A METHOD FOR SOLVING THE SYNTHESIS PROBLEM WITH SUGGESTED APPLICATIONS

By

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PREFACE

One of the most difficult problems facing the design engineer is the determination of networks which will have given response characteristics. In general, the determination of these networks, known as the synthesis problem, is extremely difficult; in fact all methods presently available are approximation methods. In this paper an entirely new approach for solving this problem is developed whereby input and output time functions are used to obtain network characteristics. Knowing this, a method is given whereby an appropriate network may be constructed.

ACKNOWLEDGEMENTS

The author is indebted to Dr. R. G. Piety of Phillips Petroleum Company who pointed out the basic mathematical relationships which led to the methods developed in this paper for solving the synthesis problem. It was his suggestion that this subject be developed as a contribution to the existing literature. The author is also indebted to Dr. H. L. Jones of Oklahoma A. & M. for his suggestions on methods for presenting the developments of this paper.

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INTRODUCTION

This Thesis will be divided into three parts, which will be explained here. In the first part, entitled "Networks From the Zero and Pole Point of View" is discussed certain properties of networks which have been extracted from the literature, and which are helpful to an understanding of this paper. It is believed that the method of presentation will give the engineer a clear picture of the significance of the application of the pole and zero theory as applied to communication networks. There is also presented a method for obtaining a duplicating or compensating network when the phase and amplitude versus frequency characteristics are known for a linear network whose component construction is unknown.

The second portion of the Thesis entitled "A Method for Solving the Synthesis Problem", deals with the Synthesis Problem. The synthesis problem may be stated in the following manner. Given a four terminal linear network whose internal construction is unknown, and given the input and output time functions, to construct a network which has the same characteristic as the given network. By using the input and output time functions, a technique is developed which allows for the construction of a plot of the phase and amplitude characteristics of the unknown network. The employment of the technique described in the first part of the paper then makes it possible to duplicate or, in some cases, compensate a given network. Several examples are given which indicate the results to be expected from this method.

The solution to the synthesis problem as developed in this paper is especially applicable to the servo-mechanism control problem. To discuss a simple example, assume a gas furnace is given and it is desired to control its temperature. For a temperature measuring devices thermocouples are used which develop an output voltage which is a function of the temperature. This voltage is to be fed to a system to control the gas valve, which in turn controls the furnace temperature. One way to derive an output time function for the thermocouples is to suddenly increase the gas pressure a known amount. This results in a unit step function, and makes a recording of the thermocouple output voltage against time. Using the techniques described in this paper, a network can be found which has the same response, on a voltage basis, to a unit step function. With this information an electrical control system can be designed to obtain an optimum control of the furnace temperature. In other words, a logical system will be developed for determining a phase and amplitude versus frequency characteristic from an input and output time function. It is also shown how to use the information to obtain correction in a control system.

To expand the discussion further, using the techniques outlined, an electrical network equivalent can be found for ANY linear system when its input and output time functions are known. For example, the displacement time function of the output of a lever system for a given input displacement time function could be known. Letting voltage equal displacement, an electrical network can be found which will have an output voltage proportional to input voltage, and these voltages would be directly proportional to the displacements in the original system. Therefore it is evident that the method is not limited to electrical systems alone, although only electrical systems will be discussed in the body of the paper.

The third part, entitled "Suggest Applications", in a discussion of the general developments applied to servomechanism design. The development is in terms of poles and zeros, and the method of attack proposed is different from that pursued in current literature. One purpose of this paper is to show that network concepts can be explained in terms of poles and zeros; and an endeavor has been made to use only this concept in each step of the discussion. It is hoped that this paper will result in a more general use of the pole and zero network concept.

It is assumed that the reader is acquainted with Fourier and LaPlacian Transforms[1] and their application to network

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[1] Cambell, G. A. and Foster, R. M., Fourier Integrals for Practical Applications

Gardner, M. F. and Barnes, J. L., Transients in Linear Systems

Goldman, Stanford, Transformation Calculus and Electrical Transients

The above books will be referred to constantly throughout this paper, and the following notation has been adopted. C&F will refer to Campbell and Foster, G&B will refer to Gardner and Barnes, and G to Goldman. If a number or a letter follows the above symbols, this indicates the transform applicable in that volume; if the symbol is followed by P. and a number, that is the page number. Although Fourier and LaPlacian Transform Tables are not always directly interchangeable (G P. 225), in this paper none of the exceptional cases arise.

analysis. An acquaintance with the Theory of Functions of a Complex Variable[2] is helpful in understanding the development of the plotting techniques described in the first part of the paper. Sufficient information has been included, however, to allow a complete understanding of the actual application of the developments. In the third part of the paper, this background is assumed in connection with the discussion of the Nyquest Stability Criterion.

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[2] Osgood, W. F., Functions of a Complex Variable.
Knopp, Konrad, Theory of Functions, Vol. I.
Guilleman, E. A., The Mathematics of Circuit Analysis.

NETWORKS FROM THE ZERO AND POLE POINT OF VIEW

NETWORKS FROM THE ZERO AND POLE POINT OF VIEW

In this part of the thesis certain properties of Fourier and LaPlacian Transforms and Functions of A Complex Variable are discussed which are useful for an understanding of the method to be developed for duplicating, or compensating, a given network when its phase and amplitude versus frequency characteristics are known. Those portions of the theory of interest to the development have been extracted from the literature. A different point of view is used from that in present literature, and it is hoped that this method will give a clear picture of the theoretical concepts.

An analogy might be made at this point concerning a man who walks into a flower shop, to buy a bouquet, and finds himself lost in the number and variety of flowers available; yet he only needs a few for his bouquet. An attempt will be made to select the proper bouquet.

In the following discussion it will be assumed that the proper way to identify networks is by the location of their poles and zeros.[1] The meaning of the terms "pole" and "zero"

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[1] Bode, Henrik W., Network Analysis and Feedback Amplifier Design

Mulligan, Jr., H. H., "The Effect of Polar and Zero Locations on the Transient Response of Linear Dynamic Systems," Proceedings of the Institute of Radio Engineers, XXXVII (May, 1949), 516.

Valley, Jr., G. E., and Wallman, Henry, Vacuum Tube Amplifiers, Chapt. VIII is especially good.

G&B P. 152.

will become clear from the discussion. Later it will become apparent that for all practical networks the network characteristics are completely identified by the location of the corresponding poles. The difficulity, as far as present literature is concerned, is that an actual picture of the situation is not clearly presented. It is very well for the mathematician to develop the theory without diagrams and models, but the engineer is often not able to visualize just what is taking place. One of the principle reasons for this is that there is no method for presenting a four dimensional system in a single drawing, so the mathematics are developed without showing pictorially what it means in network theory.

In the literature of the Theory of Functions of a Complex Variable, the general method of plotting a function of four variables is to make two-dimensional plots. Using the conventional notation, this method is developed in the following manner.

Let

$$z = x + jy$$

and let

$$w = f(z) = u + jv$$

The functions z and w are then plotted separately. If the equation is expressed in polar coordinates, and two three-dimensional plots are made, a geometrical picture of a pole and a zero is developed. That is if the equation is expressed as:

$$w = \underline{R}\, e^{j\theta}$$

and $\underline{R}$ and $e^{j\theta}$ are plotted separately.

Before proceeding further it will be necessary to develop a simple Theorem which is of fundamental importance. Let it be assumed that the voltage transfer transforms of a series of networks are known. Isolate each network from the previous one by a vacuum tube, the vacuum tube being considered as infinite input impedance, zero output impedance, and is linear. Such a tube is termed as a "perfect vacuum tube."

Theorem: If the voltage transfer transforms of a group of networks are known, when the networks are connected in series, and each network is isolated by a perfect vacuum tube, the transforms are multiplied together in the Complex S Plane.

To demonstrate this Theorem let the group of networks have the voltage transfer transform $F_1(s)$, $F_2(s)$, ... $F_n(s)$. Let the input voltage to the first network be $E(s)$, and the output voltage associated with each network be $e_1(s)$, $e_2(s)$, ... $e_n(s)$. Assume, for simplicity, that each isolating vacuum tube has a gain of M, which may be greater than, equal to, or less than unity. For network 1 the output voltage will be

$$e_1(s) = F_1(s)\ E(s)$$

Connect the second network to this system through an isolating vacuum tube, and the equation is

$$e_2(s) = e_1(s)\ M\ F_2(s) = E(s)\ M\ F_1(s)\ F_2(s)$$

and in general:

$$e_n(s) = E(s)M^{n-1}F_1(s)F_1(s)\ .\ .\ .\ F_n(s)$$

This completes the proof.

To aid this discussion, two networks have been selected and each of these networks will be developed in the S Plane.

The same notation employed by Gardner and Barnes[2] will be used throughout the discussion. Figure 1 shows two networks together with their projections in the S Plane. A resistance and capacitive network is driven by a constant voltage generator, and a resistance and inductive network is driven by a constant current generator. The output voltage is measured as shown in the diagrams. The diagram on the left represents the structures of these networks in the S Plane. These Figures may be considered as cones centered at the points -R/L and -1/RC.

These cones are developed in the following manner. Using the previously mentioned concept, neglecting the phase characteristic for the present, there results the amplitude function:

$$W = \underline{R}$$

For the RC network this will be

$$W = \underline{R} = \frac{\alpha}{\left[\omega^2 + (\alpha + \delta)^2\right]^{\frac{1}{2}}}$$

where $\alpha = 1/RC$. If the equation is expressed as a reciprocal relationship, and both sides are squared, the result being

$$\frac{1}{\underline{R}^2} = \frac{\omega^2 + (\alpha + \delta)^2}{\alpha^2}$$

This is immediately recognized as the equation for a circular cone. It develops that if the cone is cut parallel to the S Plane, the section is found to be a circle centered at $\delta = \alpha$. As the parameter $\underline{R}$ is increased, the radius of the circle will

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[2] G&B, loc. cit.

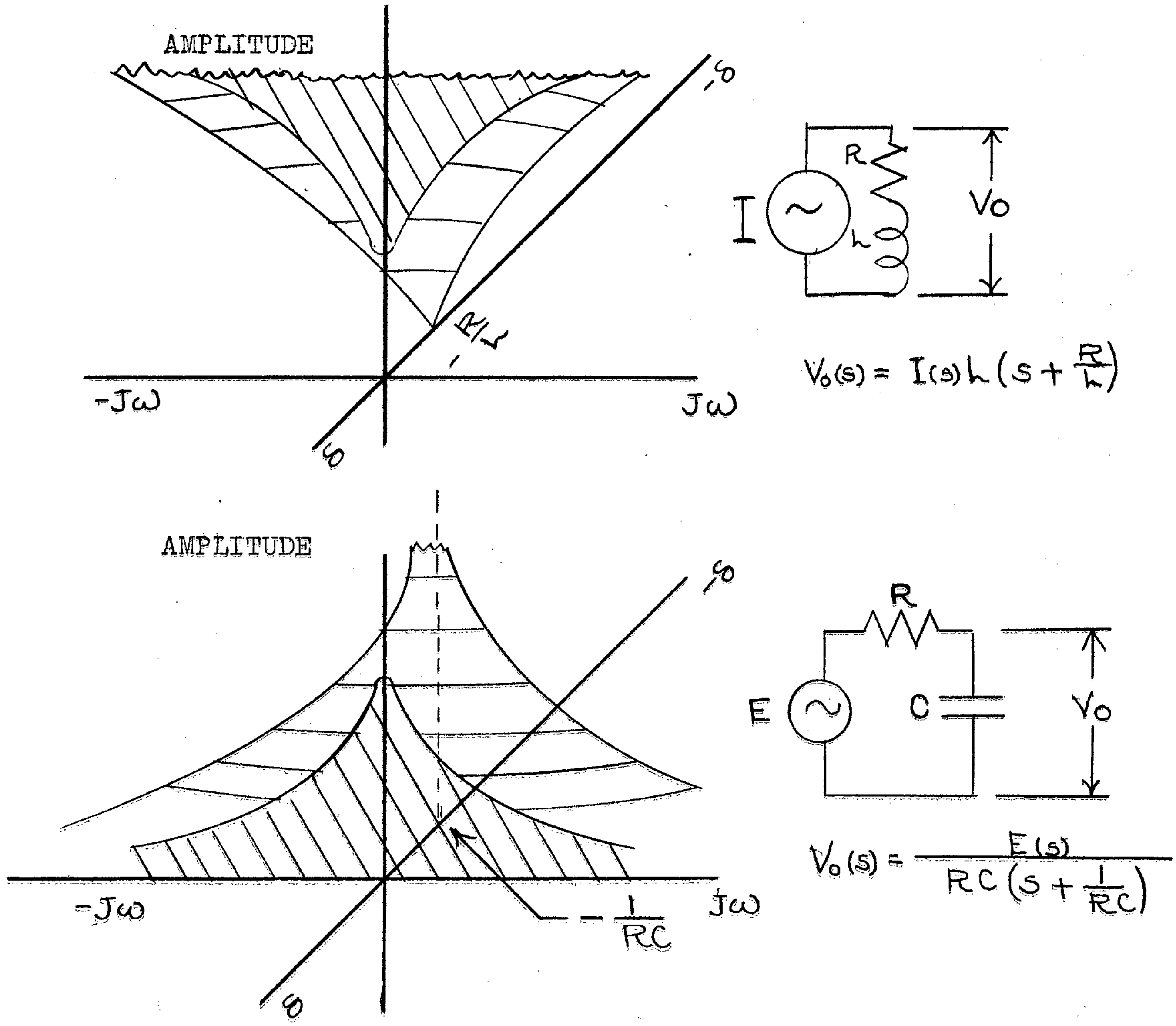
AMPLITUDE
-Jω
Jω
R/L
R
L
I
Vo
Vo(s) = I(s) L (s + R/L)
AMPLITUDE
-Jω
Jω
1/RC
R
E
C
Vo
Vo(s) = E(s) / RC(s + 1/RC)

FIGURE 1

decrease.

The resistance and inductance network may be treated in a similiar manner, and it will be found that a circular cross section results when it is cut parallel to the S Plane.

In the RL network case, at the point S = -R/L, the amplitude is zero. In the theory of functions this is known as a zero. In the RC network case, at the point S = -1/RC, the amplitude becomes infinite. In the function theory this is known as a pole. Both zeros and poles are classified as singularities. As far as networks in the S Plane are concerned, to determine zeros and poles, it is only necessary to examine the equation to determine the values of S at which the equation become zero or infinite. In more complicated networks several poles, or zeros, may exist at a single point. The number of poles or zeros is equal to the order of the pole, or zero.

Returning to Figure 1, a cross-hatched area is shown in each of the projections in the S Plane. This cross-hatched area is a "cut" on the jω axis. This cut represents the steady state amplitude response of the network. As the location of the pole or zero is varied, the cut is made in a different place on the cone, and the response curve will change. Consider the RC network. This network is a high frequency cut network. If the capacity is increased, leaving the resistance unchanged, the frequency for a given attenuation is lower. In the S Plane, this moves the cone forward, the sides becoming steeper when the cut is made on the jω axis. In the RL network case, it is seen that varying the Q of the coil moves the cone back and forth on the real axis. When a cut is made on the jω axis, it is seen that the response varies with the Q of the coil.

From the previous discussion one distinct advantage of treating networks in terms of poles and zeros is apparent. If networks are pictured as shown in Figure 1, it is seen that poles and zeros have a geometrical significance which bears a direct relationship to their name. For a given type of network, the cones will always have the same shape. This is the principle employed to find the steady-state response curves for networks using an electrolytic tank.[3]

One other point is to be noted when dealing with networks from the pole and zero point of view. Negative as well as positive frequencies are involved. Negative frequencies are the result of the mathematics and are not physically realizable.[4] It should be pointed out, however, that all physical networks projected onto the S Plane have their frequency characteristics projected as an image in the negative frequency region. Networks involving resistance, inductance, and capacity may have conjugate poles, one lying in the negative frequency region,

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[3] Huggins, W. J., "A Note on Frequency Transformations for use with the Electrolytic Tank," Proceedings of the Institute of Radio Engineers, XXXVI (March, 1948), 421.

[4] In certain special cases negative frequencies are useful from the computational standpoint, but this will not be discussed here.

the image of the one in the positive frequency region. If the cones for a network with conjugate poles are developed, it will be found that they also have a circular cross-section parallel to the S-Plane. In this case, however, it will be found that the center of each circle is different for each cut.

There are certain other advantages of thinking of networks in terms of poles and zeros. In the S-Plane, σ has the dimensions of a resistance, and all passive networks should have poles in the $-\sigma$ region. This is what mathematicians call the negative half-plane. If a pole lies in the $+\sigma$ region, the positive half-plane, this indicates a power source. In a general way, it can be stated that amplifiers should be designed to have their poles in the negative half-plane, and oscillators with their poles in the positive half-plane.[5]

In an earlier part of this paper it was mentioned that four dimensions were needed to fully represent networks. Although the location of the poles and zeros completely define any network, they have a four dimensional representation. No physical method is available to portray graphically this information. It has been shown how to obtain the amplitude charac-

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[5] In the case of amplifiers this is a sufficient but not a Necessary and Sufficient Condition. In the case of oscillators, this is a necessary condition, but not a Necessary and Sufficient Condition. The actual conditions may be determined from Nyquest's Stability Criterion.

Valley and Wallman, loc. cit.

Bode, loc. cit.

teristic by taking a cut on the jω axis of the magnitude function. The problem of phase relationships has not been considered. It will be shown that it is possible to construct a Figure in the S-plane representing phase similar to that which the represents amplitude. In the case of phase, however, the Figure is not solid like the cone, but has the shape of a warped sheet. This sheet can be moved back and forth in the same manner as the cones, a cut on the jω axis determining the phase for a given network.

Consider the RC network of Figure 1. The equation for the phase of this network in the complex S-Plane is

$$\theta = \mathrm{Tan}^{-1} - \frac{\omega}{\alpha + \delta}$$

This case is illustrated in Figure 2, which shows a phase sheet in the upper half-plane, its image in the lower half-plane not being shown for reasons of clarity. The cross-hatched area represents a cut on the positive frequency axis and is the phase characteristic for this particular value of RC.

Consider the two networks of Figure 1 again. When the RC network is connected to the output of the RL network through a perfect vacuum tube, then by the theorm developed earlier, the two networks are multiplied together. The result will be a constant if R/L and 1/RC are properly chosen, that is, compensation is obtained for the RL network. Another advantage of using poles and zeros to classify a network is now apparent. To compensate any network, where the equation of the network is known, find another network which has the same number of zeros

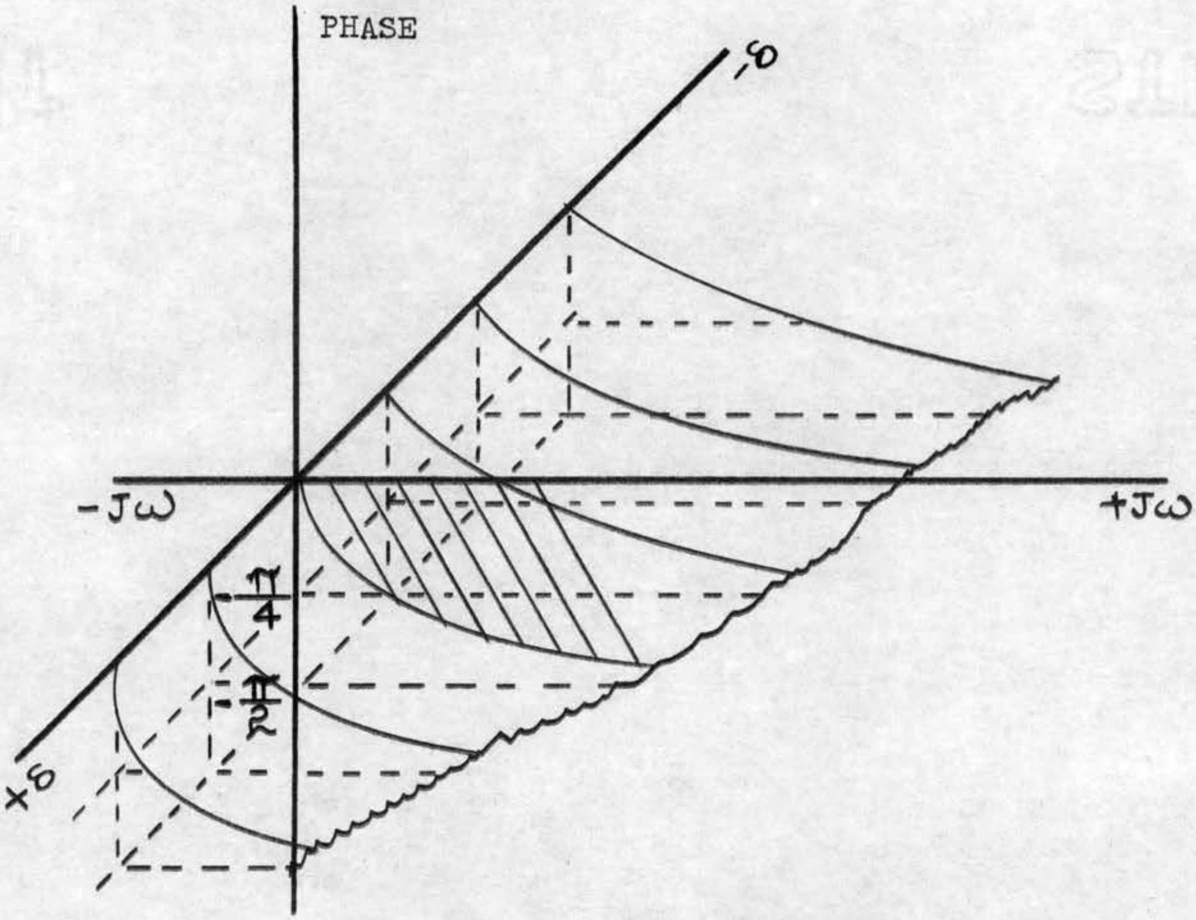

FIGURE 2

as the network to compensated has poles, and these zeros the same location as the poles. Moreover, this network must have the same number of poles as the network to be compensated has zeros, and these poles are to be located at the same points as the zeros.

Actually this idea comes directly from the Theory of Functions of a Complex Variable.[6] There it is shown that any rational complex fraction of the form

$$R(z) = \frac{G(z)}{F(z)}$$

where G(z) and F(z) are polynominals in z, that the function R(z) is completely characterized by the location of isolated poles. R(z) is assumed regular, which is always true of network transfer functions.

It is also shown that if G(z) has a root of the form (z - a) and F(z) has a root of the form (z - a), the number of these isolated poles, as determined by the polynominal of F(z), is reduced by one. This is called a "removable singularity."[7] Full compensation is merely the creation of a sufficient number of removable singularities to take care of every pole of F(z). The actual application of this in practice will be discussed more fully in the section entitled "Suggested Applications."

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6 For example, Osgood, Chapter VI, loc. cit.

7 In other developments of Functions of a Complex Variable, a removable singularity is called a doubtful point. The term removable singularity appears more appropriate from the engineering standpoint, inasmuch as it indicates what happens.

The two networks of Figure 1 were not chosen at random, but were chosen to show particular points about compensation. Assume a constant voltage generator feeds the RC network; couple the output to the grid of a pentode considered as a perfect constant current amplifier; if the pentode has the proper value of R/L in its output circuit, then compensation is obtained. Sometimes it is possible to find a constant current network which has the proper compensation characteristics, but not a voltage network having the desired characteristics. The above technique indicates how to handle this situation. It is also noted from the theorm on the addition of networks by means of isolating vacuum tubes, that the gain of the tube does not affect the compensation; it is only the location of the poles and zeros. Gain merely enters as a factor which can be taken care of by either positive or negative attenuation, depending upon the final use of the output voltage. For this reason, the gain term is sometimes referred to as sensitivity.

Although very simple networks were chosen for illustration, these same ideas may be carried over to more complicated networks. The above compensation theorm is general regardless of the complexity of the network.

There is a method for using the above ideas to either duplicate or compensate a given network when the amplitude and phase versus frequency characteristics for a network are given. Dr. R. G. Piety of Phillips Petroleum Company is responsible for the basic ideas underlying this method.

Consider the voltage transfer transform of a generalized network. The equation is

$$F(s) = \frac{\prod_{i=1}^{n} (s+\lambda_i) \prod_{J=1}^{m} (s^2 + 2\alpha_J s + \alpha_J^2 + \beta_J^2)}{\prod_{K=1}^{r} (s+\lambda_K) \prod_{\varsigma=1}^{S} (s^2 + 2\alpha_\varsigma s + \alpha_\varsigma^2 + \beta_\varsigma^2)}$$

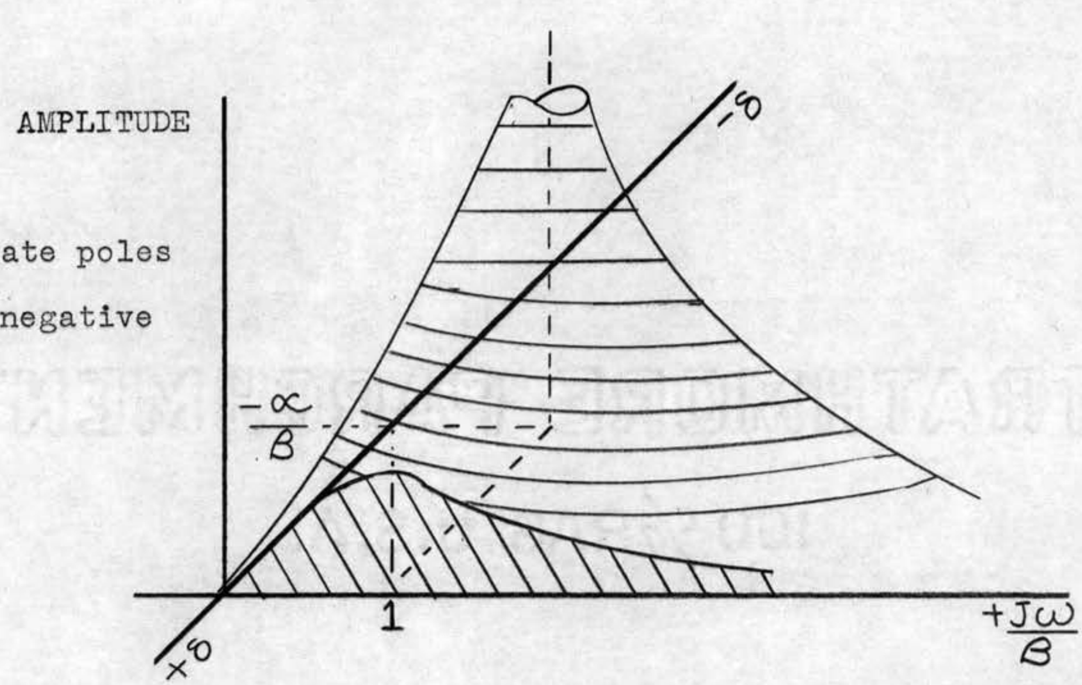
This equation may be reduced to the dimensionless form in the following manner,

$$F(s) = \frac{\prod_{i=1}^{n} \lambda_i\left(\frac{s}{\lambda_i}+1\right) \prod_{J=1}^{m} \beta_J^2\left(\frac{s^2}{\beta_J^2} + \frac{2\alpha_J s}{\beta_J^2} + \frac{\alpha_J^2}{\beta_J^2}+1\right)}{\prod_{K=1}^{r} \lambda_K\left(\frac{s}{\lambda_K}+1\right) \prod_{\varsigma=1}^{S} \beta_\varsigma^2\left(\frac{s^2}{\beta_\varsigma^2} + \frac{2\alpha_\varsigma s}{\beta_\varsigma^2} + \frac{\alpha_\varsigma^2}{\beta_\varsigma^2}+1\right)}$$

Figure 3 shows the results of this procedure in the case of an elementary network. In this Figure are shown two three dimensional drawings; one case with a pole on the negative real axis, and one showing a single conjugate pole. The cut on the $j\omega$ axis is now dimensionless, and is shown by the cross-hatched area. The curve shown for the pole on the negative real axis may be thought of as a series RC network connected to a constant voltage source, with the output voltage taken across the condenser. The cross-hatched area defines the amplitude for this voltage. The cross-hatched area of the single conjugate pole would represent the voltage across a parallel RLC network connected to a constant current source.

Now consider the voltage transfer transform of any elementary network, and express it in polar form, first making the network dimensionless by use of the above technique. If the

One of conjugate poles lying in the negative half-plane.



Pole lying on the negative real axis in the negative half-plane.

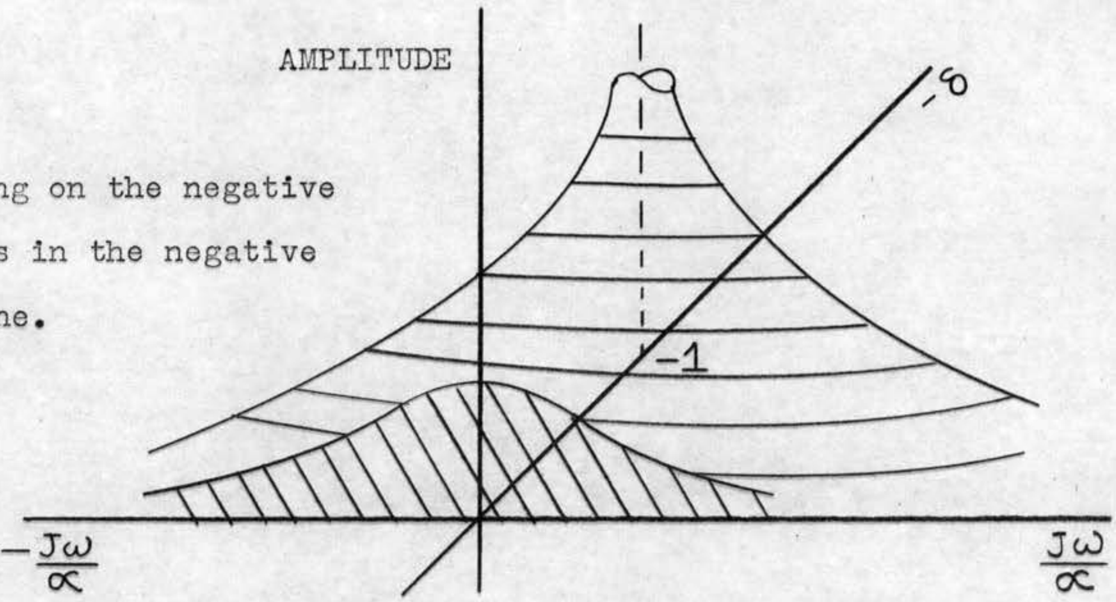


Figure 3

natural logarithm of this equation is taken, there results

$$\mathrm{Log}_e F(s) = \mathrm{Log}_e \underline{R} + j\theta$$

where

$$\underline{R} = (\mathrm{real}^2 + \mathrm{imaginary}^2)^{1/2}$$

$$\theta = \mathrm{Tan}^{-1} \frac{\mathrm{imaginary}}{\mathrm{real}}$$

It is desirable to express attenuation in decibels instead of nepers. This may be accomplished by taking the logarithm of the amplitude to the base 10, and multiplying the result by 10. The equation for attenuation in decibels is,

$$F(s)_{db} = 10\mathrm{Log}_{10}\,(\mathrm{real}^2 + \mathrm{imaginary}^2)$$

Obviously a series of these terms can be added by the procedure indicated in the previous theory to obtain a composite amplitude function.

Figure 4 has been prepared to show certain basic networks, together with the equations for constructing a series of amplitude and phase curves. Although all of the networks consist of resistive and capacitive elements, the procedure to be described is not limited to RC networks alone. Networks containing conjugate poles may also be used for constructing these curves, using the procedure outlined previously. These networks were merely selected for demonstration purposes. Also shown in Figure 4 are certain possible compensating networks. What is meant by "possible" will be explained at the end of this section.

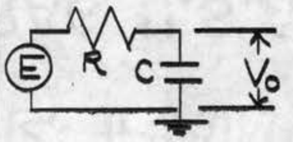
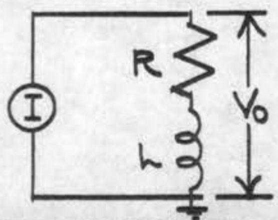
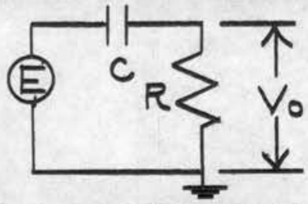
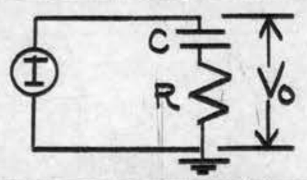
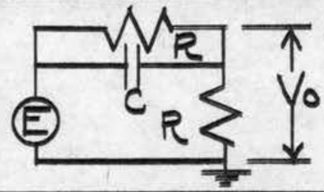
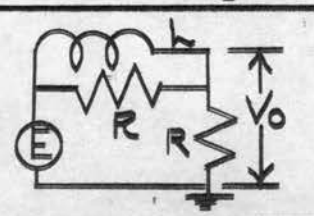
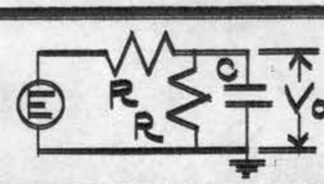
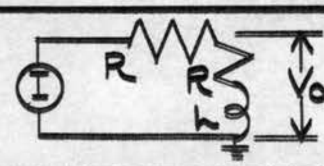
NETWORK	BASIC EQUATION	VALUE OF α	ATTENUATION EQUATION DECIBELS	PHASE EQUATION
	$\frac{1}{\frac{s}{\alpha}+1}$	$\frac{1}{RC}$	$-10\log(\omega^2+1)$	$\tan^{-1}-\omega$
	$R(\frac{s}{\alpha}+1)$	$\frac{R}{L}$		
	$\frac{\frac{s}{\alpha}}{\frac{s}{\alpha}+1}$	$\frac{1}{RC}$	$20\log\omega -$ $10\log(\omega^2+1$	$\tan^{-1}\frac{1}{\omega}$
	$\frac{R(\frac{s}{\alpha}+1)}{\frac{s}{\alpha}}$	$\frac{1}{RC}$		
	$\frac{\frac{s}{\alpha}+1}{\frac{s}{\alpha}+2}$	$\frac{1}{RC}$	$10[\log(\omega^2+1)$ $-\log(\omega^2+4)$	$\tan^{-1}\omega -$ $\tan^{-1}\frac{\omega}{2}$
	$\frac{\frac{s}{\alpha}+2}{\frac{s}{\alpha}+1}$	$\frac{R}{2L}$		
	$\frac{1}{\frac{s}{\alpha}+2}$	$\frac{1}{RC}$	$-10\log(\omega^2+4)$	$-\tan^{-1}\frac{\omega}{2}$
	$R(\frac{s}{\alpha}+2)$	$\frac{R}{2L}$		

FIGURE 4
FOUR NETWORKS FOR CONSTRUCTING
UNIVERSAL RESPONSE
CURVES

Figure 5 shows plots of the amplitude functions for three of these curves. The procedure for constructing these curves is to let $\alpha = 1$. A curve is then constructed of attenuation versus ω, this curve being marked at the point $\omega = 1$. These curves may be constructed of lucite or any other suitable material. It is obvious that the curves must be designed for the particular semi-log paper on which plots of network amplitude functions are to be made.

The step by step procedure for using these network curves is given below:

1. Plot an attenuation versus $\frac{\omega}{\alpha}$ curve for the unknown network which is to be duplicated. The $\frac{\omega}{\alpha}$ ratio should be so selected that the main points of interest are centered about the point, $\frac{\omega}{\alpha} = 1$.

2. Select one curve which approaches the one to be duplicated, sliding it to the right and left until the most satisfactory location is obtained. Movement in the vertical direction is permitted inasmuch as duplication is only to be within some amplitude function. Note the actual attenuation of the network curve at this point. Add the values shown by the attenuation curve to those obtained from the network attenuation curve through these points. As a result one network of this type will be used in the duplication process. It is necessary to note the point where $\omega = 1$ of our network curve falls on the $\frac{\omega}{\alpha}$ scale. The α chosen for construction of the curve to be duplicated is known. From this the $\frac{\omega}{\alpha}$ ratio can be

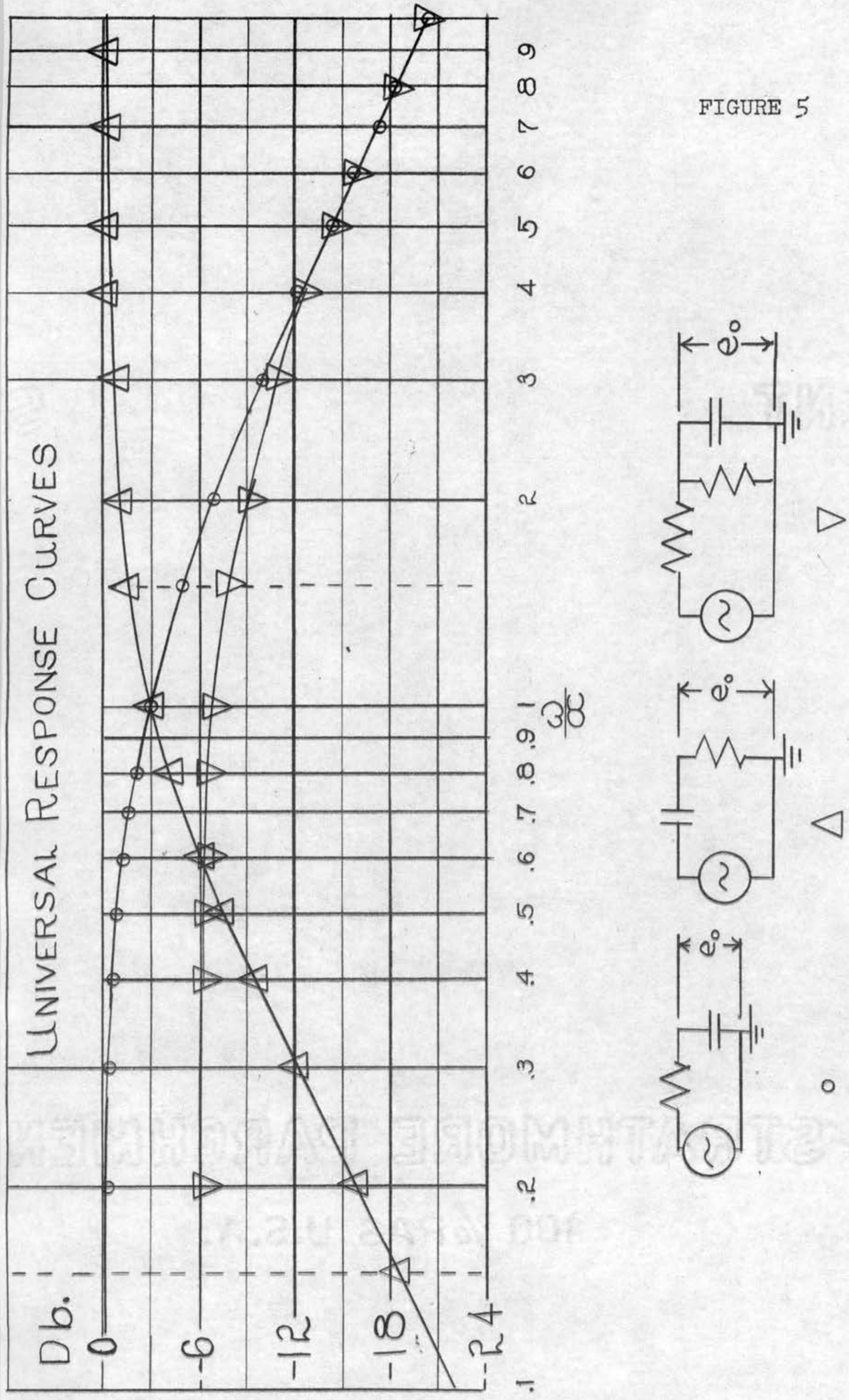

UNIVERSAL RESPONSE CURVES
Db.
0
-6
-12
-18
-24
.1
.2
.3
.4
.5
.6
.7
.8
.9
1
2
3
4
5
6
7
8
9
ω/α
e₀
e₀
e₀

FIGURE 5

determined for the duplicating network. Suppose the value, $\alpha = 100$, has been selected for the network to be duplicated, and it was found the reference point of a standard network curve fell on the point $\frac{\omega}{\alpha} = 1.2$. This would mean that when this network was constructed, the parameter of α must equal 1.2 x 100.

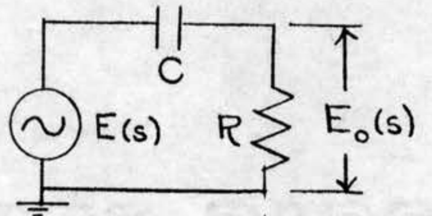
3. Continue in this manner, using different attenuation curves in the manner indicated above, until the reconstructed original curve is a straight line.

4. If just the amplitude characteristic is desired, sufficient information is available to construct a duplicating network on an amplitude basis. If exact duplication is desired, then phase curves must be used in conjunction with the amplitude curves. It is noted that these two curves cannot be used independently of one another.

5. A series of networks can be constructed, as found by the above procedure, isolating each one from its predecessor by a vacuum tube. The theorm shows that if the phase and amplitude characteristic have been duplicated exactly, then within the range of some amplitude function the equation of the duplicating network must be identical with that for the original network.

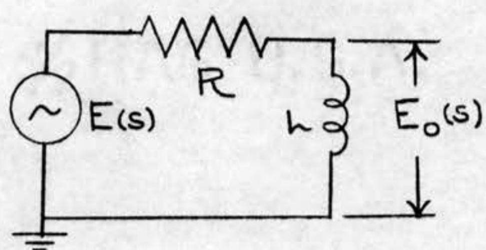
One thing should be pointed out at this time in regards to the networks shown in Figure 4. The first two networks have identical curvatures, and consequently only one curve is necessary for the two cases. Reversal of the one curve will result in the curve for the opposite type network.

There is an obvious point about networks which was hinted when Figure 1 was discussed, and is also associated with the terms for the possible compensating networks mentioned previously. Consider the two networks shown in Diagram A and B, the output voltage transforms for which are given,



$$E_o(s) = \frac{E(s)\,SRC}{SRC+1}$$

DIAGRAM A



$$E_o(s) = \frac{E(s)\,S\frac{L}{R}}{S\frac{L}{R}+1}$$

DIAGRAM B

In these networks, if RC = L/R, it is seen that it is impossible to differentiate one network from the other by means of the output voltages. Although this is a simple example, it shows that there is no uniqueness for networks; that is, many networks will have the same response characteristic, although the actual elements and method of construction are different. In complicated networks where economy of manufacture enters, this property of networks is not just of academic interest. Sometimes a little additional work will result in real economies. It was for this reason, also, that RC networks were chosen for the demonstration networks. They are cheap and easily constructed.

A METHOD FOR SOLVING THE SYNTHESIS PROBLEM

A METHOD FOR SOLVING THE SYNTHESIS PROBLEM

In this section a method for solving the synthesis problem in networks is developed. Some computed examples using the technique developed herein are also given. The method by which this result is to be accomplished is by the use of what might be called a new type of operational algebra, due to Dr. R. G. Piety of Phillips Petroleum Company. This operational method was specifically developed to find auto-correlation and cross-correlation functions[1] in order to aid in the interpretation of siesmograph records. This problem will not be discussed here, although the concepts developed for its solution can be extended to the network synthesis problem. The method will give the phase and amplitude characteristic of any linear four-terminal network when the input and output time functions are known. As was pointed out in the introduction, the method will give an anlog electrical network for any linear system when the input and output time functions are known.

The synthesis problem is to find a network which will have the same response as a given system when the input time function and the output time function are known. For example, a sealed four-terminal network might be given, and only its input and output time functions known. It is desired to either duplicate, or compensate, this network. If the phase and amplitude versus

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[1] James, Nichols, and Phillips, Theory of Servomechanisms.

Wiener, Norbert, Extrapolation, Interpolation and Smoothing of Stationary Time Series.

frequency characteristic can be determined from this information, the methods described in the first part of this paper can be used to duplicate, or compensate, the network. As was pointed out, however, the duplicate may not have the same arrangement of components in its construction.

For the type of systems to which it is applicable, the method of synthesis discussed in this paper is straightforward. Although a knowledge of the properties of Fourier and LaPlacian Transforms is necessary for a thorough understanding of the technique, this is not necessary, however, in order to be able to use the technique. The method may be compared to the problem of using log tables, as compared to understanding the theory underlying their general development.

Several systems are available for describing the properties of a network, the most conventional one being to show the relationship between phase and amplitude versus frequency. Another method, not quite so well known, but of equal value, is the concept of indicial admittance.[2] Idicial admittance is defined as the time function of the current which enters an impedance in response to a unit step function of voltage. It can be shown that all the characteristics of a network may be described in terms of indicial admittance, including the voltage

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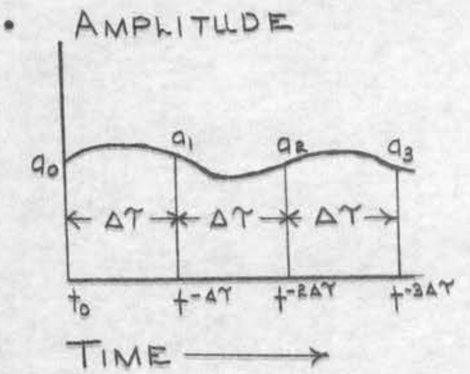
[2] Karen and Biot; Mathematical Methods in Engineering.

Carson, J. R., Electrical Circuit Theory and Operational Calculus.

G P. 97.

output across some impedance in the network due to a unit step function of voltage input. The indicial admittance concept is used in this part of the paper. As used here it is the situation existing when a unit step function of voltage is applied to two terminals of a four terminal network, and the output voltage measured against time.

The method of synthesis developed here is not limited to unit step function inputs, as will become clear from the following development. This type of input was chosen as a convenient means for showing the theory of development. In passing, it might be mentioned that the only reason for introducing indicial admittance here is that the previously cited references will show that it is a unique method for defining a network. There are other unique methods besides the two mentioned in the previous paragraph, for example, by the location of the poles and zeros as shown in the first part of the paper.



The operational algebra for use in solving the synthesis problem will nwo be developed. Assume a time series, as shown at the right, starting at t = 0. Divide this time series into equal intervals of time $\Delta\Upsilon$, designating the amplitude at t_0 as a_0, at t_1 as a_1, . . . t_n as a_n. The time series can be described in the following manner:

$$a_0 + a_1 t^{-\Delta\Upsilon} + a_2 t^{-2\Delta\Upsilon} + \ldots + a_n t^{-n\Delta\Upsilon}$$

where each of the terms has the following significance:

$\Delta\tau$ is the interval between consecutive points in the time series

$t-n\Delta\tau$ indicates a time to be associated with each amplitude, delayed $n\Delta\tau$ units from the origin.

a_n is the amplitude of the time seriesat the time $t-n\Delta\tau$.

Let a_n represent an amplitude associated with a unit impulse[3] occuring at $t-n\Delta\tau$. The basic property of a unit impulse, associated with an amplitude function, which makes it useful in this method will become clear as the development proceeds.

A method for representing a time series of any complexity at equally spaced discrete points is now available.[4] Let it be further assumed that all algebraic operations of any power series apply to the above series; that is, the commutative, associative, and distributive laws hold. That such an assumption is proper will develop later.

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3 G&B P.255.

C&F P. 7.

G P. 100.

4 This method of representing a time series has the Fourier Transform $a_n e^{-j\omega n\Delta\tau}$

G&B P. 10.

C&F P. 207.

G j.

There is a theorem which states[5], "that any function of time f(t) which contains no frequency components greater than W_o cps is uniquely determined by the values of f(t) at any set of sampling points spaced $1/2W_o$ second apart." It is seen from this theorem that the specified impulses can uniquely determine the time series within some upper frequency limit. In other words, the time series is assumed to be made up of frequencies with no component higher than $\frac{1}{2\Delta\tau}$.

Assume that the output time series from some four-terminal network, due to a unit step function input, is known. The expression for this time series in terms of the operational algebra is

$$f(t) = \sum_n a_n t^{-n}$$

where $\Delta\tau$ is chosen in a manner to be described later.

The conventional mathematical expression used in LaPlacian and Fourier Network Transform Theory is:

(Network Transform)(Input Transform) = Output Transform

In this paper it is desired to obtain the network frequency function. From the above relationship, the Fourier Transform is

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5 Oliver, Pierce and Shannon, "The Philosophy of PCM," Proceedings of the Institute of Radio Engineers, XXXVI (November, 1948), 1130.

This theorem was developed much earlier, but apparently these authors are not acquainted with this work.

Ferrar, W. L., "On the Cardinal Function Interpolation Theory," Proceedings of the Royal Society of Edinburgh, 45 (1925), 269-282.

$$\text{Network Frequency Function} = \frac{\text{Fourier Transform (Output Time Series)}}{\text{Fourier Transform (Input Time Series)}}$$

It is desired to express the unknown network in terms of an amplitude and phase versus frequency relationship. It will be shown that the above expression does just this.

When the same $\Delta\Upsilon$ is selected for the unit step function[6] as for the network output time series, the unit step function in terms of the operational algebra is

$$f(t) = \sum_{m}^{\infty} t^{-m\Delta\Upsilon}$$

when this is divided into the output time series, the result is

$$f(t)' = \sum_{n} \sum_{m} a_{n-m} \, t^{-(n-m)\Delta\Upsilon} \qquad m \leq n$$

the Fourier Transform of the above equation[7] can be found in tables, and the result is

$$f(\omega)' = \sum_{n}\sum_{m} a_{n-m} \exp{-j\omega(n-m)} \qquad m \leq n$$

The above equation is immediately recognized as a Complex Fourier Series. Expressed in a more familiar form, this equation is

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6 The unit step function contains frequencies of all orders. The output from any network whose input is a unit step function, can be looked upon made up of those particuliar frequencies it allows to pass, and the operations the network performs on these frequencies.

Goldman, Stanford, Frequency Analysis, Modulation, and Noise, p. 124.

7 G&B 10

C&F 207

G j

$$f(\omega)' = \sum_n \sum_m a_{n-m}((\mathrm{Cos}\ \omega\ \Delta\Upsilon(n-m) - j\mathrm{Sin}\ \omega\Delta\Upsilon(n-m)))$$

This is a function made up of real and imaginary parts which describes the network. It is also apparent that the operational algebra is merely a symbolic notation which indicates the method of operation. This algebra is handled in the same manner as any power series; the symbolism adopted merely helps in establishing a clear cut method of operation.

There is a point about the sampling theorm upon which the entire development hinges. When samples are taken by unit impulses, it is necessary to reconstruct the function, using the same notation as Oliver et al,[8] in the following form:

$$f(t) = f\left(\frac{n}{2W_0}\right)\frac{\mathrm{Sin}\ \pi\ (2W_0\ t - n)}{\pi\ (2\ W_0\ t - n)}$$

This function may be expressed as a series of functions of the form $\frac{\mathrm{Sin}\ \omega t}{\omega t}$ centered at each sampling point. The transform of this function is:[9]

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8 Oliver, Pierce, and Shannon, op. cit.

The above authors assign no name to this function, but Ferrar, op. cit. calls it a Cardinal Function. This term will be used henceforth in this paper to describe this function. There is another article:

Hardy, G. J., "On An Integral Equation," Proceedings London Math. Soc., (1909), 445-472.

Hardy calls this function an "M Function".

9 C&F 882.1

G&B 3.01

$$\frac{1}{\omega} \text{Tan}^{-1} \frac{\omega}{p}$$

A time series expressed in the above manner will have the Fourier Transform

$$F(\omega) = \frac{\text{Tan}^{-1}}{\omega} \frac{\omega}{p} \sum_{n} a_n \exp - j\omega n$$

When two time series are divided, the inverse tangent functions will cancel. This completes the proof of the procedure. It is to be noted that this proof fails unless the $\Delta \Upsilon$ for the input and output time functions are identical.

There is a question about the accuracy of the method developed in this paper, that is, over what frequency range is the method accurate? The previously quoted Theorem of Oliver, Pierce and Shannon indicates that the highest frequency present in the Cardinal Function, call it f_c, is

$$f_c = \frac{1}{2 \Delta \Upsilon}$$

In words, it is as if the output time function from the network were passed through a perfect filter with a cut-off frequency f_c.

There is a Theorem called the Paley-Wiener Criterion for Realizable Filters[10] which shows that a perfect filter is impossible to realize in practice. In terms of physical networks, evaluated by the foregoing technique, this means that as the plotted frequency approaches f_c, the Complex Fourier Series

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10 Valley and Wallman, p. 721-727, loc. cit.

Amplitude terms approach zero more rapidly than in the actual network. To show this, when the Complex Fourier Series is evaluated at $f = 1/2\Delta\tau$, the equation is

$$a_n(\text{Cos } 2\pi fn\Delta\tau - j\text{Sin } 2\pi fn\Delta\tau) = \sum_n a_n(-1)^n$$

If a sufficient number of terms are present in the above expression, it is a general property of Fourier Series that $a_n \cong a_{n-1}$. That is, alternate terms of the series tend to cancel and the sum will be small. Reasoning physically from the above discussion, it is seen that the highest frequency f_h which can plotted must satisfy the relationship

$$f_h < \frac{1}{2\Delta\tau}$$

In actual practice it has been found that the following relationship is satisfactory

$$f_h < \frac{1}{10\Delta\tau}$$

The above relationship was used in plotting all the examples in this paper. In a sample computation, it is shown that the curves of amplitude and phase versus frequency lose their smoothness when this relationship is exceeded. It has been ascertained by making a number of computations that this relationship must hold. No rigorous proof of this relationship has yet been found although it is noted that $10 \cong 2\pi$. It is an experimental value found to hold in practice. Just how well this relationship holds will be indicated when the Theoretical versus Computed network characteristics are discussed.

From the previous discussion of the Complex Fourier Series for finding the phase and amplitude versus frequency characteristic of an unknown network, it is apparent that the smaller $\Delta\Upsilon$ is chosen, the closer will be the approximation in a given frequency range. It is also observed that the work necessary to obtain a solution will be in direct proportion to the ratio of chosen $\Delta\Upsilon$s.

There is a convenient transform which will aid in all calculations involving the actual determination of the phase and amplitude characteristics from the Complex Fourier Series. This transform is the scale change transform[11]

$$\mathfrak{F}\, f\left(\frac{t}{a}\right) = aF(aj\omega)$$

The above transform states the relationship for a scale change. The advantage of this transform is in determining the phase and amplitude versus frequency components from the Complex Fourier Series. To find the phase and amplitude versus frequency characteristic for the unknown network, the Complex Fourier Series must be evaluated for specific frequencies. Using the above transform, a table may be prepared with a fixed series of relationships, and this table used for all computations.

In all the illustrative computed examples, $\Delta\Upsilon$ was made equal to 0.1 seconds by the application of the scale change

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11 C&F 205
G&B 8
G k

transform. The following frequencies were chosen for computational purposes; 0.000, 0.139, 0.278, 0.418, 0.556, 0.695, 0.834, and 0.973 cycles per second. With the above frequencies, $n\Delta\tau\omega$ will be 5° or some integer multiple thereof. It was for this reason this particular relationship was chosen. It is noted that the highest frequency chosen, 0.973 cycles per second and the time interval 0.1 seconds, just satisfy the stated relationship between $\Delta\tau$ and f_h.

A two place trigometric table was constructed using these relationships to solve all the illustrative networks. A portion of this table is shown in Figure 6 to indicate the method of construction. It is not necessary, obviously, to use in the relationships shown here. Other relationships can be used. It should be noted, however, that the fact 5° is the basic angle makes the chart rather easy to prepare. In any event, it is recommended that whatever basic angle is chosen, that it have an integer relationship with 90°.

In the discussion of actual network computations, a low pass prototype "T" filter network is used as an illustration. Although the Q of the inductance was only 25, it was necessary to evaluate 45 real and 45 imaginary terms for each frequency amplitude relationship plotted. Where conjugate poles and low damping are present in the network for which solution is sought, the actual chart must have an even greater number of terms. The table constructed for the computations of this paper had fifty terms.

The results of some actual computations will now be given,

f		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.139	SIN	-.087	-.17	-.26	-.34	-.42	-.50	-.57	-.64	-.71
	COS	1.000	.98	.97	.94	.91	.87	.82	.77	.71
0.278	SIN	-.17	-.34	-.50	-.64	-.77	-.87	-.94	-.98	-1.0
	COS	.98	.94	.87	.77	.64	.50	.34	.17	0.0
0.418	SIN	-.26	-.50	-.71	-.87	-.97	-1.0	-.97	-.87	-.71
	COS	.97	.87	.71	.50	.26	0.0	-.26	-.50	-.71
0.556	SIN	-.34	-.64	-.87	-.98	-.98	-.87	-.64	-.34	0.0
	COS	.94	.77	.50	.17	-.17	-.50	-.77	-.94	-1.0
0.695	SIN	-.42	-.77	-.97	-.98	-.82	-.50	-.087	.34	.71
	COS	.91	.64	.26	-.17	-.57	-.87	-1.00	-.94	-.71
0.834	SIN	-.50	-.87	-1.0	-.87	-.50	0.0	.50	.87	1.0
	COS	.87	.50	0.0	-.50	-.87	-1.0	-.87	-.50	0.0
0.973	SIN	-.57	-.94	-.97	-.64	-.087	.50	.91	.98	.71
	COS	.82	.34	.26	-.77	-1.00	-.87	-.42	.17	.71

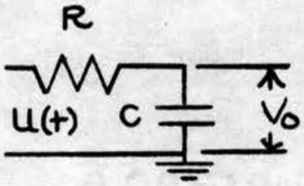
SAMPLE OF CHART USED IN DETERMINING
FOURIER SERIES COEFFICIENTS

<u>FIGURE 6</u>

using the previously develop techniques. In Figure 7 is shown four networks with the values of the components for which solution was obtained. Obviously other values could have been selected for the components, but this would have made no difference because all have been computed using the scale change Transform. Also shown is the equation giving their response to a unit step function, U(t). The output equations were solved by the LaPlacian Transform method, the particular transforms applicable being indicated by the method adopted in this paper.

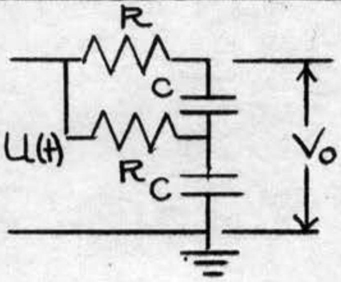
The method used to obtain these solutions was to compute the network response to the unit step function input, U(t), and then plot this result. The time series for the output time function was taken from the plot. The reason for doing the computations in this manner, was that by plotting the output time function and then taking the values for computation from the curve, the procedure would be similar to that in a practical case. In a practical case the output would be taken from an oscilloscope or some similar device. This is also the reason for only using two place trigometric tables for making computations. It was assumed that in a practical case no values could be read better than two places.

Figures 8 to 11 show the response of the networks to a unit step function, and also the theoretical amplitude and phase versus frequency characteristic. The computed phase and amplitude versus frequency characteristic, using the Complex Fourier Series, is also shown. Network Number 1 is a high cut network, and it has been plotted for two values of $\Delta\tau$. This shows



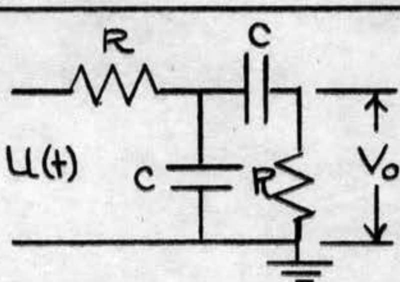
NETWORK NO. 1

$RC = 0.2$
$V_O = 1 - e^{-5t}$
G&B 1.105
C&F 448
G 11



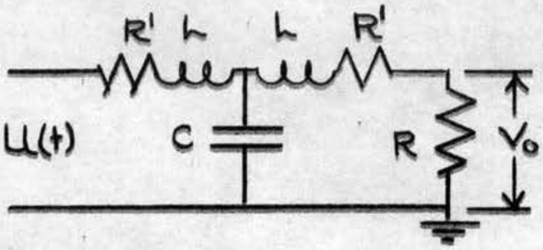
NETWORK NO. 2

$RC = 1$
$V_O = 1 - 1.14e^{-2.62t} + .17e^{-.382t}$
G&B 1.109
C&F 452 & 453
G 14



NETWORK NO. 3

$RC = 0.2$
$V_O = 0.446(e^{-1.9t} - e^{-13.1t})$
G&B 1.105
C&F 448
G 11



NETWORK NO. 4

$R' = 40$ $R = 1000$
$L = 0.5$ $C = 10^{-6}$
$F_c = 318$ cps $Q = 25$
$V_O = .926 - .7e^{-1400t} + .669e^{-339t} \sin(1720t - 159.3^{\circ})$
G&B 1.319

FIGURE 7

NETWORK NO. 1

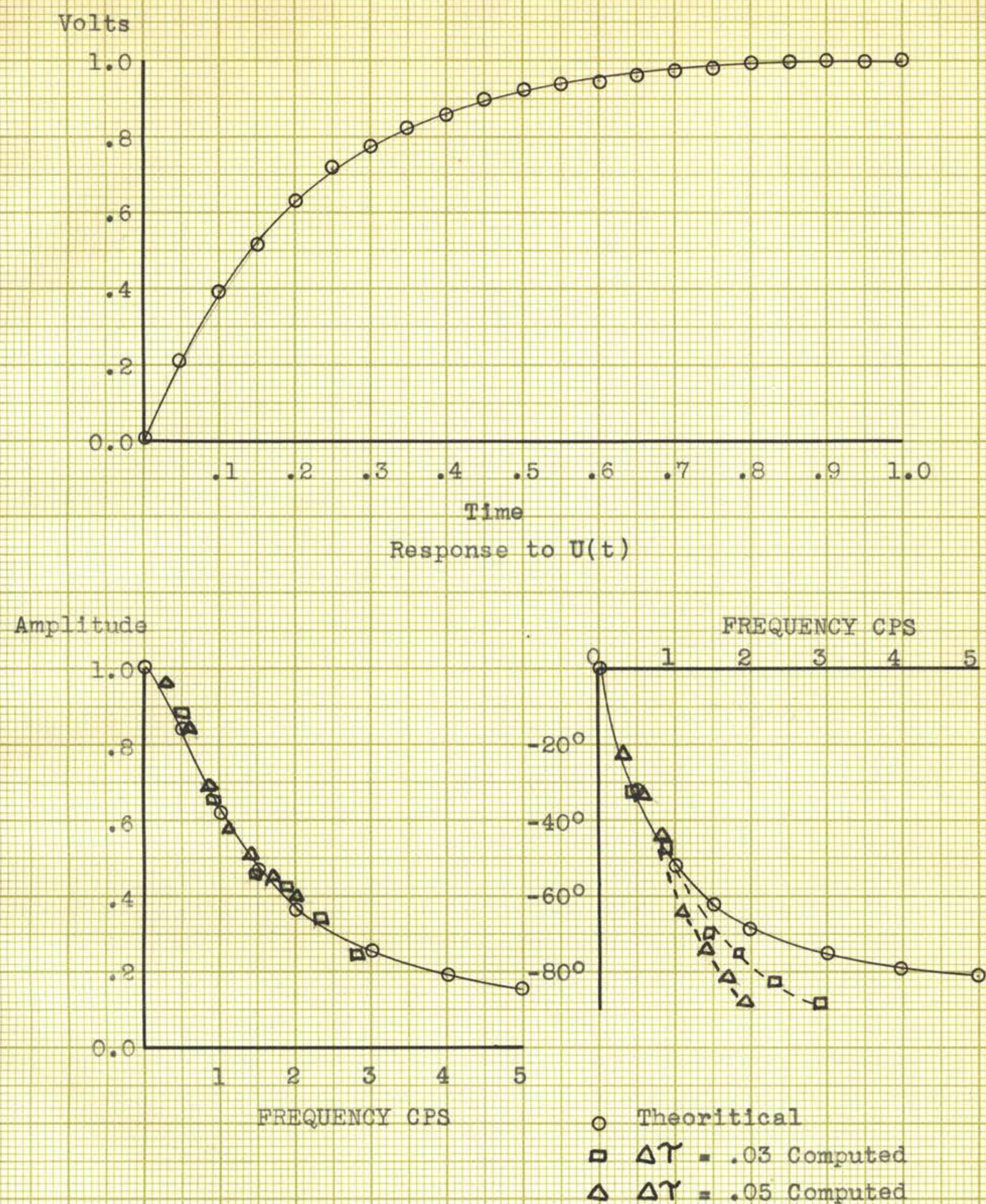


FIGURE 8

NETWORK NO. 2

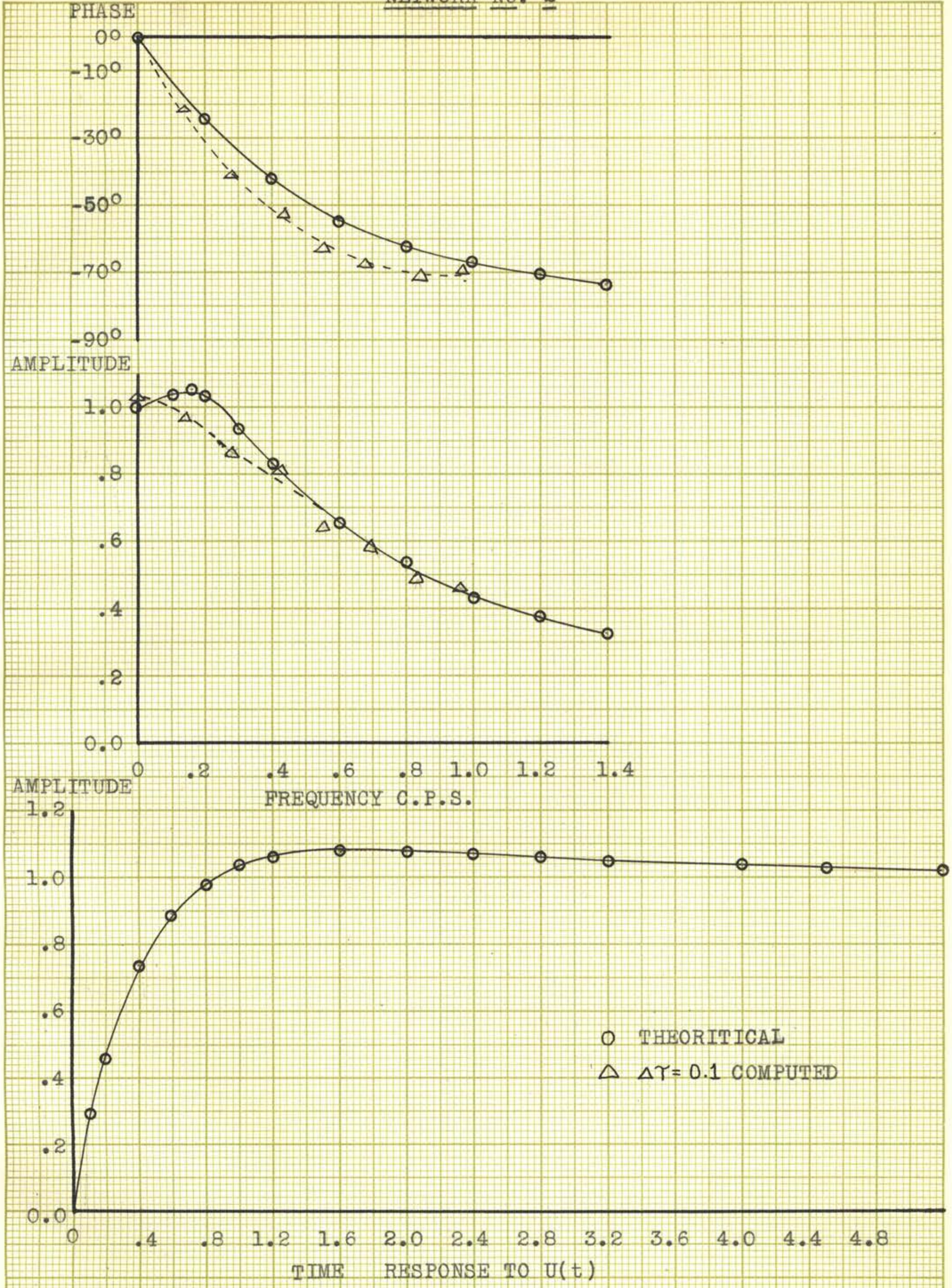


FIGURE 9

NETWORK NO. 3

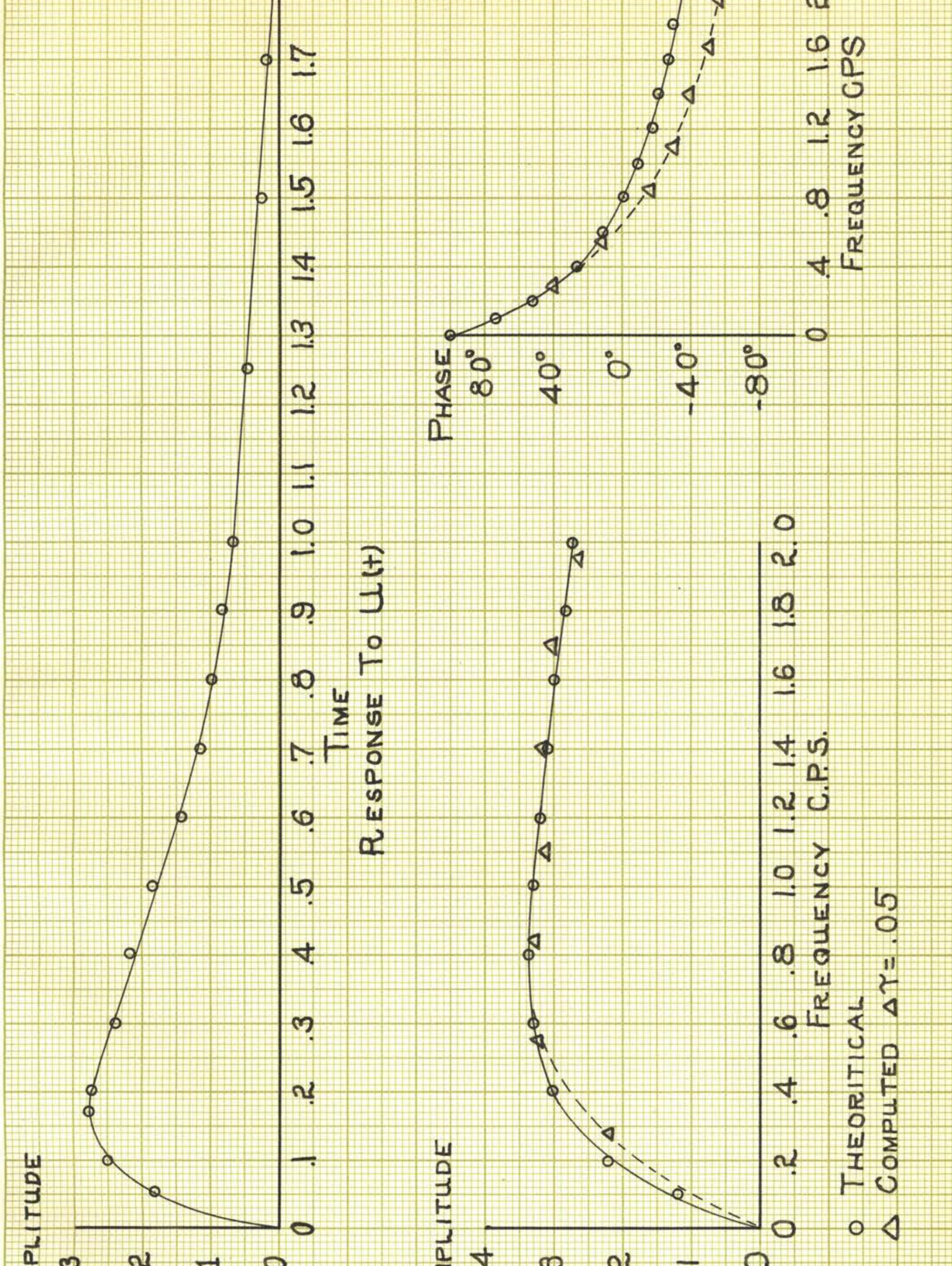


FIGURE 10

NETWORK NO. 4

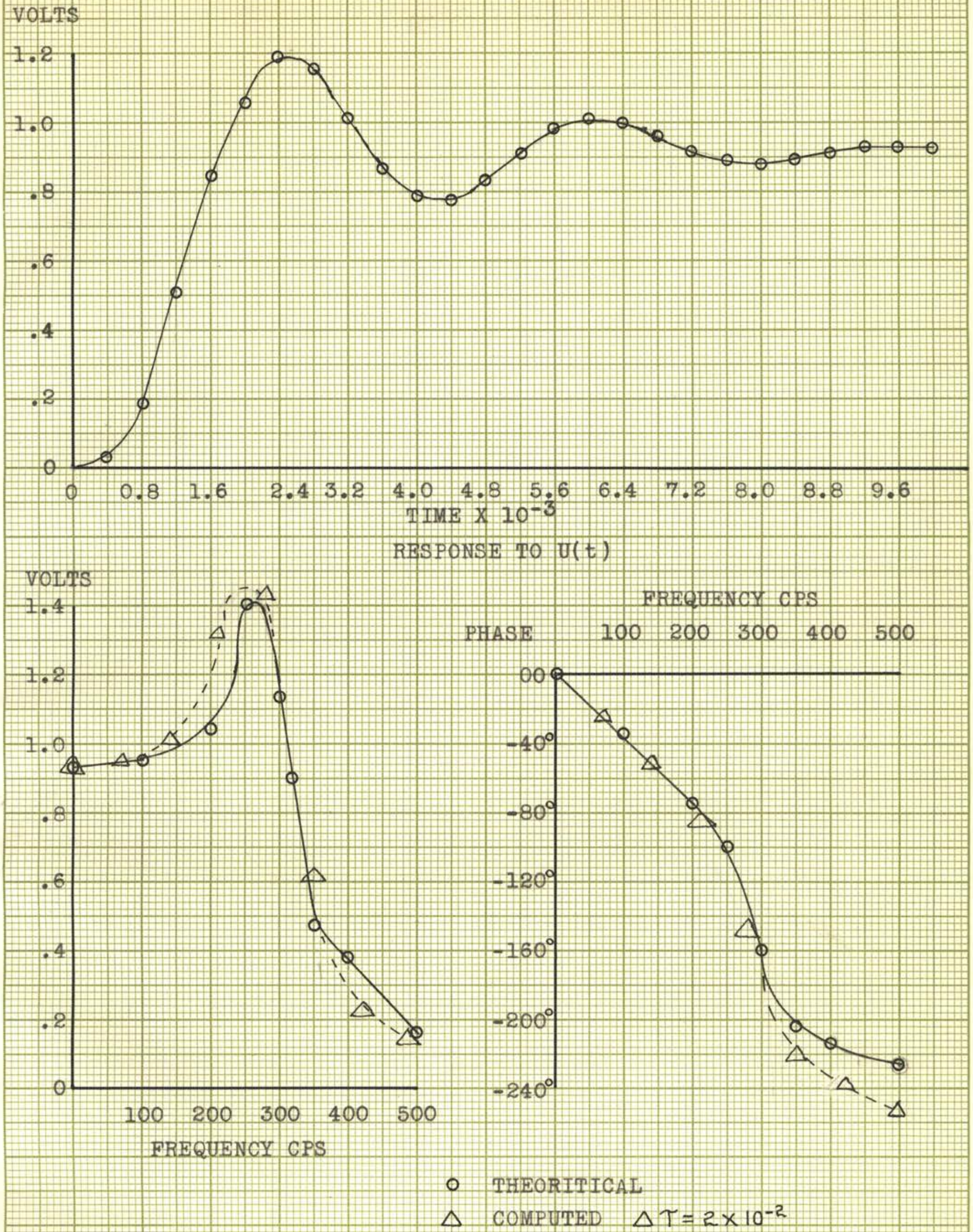


FIGURE 11

that the smaller $\Delta\tau$ is chosen, the nearer to the theoretical curves the computed curves approach. Of course, more terms are necessary to compute the curve for the smaller $\Delta\tau$.

Network Number 2 was chosen because it has certain unique features for a network having poles on the negative real axis. It possess overshot when the unit step function is applied, and the steady state response has a peak at 0.189 cycles per second which is greater than the input voltage. It is noted that the computations did not indicate this peak, although the phase characteristic correspond fairly well.

Network Number 3 is a band pass network, a typical network in a resistance coupled vacuum tube amplifier. Its response to a unit step function starts at zero and returns to zero. It was chosen for this reason.

Network Number 4 is a prototype "T" low pass filter. Its characteristics were so chosen as to aid in determining the theoretical response to a unit step function. Again this is of no importance as far as using the computational technique is concerned, because the scaling factor was used in determining the characteristics. It is noted that the network possess conjugate complex poles, and hence "rings", as can be seen from the response plot. This practical network was chosen to indicate that the methods of this paper are not restricted to those cases where poles are on the real axis.

The complete computation of Network Number 1 will now be carried out for $\Delta\tau = .05$ to show how to use the previously developed technique. This computation will be the only one completely carried through; however, all other computations were made in the same manner.

In Figure 12 is shown a method for carrying through the division. Column 1 is the time and Column 2 is the output voltage associated with the time taken from the response curve. Column 3 is the results of the division, and is obtained by subtracting each term in Column 2 from the previous one. That this is actually division can be readily proven by the reader by setting up the actual equations and dividing in the ordinary manner.[12] This is merely a short cut method which saves paper and time. The reason the method indicated here is practical is that the amplitude of each unit impulse of the time series representing the unit step function has the same value. Column 4 is the value of $\Delta\tau$ used to make the computations with the aid of the prepared table of trigometric functions. It is noted that the scaling factor is two in this case. That is, after computing a frequency component with the aid of the table, say 0.139 cycles per second, when the frequency is plotted, this value must be multiplied by two. In this case 2 x 0.139 = 0.278 cycles per second.

In Figure 13 is shown the complete computations for the terms of the Complex Fourier Series. Each term in the table has been multiplied by 100. One additional frequency, 1.11 cycles

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12 In this example, all the Complex Fourier Series terms are positive, however, in all the other examples, negative terms appeared. Division is, never-the-less, carried out in a similar manner.

TIME	OUTPUT VOLTS	COMPLEX FOURIER TERMS	$\Delta\tau$
0.00	0.00	0.00	0.0
0.05	.21	.21	0.1
0.10	.39	.18	0.2
0.15	.52	.13	0.3
0.20	.63	.11	0.4
0.25	.72	.09	0.5
0.30	.78	.06	0.6
0.35	.82	.04	0.7
0.40	.86	.04	0.8
0.45	.89	.03	0.9
0.50	.92	.03	1.0
0.55	.94	.02	1.1
0.60	.95	.01	1.2
0.65	.96	.01	1.3
0.70	.97	.01	1.4
0.75	.98	.01	1.5
0.80	.99	.01	1.6
0.85	.99	.00	1.7
0.90	1.00	.01	1.8
0.95	1.00	.00	1.9

FIGURE 12

f	nr	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.8	Σ	|Σ|	Θ
0	sin	0																			
[illegible]	cos	21	18	13	11	9	6	4	4	3	3	2	1	1	1	1	1	1	100	100	0°
.139	sin	-1.8	-3.1	-3.4	-3.7	-3.8	-3	-2.3	-2.6	-2.1	-2.3	-1.6	-.9	-.9	-.9	-1	-1	-1	-36.4	95.5	
(.278)	cos	21	17.7	12.6	10.4	8.2	5.2	3.3	3.1	2.1	1.9	1.1	.5	.4	.3	.3	.2	0	88.3		-22.3°
.278	sin	-3.6	-6.1	-6.5	-7	-6.9	-5.2	-3.8	-3.9	-3	-2.9	-1.9	-.9	-.8	-.6	-.5	-.3	0	-53.8	82.5	
(.556)	cos	20.6	16.9	11.3	8.5	5.8	3	1.4	.7	0	-.5	-.7	-.5	-.6	-.8	-.8	-.9	-1	62.1		-40°
.418	sin	-5.5	-9	-9.2	-9.6	-8.7	-6	-3.8	-3.5	-2.1	-1.5	-.5	0	.3	.5	.7	.9	1	-56	68.5	
(.836)	cos	20.4	15.7	9.2	5.5	2.3	0	-1.4	-2	-2.1	-2.3	-1.9	-1	-1	-.9	-.7	-.5	0	39.3		-54°
.556	sin	-7.1	-11.5	-11.3	-10.8	-8.8	-5.2	-2.6	1.4	0	1.1	1.3	.9	1	1	.9	.6	0	-52	57.9	
(1.11)	cos	19.5	13.9	6.5	1.9	-1.5	-3	-3.1	-3.8	-3	-2.8	-1.5	-.5	-.2	.2	.5	.8	1	24.9		-64.4°
.695	sin	-8.8	-13.9	-12.6	-10.8	-7.4	-3	-.4	1.4	2.1	2.8	2	.9	.6	.2	-.3	-.6	-1	-48.8	51.3	
(1.39)	cos	19.1	11.5	3.4	-1.9	-5.1	-5.2	-4	-3.8	-2.1	-1	.2	.5	.8	1	1	.8	0	15.2		-72.8°
.834	sin	-10.5	-15.7	-13	-9.6	-4.5	0	2	3.5	3	2.6	1	0	-.5	-.9	-1	-.9	0	-44.5	45.5	
(1.67)	cos	18.3	9	0	-5.5	-7.8	-6	-3.5	-2	0	1.5	1.7	1	.9	.5	0	-.5	1	6.7		-81.5°
.973	sin	-12	-16.9	-12.6	-7	-.9	3	3.6	3.9	2.1	.5	-.8	-.9	-1	-.8	-.3	.3	1	-38.8	39	
(1.95)	cos	17.2	6.1	-3.4	-8.5	-9	-5.2	-1.7	.7	2.1	2.9	1.8	.5	-.1	-.6	-1	-.9	0	.9		86.8°
1.11	sin	-13.4	-17.6	-11.3	-9.6	3.1	5.2	3.9	2.6	0	-1.9	-2	-.9	-.3	.3	.9	1	0	-39	39.6	
(2.22)	cos	16.2	3.1	-3.4	-5.5	-8.4	-3	.7	3.1	3	2.3	.3	-.5	-.9	-.9	-.5	.2	1			

FIGURE 13

per second, is included. This frequency does not satisfy the relationship between f_h and $\Delta\Upsilon$. It is included to indicate what happens to the approximation when the relationship between $\Delta\Upsilon$ and f_h is not maintained. It is seen that if this term were plotted, the curves showing the relationship between phase and amplitude versus frequency will no longer have a smooth relationship.

It has possibly occurred to the reader that this development of input functions in terms of the unit step function is more involved than necessary. Why not use the unit impulse as the input time series? The unit impulse has the Fourier Transform of Unity,[13] hence the amplitudes of the output time series, taken from the plot of the output time function, are the amplitude terms of the Complex Fourier Series. This point will now be discussed.

Assume the network voltage transfer transform is, as usual in practice, a rational proper fraction. For simplicity, futher assume that the network has only first order poles. Now a proper rational fraction can be factored into a series of partial fractions. In network theory, the result of each factors response to an input function can be summed to obtain the output time function. These factors will have only three forms:[14]

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13 C&F 403.1

G&B 1.01

G 1

14 Barnard, S. and Child, J. J., Higher Algebra

(1) $\frac{K}{S + a}$ (2) $\frac{K'}{(S + a)^2 + B^2}$ (3) $\frac{K''S}{(S + a)^2 + B^2}$

In words, any network, regardless of complexity, can be represented by three fundamental networks, if it is a rational proper fraction and all poles are of the first order.[15] In the case assumed, first order poles, only the above networks need be discussed.

Now a unit impulse is approached in practice by a single square wave if the width of the wave is small compared to the shortest time constant of the circuit, and further, this time must be short compared to the recipricol of the frequency of the highest mode. If the square wave satisfies the above characteristics, and has unit area, it may be considered a unit impulse. Waidelich has prepared a table of LaPlacian Transforms showing network responses to this type of input.[16] A portion of this table is shown in Figure 14, giving the response characteristics of the fundamental networks discussed above. The response to a unit impulse is also shown in the table.

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[15] As a matter of interest, it can be shown that these are the only networks necessary to duplicate another network, regardless of complexity, if the above networks are allowed to occur n times. Using adding circuits, and the Theorm developed in the first section of this paper, it is seen that another method for duplicating an unknown network is indicated. The first network is recognized as a series RC network, the voltage taken across the condenser; the other two networks are series RLC networks. In the first equation, the voltage is taken across the condenser; in the second case, across the resistance.

[16] Waidelich, D. L., "Response of Circuits to Steady-State Pulses," Institute of Radio Engineers, XXXVII (December, 1949), 1396.

F(s)	RESPONSE UNIT IMPULSE	RESPONSE APPROXIMATION UNIT IMPULSE (Waidelich)
$\frac{1}{S + a}$	exp-at G&B 1.102 C&F 438 G 10	$\frac{\text{exp-at}}{1 - \text{exp} - aT}$
$\frac{1}{(S + a)^2 + B^2}$	$\frac{\text{exp-at sinBt}}{B}$ G&B 1.303 C&F 448.1 G 20	$\frac{\text{exp-(a)(t-T)}}{2B}$ x $\frac{\text{sinBt - exp-aTsinB(t-T)}}{\text{coshaT - cosBt}}$
$\frac{S}{(S + a)^2 + B^2}$	$\frac{(a^2 + B^2)^{1/2}\text{ exp-at}}{B}$ x sin(Bt + C) $C = \tan^{-1} B/-a$ G&B 1.303 C&F 449.1 G 21	$\frac{\text{exp-a(t-T)}(a^2 + B^2)^{1/2}}{2B}$ x $\frac{\text{cos(Bt+C) - exp-aTcos(B(t-T)+C)}}{\text{cosh aT - cosh Bt}}$

FIGURE 14

Ernest Frank[17] discusses the case of elementary network response to a pulse of unit height and varying width. It is shown that as the time of duration of the pulse decreases, the amplitude of the response decreases. This, of course, can be seen from the Transforms of Figure 14. The above case does not approach the criterion of the unit impulse, as defined previously. If the stated conditions can be satisfied, the use of the unit impulse in this method of analysis has the advantage described previously in determining the coefficients of the Complex Fourier Series.

The requirements on the approximation for a unit impulse has certain definite limitations in actual practice. These limitations can best be described by noting that as the width of the pulse is decreased, the amplitude must increase. For short time constant circuits, or those having high frequency modes of response, the voltage to be applied can exceed the rating of the elements of the system. Of course a scaling factor can be used in some cases, say using a tenth amplitude unit impulse. The requirement on the width of the pulse is another disadvantage when using this concept, in-as-much as it is assumed that the characteristics of the network are unknown.

It is believed that in most practical applications the unit step function, which was used for all sample calculations, is probably the most practical input to use. It is easy to generate, and from an analytical standpoint, it has been developed to the

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17 Frank, Ernest, Pulsed Linear Networks

point where its characteristics for most systems is quite well known.

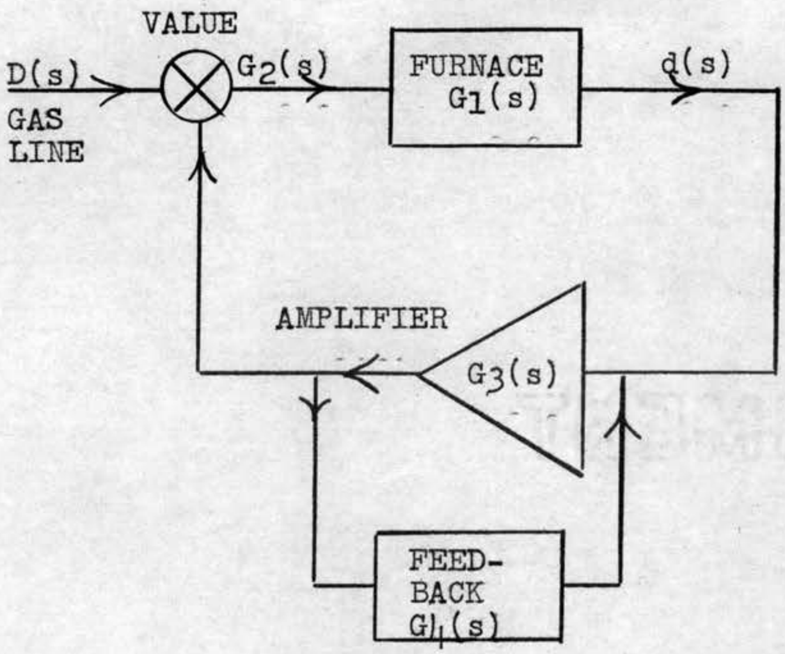
SUGGESTED APPLICATIONS

SUGGESTED APPLICATIONS

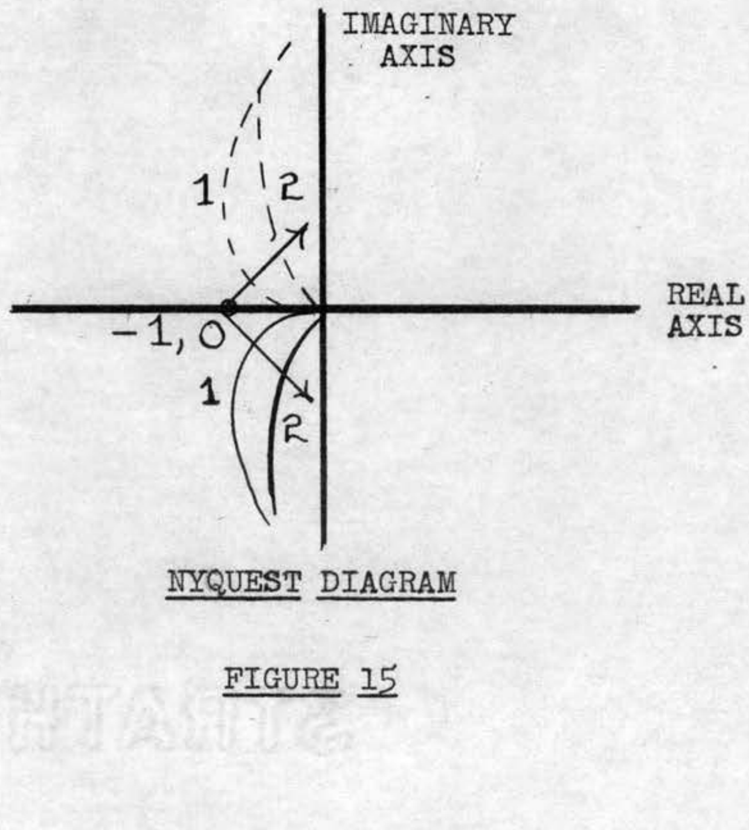
The previously discussed synthesis method is not so useful in the applications shown; that is, if a network is given, a phase amplitude characteristic is obtainable directly from the network. The technique should be more useful in finding electrical network equivalents for linear systems--that is, electrical analogs. One practical application where the previous developments should prove especially useful is in servomechanism design where electrical control systems are used. In general, the design engineer is presented with the system which has to be controlled. For example, the airplane, the ship, the rocket, and so forth, have already been designed, and it is desired to construct a servomechanism control system. The methods developed in this paper should prove especially helpful in these situations. Given a system, a test could be devised which would give input and output time series, and from this information an electrical network analog of the system could be constructed. The design of the control system can proceed from this point.

Certain parts of the previous developments will be discussed from this point of view. It is always easier to discuss principles with an actual example, so the furnace which was discussed in the introduction will be used as an example. A schematic diagram of this furnace is shown in Figure 15.

Imagine the following situation. A gas furnace is given and it is desired to maintain the temperature of the furnace at some predetermined value. It will be assumed that the entire



FURNACE SCHEMATIC



NYQUEST DIAGRAM

FIGURE 15

system is linear in the temperature range of operation. To control the gas supply, a soloniod valve is used which is so designed that it changes the gas supply proportional to the current in the coil. Further, this current is supplied by a pentode, a constant current device. The temperature of the oven is measured by thermocouples, and the design is such that the reference temperature produces zero volts in the output. To use the procedure of this paper, apply a unit step function of voltage to the grid of the pentode which against time, obtaining an output time series. Using the synthesis techniques, curves of amplitude and phase versus frequency may be plotted. Using the techniques described in the first part of this paper, an approximate duplicating network may be found. The transform of these networks are represented by $G_2(s)$ and $G_1(s)$ in Figure 15.

For the present assume that the feedback system for the amplifier is not connected into the system. If D(s) is the disturbance transform, that is the variation in the gas supply from that required to cause zero volts at the thermocouples, and d(s) is the variation transform of the system, that is, the error, the equation for the system is[1]

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[1] This equation is so well known it will not be derived here. Consult any of the references given below.

James Nichols, Phillips, loc. cit.

McColl, LeRoy A., Fundamental Theory of Servomechanisms

Brown, Gordon S., and Campbell, Donald P., Principles of Servomechanisms.

$$d(s) = \frac{D(s)\ G_1(s)}{1 + G_1(s) \cdot G_2(s) \cdot G_3(s)}$$

Assume the amplifier is flat and has a gain M, the equation becomes

$$d(s) = \frac{D(s)\ G_1(s)}{1 + M\,G_1(s) \cdot G_2(s)}$$

An examination of this equation shows that even in this simple system the response will be quite complicated. The valve, $G_2(s)$, has resistance, inductance, mass, damping, and negative compliance. Nothing can be said about the furnace, $G_1(s)$, although it should have some fairly simple long time constant electrical equivalent network. In any case it is apparent that no operations can be performed on the numerator (the furnace was given to the designer) and all operations must be performed on the denominator.

An examination of the equation for this systems indicates that design characteristics might be best applied to the amplifier $G_3(s)$. Assume that the amplifier is designed, as suggested in the first part of this paper, so as to create removable singularities in $G_1(s)$ and $G_2(s)$. The equation now becomes

$$d(s) = \frac{D(s)\ G_1(s)}{1 + M}$$

It is seen from this equation that $G_1(s)$ is never reduced to zero, it can only approach zero. It is also noted that the same characteristic response for the gas furnace exists before the control system was added, but reduced in amplitude. It is also seen that the characteristics of the furnace have been greatly simplified by the creation of the removable singularities.

There are some limitations to the removable singularities compensation technique. In practice this technique can not be applied in as easy a manner as first appears. The reason is obvious if it is noted that the transfer function of planer networks are normally rational proper fractions, that is, there are more poles and zeros. The reasons is obvious if it is remembered that the determinent of the transfer function of a network has a row and column missing in the numerator as compared to the denominator.[2] This does not mean, however, that nothing has been accomplished by introducing these ideas; at least the direction to pursue in making a design is clearly indicated.

In all the literature of servomechanism design, stability is always stated in terms of the Nyquest Stability Criterion. Certain observations will now be made concerning this criterion.[3] It is desirable to discuss general design techniques in current practice, and point out a different direction in view of the developments of this paper.

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[2] Another implication of this statement is that the best that can be hoped for by adding another mesh to an existing network, is to substitute one pole for another, and further, only the mutual elements between the added and existing mesh are useful in creating this removable singularity.

[3] James, Nichols, and Phillips, loc. cit.

Brown and Campbell, loc. cit.

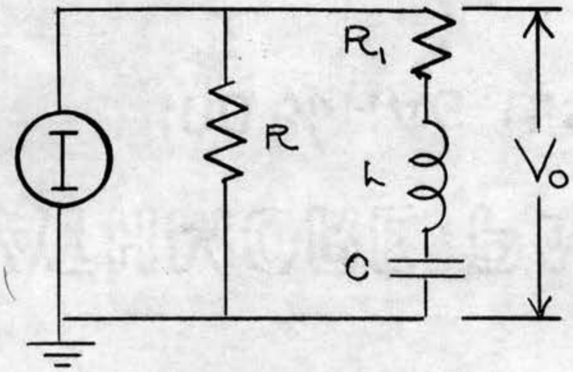
Bode, loc. cit.

In Figure 15 is shown a portion of a Nyquest Diagram. Assume the solid lines are for positive frequency and the dotted lines are for negative frequency. The Number 1 is a hypothetical case for a servomechanism. It is desired to increase the stability of this system, as in its present form it approaches the point -1.0 too closely. The conventional method for doing this is to add another network, or change the gain so as to shift the curves in the arrow direction, giving curve Number 2. This later curve is further away from the point -1.0, hence the system is more stable.

It was shown in the last section that any network transfer function can be broken down into the sum of three fundamental types of networks. Using the principles enumerated in this paper, it would appear better to examine this sum and ascertain which pole is causing the difficulty, and create a removable singularity by adding a proper zero for this particular pole. Or it may be possible to create a removable singularity for this pole, and add another pole which does not have the undesirable properties of the removed pole.

In other words, the corrective network used should have at least the same number of zeros as poles. In the creation of removable singularities, it would be desirable to only add zeros, but as was indicated previously, this can only be done in theory. The best that can be expected is that a zero and a pole will be added in the same operation.

Two possible networks meeting the criterion mentioned above are shown in Figure 16. Network Number 1 will cancel a conjugate



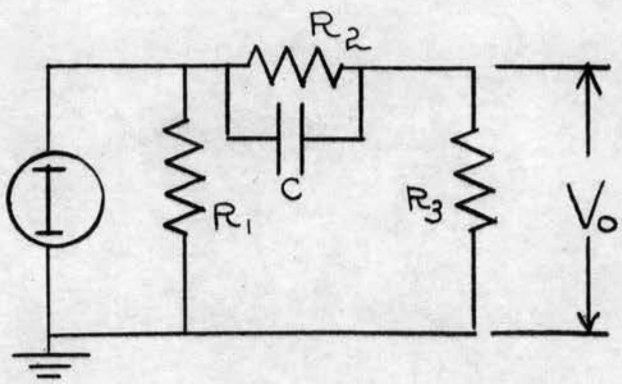
NETWORK NO. 1

$$V_o(s) = \frac{I(s)\,R\left[(s+\alpha)^2 + B^2\right]}{(s+\lambda)(s+\delta)}$$

$$\alpha = \frac{R_1}{2L} \qquad B^2 = \frac{1}{LC} - \left(\frac{R_1}{2L}\right)^2$$

$$\lambda = \frac{R_1+R}{2L} - \left[\left(\frac{R_1+R}{4L}\right)^2 - \frac{1}{LC}\right]^{\frac{1}{2}}$$

$$\delta = \frac{R_1+R}{2L} - \left[\left(\frac{R_1+R}{4L}\right)^2 - \frac{1}{LC}\right]^{\frac{1}{2}}$$



NETWORK NO. 2

$$V_o(s) = \frac{R_1 R_3\, I(s)\left(s + \frac{1}{R_2 C}\right)}{(R_1+R_3)\left(s + \frac{1}{C}\left[\frac{1}{R_2} + \frac{1}{R_1+R_3}\right]\right)}$$

FIGURE 16

pole if it is properly designed. Depending upon the selection of R and R_1, the zero can be made conjugate, but the pole will be of second order or two poles on the negative real axis. That is, a damped pole may be substituted for an undamped (conjugate) pole.

Network Number 2 will cancel a real pole and it is further noted that it causes a shorter time constant to be substituted for the removed pole. That is, the pole is moved further back on the negative real axis. In other words, the time response of the system has been decreased.

Some additional remarks will now be made about the creation of removable singularities. It is easier to take an actual example to illustrate the following point. Assume that a given network has the following voltage transfer transform:

$$E_o(s) = \frac{E_i(s)\ (s + a)}{(s + b)}$$

where E_o is the output voltage and E_i is the input voltage. The response of this network to a unit step function input is[5]

$$E_o = \frac{E_i}{b}\ (a - (a - b)\ \exp -bt)$$

This network has a surge at $t = 0$, and it is desired to remove this surge. One way to remove this surge is to add another network, and a simple one to add would be a pole at $s = -c$. The

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[5] G&B 1.107.

G 2 & 10.

equation becomes

$$E_o(s) = \frac{E_i(s)\ (s + a)}{(s + b)(s + c)}$$

Let c = na, and find the response of this network to a unit step function.[6] The output voltage is

$$E_o = E_i \left[\frac{1}{bn} - \frac{(a - b) \exp -bt}{b(na - b)} - \frac{(1 - n) \exp -nat}{n(a - b)} \right]$$

It is observed that adding the other pole has complicated the response of the system, but that by the proper choice of design parameters, the surge can be reduced. If n = 1, that is, a = c, which is the case for a removable singularity, the response to a unit step function is

$$E_o = \frac{E_i}{b} (1 - \exp{-bt})$$

It is seen that in this case, the surge has been removed, and the response has been simplified.

In the last term in equation above where n = 1, it is observed that the contribution of this term is markedly reduced in the case where the relationship a = c is only approximate. In practical systems, there are always tolerances, and the characteristics of components are influenced by such things as humidity, temperature, pressure, and other imponderables. However, the above discussion indicates marked advantages are to be gained by creating removable singularities, even if the relation-

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[6] G&B 1.109; G 14.

ship between the zero and pole are only approximate. Ordinarily the existing literature indicates the addition of additional networks to change response characteristics, but the creation of an additional pole adds a term in the response of the system, thus increasing the complexity of the response. Rather, do not add any poles or zeros to a network to change its characteristic; always add networks which will create removable singularities.

It can also be shown by carrying out computations in a manner similar to that above, that conjugate poles and zeros can be handled by a similar technique, and further, that if the relationship between the pole and zero is not exact, definite benefits are to be had in reducing the response of the conjugate term.

In Figure 15 a feedback circuit has been indicated in connection with the control amplifier. This circuit was introduced to show certain points about control systems, in view of the general developments of this paper. In using a feedback circuit in an amplifier where steady state conditions prevail, it is common practice to so construct the feedback circuit that it has the same loss characteristic as the desired over all gain characteristic.[7] In terms of poles, this means that the feedback circuit should have the same voltage transfer transform as that which it is desired to correct. In considering transient response, however, any poles added in the feedback circuit will increase the complexity of the response because of the added

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[7] Terman, E. T., Radio Engineer's Handbook, p. 395.

poles. This can be demonstrated by elementary mathematics by considering two cases of a feedback amplifier; one with a pole in the feedback circuit, and one without. All servomechanisms, however, must control transients; if steady state conditions prevailed, there would be no need for a servomechanism. In other words, in designing a feedback circuit for the amplifier of this system, or any other system where the transient response is important, the feedback system should not contain any singularities.[8] The most benefit will be derived when the feedback is real.

The effect of real feedback in any amplifier is to change the location of all the poles. It is difficult to make any general statements without considering actual circuits. Computations for actual circuits however shows that real feedback results, in most cases, in an improvement in the reproduction of transient response of the system.

Brown and Campbell[9] analyze the case where positive feedback is used to overcome certain undesirable characteristics in a servomechanism. In their development it is shown that in theory marked gains in response can be anticipated using positive feedback. It is also shown, however, that the demands made on circuit parameters are such that this type of feedback is not practical.

The only point to note in the above discussion is that

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[8] James, Nichols, and Phillips, loc. cit., p. 63.

[9] Brown and Campbell, loc. cit.

networks in the feedback loop of an amplifier will not create removable singularities; but that in general real feedback will improve the response of the system.

In this section it has been shown how to use all the precepts of the paper in the discussion of a simple gas furnace problem. The concepts used in the discussion of the problem were all from the zero and pole point of view. It was shown that this is sufficient to determine the characteristics of the system. It was also shown that the creation of removable singularities is the proper design procedure, and further, that these removable singularities need not be perfect in order to obtain an improvement in response characteristics.

CONCLUSIONS

CONCLUSIONS

In this thesis a new method for solving the Synthesis Problem has been developed. Given an input and output time function for any linear system, it is shown how to obtain a phase and amplitude versus frequency characteristic. This identifies an electrical network. It is also shown how to obtain a network which will approximate the unknown network by means of elementary networks when the phase and amplitude versus frequency characteristic is known. When this network has been found, in many practical cases, a network which will compensate the unknown system is also known.

The work involved in obtaining these solutions is lengthy, but the procedure is straightforward. The method has a firm theoretical basis, hence some idea of the accuracy, or goodness of the solution is known before computations are started.

Because it gives results in terms of electrical network equivalents, the method is expecially applicable to problems in servomechanism design. In this way the design of electrical control systems proceeds in an orderly manner. It is also indicated that such concepts as velocity and acceleration control functions are not necessary in a discussion of servomechanism design. All developments in this paper are from the zero and pole point of view, with a slightly different concept than that given in current literature. It is also shown that zeros and poles can be given a geometrical significance corresponding to their mathematical name. It is further shown that zero and pole concepts are sufficient unto themselves for the identification of network characteristics.

This paper is merely an introduction showing the power of the Theory of Functions of a Complex Variable in solving network problems. Many phases of continued development are pointed out. To mention a few, the actual problem of changing the Nyquist Stability Diagram by means of removable singularities should be developed to a point where graphical solutions are possible. It was also pointed out that only three fundamental networks were needed to duplicate any network. This indicates another method for obtaining equivalent networks. The addition of circuits is well known in the art, and it follows that the fundamental networks would be used in the process. To obtain equivalent networks in this manner, response curves could be constructed, of lucite say, and used in the same manner as was done in this paper. However, direct ratios of input and output voltages would be used instead of the logarithm of this ratio.

It was also pointed out in a footnote that with a given network, addition of another mesh to the network to accomplish compensation, can at best remove one pole from the system, and substitute another pole, and further, only the mutual element between these meshes would contribute to this relationship. The actual conditions necessary to cause this substitution could be worked out from determinant theory.

Recently it has been recognized that the computations and the methods for synthesizing a network as developed in this paper, can be markedly simplified. The point which was recognized is in relation to the representation of a network transfer function by its Complex Fourier Series. It was shown in the second section of this paper that the voltage transfer function could be expressed as

$$F(\omega) = a_o + \sum_{n} a_n(\text{Cos } n \Delta\tau\, \omega - j\text{Sin } n\Delta\tau\, \omega)$$

When this equation is carefully examined, it is evident that if the real or imaginary components are known, the other component is uniquely determined. In other words, it is only necessary to use either the real or imaginary component alone to express a networks voltage transfer characteristic.

Bode shows that for a minimum phase shift network[1] that when the attenuation characteristic is known, the phase characteristic is unique. Stated in terms applicable to the developments of this paper, if the real terms of a network transfer function are known, then the imaginary terms are uniquely determined.

The technique used in this paper for determining network characteristics from input and output time series, envolved the computation of the terms of a Complex Fourier Series. In the examples of this paper, the Complex Fourier Series was evaluated for both the real and imaginary terms. From the previous statements it is apparent that it is in reality only necessary to evaluate one series, say that for the real terms. This will cut

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[1] Bode, loc. cit.

A minimum phase shift network is one whose poles are all located in the left half-plane.

the computations necessary for determining the characteristics of the unknown network in half. It is further evident that in determining an equivalent network by means of dimensionless network curves, that these curves need only be developed for their real part.

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