SOME FIXED POINT THEOREMS FOR COMPACT CONTINUA IN METRIC SPACES

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A famous unsolved problem is this: Does every continuous transformation of a bounded continuum M in the plane onto a subset of itself, where M does not separate the plane, leave some point of M invariant? The question has been answered in the affirmative for most continua which are not indecomposable. A solution of the problem for several simple types of continua is given here. These include the 1-cell, the 2-cell, and the n-cell, or respectively, any homeomorphic image of the unit interval, the unit circle, or the unit sphere in Euclidean n-space. In addition it is shown that locally connected continua which do not contain a simple closed curve, variously called dendrites, acyclic curves, or trees, possess the fixed point property.

First the following lemma is established:

Lemma 1.

If a point set M has the property that every continuous transformation of M into itself leaves some point of M invariant, and if M' is any homeomorphic image of M, then M' has this property.

Proof: Let T be a homeomorphism which carries M into M', and T' any continuous transformation of M' into itself. Let T'' be a transformation on M defined as follows: $T''(x) = T^{-1}(T'(T(x)))$. That T'' is a continuous transformation of M into itself follows from the fact that T^{-1}, T', and T are continuous transformations. But by hypothesis some point x_0 of M is invariant under the transformation T'', so that $T''(x_0) = x_0$. Therefore:

$$T''(x_0) = x_0 = T^{-1}(T'(T(x_0))), \text{ so that:}$$

$$T(x_0) = T(T^{-1}(T'(T(x_0)))) = T'(T(x_0)).$$

But then $T(x_0)$ is a point of M' which is invariant under T'. The lemma is therefore true.

Theorm 1.

If M is a 1-cell, then every continuous transformation T of M into itself leaves some point of M invariant.

Proof: A 1-cell is any homeomorphic image of the unit interval I. By Lemma 1 it is sufficient to show that every continuous transformation T of I into itself leaves some point of I invariant.

Assume that T is a continuous transformation of I into itself which leaves no point of M invariant. Let H be the set of all points in I such that $x < T(x)$ where x and T(x) are the coordinants of the point and its image. Let K be the set of points for which $x > T(x)$. That H is closed may be shown as follows. Suppose that x_0 is a limit point of H which is not in H. Let (x_i) be a sequence in H converging to x_0. Then since T is continuous, the sequence $(T(x_i))$ converges to $T(x_0)$. By assumption $x_0 > T(x_0)$. Let E_1 and E_2 be disjoint neighborhoods of radius ϵ containing x_0 and $T(x_0)$ respectively. Then for some integer k if $n > k$, x_n is in E_1, or $x_0 - \epsilon < x_n < x_0 + \epsilon$, and $T(x_n)$ is in E_2, or $T(x_0) - \epsilon < T(x_n) < T(x_0) + \epsilon$. But $T(x_0) + \epsilon < x_0 - \epsilon$, and hence $T(x_n) < x_n$ for $n > k$. This contradicts the fact that x_n is in H, for which by definition, $x_n < T(x_n)$. Therefore, the assumption that

H is not closed is false, and hence H is closed. Similarly K is closed. Since x and T(x) are by assumption distinct for each x, every point of the continuum I is in either H or K. Then H and K have a point or a limit point in common. Being closed sets they, therefore, have a point z in common. But this means that $T(z) < z$ and $T(z) > z$, which is impossible. Hence the assumption that for each x, x and T(x) are distinct, is false, so that I has the fixed point property.

In the next theorm use is made of the following definition:

A continuous transformation T of a point set X into a point set Y is said to be inessential if there exists a function F(p,t), continuous in p and t separately, and a point z such that:

1. For each fixed t in the unit interval I, F(p,t) is a continuous transformation of X into Y.

2. For a fixed point p in X, F(p,t) is a continuous transformation of I into Y.

3. $F(p,0) = T(p)$ for every p in X.

4. $F(p,1) = z$ for every p in X.

Theorm 2.

If M is a 2-cell, then every continuous transformation T of M into itself leaves some point of M invariant.

Proof: First, it is shown that under the assumption that no point of M is left invariant, there exists a continuous mapping T' of M into its boundary S_2 which leaves each point of the boundary fixed. If x and T(x) are distinct points, then a ray from T(x) through x is uniquely defined. Let T'(x) be a transformation which carries a point x in M into a point y on the boundary, where y is the intersection of the ray from T(x) through x with the boundary of M.

If x is on the boundary, then the intersection of the ray from $T(x)$ through x is x, so that $T'(x)$ leaves the points of the boundary fixed. If x_0 is a point of M and (x_i) a sequence converging to x_0, then the sequence $(T(x_i))$ converges to $T(x_0)$. Hence the ray from $T(x_i)$ to x_i converges to the ray from $T(x_0)$ to x_0, and the points on the boundary $T'(x_i)$ converge to $T'(x_0)$. $T'(x)$ is therefore continuous.

Second, it follows that if there is a continuous transformation T' of M into its boundary S_2 which leaves each point of the boundary fixed, then the identity transformation on the boundary is inessential. For consider the function $F(x,t) = T'((1-t)x)$, where x is a point on the boundary and $(1-t)x$ is the point in M obtained by multiplying the components of x by $(1-t)$. It is clear that the function is continuous in both variables, that $F(x,0) = T'(x) = x$, and that $F(x,1) = T'(0)$ for each x. Hence by definition the identity transformation is inessential.

Third, it is shown that the identity transformation of the boundary circle S_2 of M is not inessential. For, suppose that the identity transformation is inessential. Then there is a function $F(x,t)$ and a point z satisfying the conditions in the definition.

Let $\sigma(p,q)$ be a real valued function on the circle S_2 defined as follows: $\sigma(p,q)$ is the length of the smallest arc from p to q, positive if measured in a counter clockwise direction on S_2, and negative if measured in a clockwise direction. This function is uniquely defined for those points p and q for which we use it as is shown below. It follows from the continuity of $F(x,t)$ that, given any positive number ϵ, there are numbers δ_1 and δ_2, such that, if $d(x,y) < \delta_1$ and $|t_1 - t_2| < \delta_2$, then $|\sigma(F(x,t_1), F(y,t_2))| < \epsilon$. From this, given any positive number ϵ, there exists an N_ϵ, namely the least integer greater than $1/\delta_2$ in the statement above, such that if $n > N_\epsilon$ and if $t_i = i/n$, $i = (0,1,2,\ldots,n)$, then $|\sigma(F(x,t_i), F(x,t_{i-1}))| < \epsilon$ for every x. Letting $\epsilon = \pi/2$, there is then an $N_{\pi/2}$, for which $|\sigma(F(x,t_i) - F(x,t_{i-1}))| < \pi/2$. But then for $n' > N$, there is only one smallest arc from $F(x,t_i)$ to $F(x,t_{i-1})$ for each i and x, so that $\sigma(F(x,t_i), F(x,t_{i-1}))$ is unique, and is therefore a continuous function of x for each i.

Let $V(x) = \sum_{i=1}^{n'} \sigma(F(x,t_i), F(x,t_{i-1}))$, where n' is a fixed number greater than $N_{\pi/2}$. The function V(x), then, is the sum of a finite number of continuous functions of x, and therefore is itself a continuous function of x.

The function V(x) is a continuous real valued function which gives the variation of x as t takes on values from 0 to 1 in the prescribed manner. Since $F(z,0) = z$ and $F(z,1) = z$, V(z) has the value $2n\pi$, where n is one of the numbers $0, \pm1, \pm2, \ldots$. Further, it is clear that z is the only x on S_2 for which V(x) is an integral multiple of 2π.

For some points x on the circle, $V(x) < 2n\pi$. If not, then for every point x, $V(x) \geqslant 2n\pi$. Let x_n be a point such that $\sigma(z,x_n) = -1/n$. The sequence (x_n) converges to z. Further $V(x_n) = 2n\pi + 2\pi - 1/n$, for each n, since $V(x_n) > 2n\pi$, and the length of the arc from x to z in a positive direction is $(2\pi - 1/n)$. The sequence $(V(x_n))$, then converges to $2n\pi + 2\pi = 2(n+1)\pi$. But since $V(z) = 2n\pi$, $(V(x_n))$ does not converge to V(z), and V(x) is not continuous. This is a contradiction. Therefore the assumption that for every x, $V(x) \geqslant 2n\pi$ is false, so that for some x, $V(x) < 2n\pi$. Similarly for some y, $V(y) > 2n\pi$.

Let H be the set of all x on the circle for which $V(x) \leqslant 2n\pi$, and K the set of all y for which $V(y) \geqslant 2n\pi$. The set H is closed, for if x_0 is a boundary point of H and (x_i) a sequence in H converging to x_0, then by continuity $(V(x_i))$ converges to $V(x_0)$. But $V(x_i) \leq 2n\pi$ for each i, so that $V(x_0) \leq 2n\pi$, which means that x_0 belongs to H. Similarly the set K is closed. Clearly every point of the circle is in one of the closed sets H or K. Hence the sets H and K must have at least two boundary points in common on S_2. But then for some point x' distinct from z, V(x') is in both H and K, and therefore $V(x') = 2n\pi$, which is impossible. Therefore the assumption that the identity transformation on S_2 is inessential is false.

Hence the identity transformation is not inessential, contradicting the result proved in part two of the proof under the assumption that no point is left invariant under T. Hence the theorm is true.

The fixed point theorm for an n-cell may be proved in an analogous way.

Theorm 3.

If M is an n-cell and if T is any continuous transformation of M into itself, then some point of M is left invariant under the transformation T.

Proof: An n-cell is the homeomorphic image of the region R_{n+1} in E_{n+1} space bounded by the unit S_n sphere. Suppose that T is any continuous transformation of R_{n+1} into itself such that for each x, T(x) and x are distinct. Then there is a continuous mapping T'(y) of the interior of R_{n+1} into its boundary S_n which leaves each point of the boundary fixed. For let T'(x) carry x into the point on the boundary S_n of R_{n+1} which is the intersection of the ray from T(x) through x with the boundary. It follows then that the identity mapping of S_n is inessential, for let F(x,t) be a function defined as follows:

$F(x,t) = T'((1-t)x)$, where x is on the n-sphere. Then the function $F(x,t)$ is continuous, and $F(x,0) = x$, and $F(x,1) = z$. But this contradicts the fact that the identity transformation is not inessential.[1] Therefore the theorm is true.

The next theorm shows that certain locally connected continua possess the fixed point property.

A point set M is said to be locally connected at a point p, if p belongs to M and every open subset of M that contains p contains a connected open subset of M containing p.

If M is a locally connected continuum, then M is a connected im kleinem inner limiting set.[2] M may then be considered a Moore space satisfying axioms 0, 1, and 2, in which regions are open connected sets.[3] Hence if p and q are any two points of such a region, then there is an arc from p to q contained in the region.[4] Use is made of this in the following lemmas and theorms.

Lemma 2.

If M is a locally connected continuum which contains no simple closed curve, then M has the property that one and only one arc exists between any two of its points.

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[1]Hurewicz, Witold and Wallman, Henry, Dimension Theory, pp. 37-39.

[2]Moore, R. L., "Foundations of Point Set Theory," American Mathematical Society Colloquium Publications, XIII, p. 94.

[3]Ibid., p. 96.

[4]Ibid., p. 86.

Proof: That M has the property that one and only one arc exists between any two of its points may be shown as follows: Suppose p and q are two points of M. Then, since M is locally connected, it is a connected space in which regions are open connected sets, so that there exists an arc from p to q contained in M. Further, there cannot exist two distinct arcs from p to q. For suppose paq and pbq are two distinct arcs from p to q. The intersection of the two arcs is a closed set, and the component of the intersection which contains p is a closed set. Consequently there is a last point r of this component in the order from p to q on paq and pbq. There is also a first point s of the remainder of the intersection.[5] The arcs rs on paq and rs' on pbq have only r and s in common, thus forming a simple closed curve in M. This contradicts an hypothesis of the theorm, and the assumption that two distinct arcs exist from p to q is false. The lemma is therefore true.

Let M be a continuum having the property that one and only one arc exists between any two of its points. Then a branch Cy with respect to a point y of an arc AB of M is the set of all points x in M for which the arc xy contains no point of AB except y.

Lemma 3.

If M is a continuum having the property that one and only one arc exists between any two of its points, and M contains no simple closed curve, and if AB is an arc in M, then every point in M not in AB belongs to one and only one branch.

Proof: To show that a point p in M not on AB belongs to at least one branch, consider the arc pB from p to B. Let p' be the first

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[5] Ibid., Theorm 64(a), p. 45.

point of pB on the arc AB in the order from p to B. Then the arc pp' contains no point of AB except p'. Hence p belongs to the branch Cp'. Now suppose that p belongs to two branches Cx and Cy, where x and y are distinct. Let p' be the first point of the intersection of the arcs yp and xp on xp in the order from x to p. Then the arcs yp', xp', and xy form a simple closed curve in M, contradicting an hypothesis of the lemma.

Lemma 4.

If M is a locally connected continuum and M contains no simple closed curve, and if AB is an arc from A to B in M, and Cx and Cy are distinct branches with respect to AB, then Cx and Cy are mutually separated closed sets.

Proof: Suppose on the contrary that Cx and Cy are branches which are not mutually separated. Let p be a limit point of Cx contained in Cy. Since M is locally connected, it may be considered a Moore space satisfying axioms 0, 1, and 2, in which regions are open connected sets. Let g be a region containing p. Then from the assumption, g contains a point x' of Cx and y' of Cy. There is an arc from x' to y' in g. Since x' is a point of Cx and y' is contained in an arc containing x', y' is contained in an arc containing x and hence belongs to both Cx and Cy, contradicting Lemma 3. Similarly Cx cannot contain a limit point of Cy. Hence Cx and Cy are mutually separated sets. That the sets are closed then is evident from the fact that M is closed.

Lemma 5.

If M is a locally connected compact continuum and M contains no simple closed curve, and if T is a continuous transformation of M into itself and AB is an arc in M, then there is a point x in AB such that either $T(x)$ is identical with x or $T(x)$ is contained in the branch Cx.

Proof: Suppose that for every x in AB, x and $T(x)$ are distinct and $T(x)$ is not contained in Cx. Consider a division of the arc AB into two sets H and K. Let x be in H if, in case $T(x)$ is on AB, x precedes $T(x)$, or, in case $T(x)$ is not on AB, x precedes y, where $T(x)$ is in the branch Cy. Let x be in K if x follows $T(x)$ when $T(x)$ is on AB, or follows y when $T(x)$ is in the branch Cy.

It is clear that neither H nor K is vacuous, since H contains A and K contains B. Further, every point of AB is in either H or K. For if x is in AB, then $T(x)$ is distinct from x by assumption, and if $T(x)$ is on AB, $T(x)$ must either follow or precede x. If, on the other hand, $T(x)$ is not on AB, then by Lemma 3, $T(x)$ is in Cy for one and only one point y, which is distinct from x by assumption, and x must either precede or follow y.

The set H is closed. If not, there is a limit point p_0 of H which is not in H. Let (p_i) be a sequence in H converging to p_0. Consider the sequence (y_i), where $y_i = T(p_i)$, $(i = 0,1,2,\ldots)$, if $T(p_i)$ is on AB, or is the point y for which $T(y_i)$ is in the branch Cy, in case $T(p_i)$ is not on AB. That the sequence (y_i) converges to y_0 can be shown as follows: First, suppose that $T(p_0)$ is a point of AB. Let R be any region containing y_0. Then there exists an integer k such that for $n > k$, $T(p_n)$ is contained in R. But R is connected and hence contains the arc from $T(p_n)$ to $T(p_0)$. Further, y_n lies on this arc, since otherwise there would exist two distinct arcs from $T(p_n)$ to $T(p_0)$, contradicting Lemma 2. Consequently, for $n > k$, y_n lies in the region R, and y_n converges to $y_0 = T(p_0)$. On the other hand, suppose that $T(p_0)$ is not a point of AB. Then

$T(p_0)$ is in Cy_0. Let R be a region containing $T(p_0)$ which does not contain y_0. Since R is connected, by Lemma 4, it is a subset of Cy_0. But $T(p_i)$ for $i > k$ for some integer k is contained in Cy_0. This means that $y_i = y_0$ for $i > k$, and the sequence (y_i) thus converges to y_0.

As p_0 was assumed to be a limit point of H which was not in H, p_0 is in K. By definition of K, then, y_0, which is $T(p_0)$ in case $T(p_0)$ is on AB, or a branch point if not, precedes p_0 in the order from A to B on AB. However, p_i precedes y_i for every i, since the p_i are in H. There exist disjoint intervals on AB; I_1 containing y_0 and I_2 containing p_0, having the property that if x belongs to I_1 and y belongs to I_2, then x precedes y. Since y_0 is a sequential limit point of the sequence (y_i) the interval I_1 contains y_n for $n > k$ for some integer k. Similarly the interval I_2 contains (p_i) for $n > k'$ for some k'. But then for $n > k$, and $n > k'$, y_n precedes p_n, which contradicts the fact that p_n belongs to H. Therefore the set H is closed.

Similarly the set K is closed. From this and the fact that H and K include all of AB, it follows that H and K have a point p in common. But this is impossible, as distinct points p and T(p) on AB, or p and y for T(p) in the branch Cy, cannot both follow and precede each other. Hence the lemma is true.

Theorm 4.

If M is a locally connected continuum, and if M contains no simple closed curve, then every continuous transformation T of M into itself leaves some point of M invariant.

Let A and B be two points of M and let A_1 designate the arc AB. Suppose that no point of M remains invariant under the transformation T.

Then by Lemma 5 there is a point s_1 in A_1 such that $T(s_1)$ is contained in the branch Cs_1. Let A_2 be an arc in Cs_1 from $T(s_1)$ to s_1. Then A_1 and A_2 have only the point s_1 in common. By Lemma 5, again, there is a point s_2 in A_2 such that $T(s_2)$ is contained in Cs_2. But s_2 is distinct from s_1, since $T(s_1)$ belongs to the arc A_2. Further, Cs_1 and Cs_2 with respect to A_2 have no point or limit point in common by Lemma 4, so that A_1 and A_2 have no point or limit point in common except s_1. In general then a sequence of points (s_i) and a sequence of arcs (A_i) are obtained such that $A_jA_i = 0$ except when $j = i + 1$, when the sets have only the point s_i in common. Further for each i, A_i contains $T(s_i)$.

Since the sequence (s_i) is infinite, there is a point s_0 and a subsequence (s_{n_i}) such that the sequence (s_{n_i}) converges to s_0. Hence the corresponding sequence of arcs (A_{n_i}) has a limiting set A_0, which is a continuum.[6] By continuity of T, the sequence $(T(s_{n_i}))$ converges to $T(s_0)$. That $T(s_0) = S_0$ can be shown as follows:

Assume that $T(s_0)$ and s_0 are distinct points. Let D be a connected domain containing $T(s_0)$ whose closure does not contain s_0. Then there is an integer k such that for $n_i > k$, s_{n_i} is not in $\bar{D}$, and $T(s_{n_i})$ is contained in D. Let A_i' be the component of A_{n_i} containing $T(s_{n_i})$ in $\bar{D}$. Then (A'_i) is an infinite sequence of mutually exclusive continua. Further, since A_{n_i} contains s_{n_i} not in $\bar{D}$, the boundary of D contains a point of A_i' for each i. This is a sufficient condition that M be not connected im kleinem at $T(s_0)$, and hence not locally connected at $T(s_0)$.[7] This contradicts an hypothesis of the theorm. Therefore, the assumption that $T(s_0)$ is distinct from s_0 is false, and the theorm is true.

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[6] Ibid., Theorm 42, p. 28.

[7] Ibid., Theorm 8, p. 95.

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