THEOREMS PROVED ON THE BASIS OF MOORE'S AXIOMS 0-2 RELATED TO SPECIAL CONTINUA AND SOME OF THEIR SPECIAL SUBCONTINUA

By

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Preface

This brief study is intended to prove some theorems related to special continua and some of their special subcontinua based upon Moore's Axioms 0-2. It is seen to be advantageous in the study of continua of condensation of a continuum M to introduce original definitions defining α, β, and $\alpha\beta$-continua of condensation of M as a classification of continua of condensation of M. A more general definition of an "arc" than that given in Definition M18 is introduced in Definition 4 as an α-arc.

A few theorems are proved that are closely related to neither continua of condensation of a continuum nor α-arcs but which may be of interest to the reader.

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Theorems Proved on the Basis of Moore's Axioms 0-2 Related to Special Continua and Some of Their Special Subcontinua

For the benefit of the reader, Moore's axioms 0-2 are listed below with definitions and theorems which are either employed in the proof of the theorems in this paper or listed as references for the purpose of clarification.

Axioms:[1]

Axiom 0. Every region is a point set.

Axiom 1. There exists a sequence $G_1, G_2, G_3, \cdots$ such that (1) for each n, G_n is a collection of regions covering S, (2) for each n, G_{n+1} is a subcollection of G_n, (3) if R is any region whatsoever, X is a point of R, Y is a point of R either identical with X or not, then there exists a natural number m such that if g is any region belonging to the collection G_m and containing X then $\overline{g}$ is a subset of $(R - Y) + X$, (4) if $M_1, M_2, M_3, \cdots$ is a sequence of closed point sets such that for each n, M_n contains M_{n+1} and, for each n, there exists a region g_n of the collection G_n such that M_n is a subset of $\overline{g}_n$, then there is at least one point common to all the point sets of the sequence $M_1, M_2, M_3, \cdots$.

Axiom 2. If P is a point of a region there exists a nondegenerate connected domain containing P and lying wholly in R.

Definitions:[2]

Definition M1. A point P is said to be a limit point of a point set M if every region that contains P contains at least one point of M distinct from P.

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[1] R. L. Moore, Foundations of Point Set Theory, pp. 5, 6, 86.

[2] Ibid., pp. 5, 11, 17, 21, 28, 34, 38-39, 47-48, 75-76, 81, 94, 103-104, 106, 128.

Definition M2. A point set is said to be <u>closed</u> if it contains all its limit points.

Definition M3. The boundary of a point set M is the set of all points X such that every region that contains X contains at least one point of M and at least one point which does not belong to M. (A point of the boundary of M is called a boundary point of M.)

Definition M4. Two point sets are said to be mutually exclusive if they have no point in common.

Definition M5. Two point sets are said to be <u>mutually separated</u> if they are mutually exclusive and neither of them contains a limit point of the other one.

Definition M6. A point set is said to be <u>compact</u> if every infinite subset of M has at least one limit point.

Definition M7. The point set D is said to be a <u>domain</u> if for each point p of D there exists a region containing p and lying in D.

Definition M8. A subset K of a point set M is said to be an <u>open subset</u> of M if for each point p of K there exists a region R containing p such that R • M is a subset of K.

Definition M9. An open subset of M is also called a <u>domain with respect to M</u>.

Definition M10. A point set is said to be <u>locally compact</u> if, for each point p of M, there is a compact open subset of M containing p.

Definition M11. A point set is said to be <u>connected</u> if it is not the sum of two mutually separate point sets.

Definition M12. A point set which is both closed and connected is called a <u>continuum</u>.

Definition M13. A <u>maximal</u> connected subset of a point set M is a connected subset of M which is not a proper subset of any other connected subset of M. A maximal connected subset of a point set is also called a <u>component</u> of that point set.

Definition M14. The continuum M is said to be an <u>irreducible continuum about</u> the point set N if M contains H but no proper subcontinuum of M contains H.

Definition M15. If α is a sequence of point sets M_1, M_2, M_3, . . . then by the *limiting set* of α is meant the set of all points P such that if R is a region containing P there exist infinitely many positive integers n such that M contains a point of R.

Definition M16. If H, K and T are proper subsets of the connected point set M then T is said to *separate H from K in M* if M - T is the sum of two mutually separated point sets containing H and K respectively.

Definition M17. If K is a proper subset of the connected point set M and M - K is not connected, then M is said to be *disconnected by the omission of K*, or to be *disconnected by K*, or to be *separated by K*, and K is called a *cut set* of M; and, if K is a point, it is called a *cut point* of M, and, if it is a continuum, it is called a *cut continuum* of M.

Definition M18. If A and B are two distinct points, a *simple continuous arc* from A to B is a closed, connected and compact point set which contains A and B and which is disconnected by the omission of any one of its points except A and B. A simple continuous arc is sometimes called merely an arc. The statement "AB is an arc" is to be interpreted as meaning that AB is an arc from A to B.

Definition M19. A *simple closed curve* is a nondegenerate compact continuum which is disconnected by the omission of any two of its points.

Definition M20. A point set is said to be *degenerate* if it consists of only one point. Otherwise it is said to be *nondegenerate*.

Definition M21. If M is a continuum, a *composant* of M is a point set K such that, for some point P of M, K is the set of all points X such that there is a proper subcontinuum of M containing both P and K.

Definition M22. The continuum M is said to be *indecomposable* if it is not the sum of two continua both distinct from it.

Definition M23. The point set M is said to be an *inner limiting* set if there exists a sequence of domains D_1, D_2, D_3,. . . such that (a) for every n,

D_n contains D_{n+1}, (b) M is the intersection of the domains of this sequence.

Definition M24. The point set M is said to be connected im kleinen at the point O if O belongs to M and for every open subset D of M that contains O there exists an open subset of M which contains O and which is a subset of a component of D. If the point set M is connected im kleinen at every one of its points it is said to be connected im kleinen.

Definition M25. The point set M is said to be locally connected at the point O if O belongs to M and every open subset of M that contains O contains a connected open subset of M containing O. If the point set M is locally connected at each of its points it is said to be locally connected.

Definition M26. A connected im kleinen continuum is called a continuous curve.

Definition M27. An open curve is a locally compact continuum which is separated into two connected sets by the omission of any one of its points.

Definition M28. If O is a point, a ray from O is a locally compact continuum M containing O and such that (1) M - O is connected, (2) if P is any point of M distinct from O then M - P is the sum of two mutually separated connected point sets.

Definition M29. The continuum K is said to be a continuum of condensation of the continuum M if K is a nondegenerate subset of M and every point of K is a limit point of M - K.

Definition M30. The point set M is said to be arcwise connected if every two points of M are the extremities of an arc lying wholly in M.

Theorems:[3]

Theorem M1. No point of a region is a boundary point of that region.

Theorem M2. If p is a limit point of the point set M then every region that contains p contains infinitely many points of M and, indeed, there exists an infinite sequence of points belonging to M and all distinct

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[3] Ibid., pp. 6-8, 17, 21, 28, 33, 75-77, 81, 86, 94-96, 101, 106.

from each other and from p such that p is a sequential limit point of this sequence.

Theorem M3. If a point is not a limit point of any one of a finite number of point sets then it is not a limit point of their sum.

Theorem M4. If H and K are two mutually separated point sets, every connected subset of H + K is a subset of either H or of K.

Theorem M5. If M is a connected point set and L is a point set consisting of M together with some or all of its limit points, then L is connected.

Theorem M6. If G is a collection of connected point sets and one of them contains a limit point of each of the others then the sum of all the point sets of the collection G is connected.

Theorem M7. If H and K are mutually exclusive closed subsets of the compact continuum M then there is a subcontinuum of M that is irreducible from H to K.

Theorem M8. If $M_1, M_2, M_3, \cdots$ is a sequence of connected point sets such that the point set consisting of $M_1 + M_2 + M_3 + \cdots$ together with all its limit points is compact and there exists a convergent sequence of points $A_1, A_2, A_3, \cdots$ such that, for each n, A_n belongs to M_n, then the limiting set of the sequence $M_1, M_2, M_3, \cdots$ is a continuum.

Theorem M9. If T is a connected subset of the connected point set M and M − T is the sum of two mutually separated point sets H and K then H + T and K + T are connected.

Theorem M10. If K is a composant of a compact continuum M, every point of M is a limit point of K.

Theorem M11. Every composant of a compact continuum M is the sum of a countable number of proper subcontinua of M.

Theorem M12. In order that the continuum M should be indecomposable it is necessary and sufficient that every proper subcontinuum of M should be a continuum of condensation of M.

Theorem M13. No two composants of an indecomposable continuum have a point in common.

Theorem M14. Every compact indecomposable continuum has uncountably many composants.

Theorem M15. In order that the compact continuum M should be indecomposable it is necessary and sufficient that there should exist three distinct points such that M is irreducible between each two of them.

Theorem M16. Every closed point set is an inner limiting set.

Theorem M17. If A and B are distinct points of a connected domain D there exists a simple continuous arc from A to B that lies wholly in D.

Theorem M18. The point set M is connected im kleinen at the point O if it is locally connected at that point; and if M is connected im kleinen at every point of some open subset of M that contains O then M is locally connected at O.

Theorem M19. If O is a point of the locally compact continuum M and M is not connected im kleinen at the point O then, if R is a region containing O, there exist a connected domain D containing O and lying in R, an infinite sequence of points $O_1, O_2, O_3, \cdots$ converging to O, and an infinite sequence of mutually exclusive continua $M_1, M_2, M_3, \cdots$ such that (1) $M \cdot \overline{D}$ is compact, (2) for each n, M_n is a component of $M \cdot \overline{D} - O$ containing O_n and a point of the boundary of D, (3) the sequence $M_1, M_2, M_3, \cdots$ converges to a subset of M which contains O.

Theorem M20. Suppose M is a continuous curve or, indeed, any connected and connected im kleinen inner limiting set. If M is regarded as a space and the term "region" is interpreted to mean a connected open subset of M, then, with respect to this interpretation of "point" and "region", Axioms 1 and 2 are satisfied and "limit point" is invariant under this change.

Theorem M21. If T is a subset of the point set M and M − T is the sum of the two mutually separated connected point sets H and K and is the sum of two mutually separated sets H' and K', then one of the sets H' and K' is H and the other one is K.

Theorem M22. Every arc, simple closed curve, open curve or ray is a continuous curve.

In the brief study presented herein, an attempt is made to give a slightly more detailed treatment than some texts offer of the special subcontinua, continua of condensation of a continuum, with respect to subclassifications and their usefulness in determining additional properties of that continuum of which they are continua of condensation. By assuming that continua have certain properties such as connectedness im kleinen and local connectedness (Definitions M24 and M25 respectively), one may state whether or not such continua have continua of condensation of themselves and under what subclassifications they fall. Also, under the assumption that continua have certain other properties such as compactness and irreducibility from a point A to another point B, one may determine the existence of a simple continuous arc AB from A to B. A few theorems appear here that either break the continuity of the study or are remotely related to the major theme "continua of condensation of a continuum" but which are nevertheless interesting. Examples are given whenever it is thought necessary for clarification. The reader may, of course, construct those to suit his fancy whether simpler or more complicated.

The special subcontinua of a continuum M, continua of condensation, which are defined by Definition M29 are interesting in that their properties give information regarding M. It is suggested upon consideration of various examples (see examples 1 and 2 below) that there exist various classifications of continua of condensation of a continuum. I define α, β, and $\alpha\beta$ -continua of condensation of a continuum in Definition 1-3. Examples 1-3 are examples of α, β, and $\alpha\beta$ -continua of condensation of a continuum, respectively.

Definition 1. If for each point p of a continuum of condensation K of a continuum M, there exists no open subset of M containing p which has a component containing p which is a subset of K; then K is said to be an α-continuum of condensation of M.

Definition 2. If for each point p of a continuum of condensation K of a continuum M, there exists an open subset D of M containing p and a component of D containing p which is a subset of K; then K is said to be a β-continuum of condensation of M.

Definition 3. The continuum K is said to be an $\alpha\beta$-continuum of condensation of the continuum M if K is a continuum of condensation of M and if there exists points p and q of K such that (i) no open subset of M containing p contains a component which is a subset of K and (ii) there exists an open subset D of M containing q and a component of D containing q which is a subset of K. p is called an α-point of K and q a β-point of K.

In the following examples, it will be understood that the space is the Euclidean plane. Cartesian coordinates are used.

Example 1. Let R be the collection of line segments with end points (0,0), (1,0); (1,0), (1,1); (1,1), (0,1); and (0,1), (0,0). Let γ denote the sequence of line segments R_1, R_2, R_3, $\cdots$ such that for each positive integer n, R_n has end points $(0, 1/2^n)$ and $(1, 1/2^n)$. For each integer $n > 0$, let R_n' denote a collection of line segments with end points $(0, k/2^n)$ and $(1, k/2^n)$, $k = 1, 3, 5, \cdots, 2^n - 1$. Let M denote the sum of R, γ, and R_1', R_2', R_3', $\cdots$. The line segment with end points (0,0) and (0,1) is an example of an α-continuum of condensation of the continuum M.

Example 2. Let M be the continuum consisting of that part of the graph of $y = \sin 1/x$ for which $0 < x \leq 1$ and that part of the y-axis for which $0 \leq |y| \leq 1$. The line segment with end points (0,1) and (0, − 1) is an example of a β -continuum of condensation of the continuum M.

Example 3. Let M be the continuum consisting of rays from the origin making angles $1/n$, $n = 1, 2, 3, \ldots$ with the positive x-axis. That part (denote by K) of the x-axis for which $x \geq 0$ is an example of an $\alpha\beta$ -continuum of condensation of the continuum M. The origin is an example of an α -point of K. A point x for which $x > 0$ is a β -point of K.

Theorem 1. If K is a proper nondegenerate subcontinuum of a continuum M and K is a continuum of condensation of M, then K is one of the three classifications of continua of condensation— α, β, $\alpha\beta$.

Proof.

1. If p is some point of K, then either (i) there exists an open subset D of M containing p such that no component of D containing p is a subset of K or (ii) not.

2. If (i) is true for each point of K, then K is an α -continuum of condensation of M by Definition 1.

3. If (ii) is the case for each point of K, then K is a β -continuum of condensation of M by Definition 2.

4. If (i) is true for a point p of K and (ii) is true for a point q of K, then K is an $\alpha\beta$ -continuum of condensation of M by Definition 3.

The theorem is proved.

Theorem 2. If K is an $\alpha\beta$ -continuum of condensation of M and p is a point of K, then p is either an α -point or a β -point of K.

Proof.

1. Either (i) there exists an open subset D of M such that no component of D containing p is a subset of K or (ii) not.

2. If (i) is true, then, by Definition 3, p is an α -point of K.

3. If, on the other hand, (ii) is the case, then p is a β -point of

K by Definition 3.

Theorem 2 is proved.

An example of the usefulness of the subclassification of continua of condensation of a continuum may be seen in Theorem 3 below.

Theorem 3. If M is a connected im kleinen continuum, then no proper nondegenerate subcontinuum of M is either a β -continuum of condensation of M or an $\alpha\beta$ -continuum of condensation of M.

Proof.

1. Assume that there exists a proper nondegenerate subcontinuum K of M that is a β -continuum of condensation of M.

2. Let p denote a point of K. By Definition 2, there exists an open subset D of M containing p such that D contains a component containing p which is a subset of K.

3. By hypothesis, M is connected im kleinen. Hence, by Definition M24 there exists an open subset Q of M containing p such that Q is a subset of a component of D.

4. By Definition M8, there exists a region R containing p such that $R \cdot M \subset D$. Hence, by Definition M1, p is not a limit point of M - K contrary to Definition M29 that every point of K is a limit point of M - K. The assumption of step 1 is therefore false.

5. A proof similar to steps 1-4 employing Definition 3 proves the second conclusion.

The continuum M of example 1 is an example of a connected im kleinen continuum which possesses a proper subcontinuum K of M which is an α -continuum of condensation of M. Thus, under the hypothesis of Theorem 3, the theorem is not true if the conclusion is strengthened to read no proper nondegenerate subcontinuum of M is a continuum of condensation of M.

Theorem 4. If O is a point of a locally compact continuum M and M is not connected im kleinen at O, then there exists a proper subcontinuum K of M containing O which is a continuum of condensation of M.

Proof.

1. By Theorem M19, if R is a region containing O, there exists a connected domain D containing O and lying in R, an infinite sequence of points $O_1, O_2, O_3, \cdots$ converging to O, and an infinite sequence of mutually exclusive continua $M_1, M_2, M_3, \cdots$ such that (1) $M \cdot \overline{D}$ is compact, (2) for each n, M_n is a component of $M \cdot \overline{D} - O$ containing O_n and a point of the boundary of D, (3) the sequence $M_1, M_2, M_3, \cdots$ converges to a subset K of M which contains O.

2. Let R be a region containing O such that M is not contained wholly in R. Then, K is a proper subset of M.

3. By Theorem M8, K is a continuum.

4. Let $O_1', O_2', O_3', \cdots$ be an infinite sequence of points such that for each positive integer n, O_n' is a point of M and a point of the boundary of D. The existence of the sequence $\{O_n'\}$ is assured by step 1. Also, by step 1, $M \cdot \overline{D}$ is compact and $M \cdot \overline{D} \supset \{O_n'\}$.

5. Consider the point set $M' = \sum_{n=1}^{\infty} O_n'$. By step 1, M_n is an infinite sequence of mutually exclusive continua. Hence, by Definition M4, O_i' is distinct from O_j' for each positive integer i and j with the exception $i = j$. Thus, M' is an infinite subset of $M \cdot \overline{D}$.

6. By Definitions M6 and M2, there exists at least one point p of $M \cdot \overline{D}$ such that p is a limit point of M'. By Theorem M2 and Definition M15, p belongs to K. If R is a region containing p, then by Definition M1, R contains at least one point of M'. R contains a boundary point of $M \cdot \overline{D}$ since every point of M' is a point of the boundary of $M \cdot \overline{D}$. By

Definition M3, p is a boundary point of $M \cdot \bar{\bar{D}}$. By Theorem M1 and the above, p is distinct from O.

7. Hence, by steps 2-6 and the above, K is a proper nondegenerate subcontinuum of M. By Definition M29, K is a continuum of condensation of M.

The theorem is proved.

If M is a locally compact continuum and O is a point of M, it is interesting to observe that although it is necessary that there exist a proper subcontinuum K of M which is a continuum of condensation of M in order that M not be connected im kleinen at O, it is not sufficient. The continuum M of example 1 which is connected im kleinen and locally compact is an example to show that it is not sufficient. If the condition is altered to read β -continuum of condensation of M, then the condition is sufficient as Theorem 1 shows. It will now be shown that this requirement is not necessary. If we rotate the continuum M of example 1 in a counter-clockwise manner through an angle $\pi/2$ about the origin and add to it that part of the graph of $y = \sin 1/x$ for which $0 < x \leq 1$ and that part of the y-axis for which $0 \geq y \geq -1$, we obtain a locally compact continuum which is not connected im kleinen at any point of the y-axis for which $0 \leq |y| \leq 1$. This shows that the altered condition that M contain a proper subcontinuum which is a β -continuum of condensation of M containing O in order that M not be connected im kleinen at O is not necessary.

<u>Theorem 5</u>. If M is a locally compact continuum, p is a point of M, and no subcontinuum N of M containing p contains a proper subcontinuum K containing p such that (i) K is an α -continuum of condensation of M or (ii) K is an $\alpha\beta$ -continuum of condensation of M such that p is an

α-point of K, then a necessary and sufficient condition that M not be connected im kleinen at p is that there exist a proper subcontinuum K of M containing p such that K is either a β -continuum of condensation of M or an $\alpha\beta$ -continuum of condensation of M such that p is a β -point of K.

Proof.

The condition is necessary, for:

1. By Theorem 4, there exists a proper subcontinuum K of M containing p which is a continuum of condensation of M.

2. By the hypothesis and Theorems 1-2, K is either a β -continuum of condensation of M or an $\alpha\beta$ -continuum of condensation of M such that p is a β -point of K.

The condition is sufficient by Theorem 3.

Theorem 6. If M is a locally compact continuum and no subcontinuum N of M contains a proper subcontinuum K such that K is an α -continuum of condensation of M, then a necessary and sufficient condition that M be a continuous curve is that no proper subcontinuum of M be a β -continuum or an $\alpha\beta$ -continuum of condensation of M.

Proof.

1. By Definition M26 and Theorem 3, the condition is necessary.

2. Assume that the condition is not sufficient. Then, there exists a point p of M such that M is not connected im kleinen at p.

3. By Theorem 5, there exists a subcontinuum K of M such that K is either a β -continuum of condensation of M or an $\alpha\beta$ -continuum of condensation of M such that p is a β -point of K. This is contrary to the hypothesis that no proper subcontinuum of M be a β - or an $\alpha\beta$ - continuum of condensation of M. The assumption of step 2 is false.

The condition is sufficient.

Theorem 7. If M is an arc, simple closed curve, open curve, or ray, then M contains no proper nondegenerate subcontinuum which is a β or an $\alpha\beta$ -continuum of condensation of M.

Proof.

1. By Theorem M23, M is a continuous curve. Hence, by Definition M26, M is connected im kleinen.

2. By Theorem 3 and step 1, the theorem is true.

Theorem 8. If M is a continuum, K is a proper nondegenerate subcontinuum of M, and K is a continuum of condensation of M; then K contains no domain.

Proof.

1. Assume that K contains a domain D.

2. By Definition M7, if p is a point of D, there exists a region R containing p such that R is a subset of D. Hence, by Definition M1, p is not a limit point of M - K. This is contrary to Definition M29 that every point of K is a limit point of M - K. Therefore, the assumption of step 1 is false.

The theorem is proved.

As a consequence of Theorem 8 and Theorem M12, the following theorem is proved.

Theorem 9. If M is an indecomposable continuum, then M contains no domain.

Proof.

1. Assume that M contains a domain D.

2. By axioms 1 and 2, there exists a nondegenerate connected domain D' containing a point p of D such that $\overline{D}'$ is a subset of D.

3. By Theorems M1 and M5, $\overline{D'}$ is a proper subcontinuum of M.

4. By hypothesis, M is indecomposable. Hence, by Theorem M12, $\overline{D'}$ is a continuum of condensation of M.

5. $\overline{D'}$ contains a domain, namely D', contrary to Theorem 8. The assumption of step 1 is therefore false.

The theorem is proved.

Conditions for the existence of a simple continuous arc from a point A to a point B, A and B distinct, are given in terms of locally connected continua, irreducible continua, and continua of condensation of a continuum in the following theorems.

Theorem 10. If M is a locally connected continuum irreducible from a point A to another point B, then M is an arc from A to B.

Proof.

1. By Theorem M18, M is connected im kleinen. By Theorem M16, M is an inner limiting set. Hence, by Theorem M20, we may regard M as a space and interpret the word "region" to mean a connected open subset of M.

2. If p is a point of M, then there exists a "region" R containing p such that R is a subset of M — by axiom 1, Definition M25, and step 1. By Definition M7, M is a "domain". M is a connected "domain" since M is a continuum and connected by Definition M12.

3. By Theorem M17, there exists a simple continuous arc AB from A to B that lies wholly in M.

4. $M \equiv AB$. Otherwise, M is not irreducible from A to B contrary to the hypothesis.

The theorem is proved.

Theorem 11. If M is a locally connected continuum containing two distinct points A and B, then every point that separates A from B in M lies on an arc AB from A to B.

Proof.

1. By Theorem M18, M is connected im kleinen.

2. By Theorem M16, M is an inner limiting set.

3. By Theorem M20, if M is regarded as a space, the term "region" may be interpreted to mean connected open subset of M. The existence of such "regions" is established by the hypothesis that M is locally connected and Definition M25.

4. M is a connected "domain" since M is a sum of "regions" and connected by the hypothesis that M is a continuum. By Theorem M17, there exists an arc AB from A to B lying wholly in M.

5. Assume that there exists a point p of M that separates A from B in M but does not lie on the arc AB. Then $M - p = S_A + S_B$ where S_A and S_B are mutually separated point sets containing A and B, respectively, by Definition M16.

6. By Theorem M4, AB is either a subset of S_A or S_B which is impossible. Hence, the assumption of step 5 is false.

The theorem is proved.

Theorem 12. A nondegenerate locally connected continuum is arcwise connected.

Proof.

1. Let A and B be two distinct points of the nondegenerate locally connected continuum M.

2. Steps 1-4 of Theorem 11 are valid under the hypothesis of Theorem 12. Hence, by Definition M30, M is arcwise connected.

Theorems 13 and 14 below have significance in themselves, but their main purpose in this study is their application in the proof of Theorem 15. The conclusion of Theorem 15 is interesting in view of the hypothesis.

Theorem 13. If M is a compact irreducible continuum from A to B and from A to C where A, B, and C are distinct points of M, then some nondegenerate proper subcontinuum of M is a continuum of condensation of M.

Proof.

1. Either (i) M is irreducible from C to B or (ii) not.

2. Case (i) M is irreducible from C to B (in addition to the fact that M is irreducible from A to B and from A to C as stated in the hypothesis).

(a) By Theorem M15, M is an indecomposable continuum. Hence, by Theorem M12, every proper subcontinuum of M is a continuum of condensation of M.

(b) By (a) and the hypothesis that M is compact, it follows from Theorem M14 that M has uncountably many composants. From Theorem M10, it follows that no composant of M is a degenerate point set since if K is a composant of M, $\overline{K} = M$. By Theorem M11, every composant of M is the sum of a countable number of proper subcontinua of M. Some nondegenerate proper subcontinuum of M is a continuum of condensation of M — by the above and Definition M19.

3. Case (ii) M is not irreducible from C to B.

(a) By Theorem M7, there exists a subcontinuum N_{CB} of M irreducible from C to B. N_{CB} is a proper subcontinuum of M since M is reducible from C to B — by Definition M14.

(b) Assume that $M - N_{CB}$ is not connected. Then, by Definition M11, $M - N_{CB} = S_1 + S_2$ where S_1 and S_2 are mutually separated point sets.

A does not belong to N_{CB} — by Definition M14, since M is irreducible from A to B. Therefore, either A belongs to S_1 or A belongs to S_2. For convenience, let $S_A = S_1$ contain A.

(c) By Theorem M9, $S_A + N_{CB}$ is connected. No limit point of $N_{CB} + S_A$ belongs to S_2 — by steps (a) and (b) and Theorem M3 that $\overline{N_{CB} + S_A} = \overline{N}_{CB} + \overline{S}_A$.

(d) By Theorem M5 and (c) above, $\overline{N_{CB} + S_A}$ is a proper subcontinuum of M containing A + B contrary to the hypothesis that M is irreducible from A to B. The assumption of step (b) is false. Hence, $M - N_{CB}$ is connected.

(e) By Theorem M5 and Definition M12, $\overline{M - N_{CB}}$ is a continuum. $\overline{M - N_{CB}}$ is compact — by hypothesis.

(f) Assume that $\overline{M - N_{CB}}$ contains no point of N_{CB}. Then $\overline{M - N_{CB}}$ N_{CB} is the sum of two mutually exclusive closed point sets contrary to the hypothesis that M is a continuum. Therefore, $\overline{M - N_{CB}}$ contains some point x of N_{CB}.

(g) If x = C or x = B, then $\overline{M - N_{CB}}$ contains N_{CB}. Otherwise, $\overline{M - N_{CB}}$ is a proper subcontinuum of M containing A + B or A + C contrary to the hypothesis that M is irreducible from A to B and from A to C. Therefore, every point of N_{CB} is a limit point of $M - N_{CB}$. By Definition M29, N_{CB} is a continuum of condensation of M. N_{CB} is a nondegenerate proper subcontinuum of M — by step (a). Hence, the theorem is true.

(h) If no point x of N_{CB} belonging to $\overline{M - N_{CB}}$ is B or C, then $\overline{M - N_{CB}}$ is a proper subcontinuum of M that contains neither B nor C.

(i) By Theorem M7, there exists a subcontinuum N_{Ax} of $\overline{M - N_{CB}}$ irreducible from A to x. x belongs to $(N_{CB}).(\overline{M - N_{CB}})$. Also, by Theorem M7, there exist subcontinua N_{xC} and N_{xB} of N_{CB} irreducible from x to C and from x to B, respectively.

(j) If $N_{xB} \equiv N_{CB} \equiv N_{xC}$, then by Theorems M15 and M12, every proper subcontinuum of N_{CB} is a continuum of condensation of N_{CB} and hence of M. The existence of proper subcontinua of N_{CB} is established by an argument similar to 1 (b). The theorem is therefore true for $N_{xB} \equiv N_{CB} \equiv N_{xC}$.

(k) Assume that either N_{xB} or N_{xC} is a proper subcontinuum of N_{CB}.

(l) If N_{xB} is a proper subcontinuum of N_{CB}, then N_{xB} does not contain C. Otherwise, N_{CB} is not irreducible from C to B contrary to step (a). By Theorems M3 and M6, $N_{Ax} + N_{xB}$ is a continuum containing A + B. Neither N_{xB} nor N_{Ax} contains C. Therefore, $N_{Ax} + N_{xB}$ is a proper subcontinuum of M. This is contrary to the hypothesis that M is irreducible from A to B. A similar argument leads to a similar contradiction assuming that N_{xC} is a proper subcontinuum of N_{CB}. Therefore, the assumption of step (k) is false. We obtain under the restrictions of (h) that $N_{xB} \equiv N_{CB} \equiv N_{xC}$. The theorem is true by the argument of step (j).

(m) Having considered all possible cases in steps (g) and (h) for a point x to be a point of $(N_{CB}) \cdot (\overline{M - N_{CB}})$, it follows from the results of those considerations that the theorem is true.

<u>Theorem 14</u>. If M is a compact irreducible continuum from a point A to another point B and no nondegenerate subcontinuum of M is a continuum of condensation of M and if C is any point of M not A or B, then there exist proper subcontinua of M irreducible from A to C and from C to B.

Proof.

1. Assume that there exists no proper subcontinuum of M that contains both A and C.

2. By step 1 and Definition M14, M is irreducible from A to C.

3. M is compact by hypothesis. Hence, M is a compact irreducible continuum from A to B and from A to C. M satisfies the hypothesis of Theorem 13. Hence, by Theorem 13, some proper subcontinuum of M is a continuum of condensation of M. This is contrary to the hypothesis that no proper subcontinuum of M is a continuum of condensation of M. Therefore, the assumption of step 1 is false.

4. By step 3 and Theorem M7, there exists a proper subcontinuum of M irreducible from A to C.

5. By an argument similar to that of steps 1-4, there exists a proper subcontinuum of M irreducible from C to B.

6. Combining steps 4 and 5, we have the desired result.

<u>Theorem 15</u>. If M is a compact irreducible continuum from a point A to another point B and no nondegenerate proper subcontinuum of M is a continuum of condensation of M, then M is an arc from A to B.

Proof.

1. Let C be any point of M not A or B.

2. By Theorem 14, there exist proper subcontinua N_{AC} and N_{CB} irreducible from A to C and from C to B, respectively.

3. By Theorem M3 and M6, $N_{AC} + N_{CB}$ is a continuum. Since $N_{AC} \supset A$ and $N_{CB} \supset B$, $N_{AC} + N_{CB} \supset A + B$.

4. By hypothesis, M is irreducible from A to B. By Definition M14 and step 3, $M = N_{AC} + N_{CB}$.

5. Assume that N_{AC} and N_{CB} have at least one point x in common other than C.

6. By Theorem 14, there exist proper subcontinua of N_{AC} and N_{CB},

designated by N_{Ax} and N_{xB}, irreducible from A to x and from x to B respectively.

7. By steps 4 and 6 and Definition M14, neither N_{Ax} nor N_{xB} contains C.

8. By Theorems M3 and M6, and step 7, $N_{Ax} + N_{xB}$ is a proper subcontinuum of M containing A + B. This is contrary to the hypothesis that M is irreducible from A to B. Thus, the assumption of step 5 is false. Hence, the only point that N_{AC} and N_{CB} have in common is C.

9. By step 3, $M = N_{AC} + N_{CB}$. Consider $M - C = (N_{AC} - C) + (N_{CB} - C)$. By step 8, no point of $N_{AC} - C$ is a point of $N_{CB} - C$. Similarly, no point of N_{CB} is a point of $N_{AC} - C$. Since N_{CB} is closed, no point of N_{AC} other than C is a limit point of N_{CB}. Otherwise, N_{CB} would contain a point of N_{AC} other than C. Similarly, no point of N_{CB} is a limit point of N_{AC} other than C. In conclusion, $(N_{AC} - C)$ and $(N_{CB} - C)$ are mutually separated point sets — by Definition M5. Hence, by Definition M17, M is disconnected by C.

10. By hypothesis, M is a connected and compact point set containing A + B. Since C was any point of M not A or B, M is disconnected by the ommission of any one of its points not A or B. Hence, by Definition M18, M is a simple continuous arc from A to B.

Definitions M27 and M28 define <u>open curve</u> and <u>ray</u>, respectively. The following theorem is one of two consecutive theorems which are miscellaneous items but may be of some interest in the study of open curves and indecomposable continua, respectively.

<u>Theorem 16</u>. If p is a point of an open curve M, then M is the sum of two rays having only the point p in common.

Proof.

1. $M - p = H + K$, two mutually separated connected point sets —by Definition M27. $p + K$ and $p + H$ are connected by Theorem M9. Thus, both $p + K - p$ and $p + H - p$ are connected.

2. Let h and k be points of H and K, respectively. Also, let $R_1 = p + K$ and $R_2 = p + H$.

3. Assume that $R_1 - k$ is connected. Then $R_1 + R_2 - k = M - k$ is connected contrary to Definition M27. Thus, $R_1 - k = S_1 + S_2$ — two mutually separated point sets. For convenience, let p belong to S_1. Then $M - k = (R_2 + S_1) + S_2$, the sum of two mutually separated point sets. $M - k = H' + K'$, two mutually separated connected point sets by Definition M27. By Theorem M21, either H' or K' is $R_2 + S_1$ and the other is S_2. Hence, both $R_2 + S_1$ and S_2 are connected.

4. Assume that S_1 is not connected. Then $S_1 = S_1' + S_1''$ by Definition M11. Let S_1' contain p, for convenience. Then, $R_2 + S_1'$ and S_1'' are mutually separated point sets by steps 2 and 3. This is contrary to Definition M27. Hence, S_1 is connected.

5. By Definition M28, R_1 is a ray from p. Similary, R_2 is a ray from p. From steps 1 and 2, R_1 and R_2 have only the point p in common.

The theorem is proved.

Theorem 17. If H is a proper subcontinuum of an indecomposable continuum M, then $M - H$ is connected.

Proof.

1. Assume that $M - H$ is not connected. Then, by Definition M11, $M - H = S_1 + S_2$ where S_1 and S_2 are two mutually separated point sets.

2. H is connected by hypothesis and separates S_1 from S_2 in M by

Definition M16. Hence, by Theorem M9, both $H + S_1$ and $H + S_2$ are connected.

3. No point of S_2 is a limit point of S_1 — by step 1. Since H is closed, no point of S_2 is a limit point of H. M is closed by Definition M12. Thus, $\overline{H + S_1} = \overline{H} + \overline{S}_1$ by Theorem M3.

4. By Theorem 5 and Definition M11, $H + S_1$ and $H + S_2$ are proper subcontinua of M. $(H + S_1) + (H + S_2) = M$ — by step 1. Hence, M is decomposable contrary to the hypothesis and Definition M22.

The theorem is proved.

A more general definition of an "arc" than that given in Definition M18 is that which I define in the following definition.

<u>Definition 4</u>. If A and B are two distinct points, an <u>α-arc</u> from A to B is a continuum M such that M contains $A + B$ and every point of $M - (A + B)$ separates A from B in M.

<u>Example 4</u>. Let S denote a subspace of the Euclidean plane such that S plus that part of the y-axis for which $0 < |y| \leq 1$ is the Euclidean plane. Now let M denote that part of the graph of $y = \sin 1/x$ for which $0 < x \leq 2/\pi$ plus the origin. If A denotes the origin and B the point $(2/\pi, 0)$, then M is an α-arc from A to B. M is not locally compact.

An aposyndetic continuum[4] is defined as follows:

> The point set M is said to be aposyndetic at the point P if P belongs to M and for each point X of M distinct from P there exists an open subset of M which contains P and belongs to a connected and relatively closed subset of M lying in $M - X$. A point set which is aposyndetic at each of its points is said to be aposyndetic.

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[4] F. Burton Jones, "Aposyndetic Continua and Certain Boundary Problems," <u>American Journal of Mathematics</u>, LXIII (July, 1941), 545.

Then the continuum M in example 4 above is aposyndetic but neither connected im kleinen nor semi-locally connected. The definition:[5]

> A connected set M will be said to be semi-locally connected (s.l.c.) at a point X of M provided that for any $\epsilon > 0$ there exists a neighborhood V of X in M of diameter less than ϵ such that M - V has only a finite number of components. If M is s.l.c. at each of its points, it is said to be s.l.c.

has not been used previously in this study. It is interesting to note that if an α -arc is locally connected (and hence both connected im kleinen and aposyndetic), it is a simple continuous arc. This is proved in the following theorem as a consequence of Theorem 11.

Theorem 18. If M is a locally connected α -arc from a point A to another point B, then M is a simple continuous arc from A to B.

Proof.

1. By Theorem 11, every point that separates A from B in M lies on an arc AB from A to B.

2. By Definition 4, every point of M - (A + B) separates A from B in M. Hence, M = AB is a simple continuous arc.

Other simple theorems concerning α -arcs are the following:

Theorem 19. If M is an α -arc from a point A to another point B, then M is irreducible from A to B.

Proof.

1. Assume that M is not irreducible from A to B. Then there exists a proper subcontinuum N of M containing A + B —by Definition M14.

2. By step 1, M - N is non vacuous. Let p be a point of M - N. By Definition 4, p separates A from B. Therefore, $M - p = S_A + S_B$ two

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[5] G. T. Whyburn, Analytic Topology, p. 19.

mutually separated point sets containing A and B, respectively, by Definition M16.

3. By Theorem M4, N is a subset of either S_A or S_B which is impossible. The assumption of step 1 is therefore false.

The theorem is proved.

A definition and theorem of F. Burton Jones follow for use in the proof of Theorems 20 and 21, respectively.

Definition J.[6]

> A continuum M is said to be freely decomposable provided that if A and B are distinct points of M then M is the sum of two continua neither of which contains both A and B.

Theorem J. "In order that a continuum be aposyndetic it is necessary and sufficient that it be freely decomposable."[7]

Theorem 20. An α-arc AB from a point A to another point B is freely decomposable.

Proof.

1. Assume that AB is not freely decomposable.

2. By Definition 4, if p is a point of AB - (A + B), then AB - p $S_A + S_B$ where S_1 and S_2 are two mutually separated point sets containing A and B, respectively. By Theorem M9 and Definition M12, $S_A + p$ and $S_B + p$ are proper subcontinua of AB.

3. AB = (S_A + p) + (S_B + p) — by step 2. Hence, by Definition M22, AB is decomposable.

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[6] Jones, op. cit., LXIII, 547.

[7] Ibid., p. 548.

4. By steps 1-2 and Definition J, AB is the sum of two continua R and S one of which contains both A and B.

5. By Theorem 19, AB is irreducible from A to B. Hence, by Definition M14, the conclusion of step 4 is impossible.

The theorem is proved.

Theorem 21. An α-arc AB from a point A to another point B is aposyndetic.

Proof.

By Theorem 20 and Theorem J, AB is aposyndetic.

There is evidence that with sufficient time, one could obtain important theorems concerning continua of condensation of a continuum and α-arcs. Perhaps the theorems proved in this study, are true in more general spaces than those satisfying Moore's Axioms 0-2 upon which they are based. An investigation which would require more time than is allowed for the study given here would reveal the answer. Example 4 and Theorems 18-21 were given to show that an α-arc is more general than a simple continuous arc and that theorems may be obtained concerning its properties. It is concluded from the theorems proved that the subclassification of continua of condensation of a continuum is advantageous. Also, it is regrettable that time does not permit a more extensive study of the properties of continua that may be obtained with the aid of definitions 1-4 based upon Moore's Axioms 0-2 as well as axioms for other spaces.

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