A THEORETICAL STUDY OF IMPEDANCE MATCHING AS APPLIED TO SURGE SUPPRESSION INSTRUMENTS IN HYDRAULIC SYSTEMS

By

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TABLE OF SYMBOLS

a = Velocity of propagation

α = Attenuation constant

A = Area of the pipe

β = Phase constant

wC = Capacitance per unit length

c = Frequency of vibration

D = Diameter of the pipe

e = Thickness of the pipe

E = Modulus of elasticity

f = Friction coefficient

$f(Q)$ = A function of Q

$g(Q)$ = A function of Q

g = Acceleration constant of gravity

γ = Propagation constant

j = $\sqrt{-1}$

K = Bulk modulus of the fluid

x_1 = Length of the pipe

wL = Inertiance per unit length

Γ = Reflection coefficient

n = Length of the crank arm

N_r = Reynolds number

ϕ = Angle of pressure lag

ρ = Density

P_x = Pressure at any point x distance from the end

P_a = Average pressure

P_m = Amplitude of pressure variation

P_r = Receiving end pressure

P_s = Sending end pressure

Q_x = Rate of flow at any point

Q_a = Average flow

Q_m = Amplitude of flow variation

Q_r = Receiving end flow

Q_s = Sending end flow

r = Radius of crank arm

R = Frictional resistance per unit length

Re = Symbol signifying real value of a complex number

S_t = Hoop tension stress

t = Time in seconds

u = Viscosity

v_x = Velocity at any point

v_a = Average velocity

v_m = Amplitude of velocity variations

V = Volume

w = Angular velocity radians per second

x = Distance from the end of the pipe

y = Pressure head, feet of fluid

Y = jwC

Z = Series impedance per unit length

Z_1 = jwL

Z_c = Characteristic impedance

Z_x = Impedance at any point x

Z_r = Impedance at the receiving end

Z_s = Impedance at the sending end

CHAPTER I

INTRODUCTION

The use of an electrical analogy in computing the behavior of surges in pipelines is a new approach to find the most practical way to arrange pumps and surge suppression devices where they will cause the least damage.

It is thought that since the method of determining the parameters in an electrical network is a mathematical process, that process or a similar one can be used in the analysis of the pipeline phenomenon. The basic equations of heat flow, fluid flow, and electrical flow are of the type called Leplace's equation. The solution of this type equation would be similar for any of the above types of flow, but the interpretation by the individual is different. The electrical engineer interprets his results in terms of volts and amperes whereas the hydraulic engineer thinks in terms of pressures and velocities.

The electrical engineers have, as a necessity, had to thoroughly and completely solve and interpret the results of variable flow. Perhaps some of their results will be useful in limiting pulsations in a pipeline.

CHAPTER II

PREVIOUS INVESTIGATIONS

The phenomenon of variable flow was first noted when water was pumped in a closed conduit or was allowed to flow in penstocks. The first notable explanations were by Joukowsky[1] and Allievi[2]. There were then a great many investigations performed and as many results as investigators were found. To remedy this the hydraulic Division of A. S. M. E. appointed a committee on water hammer. The report of this committee is contained in the Symposium on Water Hammer[3] presented at the Century of Progress Exposition, Chicago, Illinois, June 30, 1933. Most all of the above work was done with emphasis on water hammer of opening or closing of a valve. The investigations of variable flow (pulsations) in a pipeline due to the periodic motion of a piston was first published by John Goodman[4]. The results of his paper created interest in the subject.

J. W. Squire[5] of Service Pipeline Company presented a valuable paper to the Petroleum Engineering Conference of the A. S. M. E. in 1948. His

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[1] Joukowsky, "Water Hammer," Proceedings American Water Works Association, 1904, p. 341, Translated by O. Simon.

[2] Lorenzo Allievi, "General Theory of Perturbed Flow of Water in Pressure Conduit," Milan 1903, Translated by E. E. Halmos 1925.

[3] Symposium on Water Hammer, Published in 1933 by The Hydraulics Division of A. S. M. E.; Second Edition 1949 by A. S. M. E.

[4] John Goodman, "Hydraulic Experiments on a Plunger Pump." Proceedings Institute of Mechanical Engineers, (February 20, 1903), pp. 123-197.

[5] J. W. Squire, "Pressure Surges and Vibrations in Reciprocating Pump Piping," World Oil Vol. 128, No. 12, (March 1949), pp. 171-182.

paper gave a background of the accomplishment of previous investigations and correlated their findings with his. He pointed out the inefficiency of the surge removal instruments of that time at high speeds. His recommendations were to redesign the valve and pump connections so as to reduce the surge at the start.

Results obtained by the Division of Engineering Research, Oklahoma Agricultural and Mechanical College, gave impetus to the belief that the same general results are obtained in pipelines as those obtained in electrical transmission lines. Results of present investigations in the Oklahoma Institute of Technology are not available for inclusion in this thesis.

CHAPTER III

OBJECTIVE

The objective of this investigation is to arrive at a basis for determining the required characteristics of surge suppression equipment which effects the maximum suppression of pressure variations in fluid systems. Determination of the required characteristics may depend upon measurement of system constants that are not at present readily measureable or are at present ignored. It is also possible that certain factors thought to be of prime importance may not have the magnitude or affect commonly attributed to these factors.

This work will be limited to the theoretical aspects only. Verification and evaluation will be the basis of further study.

CHAPTER IV

DERIVIATION OF FLOW RELATIONS

1. Derivation of Basic Differential Equations.

Figure 1

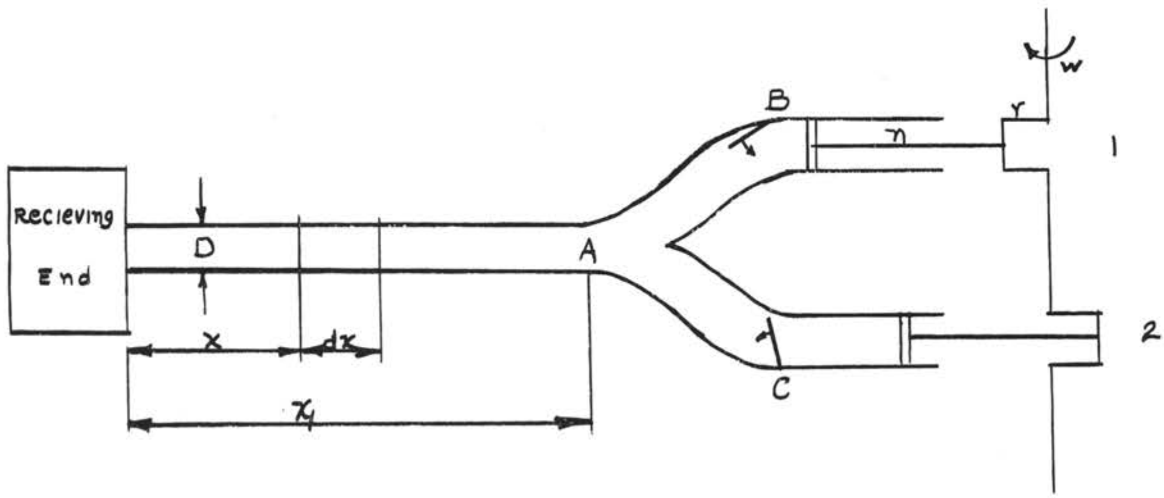

Fig. 1 is the horizontal view of the system which will be used in this derivation. The pipeline of length x_1 is filled with a fluid with an average density ρ, an average pressure P_a, and flowing with an average velocity of v_a. The average flow is then $Q_a = v_a A$.

The pump is operating upstream and the pistons are moving with a motion that can be calculated by use of kinetics. In any case the motion is some function of time. The rate of flow is also a function of time. There will be a pressure which will be a function of time.

A summation of forces at the section will give:

$$\sum F_x = \text{Mass x acceleration}$$

Whereby $$A\ dP - ARQdx = \rho \frac{dv}{dt} Adx \qquad (1)$$

Since $$\frac{Adv}{dt} = \frac{dQ}{dt} \quad \text{and } dP = \frac{\partial P}{\partial x} dx$$

Then; $$\frac{\partial P}{\partial x} = R\ Q + \frac{\rho}{A} \frac{dQ}{dt} \qquad (2)$$

It is now noted that another relation can be obtained by considering the elemental ring of length dx. The volume of fluid stored in length dx during time dt due to the elasticity of the pipe is as follows:

Let dV_P = the volume of fluid stored in the pipe

Then, $$dV_P = \left[\frac{\pi}{4}(D + dD)^2 - \frac{\pi}{4} D^2\right] dx = \frac{\pi D\ dD\ dx}{2} \qquad (3)$$

and $$S_t = \frac{\partial P}{\partial t} dt \frac{D}{2e}, \quad S_t = \frac{dD}{D} E \qquad (4)$$

therefore, $$dD = \frac{D}{2\ e} \frac{D}{E} \frac{\partial P}{\partial t} dt \qquad (5)$$

Substituting eq. (5) into eq. (3) gives,

$$dV_P = \frac{AD}{Fe} \frac{\partial P}{\partial t} dt\ dx \qquad (6)$$

Let dV_f = The volume of fluid stored in the section due to the compressibility of the fluid.

The bulk modulus of the fluid "K" is defined as:

$$K = \frac{\text{increase in Pressure}}{\text{Diminution in Volume}}$$

or $$K = \frac{\frac{\partial P}{\partial t} dt}{\frac{dV_f}{V}} = \frac{Adx}{dV_f} \frac{\partial P}{\partial t} dt \qquad (7)$$

$$dV_f = \frac{A}{K} \frac{\partial P}{\partial t} dt\ dx \qquad (8)$$

$$dV_{total} = dV_P + dV_f = A\left[\frac{1}{K} + \frac{D}{e\,E}\right]\frac{\partial P}{\partial t}\ dt\ dx \tag{9}$$

The total change in volume must be equal to the difference of the volume of fluid flowing at the ends of the section or $\frac{\partial Q}{\partial X}\,dx\,dt = dV_t$

$$\frac{\partial Q}{\partial x}\,dx\,dt = A\left[\frac{1}{K} + \frac{D}{e\,E}\right]\frac{\partial P}{\partial t}\ dt\ dx \tag{10}$$

resulting in

$$\frac{\partial Q}{\partial x} = A\left[\frac{1}{K} + \frac{D}{E\,e}\right]\frac{\partial P}{\partial t} \tag{11}$$

The similarity of eqs. (2) and (11 to those given by Alleivi[6] are of great significance for if P = gy and R = 0 they are identical and reduce to:

$$\frac{\partial y}{\partial x} = \frac{1}{g}\frac{dv}{dt} \tag{12}$$

$$\frac{\partial v}{\partial t} = \frac{g}{a^2}\frac{dy}{dt} \tag{13}$$

Which are those given by Allievi[7].

2. Solution of Differential Equations.

For convenience let $L = \frac{\rho}{A}$ and $C = A\left[\frac{1}{K} + \frac{D}{E\,e}\right]$

The eqs. (2) and (11) become:

$$\frac{\partial P}{\partial x} = RQ + L\frac{dQ}{dt} \tag{14}$$

$$\frac{\partial Q}{\partial x} = C\frac{\partial P}{\partial t} \tag{15}$$

A solution to eqs. (14) and (15) may be found if $\frac{dQ}{dt}$ and $\frac{\partial P}{\partial t}$ can be evaluated. This can be done by consideration of the motion of the fluid

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[6] Op. cit.

[7] Op. cit

at the pump.

In order to obtain results of a basic nature certain assumptions must be made.

1. The motion of the fluid at point "A" Fig. 1 has the same motion as either piston. At the instant shown piston (1) has completed its power stroke and started on the suction stroke. The valve "B" is beginning to close as shown by the arrow. Piston (2) is starting its power stroke and the valve at "C" is opening.

(a) Assume valves "B" and "C" open or close instantaneously.

(b) Assume that the pipe is curved such that the motion of "A" is the motion at either piston.

2. The piston rod is long in comparison to the radius of the crank shaft.

3. The pipeline is ideal and does not vary in thickness, diameter, or elevation throughout its length.

Figure 2

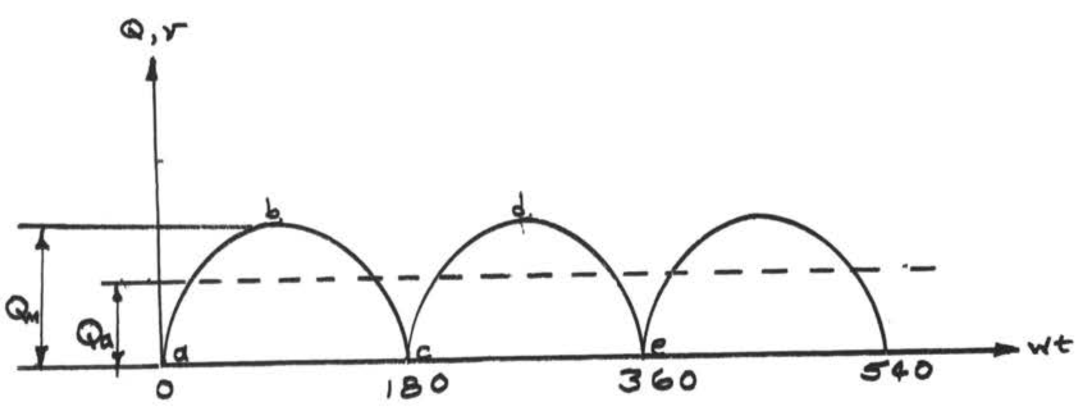
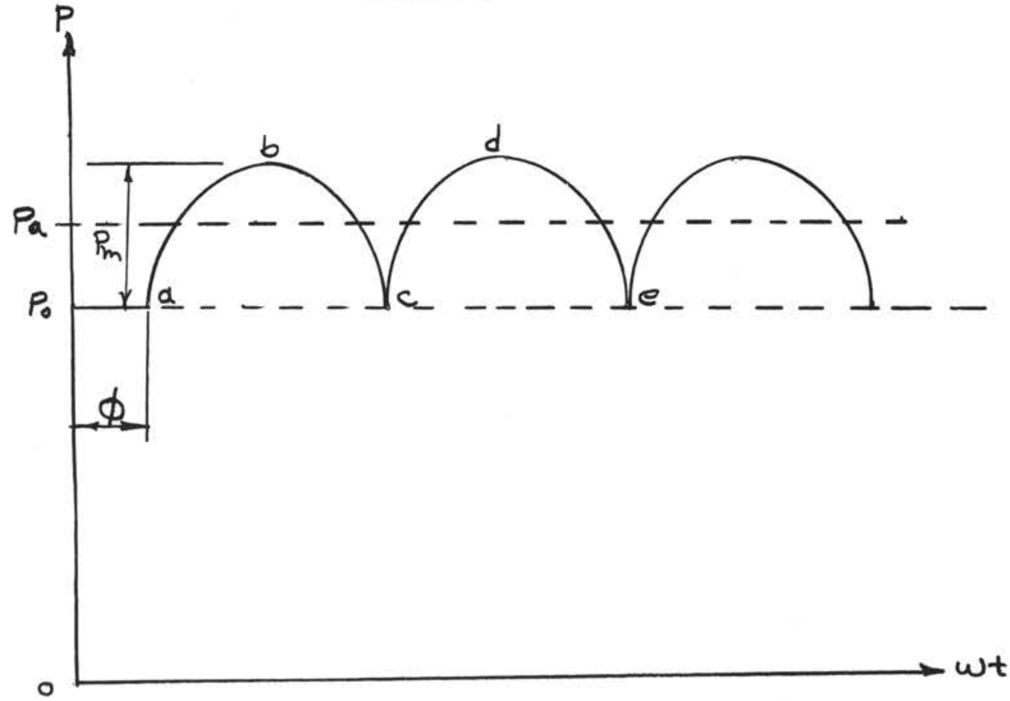


Figure 3



As piston (2) starts its motion the fluid at "A" will have a velocity that varies as a sine curve throughout the action or for 180° Fig. 2, a b c. The resultant pressure will be of the same nature, Fig. 3, a b c, but displaced by the amount of the static head represented by Po and displaced in phase by a certain angle. That angle ϕ is the amount of time that it takes for the pressure to act after the piston is in motion.

In Fig. 2, c d e is the velocity curve for piston (1) during its power stroke. In a like manner Fig. 3, c d e is the resulting pressure. The average flow Q_a is $\frac{2}{\pi} Q_m$ as given by the following:

$$Q_a = \frac{1}{\pi} \int_0^{\pi} Q_m \operatorname{Sin} wt \, d(wt) = \frac{1}{\pi} \Big[- Q_m \operatorname{Cos} wt \Big]_0^{\pi} = \frac{2}{\pi} Q_m$$

and the average pressure is given by $Pa = Po + \frac{2}{\pi} Pm$.

By use of the Fourier Series a value of V, Q and P may be obtained the results of which are shown below.

$$v = \frac{2}{\pi} v_m + \frac{\pi - 2}{\pi} v_m \operatorname{Cos}(2wt + 180) + \text{higher harmonics} \tag{16}$$

$$Q = \frac{2}{\pi} Qm + \frac{\pi - 2}{\pi} Qm \operatorname{Cos}(2wt + 180) + \text{higher harmonics} \tag{17}$$

$$P = P_o + \frac{2}{\pi} Pm + \frac{\pi - 2}{\pi} Pm \operatorname{Cos}(2wt - \phi + 180) + \text{higher harmonics} \tag{18}$$

Figure 4

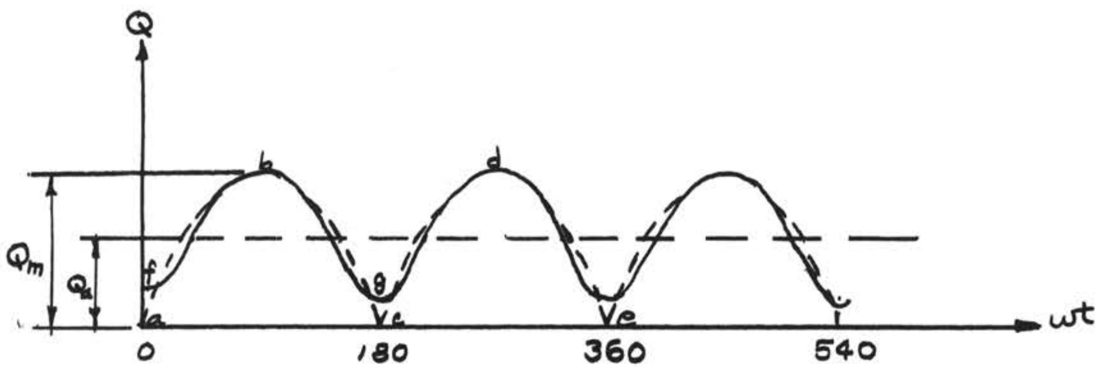


Fig. 4 is a plot of eq. (17) and as can be seen it is a fairly close approximation. The average value of curve a b c is shown below.

$$Qa = \frac{1}{\pi}\int_0^{\pi}\left[\frac{2}{\pi} Qm + \frac{\pi - 2}{\pi} Qm \operatorname{Cos}(2wt + 180)\right] d(wt)$$

$$Qs = \frac{1}{\pi}\left[\frac{2}{\pi} Qm\ wt\right]_0^{\pi} = \frac{2}{\pi} Qm \tag{19}$$

Assuming that eqs. (17) and (18) are correct as far as average values are concerned and by use of the convenient notation of rotating vectors a workable solution of eqs. (14) and (15) is obtained as follows.

Consider a vector $\bar{A}$ where, $\bar{A} = \cos\theta + j \sin\theta$

$j^2 = -1$, $j^3 = -j$, and $j^4 = 1$. Also consider the real and imaginary values where:

$\mathrm{Re}\,\bar{A} = \cos\theta$ and $\mathrm{im}\,\bar{A} = \sin\theta$

Vector $\bar{A}$ can be expressed as an exponential $\bar{A} = e^{j\theta}$ and $\mathrm{Re}\bar{A} = \mathrm{Re}\,e^{j\theta}$

With the above in mind, eqs. (17) and (18) are as follows:

$$Q = \frac{2}{\pi} Qm + \mathrm{Re}\,\frac{\pi - 2}{\pi} Qm\, e^{j(2wt + 180)} \tag{20}$$

$$P = P_o + \frac{2}{\pi} Pm + \mathrm{Re}\,\frac{\pi - 2}{\pi} Pme^{j(2wt - \phi + 180)} \tag{21}$$

Since both Q and P are shifted in phase 180° or one complete wave length ($\lambda = \pi$), and the period of vibration of curve f b g of Fig. 4 is half the period of vibration of curve a b c, then eqs. (20) and (21) can be rewritten:

$$Q = Q_1 + R_e Q_2\, e^{jw_1 t} \tag{22}$$

$$P = P_1 + R_e P_2\, e^{j(w_1 t - \theta)} \tag{23}$$

where $Q_1 = \frac{2}{\pi} Q_m$, $Q_2 = \frac{\pi - 2}{\pi} P_m$ Q_m

Remembering that w_1 is the angular velocity of the new function and is related to the angular velocity of the crank by $w_1 = 2w$. For convenience the subscript will be dropped.

Since eqs. (20) - (23) are symbolic, mathematical operations must not be performed on them without first determining whether the results produced have a real significance. The chief operations that can be performed on those equations are differentation and intergration. With this fact in mind it is now possible to find expressions for $\frac{\partial P}{\partial t}$ and $\frac{d\,Q}{d\,t}$ as shown below.

$$\frac{d\,P}{d\,t} = ReP_2jwe^{j(wt\,-\,\emptyset)} = jwP_2 = jwP - jwP_1 \tag{24}$$

$$\frac{d\,Q}{d\,t} = R_eQ_2jwe^{jwt} = jwQ - jwQ_1 \tag{25}$$

By substituting eqs. (24) and (25) into (14) and (15) the following set of differential equations are obtained.

$$\frac{\partial P}{\partial x} = RQ + jwLQ - jwQL = (R + jwL)Q - jwLQ_1 \tag{26}$$

$$\frac{\partial Q}{\partial x} = jwCP - jwCP_1 \tag{27}$$

Let $R + jwL = Z$, $jwL = Z_1$, and $jwC = Y$

Then,

$$\frac{\partial P}{\partial x} = ZQ - Z_1Q_1 \tag{28}$$

$$\frac{\partial Q}{\partial x} = YP - YP_1 \tag{29}$$

There are various ways to solve the above equations. The method used is similar to the method of solving electrical transmission line equations. By taking the partial of the eqs. (28) and (29) with respect to x and substituting into the new equations the values from eqs. (28) and (29) the following results are obtained:

$$\frac{\partial}{\partial x}\left(\frac{\partial P}{\partial x}\right) = \frac{\partial^2 P}{\partial x^2} = Z\,\frac{\partial Q}{\partial x} \tag{30}$$

and

$$\frac{\partial^2 Q}{\partial x^2} = Y\,\frac{\partial P}{\partial x} \tag{31}$$

$$\frac{\partial^2 P}{\partial x^2} = YZP - YZP_1 \tag{32}$$

These equations are linear and of the second order, the solutions of which are of the form,

$$P = C_1e^{mx} + C_2e^{-mx} + P_P \tag{34}$$

Where m is the solution of the auxiliary equation $m^2 = ZY$, C_1 and C_2 are constants, and P_P is the particular solution and is equal to P_1.

$$P_x = C_1e^{\sqrt{ZY}\,x} + C_2e^{-\sqrt{ZY}\,x} + P_1 \tag{35}$$

By differentiating (35) with respect to x.

$$\frac{\partial P}{\partial x} = C_1 \sqrt{ZY}\, e^{\sqrt{ZY}\,x} - C_2\sqrt{ZY}\, e^{-\sqrt{ZY}\,x}$$

$$Q_x = C_1\frac{\sqrt{ZY}}{Z}\, e^{\sqrt{ZY}\,x} - C_2\frac{\sqrt{ZY}}{Z}\, e^{-\sqrt{ZY}\,x} + \frac{Z_1Q_1}{Z} \tag{36}$$

$$\text{Let } Z_c = \sqrt{\frac{Z}{Y}} \quad (37) \qquad \text{and} \quad \gamma = \sqrt{ZY} \tag{38}$$

Also let P_r and Q_r be the pressure and volume current at the end of the pipe (receiving end) where x = o.

Substituting those values into eqs. (35) and (36)

when $\quad x = 0$

$$P_r = C_1e^0 + C_2e^0 + P_1 = C_1 + C_2 + P_1$$

and $$Z_cQ_r - \frac{Z_1Q_1}{\gamma} = C_1 - C_2$$

and $$C_1 = \frac{(P_r - P_1) + (Z_cQ_r - \frac{Z_1\,Q_1}{\gamma})}{2} \tag{39}$$

$$C_2 = \frac{(P_r - P_1) - (Z_cQ_r - \frac{Z_1Q_1}{\gamma})}{2} \tag{40}$$

Substituting the values of C_1 and C_2 into (35) and (36)

$$P_x = \frac{(P_r - P_1) + (Q_rZ_c - \frac{Z_1}{\gamma}Q_1)}{2}\, e^{\gamma x} + \frac{(P_r - P_1) - (Q_rZ_c - \frac{Z_1}{\gamma}Q_1)}{2}\, e^{-\gamma x} + P_1 \tag{41}$$

by rearranging and using the following relations

$$\text{Cosh}\,\gamma x = \frac{e^{\gamma x} + e^{-\gamma x}}{2} \quad + \quad \text{Sinh}\,\gamma x = \frac{e^{\gamma x} - e^{-\gamma x}}{2}$$

Therefore:

$$P_x = (P_r - P_1)\ \text{Cohs}\,\gamma x + (Q_r Z_c - \frac{Z_1}{\gamma} Q_1)\ \text{Sinh}\ \ x + P_1 \qquad (42)$$

and similarly,

$$Q_x = \tfrac{1}{2}\left[\frac{(P_r - P_1)}{Z_c} + (Q_r - \frac{Z_1 Q_1}{Z})\right] e^{\gamma x} - \tfrac{1}{2}\left[\frac{P_r - P_1}{Z_c} - (Q_r + \frac{Z_1 Q_1}{Z})\right] e^{-\gamma x} + \frac{Q_1 Z_1}{Z}$$

$$(43)$$

and

$$Q_x = (Q_r - \frac{Z_1 Q_1}{Z})\text{Cosh}\,\gamma x + \frac{P_r - P_1}{Z_c}\ \ \text{Sinh}\,\gamma x + \frac{Q_1 Z_1}{Z} \qquad (44)$$

if $x = \chi_1$ P_s and Q_s are the sending end pressure and flow. Then

$$P_s = (P_r - P_1)\ \text{Cosh}\,\gamma x_1 + (Q_r Z_c - \frac{Z_1}{\sqrt{ZY}} Q_1)\ \text{Sinh}\,\gamma X_1 + P_1 \qquad (45)$$

$$Q_s = (Q_r - \frac{Z_1 Q_1}{Z})\ \text{Cosh}\,\gamma\chi_1 + \frac{P_r - P_1}{Z}\ \ \text{Sinh}\,\gamma x_1 + \frac{Q_1 Z_1}{Z} \qquad (46)$$

By rearranging eqs. (45) and (46)

$$P_r - P_1 = (P_s - P_1)\ \text{Cosh}\,\gamma x_1 - (Z_c Q_r - \frac{Z_1 Q_1}{\sqrt{ZY}})\ \text{Sinh}\,\gamma x_1 \qquad (47)$$

and

$$Q_r - \frac{Z_1}{Z} Q_1 = (Q_s - \frac{Z_1}{Z} Q_1)\ \text{Cosh}\,\gamma x_1 - \frac{P_s - P_1}{Z_c}\ \ \text{Sinh}\,\gamma x_1 \qquad (48)$$

3. Explanation of Equations.

In order to give some physical interpretation to eqs. (41) and (43) they can be thought of as travelling waves in the pipeline. For convenience they can be abbreviated as follows.[8]

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[8] Kimbark, E. W. Electrical Transmission of Power and Signals. John Wiley and Sons: Chapter 6, pp. 92-134.

$$\text{Let } P^+ = P_r^+ e^{\gamma x} = \tfrac{1}{2}\left[(P_r - P_1) + (Q_r Z_c - \frac{Z_1 Q_1}{\gamma})\right] e^{\gamma x} \tag{49}$$

$$P^- = P_r^- e^{-\gamma x} = \tfrac{1}{2}\left[(P_r - P_1) - Q_r Z_c + \frac{Z_1 Q_1}{\gamma}\right] e^{-\gamma x} \tag{50}$$

$$Q^+ = Q_r^+ \; e^{\gamma x} = \tfrac{1}{2}\left[\frac{(P_r - P_1)}{Z_c} + Q_r - \frac{Z_1 Q_1}{Z}\right] e^{\gamma x} \tag{51}$$

$$Q^- = Q_r^- \; e^{-\gamma x} = -\tfrac{1}{2}\left[\frac{P_r - P_1}{Z_c} - (Q_r - \frac{Z_1 Q_1}{Z})\right] e^{-\gamma x} \tag{52}$$

Then

$$P_x - P_1 = P^+ + P^- = P_r^+ \; e^{\gamma x} + P_r^- \; e^{-\gamma x} \tag{53}$$

$$Q_x - Q_1 = Q^+ + Q^- = Q_r^- \; e^{\gamma x} + Q_r^- \; e^{-\gamma x} \tag{54}$$

It is now clear that P_r^+ and P_r^- are components of the receiving end pressures and likewise Q_r^+ amd Q_r^- are components of the receiving end flow. It is now necessary to examine in detail the nature of the exponential functions occurring in the above equations. As defined by eq. (38)

$$\gamma = \sqrt{ZY} = \sqrt{(R + jwL)(jwC)}$$

Since both Z and Y are complex $\sqrt{ZY}$ will be complex also, and may be expressed as: $\gamma = \alpha + j\beta$ where α and β are constants. (55)

$$\text{then} \quad e^{\gamma x} = e^{\alpha x} \, e^{j\beta x} \tag{56}$$

$$e^{-\gamma x} = e^{-\alpha x} \, e^{-j\beta x} \tag{57}$$

Now consider $e^{\alpha x}$. As x increases the product αx increases and $e^{\alpha x}$ increases, therefore miltiplication of P_r^+ by $e^{\alpha x}$ changes the magnitude only.

The second factor $e^{j\beta x} = \mathrm{Cos}\beta x + j\,\mathrm{Sin}\beta x = \bar{1} \; \underline{/\beta x}$ always has a constant magnitude but has a phase angle which is directly proportional to x. Therefore multiplication of P_r^+ by $e^{j\beta x}$ changes the phase of P_r^+ only and multiplication by $e^{\gamma x}$ changes both phase and magnitude.

As we go away from the receiving end (or toward the sending end), in each unit of distance the pressure $P_r^{+\,\gamma x}$ is increased in magnitude by the constant factor e^{α} and is advanced in phase by the constant increment β radians. Since a traveling wave is characterized by a retardation of phase the first term of eq. (53) $P^+ = P_r^+ e^{\gamma x}$ is a wave traveling from the pump toward the receiving end. Since $e^{-\gamma x}$ is the reciprocal of $e^{\gamma x}$ it decreases the phase from the receiving end to the source and $P^- = P_r^- e^{-\gamma x}$ is a wave traveling toward the source.

It should now be evident that in eq. (41) P_x is the sum of two waves, the incident wave and the reflected wave, where the incident wave is traveling away from the source and the reflected wave is reflected from the receiving end back toward the source.

The above results are similar to those found by Allievi, for he found the surge pressure was the sum of two traveling waves, thus

$$P_x = F_1(x + at) + F_2(x - at) + P_1 \tag{58a}$$

$$Q_x = F_1(x + at) - F_2(x - at) + Q_1 \tag{58b}$$

The change in phase of a traveling wave in unit distance is β radians; in distance x, it is βx radians. The wave length λ is defined as the distance in which the phase changes by a whole revolution, or by 2π radians. Hence

$$\beta\lambda = 2\pi \text{ radians} \tag{59}$$

or

$$\lambda = \frac{2\pi}{\beta} \quad \text{ft. per cycle} \tag{60}$$

$$\beta = \frac{2\pi}{\lambda} \quad \text{radians per ft.} \tag{61}$$

Since

$$a = \lambda c, \tag{62}$$

where "a" is the velocity of propagation in ft. per sec. and c is the frequency in cycles per second,

then

$$a = \frac{2\pi}{\beta} c = \frac{w}{\beta} \tag{63}$$

In eqs. (41) and (43) there are four traveling waves; two pressure waves and two flow waves or two forward and two backward waves. The ratio of the forward pressure to the forward flow is

$$\frac{P^+}{Q^+} = \frac{(P_r - P_1) + (Q_r Z_c - \frac{Z_1 Q_1}{\gamma}) Z_c}{P_r - P_1 + Z_c (Q_r - \frac{Z_1 Q_1}{Z})} = Z_c \tag{64}$$

The ratio of the backward pressure to the backward flow is

$$\frac{P^-}{Q^-} = -Z_c \tag{65}$$

Equations (64) and (65) show that the ratio of forward pressure to forward flow is Z_c and is independent of the terminal conditions. This value is the characteristic impedance of the pipeline and its value is, by eq. (37)

$$Z_c = \sqrt{\frac{Z}{Y}} = \sqrt{\frac{R + jwL}{jwC}} \quad \frac{\# - \text{Sec}}{\text{ft}^5} \tag{37}$$

If $R = 0$ or is small in comparison to wL, then Z_c is a constant and real. $Z = jwL = Z_1$ and $\frac{Z_1}{\gamma} = \frac{Z}{\sqrt{ZY}} = Z_c$

Define Γ as the reflection coefficient. It is the ratio of the reflected wave to the incident wave.

Defining $Z_x = \frac{P_x - P_1}{Q_x - Q_1}$ (66)

therefore $(P_r - P_1) = Z_r\ (Q_r - Q_1)$ and substituting these values into eqs. (49) - (52) and taking the ratio of reflected to incident waves there results

$$\Gamma = \frac{P^-}{P^+} = \frac{Z_r - Z_c}{Z_r + Z_c} \tag{67}$$

In a like manner by using Q^+, Q^-

$$\Gamma = \frac{Q^-}{Q^+} = -\frac{Z_r - Z_c}{Z_c \quad Z_r} \tag{68}$$

When $Z_r = 0$, the pressure surge is totally reflected with reversal of sign ($\Gamma = -1$). The flow is totally reflected, but no reversal of sign occurs ($\Gamma = 1$).

When $Z_r = \infty$ the pressure surge is reflected with no reversal of sign, but the flow will have a reversal of sign.

When $Z_r = Z_c$, $\Gamma = 0$, and the incident wave is totally absorbed by the end conditions. This is the condition which those interested in supression of surges strive to obtain in their equipment. If this match is not present there is a great possibility of progressively increasing pressures. The ratio of Z_r to Z_c is an important parameter in the analysis of the pipeline.

P_1 and Q_1 are the vertical displacements of the pressure and flow waves from the wt axis.

From eq. (66) the sending end impedance is defined as

$$Z_s = \frac{P_s - P_1}{Q_s - Q_1} \tag{69}$$

When $Z_c = Z_r$ (no reflected wave) from eqs. (47) and (48) and R = 0,

then $$Z_c = Z_r = Z_c \frac{(P_s - P_1)\operatorname{Cosh}\gamma x_1 - Z_c(Q_s - Q_1)\operatorname{Sinh}\gamma x_1}{Z_c(Q_s - Q_1)\operatorname{Cosh}\gamma x_1 - (P_s - P_1)\operatorname{Sinh}\gamma x_1} \tag{70}$$

Therefore by rearranging and solving for Z_s the following relation is obtained:

$$Z_s = Z_c \tag{71}$$

The significance of eq. (71) is that if there is no surge the sending end impedance is also matched with the characteristic impedance.

Eq. (70) may be rearranged and in terms of impedances only

$$\frac{P_r - P_1}{Q_r - Q_1} = Z_r = Z_c \frac{Z_s \operatorname{Cosh}\gamma x_1 - Z_x \operatorname{Sinh}\gamma x_1}{Z_c \operatorname{Cosh}\gamma x_1 - Z_s \operatorname{Sinh}\gamma x_1} \tag{72}$$

and from eqs. (45) and (46) and R = 0

$$\frac{P_s - P_1}{Q_s - Q_1} = Z_s = Z_c \frac{Z_r \operatorname{Cosh}\gamma x_1 + Z_c \operatorname{Sinh}\gamma x_1}{Z_c \operatorname{Cosh}\gamma x_1 + Z_r \operatorname{Sinh}\gamma x_1} \tag{73}$$

4. Explanation of Assumed and Derived Parameters.

a. Flow Resistance.

The resistance to flow has been assumed to vary linearly with the flow. This is rute only when the flow is laminar, other cases will be
right

investigated later in this paper.

In eq. (1) for laminar flow R has the value shown below when $f = \frac{64}{N_r}$,

$N_r = \frac{Dv\rho}{u}$, $Q_r = f\frac{\rho v^2}{2\,D} = \frac{128\ u\ Q}{\pi\ D^4}$ and

$$R = \frac{128}{\pi}\ \frac{u}{D^4}\ , \quad \frac{\text{lb - Sec}}{\text{ft}^5} \text{ per ft. of length} \tag{74}$$

Equation (74) gives the value of R which is a property of the fluid and the pipe size.

b. Flow Inertance.

The inertance per unit length is given by

$$wL = \frac{w\rho}{A} \quad \frac{\text{lb. Sec}}{\text{ft}^5} \text{ per ft. of length} \tag{75}$$

The inertance is dependent on the speed of the pump, the fluid flowing, and the pipe.

c. Capacitance.

The capacitance per unit length is given by

$$wC = wA\ \left(\frac{1}{K} + \frac{1}{E}\ \frac{D}{e} \right) \frac{\text{ft}^5}{\text{lb - Sec}} \text{ per unit length} \tag{76}$$

and it like the inertance, is dependent on the speed, fluid, and pipe.

d. Characteristic Impedance.

The characteristic impedance is derived as

$$Z_c = \sqrt{\frac{R + jwL}{jwC}} \qquad \frac{\text{lb - Sec}}{\text{ft}^5} \tag{77}$$

and is independent of the length of the pipeline. By rearranging the numerator of eq. (77) the ratio $\frac{R}{wL}$ will be used for a comparison with unity (coefficient of j).

$$j + \frac{R}{wL} \tag{78}$$

e. Propagation Constant.

The propagation constant γ is derived as $\gamma = \sqrt{(R + jwL)(jwC)}$ (79)

By rearranging the first term of eq. (79) the relation (78) is obtained.

f. Comparison of different fluids at a specified temperature (70° F) and small diameter pipe (4") gives the following results.

TABLE I

Fluid	Sp. Gr.	Viscosity	Density	R	wL	$\frac{R}{wL}$	% Diff.
Water	1	2.04×10^{-5}	1.93	0.067	222	0.0003	0.03
Crude oil	0.86	2.4×10^{-4}	1.4	0.46	183	0.0025	0.25
Crude oil	0.93	1×10^{-3}	1.8	3.3	206	0.016	1.6

For larger diameter pipes or for a higher speed the frictional resistance will have an even smaller effect on either Z_c or γ.

For an increasing viscosity (decreasing temperature) the frictional resistance will become more noticeable, but the value may be found by using eq. (74).

g. If R is small the following relations are obtained

$$Z_c = \sqrt{\frac{jwL}{jwC}} = \sqrt{\frac{L}{C}} \tag{80}$$

$$\gamma = \sqrt{(jwL)\ (j\ w\ C)} = jw\sqrt{LC} \tag{81}$$

Where Z_c is real and γ is pure imaginary and since $\gamma = \alpha + j\beta$, then $\alpha = 0$ and

$$\beta = w\sqrt{LC} \tag{82}$$

and from eq. (63)

$$a = \frac{w}{\beta} = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\rho\left(\frac{1}{K} + \frac{1}{E}\frac{D}{e}\right)}} \tag{83}$$

and $$\lambda = \frac{2\pi}{\beta} = \frac{2\pi a}{w} \tag{60}$$

The last relation in (83) is the commonly accepted value of the velocity of propagation of the wave when $R = 0$.

If eqs. (80) and (81) hold, then eqs. (42) and (44) become

$$P_x = (P_r - P_1) \text{ Cosh } j\beta x + Z_c \ (Q_r - Q_1) \text{ Sinh } j\beta x + P_1$$

and

$$P_x{}^{9} = (P_r - P_1) \text{ Cos}\beta x + jZ_c \ (Q_r - Q_1) \text{ Sin}\beta x + P_1 \tag{84}$$

$$Q_x{}^{10} = (Q_r - Q_1) \text{ Cos}\beta x + j \ (\frac{P_r - P_1}{Z_c}) \text{ Sin}\beta x + Q_1 \tag{85}$$

Eqs. (84) and (85) give a more usable function to calculate P_x and Q_x.

h. For water in a 4 in. pipe with the value of $L = 22.2$, $C = 2.58 \times 10^{-9}$, and $w = 10$ rad. per sec.,

then $\beta = w\sqrt{LC} = 2.4 \times 10^{-3}$ rad. per ft. of length

and "a" $= \frac{w}{\beta} = \frac{1}{\sqrt{LC}} = 4150$ ft/ sec. The wave length $\lambda = \frac{2\pi}{\beta} = 2620$ ft.

i. Terminal Impedance.

From examination of eqs. (17) and (18) it is easy to see that $P_s - P_1 = P_2 \text{ Cos } (2wt + 180 - \phi)$ and $Q_s - Q_1 = Q \text{ Cos } (2wt + 180)$.

If this is true then,

$$Z_s = (\frac{P_2}{Q_2}) \ \frac{\text{Cos } (2wt + 180 - \phi)}{\text{Cos } (2wt + 180)} \tag{86}$$

If there is no lag between the motion of the piston and the pressure rise ($\phi = 0$), Z_s is the ratio of the amplitudes of the pressure and flow curves. This may be in either of two forms.

$$Z_s = \frac{\pi - 2}{\pi} P_m \Big/ \frac{\pi - 2}{\pi} Q_m = \frac{P_m}{Q_m} \tag{87}$$

Z_s may be calculated from eq. (87) if the amplitude of the pressure and flow

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[9] Cosh $j\beta x = \text{Cos}\beta x$ and Sinh $j\beta x = j \text{ Sin}\beta x$

[10] *Ibid.*

curves are known. The pressure wave may be found by use of an oscillcgraph. The flow amplitude may be calculated from the original equation of motion of the piston, $Q = (Arw) \text{ Sin } wt = Q_m \text{ Sin } wt$

If the phase angle $\emptyset$ is not zero the value of Z_s is calculated from eq. (86).

In any case Z_r may be found from eq. (72).

5. Effects of Turbulent Flow.

The possibility of the assumption in eq. (1) that the pressure gradient varies linearily with the flow is in error when the Reynolds number is between 2000 and 10^6. Since the pressure gradient varies with $Q^{1.75}$ eq. (1) is not readily solved. For the purpose of obtaining a reasonable workable solution, assume that the pressure varies with Q^2. The eqs. (26) and (27) become:

$$\frac{\partial P}{\partial x} = RQ^2 + jwLQ - jwLQ_1 \qquad (88)$$

$$\frac{\partial P}{\partial x} = jwCP - jwCP_1 \qquad (89)$$

where R in eq. (88) does not have the same value as before.

In a manner similar to the solution of eqs. (26) and (27) taking the partial of eq. (88) with respect to x gives,

$\frac{\partial^2 P}{\partial x^2} = (2RQ + jwL) \frac{\partial Q}{\partial x}$ and by substituting into eq. (89) obtain

$$\frac{\partial^2 P}{\partial x^2} = jwC\ (2RQ + jwL)\ (P - P_1) \qquad (90)$$

For simplicity let $\gamma = jwC\ (2RQ = jwL)$ and the solution of eq. (90) is,

$$P = f_1(Q)\ e^{\gamma x} + f_2(Q)\ e^{-\gamma x} + P_1 \qquad (91)$$

Now take the partial of eq. (91) with respect to x, substitute that value into eq. (88) and obtain eq. (92) as shown below:

$$\frac{\partial P}{\partial x} = f_1(Q)\gamma e^{\gamma x} - f_2(Q)\gamma e^{-\gamma x} \qquad \text{and}$$

$$1/\gamma \left[RQ^2 + jwL\,(Q - Q_1)\right] = f_1\,(Q)\,e^{\gamma x} - f_2\,(Q)\,e^{-\gamma x} \tag{92}$$

Let $g(Q) = 1/\gamma \left[RQ^2 + jwL\,(Q - Q_1)\right]$ and eq. (92) reduces to

$$g(Q) = f_1(Q)\,e^{\gamma x} - f_2(Q)\,e^{-\gamma x} \tag{93}$$

Next consider a method of finding the values of $f_1(Q)$ and $f_2(Q)$ by use of the following boundary conditions; when $x = 0$, $P = P_r$, $Q = Q_r$, and $g(Q) = g(Q_r)$. By substituting these values into eqs. (91) and (92) and solving for $f_1(Q)$ and $f_2(Q)$ eq. (94) is obtained.

$$f_1(Q) = \tfrac{1}{2}\left[(P_r - P_1) + g(Q_r)\right] \quad \text{and} \quad f_2(Q) = \tfrac{1}{2}\left[(P_r - P_1) - g(Q_r)\right] \tag{94}$$

By substituting the above values into eqs. (91) and (93) the following results are obtained:

$$P - P_1 = \tfrac{1}{2}\left[(P_r - P_1) + g(Q_r)\right] e^{\gamma x} + \tfrac{1}{2}\left[(P_r - P_1) - g(Q_r)\right] e^{-\gamma x} \tag{95a}$$

$$g(Q) = \tfrac{1}{2}\left[(P_r - P_1) + g(Q_r)\right] e^{\gamma x} - \tfrac{1}{2}\left[(P_r - P_1) - g(Q_r)\right] e^{-\gamma x} \tag{95b}$$

and rearranging in terms of the hyperbolic functions

$$P - P_1 = (P_r - P_1)\,\mathrm{Cosh}\,\gamma x + g(Q_r)\,\mathrm{Sinh}\,\gamma x \tag{96a}$$

$$g(Q) = g(Q_r)\,\mathrm{Cosh}\,\gamma x + (P_r - P_1)\,\mathrm{Sinh}\,\gamma x \tag{96b}$$

The above equations are similar to those obtained before.

The constants L and C have the same values as given in eqs. (75) and (76), but R has a new value, it is $R = \frac{f\rho}{2\,D\,A^2}$ (97)

By rearranging the terms of $g(Q)$ and γ a comparison of the values of the ratios $\frac{QR}{wL}$ and $\frac{2QR}{w\,L}$ with unity will be shown to have little effect in the determination of P and Q.

From Table II for increasing Reynolds' numbers and decreasing f the product QR increases slowly and the ratio $\frac{QR}{wL}$ remains small in relation to j until a large value of N_r is reached. In this table w is considered constant, but in reality it would increase as the flow Q increased resulting in a further reduction of $\frac{QR}{wL}$.

TABLE II

N_r	f	Q	R	QR	QR/wL
5000	0.04	0.0475	0.049	0.0023	0.000093
8000	0.035	0.076	0.043	0.0032	0.000031
19000	0.03	0.18	0.037	0.0067	0.00027
50000	0.025	0.475	0.031	0.015	0.00059
120000	0.02	1.14	0.025	0.028	0.0011
10^6	0.015	9.5	0.019	0.176	0.0071
4×10^6	0.0125	38.0	0.015	0.585	0.0236
10^7	0.0115	95.0	0.014	1.35	0.055

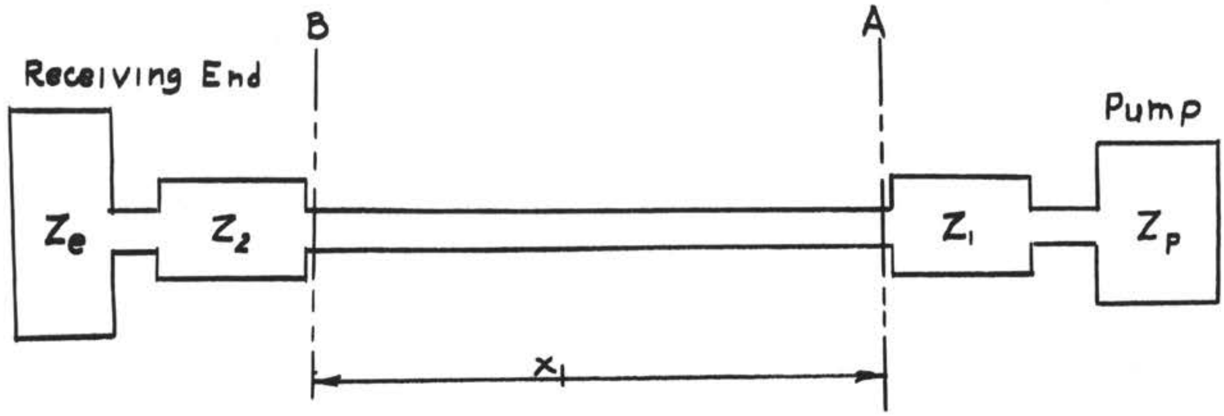
By neglecting the product QR or assuming that it does not affect the problem, eqs. (95) and (96) reduce to eqs. (84) and (85). The elimination of the product QR is not advisable for all fluids until a thorough investigation of the fluid, pipe, and pump has been carried out.

It is thought that the best procedure to follow is to start with eq. (5) and decide which factors have the most influence, then an analysis of the particular installation can be made with the use of the preceeding derived relations.

6. Impedance Matching.

It has been shown by eqs. (64) and (67) that the characteristic impedance is independent of end conditions and where there is no reflection, ($\Gamma_r = 0$) $Z_c = Z_r$. Since it is desirous to eliminate the pressure surge some device should be placed at the end of the pipe with an impedance which will match that of the characteristic impedance.

Figure 5



That is $Z_c = Z_2 \longrightarrow Z_e$ Fig. 5.

In a like manner the impedance of the pump must be matched with Z_1 to give Z_c. Such that $Z_c = Z_p \longrightarrow Z_1$

If that is done, at section A looking toward the pump, the impedance is Z_c. At section B looking toward the end, the impedance is Z_c and, of course, at any point between A and B the impedance is Z_c.

CHAPTER V

INTERPRETATION OF RESULTS AND RECOMMENDATIONS

The frictional resistance should be investigated for each individual installation, especially where the fluid viscosity is high or the temperature is low. The results of Tables I and II are by no means a thorough investigation of the frictional influence, nor were they intended to be conclusive. The purpose of Tables I and II are indications of the effects of resistance in that range.

Using the reasoning in Chapter IV, Section 6 it seems logical that an impedance match of the pump and of any particular surge removal device may be found if a valve was placed at section "A" in Fig. 5. This could be accomplished by closing the valve and varying either the speed of the pump or pressure in the surge suppressor until the fluctuations of pressure were removed. The results of this test would help to predict the actions of the surge suppressor when installed in any given system at either end of the pipe.

It is the author's belief that if the values of impedances of the system are evaluated then a system of electrical impedances can be constructed which will help predict the results of any particular pipeline system. The electrical network is recommended for two reasons. The cost of building an electrical network is much smaller than if an actual pipeline were used, and the analysis of electrical networks has been developed to a high degree of accuracy.

Considerable theoretical work should be done on determining methods of building the required impedance in surge suppression devices.

Further work should be done to determine the character of matching impedances as the wave length varies or as the line terminates on fractional wave lengths. It is known that in electrical transmission line theory that the matching impedance changes from a capacitance to an inertiance each quarter wave length.

Theoretical work should be verified in the laboratory and later the laboratory results should be verified in actual field systems.

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