned}
\frac{\partial r}{\partial a} &= \frac{\partial(ta^m t^{-1} a^{-n})}{\partial a} \\
&= \frac{\partial t}{\partial a} + t\frac{\partial(ta^m t^{-1} a^{-n})}{\partial a} \\
&= 0 + t\Big[\frac{\partial(a^m)}{\partial a} + a^m \frac{\partial(t^{-1}a^{-n})}{\partial a}\Big] \\
&= t\Big[(1 + a + \cdots + a^{m-1}) + a^m\Big(\frac{\partial(t^{-1})}{\partial a} + t^{-1}\frac{\partial(a^{-n})}{\partial a}\Big)\Big] \\
&= t\Big[(1 + a + \cdots + a^{m-1}) + a^m\big(0 + t^{-1}(-a^{-1} - a^{-2} - \cdots - a^{-n})\big)\Big] \\
&= t\Big[(1 + a + \cdots + a^{m-1}) + a^m t^{-1}(-a^{-1} - a^{-2} - \cdots - a^{-n})\Big] \\
&= t + ta + \cdots + ta^{m-1} - ta^m t^{-1}a^{-1} - ta^m t^{-1} a^{-2} - \cdots - ta^m t^{-1} a^{-n}
\end{aligned}$$

Therefore,

$$\frac{\partial R}{\partial(a,t)} := [\, t + ta + \cdots + ta^{m-1} - ta^m t^{-1} a^{-n} - ta^m t^{-1} a^{-n+1} - \cdots - ta^m t^{-1} a^{-1} \quad 1 - ta^m t^{-1} \,]$$

Using the relator, r, the image of this matrix under our representation ρ becomes

$$\rho \circ \frac{\partial R}{\partial(a,t)} = \left[\, T_\rho + T_\rho A_\rho + \cdots + T_\rho A_\rho^{m-1} - I_k - A_\rho - A_\rho^2 - \cdots - A_\rho^{n-1} \quad I_k - A_\rho^n \,\right]$$

Now, set $B_1 = (b_{ij}^{(1)}), B_2 = (b_{ij}^{(2)}) \in M_k(\mathbb{C})$ and compute the kernel of $D(B_1, B_2) = \left(\rho \circ \frac{\partial R}{\partial(a,t)}\right) \cdot (B_1, B_2)$, which is given by:

$$(T_\rho + T_\rho A_\rho + \cdots + T_\rho A_\rho^{m-1} - I_k - A_\rho - A_\rho^2 - \cdots - A_\rho^{n-1}) \cdot B_1 + (I_k - A_\rho^n) \cdot B_2$$

The dot action on B_1 and B_2 is just conjugation where, for example, we would have:

$$(A + B) \cdot C = ACA^{-1} + BCB^{-1}$$

Thus, we compute the kernel of

$$\begin{gathered}
\Big(T_\rho B_1 T_\rho^{-1} + T_\rho A_\rho B_1 A_\rho^{-1} T_\rho^{-1} + \cdots + T_\rho A_\rho^{m-1} B_1 A_\rho^{1-m} T_\rho^{-1} - B_1 - A_\rho B_1 A_\rho^{-1} - \\
A_\rho^2 B_1 A_\rho^{-2} - \cdots - A_\rho^{n-1} B_1 A_\rho^{1-n}\Big) + \Big(B_2 - A_\rho^n B_2 A_\rho^{-n}\Big)
\end{gathered}$$

As a result, we obtain the following formulas for the entries of $D(B_1, B_2) = \big(d_{ij}\big)$, recalling the cyclic permutation associated to T_ρ was $\sigma = \big(1\,2\,3 \cdots k\big)$:

$$
\begin{aligned}
d_{11} &= mb^{(1)}_{kk} - nb^{(1)}_{11} \\
d_{ii} &= mb^{(1)}_{i-1,i-1} - nb^{(1)}_{ii}, \; 2 \le i \le k \\
d_{ij} &= \Big(1 + \frac{\lambda_{\sigma^{-1}(i)}}{\lambda_{\sigma^{-1}(j)}} + \cdots + \frac{\lambda^{m-1}_{\sigma^{-1}(i)}}{\lambda^{m-1}_{\sigma^{-1}(j)}}\Big) b^{(1)}_{\sigma^{-1}(i)\sigma^{-1}(j)} - \Big(1 + \frac{\lambda_i}{\lambda_j} + \cdots + \frac{\lambda_i^{n-1}}{\lambda_j^{n-1}}\Big) b^{(1)}_{ij} + \Big(1 - \frac{\lambda_i^n}{\lambda_j^n}\Big) b^{(2)}_{ij}
\end{aligned}
$$

In searching for the dimension of the kernel of this map, we set $d_{ij} = 0$ for all i and j. We see $b^{(1)}_{kk} = (\frac{m}{n})b^{(1)}_{11} = (\frac{m}{n})^2 b^{(1)}_{22} = \cdots = (\frac{m}{n})^k b^{(1)}_{kk}$ and so $b^{(1)}_{kk} = 0$, for all $1 \le i \le k$.

Moreover, setting d_{ij} equal to zero in the third formula above shows us the $k^2 - k$ non-diagonal entries of matrix B_1 determine the non-diagonal entries of B_2. This is true as a result of Lemma 4.4, given below, which guarantees the coefficients of all the $b^{(2)}_{ij}$ are non-zero. Also, the k diagonal entries of B_2 are freely chosen since they do not appear at all in the expression for the diagonal entries of $D(B_1, B_2) = \big(d_{ij}\big)$. Thus, the dimension of our solution space is indeed $(k^2 - k) + k = k^2$. □

Continuing with our $BS(2,3)$ dimension $k = 3$ example, we compute and verify $\dim_{\mathbb{C}}\Big(T_\rho(\mathcal{R}_3(BS(2,3)))\Big) = \dim_{\mathbb{C}}\Big(Z^1(BS(2,3), Ad \circ \rho)\Big) = 3^2 = 9$. As in the proof of Theorem 4.2, Fox Calculus gives us

$$
\begin{aligned}
\frac{\partial R}{\partial(a,t)} &:= \begin{bmatrix} \frac{\partial r}{\partial a} & \frac{\partial r}{\partial t} \end{bmatrix} \\
&= \begin{bmatrix} t + ta - ta^2t^{-1}a^{-1} - ta^2t^{-1}a^{-2} - ta^2t^{-1}a^{-3} & 1 - ta^2t^{-1} \end{bmatrix}
\end{aligned}
$$

And so, when we compose this matrix with our representation, ρ, we take advantage of the group relator to obtain

$$\rho\circ\frac{\partial R}{\partial(a.t)} = \left[\, T_\rho + T_\rho A_\rho - T_\rho A_\rho^2 T_\rho^{-1} A_\rho^{-1} - T_\rho A_\rho^2 T_\rho^{-1} A_\rho^{-2} - T_\rho A_\rho^2 T_\rho^{-1} A_\rho^{-3} \quad I_3 - T_\rho A_\rho^2 T_\rho^{-1} \,\right]$$
$$= \left[\, T_\rho + T_\rho A_\rho - A_\rho^2 - A_\rho - I_3 \quad I_3 - A_\rho^3 \,\right]$$

Again, we wish to compute (the dimension of) the kernel of the matrix $D(B_1, B_2) = \left(\rho\circ\frac{\partial R}{\partial(a.t)}\right)\cdot(B_1, B_2)$, which is given by the matrix

$$\begin{pmatrix} 2b_{33}^{(1)} - 3b_{11}^{(1)} & \left(\frac{\lambda_1+\lambda_3}{\lambda_1}\right)b_{31}^{(1)} - \left(\frac{\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2}{\lambda_2^2}\right)b_{12}^{(1)} + \left(\frac{\lambda_2^3-\lambda_1^3}{\lambda_2^3}\right)b_{12}^{(2)} & \left(\frac{\lambda_2+\lambda_3}{\lambda_2}\right)b_{32}^{(1)} - \left(\frac{\lambda_1^2+\lambda_1\lambda_3+\lambda_3^2}{\lambda_3^2}\right)b_{13}^{(1)} + \left(\frac{\lambda_3^3-\lambda_1^3}{\lambda_3^3}\right)b_{13}^{(2)} \\ \left(\frac{\lambda_1+\lambda_3}{\lambda_3}\right)b_{13}^{(1)} - \left(\frac{\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2}{\lambda_1^2}\right)b_{21}^{(1)} + \left(\frac{\lambda_1^3-\lambda_2^3}{\lambda_1^3}\right)b_{21}^{(2)} & 2b_{11}^{(1)} - 3b_{22}^{(1)} & \left(\frac{\lambda_1+\lambda_2}{\lambda_2}\right)b_{12}^{(1)} - \left(\frac{\lambda_2^2+\lambda_2\lambda_3+\lambda_3^2}{\lambda_3^2}\right)b_{23}^{(1)} + \left(\frac{\lambda_3^3-\lambda_2^3}{\lambda_3^3}\right)b_{23}^{(2)} \\ \left(\frac{\lambda_2+\lambda_3}{\lambda_3}\right)b_{23}^{(1)} - \left(\frac{\lambda_1^2+\lambda_1\lambda_3+\lambda_3^2}{\lambda_1^2}\right)b_{31}^{(1)} + \left(\frac{\lambda_1^3-\lambda_3^3}{\lambda_1^3}\right)b_{31}^{(2)} & \left(\frac{\lambda_1+\lambda_2}{\lambda_1}\right)b_{21}^{(1)} - \left(\frac{\lambda_2^2+\lambda_2\lambda_3+\lambda_3^2}{\lambda_2^2}\right)b_{32}^{(1)} + \left(\frac{\lambda_2^3-\lambda_3^3}{\lambda_2^3}\right)b_{32}^{(2)} & 2b_{22}^{(1)} - 3b_{33}^{(1)} \end{pmatrix}$$

From this matrix, we see $b_{33}^{(1)} = \frac{3}{2}b_{11}^{(1)} = \frac{3}{2}\left(\frac{3}{2}b_{22}^{(1)}\right) = \frac{9}{4}b_{22}^{(1)} = \frac{9}{4}\left(\frac{3}{2}b_{33}^{(1)}\right) = \frac{27}{8}b_{33}^{(1)}$ and so we obtain $b_{33}^{(1)} = 0$. Thus, $b_{11}^{(1)} = b_{22}^{(1)} = b_{33}^{(1)} = 0$. Moreover, we see that the entries $b_{ij}^{(2)}$ can be solved for in terms of the $b_{ij}^{(1)}$:

$$\begin{aligned}
b_{12}^{(2)} &= \left(\left(\frac{\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2}{\lambda_2^2}\right)b_{12}^{(1)} - \left(\frac{\lambda_1+\lambda_3}{\lambda_1}\right)b_{31}^{(1)}\right)\Big/\left(\frac{\lambda_2^3-\lambda_1^3}{\lambda_2^3}\right) \qquad (**)\\
b_{13}^{(2)} &= \left(\left(\frac{\lambda_1^2+\lambda_1\lambda_3+\lambda_3^2}{\lambda_3^2}\right)b_{13}^{(1)} - \left(\frac{\lambda_2+\lambda_3}{\lambda_2}\right)b_{32}^{(1)}\right)\Big/\left(\frac{\lambda_3^3-\lambda_1^3}{\lambda_3^3}\right)\\
b_{21}^{(2)} &= \left(\left(\frac{\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2}{\lambda_1^2}\right)b_{21}^{(1)} - \left(\frac{\lambda_1+\lambda_3}{\lambda_3}\right)b_{13}^{(1)}\right)\Big/\left(\frac{\lambda_1^3-\lambda_2^3}{\lambda_1^3}\right)\\
b_{23}^{(2)} &= \left(\left(\frac{\lambda_2^2+\lambda_2\lambda_3+\lambda_3^2}{\lambda_3^2}\right)b_{23}^{(1)} - \left(\frac{\lambda_1+\lambda_2}{\lambda_2}\right)b_{12}^{(1)}\right)\Big/\left(\frac{\lambda_3^3-\lambda_2^3}{\lambda_3^3}\right)\\
b_{31}^{(2)} &= \left(\left(\frac{\lambda_1^2+\lambda_1\lambda_3+\lambda_3^2}{\lambda_1^2}\right)b_{31}^{(1)} - \left(\frac{\lambda_2+\lambda_3}{\lambda_3}\right)b_{23}^{(1)}\right)\Big/\left(\frac{\lambda_1^3-\lambda_3^3}{\lambda_1^3}\right)\\
b_{32}^{(2)} &= \left(\left(\frac{\lambda_2^2+\lambda_2\lambda_3+\lambda_3^2}{\lambda_2^2}\right)b_{32}^{(1)} - \left(\frac{\lambda_1+\lambda_2}{\lambda_1}\right)b_{21}^{(1)}\right)\Big/\left(\frac{\lambda_2^3-\lambda_3^3}{\lambda_2^3}\right)
\end{aligned}$$

The six non-diagonal entries of matrix B_1 therefore determine the non-diagonal entries of B_2 in the search for the kernel of $D(B_1, B_2)$ and so those six entries are freely chosen. Moreover, the diagonal entries of B_1 must be zero, and so the three diagonal entries of B_2 are freely chosen. Thus, the dimension of our solution space, and hence our tangent space, is $\dim_{\mathbb{C}}\Big(Z^1(BS(2,3), Ad\circ\rho)\Big) = 6 + 3 = 3^2 = 9 = \dim_{\mathbb{C}}\Big(T_\rho(\mathcal{R}_3(BS(2,3)))\Big)$.

Corollary 4.3 *If ρ is a simple k-dimensional representation of $BS(m,n)$ where $\rho(a)$ is diagonal with distinct non-zero complex entries and $\rho(t)$ cyclically permutes those diagonal elements by conjugation, then* $\dim_{\mathbb{C}}\Big(H^1(BS(m,n), Ad\circ\rho)\Big) = 1$.

Proof Since our representation, ρ, is simple we have

$$\dim_{\mathbb{C}}\Big(B^1(BS(m,n), Ad\circ\rho)\Big) = k^2 - 1.$$

In the theorem above, we computed

$$\dim_{\mathbb{C}}\Big(Z^1(BS(m,n), Ad\circ\rho)\Big) = k^2.$$

For $H^1(BS(m,n), Ad\circ\rho) = Z^1(BS(m,n), Ad\circ\rho)/B^1(BS(m,n), Ad\circ\rho)$ we have $\dim_{\mathbb{C}}\Big(H^1(BS(m,n), Ad\circ\rho)\Big) = (k^2) - (k^2-1) = 1$. □

In addition to the above result, we can easily find a basis for $H^1(BS(m,n), Ad\circ\rho)$ using Proposition 3.18. That result, and the details therein, indicate to us a more concrete description of $B^1(BS(m,n), Ad\circ\rho)$ as the image of a particular map. Specifically, we consider the following map and note the reference $(**)$ is to the set of equations found in the proof of Theorem 4.2:

$$\Phi : M_k(\mathbb{C}) \longrightarrow T_\rho(\mathcal{R}_k(BS(m,n))) = \Big\{(B_1, B_2) \in M_k(\mathbb{C})^{(2)} : (**)\Big\}$$

This map is given by $C \in M_k(\mathbb{C}) \mapsto \Big(\rho(a)C\rho(a)^{-1} - C, \rho(t)C\rho(t)^{-1} - C\Big)$. The image of this map is $B^1(BS(m,n), Ad\circ\rho)$ and so to find a basis for $H^1(BS(m,n), Ad\circ\rho)$ we merely need to find one element of $T_\rho(\mathcal{R}_k(BS(m,n))) = Z^1(BS(m,n), Ad\circ\rho)$ that is not an element of

$B^1(BS(m,n), Ad \circ \rho)$. We find such an example in the vector $\left((0), \begin{pmatrix} I_1 & 0 \\ 0 & 0 \end{pmatrix}\right) \in M_k(\mathbb{C})^{(2)}$. Other such elements can be found in the form $\left((0), \alpha \cdot \begin{pmatrix} I_l & 0 \\ 0 & 0 \end{pmatrix}\right)$, where $l < k$ and $\alpha \in \mathbb{C}^*$.

The above theorem was stated with what is hoped to be the weakest possible set of hypotheses. As we see in the following lemma, the simple representations of Theorem 4.1 satisfy the hypotheses of Theorem 4.2 so those computations do indeed apply to that special type of simple representation.

Lemma 4.4 *Suppose $k, m, n \in \mathbb{N}$, with n odd, $n > m$, $\gcd(m,n) = 1$ and $\mu = 2(n^k - m^k)$. Also, let θ be a primitive μ^{th} root of unity. Moreover, let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be distinct non-zero complex numbers where, for $i = 1, \ldots, k$, $\lambda_i = \theta^{\alpha_i}$, $1 \leq \alpha_i < \mu$. Then $i \neq j$ imples $\lambda_i^n \neq \lambda_j^n$.*

Proof Suppose the statement is not true. Then there exist $i \neq j$ such that $\lambda_i^n = \lambda_j^n$. Without loss of generality, we may assume $\alpha_i > \alpha_j$. We observe

$$1 = \frac{\lambda_i^n}{\lambda_j^n} = \frac{\theta^{n\alpha_i}}{\theta^{n\alpha_j}} = \theta^{n(\alpha_i - \alpha_j)}$$

and recall next that $1 \leq \alpha_i, \alpha_j < \mu$. Thus $\alpha_i - \alpha_j < \mu$. Since θ is a primitive μ^{th} root of unity, we see the following congruence must hold:

$$n(\alpha_i - \alpha_j) \equiv 0 \pmod{\mu}.$$

Now, we use the fact that n odd implies $\gcd(n, \mu) = 1$. This statement was proved in the Lemma used in the proof of Theorem 4.1. Thus, since $\gcd(n, \mu) = 1$, we are guaranteed the existence of $n^{-1} \pmod{\mu}$. Thus,

$$\begin{aligned} n(\alpha_i - \alpha_j) &\equiv 0 && \pmod{\mu} \\ n^{-1} \cdot n(\alpha_i - \alpha_j) &\equiv n^{-1} \cdot 0 && \pmod{\mu} \\ \alpha_i - \alpha_j &\equiv 0 && \pmod{\mu} \\ \alpha_i &\equiv \alpha_j && \pmod{\mu} \end{aligned}$$

This last congruence implies equality between α_i and α_j. However, this contradicts the computations from Theorem 4.1, which observed that $i \neq j$ implies $\alpha_i \neq \alpha_j$. Thus, we have $\lambda_i^n \neq \lambda_j^n$, the desired conclusion. □

Deformations of $BS(m,n)$

Recall from Corollary 4.3 the fact $\dim_{\mathbb{C}}\Big(H^1(BS(m,n), Ad \circ \rho)\Big) = 1$, where ρ is the special type of simple representation of $BS(m,n)$ where $\rho(a) = A_0$ is diagonal with distinct non-zero complex entries and $\rho(t) = T_0$ is the monomial matrix corresponding to the cyclic permutation $(1\,2\,3 \cdots k)$ under conjugation. We now discuss the fact that ρ is a deformable representation.

Note we employ this term in a similar manner as in [AM]. Thus, we can equivalently say if there is a curve $C \subset S_n(\Gamma)$ on which the equivalence class (under conjugation) $[\rho]$ is a point, then we say (the class of) ρ *deforms along curve* C. Our original representation ρ lies on a curve in $S_n(\Gamma)$ defined by $\alpha \longmapsto [\rho_\alpha]$, where $\alpha \in \mathbb{C}^*$, $\rho_\alpha(a) = A_0$ and $\rho_\alpha(t) = \alpha \cdot T_0$. Specifically, ρ lies on the continuous curve which is the closure of the image of this map.

Lemma 4.5 *If* $\alpha \neq \beta \in \mathbb{C}^*$, *then* $[\rho_\alpha] \neq [\rho_\beta]$.

Proof If $\rho_\beta \in [\rho_\alpha]$, then there would exist $B \in GL_k(\mathbb{C})$ such that $B \in C_{GL_k(\mathbb{C})}(A_0)$, the centralizer of A_0, and such that $B(\alpha \cdot T_0)B^{-1} = \beta \cdot T_0$. However, $C_{GL_k(\mathbb{C})}(A_0) = \{\gamma \cdot I_k : \gamma \in \mathbb{C}^*\} \simeq \mathbb{C}^*$ and thus clearly no matrix from $C_{GL_k(\mathbb{C})}(A_0)$ can satisfy the second condition previously stated. Thus, $\rho_\beta \notin [\rho_\alpha]$. i.e., $[\rho_\beta] \neq [\rho_\alpha]$. □

Now that we know ρ is deformable, we can look ahead to seeing a *formal deformation* for it. To that end, we create a curve of simple representations in an alternate but helpful way. Define the curve, C, and thus a *geometric deformation* of ρ, via the map Ψ:

$$\Psi : \mathbb{C}^* \longrightarrow \mathcal{R}_k^s(BS(m,n)) : \alpha \mapsto \rho_\alpha = \begin{cases} a \mapsto A_0 \\ t \mapsto \alpha \cdot T_0 \end{cases}$$

As a result, we recover our original representation at $\alpha = 1$ and note that it acts as our basepoint for this curve of simple representations. More importantly, we may use the result that every simple representation with a geometric deformation has a non-trivial formal deformation. Thus, by definition, there exists a representation $\widehat{\rho}_x : BS(m,n) \longrightarrow GL_k(\mathbb{C}[[x]])$ which is a non-trivial formal deformation of ρ in that $\widehat{\rho}\big|_{x=0} = \rho_1$. This is sometimes denoted $\widehat{\rho}_x \equiv \rho_1 \pmod{x}$. Moreover, for any $\gamma \in BS(m,n)$, we have

$$\begin{aligned} \widehat{\rho}_x(\gamma) &= \rho^{(0)}(\gamma) + \rho^{(1)}(\gamma)x + \rho^{(2)}(\gamma)x^2 + \cdots \\ &= \rho^{(0)}(\gamma)\left[I_k + \rho^{(0)}(\gamma)^{-1}\rho^{(1)}(\gamma)x\right] + \rho^{(2)}(\gamma)x^2 + \cdots \end{aligned} \quad , \qquad (1)$$

where $\rho^{(i)}(\gamma) \in \mathcal{R}_k(BS(m,n))$, for all $i \in \mathbb{Z}_{\geq 0}$. This indicates to us that $\rho^{(0)}(a) = A_1 = \rho_1(a)$ and $\rho^{(0)}(t) = T_1 = \rho_1(t)$, the images of a and t under our original representation.

As an aside, we see from the canonical surjections $\mathbb{C}[[x]] \longrightarrow \mathbb{C}[[x]]/\langle x^i \rangle$ that we have a family of representations

$$\left\{ \widehat{\rho}_i : BS(m,n) \longrightarrow GL_k(\mathbb{C}[[x]]/\langle x^{i+1} \rangle) : i \in \mathbb{Z}_{\geq 0} \right\},$$

and that, for $i > j$, $\widehat{\rho}_i \equiv \widehat{\rho}_j \, (\bmod\, x^{j+1})$. We call $\widehat{\rho}_i$ a *lifting of ρ to level i.*

Now, in order for $\widehat{\rho}_x$ to blend with the image of Ψ, we need the following equations to hold:

$$\begin{cases} \widehat{\rho}_x(a) = A_1 + \mathbf{0}_k x + \mathbf{0}_k x^2 + \cdots \\ \widehat{\rho}_x(t) = T_1 + T_1 x + \mathbf{0}_k x^2 + \cdots \end{cases}$$

Thus, the following equations, arising from (1) above, must hold:

$$\begin{cases} \rho^{(0)}(a)^{-1}\rho^{(1)}(a) = A_1^{-1}\rho^{(1)}(a) = \mathbf{0}_k \\ \rho^{(0)}(t)^{-1}\rho^{(1)}(t) = T_1^{-1}\rho^{(1)}(t) = I_k \end{cases}$$

These comments arise because in order to calculate the tangential representation, we consider the lifting of ρ to level 1. As a result of the equations above, we see the matrix pair $(\mathbf{0}_k, I_k)$ is the tangential representation corresponding to the representation ρ. In fact, this tangential representation is what truly indicates to us that our geometric deformation Ψ relies on perturbing the image of the group generator t, and is little related to the image of generator a.

As a result, we see that our geometric deformation that perturbs the image of t with elements in $\mathbb{C}^*$ accounts for our computation of $\dim_{\mathbb{C}}\Big(H^1(BS(m,n), Ad\circ\rho)\Big) = 1$. This result is important because it gives us the basis for some conclusions regarding the dimensions of $\mathcal{R}_k(BS(m,n))$ (the representation *scheme*), $R_k(BS(m,n))$ (the representation *variety*) and $S_k(BS(m,n))$, (the classes of simple representations).

We note that since our representation ρ is simple, the dimension of its orbit is $k^2 - 1$. Thus, we have $\dim_\rho\Big(R_k(BS(m,n))\Big) \geq k^2 - 1$. However, since ρ has a non-trivial deformation, we have $\dim_\rho\Big(R_k(BS(m,n))\Big) \geq (k^2-1)+1 = k^2$. We now recall the following inequalities from Chapter 3:

$$\dim_\rho\Big(\mathcal{R}_k(\Gamma)\Big) \leq \dim_{\mathbb{C}}\Big(T_\rho(R_k(\Gamma))\Big) \leq \dim_{\mathbb{C}}\Big(T_\rho(\mathcal{R}_k(\Gamma))\Big) = \dim_{\mathbb{C}}\Big(Z^1(\Gamma, Ad\circ\rho)\Big)$$

As a result, we see that all four of these values are equal to k^2, thus assuring us of the following corollaries:

Corollary 4.6 *If $k, m, n \in \mathbb{N}$ with $\gcd(m,n) = 1$, and ρ is a simple k-dimensional representation of $BS(m,n)$ where $\rho(a)$ is diagonal with distinct non-zero complex entries and $\rho(t)$ cyclically permutes those diagonal elements by conjugation, then*

$$\dim_\rho\Big(\mathcal{R}_k(BS(m,n))\Big) = \dim_\rho\Big(R_k(BS(m,n))\Big) = k^2.$$

Moreover, this implies

$$\dim_{[\rho]}\Big(S_k(BS(m,n))\Big) = 1.$$

Corollary 4.7 *If* $k, m, n \in \mathbb{N}$ *with* $\gcd(m, n) = 1$, *and* ρ *is a simple* k*-dimensional representation of* $BS(m, n)$ *where* $\rho(a)$ *is diagonal with distinct non-zero complex entries and* $\rho(t)$ *cyclically permutes those diagonal elements by conjugation, then both* $\mathcal{R}_k(BS(m, n))$ *and* $S_k(BS(m, n))$ *are non-singular at* ρ.

Corollary 4.8 *If* $k, m, n \in \mathbb{N}$ *with* $\gcd(m, n) = 1$, *then the scheme,* $\mathcal{R}_k(BS(m, n))$, *is reduced at any simple representation,* ρ, *of* $BS(m, n)$ *where* $\rho(a)$ *is diagonal with distinct non-zero complex entries and* $\rho(t)$ *cyclically permutes those diagonal elements by conjugation.*

This last corollary is true because if ρ is non-singular on $\mathcal{R}_k(BS(m, n))$, then the local ring of ρ on $\mathcal{R}_k(BS(m, n))$ has no nilpotents.

Chapter 5 — Computational Examples

In this chapter, examples will be given in which we compute (dimension of) the coordinate ring, $\mathcal{A}_k(\Gamma)$, for several different examples of groups Γ and for several different values of k. All of the dimensional computations were attempted using the computer algebra system *CoCoA*, which can be found and downloaded free from the website

```
http://cocoa.dima.unige.it/
```

The CAS *Macaulay 2* was also used for several examples. However, it should be noted that the systems performed almost equally. The examples *CoCoA* was successful at computing were the same as those *Macaulay 2* was capable of computing. Moreover, both *CoCoA* and *Macaulay 2* were unsuccessful on the same set of examples. We also note here that the matrix algebra computations were performed in the *Mathematica©*. Those computations will be omitted. In their stead will be exposition on some of the different aspects of computing some of the individual examples.

Example 5.1 We first show a successful computation involving $F_2 = \langle x, y : - \rangle$. We calculate $\dim\Big(\mathcal{A}_3(F_2)\Big)$. This is perhaps the easiest example to compute, as there are no relators in the presentation of the group. We accomplish this by using matrices

$$\rho(x) = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \quad \text{and} \quad \rho(y) = \begin{pmatrix} j & k & l \\ m & n & o \\ p & q & r \end{pmatrix}$$

and inverting the determinants of those matrices by using the polynomials

$$s\Big(\det\big(\rho(x)\big)\Big) - 1\,.$$
$$t\Big(\det\big(\rho(y)\big)\Big) - 1$$

As a result, the ideal we mod out involves only the polynomials above. Now, we see the *CoCoA* output from the computation:

```
--------------------------------
-- The current ring is R ::= Q[x,y,z];
--------------------------------
Use
R::=Q[a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t];

A:=s(a(ei-fh)-b(di-fg)+c(dh-ge))-1;
B:=t(j(nr-oq)-k(mr-op)+l(mq-np))-1;

I:=Ideal(A,B);

Dim(R/I);
18
--------------------------------
Time T:=Dim(R/I);
Cpu time = 0.60, User time = 0
```

Example 5.2 Our second example will be a successful computation involving the group $\Gamma = \mathbb{Z}^2 = \langle x, y : xyx^{-1}y^{-1}\rangle$. What we note in this example is the relator: we will strictly adhere to the constructive method of the proof of Proposition 3.3 to compute $\dim\Big(\mathcal{A}_2(\mathbb{Z}^2)\Big)$. Specifically, we use the relator explicitly in the form $xyx^{-1}y^{-1}$. The next example will take a different tack.

As before, we use matrices

$$\rho(x) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad \rho(y) = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

and invert the determinants of those matrices by using the polynomials

$$s\Big(\det(\rho(x))\Big) - 1 .$$
$$t\Big(\det(\rho(y))\Big) - 1$$

```
-------------------------------
-- The current ring is R ::= Q[x,y,z];
-------------------------------
Use R::=Q[a,b,c,d,e,f,g,h,s,t];

A:=s(ad-bc)-1;
B:=t(eh-fg)-1;

C:=st((dh+bg)(ae+bg)-(df+be)(ce+dg))-1;
D:=st((dh+bg)(af+bh)-(df+be)(cf+dh));
E:=st((cf+ae)(ce+dg)-(ch+ag)(ae+bg));
F:=st((cf+ae)(cf+dh)-(ch+ag)(af+bh))-1;

I:=Ideal(A,B,C,D,E,F);

Dim(R/I);
6
-------------------------------
Time T:=Dim(R/I);
Cpu time = 0.60, User time = 0
```

Example 5.3 Here is the same successful $\dim\Big(\mathcal{A}_2(\mathbb{Z}^2)\Big) = 6$ computation as in the previous example. However, the polynomials involved are a bit simpler. One realizes that instead of being required to invert matrices because of the relator $xyx^{-1}y^{-1}$, one can use the relator in the form $xy = yx$. This is obviously not possible for any given relator which involves inverses, but these computations are generally more "user-friendly" when the relator can be rewritten without any inverses.

We now see the (equivalent) computations leading to the same result above, that $\dim\Big(\mathcal{A}_2(\mathbb{Z}^2)\Big) = 6$:

```
--------------------------------
-- The current ring is R ::= Q[x,y,z];
--------------------------------
Use R::=Q[a,b,c,d,e,f,g,h,s,t];

A:=s(ad-bc)-1;
B:=t(eh-fg)-1;

C:=bg-cf;
D:=af-be-df+bh;
E:=ce-ag+dg-ch;
F:=cf-bg;

I:=Ideal(A,B,C,D,E,F);

Dim(R/I);
6
--------------------------------
Time T:=Dim(R/I);
Cpu time = 0.60, User time = 0
```

Example 5.4 Now, we employ the same method as in Example 5.3 to compute $\dim\Big(\mathcal{A}_3(\mathbb{Z}^2)\Big) = 12$. We view the relator as the equation $xy = yx$. As usual, we use matrices

$$\rho(x) = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \quad \text{and} \quad \rho(y) = \begin{pmatrix} j & k & l \\ m & n & o \\ p & q & r \end{pmatrix}$$

and invert the determinants of those matrices by using the polynomials

$$s\Big(\det(\rho(x))\Big) - 1\,.$$
$$t\Big(\det(\rho(y))\Big) - 1$$

We obtain:

```
-------------------------------
The current ring is R ::= Q[x,y,z];
-------------------------------
Use R::=Q[a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t];

A:=s(-ceg+bfg+cdh-afh-bdi+aei)-1;
B:=t(-lnp+kop+lmq-joq-kmr+jnr)-1;

C:=-dk-gl+bm+cp;
D:=-bj+ak-ek-hl+bn+cq;
E:=-cj-fk+al-il+bo+cr;

F:=dj-am+em-dn-go+fp;
G:=dk-bm-ho+fq;
H:=dl-cm-fn+eo-io+fr;

I:=gj+hm-ap+ip-dq-gr;
J:=gk+hn-bp-eq+iq-hr;
K:=gl+ho-cp-fq;

L:=Ideal(A,B,C,D,E,F,G,H,I,J,K);

Dim(R/L);
12
-------------------------------
Time T:=Dim(R/L);
Cpu time = 131.30, User time = 14
```

Example 5.5 Now we compute some examples with a few Baumslag-Solitar groups. Our main group will, of course, be $BS(2,3) = \langle a, t : ta^2 = a^3t \rangle$. We will see very quickly that the polynomials involved in computing $\dim\Big(\mathcal{A}_k(BS(2,3))\Big)$ will have increasingly many terms as k becomes large, despite using the "nicer" version of the relator. Here, we compute $\dim\Big(\mathcal{A}_2(BS(2,3))\Big)$.

```
-------------------------------
-- The current ring is R ::= Q[x,y,z];
-------------------------------
Use R::=Q[a,b,c,d,e,f,g,h,s,t];

A:=s(ad-bc)-1;
B:=t(eh-fg)-1;

C:=a(e^2-e^3-2efg-fg(-1+h))+bg(e+h)-cf(e^2+fg+eh+h^2);
D:=f(ae+bg)+h(af+bh)-b(e^3+2efg+fgh)-df(e^2+fg+eh+h^2);
E:=e(ce+dg)+g(cf+dh)-ag(e^2+fg+eh+h^2)-c(efg+2fgh+h^3);
F:=cf(e+h)-bg(e^2+fg+eh+h^2)+d(fg-efg-2fgh+h^2-h^3);

I:=Ideal(A,B,C,D,E,F);

Dim(R/I);
4
-------------------------------
Time T:=Dim(R/I);
Cpu time = 10.40, User time = 1
```

Example 5.6a In this example, we note the computation of $\dim\Big(\mathcal{A}_3(BS(2,3))\Big)$, in full generality, was unsuccessful. As in the other examples, we used matrices

$$\rho(a) = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \quad \text{and} \quad \rho(t) = \begin{pmatrix} j & k & l \\ m & n & o \\ p & q & r \end{pmatrix}$$

However, the polynomials involved in this computation become exceedingly long, and thus *CoCoA* was unable to complete the computation. Moreover, an attempt to use *Macaulay 2* was also unsuccessful. For completeness, we show the *CoCoA* input:

```
-------------------------------
-- The current ring is R ::= Q[x,y,z];
-------------------------------
Use R::=Q[a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t];

A:=s(-ceg+bfg+cdh-afh-bdi+aei)-1;
B:=t(-lnp+kop+lmq-joq-kmr+jnr)-1;

C:=-(a(a^2+bd+cg)+d(ab+be+ch)+g(ac+bf+ci))j+a(aj+dk+gl)+
d(bj+ek+hl)+g(cj+fk+il)-(b(a^2+bd+cg)+e(ab+be+ch)+
h(ac+bf+ci))m-(c(a^2+bd+cg)+f(ab+be+ch)+i(ac+bf+ci))p;

D:=-(a(a^2+bd+cg)+d(ab+be+ch)+g(ac+bf+ci))k+b(aj+dk+gl)+
e(bj+ek+hl)+h(cj+fk+il)-(b(a^2+bd+cg)+e(ab+be+ch)+
h(ac+bf+ci))n-(c(a^2+bd+cg)+f(ab+be+ch)+i(ac+bf+ci))q;

E:=-(a(a^2+bd+cg)+d(ab+be+ch)+g(ac+bf+ci))l+c(aj+dk+gl)+
f(bj+ek+hl)+i(cj+fk+il)-(b(a^2+bd+cg)+e(ab+be+ch)+
h(ac+bf+ci))o-(c(a^2+bd+cg)+f(ab+be+ch)+i(ac+bf+ci))r;

F:=-(a(ad+de+fg)+d(bd+e^2+fh)+g(cd+ef+fi))j-(b(ad+de+fg)+
e(bd+e^2+fh)+h(cd+ef+fi))m+a(am+dn+go)+d(bm+en+ho)+
g(cm+fn+io)-(c(ad+de+fg)+f(bd+e^2+fh)+i(cd+ef+fi))p;

G:=-(a(ad+de+fg)+d(bd+e^2+fh)+g(cd+ef+fi))k-(b(ad+de+fg)+
e(bd+e^2+fh)+h(cd+ef+fi))n+b(am+dn+go)+e(bm+en+ho)+
h(cm+fn+io)-(c(ad+de+fg)+f(bd+e^2+fh)+i(cd+ef+fi))q;

H:=-(a(ad+de+fg)+d(bd+e^2+fh)+g(cd+ef+fi))l-(b(ad+de+fg)+
e(bd+e^2+fh)+h(cd+ef+fi))o+c(am+dn+go)+f(bm+en+ho)+
i(cm+fn+io)-(c(ad+de+fg)+f(bd+e^2+fh)+i(cd+ef+fi))r;

I:=-(a(ag+dh+gi)+d(bg+eh+hi)+g(cg+fh+i^2))j-(b(ag+dh+gi)+
e(bg+eh+hi)+h(cg+fh+i^2))m-(c(ag+dh+gi)+f(bg+eh+hi)+
i(cg+fh+i^2))p+a(ap+dq+gr)+d(bp+eq+hr)+g(cp+fq+ir);

J:=-(a(ag+dh+gi)+d(bg+eh+hi)+g(cg+fh+i^2))k-(b(ag+dh+gi)+
e(bg+eh+hi)+h(cg+fh+i^2))n-(c(ag+dh+gi)+f(bg+eh+hi)+
i(cg+fh+i^2))q+b(ap+dq+gr)+e(bp+eq+hr)+h(cp+fq+ir);

K:=-(a(ag+dh+gi)+d(bg+eh+hi)+g(cg+fh+i^2))l-(b(ag+dh+gi)+
e(bg+eh+hi)+h(cg+fh+i^2))o-(c(ag+dh+gi)+f(bg+eh+hi)+
i(cg+fh+i^2))r+c(ap+dq+gr)+f(bp+eq+hr)+i(cp+fq+ir);

M:=Ideal(A,B,C,D,E,F,G,H,I,J,K);

Dim(R/M);
```

Example 5.6b In the second part of this example, we attempt to compute $\dim\left(\mathcal{A}_3(BS(2,3))\right)$ in a special case. We restrict our computation to the case where the image of generator a is diagonal. In other words, the matrices we use in this example are

$$\rho(a) = \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{pmatrix} \quad \text{and} \quad \rho(t) = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

Thus, we obtain

```
--------------------------------
-- The current ring is R ::= Q[x,y,z];
--------------------------------
Use R::=Q[a,b,c,d,e,f,g,h,i,x,y,z,s,t];

A:=s(xyz)-1;
B:=t(bfg-ceg+cdh-afh-bdi+aei)-1;

C:=-ax^2(x-1);
D:=b(y^2-x^3);
E:=c(z^2-x^3);

F:=d(x^2-y^3);
G:=-ey^2(y-1);
H:=f(z^2-y^3);

I:=g(x^2-z^3);
J:=h(y^2-z^3);
K:=-iz^2(z-1);

L:=Ideal(A,B,C,D,E,F,G,H,I,J,K);

Dim(R/L);
9
--------------------------------
Time T:=Dim(R/L);
Cpu time = 37.90, User time = 4
```

Example 5.7 Our next-to-last example is of an *un-successful* computation of $\dim\left(\mathcal{A}_4(BS(2,3))\right)$. We learn our lesson from the above examples and again perform this computation in the case where the image of generator a is diagonal. However, this computation failed, possibly due to the number of polynomials involved, regardless of their numbers of terms.

```
-------------------------------
-- The current ring is R ::= Q[x,y,z];
-------------------------------
Use W::=Q[a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,x,y,z,w,q,r];

A:=q(xyzw)-1;
B:=r(dgjm-chjm-dfkm+bhkm+cflm-bglm-dgin+chin+dekn-ahkn-
celn+agln+dfio-bhio-dejo+ahjo+belo-aflo-cfip+bgip+cejp-
agjp-bekp+afkp)-1;

C:=ax^2(1-x);
D:=b(y^2-x^3);
E:=c(z^2-x^3);
F:=d(w^2-x^3);

G:=e(x^2-y^3);
H:=fy^2(1-y);
I:=g(z^2-y^3);
J:=h(w^2-y^3);

K:=i(x^2-z^3);
L:=j(y^2-z^3);
M:=kz^2(1-z);
N:=l(w^2-z^3);

O:=m(x^2-w^3);
P:=n(y^2-w^3);
R:=o(z^2-w^3);
S:=pw^2(1-w);

T:=Ideal(A,B,C,D,E,F,G,H,I,J,K,L,M,N,O,P,R,S);

Dim(W/T);
Time U:=Dim(W/T);
```

Example 5.8 Our last example is to attempt to compute $\dim\Big(\mathcal{A}_2\big(BS(3,5)\big)\Big)$, where $BS(3,5) = \langle a, t : ta^3 = a^5 t \rangle$. Despite the fact that we are looking at two-dimensional representations, we see that the computations involve some pretty large polynomials:

```
--------------------------------
-- The current ring is R ::= Q[x,y,z];
--------------------------------

Use R::=Q[a,b,c,d,e,f,g,h,s,t];

A:=s(ad-bc)-1;
B:=t(eh-fg)-1;

C:=bg(e^2+fg+eh+h^2)-cf(e^4+3e^2fg+f^2g^2+e^3h+4efgh+
e^2h^2+3fgh^2+eh^3+h^4)-a(e^5+e^3(-1+4fg)+3e^2fgh+
fgh(-1+2fg+h^2)+efg(-2+3fg+2h^2));

D:=-b(e^5+4e^3fg+3e^2fgh+efg(-1+3fg+2h^2)+h(-
2fg+2f^2g^2-h^2+fgh^2))-
f(a(e^2+fg+eh+h^2)+d(e^4+3e^2fg+f^2g^2+e^3h+
4efgh+e^2h^2+3fgh^2+eh^3+h^4));

E:=-c(e^3(-1+fg)+2e^2fgh+efg(-2+2fg+3h^2)+h(-
fg+3f^2g^2+4fgh^2+h^4))-g(-
d(e^2+fg+eh+h^2)+a(e^4+3e^2fg+f^2g^2+e^3h+
4efgh+e^2h^2+3fgh^2+eh^3+h^4));

F:=cf(e^2+fg+eh+h^2)-bg(e^4+3e^2fg+f^2g^2+e^3h+4efgh+
e^2h^2+3fgh^2+eh^3+h^4)-d(-efg+e^3fg+2ef^2g^2-2fgh+
2e^2fgh+3f^2g^2h+3efgh^2-h^3+4fgh^3+h^5);

I:=Ideal(A,B,C,D,E,F);

Dim(R/I);
4
--------------------------------
Time T:=Dim(R/I);
Cpu time = 417.50, User time = 42
```

What we see from these last several examples is that there is a fine line dividing the collection of successful computations from the unsuccessful ones. Moreover, there is no clear-cut culprit as to whether the number of polynomials or the size of the polynomials is at fault in the unsuccessful cases.

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