A COMBINATION OF MATRIX AND LAPLACE TRANSFORM METHODS IN THE MATHEMATICAL THEORY OF AIRCRAFT FLUTTER

A COMBINATION OF MATRIX AND LAPLACE TRANSFORM METHODS IN THE MATHEMATICAL THEORY OF AIRCRAFT FLUTTER

By

CROSMAN JAY CLARK

Bachelor of Arts

Oklahoma Agricultural and Mechanical College

Stillwater, Oklahoma

1946

Submitted to the Department of Mathematics

Oklahoma Agricultural and Mechanical College

In Partial Fulfillment of the Requirements

for the Degree of

MASTER OF SCIENCE

1948

APPROVED BY:

E. F. Allen
Chairman, Thesis Committee

O. H. Hamilton
Member of the Thesis Committee

Ainsley H Diamond
Head of the Department

D. C. McIntosh
Dean of the Graduate School

PREFACE

One of the greatest practical problems in the design of modern aircraft is the phenomenon of flutter. Test flights may reveal flutter in wing, aileron, fuselage or tail assembly unless it is carefully considered at the time of preliminary design. The purpose of this paper is to present the fundamental nature of, and the basic methods of, handling the analysis of this problem of flutter in preliminary design.

This paper will also include a method of solving systems of equations that arise in the analysis by a combination of matrix and Laplace transform methods.

References to the bibliography throughout the text are indicated by bracketed numbers followed by the page numbers.

I wish to express my gratitude to Mr. and Mrs. L. J. Fila lately associated with the Lockheed and Glenn L. Martin aircraft companies and Professor E. F. Allen of the Mathematics Department for their aid in the final preparation of this paper.

January, 1948 C. J. C.

Stillwater, Oklahoma

TABLE OF CONTENTS

INTRODUCTION

In view of the recent interest and development in the design of high speed aircraft, it seems proper to bring out the general methods employed in dealing with one of the most important design problems involved, that of aircraft flutter. The vibrations occurring in flutter phenomena can often lead to loss of control or to structural failure in such aircraft parts as wing, aileron, fuselage, and tail. The increasing size and cost of aircraft, the danger of actual flight testing, and the difficulty with which idealized wind tunnel experiments are carried on makes it imperative that there be developed analytical methods whereby the flutter characteristics can be accurately predicted. All of the theoretical computations and experimental measurements on aircraft vibration have the one basic objective in mind, and that is to establish the maximum safe air speed of the aircraft.

For convenience, this paper will deal for the most part with the specialized problem of flutter in the wing or wing-aileron structure. Such a problem is also discussed in [5, pp. 220-228]. It is known that when such a structure is restrained to an initial position of equilibrium, it may become unstable under certain conditions of motion. The theory of small oscillations is, in general, an approximate theory of the motion of a mechanical system in the neighborhood of an equilibrium position and therefore is used in the analysis of flutter. We shall consider the wing and aileron as our mechanical system. The wing is considered as an elastic structure clamped to the airplane fuselage which is considered as a rigid base. The wing is then taken to have two degrees of freedom, corresponding to bending and twisting and a third degree of freedom representing the relative deflection of the aileron. Still more recent developments in aircraft design require an addition of a fourth degree of freedom, that of relative tab deflection. However, in our present discussion, we shall consider the system to have only two degrees, namely the angular deflection q^1 of the wing corresponding to twisting, and the angular deflection q^2 of the aileron relative to the direction of flight.

There are three kinds of forces affecting such a system: (1) the inertia forces, (2) the restraining forces, and (3) the aerodynamic forces. The aerodynamic forces are defined to be the forces due to the air pressure acting on the wing or wing-aileron and are functions of the air speed. Since the critical speed for flutter is defined to be the lowest forward speed of the aeroplane for which free oscillations are steady, we shall employ Lagrange's equations of motion for a system performing free oscillations. These equations contain statements regarding the equilibrium of the above mentioned forces.

The mechanical system is treated as non-conservative and this non-conservative nature arises in two distinct ways. First, where mechanical energy is actually transformed into heat, that is, where there is a damping effect and energy is dissipated, and the other is due to the method of analysis alone. For example we chose our system to consist of the wing and aileron alone and not the surrounding air, it is non-conservative in the sense that it can absorb energy from or lose energy to the nearby air. The Lagrangian equations of motion of a general non-conservative system of n degrees of freedom performing free oscillations can be written in the form

$$(1) \quad a_{ij} \frac{d^2 q^j(t)}{dt^2} + b_{ij} \frac{d q^j(t)}{dt} + c_{ij} q^j(t) = 0$$

where j is a dummy summation index and both i and j have the range 1 to n. The coefficients a_{ij}, b_{ij}, c_{ij} are computed from a large number of aerodynamic constants and parameters of the aircraft structure. Some of these aerodynamic factors are wing density, location of stiffness axis (elastic axis) of the wing, location of centers of gravity of the wing and aileron, aileron length, and chord length of wing. There are various methods of solving equations of the form (1), and at this point I shall give the matrix-Laplace transform method.

MATRIX-LAPLACE TRANSFORM METHOD

Define A, B, C to be square matrices composed of elements a_{ij}, b_{ij}, c_{ij} respectively, and q, $\dot{q}$, $\ddot{q}$ to be column matrices comprising the elements q^i, $\frac{dq^i}{dt}$, $\frac{d^2 q^i}{dt^2}$ respectively. Then the n equations (1) can be written as one matric differential equation

(2) $A\ddot{q} + B\dot{q} + Cq = 0$

Since A contains the inertial properties and C contains the stiffness properties of the structure, matrices A and C are referred to as the inertial and stiffness matrices respectively. The terms involving the velocities are due to the damping forces, and therefore B is properly called the damping matrix.

The Laplace transform of a function $q^i(t)$ is a function $\bar{q}^i(p) = \int_0^\infty e^{-pt} q^i(t)\,dt$ and we shall let $\bar{q}$ denote a column matrix with elements $\bar{q}^i$.

Equation (2) is equivalent to

(3) $\ddot{q} + A^{-1} B\dot{q} + A^{-1} Cq = 0$

after premultiplying by A^{-1}. Since A arises from the kinetic energy, A^{-1} exists. If we now take the Laplace transform of each term we obtain symbolically the equation

$$\bar{\ddot{q}} + A^{-1} B\bar{\dot{q}} + A^{-1} C\bar{q} = 0$$

and upon performing the indicated operation,

$$-(q_1 + pq_0) + p^2 \bar{q} - A^{-1} Bq_0 + pA^{-1} B\bar{q} + A^{-1} C\bar{q} = 0.$$

Here p is a scalar and q_0, q_1 are composed of the elements $q(0)$ and $\dot{q}(0)$ respectively, where $q(0) = 0$ in our case, since $q^i(0)$ is a stable equilibrium point. A rearrangement gives

(4) $(p^2 + I + pA^{-1} B + A^{-1} C)\bar{q} = q_1$

where I is a unit matrix. We may obtain from this

(5) $\bar{q} = (p^2 I + pA^{-1} B + A^{-1} C)^{-1} q_1.$

After the matrix multiplication of the right hand member of equation (4), one may obtain from the resulting column matrix the subsidiary equations and from these using a table of Laplace transforms obtain the solutions in the form

$$(6) \qquad q = \sum_{1}^{n} a^r \sin(w_r t + \theta_r)$$

where a^r is a column matrix of constants.

A simple numerical example will suffice to demonstrate the machinery of the computations. Consider a system to be represented by the following differential equations of motion.

$$(7) \qquad 3\frac{d^2 q^1}{dt^2} + 2q^1 + 5q^2 = 0$$

$$4\frac{d^2 q^2}{dt^2} + q^1 + 4q^2 = 0$$

where $q^1(0) = q^2(0) = \frac{d q^2}{dt} = 0$

and $\frac{d q^1}{dt} = r$

Equations(7) can be written in the matrix form

$$\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} \begin{Bmatrix} \ddot{q}^1 \\ \ddot{q}^2 \end{Bmatrix} + \begin{bmatrix} 2 & 5 \\ 1 & 4 \end{bmatrix} \begin{Bmatrix} q^1 \\ q^2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Thus

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 5 \\ 1 & 4 \end{bmatrix}$$

$$A^{-1} C = \begin{bmatrix} 2/3 & 5/3 \\ 1/4 & 1 \end{bmatrix}, \quad (p^2 I + A^{-1} C) = \begin{bmatrix} p^2 + 2/3 & 5/3 \\ 1/4 & p^2 + 1 \end{bmatrix}$$

$$(p^2 I + A^{-1} C)^{-1} = \frac{1}{(p^2 + 2/3)(p^2 + 1) - 5/12} \begin{bmatrix} p^2 + 1 & -5/3 \\ -1/4 & p^2 + 2/3 \end{bmatrix}$$

Now from the initial conditions $q_1 = \begin{pmatrix} r \\ 0 \end{pmatrix}$

$$\therefore \bar{q} = \frac{12r}{(6p^2 + 1)(2p^2 + 3)} \begin{Bmatrix} (p^2 + 1) \\ -1/4 \end{Bmatrix}$$

$$\bar{q}^1 = \frac{12r\,(p^2 + 1)}{(6p^2 + 1)(2p^2 + 3)} = \frac{3r}{4} \left(\frac{1}{2p^2 + 3} + \frac{5}{6p^2 + 1} \right)$$

$$\bar{q}^2 = -\frac{3r}{(6p^2 + 1)(2p^2 + 3)} = \frac{3r}{8} \left(\frac{1}{2p^2 + 3} - \frac{3}{6p^2 + 1} \right)$$

Using a table of Laplace transforms we obtain the solutions

$$q^1 = \frac{3r}{4\sqrt{6}} \left(\sin\sqrt{3/2}\; t + 5 \sin \frac{1}{\sqrt{6}}\; t\right) \tag{8}$$

$$q^2 = \frac{3r}{8\sqrt{6}} \left(\sin\sqrt{3/2}\; t - 3 \sin \frac{1}{\sqrt{6}}\; t\right)$$

STABILITY CONDITIONS

The determinant of the matrix that premultiplies $\overline{q}$ in equation (4) will be observed to be identical with the determinatal or frequency equation[1] of the classical method of solving equations (1). Let us examine for a moment this frequency equation $\left| w^2 I + w A^{-1} B + A^{-1} C \right| = 0$. It is in general of degree 2n, where n is the number of degrees of freedom. The roots w are in general complex and occur in conjugate complex pairs, so the solution will be of the form

$$(9) \quad q = \sum_{1}^{2n} a^r e^{w_r t} \qquad \text{where } w_r = c_r + i d_r$$

Because of conjugate complex pairs, the solution may be written in the form

$$q = \sum_{1}^{n} a^r e^{w_r t} + \sum_{1}^{n} \overline{a}^r e^{\overline{w}_r t}$$ where the bar represents the conjugate

$$(10) \quad q = \sum_{1}^{n} b^r e^{c_r t} \cos (d_r t + \phi_r) \qquad b^r = 2e^{i\phi_r} a^r$$

If any of the c_r are positive, we obtain at least one mode of oscillation with increasing amplitude and the system is unstable.

A method by which one could obtain stability conditions or determine if a system is stable is the application of Routh's tests to the determinantal equation, since a necessary and sufficient condition for the real parts of all the roots to be negative is that all the test functions shall be positive.[2]

Consider now the differential equations of motion of the simplified flutter problem that was discussed in the introduction. The damping forces will be

---

[1] [7, p. 204]

[2] E. J. Routh, Advanced Rigid Dynamics, London 6th ed. (1905) pp. 297-301

Fraser, Duncan, "On the Criteria for the Stability of Small Motions," Proc. Roy. Soc. Series A, Vol. 124, p. 642 (1929) neglected[3] since the coefficients of the linear functions of the velocities are complicated functions of the frequency and their inclusion does not bring any new aspect into the analysis. Using q^1 and q^2 as the generalized coordinates and the generalized forces in this case as moments, we obtain the equations

(11)
$$a_{11}\frac{d^2 q^1}{dt^2} + a_{12}\frac{d^2 q^2}{dt^2} + k_{11} q^1 + k_{12} q^2 = 0$$

$$a_{12}\frac{d^2 q^1}{dt^2} + a_{22}\frac{d^2 q^2}{dt^2} + k_{21} q^1 + k_{12} q^2 = 0$$

The coefficients[4] are given by

$$a_{11} = I_1 + m_2 a^2$$
$$a_{12} = m_2 a s_2$$
$$a_{22} = m_2 i_2^2$$

where

I_1 = moment of inertia of the wing about a fixed axis (elastic axis)

m_2 = mass of aileron

a = distance between elastic axis of wing and hinge axis of aileron

i_2 = radius of gyration of aileron

s_2 = distance of center of gravity from hinge axis

We shall assume no dynamic coupling between the coordinates q^1 and q^2 and therefore $s_2 = 0$. For an analysis of coupled modes see [2]. The k_{ij} are functions of the air speed as they result from aerodynamic forces.

Writing equations(11) in matrix form with $a_{12} = 0$

(12)
$$\begin{bmatrix} a_{11} & 0 \\ 0 & a_{12} \end{bmatrix} \begin{pmatrix} \ddot{q}^1 \\ \ddot{q}^2 \end{pmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{pmatrix} q^1 \\ q^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

---

3 The system, however, is still non-conservative. See p. 2

4 See discussion of coefficients on p. 2

From equation (12) we can write equation (4) for our specialized problem as

$$\begin{bmatrix} p^2 + \frac{k_{11}}{a_{11}} & \frac{k_{12}}{a_{11}} \\ \frac{k_{21}}{a_{22}} & p^2 + \frac{k_{22}}{a_{22}} \end{bmatrix} \begin{pmatrix} \bar{q}^1 \\ \\ \bar{q}^2 \end{pmatrix} = \begin{pmatrix} r_1 \\ \\ 0 \end{pmatrix}$$

for convenience assume $\dot{q}^2(0) = 0$ and $\dot{q}^1(0) = r_1$

$$\begin{pmatrix} \bar{q}^1 \\ \\ \bar{q}^2 \end{pmatrix} = \frac{r_1}{f(p)} \begin{pmatrix} p^2 + \frac{k_{22}}{a_{22}} \\ \\ - \frac{k_{21}}{a_{22}} \end{pmatrix}$$

where $f(p) = p^4 + p^2 \left(\frac{k_{11}}{a_{11}} + \frac{k_{22}}{a_{22}} \right) + \frac{k_{11}\, k_{22}}{a_{11}\, a_{22}} - \frac{k_{12}\, k_{21}}{a_{11}\, a_{22}}$

$$f(p) = (p^2 + g)(p^2 + h)$$

(13) for g and h $= \frac{(w_{11} + w_{22})2}{2} \pm \sqrt{\frac{(w_{11}^{\,2} + w_{22}^{\,2})^2}{4} - w_{11}^{\,2}\, w_{22}^{\,2} + \frac{k_{12}\, k_{21}}{a_{11}\, a_{22}}}$

where $w_{11}^{\,2} = \frac{k_{11}}{a_{11}}$ $\quad w_{22}^{\,2} = \frac{k_{22}}{a_{22}}$

The solution will contain a linear combination of the $\sin \sqrt{h}$ and $\sin \sqrt{g}$. The inverse transform requires $\sqrt{h}$ and $\sqrt{g}$ to be real and this will be true upon examining equations (13) when the following conditions are satisfied

(14) $\quad (w_{11}^{\,2} - w_{22}^{\,2})^2 + 4 \frac{k_{12}\, k_{21}}{a_{11}\, a_{22}} > 0$

$$\frac{w_{11}^{\,2} + w_{22}^{\,2}}{2} > \sqrt{\frac{(w_{11}^{\,2} + w_{22}^{\,2})^2}{4} - w_{11}^{\,2}\, w_{22}^{\,2} + \frac{k_{12}\, k_{21}}{a_{11}\, a_{22}}}$$

These same conditions are obtained by the expansion of the determinantal equation, which is essentially what was done above. For a discussion of that method and the physical meaning of these stability conditions see [5, pp. 225-227].

As the number of degrees of freedom is increased, the expansion of the determinatal equation is increasingly more difficult. Therefore a graphical method of determining the critical speeds is used. For any speed below this critical value the oscillations resulting from any given initial disturbance eventually die away; at the actual critical speed the motion tends to become simply sinusoidal; while for all speeds over a certain range whose lower limit is the critical speed, oscillations occur which increase to an indefinitely large amplitude however small the initial disturbance may be.

From the above it is clear that at the critical speed for flutter the determinantal equation will have at least one pair of conjugate pure imaginary roots. These are sometimes referred to as critical roots. The diagramatic method by which the flutter speed of an aeroplane can be represented consists of showing the variation of the damping factor c with the air speed for the several constituents of the motion. A few graphs[5] will illustrate this procedure.

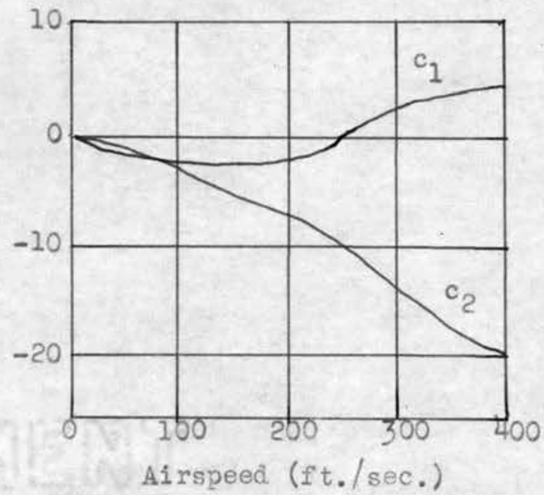


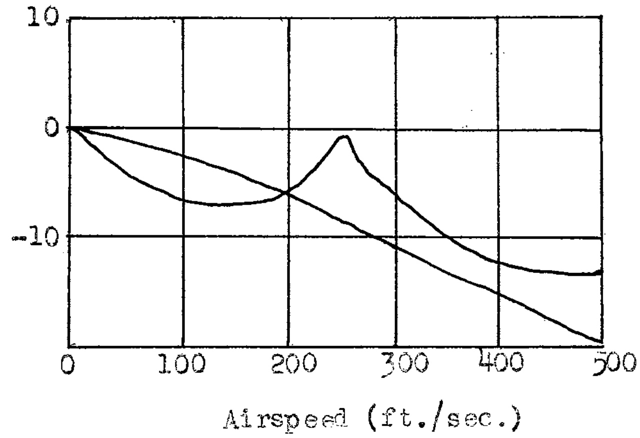
Fig. 1

The curve c_1 of Fig. 1 represents the damping factor of one oscillatory constituent which is damped for all speeds less than 240 ft/sec. and grows indefinitely large for all speeds greater than this value, while c_2 of the second constituent remains damped for all speeds.

---

[5] These graphs were obtained from [4, p.358]

Fig. 2



Both of the constituents of the oscillation in Fig. 2 remain damped throughout the range covered by the diagram, however there is a very small margin of stability in the vicinity of 250 ft/sec.

MATRIX ITERATION METHOD

The method by which one can obtain the frequencies of the several constituent motions depends on the complexity of the problem at hand. For a small number of degrees of freedom[6] (3 or less) the Laplace transform-matrix method may be employed. There is an approximate numerical method called the matrix iteration method which handles large numbers of degrees of freedom and also is useful in taking care of corrections due to our initial assumptions. A brief outline of this method will be given.

We shall first replace the one second-order matric differential equation (3) with the two first-order equations

$$(15)\qquad \begin{aligned} \dot{q} &= r \\ \dot{r} &= -A^{-1}C\,q - A^{-1}B\,\dot{q} \end{aligned}$$

Define matrices s and U by the following

$$s = \begin{Bmatrix} q \\ r \end{Bmatrix}$$

$$U = \begin{bmatrix} 0 & I \\ -A^{-1}C & -A^{-1}B \end{bmatrix}$$

where 0 and I are zero and unit matrices of such an order as to make U a square matrix. Then equations (15) can be written as one first-order matric differential equation[7].

$$(16)\qquad \dot{s} = Us$$

---

[6] See [10] for a discussion on the advisability of employing large numbers of degrees of freedom.

[7] The Laplace transform may be applied at this point rendering

$$\bar{s} = (p\,I - U)^{-1}\,s_0$$

and using the first n elements, we obtain the same results as before.

We now seek solutions of the type

(17) $s = a e^{wt}$ w is a scalar

Substituting equation (17) in (16) and we obtain

(18) $Ua = wa.$

We must now get values of w and a that will satisfy equation (18). Consider now the recurrence relation

$$U a_{r-1} = w a_r \qquad [r = 0, 1, 2, \ldots]$$

where a_0 is an arbitrarily given column matrix. Now by a successive use of this recurrence formula we can express a_r in terms of a_0 thusly

$$U^r a_0 = (w^r) a_r.$$

It can be shown that for large r the ratio of the elements of the column matrix $U^r a_0$ to the corresponding elements of the column matrix $U^{r-1} a_0$ is approximately a constant equal to w_1, where w_1 is the greatest frequency of our oscillating system. The fundamental frequency and the intermediate overtones and their corresponding amplitudes can also be obtained by a method consisting of reducing the numbers of degrees of freedom and repeating the iteration process. The iteration process is a systematic method of ironing out the errors in the assumed column matrix. For a detailed account of this method along with aids in computing the real and imaginary parts of the complex characteristic roots see [3].

AN EXPERIMENTAL METHOD

In conclusion, it might be of some interest to observe briefly one of the methods used in obtaining experimental results against which analytical methods are checked.

Vibration equipment[8] capable of recording a number of positions simultaneously is used so that frequency and relative phase and amplitude of various points on the structure can be obtained so as to define deflection curves of the vibrating structure. These deflection curves will correspond to the column matrix obtained in the iteration method.

The wing is analyzed by placing pickups along two lines, one near the leading edge and the other near the trailing edge, in order to determine two deflection curves for the wing. From the two curves it is possible to determine the amount of bending and torsion present at each wing station. The pickups are so placed on the wing as to measure the motion perpendicular to the surface. For studying wing fore-and-aft modes of vibration, both vertical and horizontal components are studied to determine the amplitude as well as the direction of motion. The exciter used is a rotating unbalanced weight driven by a variable-speed transmission through a flexible drive shaft. The output of each pickup is put into a separate amplifier and then into a multielement recording oscillograph so the frequency, amplitude, and phase relation of each wing station studied can be recorded simultaneously, thus determining the wing deflection curves for each mode of vibration.

---

[8] For a detailed list of such equipment and its range, see [2, p. 369-370].

SUMMARY

In the practical sense flutter means an oscillation which grows and finally breaks the structure. The problem of flutter is concerned with the motion of a mechanical system in the neighborhood of an equilibrium position and therefore the theory of small oscillations is employed in the analysis of flutter. In the theory, critical values are determined when free oscillations are steady. Therefore the Lagrangian equations of motion for a system performing free oscillations are set up using as generalized coordinates the degrees of freedom described by the bending and twisting of the wing and the relative deflections of the aileron and tab. The generalized forces used are the moments due to inertial, retraining, and dynamical forces.

The solution of Lagrange's equations of motion is accomplished by writing them as one matrix differential equation and applying the method of Laplace transforms to this matrix equation. After the matrix subsidary equation is obtained, the single equations that make up the matrix equation are solved by taking the inverse transform.

Due to the complexity of the inversion of a matrix involving elements containing a variable, the method at its present development is limited in the number of degrees of freedom that can be employed. The author is at present working on the derivation of formulae and methods whereby more degrees can be handled. Also work is being done on the extension of the Laplace-transform theory so that formulae may be used for obtaining the inverse transform of the subsidary matrix equation directly.

Stability conditions are determined by obtaining the conditions for which any one of the constituent oscillations will not be a growing oscillation. For this procedure, Routh's tests or direct expansion of the determinantal equation may be employed.

The matrix iteration method is also used in solving the differential equations of motion, and a graphical method showing the variation of the damping factors of the constituent motions with the air speed is employed in determining the critical flutter speed.

BIBLIOGRAPHY

[1] Carslaw, H. S.; Jaeger, J. C. Operational Methods in Applied Mathematics. London: Oxford University Press, 1943.

[2] Critchlow, E. F. "Measurement and Prediction of Aircraft Vibration," S.A.E. Journal (Transactions), (52), 8, (August, 1944), 368-379.

[3] Duncan, W. J.; Collar, A. R. "Matrices Applied to the Motions of Damped Systems," Phil. Mag., IXX (1935), 197-219.

[4] Frazer, R. A.; Duncan, W. J.; Collar, A. R. Elementary Matrices and Some Applications to Dynamics and Differential Equations. New York: The MacMillan Company, 1946.

[5] von Karman, T.; Biot, M. A. Mathematical Methods in Engineering, New York and London, McGraw-Hill Book Company, Inc., 1940

[6] Michal, A. D. Matrix and Tensor Calculus with Applications to Mechanics, Elasticity, and Aeronautics. New York: John Wiley and Sons, Inc., 1947.

[7] Pipes, L. A. Applied Mathematics for Engineers and Physicists, New York and London: McGraw-Hill Book Company, Inc., 1946.

[8] Theodorson, T. "General Theory of Aerodynamic Instability and the Mechanism of Flutter," N.A.C.A. Technical Report No. 496, 1934.

[9] Theodorson, T.; Garrick, I. E. "Mechanism of Flutter, A Theoretical and Experimental Investigation of the Flutter Problem," N.A.C.A. Technical Report No. 685, 1940.

[10] Wasserman, L. " Comments on Paper by E. F. Critchlow," S.A.E. Journal (52), 8 (August, 1944) 379.

Typist:

Luella Lane