TRANSIENTS IN ELECTRICAL FILTERS

By

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Diploma from the

Federal Institute of Technology

Zurich, Switzerland

1950

Submitted to the Faculty of the Graduate School of

the Oklahoma Agricultural and Mechanical College

in Partial Fulfillment of the Requirements

for the Degree of

MASTER OF SCIENCE

1951

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PREFACE

The purpose of the present paper was to investigate the influence discontinuous input voltages have on the response of the various basic filters. Although the literature on the steady-state response of these filters is very numerous, scarcely any information could be obtained concerning the transient response. The attempt was, therefore, made to compute the transient responses of the basic filters taking into account losses in the circuit elements. The results have been obtained by application of the theory of Fourier transformation and Heaviside's theorem.

The author would like to express his thanks to his adviser, David L. Johnson, for his many valuable ideas and suggestions and especially the careful reading of the manuscript and for his advice in questions concerning the language.

Special thanks are due to Mrs. Fern B. Hall for the typing of this thesis with its many formulas.

TABLE OF CONTENTS

	Page
PREFACE	iii
LIST OF ILLUSTRATIONS	vi
Chapter	
I. THE IDEAL BAND-PASS FILTER	1
II. TRANSIENTS IN T AND II SECTIONS.	12
1. Basic Theory of Four-Terminal Networks	12
2. The T-Type Filter.	13
(a). Network equations for the T-section.	13
(b). Wave parameters of a T-section	15
(c). The characteristic impedance of the T-section.	17
(d). The matching problem for basic filters	18
(e). The propagation constant g.	19
(f). Transient response of T-section.	21
(g). Transients in conventional filters	24
Low-pass filter.	24
High-pass filter	27
Band-pass filter	31
Band-elimination filter.	35
3. The II-Type Filter	37
(a). Network equations.	37
(b). The wave parameters.	38
(c). Transient response of II-section	39
III. THE INFLUENCE OF LOSS RESISTANCE	42
IV. APPLICATION OF THE THEORY AND CONCLUSIONS.	45
1. The Transient Response of a Low-Pass Filter.	45

Chapter	Page
(a). The ideal low-pass filter.	45
(b). The physical low-pass filter without losses. . .	45
(c). The low-pass filter with losses.	48
2. The Transient Response of a High-Pass Filter	48
(a). The ideal high-pass filter	48
(b). The physical high-pass filter without losses . .	49
(c). The high-pass filter with losses	49
3. The Transient Response of the Band-Pass Filter	51
(a). The ideal band-pass filter	51
(b). The physical band-pass filter.	51
4. Conclusions.	54
BIBLIOGRAPHY.	56

LIST OF ILLUSTRATIONS

Figure		Page
1.	Amplitude and phase characteristic of an ideal network . . .	3
2.	Path of integration.	4
3.	Path of integration for the first term	6
4.	Four-terminal network.	13
5.	T-section.	14
6.	Characteristic impedance of a T-section.	18
7.	Phase and attenuation factors of a T-section	20
8.	Conventional low-pass filter	24
9.	Response of the low-pass filter to a unit step	27
10.	Conventional high-pass filter.	27
11.	Response of a high-pass filter to a unit step.	31
12.	Conventional band-pass filter.	31
13.	Band elimination filter.	35
14.	II-section	37
15.	Characteristic impedance of a II-section	38
16.	Reduced II-section	40
17.	Transient response of the ideal low-pass filter.	47
18.	Transient response of the physical low-pass filter	47
19.	Transient response of the ideal high-pass filter	50
20.	Transient response of the physical high-pass filter. . . .	50
21.	Transient response of the ideal band-pass filter	53
22.	Transient response of the physical band-pass filter. . . .	53
23.	Transient response of the ideal band-elimination filter. . .	53

Chapter I

The Ideal Band-Pass Filter

The subject of this chapter is to compute the transient that will be expected at the output of a so called idealized filter; i.e., a filter which shows neither amplitude nor phase distortion for signals which are composed of frequencies lying inside the pass band of the filter. In spite of the fact that this filter is able to transfer any signal of this nature without distortion (this restriction demands that the signal be periodic) there will be a transient at the switching instant, for the theory of Fourier transformation shows that every switching process inserts a frequency band of infinite broadness. Because the purpose of the filter is to be frequency discriminative one would expect a transient at the output of the filter at the switching instant.

Two possible methods of analysis of the problem are Fourier transformation and Laplace transformation. Of these the theory of Fourier analysis is more convenient. The more general theory of Laplace transformation would have to be specialized in such a way that there would be only a formal difference in the definitions of these transformations.

Fourier transformation $$f(j\omega) = \frac{1}{2\pi}\int_0^\infty f(t)e^{-j\omega t}\,dt$$

Laplace transformation $$f(s) = \int_0^\infty f(t)e^{-st}dt$$

The Fourier transformation is much more illustrative for, instead of the complex factor s which has no physical significance, the parameter ω in the Fourier transformation has the significance of a frequency, which is a very familiar concept to engineers and physicists.

The first question that arises now is, what are the properties of an electrical network that shows no distortion at all; i.e., a network that reproduces any input signal with only a change in amplitude even though the signal may be composed of frequencies inside a frequency band extending from zero to infinity (including discontinuities). According to the Fourier transformation theory any input signal of this nature may be written in the following way:

$$f(t) = \int_{-\infty}^{\infty} f(j\omega)e^{j\omega t}d\omega \qquad 1.1$$

The action of the filtering process is mathematically expressed as a multiplication of the frequency response and a term expressing the frequency dependence of the transfer through the filter. Frequency response always includes both amplitude and phase response.

$F_2(j\omega) = F_1(j\omega)\ A(j\omega)$. $A(j\omega)$ may be complex and may be written in the form

$$A(j\omega) = |A(\omega)|\ e^{j\varphi(\omega)}$$

or $\quad A(j\omega) = X(\omega) + jY(\omega),$

where $\quad A^2 = X^2 + Y^2 \quad$ and $\varphi = \tan^{-1}(\frac{Y}{X}).$

Equation 1.1 changes now into

$$f_2(t) = \int_{-\infty}^{+\infty} F_1(j\omega)|(A)|e^{j\varphi(\omega)}\ e^{j\omega t}\ d\omega \qquad 1.2$$

If the signal is allowed to have a time delay and an amplitude distortion at the output, the form of the output has to be:

$$f_2(t) = K\, f_1(t-t_o) = K\int_{-\infty}^{+\infty} F_1(j\omega)\; e^{j\omega(t-t_o)}\, d\omega$$

$$= \int_{-\infty}^{+\infty} F_1(j\omega)\; K\, e^{-j\omega t_o}\; e^{j\omega t}\; d\omega \qquad 1.3$$

A comparison of equation 1.2 and 1.3 gives:

$$A = K \qquad\qquad \varphi(\omega) = -\omega t_o$$

The amplitude characteristic of the filter has to be a constant, and the phase characteristic has to show a phase delay proportional to frequency, as shown in Figure 1.

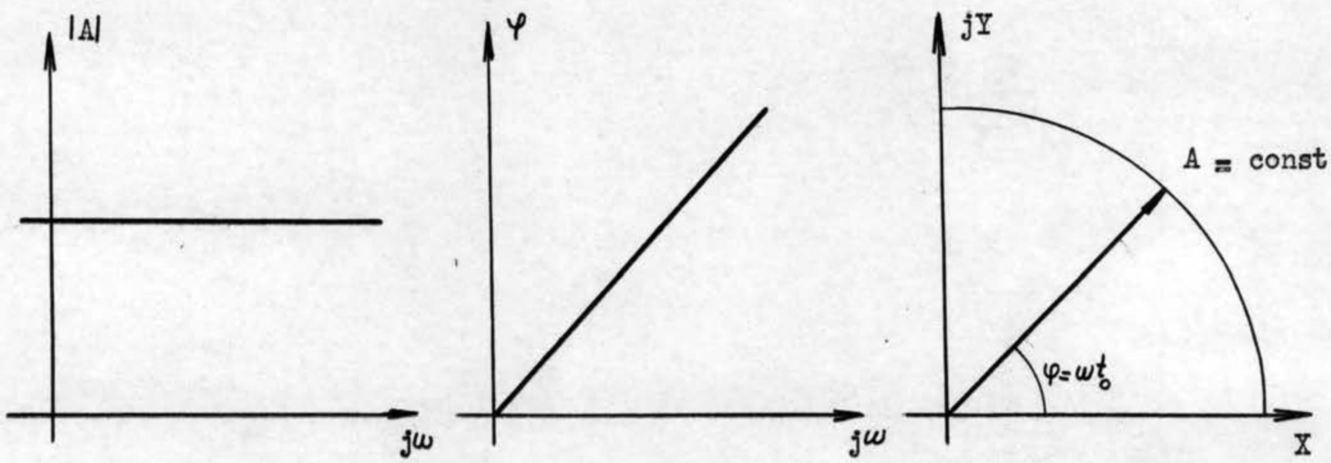


Fig. 1. Amplitude and phase characteristic of an ideal network.

Another property of the ideal network is found very easily. If an arbitrary phase angle ω_o is added to the phase characteristic, the influence on the output signal consists only of a change in time delay. This can be proved in the following way:

$$A(j\omega) = K\, e^{-j(\omega t + \varphi_o)} = K\, e^{-j\omega(t + \frac{\varphi_o}{\omega})}$$

It is therefore not essential how the phase characteristic is assumed. It has only to be a straight line and its position in the $\varphi - \omega$ plane can be arbitrary. As a next step the constant-amplitude characteristic shall be narrowed to a frequency band of the broadness $\omega_2 - \omega_1$. There shall be a unit step inserted at the input of this network. The frequency distribution of the unit step is computed in the following way:

$$S(j\omega) = \frac{1}{2\pi}\int_0^{\infty} f(t)\, e^{-j\omega t}\, dt \qquad f(t) = U(t)\, E$$

$$= \frac{1}{2\pi}\int_0^{\infty} E e^{j\omega t}\, dt = \frac{1}{2\pi j\omega} E$$

This voltage inserted on the ideal filter gives the frequency distribution of the output:

$$S(j\omega) = \frac{EK}{2\pi j\omega}\, e^{-j\omega t_o}$$

In the exponent appears a negative sign in order to express a time delay. The reverse transformation of this expression is

$$f(t) = \frac{EK}{2\pi j}\int_{-\infty}^{+\infty} \frac{e^{-j\omega t_o}\, e^{j\omega t}}{\omega}\, d\omega$$

In order to consider that the band filter only transfers frequencies from ω_1 to ω_2 only a part of the whole path of integration has to be taken. Because the integration has to be done from $-\infty$ to $+\infty$ the frequencies $-\omega_1$ and $-\omega_2$ must also be considered. The path of integration consists therefore of the following two pieces: (Figure 2)

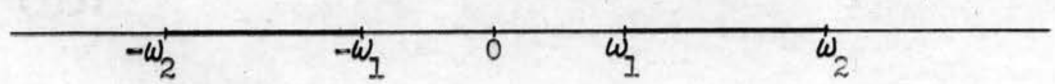


Fig. 2. Path of integration.

In all integrals of this form it must be noted whether there are poles of the integrand on the path of integration. Although the integrand is zero (for $A(j\omega) = 0$) at the point $\omega = 0$, this point is critical and determines in a way the behaviour of the transferred signal. The path of integration must therefore be interpreted in another way. It shall be split up in the following way.

First path $\quad -\omega_2 < \omega < \omega_2$

Second path $\quad -\omega_1 < \omega < \omega_1$

A subtraction of these paths gives the same path as was found earlier. The transformation integral changes now to the following two integrals:

$$f(t) = \frac{EK}{2\pi j}\int_{-\omega_2}^{-\omega_1} \frac{e^{j\omega(t-t_0)}}{\omega}\,d\omega + \frac{EK}{2\pi j}\int_{\omega_1}^{\omega_2} \frac{e^{j\omega(t-t_0)}}{\omega}\,d\omega$$

This may be written in the following way:

$$f(t) = \frac{EK}{2\pi j}\left[\int_{-\omega_2}^{+\omega_2} \frac{e^{j\omega(t-t_0)}}{\omega}\,d\omega - \int_{-\omega_1}^{+\omega_1} \frac{e^{j\omega(t-t_0)}}{\omega}\,d\omega\right] \qquad 1.4$$

Both of these integrals are of the same kind:

$$\int_{-\alpha}^{\alpha} \frac{e^{j\omega(t-t_0)}}{\omega}\,d\omega$$

With a transformation of variables this is transformed to

$$\omega(t-t_0) = u \qquad d\omega = \frac{du}{t-t_0}$$

and with $\quad \alpha(t-t_0) = b$

$$\int_{-b}^{b} \frac{e^{ju}}{u}\,du$$

This may be written as

$$\int_{-b}^{+b} \frac{\cos u}{u}\, du + j \int_{-b}^{+b} \frac{\sin u}{u}\, du \qquad 1.5$$

This integral can be split up and each term treated individually. The first term shows a pole at $u = 0$. It is therefore finite only when the point zero is excluded; i.e., when the path of integration changes in the following way: (Fig. 3)

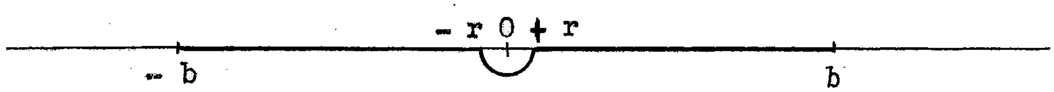
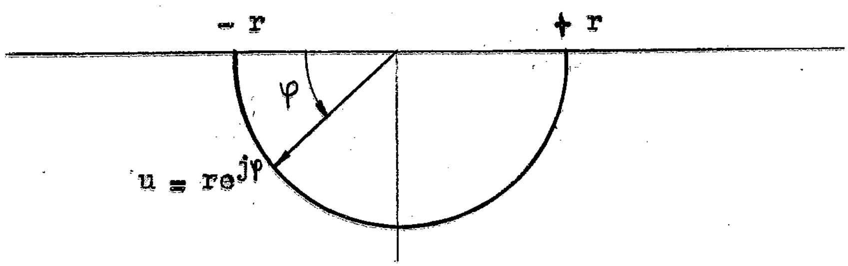


Fig. 3. Path of integration for the first term.

The way excluding zero shall be assumed to be a semicircle, but it could be another path, as can be proved with the aid of the theory of residues. Because the integrand is an odd function, the two integrations from $-b$ to $-r$ and r to b cancel each other. The integral changes therefore to

$$\int_{-b}^{+b} \frac{\cos u}{u}\, du = \lim_{r=0} \int_{\smile} \frac{\cos u}{u}\, du \qquad \begin{aligned} u &= r\, e^{j\varphi} \\ du &= jre^{j\varphi} d\varphi \\ \lim_{r=0}\ \cos re^{j\varphi} &= 1 \end{aligned}$$

Whereby the path of integration consists only of the following curve:



The integral, therefore, transforms into:

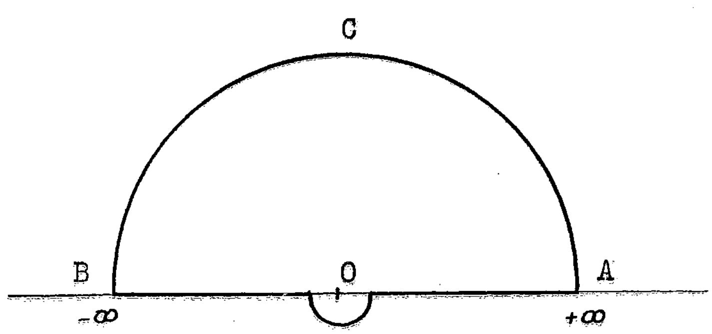
$$\int_{-b}^{+b} \frac{\cos u}{u}\, du = j \int_{0}^{\pi} \frac{re^{j\varphi}}{re^{j\varphi}}\, d\varphi = j \int_{0}^{\pi} d\varphi = j\pi$$

Principally there exists another approach to the problem. The function of complex variables gives the following theorem, called Cauchy's theorem:

Given a function which is analytic on a closed path. The so-called residues of this function are defined as follows:

$$R = \frac{1}{2\pi j} \oint \frac{f(z)dz}{z - z_o}$$ where z_o is the value of z at a point where $f(z)$ has a pole. The value of this residue may be found as the coefficient of the first negative power of the power series of $f(z)$ developed in the neighborhood of $z = z_o$.

The theorem of Cauchy states now that the integral once around the closed contour is equal to the sum of the residues inside this path of integration. The idea of the calculus of residues is to change the path of integration along the real axis to the path shown in the figure below.



For most practical applications it can be proved that the integral along the path ACB tends to zero, as the radius of the half circle approaches infinity. This path, therefore, contributes nothing to the total integral. According to Cauchy's theorem the integral from A to B must, therefore, be equal to the sum of the residues on the left half plane. Unfortunately Cauchy's theorem cannot be applied to this problem, for a prerequisite for the application of Cauchy's theorem is that the function, the residue of which shall be evaluated, be analytic on the boundary. This is not the case for the present problem, because there are two places on the u axis where the integrand shows a discontinuity; namely, at the frequency limits of the ideal filter.

The second term in equation 1.5 is:

$$\int_{-b}^{+b} \frac{\sin u}{u} \, du$$

This integral has no pole at $u = 0$ since $\lim_{u = o} \frac{\sin u}{u} = 1$. This may be proved with the aid of Bernoulli-Hopital's theorem, which states that

$$\lim_{x = o} \frac{f(x)}{g(x)} = \lim_{x = o} \frac{f'(x)}{g'(x)}$$

In this case: $$\lim_{u = o} \frac{\sin u}{u} = \lim_{u = o} \frac{\cos u}{1} = 1$$

The integrand is an even function and, therefore, the integral may be written:

$$\int_{-u}^{u} \frac{\sin u}{u} \, du = 2 \int_{o}^{u} \frac{\sin u}{u} \, du$$

This integral is known as the "sine-integral"

$$\int_{o}^{x} \frac{\sin x}{x} \, dx = Si(x)$$

All these integrals inserted into equation 1.4 give:

$$f(t) = \frac{EK}{\pi} \left[Si_2(t - t_o) - Si_1(t - t_o) \right]$$

The slopes of the first term and the second term are

$$\frac{E\omega_1}{\pi} \text{ and } \frac{E\omega_2}{\pi}$$

As readily seen, the higher the frequency is, the steeper is the slope of these terms.

The slope of the combination of the two components is, therefore:

$$\operatorname{tg}\varphi = \frac{E\omega_1}{\pi} - \frac{E\omega_2}{\pi} = \frac{E}{\pi}(\omega_1 - \omega_2) = \frac{E}{\pi}\,\Delta\omega$$

The slope of the curve at the steepest point is only a function of the band width of the filter and is not dependent on the position of the pass band on the frequency axis. In order to illustrate this more theoretical derivation there will be given some special cases in Chapter 4.

The response of a high-pass filter can be found when ω_2 in equation 1.4 approaches infinity. $\mathrm{Si}\,\omega_2(t - t_o)$ then becomes a step function with the amplitude $\frac{\pi}{2}$ for positive t and $-\frac{\pi}{2}$ for negative t. The response of the high-pass filter is, therefore:

$$f(t) = \frac{EK}{\pi}\left[\frac{\pi}{2} - \mathrm{Si}\,\omega_1(t - t_o)\right] \qquad 1.6$$

It must be noted that the computed output of the filter never contains a direct-current component, because the chosen path of integration carefully avoided the point $\omega = 0$. In most applications only the shape of the transient response is interesting. For other cases the function $\frac{EK}{2}\,U(t)$ must be added to the output function if the frequency $\omega = 0$ is transmitted through the filter. In the calculation of the response of the high-pass filter use could be made of the fact that all the frequencies which have been retained by the low-pass filter go through the high-pass filter. The sum of the responses of a high-pass filter and of a low-pass filter with the same limit-frequencies must, therefore, be equal to the undistorted transferred input voltage $EKU(t)$, or in other words:

$$f_2(t)_{HP} = Kf_1(t) - f_2(t)_{LP}$$

$$f_2(t)_{HP} = EK\left[U(t) - \frac{1}{\pi}\,\mathrm{Si}\,\omega_1(t - t_o)\right]$$

$$= \frac{EK}{\pi}\left[\pi - \mathrm{Si}\,\omega_1(t - t_o)\right]$$

This result is equal to equation 1.6 with the exception of the direct current $\frac{EK}{2}$ which was added to it. This way of calculation is valid only for the ideal low-pass and high-pass filters. It may be applied to the ideal band-pass filter and the ideal band-elimination filter. Because the band-elimination filter transfers the frequency $\omega = 0$, a direct current component $\frac{EK}{2}$ must be added to the solution obtained with the aid of the contour integration. In Chapter 4 use will be made of the relation between the band-pass filter and the band-elimination filter. This way of computation is much simpler and gives immediately the correct answer including direct-current components. The only discontinuity that was considered until now was the unit-step function. The response of a network to a unit-step gives the possibility of calculating the response of the same network to any other function with a discontinuity; i.e., the sudden application of a sine- or cosine-function at the input terminals of the network. The relation between these two responses is given in terms of an expression which is analogous to the superposition integral in the theory of Laplace transformations. The derivation for the application of the superposition integral to the Fourier transformation is exactly the same as in the theory of Laplace transformations.
The result is:

$$V_2(t) = \frac{d}{dt}\int_0^t V_1(t)\;V_o(t - \tau)\;d\,\tau$$

$V_o(t)$ is the response of the filter to a step function at the input terminals. The engineer usually contents himself with attempting to make the transient to a step function as small as possible and assumes that a minimal response to a step function results in a minimal response to any discontinuous function.

Chapter II

Transients in T and II Sections

1. Basic Theory of Four-Terminal Networks

The common theory of passive linear four-terminal networks shows that the input voltage and input current are merely linear combinations of output current and output voltage.

$$V_1 = a_{11}V_2 + a_{12}I_2$$
$$I_1 = a_{21}V_2 + a_{22}I_2 \qquad 2.1$$

These so-called chain equations are much more adequate for filter problems, because a connection of two four-terminal networks in cascade results only in a matrix-multiplication of the two chain equations. The a_{ik} are functions of the nature of the four-terminal network. They fulfill the relation

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = 1 \qquad 2.2$$

If the four-terminal network is symmetrical; i.e., it does not change its character at an interchange of input terminals and output terminals, it will be shown that $a_{11} = a_{22}$.

As a proof, solve the equations 2.1 for V_2 and I_2. Taking into account the fact that both V_2 and I_2 have the wrong direction when the four-terminal network is looked at from the output terminals, the result is:

$$V_2 = \begin{vmatrix} V_1 & a_{12} \\ I_1 & a_{22} \end{vmatrix} = a_{22}V_1 - a_{12}I_1$$

$$-I_2 = - \begin{vmatrix} a_{11} & V_1 \\ a_{21} & I_1 \end{vmatrix} = a_{21}V_1 - a_{11}I_1$$

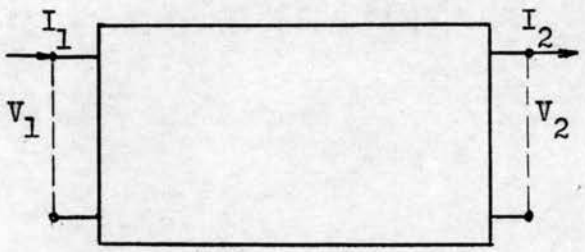


Fig. 4. Four-terminal network.

The minus signs appear because the currents are flowing away from the load and into the voltage source. A change in direction of the two currents changes the equations into the form

$$\begin{aligned} V_2 &= a_{22}V_1 + a_{12}I_1 \\ I_2 &= a_{21}V_1 + a_{11}I_1 \end{aligned} \qquad 2.3$$

A comparison of the equations 2.3 with 2.1 shows that the equation $a_{II} = a_{22}$ must be fulfilled, if the four-terminal network is symmetrical.

2. <u>The T-Type Filter</u>

(a) <u>Network equations for the T-section</u>.

A T-section is shown in Figure 5.

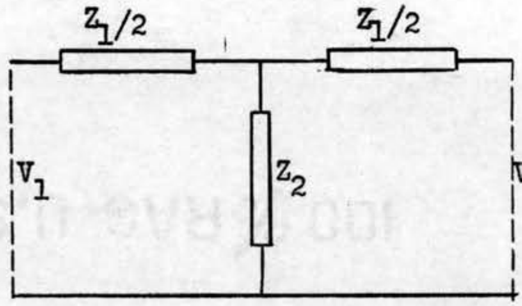


Fig. 5. T-section.

The loop equations for this network are given by

$$V_1 = (\tfrac{1}{2}Z_1 + Z_2)\ I_1 - Z_2 I_2$$
$$V_2 = Z_2 I_2 - (\tfrac{1}{2}Z_1 + Z_2)\ I_2 \qquad 2.4$$

From the second equation the value I_1 can be computed

$$I_1 = \frac{1}{Z_2}\left[\ V_2 + (\tfrac{1}{2}Z_1 + Z_2)\ I_2\right]$$

This value inserted into the first of the equations 2.4 gives

$$V_1 = \frac{1}{Z_2}(\tfrac{1}{2}Z_1 + Z_2)\ V_2 + \frac{1}{Z_2}(\tfrac{1}{2}Z_1 + Z_2)^2\ I_2 - Z_2 I_2$$

$$I_1 = \frac{1}{Z_2}\ V_2 + \frac{1}{Z_2}(\tfrac{1}{2}Z_1 + Z_2)\ I_2$$

From these two equations the network parameters are evaluated in the following way

$$a_{11} = 1 + \frac{Z_1}{2Z_2} \qquad a_{12} = \frac{Z_1^{\ 2}}{4Z_2} + Z_1 + Z_2 - Z_2$$
$$a_{21} = \frac{1}{Z_2} \qquad a_{22} = a_{11} \qquad 2.5$$

(b) Wave parameters of a T-section.

As the network parameters show, there are four constants that determine the nature of any four-terminal network. But in addition to these four network parameters there are two additional equations which have to be fulfilled, when the network is symmetrical and passive, namely, the determinant of the coefficients a_{ik} (equation 2.2) must be equal to 1. Furthermore $a_{11} = a_{22}$.

For the symmetrical and passive four-terminal network there are, therefore, only two parameters required. In the case of the T-section the impedances Z_1 and Z_2 may be considered as such parameters. It is a general custom to chose two parameters that seem to have more practical value. They are called the wave parameters g and Z_o.

g is called the propagation constant

Z_o is called the characteristic impedance

The propagation constant g is a measure of the relation between input power and output power. The characteristic impedance Z_o is identical with the generally used characteristic impedance. The exact derivation for these and the following relations may be found in textbooks dealing with theory of networks.[1]

In the same way as the network parameters a_{ik} were evaluated in terms of Z_1 and Z_2, they may also be evaluated in terms of g and Z_o. The theory of communication networks gives the following solution:

$$V_1 = \text{Cos h } (g)\, V_2 + Z_o \text{ Sin h } (g)\, I_2$$

$$I_1 = \text{Sin h } (g)\, \frac{V_2}{Z_o} + \text{Cos h } (g)\, I_2 \qquad 2.6$$

---

1 See for example Goldman: Transformation Calculus and Electrical Transients, p. 399.

These two equations are the most important equations of the theory of four-terminal networks, because they relate the network parameters, which may be measured at any four-terminal network, with g and Z_o. This measurement consists of a measurement of input voltage and input current in the case of short circuited output and open circuited output. Equation 2.1 shows for the different cases:

short circuited output: $V_2 = 0$

$$a_{12} = \frac{V_1}{I_2} \qquad a_{22} = \frac{I_1}{I_2}$$

open circuited output: $I_2 = 0$

$$a_{11} = \frac{V_1}{V_2} \qquad a_{21} = \frac{I_1}{V_2}$$

The network parameters are therefore:

$$a_{11} = \text{Cos h } (g) \qquad a_{12} = Z_o \text{ Sin h } (g)$$

$$a_{21} = \frac{\text{Sin h } (g)}{Z_o} \qquad a_{22} = \text{Cos h } (g)$$

And from these equations it is easy to evaluate:

$$Z_o = \sqrt{\frac{a_{12}}{a_{21}}} \qquad 2.7$$

Since $\text{Cos h } (g) - 1 = a_{11} - 1 = 2 \text{ Sin h}^2 \left(\frac{g}{2}\right)$

$$\text{Sin h } \left(\frac{g}{2}\right) = \sqrt{\frac{a_{11} - 1}{2}} \qquad 2.8$$

The relation $2 \text{ Sin h}^2 \left(\frac{x}{2}\right) = \text{Cos h } (x) - 1$ may be proved as follows:

$$2 \operatorname{Sin} h^2 \left(\frac{x}{2}\right) = 2\left[\frac{e^{x/2} - e^{-x/2}}{2}\right]^2 = 2\,\frac{e^x - 2 + e^{-x}}{4}$$

$$= \frac{e^x + e^{-x}}{2} - 1 = \operatorname{Cos} h\,(x) - 1$$

The equations 2.7 and 2.8 are valuable for any four-terminal network. An application to the T-section gives:

$$Z_o = \sqrt{\frac{a_{12}}{a_{21}}} = \sqrt{Z_1 Z_2}\sqrt{1 - \frac{Z_1}{4Z_2}} \qquad 2.9$$

and

$$\operatorname{Sin} h\left(\frac{g}{2}\right) = \sqrt{\frac{a_{11} - 1}{2}} = \sqrt{\frac{Z_1}{4Z_2}} \qquad 2.10$$

(c) <u>The characteristic impedance of the T-section.</u>

Dealing with these so-called basic filters it is usual at the present time to follow the theory developed by Zobel. A very essential prerequisite of this theory is that Z_1Z_2 be real and independent of frequency. In other words Z_1 and Z_2 are reciprocal to each other with respect to a ohmic resistance R. It is usual to write:

$$Z_1 Z_2 = R^2$$

Returning to the equation 2.9, it is easily seen that the second term of the product on the right is the only term that is dependent upon frequency. With the abbreviation

$$-\frac{Z_1}{4Z_2} = x^2$$

equations 2.9 and 2.10 change into

$$Z_o = R\sqrt{1 - x^2} \qquad 2.9\text{ a}$$

$$\text{Sin h } \left(\frac{g}{2}\right) = jx \qquad 2.10\text{a}$$

The plot of the function $Z_o = f(x)$ shows the familiar impedance curve. See Figure 6.

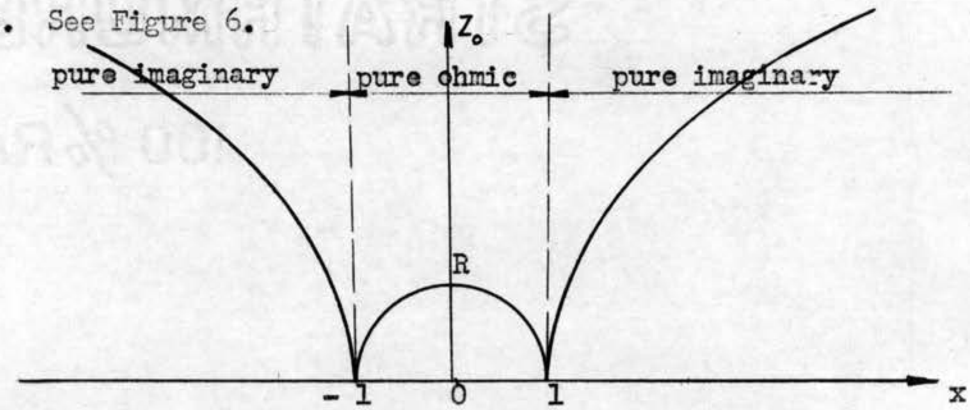


Fig. 6. Characteristic impedance of a T-section.

As far as the computation goes until now, no assumption was made with regard to the nature of Z_1 and Z_2. The terminology used gives the possibility of bringing all sorts of basic filters (low pass, band pass, high pass and the related eliminators) into a single scheme. Furthermore, all the formulas have become very simple.

(d) <u>The matching problem for basic filters</u>.

Equation 2.9a shows, that the characteristic impedance of a T-section is real inside the pass band. Nevertheless, this resistance is dependent upon frequency. It is impossible in practice to terminate filters with such frequency dependent ohmic resistances. Usually the filters are terminated with the value $R = \sqrt{Z_1 Z_2}$; i.e., the value of the characteristic impedance when $x = 0$. It can be shown that this value corresponds with the value of Z_o at the midpoint of the pass band.[2]

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[2] See Guillemin: Communication Networks, p. 321.

(e) <u>The propagation constant</u> g.

The propagation constant g is given in equation 2.10

$$\sinh\left(\frac{g}{2}\right) = jx$$

wherein g may be complex. The left side can be split up into real and imaginary terms: $g = a + jb$

$$\sinh\left(\frac{g}{2}\right) = \sinh\left(\frac{a + jb}{2}\right)$$

$$= \sinh\left(\frac{a}{2}\right) \cosh\left(\frac{jb}{2}\right) - \cosh\left(\frac{a}{2}\right) \sinh\left(\frac{jb}{2}\right)$$

$$= \sinh\left(\frac{a}{2}\right) \cos\left(\frac{b}{2}\right) - j \cosh\left(\frac{a}{2}\right) \sin\left(\frac{b}{2}\right)$$

$$= jx$$

Only the magnitude shall be considered. This equation is only possible if

$$\sinh\left(\frac{a}{2}\right) \cos\left(\frac{b}{2}\right) = 0$$

$$\cosh\left(\frac{a}{2}\right) \sin\left(\frac{b}{2}\right) = x$$

In order to fulfill the first equation the following two cases are possible:

$\sinh\left(\frac{a}{2}\right) = 0$. This means $a = 0$. But then the second equation changes into:

$$\sin\left(\frac{b}{2}\right) = x \qquad a = 0 \qquad \cosh(0) = 1 \qquad 2.11$$

In the same way as in the theory of transmission lines the factor a has the significance of an attenuation factor. From $a = 0$ it is evident that this case is for frequencies inside the pass band. In order that equation 2.11 be possible it is necessary to assume $x < 1$.

$\cos(\frac{b}{2}) = 0.$ This means, for example, $b = \pi$ but then

$$\cosh(\frac{a}{2}) = x \qquad x > 1 \tag{2.12}$$

This case is for frequencies outside the pass band.

A plot of the functions $a = f_1(x)$ and $b = f_2(x)$ shows the following picture:

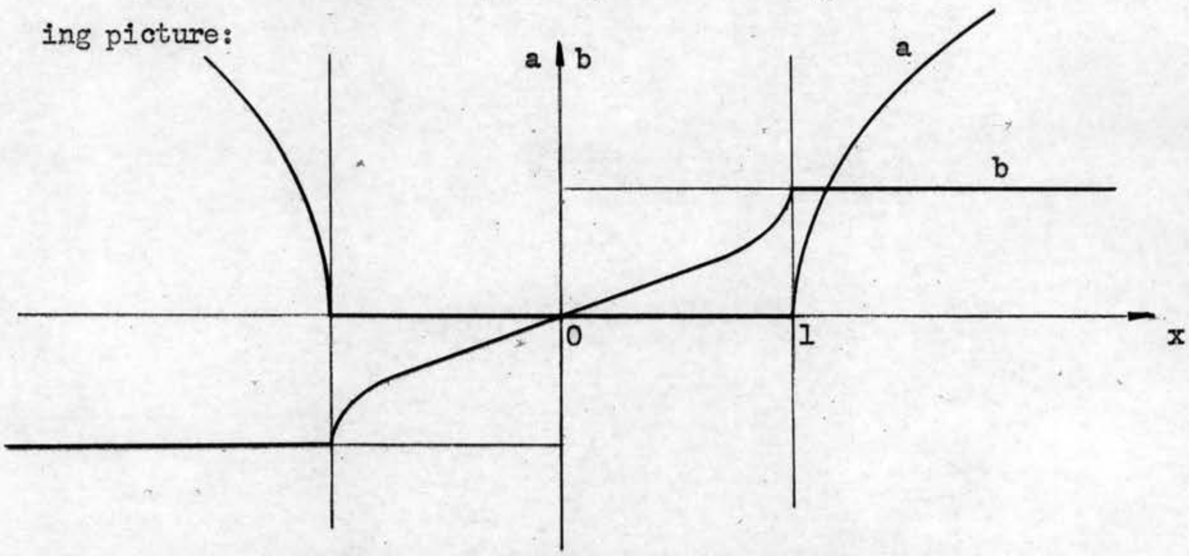

Fig. 7. Phase and attenuation factors of a T-section.

Fig. 7 shows that the phase characteristic as well as the amplitude characteristic of the T-section deviates from the ideal case. Chapter I showed that the phase characteristic would be a straight line and the amplitude characteristic a rectangular function for the ideal filter. An approach to compute the output of a four-terminal network as a result of an arbitrary input in terms of these deviations from the ideal case is discussed very briefly by Guillemin.[3]

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[3] Guillemin: Communication Networks, p. 497.

(f) Transient response of T-sections.

In this paragraph the transient response of a T-section will be computed. It was the intention of the author to keep the derivation as general as possible; i.e., not to make any assumption with regard to the nature of the circuit elements Z_1 and Z_2. Proceeding in this way it was possible to reduce the solution of a sixth-degree equation to the solution of a third and a second-degree equation. As in the case of the ideal filter there shall be a unit step voltage applied at the input terminals of the T-section.

The network equations show:

$$V_1 = a_{11}V_2 - a_{12}I_2$$

$$I_1 = a_{21}V_2 - a_{22}I_2$$

and with the relation $I_2 = \frac{V_2}{Z_o}$ which is in this case $I_2 = \frac{V_2}{\sqrt{Z_1 Z_2}}$

$$V_1 = a_{11}V_2 + \frac{V_2}{Z_o}a_{12} \quad \text{and from this}$$

$$V_2 = \frac{V_1}{a_{11} + a_{12}/Z_o}$$

Because this relation is a result obtained only by application of the methods used to obtain network equations, it is valid for the transient too, when p is changed into s. With the assumption $V_1 = \frac{1}{s}$ and using equations 2.5 for the circuit parameters, the output voltage becomes:

$$V_2 = \frac{V_1(s)}{1 + Z_1/2Z_2 + Z_1^2/4Z_2Z_o + Z_1/Z_o} \qquad 2.13$$

With the relation $Z_o = R = \sqrt{Z_1 Z_2}$ the denominator of this fraction is:

$$D = a_{11} + \frac{a_{12}}{R} = 1 + \frac{Z_1^2}{2R^3}\left(\frac{Z_1}{2} + \frac{2R^2}{Z_1}\right)$$

A division by 2 gives the following equation:

$$\tfrac{1}{2}D = \tfrac{1}{2} + \frac{Z_1^2}{4R^2} + \frac{Z_1^3}{8R^3} + \frac{Z_1}{2R}$$

As a first step the zero points of this denominator are to be found. With the substitution $\frac{Z_1}{2R} = u$ the roots of the following equation shall be computed:

$$u^3 + u^2 + u + \tfrac{1}{2} = 0 \qquad 2.14$$

Because the degree of this equation is represented by an odd number there must be at least one real root which may be found with the method of Newton.

First approximation: $u_1 = -0.5$

$$f(u) = u^3 + u^2 + u + \tfrac{1}{2} = -0.125 + 0.25 - 0.5 + 0.5 = 0.125$$

$$f'(u) = 3u^2 + 2u + 1 = 0.75 - 1 + 1 = 0.75$$

$$\delta = -\frac{f(u)}{f'(u)} = -0.15$$

Second approximation: $u_1 = -0.65$

$$f(u) = -0.274625 + 0.4225 - 0.65 + 0.5 = -0.002125$$

$$f'(u) = 1.2675 - 1.3 + 1 = 0.9675$$

$$\delta = 0.002$$

Third approximation: $u_1 = -\ 0.648$

$$f(u) = -\ 0.00018$$

As a very accurate result $u_1 = -0.648$ may be chosen. Through an eliminating process equation 2.14 may now be reduced to a second degree equation:

$$\frac{(u^3 + u^2 + u + \frac{1}{2})}{(u + 0.648)} = u^2 + 0.352u + 0.7718$$

$$\frac{-0.648u^2}{0.352u^2 + u}$$

$$\frac{-\ 0.2282u}{0.7718 + \frac{1}{2}}$$

The remaining equation is therefore:

$$u^2 + 0.352u + 0.7718 = 0$$

The roots of this equation are:

$$u_2 = -0.167 + \sqrt{0.030976 - 0.7718}$$

$$= -0.176 + j\sqrt{0.740884}$$

$$u_2 = -0.176 + j0.86075$$

and

$$u_3 = -0.176 - j0.86075$$

equation 2.13 now has the form:

$$V_2(t) = \frac{V_1(s)}{2\,\frac{D}{2}} = \frac{V_1(s)}{2(u - u_1)(u - u_2)(u - u_3)} \qquad 2.13a$$

(g). Transients in conventional filters.

Low-pass filter:

Figure 8 shows the circuit of a conventional low-pass filter.

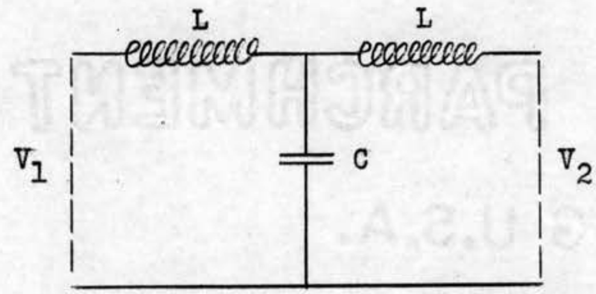

Fig. 8. Conventional low-pass filter.

The circuit elements are:

$$\tfrac{1}{2}Z_1 = Lp \qquad\qquad Z_2 = \frac{1}{pC}$$

The load R may be computed:

$$R = \sqrt{Z_1 Z_2} = \sqrt{\frac{2L}{C}}$$

The frequency band of a filter is a quality which is related only to the steady-state response of the given network. All the formulas derived from the expressions 2.10 and 2.11 for attenuation and characteristic impedance are valid only for the steady-state response of the filter. The factor p in the expressions for Z_1 and Z_2 has, therefore, to be replaced by $j\omega$. The frequency limit of any filter is given by the relation:

$$x^2 = 1 \qquad \text{and with} \qquad x^2 = \frac{Z_1}{4Z_2}$$

$$-\frac{j L j C}{2} = 1 \qquad \text{which means}$$

$$\omega_1^2 = \frac{2}{LC} \quad \text{or} \quad \omega_1 = \sqrt{\frac{2}{LC}} = \frac{1}{L}\sqrt{\frac{2L}{C}} = \frac{R}{L}$$

The roots of the denominator of equation 2.13 shall now be computed. From the last section the zero-points of the third-degree equation 2.14 are known. From

$$u_1 = \frac{Z_1}{2R} = \frac{s_1 L}{R} = -\ 0.65$$

$$u_2 = \frac{s_2 L}{R} = -\ 0.176 + j0.86075$$

$$u_3 = \frac{s_3 L}{R} = -\ 0.176 - j0.86075$$

the zero-points of the denominator D of equation 2.13 are:

$$s_1 = -\ 0.65\ \frac{R}{L} = -\ 0.65\omega_1$$

$$s_2 = -\ 0.176\ \frac{R}{L} + j0.86075\ \frac{R}{L} = -\ 0.176\omega_1 + j0.86075\omega_1$$

$$s_3 = -\ 0.176\ \frac{R}{L} - j0.86075\ \frac{R}{L} = -\ 0.176\omega_1 - j0.86075\omega_1$$

Equation 2.13, therefore, has the following form:

$$V_2(s) = \frac{A}{s(s-s_1)(s-s_2)(s-s_3)} \qquad 2.15$$

s_2 and s_3 are conjugates of each other and, therefore, equation 2.15 has the type:

$$f(s) = \frac{1}{s(s-a)\left[(s+\alpha)^2+\beta^2\right]} \qquad 2.16$$

α is the real part and β is the imaginary part of $-s_2$. Equation 2.16 shows that the solution for the low-pass filter consists of a damped oscillation and a rising exponential expressed by the product $s(s-a)$ in the denominator. It looks strange that the dissipationless circuit of Fig. 8 should have a damped oscillation as transient response. It must nevertheless be noted that the load R is coupled to the circuit and, therefore, dissipates the energy of the oscillation. The inverse transformation of equation 2.16 could not be found in the literature and had, therefore, to be computed with the aid of Heaviside's theorem.

$$F(t) = \frac{1}{a(\alpha^2 - \beta^2)} - \frac{1}{a\left[(a - \alpha)^2 + \beta^2\right]} e^{-at} \qquad 2.17$$

$$+ e^{-\ t} \quad \frac{(\alpha^2 - a\alpha - \beta^2)\sin(\beta t) - (2\alpha\beta - a\beta)\cos(\beta t)}{(\alpha^2 + \beta^2)\left[(a - \alpha)^2 + \beta^2\right]}$$

For the low-pass filter the terms a, α, β have the values:

$$a \doteq 0.648\omega_1$$

$$\alpha = 0.176\omega_1$$

$$\beta = 0.86075\omega_1$$

For the final solution this function has to be multiplied by a factor A. This factor shall now be computed:

Equation 2.13 gives:

$$V_2(t) = \frac{V_1(s)}{2\frac{1}{2}D} = \frac{1}{2s(u^3 + u^2 + u + \frac{1}{2})} \quad \frac{1}{2s(u - u_1)(u - u_2)(u - u_3)}$$

$$= \frac{1}{2s(\frac{s}{\omega_1} - u_1)(\frac{s}{\omega_1} - u_2)(\frac{s}{\omega_1} - u_3)} \quad \frac{\omega_1{}^3}{2s(s - u_1\omega_1)(s - u_2\omega_1)(s - u_3\omega_1)}$$

$$= \frac{\omega_1{}^3}{2s(s - s_1)(s - s_2)(s - s_3)}$$

The factor A is, therefore:

$$A = E\tfrac{1}{2}\omega_1^{\,3}$$

After a lengthy computation $V_2(t)$ takes the following form:

$$V_2(t) = E\left[1 - 0.8\,e^{-0.64\omega_1 t} - 0.6\cos(0.86\omega_1 t - \varphi)\right.$$

$$\text{when } \varphi = \tan^{-1}(2.82)$$

For t = zero, $V_2(t)$ must become zero, because the inductances do not allow a discontinuity to appear at the load. This relation holds in equation 2.17 which can be proved by setting t = 0.

On the other hand it can be shown that at t = 0, even the derivative with respect to time is zero, which means that the response function has a horizontal tangent at t = 0. The response of the low-pass on a unit step has, therefore, the following form.

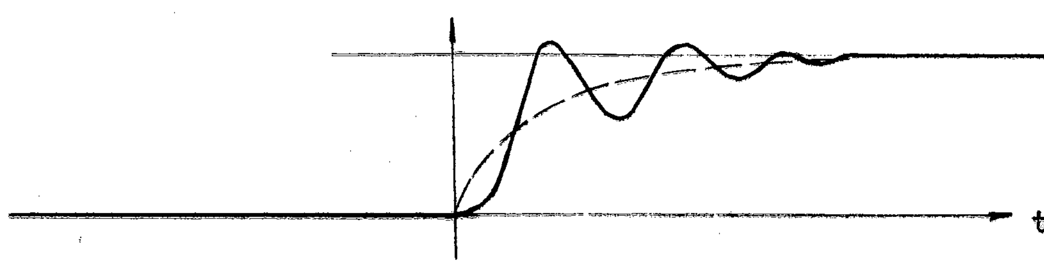

Fig. 9. Response of the low-pass filter to a unit step.

Specific examples will be given in a later section.

<u>High-pass filter</u>:

The circuit of a high-pass filter is shown in Figure 10.

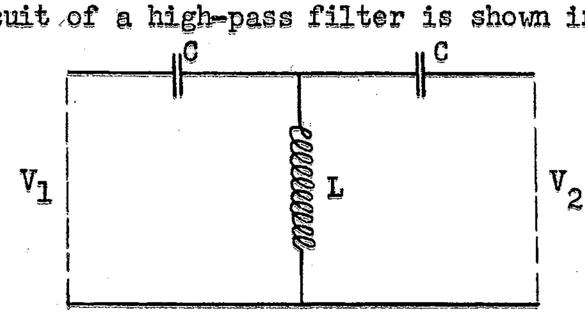

Fig. 10. Conventional high-pass filter.

For this case the circuit elements are:

$$\tfrac{1}{2}Z_1 = \frac{1}{pC} \qquad\qquad Z_2 = pL$$

and

$$R = \sqrt{\frac{2L}{C}}$$

In the same way as for the low-pass the limit frequency may be computed:

$$x^2 = 1 = \frac{1}{2\omega^2 LC} \qquad \text{and, therefore,}$$

$$\omega_1 = \sqrt{\frac{1}{2LC}} = \frac{1}{RC}$$

With $u = \frac{Z_1}{2R} = \frac{1}{sRC} = \frac{\omega_1}{s}$ the roots of the denominator of equation 2.13 become:

$$s_1 = -1.54\omega_1$$

$$s_2 = (-0.205 + j0.98)\omega_1$$

$$s_3 = (-0.205 - j0.98)\omega_1$$

The denominator of equation 2.14 has in this case the form:

$$s(u - u_1)(u - u_2)(u - u_3) = s(\frac{1}{s} - u_1)(\frac{1}{s} - u_2)(\frac{1}{s} - u_3) \qquad 2.19$$

When this expression is expanded the result is a function of the form:

$$D = s(\frac{a}{s^3} + \frac{b}{s^2} + \frac{c}{s} + \cdots) \qquad 2.20$$

The method developed in the preceding paragraphs gives the zero points of this denominator. It must be kept in mind that for this case the polynomial in the parantheses of equation 2.20 cannot be written in the product form $(s - s_1)(s - s_2) \dots\dots (s - s_n)$, where $s_1, s_2 \dots s_n$ are the roots of the equation

$$D(s) = 0 \qquad 2.21$$

This would only be true if all the exponents of s in equation 2.20 were positive or equal to zero. There is nevertheless a possibility of reducing equation 2.19 in such a way that the product form may be applied. The following general function may be assumed:

$$D(s) = c_{-n}s^{-n} + c_{-n-1}s^{-n-1} + \dots\dots c_o + c_1 s + c_2 s^2 + \dots\dots c_m s^m$$

This function can be written:

$$D(s) = \frac{1}{s^n}(c_{-n} + c_{-n-1}s + c_{-n-2}s^2 \dots\dots c_o s^n + c_1 s^{n-1} \dots\dots c_m s^{m-n})$$

The expression in the parantheses is a function which contains only exponents of s which are equal to or greater than zero and may, therefore, be expressed as a product. Because

$$(c_{-n} + c_{-n-1}s + \dots\dots) = 0$$

has the same roots as equation 2.21 (multiplication on both sides with $\frac{1}{s^n}$ it is certain that the zero points of both expressions coincide.

The following theorem is, therefore, valid

Any polynomial of the form:

$$c_{-n}s^{-n} - c_{-n-1}s^{-n-1} - \dots\dots c_o - c_1 s \dots\dots c_m s^m$$

may be written in the form:

$$\frac{c_m}{s^n}(s - s_1)(s - s_2) \dots\dots (s - s_n) \quad \text{where } s_1, s_2 \dots s_n \text{ are}$$

the roots of the equation:

$$c_{-n}s^{-n} + c_{-n-1}s^{-n-1} \;\ldots.\; c_o + c_1 s \;\ldots.\; c_m s^m = 0$$

Equation 2.20 shows that the maximal negative power of the expression inside the parantheses is three. The denominator D(s) has, therefore, the following form: (The factor c_n is equal to $\frac{1}{u_1\,u_2\,u_3} = 1$)

$$D(s) = s\frac{1}{s^3}(s - s_1)(s - s_2)(s - s_3)$$

$V_2(t)$ is, therefore, the inverse transformation of the expression

$$V_2(s) = \frac{s^2}{(s - s_1)(s - s_2)(s - s_3)}$$

where s_2 and s_3 are again conjugates of each other. This may be written as:

$$= \frac{s^2}{(s + a)[(s + \alpha)^2 + \beta^2]} \qquad 2.22$$

where α is the real and β is the imaginary part of $-s_2$, and a is equal to $-s_1$.

$$a = 1.54_{\ 1}$$

$$\alpha = 0.205_{\ 1}$$

$$\beta = 0.98_{\ 1}$$

The inverse transformation of equation 2.22 may be found in the literature[4] and is:

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4 Goldman, op. cit., p. 421, Formula 24.

$$V_2(t) = \frac{a^2}{(a-\alpha)^2 - \beta^2}\, e^{-at} + \frac{\alpha^2 + \beta^2 - 2a\alpha}{(a-\alpha)^2 + \beta^2}\, e^{-\alpha t} \cos\beta t$$

$$+ \frac{(a-\alpha)(\alpha^2 - \beta^2) - a\beta^2\alpha}{\beta\left[(a-\alpha)^2 + \beta^2\right]}\, e^{-\alpha t} \sin\beta t \qquad 2.23$$

Inserting the values of a, α, β gives:

$$V_2(t) = 0.865\, e^{-1.54\omega_1 t} + 0.138 e^{-0.205\omega_1 t} \cos 0.98\omega_1 t$$

$$- 0.612 e^{-0.205\omega_1 t} \sin 0.98\omega_1 t \qquad 2.24$$

The response of a high-pass filter shows, therefore, the following form.

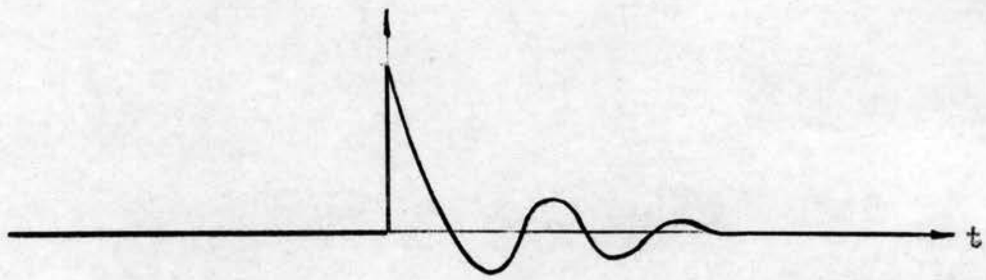

Fig. 11. Response of the high-pass filter to a unit step.

Band-pass filter:

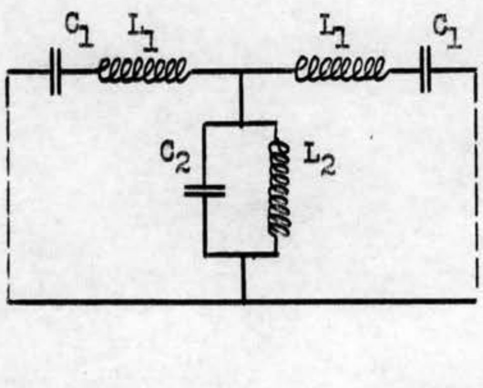

Fig. 12. Conventional band-pass filter.

The circuit elements are:

$$\tfrac{1}{2}Z_1 = Lp + \frac{1}{pC} = L\omega_o\left(\frac{p}{\omega_o} + \frac{\omega_o}{p}\right) \quad \text{with } \omega_o^2 = \frac{1}{LC}$$

$$Z_2 = \frac{R^2}{Z_1}$$

This is the way the circuit element Z_2 is usually evaluated. In most practical problems R is assumed to be 600 ohms. The limit frequencies are evaluated according to the formula

$$x = = \frac{L\omega_o\left(\frac{\omega}{\omega_o} - \frac{\omega^o}{\omega}\right)}{R} = \pm 1$$

With the abbreviation $\frac{R}{\omega_o L} = v$ and taking only positive frequencies into account, this equation may be solved for :

$$\omega_1 = \omega_o\left[\sqrt{\left(1 - \frac{v^2}{4}\right)} + \frac{v}{2}\right]$$

$$\omega_2 = \omega_o\left[\sqrt{\left(1 - \frac{v^2}{4}\right)} - \frac{v}{2}\right]$$

It can easily be shown by expansion that $\omega_1\omega_2 = \omega_o^2$; i.e., ω_o is the geometric mean of ω_1 and ω_2.

Furthermore from $u = \frac{Z_1}{2R}$ the equation may be derived:

$$u = p\frac{L}{R} + \frac{1}{pRC} = \frac{\omega_o L}{R}\left(\frac{p}{\omega_o} + \frac{\omega_o}{p}\right) \quad \text{again with } \omega_o^2 = \frac{1}{LC}$$

With $p = s$ for the transient response this may be put into the equation:

$$s^2 - uv_o s + \omega_o^2 = 0 \quad \text{when} \quad \frac{\omega_o L}{R} = \frac{1}{v} = \frac{1}{R}\sqrt{\frac{L}{C}}$$

which is very often referred to as Q and is named quality factor.
The solution of this equation gives:

$$s_1 = \omega_0 \left(\frac{uv}{2} - j \sqrt{1 - \left(\frac{uv}{2}\right)^2}\right)$$

$$s_2 = \omega_0 \left(\frac{uv}{2} + j \sqrt{1 - \left(\frac{uv}{2}\right)^2}\right)$$

When the three zeroes of u:

$$u_1 = -0.648 \qquad u_2 = -0.176 - j0.86 \qquad u_3 = -0.176 + j0.86$$

are inserted into these equations it is readily seen that there must be 6 zero points of equation 2.14 in the case of the band-pass filter. The relation between s_1, s_2 and ω_1, ω_2 are unfortunately not as simple as in the case of the low-pass and the high-pass filters. In other words, the zero points of the denominator are a function of R. This is easy to understand, because a damped resonant circuit has not the same frequency as the undamped circuit with the same capacitance and inductance. Because R is not negligible the change in frequency must be taken into account.

Equation 2.13, therefore, has the following form:

$$f(s) = \frac{s^2}{(s - s_1)(s - s_2)(s - s_3)(s - s_4)(s - s_5)(s - s_6)}$$

Because always two solutions are the conjugates of each other this may be written:

$$f(s) = \frac{s^2}{\left[(s + \alpha_1)^2 + \beta_1^2\right]\left[(s + \alpha_2)^2 + \beta_2^2\right]\left[(s + \alpha_3)^2 - \beta_3^2)\right]}$$

The inverse transformation of this expression is a function which is composed of three terms according to the three brackets in the denominator. All of these terms are built in the same way and are of the nature:

$$\left\{\frac{(\alpha_x^2-\beta_x^2)\left\{\left[(\alpha_m-\alpha_x)^2+(\beta_m^2-\beta_x^2)\right]\left[(\alpha_n-\alpha_x)^2+(\beta_n^2-\beta_x^2)\right]-2\beta_x^2(\alpha_m-\alpha_x)(\alpha_n-\alpha_x)\right\}}{\beta_x\left[(\alpha_m-\alpha_x)^2+(\beta_m+\beta_x)^2\right]\left[(\alpha_m-\alpha_x)^2+(\beta_m-\beta_x)^2\right]\left[(\alpha_n-\alpha_x)^2+(\beta_n+\beta_x)^2\right]\left[(\alpha_n-\alpha_x)^2+(\beta_n-\beta_x)^2\right]}\right.$$

$$\left.\frac{(4\alpha_x\beta_x^2)\left\{(\alpha_n-\alpha_x)\left[(\alpha_m-\alpha_x)^2+(\beta_m^2-\beta_x^2)\right]+(\alpha_m-\alpha_x)\left[(\alpha_n-\alpha_x)^2+(\beta_n^2-\beta_x^2)\right]\right\}}{\beta_x\left[(\alpha_m-\alpha_x)^2+(\beta_m+\beta_x)^2\right]\left[(\alpha_m-\alpha_x)^2+(\beta_m-\beta_x)^2\right]\left[(\alpha_n-\alpha_x)^2+(\beta_n+\beta_x)^2\right]\left[(\alpha_n-\alpha_x)^2+(\beta_n-\beta_x)^2\right]}\right\}$$

$$e^{-\alpha_x t}\sin\beta_x t \tag{2.25}$$

$$-\left\{\frac{(2\alpha_x\beta_x)\left\{\left[(\alpha_m-\alpha_x)^2+(\beta_m^2-\beta_x^2)\right]\left[(\alpha_n-\alpha_x)^2+(\beta_n^2-\beta_x^2)\right]-2\beta_x^2(\alpha_m-\alpha_x)(\alpha_n-\alpha_x)\right\}}{\beta_x\left[(\alpha_m-\alpha_x)^2+(\beta_m+\beta_x)^2\right]\left[(\alpha_m-\alpha_x)^2+(\beta_m-\beta_x)^2\right]\left[(\alpha_n-\alpha_x)^2+(\beta_n+\beta_x)^2\right]\left[(\alpha_n-\alpha_x)^2+(\beta_n-\beta_x)^2\right]}\right.$$

$$\left.-\frac{2\beta_x(\alpha_x^2-\beta_x^2)\left\{(\alpha_n-\alpha_x)\left[(\alpha_m-\alpha_x)^2+(\beta_m^2-\beta_x^2)\right]+(\alpha_m-\alpha_x)\left[(\alpha_n-\alpha_x)^2+(\beta_n^2-\beta_x^2)\right]\right\}}{\beta_x\left[(\alpha_m-\alpha_x)^2+(\beta_m+\beta_x)^2\right]\left[(\alpha_m-\alpha_x)^2+(\beta_m-\beta_x)^2\right]\left[(\alpha_n-\alpha_x)^2+(\beta_n+\beta_x)^2\right]\left[(\alpha_n-\alpha_x)^2+(\beta_n-\beta_x)^2\right]}\right\}$$

$$e^{-\alpha_x t}\cos\beta_x t.$$

There will be three expressions like this one in the final solution. The first expression may be evaluated by substituting α_1 for α_x and α_2 and α_3 for α_m and α_n. For the second expression, substitute α_2 for α_x and α_1 and α_2 for α_m and α_n. For the third, substitute α_3 for α_x

and α_1 and α_2 for α_m and α_n. For all of these expressions, the β-subscripts must be changed in the same way as the α subscripts are. As soon as there are given numbers inserted into the different terms they reduce to the form $(a \sin xt - b \cos xt)e^{-yt}$. As stated before, the final solution consists of three damped sinusoidal oscillations which are superposed. An example will be given in the last chapter.

<u>Band elimination filter</u>:

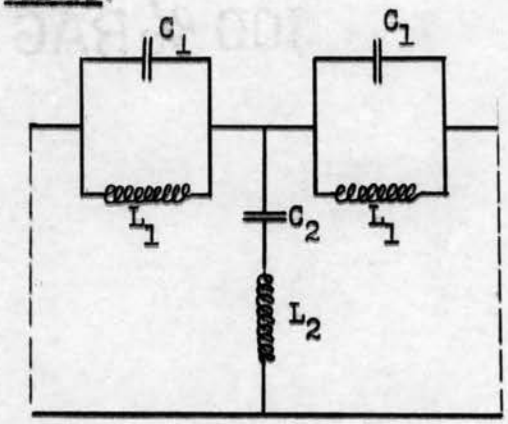

Fig. 13. Band elimination filter.

The procedure for the band-elimination filter is basically the same as for the band-pass filter. The circuit elements are:

$$\frac{1}{\frac{1}{2}Z_1} = pC - \frac{1}{Lp} = \omega_o C\left(\frac{p}{\omega_o} - \frac{\omega_o}{p}\right)$$

For this u may be computed:

$$u = \frac{Z_1}{2R} = \frac{1}{RC\omega_o\left(\frac{p}{\omega_o} - \frac{\omega_o}{p}\right)} \quad \text{and with } C\omega_o R = R\sqrt{\frac{C}{L}} = \frac{1}{v}$$

$$u = \frac{v}{\frac{p}{\omega_o} - \frac{\omega_o}{p}} = \frac{vp\omega_o}{p^2 - \omega_o^2}$$

With this and $p = s$ for the transient, equation 2.13 has a denominator of the form:

$$D(s) = s\left(\frac{vs\omega_o}{s^2 + \omega_o^2} - u_1\right)\left(\frac{vs\omega_o}{s^2 + \omega_o^2} - u_2\right)\left(\frac{vs\omega_o}{s^2 + \omega_o^2} - u_3\right)$$

In order to make this denominator a polynomial without negative exponents of s this expression must be multiplied by $(s^2 - \omega_o^2)^3$.

Equation 2.13 has, therefore, the form:

$$V_2(s) \frac{A(s^2 - \omega_o^2)^3}{s\left[(s+\alpha_1)^2 + \beta_1^2\right]\left[(s+\alpha_2)^2 + \beta_2^2\right]\left[(s+\alpha_3)^2 + \beta_3^2\right]} \qquad 2.26$$

where

$$\alpha_i = -\frac{v\omega_o}{2u_i} \qquad \beta_i = \sqrt{\omega_o^2 - \frac{v^2\omega_o^2}{4u_i^2}}$$

This expression is too complicated to be transformed directly. As soon as there are specific numbers for ω_o and α and β, the transformation is much easier.

For the sake of completeness the expressions for the limit frequencies shall be given:

$$\omega_1 = \omega_o\left(\sqrt{1 - \frac{v^2}{4}} + \frac{v}{2}\right)$$

$$\omega_2 = \omega_o\left(\sqrt{1 - \frac{v^2}{4}} - \frac{v}{2}\right)$$

The formulas turn out to be exactly equal to those for the bandpass filter.

3. The π-Type Filter

Equations for the π-section will be developed in the same way as for the T-section.

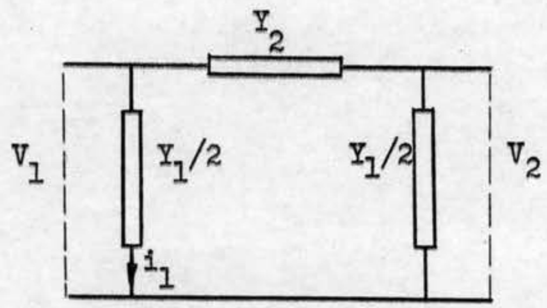

Fig. 14. π-section.

(a) Network equations

The node analysis of the circuit of Fig. 14 gives the equations:

$$V_1(\tfrac{1}{2}Y_1 + Y_2) - V_2Y_2 = I_1$$
$$V_2(\tfrac{1}{2}Y_1 + Y_2) - V_1Y_2 = - I_1 \qquad 2.27$$

From this the chain equations may be drived:

$$V_1 = (1 + \frac{Y_1}{2Y_2})V_2 + \frac{1}{Y_2}I_2$$
$$I_1 = Y_1(1 + \frac{Y_1}{4Y_2})V_2 + (1 + \frac{Y_1}{2Y_2})I_2$$

This gives the chain parameters:

$$a_{11} = 1 + \frac{Y_1}{2Y_2}\,; \qquad a_{12} = \frac{1}{Y_2}$$
$$a_{21} = Y_1(1 + \frac{Y_1}{4Y_2})\,; \qquad a_{22} = a_{11}$$

(b) The wave parameters

With the aid of equation 2.7 and 2.8 the wave parameters may be evaluated:

$$Z_o = \sqrt{\frac{a_{11}}{a_{21}}} = \frac{1}{\sqrt{Y_1Y_2}\sqrt{1 - \frac{Y_1}{4Y_2}}}$$

and again with $\frac{Y_1}{4Y_2} = -x^2$ and $Y_1Y_2 = \frac{1}{R^2}$ this is equal to

$$Z_o = \frac{R}{\sqrt{1 - x^2}}$$

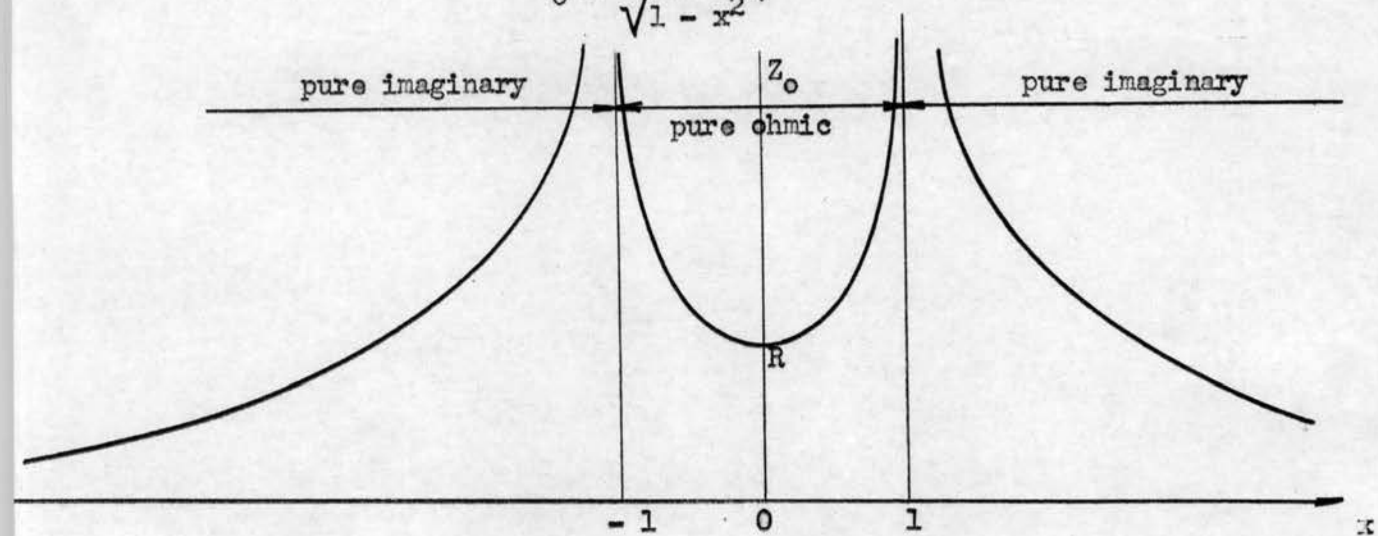

Fig. 15. Characteristic impedance of a π-section.

In the same way as for the T-section it can be shown that:

$$\text{Sin h}\left(\frac{g}{2}\right) = \sqrt{\frac{a_{11} - 1}{2}} = \sqrt{\frac{Y_1}{4Y_2}} = jx$$

This equation looks exactly the same as equation 2.10a. The plot of the function $g = f(x)$ is, therefore, the same as Fig. 7. The propagation constant for both T and π-section are, therefore, identical.

(c) Transient response of Π-section

The frequency response of the output voltage is calculated in the same way as equation 2.13 was developed. Starting at the equation:

$$V_2 = \frac{V_1}{a_{11} + \frac{a_{12}}{Z_o}}$$

with $Z_o^2 = R^2 = \frac{1}{Y_1 Y_2}$ it may be shown that

$$V_2 = \frac{V_{1(s)}}{1 + \frac{R^2 Y_1{}^2}{2} + RY_1}$$

The denominator becomes, therefore:

$$1 + \frac{R^2 Y_1{}^2}{2} + RY_1$$

Dividing by 2 and with the abbreviation: $u = \frac{RY_1}{2}$ the zero-points of this denominator are found as roots of the equation

$$u^2 + u + \tfrac{1}{2} = 0$$

$$u = -\tfrac{1}{2} \pm j\tfrac{1}{2}$$

The solution consists, therefore, of a single damped oscillation. The reason is that the given problem demands that the voltage at the input of the filter shall be a unit step independent of the input current. In other words the current i_1 through Z_1 at the input of the Π-filter shall not influence the input voltage. Under these circumstances this current i_1 has no influence on the output voltage and the circuit of the Π section may be reduced to the following:

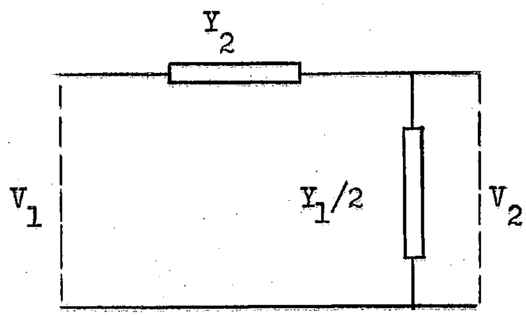

Fig. 16. Reduced π-section.

This circuit is no longer a four-terminal network, and a great part of the discriminative quality of the filter has been lost. The π-section is, therefore, not the best circuit for a voltage source as energy source. With other words, the π-section loses a great part of its filtering quality when the energy-source has a small internal resistance.

If the energy source has a high internal resistance; i.e., is comparable with a current source, the input current may be assumed to be a unit-step current.

Then the four-terminal network equations are:

$$V_1 = a_{11}V_2 + a_{12}I_2$$

$$I_1 = a_{21}V_2 + a_{22}I_2$$

With the relation $I_2 = \frac{U_2}{R}$ when the load at the output is R:

$$I_1 = a_{21}V_2 + a_{22}\frac{V_2}{R}$$

$$V_2 = \frac{I_1 R}{a_{22} + a_{21}\ R}$$

and with $R = \frac{1}{G}$ the denominator of this fraction becomes:

$$D = 1 + \frac{Y_1^2}{2G^2} + \frac{Y_1}{G} + \frac{Y_1^3}{4G^3} .$$ When this is again divided

by 2 and with the abbreviation $\frac{Y_1}{2G} = u$ this becomes:

$$\tfrac{1}{2}D = u^3 + u^2 + u + \tfrac{1}{2}$$

This is exactly the same equation as equation 2.14. When the denominators of equation 2.13 for the T-section and the π-section are compared it is readily seen that Z changed to Y and R to G. The nature of the transient response in the T-section and in the π-section is, therefore, the same. The transient of the π-section may be derived from the transient of the T-section by simply replacing Z_1 by Y_1 and R by G.

Chapter III

The Influence of Loss Resistance

Again the relation $Z_1Z_2 = R^2$ shall be fulfilled. In this paragraph will be shown what consequences this condition has on the choice of the circuit elements. The most complicated case is the one in which Z_1 is represented by a resonant circuit.

$$Z_1 = R_1 + j(\omega L_1 - \frac{1}{\omega C_1})$$

$$Y_2 = G_2 + j(\omega C_2 - \frac{1}{\omega L_2})$$

From $Z_1Z_2 = R^2$ there may be derived:

$$R_1 + j(\omega L_1 - \frac{1}{\omega C_1} = R^2\left[G_2 + j(\omega C_2 - \frac{1}{\omega L_2})\right]$$

A comparison of the real parts on both sides and of the coefficients of ω and $\frac{1}{\omega}$ gives the following three equations:

$$R_1 = R^2G_2 \qquad L_1 = R^2C_2 \qquad \frac{1}{C_1} = R^2\frac{1}{L_2}$$

If the quality factor of Y_2 is computed it may be shown that:

$$Q_2 = \frac{\omega_o L_2}{R_2} = \omega_o L_2 G_2 = \frac{\omega_o R^2 C_1 R_1}{R^2} = \omega_o C_1 R_1$$

But the expression $\omega_o CR$ is known as the quality factor of a series resonance circuit. The consequence of the condition $Z_1Z_2 = R^2$ is, therefore, that the quality factors of all circuit elements are the same. It is today common use to design filters according to this, even if there must be inserted artificial loss resistances in the form of lumped ohmic resistances.

From this derivation it may be seen that equation 2.14 is still valid for a filter which includes losses. The only assumption concerning the circuit elements made when this equation was developed was that $Z_1 Z_2 = R^2$. This is the second big advantage this form of analysis has over any other. The zero points of equation 2.13 may be found by substituting $Z_1(s)$ into the formula for u:

$$u = \frac{Z_1}{2R} \quad \text{and solving for s.}$$

Low-pass filter:

$$Z_1 = R_1 - sL$$

$$u = \frac{R_1}{R} - s\frac{L}{R} \quad \text{solved for s gives:}$$

$$s = u\omega_1 - \frac{R_1}{L} \qquad 3.1$$

Equation 2.14 gave 3 solutions for u, among which one was real. The influence of the losses is expressed by the last term on the right of equation 3.1. Because this term is always real, there is no influence of the loss upon the frequency of the oscillation. The only influence consists in an increase of the damping factor and a decrease in amplitude of the whole solution.

High-pass filter:

$$\tfrac{1}{2}Z_1 = \frac{1}{G + sC}$$

and from this again

$$s = \frac{\omega_1}{u} - \frac{G}{C} \qquad 3.2$$

and again the influence is only upon the real part.

Band-pass filter:

$$\tfrac{1}{2}Z_1 = R_1 + pL + \frac{1}{pC}$$

and

$$p = -\frac{R_1 - Ru}{2} \pm j\sqrt{\frac{1}{LC} - \left(\frac{R_1 - Ru}{2}\right)^2} \qquad 3.3$$

In this case there is an influence of R_1 upon the frequency of the oscillation. The example in Chapter 4 will show that this influence is negligible.

A similar derivation could be made for the band-elimination filter. All these cases show that there is no essential difference in the transient of a basic filter whether there is loss or not. All the transformation formulas remain the same. The difference consists only in a slightly increased damping factor of the oscillations.

Chapter IV

Application of the Theory and Conclusions

1. The transient response of the low-pass filter

(a). The ideal filter

Chapter I furnished the following final formula for the ideal band-pass filter:

$$V_2(t) = \frac{EK}{\pi} \left[Si\omega_2(t - t_o) - Si\omega_1(t - t_o)\right] \qquad 4.1$$

In the case of the low-pass filter the angular frequency ω_1 must be set equal to zero and ω_2 represents the limit frequency. Equation 4.1 then changes into:

$$V_2(t) = \frac{EK}{\pi} \; Si\omega_2(t - t_o)$$

For the transient response the time delay t_o is of no interest. Thus the final solution takes the form:

$$V_2(t) = \frac{EK}{\pi} \; Si\omega_2 t$$

Figure 17 represents the transient of an ideal low-pass filter with the limiting frequency $f_2 = 1000$ cps; i.e., $\omega_2 = 6280 \text{ sec}^{-1}$. The product EK is assumed to be equal to 1.

(b). The physical low-pass filter without loss

The transient response of a physical low-pass filter without losses shall now be computed according to the theory developed in Chapter II which lead to formula 2.18.

The following data for the low-pass filter shall be given:

limit frequency: $f_1 = 1000$ cps

i.e., $\omega_1 = 6280 \text{ sec}^{-1}$

load at output terminals

$$R = 600 \text{ ohms}$$

The circuit elements are determined according to the formula

$$\omega_1 = \frac{R}{L} \quad \text{and, therefore,}$$

$$L = \frac{R}{\omega_1} = \frac{600}{6280} = 0.0955 \simeq 0.1 \text{ henry}$$

and from $Z_1 Z_2 = \frac{L}{C} = R^2$

$$C = \frac{L}{R^2} = 0.28 \mu F$$

If the value of ω_1 is inserted into equation 2.18 the final form of the transient is:

$$V_2(t) = 1 - 0.8\, e^{-4070t} - 0.6\, e^{-1110t} \cos(810t - \varphi)$$

$$\varphi = 70^\circ 40' = 1.23 \text{ radians}$$

Figure 18 shows a plot of this function vs. time.

It is readily seen that the most essential difference between the ideal filter and the physical filter without losses consists of an oscillation at negative time at the output of the ideal filter. It seems to be a paradox that the output of the ideal filter already shows an oscillation before the signal at the input is applied. The reason for this is that the amplitude and phase characteristic of the ideal filter have been assumed to be independent of each other. In other words the ideal filter is not physically realizable. There exists a very definite relation between phase and amplitude characteristic.[5]

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[5] Goldman: op. cit., Page 128.

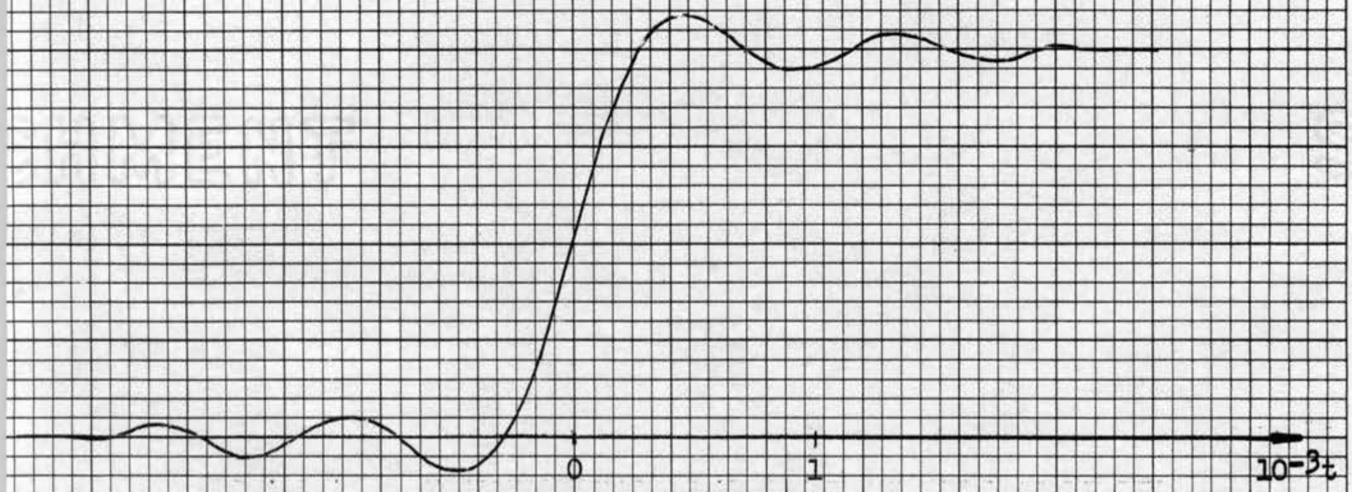

Fig. 17. Transient response of the ideal low-pass filter

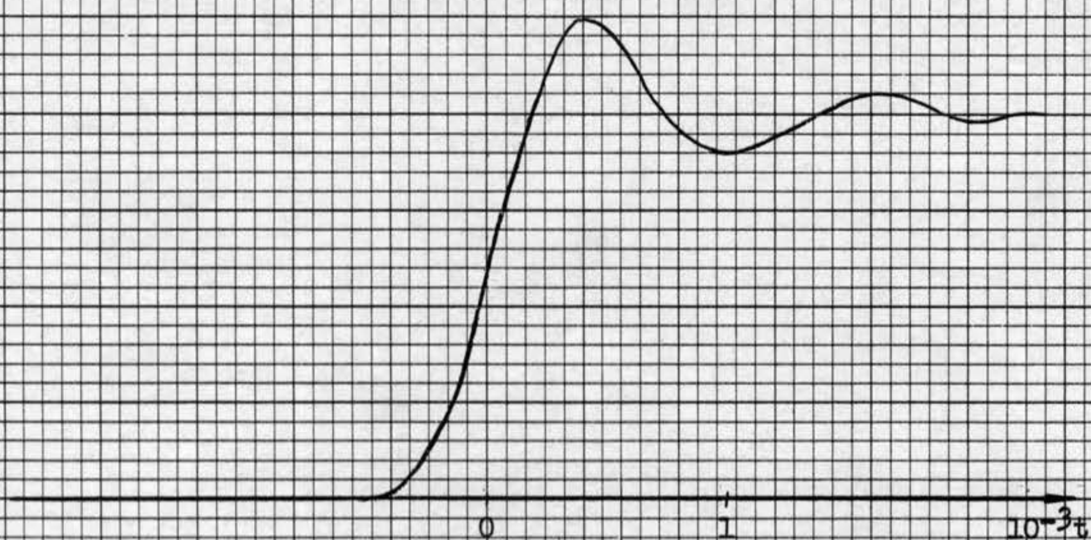

Fig. 18. Transient response of the physical low-pass filter

(c). The low-pass filter with losses

Chapter III showed that the only difference between a filter without losses and a filter with losses is a change of the real parts of the roots of equation 2.14. The low-pass filter showed the following change in s:

physical filter without loss $\quad s = u\omega_1$

physical filter with loss $\quad s = u\omega_1 - \frac{R_1}{L}$

where R_1 is the series loss resistance of the inductance. Even at very low and moderately high frequencies it is possible to build inductances with time constant $\frac{L}{R}$ smaller than $\frac{1}{5}$.

The assumed filter shall have the same data as the low-pass filter without losses and $\frac{L}{R}$, which represents the quality factor of the inductance, shall be assumed to be 0.2.

$$s_1 = u_1\omega_1 - 5.0 = 0.648(6280) - 5.0 = -4070 - 5.0$$

$$s_2 = u_2\omega_1 - 5.0 = 0.176(6280) - 5.0 - j0.86(6280)$$

$$s_3 = u_3\omega_1 - 5.0 = 0.176(6280) - 5.0 - j0.86(6280)$$

The computation shows that the influence on the real part is smaller than one half of a percent.

2. The transient response of the high-pass filter

(a). The ideal high-pass filter

From equation 1.6 there may be derived in the same way as for the low-pass filter the equation:

$$f(t) = \frac{EK}{\pi}\left(\frac{\pi}{2} - Si\,\omega_1 t\right)$$

Figure 19 shows a plot of the transient of the ideal high-pass filter. Again the product EK is assumed to be equal to 1.

(b). <u>The physical high-pass filter without losses</u>

The following data shall be given:

$$f_1 = 1000 \text{ cps}$$

$$\omega_1 = 6280 \text{ sec}^{-1}$$

Again the output terminals shall be connected to a resistance of 600 ohms.

Circuit elements:

$$\omega_1 = \frac{1}{RC}$$

and, therefore: $C = \frac{1}{\omega_1 R} = 0.274 \mu F$

and again from $Z_1 Z_2 = R^2 = \frac{L}{C}$

$$L = R^2 C = 0.36(0.274) = 0.099 \text{ henry}$$

Equation 2.24 gives with $\omega_1 = 6280 \text{ sec}^{-1}$:

$$V_2(t) = 0.865\, e^{-9700t} + 0.62\, e^{-1290t} \cos(6150t + \varphi)$$

$$\varphi = 77^\circ 20'$$

Figure 20 shows a plot of the transient of a physical high-pass filter without loss.

Again the response of the ideal filter shows an oscillation for negative time. The reason is the same as for the low-pass filter.

(c). <u>The high-pass filter with losses</u>

Chapter III gave the following equation for the roots of equation 2.14:

$s = \frac{\omega_1}{u} - \frac{G_1}{C}$ when G_1 is the shunt loss admittance of the condenser. The term $\frac{C}{G_1}$ is the time constant and represents again the

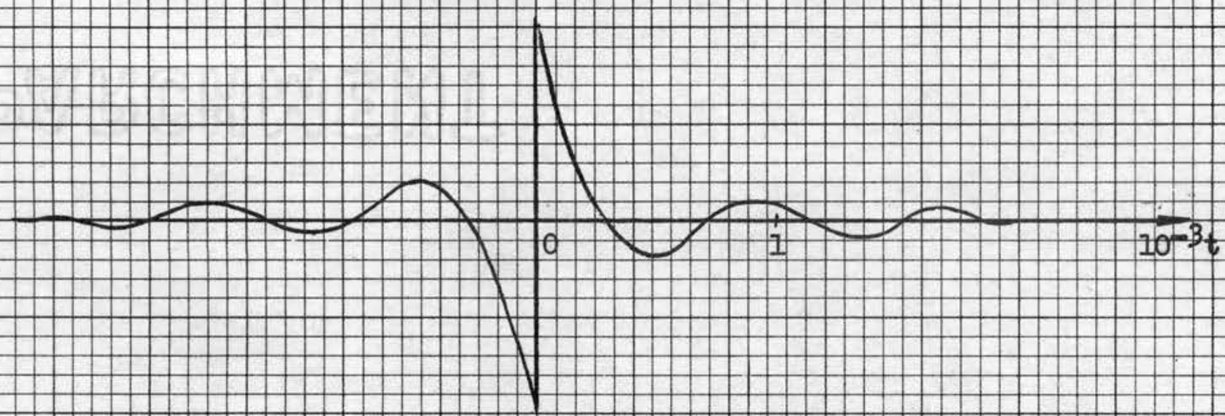

Fig. 19. Transient response of the ideal high-pass filter.

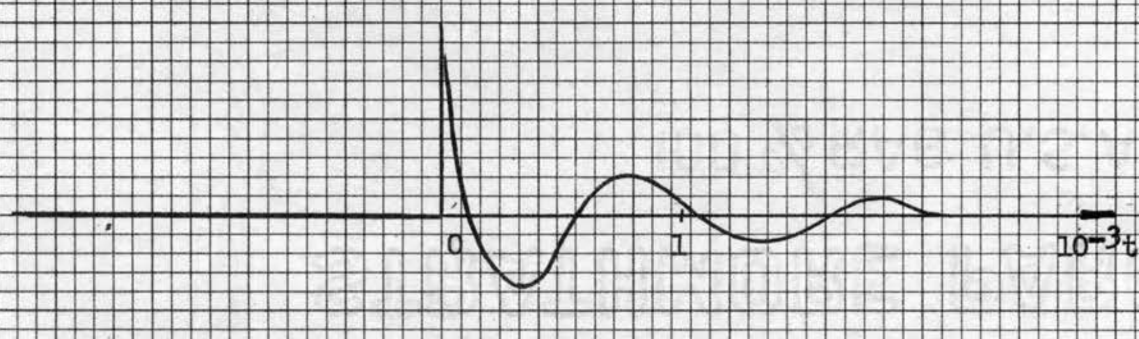

Fig. 20. Transient response of the physical high-pass filter.

quality factor of the condenser. Condensers have usually quality factors which are well above 0.2. Because Z_2 is an inductance which has to have the same quality factor as the condenser this factor will be assumed to be equal to 0.2.

The numerical values for u_1, u_2, and u_3 inserted into the equation for s shows that again the influence of the losses upon the transient response is negligible.

3. The transient response of the band-pass filter

(a). The ideal band-pass filter

The following data shall be assumed:

$f_1 = 4000$ cps $\qquad f_2 = 8000$ cps

$\omega_1 = 25000\ \mathrm{sec}^{-1} \qquad \omega_2 = 50000\ \mathrm{sec}^{-1}$

Neglecting a time delay, equation 4.1 then gives

$$V_2(t) = \frac{EK}{\pi}\,(\mathrm{Si}\,\omega_2 t - \mathrm{Si}\,\omega_1 t)$$

Figure 21 shows this function under the assumption $EK = 1$.

(b). The physical band-pass filter

Chapter II showed that the resonant frequency of the resonant circuits is equal to the geometrical mean of the two limit frequencies.

$$\omega_o^{\,2} = \omega_1\omega_2 = 12.5 \times 10^8 = \frac{1}{L_1 C_1}$$

Either L_1 or C_1 can be freely chosen. Usually L_1 is made as small as possible because the greatest part of the losses of the resonant circuit are included in the inductance. The favorable choice of L_1 is one of the problems in the design of filters which needs most experience and knowledge of the materials which are available. If L_1 is very small the influence of temperature and wiring capacities relative to L_1 is great. Furthermore, it is difficult to manufacture small inductance in mass production with good accuracy. On the other hand it is difficult to maintain a low value in loss resistance as soon as L_1 is very big, because big L_1 requires big iron cores which in return inserts losses as the result of eddy current and hysteresis. A careful choice of the core material can keep these losses small. As an example L_1 will be chosen 0.085 henry, which is obtained with a medium size ring coil of about 2 inches diameter.

$L_1 = 0.085$ henry $\qquad\qquad C_1 = 0.01\,\mu F$

$L_2 = R^2 C_1 = 0.0035$ henry $\qquad\qquad C_2 = 0.235\,\mu F$

The computation of the transient response of the high-pass filter according to the theory developed in Chapter II gives the following three terms:

$$V_2(t) = 0.35\, e^{-2300t} \sin 35400t$$

$$-0.107\, e^{-570t} \cos(38100t - 88^\circ 10')$$

$$-0.0985\, e^{-677t} \cos(32000t - 87^\circ 31')$$

A plot of this function is given in Figure 22. The computation of this response function is a very tedious work. It is readily seen that the main part of the output consists of the first term of the above expression. The second and the third term are much smaller. Their essential purpose is to bring the derivative of the output voltage with respect to time to zero at the time $t = 0$.

The influence of the losses may be computed according to the formulas developed in Chapter III. Again the influence will be found to be negligible.

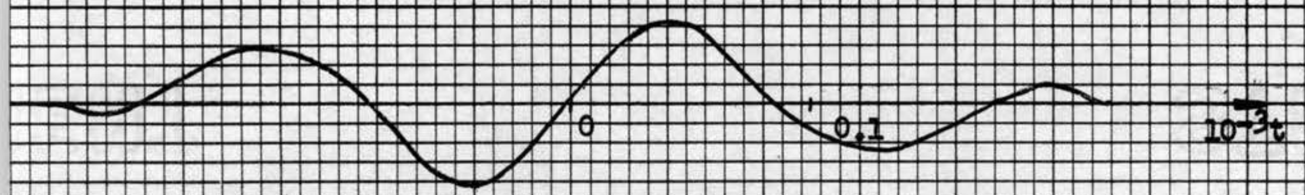

Fig. 21. Transient response of the ideal band-pass filter.

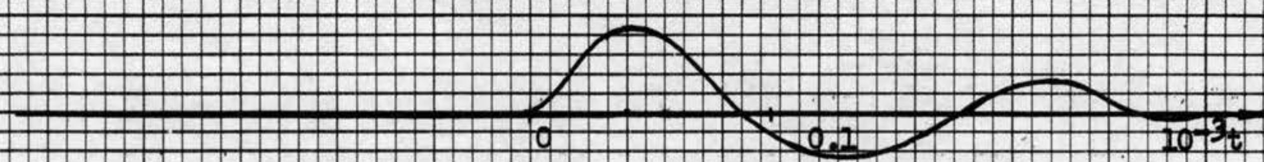

Fig. 22. Transient response of the physical band-pass filter.

Fig. 23. Transient response of the ideal band-elimination filter.

The given examples show that it is possible to substitute an ideal filter for the physical filter, when only the transient response is considered provided the following changes are made:

1. All the oscillations for negative time are to be eliminated.
2. The tangent of the transient response at the time $t = 0$ must be horizontal in the case of the low-pass filter and the band-pass filter.
3. The response of the high-pass filter for negative t must be eliminated.

The given examples show that under these assumptions the ideal filter is a very good approximation. For the band-elimination filter it can be said that the same approximation must be allowed, because of the relation between ideal and physical band-pass and ideal band-pass and ideal band-elimination filter. When the ideal band-pass filter is a good approximation for the physical band-pass filter, the ideal band-elimination filter must be a good approximation for the physical band-elimination filter.

4. Conclusions

The final conclusions that may be derived from the present paper are the following:

(a). The ideal filter as defined in Chapter I is for most practical application a good approximation.

It can be said that for the m-derived filters this approximation is even better, because these filters have a phase and an amplitude characteristic which approaches the characteristics of the ideal filter closer than the basic filters do.

(b). The influence of losses in the circuit elements are negligible for the transient response of the basic filters.

For the steady-state response of the filter the losses have to be kept so small that they have no influence on the transient response. It is, therefore, impossible to insert losses in order to keep the transient response small without disturbing at the same time the discriminative quality of the filter.

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THESIS TITLE: TRANSIENTS IN ELECTRICAL FILTERS

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