MINIMAL MINERVAS AND COMPLETELY INFLATED FORMS

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PREFACE

I deeply appreciate the supervision and guidance given me by Professor J. C. C. McKinsey in the writing of this paper.

Problems concerning minimal minervas and completely deflated information patterns arise in the theory of zero-sum two-person games with n moves. We shall show that two information patterns have the same class of minimal minervas if and only if they have the same completely deflated form. We shall treat information patterns in a purely formal way, as independent mathematical entities, making no attempt to relate them to the theory of games.

In this paper we use conventional set-theoretical notation as follows,

$\{a_1, a_2, \ldots, a_n\}$ for the set whose only members are $a_1, a_2, \ldots, a_n$,

$\langle a_1, a_2, \ldots, a_n \rangle$ for ordered n-tuples,

$\hat{x}$ for abstraction, read 'the x's such that' ,

$-$ for difference of sets,

Λ for the null-set,

$\subseteq$ for inclusion,

$\in$ for membership,

and, $\cup$ and $\cap$ for union and intersection.

We denote by I_n the set $\{1, 2, \ldots, n\}$ of the first n positive integers. We denote by $\mathcal{S}_n$ the set of all subsets of I_n. Also, if A is a subset of I_n then we denote $I_n - A$ by $-A$.

Definition 1. An information function[1] is a function F which maps I_n into $\mathcal{S}_n$ in such a way that

$$F(1) = \Lambda$$

$$F(i+1) \subseteq \{1, 2, \ldots, i\} .$$

Definition 2. If S is a subset of I_n, and F is an information function, then we call the ordered couple $\langle S, F \rangle$ a pattern of information.

__Definition 3.__ If $\langle S, F \rangle$ is a pattern of information, then by a __minerva__ with respect to $\langle S, F \rangle$ we mean a sequence $\langle i_1, i_2, \ldots, i_r \rangle$ of (at least two) elements of I_n such that:

(i) $i_1 \in F(i_2),\ i_2 \in F(i_3), \ldots, i_{r-1} \in F(i_r)$;

(ii) if $i_1 \in S$, then $i_k \in -S$ for $k = 2, \ldots, r$;

(iii) if $i_1 \in -S$, then $i_k \in S$ for $k = 2, \ldots, r$.

We call r the __length__ of the minerva, and i_1 and i_r its __terminal elements__. If $\langle j_1, j_2, \ldots, j_s \rangle$ is a subsequence of $\langle i_1, i_2, \ldots, i_r \rangle$ which is itself a minerva, then we call $\langle j_1, j_2, \ldots, j_s \rangle$ a __subminerva__ of $\langle i_1, i_2, \ldots, i_r \rangle$. A subminerva of $\langle i_1, i_2, \ldots, i_r \rangle$ is called __proper__ if it is not identical with $\langle i_1, i_2, \ldots, i_r \rangle$. A minerva is called __minimal__ if it has no proper subminervas with the same terminal elements.

__Remarks.__ It is clear from Definitions 1 and 3 that if $\langle i_1, i_2, \ldots, i_r \rangle$ is a minerva with respect to any information pattern $\langle S, F \rangle$, then,

$$i_1 < i_2 < \cdots < i_r .$$

It is also clear that $\langle i_1, i_2, \ldots, i_r \rangle$ is a minerva (or minimal minerva) with respect to $\langle S, F \rangle$ if and only if it is a minerva (or minimal minerva) with respect to $\langle -S, F \rangle$.

It is easily seen, finally, that a minerva $\langle i_1, \ldots, i_r \rangle$ is a minimal minerva if and only if

$$i_s \notin F(i_k)$$

for $s = 1, \ldots, r-2$ and $k = s+2, \ldots, r$. A minerva of length two is always a minimal minerva.

We now introduce two functions[2] J and J^*, both of which assume subsets of I_n as values. J depends on four arguments A, F, k, i, where

A is a subset of I_n, F is an information function, and k and i are elements of I_n; J^* depends on merely the three arguments A, F, and k.

Definition 4. Let A be any subset of I_n, and F any information function. Then we set:

(i) $\quad J(A,F,k,0) = \Lambda$

for k any element of I_n;

(ii) $\quad J(A,F,k,i+1) = \Lambda$

for k any element of $-A$ and $i+1$ any element of I_n;

(iii) $\quad J(A,F,k,i+1) = \hat{j}\,[j \in A, j < k, \text{ and } F(j) \subseteq F(k) \cup J(A,F,k,i)]$

for k any element of A and $i+1$ any element of I_n.

We set

$$J^*(A,F,k) = J(A,F,k,k-1) .$$

Definition 5. By an immediate deflation of an information pattern $\langle S,G\rangle$ will be meant any information pattern $\langle S,F\rangle$ for which there are integers λ and μ such that:

$$\mu \in G(\lambda) \text{ and } \mu \in J^*(S,G,\lambda) \cup J^*(-S,G,\lambda);$$

$$F(\lambda) = G(\lambda) - \{\mu\} ;$$

$$F(i) = G(i) \quad \text{for } i \neq \lambda .$$

Definition 6. An information pattern is said to be completely deflated if it does not possess any immediate deflations.

Remark. It is seen immediately from Definitions 5 and 6 that an information pattern $\langle S,F\rangle$ is completely deflated if and only if the following condition holds for all j and k in I_n:

$$\text{if } j \in J^*(S,F,k) \cup J^*(-S,F,k), \text{ then } j \notin F(k).$$

It is also clear that, starting with any information pattern $\langle S,G\rangle$ we can, by successive deflations, obtain an information pattern

$\langle S,F\rangle$ which is completely deflated; when $\langle S,F\rangle$ and $\langle S,G\rangle$ are so related, we call $\langle S,F\rangle$ a <u>completely deflated form</u> of $\langle S,G\rangle$. We shall see later that an information pattern has only one completely deflated form.

<u>Theorem 1</u>. If $\langle S,F\rangle$ is an information pattern, A any subset of I_n, and k any member of I_n, then

$$J(A,F,k,i) \subseteq J(A,F,k,i+1)$$

for $i+1<k$.

<u>Proof</u>. This will be proved by an induction on i. If $i=0$, then $J(A,F,k,i)=\Lambda$, so, clearly,

$$J(A,F,k,i) \subseteq J(A,F,k,i+1).$$

Now we wish to show that if

$$J(A,F,k,i) \subseteq J(A,F,k,i+1),$$

then

$$J(A,F,k,i+1) \subseteq J(A,F,k,i+2).$$

Let j be any member of $J(A,F,k,i+1)$. Then

$$F(j) \subseteq F(k) \cup J(A,F,k,i)$$

and hence, using the induction hypothesis,

$$F(j) \subseteq F(k) \cup J(A,F,k,i+1).$$

Hence $j \in J(A,F,k,i+2)$, as was to be shown.

<u>Lemma 1</u>. Let $\langle i_1, \ldots, i_r\rangle$ be a minimal minerva with respect to $\langle S,F\rangle$, and let i_s and i_k be any members of $\{i_1, \ldots, i_r\}$. Then

$$i_s \notin J^*(S,F,i_k) \cup J^*(-S,F,i_k).$$

<u>Proof</u>. We shall prove only that $i_s \notin J^*(S,F,i_k)$. The proof that $i_s \notin J^*(-S,F,i_k)$ would be analogous.

If $s \geqslant k$, then $i_s \geqslant i_k$ by Definitions 1 and 3, and hence $i_s \notin J^*(S,F,i_k)$ by Definition 4 (iii). Hence we need consider only the case that $s < k$. We now distinguish two cases, according as

$$i_1 \in S$$
$$i_2, i_3, \ldots, i_r \in -S$$

or

$$i_1 \in -S$$
$$i_2, i_3, \ldots, i_r \in S.$$

In the first case, we see by Definition 4 (ii) that, for $k = 2, \ldots, r$,

$$J^*(S,F,i_k) = \Lambda ,$$

so that, clearly,

$$i_s \notin J^*(S,F,i_k)$$

for $s < k$.

Thus we are left with the case that $s < k$ and

(1) $\quad i_1 \in -S$

(2) $\quad i_2, i_3, \ldots, i_r \in S.$

We shall prove the lemma for this case by an induction on s.

For the case $s = 1$, the lemma is obvious, since $J^*(S,F,i_k)$, for $k = 2, \ldots, r$, contains only members of S.

Now we wish to show that, if $i_s \notin J^*(S,F,i_k)$ for $k = s+1, \ldots, r$, then $i_{s+1} \notin J^*(S,F,i_k)$ for $k = s+2, \ldots, r$ — which is to show that

$$F(i_{s+1}) \not\subseteq F(i_k) \cup J(S,F,i_k,i_k - 2)$$

for $k = s+2, \ldots, r$. Since $\langle i_1, \ldots, i_r \rangle$ is a minimal minerva, $i_s \notin F(i_k)$ for $k = s+2, \ldots, r$, and by the induction hypothesis $i_s \notin J^*(S,F,i_k)$. Thus $i_s \notin J(S,F,i_k,i_k - 1)$, and hence, by Theorem 1,

$i_s \notin J(S,F,i_k,i_k - 2)$. But $i_s \in F(i_{s+1})$, and therefore

$$F(i_{s+1}) \nsubseteq F(i_k) \cup J(S,F,i_k,i_k - 2),$$

as was to be shown.

<u>Lemma 2</u>. Let $\langle S,F \rangle$ be an immediate deflation of $\langle S,G \rangle$ and let λ and μ be integers such that,

$$F(j) = G(j) \quad \text{for } j \neq \lambda ,$$

$$F(\lambda) \cup \{\mu\} = G(\lambda) ,$$

$$\mu \notin F(\lambda) ,$$

and suppose that δ is an integer less than μ, and that $\delta \in J(S,G,\lambda,i)$ for some i in I_n; then $\delta \in J(S,F,\lambda,i)$.

<u>Proof</u>. This will be proved by an induction on i. If $i = 0$, then

$$J(S,G,\lambda,i) = \Lambda$$

and the lemma is vacuously true. Now we wish to show that if our lemma is true for $i = k$, then it is true for $i = k+1$. By hypothesis,

$$\delta \in J(S,G,\lambda,i+1),$$

and therefore,

$$G(\delta) \subseteq G(\lambda) \cup J(S,G,\lambda,i).$$

Since $\delta < \mu < \lambda$, $F(\delta) = G(\delta)$, and therefore,

$$F(\delta) \subseteq G(\lambda) \cup J(S,G,\lambda,i).$$

Now $G(\lambda) = F(\lambda) \cup \{\mu\}$ and $\mu \notin F(\delta)$; thus we see that

$$F(\delta) \subseteq F(\lambda) \cup J(S,G,\lambda,i).$$

If δ' is any integer that belongs to $F(\delta)$ and to $J(S,G,\lambda,i)$, then $\delta' < \delta < \mu$ and by the induction hypothesis $\delta' \in J(S,F,\lambda,i)$. Hence

$$F(\delta) \subseteq F(\lambda) \cup J(S,F,\lambda,i)$$

and

$$\delta \in J(S,F,\lambda,i\ 1)$$

as was to be shown.

The proof of the next lemma, which is very similar to the proof of Lemma 2, will be omitted.

Lemma 3. Let $\langle S, F \rangle$ be an immediate deflation of $\langle S, G \rangle$, and let λ and μ be integers such that

$$F(j) = G(j) \quad \text{for } j \neq \lambda \text{ ,}$$

$$F(\lambda) \cup \{\mu\} = G(\lambda),$$

$$\mu \notin F(\lambda) \text{ ,}$$

and suppose that δ is an integer less than μ, and that $\delta \in J(-S, G, \lambda, i)$ for some i in I_n; then $\delta \in J(-S, F, \lambda, i)$.

Lemma 4. Let $\langle S, F \rangle$ be an immediate deflation of $\langle S, G \rangle$, and let λ and μ be integers such that

(i) $\quad F(j) = G(j) \quad \text{for } j \neq \lambda$,

(ii) $\quad \mu \in J^*(S, G, \lambda) \cup J^*(-S, G, \lambda)$,

(iii) $\quad \mu \notin F(\lambda)$

(iv) $\quad G(\lambda) = F(\lambda) \cup \{\mu\}$.

Then

$$\mu \in J^*(S, F, \lambda) \cup J^*(-S, F, \lambda) \text{ .}$$

Proof. We shall prove the lemma for the case that $\mu \in J^*(S, G, \lambda)$. The proof in case $\mu \in J^*(-S, G, \lambda)$ is very similar.

Since

$$\mu \in J^*(S, G, \lambda)$$

we have

(1) $\quad G(\mu) \subseteq G(\lambda) \cup J(S, G, \lambda, \lambda - 2)$.

Now let δ be an arbitrary member of $F(\mu)$. Since $\delta \in F(\mu)$, we have

(2) $\quad \delta < \mu$.

Since, moreover, by (iv) of the hypothesis of our lemma we have $\mu \in G(\lambda)$, we see that

(3) $\mu < \lambda$.

From (3), and (i) of the hypothesis, it follows that

$$F(\mu) = G(\mu),$$

so that

$$\delta \in G(\mu).$$

From (1) we therefore have

$$\delta \in G(\lambda) \cup J(S, G, \lambda, \lambda - 2),$$

so that either $\delta \in G(\lambda)$ or $\delta \in J(S, G, \lambda, \lambda - 2)$. If $\delta \in G(\lambda)$, then from (2), together with (iv), we have

$$\delta \in F(\lambda).$$

If $\delta \in J(S, G, \lambda, \lambda - 2)$, then, by (2) and Lemma 2, we see that

$$\delta \in J(S, F, \lambda, \lambda - 2).$$

Thus every element δ of $F(\mu)$ belongs either to $F(\lambda)$ or to $J(S, F, \lambda, \lambda - 2)$, so we conclude that

$$F(\mu) \subseteq F(\lambda) \cup J(S, F, \lambda, \lambda - 2),$$

and hence

$$\mu \in J^*(S, F, \lambda),$$

as was to be shown.

<u>Theorem 2</u>. If $\langle S, F \rangle$ is an immediate deflation of $\langle S, G \rangle$, then $\langle S, F \rangle$ and $\langle S, G \rangle$ have the same class of minimal minervas.

<u>Proof</u>. Since $\langle S, F \rangle$ is an immediate deflation of $\langle S, G \rangle$, there are integers λ and μ such that

$$F(i) = G(i) \quad \text{for } i \neq \lambda,$$

$$F(\lambda) \cup \{\mu\} = G(\lambda),$$

$\mu \notin F(\lambda)$,

$\mu \in J^*(S, G, \lambda) \cup J^*(-S, G, \lambda)$.

It is immediately apparent from these conditions that a sequence $\langle i_1, i_2, \ldots, i_r \rangle$, where $i_s \neq \mu$ for $s = 1, \ldots, r$, is a minimal minerva with respect to $\langle S, F \rangle$ if and only if it is a minimal minerva with respect to $\langle S, G \rangle$; and the same is true for sequences $\langle i_1, i_2, \ldots, i_r \rangle$, where $i_s \neq \lambda$ for $s = 1, \ldots, r$.

Hence we can restrict ourselves to sequences $\langle i_1, i_2, \ldots, \mu, \ldots, \lambda, \ldots, i_r \rangle$. Moreover, since by hypothesis

$\mu \in J^*(S, G, \lambda) \cup J^*(-S, G, \lambda)$,

we see by Lemma 1 that there are no such minimal minervas with respect to $\langle S, G \rangle$. Finally, by Lemma 4, we see that

$\mu \in J^*(S, F, \lambda) \cup J^*(-S, F, \lambda)$;

hence, again by Lemma 1, we see that there are no such minimal minervas with respect to $\langle S, F \rangle$.

<u>Lemma 5</u>. Let $\langle S, F \rangle$ be a completely deflated information pattern; let i_1 and i_2 be integers, both of which belong to S, and such that $i_2 \in F(i_1)$; and let r be an integer greater than 2. Then either

(A) there is an integer s satisfying $3 \leq s \leq r$, and elements $i_3, \ldots, i_s$ of I_n such that $\langle i_s, \ldots, i_3, i_2, i_1 \rangle$ is a minerva, and $i_j \notin F(i_1)$ for $j = 3, \ldots, s$, or

(B) there are elements $i_3, \ldots, i_r$ of I_n such that the r-tuple $\langle i_r, \ldots, i_3, i_2, i_1 \rangle$ satisfies the following conditions

(α) $i_j \in S$ for $j = 1, \ldots, r$,

(β) $i_j \in F(i_{j-1})$ for $j = 2, \ldots, r$,

(γ) $i_j \notin F(i_1) \cup J(S, F, i_1, i_1 - j + 1)$ for $j = 3, \ldots, r$.

Proof. This will be proved by an induction on r.

Let $r=3$. Since $\langle S,F\rangle$ is completely deflated and $i_2 \in F(i_1)$, then (cf. Remark following Definition 6)

$$i_2 \notin J(S,F,i_1,i_1-1) .$$

Therefore

$$F(i_2) \nsubseteq F(i_1) \cup J(S,F,i_1,i_1-2) ,$$

and there exists an integer i_3 such that

$$i_3 \in F(i_2)$$

and

$$i_3 \notin F(i_1) \cup J(S,F,i_1,i_1-2) .$$

If $i_3 \in -S$, then $\langle i_3,i_2,i_1\rangle$ is a minerva, and the elements i_3,i_2,i_1 satisfy condition (A). If $i_3 \in S$ then $\langle i_3,i_2,i_1\rangle$ satisfies condition (B).

Now we want to show that, if our lemma is true for $r=k$, it is also true for $r=k+1$. If condition (A) holds for $r=k$, it holds *a fortiori* for $r=k+1$; for if $s \leq k$ then certainly $s \leq k+1$. Hence we suppose that there is a k-tuple $\langle i_k, \ldots, i_2,i_1\rangle$ satisfying condition (B). Since, then

$$i_k \notin F(i_1) \cup J(S,F,i_1,i_1-k+1)$$

and hence

$$F(i_k) \nsubseteq F(i_1) \cup J(S,F,i_1,i_1-k+1-1) =$$
$$F(i_1) \cup J(S,F,i_1,i_1-(k+1)+1).$$

Thus there exists an integer i_{k+1} such that;

$$i_{k+1} \in F(i_k)$$
$$i_{k+1} \notin F(i_1) \cup J(S,F,i_1,i_1-(k+1)+1) .$$

Thus we conclude that the $k+1$ - tuple

$$\langle i_{k+1},i_k, \ldots, i_2,i_1\rangle$$

satisfies condition (A) or condition (B), according as $i_{k+1} \in -S$ or $i_{k+1} \in S$, which completes the proof.

The proof of the next lemma, which is very similar to the proof of Lemma 5, will be omitted.

Lemma 6. Let $\langle S,F \rangle$ be a completely deflated information pattern; let i_1 and i_2 be integers, both of which belong to $-S$, and such that $i_2 \in F(i_1)$; and let r be an integer greater than 2. Then either:

(A) there is an integer s satisfying $3 \leq s \leq r$ and elements $i_3, \ldots, i_s$ of I_n such that $\langle i_s, \ldots, i_3, i_2, i_1 \rangle$ is a minerva and $i_j \notin F(i_1)$ for $j = 3, \ldots, s$, or

(B) there are elements $i_3, \ldots, i_r$ of I_n such that the r-tuple $\langle i_r, \ldots, i_3, i_2, i_1 \rangle$ satisfies the following conditions

(α) $i_j \in -S$ for $j = 1, \ldots, r$,

(β) $i_j \in F(i_{j-1})$ for $j = 2, \ldots, r$,

(γ) $i_j \notin F(i_1) \cup J(-S,F,i_1, j+1)$ for $j = 3, \ldots, r$.

Lemma 7. Let $\langle i_1, i_2, \ldots, i_r \rangle$ be a minerva with respect to $\langle S,F \rangle$ such that, for $j = 1, \ldots, r-2$, $i_j \notin F(i_r)$. Then there exists a subsequence $\langle \mu_1, \mu_2, \ldots, \mu_s \rangle$ of $\langle 1, 2, \ldots, r-2 \rangle$ such that

$$\langle i_{\mu_1}, i_{\mu_2}, \ldots, i_{\mu_s}, i_{r-1}, i_r \rangle$$

is a minimal minerva with respect to $\langle S,F \rangle$.

Proof. This will be proved by an induction on r. If $r = 2$, then $\langle i_1, i_2 \rangle$ is itself a minimal minerva with respect to $\langle S,F \rangle$. Assume the lemma is true for $r \leq k$, and let $\langle i_1, i_2, \ldots, i_k, i_{k+1} \rangle$ be a minerva such that for $j = 1, \ldots, k-1$, $i_j \notin F(i_{k+1})$. If $\langle i_1, i_2, \ldots, i_k, i_{k+1} \rangle$ is not a minimal minerva then $i_\alpha \in F(i_\beta)$ for some α and β satisfying

$$1 \le \alpha \le \beta - 1 \le k - 1 .$$

Hence, if $i_{\alpha+1}, \ldots, i_{\beta-1}$ is left out, the resulting sequence will still be a minerva. This minerva is of length less than k and our lemma now follows by means of the induction hypothesis.

Lemma 8. Let $\langle S,F \rangle$ be a completely deflated information pattern; and let i_1 and i_2 be integers, both of which belong to S, or both of which belong to -S, and let $i_2 \in F(i_1)$. Then there are elements $i_3, \ldots, i_t$ of I_n such that $\langle i_t, \ldots, i_3, i_2, i_1 \rangle$ is a minimal minerva.

Proof. Taking $r = i_1 + 1$, we see that there cannot be elements $i_3, \ldots, i_r$ of I_n satisfying condition (B) of Lemma 5; for condition (B) would imply

$$i_r < i_{r-1} < i_{r-2} < \ldots < i_2 < i_1$$

and hence i_r would have to be negative, contrary to the definition of I_n.

Hence there is a minerva $\langle i_s, \ldots, i_2, i_1 \rangle$ such that:

$$i_j \notin F(i_1) \quad \text{for } j = 3, \ldots, s .$$

By Lemma 7 there then exists a subsequence $\langle \mu_1, \mu_2, \ldots, \mu_s \rangle$ of $\langle s, s-1, \ldots, 3 \rangle$ such that

$$\langle i_{\mu_1}, i_{\mu_2}, \ldots, i_{\mu_s}, i_2, i_1 \rangle$$

is a minimal minerva.

Theorem 3. If $\langle S,F \rangle$ and $\langle S,G \rangle$ are both completely deflated information patterns, and $F \neq G$, then $\langle S,F \rangle$ and $\langle S,G \rangle$ have different classes of minimal minervas.

Proof. Let j be an integer for which F and G are different. We suppose that $j \notin S$ (if $j \in -S$ the proof is similar). Without loss of generality we can assume that there exists an integer k which belongs to

$F(j)$ but not to $G(j)$. If $k \in -S$ then $\langle k, j \rangle$ is a minimal minerva with respect to $\langle S, F \rangle$. Since $k \notin G(j)$, $\langle k, j \rangle$ is not a minimal minerva with respect to $\langle S, G \rangle$. Suppose that $k \in S$; then by Lemma 8 there exist integers $i_1, i_2, \ldots, i_r$ such that $\langle i_1, i_2, \ldots, i_r, k, j \rangle$ is a minimal minerva with respect to $\langle S, F \rangle$ but not with respect to $\langle S, G \rangle$ since $k \notin G(j)$. Hence we conclude that $\langle S, F \rangle$ and $\langle S, G \rangle$ have different classes of minimal minervas.

Theorem 4. Every information pattern has a unique completely deflated form.

Proof. From a previous remark we know that every information pattern has at least one completely deflated form.

Let $\langle S, F_1 \rangle$ and $\langle S, F_2 \rangle$ be completely deflated forms of the information pattern $\langle S, F \rangle$. Then, by Theorem 2, we see that $\langle S, F_1 \rangle$ and $\langle S, F \rangle$ have the same class of minimal minervas; and similarly $\langle S, F_2 \rangle$ and $\langle S, F \rangle$ have the same class of minimal minervas. Thus $\langle S, F_1 \rangle$ and $\langle S, F_2 \rangle$ have the same class of minimal minervas, so by Theorem 3 we conclude that $F_1 = F_2$, as was to be shown.

Theorem 5. Two information patterns have the same class of minimal minervas if and only if they have the same completely deflated form.

Proof. By Theorems 2 and 3.

FOOTNOTES

1). See McKinsey [1]. This definition and the following two are due to J. C. C. McKinsey.

2). See Quine [2]. This definition is due to W. V. Quine. The definition given here differs from that given by Quine in two ways. First, in the definition of $J^*(A,F,k,i+1)$, we impose the condition that j be less than k. Second, Quine defines

$$J^*(A,F,k) = \sum_{i} J(A,F,k,i),$$

while we define

$$J^*(A,F,k) = J(A,F,k,k-1).$$

However, we shall see, by Theorem 1, that

$$J(A,F,k,k-1) = \sum_{j<k} J(A,F,k,i).$$

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Typist: Mrs. J. P. Gardner