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Abstract

It is typically the case in orthogonal image analysis that the pattern matrix is interpreted in an attempt to define the factors underlying the observed variables. It is shown that, unlike other orthogonal factor analytic models, pattern and structure are <u>not</u> the same in orthogonal image analysis. The structure for image analysis is derived for both the full and deficient rank cases. The differences between image pattern and structure, as they relate to the interpretation of factors, are demonstrated in a series of numerical examples.

PATTERN AND STRUCTURE IN APPLIED IMAGE ANALYSIS

The term factor analysis, in the popular usage is generic and refers to a class of procedures designed to determine the structure underlying a set of observed variables. In psychological research, especially in the areas of intelligence and personality assessment, measurements are taken on a large number of observed variables, and it may be desirable to account for the interrelationships among the observed variables in terms of a fewer number of underlying variables. If a reduction in the number of variables is possible, a certain economy is obtained in terms of the number of factors needed to account for a variety of observed psychological variables.

In the history of factor analysis, the predominant model has been the common factor analysis model. This model assumes that the variability of the observed variables can be represented by two parts; the variability common to all of the variables, and the variability which is specific to each variable. The common factor model attempts to account for the variability of the common parts of the observed variables in terms of a set of latent variables called factors. It is further assumed that since the latent variables form a basis for the common parts of the observed variables, there are in general fewer common factors than original variables.

In order to apply the common factor analysis to data, a determination of the unique parts of each of the original variables (the uniquenesses) is made and these uniquenesses are removed from the intercorrelations of the original (standardized) variables. The resultant covariance matrix (called the reduced correlation matrix) is then "factored" to determine the number and composition of the underlying common factors. This model, however, has a very serious limitation. The assumption that the p original variables may be exactly reproduced by r common factors (r<p) and p unique factors leads to a system of p linear equations involving p+r unknowns. The solution of such a system of linear equations is not unique (e.g., Searle, 1966, p. 138). This state of affairs has been refered to as the indeterminacy problem of common factor analysis (Guttman, 1955).

A considerable amount of research in the area of factor analysis has been done on methodology. Much of this research has been involved with finding methods for determining the uniquenesses of the observed variables. Other aspects of the research on the common factor analysis have involved the determination of alternative models which do not suffer the indeterminacy problem.

Image analysis (Guttman, 1953) is one of the alternative models. The fundamental theorem of image analysis involves the determination of the covariance matrix of those parts of the original variables which are linearly predictable from the remaining variables.

The predictable parts of the original variables are called images, and those parts which are not predictable from the other variables are called anti-images. In this formulation, each of the p original variables may be reproduced by adding its respective image and anti-image. An advantage of the image model is that it is completely determinate.

Image analysis, in addition to providing an analogous and determinate alternative to common factor analysis, has also provided a logical method for determining estimates of the common parts (and consequently the unique parts) of the observed variables in common factor analysis. McDonald (1975) has criticized the use of images as estimates of "common parts" and suggests that when images are used, the same basic problems exist as when the communalities are obtained by the methods typical of common factor analysis. There is, however, a difference between employing images as estimates of common parts in the common factor model and choosing image analysis as the basic model. This difference exists because the basic assumptions of common factor analysis and image analysis are different; the assumptions of the common factor model lead to the factor indeterminacy problem, whereas the assumptions of the image model eliminate the indeterminacy problem.

It is the purpose of this paper to detail the common factor analytic model with its assumptions and results and then use it as a basis for the development of the image analysis model. Based on the development of an analogy between the two models it is observed that there are discrepancies in the way these models are applied to data. These discrepancies are identified, and a resolution is provided along with numerical examples. Finally, it is noted that the method suggested to resolve the discrepancies in the application of image analysis posesses certain properties which relate to principal components analysis. These relationships are developed analytically.

Definition of the Models

Three of the most well known and widely used methods of determining the structure underlying a set of observed variables are common factor analysis (e.g., Thurstone, 1947), image analysis (Guttman, 1953), and principal components analysis (Hotelling, 1933). For purposes of definition and comparison, consider the following formulations of the models.

Common factor analysis (CFA). The CFA model is written as

(1) $\underline{z} = \underline{c} + \underline{u} = F\underline{x} + U\underline{v}$,

where $\underline{z}$ is a px1 vector of observed (standardized) variables, $\underline{c}$ is a px1 vector of common parts of the original variables and consists of the product of F, a pxr matrix of unknown common factor loadings (coefficients) and $\underline{x}$, an rx1 vector ($r<p$) of common factors (factor scores). In (1), $\underline{u}$ is a px1 vector of unique parts of the original variables and is the product of U, a diagonal matrix of unknown unique factor coefficients and $\underline{v}$, a px1 vector of unique factors (factor scores).

The orthogonal CFA model makes the following basic assumptions:

(i) $E(\underline{xx}') = I_{(r)}$,

(ii) $E(\underline{vv}') = I_{(p)}$,

(iii) $E(\underline{xv}') = 0$.

There are several consequences which result from these basic assumptions. It is immediately seen that the unique parts are uncorrelated;

(2) $E(\underline{uu}') = E(U\underline{vv}'U') = U^2$ (diagonal).

The common and unique parts are uncorrelated,

(3) $E(\underline{cu}') = E(F\underline{xv}'U') = 0$,

and the common parts have the following covariances

(4) $E(\underline{cc}') = E(F\underline{xx}'F') = FF' = R - U^2$.

From these relationships, the correlation matrix of the original variables may be written

(5) $R = E(\underline{zz}') = FF' + U^2$,

which has been labelled the fundamental theorem of CFA.

It is well known that the CFA model has two inherent problems. The first of these is referred to as the rotation or identifiability problem. The identifiability problem may be stated by noting that (5) may be written

(6) $R - U^2 = FF' = FTT'F' = F^*F^{*\prime}$,

where $TT' = T'T = I$ and $F^* = FT$. Thus, there are infinitely many choices of weights F^* which satisfy the model. The identifiability problem is not unique to CFA; it is a problem in all factor analytic models. It is usually resolved in practice by choosing a matrix F^* which satisfies Thurstone's simple structure criteria (Thurstone, 1947, p. 335), which, in the orthogonal case, is approximated by Kaiser's (1958b) Varimax criterion.

The second problem in common factor analysis is that the solution for the common and unique factors is not unique. The nonuniqueness of the common and unique factors is referred to as the indeterminacy problem (Guttman, 1955; Schönemann, 1971), and arises from the assumption that the unique factors are uncorrelated. Assumption (ii), in conjunction with (iii), gives rise to the indeterminacy by requiring that the r common parts, $\underline{c}$, and the p unique parts, $\underline{u}$, be solved for in a system of p linear equations. It is well known that a system of p linear equations in p+r unknowns can not have a unique solution. Indeterminacy is a serious problem for the CFA model because there is no known satisfactory resolution of the problem. There are, however, alternative models which yield determinate derived variables which are similar to the unobservable, indeterminate common factors. Two such alternative models are now discussed.

Image analysis (IA). The IA model (Guttman, 1953) is written as

$$(7)\quad \underline{z} = \underline{m} + \underline{a},$$

where $\underline{m}$ is a px1 vector of images -- the portions of the original variables linearly predictable from the p-1 remaining variables, and $\underline{a}$ is a px1 vector of anti-images -- the portions of the original variables not predictable from the p-1 other variables. The vector $\underline{m}$ may, therefore, be written

$$(8)\quad \underline{m} = W\underline{z},$$

where W is a pxp matrix of row-wise least squares regression weights for predicting the jth (j=1,2,...,p) observed variable from the p-1 other variables. Guttman (1940) showed that

$$(9)\quad W = (R - S^2)R^{-1} = (I - S^2R^{-1}),$$

where R is the correlation matrix of the original variables, and

$$(10)\quad S^2 = (\text{Diag } R^{-1})^{-1}.$$

The vector of anti-images is obtained from

$$(11)\quad \underline{a} = (I - W)\underline{z} = S^2R^{-1}\underline{z}.$$

The matrix of covariances of the images, G, is given by

$$(12)\quad G = E(\underline{mm}') = E(W\underline{zz}'W') = WRW' = R - 2S^2 + S^2R^{-1}S^2.$$

Since G is, in general, non-diagonal, it is possible to obtain a set of variables, $\underline{d}$, that form an orthogonal basis for the images. A basis for the images may be expressed as follows:

$$(13)\quad \underline{m} = B\underline{d},$$

where B is a pxp matrix of coefficients, and $\underline{d}$ is a px1 orthogonal basis scaled so that $E(\underline{dd}') = I$. Equation (13) implies that

$$(14)\quad G = E(\underline{mm}') = E(B\underline{dd}'B') = BB'.$$

However, a basis for the images is not unique since we may write (14) as

$$(15)\quad G = BB' = BTT'B' = B^*B^{*\prime},$$

where $TT' = T'T = I$. A logical basis, unique by restriction, is provided by a principal axes decomposition of the image covariance matrix G. The coefficients, B, of the images relative to the principal axes basis are provided by the decomposition

$$(16)\quad G = QD_{\lambda}Q' = BB',$$

where Q is a matrix whose columns are the eigenvectors of G, D_λ is a diagonal matrix whose entries are the eigenvalues of G, and $B = QD_\lambda^{\frac{1}{2}}$. The principal axes decomposition of G in (16) has been suggested and employed in practice (e.g., Kaiser, 1958a, 1963; Mulaik, 1972, p. 191).

Principal components analysis (PCA). A second alternative to the indeterminate CFA model is principal components analysis (Hotelling, 1933). The PCA model is given by

$$(17)\quad \underline{z} = \underline{l} + \underline{e} = V\underline{y} = V_1\underline{y}_1 + \underline{e},$$

where $V = \left[V_1 \mid V_2\right]$ is a pxp matrix of weights, and $\underline{y}' = \left[y_1' \mid y_2'\right]$ is a px1 vector of orthogonal principal components scaled so that $E(\underline{yy}') = I$, $\underline{l}$ is a px1 vector which contains a linear combination of the important principal components of $V\underline{y}$ (i.e., $V_1\underline{y}_1$), and $\underline{e}$ represents a px1 vector containing a linear combination of the trivial components of $V\underline{y}$ (i.e., $V_2\underline{y}_2$). From this formulation, it can be seen that $\underline{l}$ and $\underline{e}$ are uncorrelated by noting that

$$(18)\quad E(\underline{le}') = E(V_1\underline{y}_1\underline{y}_2'V_2') = 0,$$

because of the definition of $\underline{y}' = \left[\underline{y}_1' \mid \underline{y}_2'\right]$.

The PCA model provides a decomposition of R as

$$(19)\quad R = E(\underline{zz}') = E(V\underline{yy}'V') = VV',$$

where $V = LD_\mu^{\frac{1}{2}}$, and where L is a matrix whose columns are the eigenvectors of R and D_μ is a diagonal matrix of eigenvalues of R. Even though the form of VV' in (19) represents a unique decomposition of R, there are infinitely many choices of weights, $V^* = VT$ $(TT' = T'T = I)$, which will satisfy the model. But, as in the preceding two models, the particular choice of weights are often those which satisfy a simple structure criterion (e.g., Varimax).

Comparison of the Models

From the preceding explication of models, several points of comparison can be made. First, as previously mentioned, all of the models are subject to the rotation or identifiability problem. The matrices which satisfy the simple structure criterion, however, are the usual choices for the weight matrices. A second point of comparison involves the nature of the "residual" parts in each of the models. In the CFA codel, the residual parts are the unique parts of the original variables, and in (2) it was shown that the unique parts are uncorrelated and that their variance-covariance matrix, U^2, is diagonal and full rank. It is precisely this consequence of (ii) which causes the indeterminacy problem in the CFA model.

In the IA model, the residual parts are represented by the anti-images, $\underline{a}$. Unlike the unique parts in CFA, the anti-images are correlated as may be seen from (11), and noting that

$$(20)\quad E(\underline{aa}') = E((S^2R^{-1}\underline{z})(\underline{z}'R^{-1}S^2)) = S^2R^{-1}S^2.$$

This covariance matrix, like U^2, is full rank under the assumption that R is full rank, but is in general non-diagonal.

In PCA, the residuals, $\underline{e}$, are correlated. From the partitioning of V and $\underline{y}$, we see that $\underline{e} = V_2\underline{y}_2$, and

$$(21)\quad E(\underline{ee}') = E(V_2\underline{y}_2\underline{y}_2'V_2') = V_2V_2'.$$

This covariance matrix is deficient in rank, since the rank of V_2 is equal to p-r, the number of rejected (trivial) components of V. Thus, the relationships among the residual parts in each of the three models are different. In CFA the residuals are uncorrelated and have a full rank covariance matrix. In IA the residuals, $\underline{a}$, are correlated and yield a covariance matrix that may, or may not be full rank. The residuals, $\underline{e}$, in PCA are correlated and always have a deficient rank covariance matrix. These conditions are sufficient for avoiding an indeterminacy in IA and PCA.

A final point of comparison of the three models involves the relationships between the residual and the nonresidual parts of the original variables. In CFA, the common and unique parts are uncorrelated. This relationship was shown in (3) as a consequence of (iii). In IA, only the image and the anti-image of the jth variable are uncorrelated. In general, the ith image and the jth anti-image are correlated. From

$$(22)\quad E(\underline{ma}') = E((I - S^2R^{-1})\underline{zz}'(R^{-1}S^2)) = S^2 - S^2R^{-1}S^2,$$

it may be seen that the jth image and the jth anti-image are uncorrelated because the diagonal of $S^2R^{-1}S^2 = S^2$ implying that the diagonal elements of the covariance matrix $S^2 - S^2R^{-1}S^2$ are zero. The off-diagonal elements of $S^2 - S^2R^{-1}S^2$ are generally non-zero. Finally, regarding PCA, it was shown in (18) that the linear combinations of the important and the trivial principal components are uncorrelated (i.e., $E(\underline{y}_1\underline{y}_2') = 0$). From the relationships between the non-residual and the residual parts in each of the models, we see that in CFA, the common and unique parts are uncorrelated, the non-residual and residual parts in PCA are also uncorrelated, but images and anti-images are correlated. These relationships facilitate a comparison of the structure matrices in each of the models. This comparison is developed in the next section.

Pattern and Structure Matrices

In the CFA model, the common factor weight matrix, F, which partially reproduces observed variables as linear combinations of common factors, is called the factor pattern or the pattern matrix (Thurstone, 1947; see also Mulaik, 1972, p. 101). The factor structure, or the structure matrix, is the matrix of cross-correlations or cross-covariances among the observed variables and the (unobservable) common factors. Although these definitions pertain primarily to the CFA model, they may be generalized to the IA and PCA models.

In the three models presently under discussion, the pattern matrices are readily identified. From (1), the matrix F in the CFA model is the pattern matrix. From (17) the pattern matrix for components analysis can be seen to be V_1. From these relationships, it may be implied from combining (7) and (13) as

(23) $\underline{z} = B\underline{d} + \underline{a}$,

that the matrix B is the pattern matrix for image analysis. Thus, the pattern matrices for the three models are quite similar. One might be justified, however, in questioning the value of pattern matrices when it comes to the interpretation of underlying factors.

We note that the CFA model in (1) may be rewritten as

$$(24)\ \underline{z} - \underline{u} = F\underline{x},$$

and the PCA model in (17) may be rewritten

$$(25)\ \underline{z} - \underline{e} = V_1\underline{y}_1.$$

Thus, we see that the pattern matrices, F and V_1, are of little interpretative value in that they provide the weights on one set of latent or derived variables which reproduce yet another set of latent or derived variables. The only redeeming quality these matrices have is that in orthogonal CFA and PCA they are also structure matrices. We may see this by noting that in orthogonal CFA

$$(26)\ E(\underline{zx}') = E((F\underline{x} + \underline{u})\underline{x}') = E(F\underline{xx}') + E(\underline{ux}') = F,$$

and in PCA

$$(27)\ E(\underline{zy}_1') = E((V_1\underline{y}_1 + \underline{e})\underline{y}_1') = E(V_1\underline{y}_1\underline{y}_1' + E(\underline{ey}_1') = V_1.$$

An arguement can be made that a pattern only has interpretive value when it is simultaneously a structure.

Indeed, Brogden (1969) discusses pattern and structure in the context of oblique factor analysis and draws a clear distinction between the use of pattern and structure in factor identification. Although Brogden's paper was concerned with oblique factor analysis, his major point is relevant to any situation where pattern and structure are distinct. Brogden's major point is that given a knowledge of the original variables and no knowledge of the underlying variables (factors), the underlying variables are best interpreted by considering the correlations between underlying variables and the observed variables. Since the coefficients in the pattern matrix are weights (specifically, regression weights) on latent variables which reproduce a portion of the observed variables, they possess dubious interpretive value. The coefficients in the structure matrix, however, are directly relevant to the interpretation and/or definition of the underlying variables since they provide the only direct link between the observed variables and the underlying variables.

It is easy to show that the pattern and structure are not the same in image analysis as they are in CFA and PCA. To show this (23) may be written in the form

(28) $\underline{z} - \underline{a} = B\underline{d}$.

Thus, B, the pattern matrix in image analysis, like F and V_1, is a matrix of weights on one set of derived variables which reproduce another set of derived variables. However, it may be shown that B, unlike F and V_1, is <u>not</u> a structure matrix. To show this we use (13) to derive

(29) $E(\underline{md}') = E(B\underline{dd}') = B$.

Thus, B, is a matrix which contains covariances (correlations) between two sets of derived variables. The elements of B can be interpreted as covariances; however, the elements are not covariances among observed and derived variables, and as a result B is not a structure.

The suggestion that B is not a structure matrix for IA leads to the development of the image structure matrix. From (8) and (13) we may write

(30) $\underline{m} = W\underline{z} = B\underline{d}$;

hence,

(31) $\underline{z} = W^{-1}B\underline{d}$.

From this we obtain the structure matrix, A, as

(32) $A = E(\underline{zd}') = E(W^{-1}B\underline{dd}') = W^{-1}B$.

Because of the scaling assumptions on the variables $\underline{z}$ and $\underline{d}$, the structure matrix, A, contains the correlations among observed and derived variables. It may be the case, however, that W does not have an inverse even when the correlation matrix, R, is full rank. This may be seen by noting from (9) that $W^{-1} = R(R - S^2)^{-1}$. Thus, the existence of W^{-1} depends upon a full rank $(R - S^2)$, which may not be the case even when R is full rank. In the situation where W^{-1} does not exist, we may still obtain the structure matrix, A, for image analysis. To do this we make use of (13) and (8) to obtain

(33) $\underline{d} = (B'B)^{-1}B'\underline{m} = (B'B)^{-1}B'W\underline{z}$,

where $(B'B)^{-1}B'$ is the (uniquely determined) left-hand inverse of B. From this result we may obtain the structure matrix, A, as

(34) $A = E(\underline{zd}') = E(\underline{zz}'W'B(B'B)^{-1}) = RW'B(B'B)^{-1}$.

From (16), B may be written as $QD_\lambda^{\frac{1}{2}}$ and $(B'B)^{-1} = D_\lambda^{-\frac{1}{2}}Q'QD_\lambda^{-\frac{1}{2}} = D_\lambda^{-1}$. Equation (34) then reduces to

(35) $A = (R - S^2)BD_\lambda^{-1}$,

where D_λ^{-1} is a diagonal matrix of reciprocals of the non-zero eigenvalues of G. In the case where W and G are not full rank, the structure matrix, A, may still be obtained since D_λ can be restricted to contain only the non-zero eigenvalues of G, and Q will then be composed of the eigenvectors of G corresponding to the non-zero eigenvalues. Therefore, A, in the deficient rank case will have order pxr, where r is the rank of G ($r<p$).

At this point there is a further set of relationships which may be developed for the case of full rank R and G. From (32), (14), and (12) it may be seen that

$$(36)\ AA' = W^{-1}BB'W'^{-1} = W^{-1}GW'^{-1} = W^{-1}WRW'W'^{-1} = R,$$

which implies that A is a Gram factor of the correlation matrix, R. It was shown in the development of the principal components model, (19), that V is also a Gram factor of R. From the relationships in (36) and (19) it may be noted that A and V are both matrices of correlations among orthogonal bases and original variables, or equivalently, A and V are the weight coefficients for the two different bases for $\underline{z}$. It must therefore be the case that A and V differ by an orthogonal rotation. That is,

$$(37)\ VT_1 = A,$$

and

$$(38)\ AT_2 = V,$$

where $T_1T_1' = T_1'T_1 = T_2T_2' = T_2'T_2 = I_{(p)}$.

The intercorrelation matrix $R_{yd} = E(\underline{y}\underline{d}')$ for the two sets of components, $\underline{y}$ and $\underline{d}$, is a matrix of projections of the principal component basis, $\underline{y}$, on the image basis, $\underline{d}$, and is equal to the orthonormal pxp transformation matrix T_1. By similar reasoning the transformation matrix T_2 can be shown to equal $E(\underline{dy}') = R'_{yd} = T_1'$. In order to obtain the transformation matrices T_1 and T_2, $\underline{d}$ and $\underline{y}$ must be obtained and the expected value of their outer product taken. From (17) and (31) we see that

$$(39)\quad E(\underline{yd}') = V^{-1}W^{-1}B = V^{-1}A = T_1.$$

We now show that T_1 is orthonromal;

$$(40a)\quad T_1T_1' = V^{-1}AA'V'^{-1} = V^{-1}RV'^{-1} = V^{-1}V\ V'V'^{-1} = I,$$

$$(40b)\quad T_1'T_1 = A'V'^{-1}V^{-1}A = A'R^{-1}A = A'A'^{-1}\ A^{-1}A = I,$$

and that T_1 and T_2 perform the desired transformations;

$$(41a)\quad VT_1 = VV^{-1}A = A,$$

$$(41b)\quad AT_2 = AT_1' = AA'V'^{-1} = RV'^{-1} = VV'V'^{-1} = V.$$

Thus, in the full rank case the structure matrix, V, obtained from principal components analysis, is linearly related to the structure matrix, A, obtained from image analysis.

This relationship between A and V is of more theoretical than applied import because in applied work the researcher is usually interested in choosing r (r<p) components which satisfactorily account for the structure underlying the p observed variables in $\underline{z}$. In the situation where an approximate reduced rank solution is desired, the relationship between A and V may be examined by partitioning V and A as follows:

$$(42)\quad V = \left[V_1 \mid V_2\right],$$

and

$$(43)\quad A = \left[A_1 \mid A_2\right],$$

where V_1 and A_1 are both pxr and represent the desired approximate reduced rank solutions. Both V_2 and A_2 are dimensioned px(p-r) and represent the rejected components of V and A. Using the partitioning to rewrite (37) we obtain

$$(44)\quad VT_1 = \left[V_1 \mid V_2\right] T_1 = \left[A_1 \mid A_2\right]$$
$$= \left[V_1 \mid V_2\right]\left[\begin{array}{c|c} T_{11} & T_{12} \\ \hline T_{21} & T_{22}\end{array}\right] = \left[A_1 \mid A_2\right]$$
$$= \left[V_1T_{11} + V_2T_{21} \mid V_1T_{12} + V_2T_{22}\right] = \left[A_1 \mid A_2\right],$$

where T_{11}, T_{12}, T_{21}, and T_{22} represent the appropriate partitioning of the transformation matrix, T_1. From these results we see that A_1 (the r retained components of the basis, d) depends not only upon the r components retained from principal components analysis, V_1, but also on the p-r rejected components, V_2. Thus, the r components of V_1 cannot be used to perfectly reproduce A_1 by means of an orthogonal transformation. Furthermore, the matrix T_{11} in (44) is not guaranteed to be orthogonal, in which case $V_1T_{11} = A_1^*$ would not be an orthogonal rotation of V_1, and A_1^* does not, in general, equal A_1.

Although we have argued against the interpretation of B, it is the matrix which is usually interpreted in practice (e.g., Kaiser, 1963; Mulaik, 1972, p. 191; Veldman, 1967, p. 218). It was pointed out earlier, (28), that B is a covariance matrix. Due to the unbounded nature of covariances it might be preferable to rescale the entries of B into correlations if one is interested in interpreting B. The necessary rescaling of B may be accomplished by premultiplying

B by a diagonal matrix, D, whose jth diagonal entry is the reciprocal of the multiple correlation coefficient for predicting the jth variable from the p-1 other variables. Hence,

$$(45)\quad D = (I - S^2)^{-\frac{1}{2}},$$

and the rescaled matrix, designated B*, is given as

$$(46)\quad B^* = DB = (I - S^2)^{-\frac{1}{2}}B.$$

A Numerical Comparison of Pattern and Structure

Because the matrices B and A are different in composition, it seems likely that interpretations based on these matrices would be different. These differences are demonstrated in a series of examples. In each of the examples, the image covariance matrix, G, was obtained from the correlation matrix, R. A principal axes decomposition of G provided the image pattern matrix B. The rescaled pattern matrix, B*, and the image structure matrix, A, were then obtained as in (46) and (32). (Some of the important matrices resulting from the intermediate computational steps are labelled and presented in the Appendix.) Typically in applied factor or principal components analysis, a subset of the factors obtained are rotated and interpreted. Although there are many methods for determining how many components to rotate, we arbitrarily chose to rotate three. For purposes of interpretation in applied work, a variable whose correlation with a given factor exceeds .30 is generally considered to contribute to the definition of that factor. This conventional rule was employed in the comparison of the various matrices in the examples. Since the purpose of this paper is to point out the differences between the interpretation of the pattern matrix, B, and the structure matrix, A, the examples focus on these two matrices. The rescaled pattern matrices, B*, are also provided.

The data employed in example one were the intercorrelations of the eleven subscales of the Wechsler Adult Intelligence Scale (Wechsler, 1955, p. 16). These intercorrelations, based on the data from 150 males and 150 females, are presented in the matrix M in Table 1.

Insert Table 1 about here

The upper triangular portion of M contains the correlations between the observed variables. The diagonal elements are the squared multiple correlations for predicting each variable from the remaining variables (i.e., $I - S^2$), and the lower triangular portion contains the image covariances. The matrix S^2 may be obtained by taking 1 minus the diagonal elements of M. Thus, Table 1 contains all of the basic information necessary to conduct an image analysis. A principal axes decomposition of G yielded the matrix B, which was rescaled into B*, and the matrix A was obtained as in (32). The three matrices resulting from a Varimax rotation of B, B*, and A are presented in Table 2 and are subscripted with an r.

Insert Table 2 about here

From Table 2 it may be seen that the entries in B_r suggest a solution which contains two factors (Factors I and II) which are each defined by all but one of the variables, and a third factor defined by a small loading on variable 9. A solution which contains two general type factors is of questionable utility. A considerably different solution is obtained by an examination of A_r. In A_r, Factor I is defined by all of the variables but 9 and 11, and Factor II is defined by variables 1, 8, 9, 10, and 11. And, Factor III is defined by variables 2, 7, 8, 9, and 10.

An examination of B_r or A_r suggests that an oblique solution would probably be needed to obtain simple structure, but the point to be made here is that B_r and A_r lead to different interpretations concerning the nature and the composition of the variables underlying the observed variables.

The second example was taken from Veldman (1967, p. 222) in which a traditional image analysis was performed on a set of real, Self-Report Inventory (Bown, 1961) data (N=16). The upper triangular portion of the correlation matrix and the lower triangular portion of the image covariance matrix, to include the diagonal, are presented in Table 3 as the matrix, M. As in example one, the matrices B, B*, and A were computed and since Veldman's example contained three rotated factors, we also rotated three factors to facilitate comparison. The resultant matrices, B_r, B^*_r, and A_r, are presented in Table 4.

Insert Tables 3 and 4 about here

In Table 4, Factor I in B_r is defined by all of the variables except variables 4, 6, and 7, with variable 2 correlating highest with this factor. In A_r, however, variable 1 does not contribute to the interpretation of Factor I. There are no noteworthy differences between the interpretations of Factor II given B_r and A_r. This is not the case with Factor III. In B_r, Factor III is defined by variables 3, 4, 5, and 6. In A_r, however, only variables 3 and 7 contribute to the definition of Factor III. The interpretation of Factor III, based on B_r is considerabley different from the interpretation of Factor III provided by A_r. Additionally, A_r looks "nicer" in terms of simple structure. In A_r only two variables (3 and 7) load on two or more factors, whereas in B_r, 4 of the 8 variables load on two or more of the factors.

The third and final example was a hypothetical example taken from Harman (1967, p. 88). As in the preceding two examples, the correlations and image covariances were obtained and are presented as M in Table 5. The matrices B, B*, and A were computed and the matrices B_r, B^*_r, and A_r, resulting from the rotation of three factors, are presented in Table 6.

Insert Tables 5 and 6 about here

It may be noted from Table 6 that Factor I in B_r is defined by meaningful loadings on the first three variables. Factor I in A_r is defined by the same variables, 1, 2, and 3, with slightly larger loadings. Regarding Factor II, in B_r we see that it is defined by all six variables, whereas in A_r, Factor II is defined by variables 1, 4, 5, and 6. Hence, there is a considerable difference in the interpretation of that factor. Finally, Factor III in B_r and in A_r is defined by variable 3.

In this last example there is a difference in the identification of Factor II, and as in the second example, A_r presents a better overall appearance cf simple structure than does B_r.

Summary and Conclusions

In the present paper we have presented the basic models for common factor analysis, image analysis, and principal components analysis. We have emphasized the distinction between pattern (a matrix of weights on underlying variables which reproduce portions of the original variables) and structure (a matrix of covariances among the observed variables and the underlying variables). It is well known that pattern and structure are the same for orthogonal common factor analysis and principal components analysis. Therefore, identification or definition of the factors based on the pattern-structure matrix is an acceptable practice in these two models. For orthogonal image analysis, however, the pattern is <u>not</u> simultaneously a structure. An image structure matrix was derived, and it was argued that structure, not pattern, is most appropriate for interpreting the structure of observed data. Since the pattern and structure are different in image analysis, it is likely that the interpretations of the derived variables based on these matrices would also be different. The differences which result from interpreting image pattern versus image structure were demonstrated in three numerical examples. In all of the examples there was an important difference in the interpretation of at least one of the factors obtained. It is recommended that for applied image analysis the structure matrix as derived in this paper be interpreted because only this matrix allows one to determine the relationships between the derived variables (the image basis) and the original variables. It was also shown that, in the case of full rank G and R, the image structure matrix, A, is a Gram factor of the correlation matrix, and therefore, is linearly related to the principal components structure matrix (also a Gram factor of the correlation matrix).

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Table 1

Matrix of Correlations and Image Covariances for

11 WAIS Subscales

$$M = \begin{bmatrix}
(.76) & .70 & .66 & .70 & .53 & .81 & .57 & .67 & .58 & .62 & .45 \\
.64 & (.59) & .49 & .62 & .40 & .73 & .44 & .56 & .49 & .57 & .43 \\
.57 & .50 & (.49) & .55 & .49 & .59 & .43 & .50 & .51 & .49 & .37 \\
.67 & .58 & .51 & (.60) & .46 & .74 & .53 & .56 & .52 & .52 & .39 \\
.49 & .42 & .39 & .43 & (.35) & .51 & .41 & .39 & .39 & .47 & .30 \\
.73 & .64 & .58 & .65 & .49 & (.76) & .60 & .61 & .53 & .62 & .43 \\
.54 & .49 & .44 & .48 & .37 & .53 & (.44) & .48 & .47 & .51 & .44 \\
.60 & .54 & .50 & .53 & .41 & .60 & .47 & (.56) & .62 & .57 & .54 \\
.56 & .49 & .45 & .48 & .38 & .55 & .45 & .53 & (.56) & .58 & .61 \\
.60 & .51 & .48 & .54 & .39 & .58 & .46 & .53 & .51 & (.54) & .52 \\
.47 & .39 & .37 & .41 & .31 & .44 & .36 & .45 & .45 & .44 & (.45)
\end{bmatrix}$$

Table 2

Rotated Pattern and Structure Matrices for

11 WAIS Subscales

Matrix	Variable	Factor I	Factor II	Factor III
	1	.74	.44	.12
	2	.65	.32	.24
	3	.55	.34	.19
	4	.68	.37	.08
	5	.47	.27	.16
B_r	6	.75	.38	.17
	7	.49	.34	.25
	8	.49	.50	.25
	9	.38	.54	.33
	10	.50	.50	.15
	11	.27	.61	.07
	1	.84	.50	.13
	2	.84	.42	.31
	3	.79	.49	.27
	4	.87	.47	.11
	5	.80	.46	.28
B^*_r	6	.86	.43	.19
	7	.74	.52	.38
	8	.66	.66	.33
	9	.51	.72	.44
	10	.68	.69	.20
	11	.40	.90	.10
	1	.84	.34	.21
	2	.66	.12	.50
	3	.61	.28	.27
	4	.77	.29	.15
	5	.53	.20	.23
A_r	6	.89	.18	.29
	7	.47	.26	.48
	8	.43	.51	.49
	9	.27	.57	.64
	10	.48	.57	.30
	11	.19	.86	.17

Table 3

Matrix of Correlations and Image Covariances

for Veldman's Data

$$M = \begin{bmatrix} (.61) & .18 & .47 & .27 & .24 & -.39 & .56 & -.03 \\ .30 & (.68) & .62 & .55 & .55 & -.22 & .02 & .39 \\ .30 & .42 & (.61) & .26 & .32 & -.05 & .27 & .31 \\ .21 & .37 & .36 & (.41) & .35 & -.37 & .14 & .26 \\ .08 & .40 & .32 & .32 & (.56) & -.21 & -.30 & .11 \\ -.25 & -.16 & -.21 & -.27 & -.15 & (.33) & -.28 & -.11 \\ .34 & -.02 & .22 & .10 & -.05 & -.19 & (.56) & -.01 \\ .05 & .31 & .19 & .19 & .18 & -.01 & .02 & (.22) \end{bmatrix}$$

Table 4

Rotated Pattern and Structure Matrices for

Veldman's Data

Matrix	Variable	Factor I	Factor II	Factor III
B_r	1	.33	.66	-.00
	2	.76	.09	.28
	3	.36	.33	.47
	4	.28	.17	.49
	5	.32	-.11	.60
	6	-.02	-.33	-.34
	7	-.15	.67	.10
	8	.35	-.01	.15
B^*_r	1	.43	.85	-.00
	2	.92	.11	.35
	3	.46	.43	.61
	4	.44	.27	.77
	5	.43	-.15	.80
	6	-.03	-.58	-.60
	7	-.20	.89	.13
	8	.75	-.03	.31
A_r	1	.15	.90	.00
	2	.92	.18	-.03
	3	.59	.36	.52
	4	.58	.30	.07
	5	.77	-.06	.02
	6	-.21	-.48	-.02
	7	-.25	.80	.38
	8	.43	.01	.06

Table 5

Matrix of Correlations and Image Covariances

Harman's Illustrative Example

$$M = \begin{bmatrix} (.66) & .72 & .75 & .49 & .42 & .28 \\ .64 & (.66) & .78 & .42 & .36 & .24 \\ .61 & .61 & (.69) & .35 & .30 & .20 \\ .38 & .37 & .39 & (.32) & .42 & .28 \\ .34 & .32 & .33 & .25 & (.25) & .24 \\ .23 & .22 & .22 & .18 & .16 & (.12) \end{bmatrix}$$

Table 6

Rotated Pattern and Structure Matrices for

Harman's Illustrative Example

Matrix	Variable	Factor I	Factor II	Factor III
B_r	1	.67	.41	.20
	2	.69	.35	.22
	3	.53	.37	.53
	4	.23	.46	.20
	5	.22	.42	.11
	6	.14	.30	.06
B^*_r	1	.82	.50	.24
	2	.85	.43	.27
	3	.63	.44	.63
	4	.40	.81	.35
	5	.43	.84	.22
	6	.42	.87	.18
A_r	1	.83	.43	-.00
	2	.91	.21	.06
	3	.80	.16	.57
	4	.20	.83	.16
	5	.21	.71	.02
	6	.12	.53	-.01

APPENDIX

INTERMEDIATE COMPUTATIONAL MATRICES

INTERMEDIATE COMPUTATIONAL MATRIX

R INVERSE

FOR 11 WAIS SUBSCALES

4.170	-0.590	-0.750	-0.330	-0.260	-1.450	-0.210	-0.700	-0.170	-0.160	0.140
-0.590	2.460	0.060	-0.270	0.080	-0.990	0.230	-0.130	0.000	-0.300	-0.180
-0.750	0.060	1.960	-0.190	-0.310	-0.110	0.020	-0.020	-0.280	-0.050	0.000
-0.330	-0.270	-0.190	2.520	-0.110	-0.970	-0.210	-0.150	-0.230	0.090	0.080
-0.260	0.080	-0.310	-0.110	1.550	-0.170	-0.120	0.060	-0.030	-0.270	0.030
-1.450	-0.990	-0.110	-0.970	-0.170	4.230	-0.550	-0.090	0.160	-0.330	0.090
-0.210	0.230	0.020	-0.210	-0.120	-0.550	1.780	-0.040	-0.090	-0.200	-0.260
-0.700	-0.130	-0.020	-0.150	0.060	-0.090	-0.040	2.290	-0.490	-0.190	-0.370
-0.170	0.000	-0.280	-0.230	-0.030	0.160	-0.090	-0.490	2.250	-0.350	-0.680
-0.160	-0.300	-0.050	0.090	-0.270	-0.330	-0.200	-0.190	-0.350	2.150	-0.310
0.140	-0.180	0.000	0.080	0.030	0.090	-0.260	-0.370	-0.680	-0.310	1.820

INTERMEDIATE COMPUTATIONAL MATRIX

B

FOR 11 WAIS SUBSCALES

0.850	-0.110	-0.090	-0.080	-0.070	-0.020	-0.020	0.020	-0.040	0.000	0.020
0.740	-0.120	0.070	-0.080	0.050	0.060	0.080	0.010	-0.020	0.020	-0.010
0.680	-0.050	0.030	0.100	0.090	-0.060	-0.060	0.040	-0.040	0.000	0.000
0.760	-0.130	-0.090	-0.050	0.030	0.000	-0.020	-0.040	0.000	-0.020	-0.020
0.570	-0.060	0.030	0.030	-0.070	-0.140	0.050	-0.040	0.010	0.010	0.000
0.840	-0.160	-0.020	0.130	-0.070	0.080	0.020	0.020	0.040	0.000	0.000
0.640	0.000	0.100	-0.070	0.050	-0.040	0.000	0.030	0.060	-0.010	0.010
0.730	0.110	0.050	0.060	0.070	0.030	0.030	-0.060	-0.020	-0.010	0.010
0.690	0.230	0.120	-0.020	-0.110	0.020	-0.040	0.010	-0.020	-0.010	-0.010
0.710	0.100	-0.050	-0.020	0.030	0.030	-0.090	-0.030	0.030	0.030	0.000
0.580	0.300	-0.130	0.010	0.020	-0.020	0.070	0.030	0.010	0.000	0.000

INTERMEDIATE COMPUTATIONAL MATRIX

B STAR

FOR 11 WAIS SUBSCALES

0.980	-0.120	-0.100	-0.090	-0.080	-0.020	-0.020	0.020	-0.040	-0.010	0.020
0.970	-0.160	0.090	-0.110	0.070	0.070	0.100	0.020	-0.030	0.030	-0.010
0.970	-0.080	0.040	0.150	0.130	-0.090	-0.080	0.060	-0.060	0.000	-0.010
0.970	-0.170	-0.120	-0.070	0.030	0.000	-0.030	-0.050	0.000	-0.020	-0.030
0.950	-0.100	0.050	0.060	-0.110	-0.230	0.080	-0.070	0.020	0.020	0.000
0.960	-0.180	-0.030	0.150	-0.080	0.100	0.020	0.020	0.040	0.000	0.000
0.970	-0.010	0.150	-0.110	0.080	-0.060	0.000	0.050	0.100	-0.020	0.010
0.970	0.150	0.060	0.070	0.090	0.040	0.040	-0.090	-0.030	-0.020	0.020
0.920	0.310	0.160	-0.030	-0.150	0.030	-0.050	0.010	-0.030	-0.010	-0.020
0.980	0.130	-0.070	-0.030	0.040	0.040	-0.130	-0.040	0.040	0.030	0.000
0.860	0.450	-0.200	0.010	0.030	-0.030	0.110	0.050	0.010	0.000	-0.010

INTERMEDIATE COMPUTATIONAL MATRIX

W

FOR 11 WAIS SUBSCALES

0.000	0.140	0.180	0.080	0.060	0.350	0.050	0.170	0.040	0.040	-0.030
0.240	0.000	-0.020	0.110	-0.030	0.400	-0.090	0.050	0.000	0.120	0.070
0.380	-0.030	0.000	0.100	0.160	0.060	-0.010	0.010	0.140	0.030	0.000
0.130	0.110	0.080	0.000	0.040	0.390	0.080	0.060	0.090	-0.040	-0.030
0.170	-0.050	0.200	0.070	0.000	0.110	0.080	-0.040	0.020	0.170	-0.020
0.340	0.230	0.030	0.230	0.040	0.000	0.130	0.020	-0.040	0.080	-0.020
0.120	-0.130	-0.010	0.120	0.070	0.310	0.000	0.020	0.050	0.110	0.140
0.300	0.060	0.010	0.060	-0.030	0.040	0.020	0.000	0.210	0.080	0.160
0.070	0.000	0.130	0.100	0.010	-0.070	0.040	0.220	0.000	0.160	0.300
0.070	0.140	0.020	-0.040	0.130	0.160	0.090	0.090	0.160	0.000	0.140
-0.080	0.100	0.000	-0.040	-0.010	-0.050	0.140	0.200	0.370	0.170	0.000

INTERMEDIATE COMPUTATIONAL MATRIX

W INVERSE

FOR 11 WAIS SUBSCALES

-2.890	1.400	1.660	3.110	0.560	0.030	-2.180	-0.990	1.570	-0.610	-0.020
2.370	1.880	-1.550	-4.300	0.160	1.200	-2.160	1.130	-2.000	3.950	-0.560
3.550	-1.960	0.150	-0.870	1.840	-1.570	-1.220	2.670	0.250	-0.670	-1.530
5.160	-4.210	-0.670	-2.350	-3.040	1.750	5.060	4.140	-1.270	-4.330	-0.520
1.520	0.250	2.330	-4.950	1.060	0.100	0.590	-2.950	-1.540	5.310	-0.580
0.030	0.700	-0.730	1.050	0.040	-0.170	0.870	-0.230	-0.510	0.290	-0.120
-5.100	-2.990	-1.350	7.170	0.520	2.060	2.520	-2.420	1.940	-1.670	1.260
-1.790	1.220	2.280	4.550	-1.990	-0.420	-1.880	-3.290	3.940	-3.020	1.670
2.910	-2.180	0.220	-1.430	-1.060	-0.970	1.530	4.020	-1.710	-1.240	0.660
-1.190	4.520	-0.610	-5.060	3.820	0.570	-1.380	-3.210	-1.300	4.560	1.500
-0.050	-0.750	-1.650	-0.720	-0.500	-0.270	1.230	2.100	0.810	1.780	-1.320

INTERMEDIATE COMPUTATIONAL MATRIX

W INVERSE B

FOR 11 WAIS SUBSCALES

0.910	-0.200	0.100	0.220	0.100	0.020	0.000	-0.080	-0.140	0.010	-0.200
0.780	-0.140	-0.270	0.060	-0.120	0.430	0.010	-0.110	-0.050	0.230	0.180
0.710	-0.080	0.000	0.030	-0.360	-0.410	-0.120	-0.270	-0.290	-0.050	0.130
0.800	-0.220	0.120	0.100	-0.110	0.080	0.090	0.240	0.080	-0.330	0.300
0.600	-0.100	-0.020	0.130	0.070	-0.290	-0.560	0.310	0.120	0.280	0.130
0.890	-0.310	-0.050	-0.280	0.100	0.020	0.050	-0.040	0.090	0.000	-0.010
0.680	0.070	-0.200	0.100	-0.170	-0.100	-0.050	-0.090	0.550	-0.210	-0.290
0.780	0.270	-0.080	-0.140	-0.130	0.130	-0.040	0.300	-0.290	-0.100	-0.280
0.740	0.470	-0.190	0.070	0.300	-0.140	0.080	-0.060	-0.100	-0.140	0.190
0.770	0.210	0.110	0.000	-0.110	-0.150	0.320	0.130	0.160	0.420	0.020
0.630	0.540	0.360	-0.150	-0.010	0.220	-0.210	-0.220	0.130	-0.010	0.070

INTERMEDIATE COMPUTATIONAL MATRIX

R INVERSE

FOR VELDMAN'S DATA

2.570	0.920	-1.120	-0.260	-0.970	0.540	-1.250	0.280
0.920	3.090	-1.580	-0.950	-1.050	0.270	-0.250	-0.300
-1.120	-1.580	2.560	0.400	0.000	-0.620	-0.280	-0.380
-0.260	-0.950	0.400	1.690	-0.120	0.260	-0.150	-0.150
-0.970	-1.050	0.000	-0.120	2.270	0.220	1.330	0.200
0.540	0.270	-0.620	0.260	0.220	1.490	0.320	0.180
-1.250	-0.250	-0.280	-0.150	1.330	0.320	2.290	0.100
0.280	-0.300	-0.380	-0.150	0.200	0.180	0.100	1.280

INTERMEDIATE COMPUTATIONAL MATRIX

B

FOR VELDMAN'S DATA

0.510	-0.460	-0.270	0.200	0.130	-0.070	-0.070	0.010
0.700	0.290	-0.300	0.020	-0.030	0.060	0.090	-0.010
0.680	-0.030	0.090	-0.260	-0.070	-0.270	0.000	0.000
0.560	0.080	0.160	0.090	-0.170	0.080	-0.110	-0.010
0.520	0.380	0.240	0.050	0.290	0.030	0.000	0.000
-0.370	0.200	-0.220	-0.270	0.140	-0.020	-0.090	-0.010
0.270	-0.620	0.110	-0.220	0.070	0.170	0.040	-0.010
0.310	0.180	-0.120	-0.180	-0.070	0.190	-0.040	0.020

INTERMEDIATE COMPUTATIONAL MATRIX

B STAR

FOR VELDMAN'S DATA

0.660	-0.580	-0.350	0.250	0.160	-0.080	-0.080	0.010
0.850	0.360	-0.360	0.020	-0.040	0.070	0.110	-0.020
0.870	-0.040	0.110	-0.330	-0.090	-0.350	0.000	0.000
0.880	0.130	0.260	0.140	-0.260	0.130	-0.180	-0.020
0.700	0.500	0.320	0.060	0.390	0.040	0.000	0.000
-0.650	0.350	-0.380	-0.470	0.240	-0.040	-0.160	-0.020
0.360	-0.830	0.150	-0.300	0.100	0.230	0.050	-0.010
0.660	0.390	-0.260	-0.390	-0.160	0.400	-0.080	0.050

INTERMEDIATE COMPUTATIONAL MATRIX

W

FOR VELDMAN'S DATA

0.000	-0.360	0.440	0.100	0.380	-0.210	0.480	-0.110
-0.300	0.000	0.510	0.310	0.340	-0.090	0.080	0.100
0.440	0.620	0.000	-0.160	0.000	0.240	0.110	0.150
0.150	0.560	-0.240	0.000	0.070	-0.150	0.090	0.090
0.430	0.460	0.000	0.050	0.000	-0.100	-0.580	-0.090
-0.360	-0.180	0.420	-0.170	-0.150	0.000	-0.210	-0.120
0.550	0.110	0.120	0.070	-0.580	-0.140	0.000	-0.040
-0.220	0.240	0.300	0.120	-0.160	-0.140	-0.080	0.000

INTERMEDIATE COMPUTATIONAL MATRIX

W INVERSE

FOR VELDMAN'S DATA

0.070	0.870	0.130	0.450	0.410	0.450	1.110	-2.300
0.720	-1.710	0.970	-0.910	0.440	-1.400	-1.260	3.620
0.130	1.170	0.520	0.160	0.020	1.350	0.690	-0.890
0.690	-1.670	0.250	-4.190	0.750	-4.870	-1.360	5.860
0.470	0.590	0.020	0.550	0.630	0.290	-0.710	-1.370
0.770	-2.900	2.310	-5.510	0.440	-3.750	-2.430	5.670
1.250	-1.700	0.770	-1.010	-0.710	-1.580	-0.640	2.750
-4.610	8.700	-1.770	7.690	-2.430	6.590	4.900	-13.200

INTERMEDIATE COMPUTATIONAL MATRIX

W INVERSE B

FOR VELDMAN'S DATA

0.630	-0.620	0.220	0.100	0.330	-0.190	0.110	-0.080
0.850	0.400	0.070	-0.190	-0.190	0.030	-0.150	0.150
0.750	-0.040	-0.410	-0.430	0.220	-0.150	0.020	-0.060
0.650	0.090	0.000	0.260	-0.160	0.440	0.450	0.280
0.590	0.490	-0.040	0.510	0.220	-0.320	-0.040	-0.040
-0.450	0.250	-0.100	-0.440	-0.040	-0.560	0.360	0.310
0.310	-0.850	-0.170	-0.030	-0.350	-0.050	-0.110	0.130
0.370	0.230	-0.060	-0.220	-0.490	0.050	0.300	-0.650

INTERMEDIATE COMPUTATIONAL MATRIX

R INVERSE

FOR HARMAN'S EXAMPLE

2.950	-0.690	-1.370	-0.490	-0.330	-0.170
-0.690	2.950	-1.640	-0.240	-0.160	-0.080
-1.370	-1.640	3.200	0.170	0.120	0.060
-0.490	-0.240	0.170	1.470	-0.340	-0.170
-0.330	-0.160	0.120	-0.340	1.330	-0.120
-0.170	-0.080	0.060	-0.170	-0.120	1.130

INTERMEDIATE COMPUTATIONAL MATRIX

B

FOR HARMAN'S EXAMPLE

0.790	0.090	-0.120	-0.010	-0.100	-0.010
0.790	0.150	-0.090	0.030	0.090	0.000
0.790	0.060	0.240	-0.030	-0.010	0.010
0.500	-0.220	0.020	0.130	0.000	-0.010
0.440	-0.200	-0.050	-0.120	0.030	-0.030
0.300	-0.150	-0.050	-0.020	-0.010	0.060

INTERMEDIATE COMPUTATIONAL MATRIX

B STAR

FOR HARMAN'S EXAMPLE

0.980	0.110	-0.150	-0.010	-0.120	-0.010
0.970	0.190	-0.120	0.040	0.110	0.010
0.950	0.070	0.290	-0.040	-0.010	0.010
0.890	-0.380	0.040	0.230	-0.010	-0.030
0.880	-0.390	-0.090	-0.230	0.070	-0.050
0.870	-0.430	-0.140	-0.060	-0.020	0.170

INTERMEDIATE COMPUTATIONAL MATRIX

W

FOR HARMAN'S EXAMPLE

0.000	0.230	0.460	0.170	0.110	0.060
0.230	0.000	0.550	0.080	0.050	0.030
0.430	0.510	0.000	-0.050	-0.040	-0.020
0.330	0.160	-0.120	0.000	0.230	0.120
0.250	0.120	-0.090	0.250	0.000	0.090
0.150	0.070	-0.050	0.150	0.100	0.000

INTERMEDIATE COMPUTATIONAL MATRIX

W INVERSE

FOR HARMAN'S EXAMPLE

-1.960	1.840	0.600	0.310	0.410	0.690
1.840	-1.640	1.260	-0.160	-0.210	-0.350
0.550	1.160	0.010	-0.280	-0.370	-0.610
0.630	-0.310	-0.600	-1.660	1.760	2.940
0.920	-0.460	-0.880	1.940	-3.220	4.300
1.810	-0.900	-1.730	3.820	5.070	-10.600

INTERMEDIATE COMPUTATIONAL MATRIX

W INVERSE B

FOR HARMAN'S EXAMPLE

0.910	-0.110	0.160	0.040	0.360	0.050
0.890	0.110	0.260	-0.100	-0.330	-0.030
0.880	0.450	-0.130	0.050	0.040	-0.020
0.590	-0.440	-0.450	-0.480	-0.040	0.140
0.520	-0.450	-0.280	0.540	-0.260	0.290
0.350	-0.360	-0.200	0.130	-0.030	-0.830