SOME DEVELOPMENTS AND APPLICATIONS OF A NEW APPROXIMATION METHOD FOR PARTIAL DIFFERENTIAL EIGENVALUE PROBLEMS

by

A. K. Jennings

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Thesis and Abstract Approved:

N. Aronszajn
Thesis Advisor

Herman W. Smith
Faculty Representative

D. C. McIntosh
Dean of the Graduate School

Preface

The approximation method to be discussed and applied in this paper was originally presented by N. Aronszajn in his seminar on Hilbert space theory at Oklahoma A. and M. College in the spring of 1950. Since that time the method has been further discussed and analyzed and a preliminary report on a forthcoming paper by N. Aronszajn and the author was presented by the author before the American Mathematical Society at the Chicago meeting on April 28, 1951.

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1. Introduction. We shall consider a differential problem which is of such a type that we can replace this differential problem by an equivalent variational problem. The new method to be discussed here will then be applied to the variational problem -- in much the same way as the Rayleigh-Ritz and Weinstein methods are actually applied to an equivalent variational problem (see [1]).[1] In making the transition from the differential problem to the variational problem we shall use many of the results presented by N. Aronszajn in [2], although we shall not always refer to them explicitly.

To begin with we shall consider the differential eigenvalue problem

(1a) $$Au = \mu Bu \quad \text{in} \quad D ,$$

(1b) $$\Lambda_i u = 0 \quad \text{on} \quad S ,$$

where S is the boundary of a domain D in ν-dimensional space, A and B are elliptic positive differential operators of orders 2t and 2t' respectively, $t > t'$, and $\{\Lambda_i\}$ is a system of t linear

1. Numbers in brackets refer to the references at the end of the paper.

differential boundary operators of orders less than or equal $2t-1$. [2] It is well known that the differential problem (1) is equivalent, in the usual cases, to the variational problem

$$\mu = \min \frac{\int_D Au\,\bar{u}\,d\omega}{\int_D Bu\,\bar{u}\,d\omega} ,$$

where the function u varies in an appropriate class of admissible functions, usually 2t times continuously differentiable and satisfying the boundary conditions (1b).

For our future considerations it is important that we describe the equivalent variational problem more precisely. For this purpose we shall introduce the class $\mathcal{K}$ of functions $u \varepsilon C^{(2t)}$ in $\bar{D}$ and satisfying the boundary conditions $\Lambda_i u = 0$ on S, and in this class we define the two (hermitian) bilinear forms

$$(2) \qquad \mathcal{A}(u,v) = \int_D Au\,\bar{v}\,d\omega , \quad u,v \,\varepsilon\, \mathcal{K} ,$$

$$(3) \qquad \mathcal{B}(u,v) = \int_D Bu\,\bar{v}\,d\omega , \quad u,v \,\varepsilon\, \mathcal{K} .$$

The variational problem described by the above formula is then the quotient of the corresponding quadratic forms $\mathcal{A}(u,u)$ and $\mathcal{B}(u,u)$ considered in the class $\mathcal{K}$. In the classical problems the minimum of this quotient is actually attained in the class $\mathcal{K}$ and this variational problem is truly equivalent to the differential prob-

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2. The methods to be discussed may also be generalized to include any self-adjoint operator B of smaller order than 2t. There must also be some additional restrictions on the boundary S and the operators Λ_i, which are analyzed more fully in [2] -- these conditions are all satisfied in the usual problems considered.

lem (1). However there is no reason to suspect that there will always be a minimizing function in the class $\mathcal{K}$, and even when there is we may wish to consider an auxiliary problem where this is no longer true.

Before we can rigorously analyze the variat onal problem we must transform the quadratic forms $\mathcal{O}(u,u)$ and $\mathcal{L}(u,u)$ into expressions which are 'formally positive' quadratic forms (as discussed in [2], [6], and [7]) [3].

(4) $$\mathcal{O}(u,u) = \int_D \sum |A_k u|^2 \, d\omega + \int_S \sum |\Omega_j u|^2 \, ds,$$

(5) $$\mathcal{L}(u,u) = \int_D \sum |B_k u|^2 \, d\omega + \int_S \sum |\Theta_j u|^2 \, ds,$$

where the operators A_k are of orders less than or equal t, the operators B_k are of orders less than or equal t', and the operators Ω_j and Θ_j are boundary operators of orders less than or equal t-1 and t'-1 respectively. The quadratic forms given by (4) and (5) are equivalent <u>in $\mathcal{K}$</u> to those corresponding to (2) and (3) since they differ for any function in $C^{(2t)}$ only by boundary integrals which will vanish for functions satisfying <u>all</u> of the boundary conditions. The quadratic form $\mathcal{O}(u,u)$ as given by (4) will now be positive <u>definite</u> for all functions which satisfy only the boundary conditions of orders less than or equal t-1 (this property is discussed in [2] and depends essentially on the

3. These representations are assured by the assumptions concerning the operators A, B, Λ_i, and the boundary S. It should be recalled that some of the assumptions about the boundary S and the operators Λ_i are not mentioned explicitly but are presented in [2].

existence and regularity of a Green's function). We may now consider a norm in the space $\mathcal{K}$ as defined by $\|u\|^2 = \mathcal{A}(u,u)$. With this quadratic norm $\mathcal{K}$ has the character of an 'incomplete' Hilbert space and we can consider its functional completion $\bar{\mathcal{K}}$. Our purpose in transforming $\mathcal{A}$ and $\mathcal{L}$ into the representations (4) and (5) was to enable us to form the functional completion of $\mathcal{K}$, to which (4) and (5) can immediately be extended. The functions of the (complete) Hilbert space $\bar{\mathcal{K}}$ will still satisfy the 'stable' boundary conditions, i.e. of orders less than or equal t-1, but they need not satisfy the 'unstable' boundary conditions, i.e. of orders greater than or equal t, (the terms 'stable' and 'unstable' then having an obvious significance).

We can now say in general that the variational problem

$$\mu = \inf_{\mathcal{K}} \frac{\mathcal{A}(u,u)}{\mathcal{L}(u,u)} \tag{6}$$

is equivalent to the variational problem

$$\mu = \min_{\bar{\mathcal{K}}} \frac{\mathcal{A}(u,u)}{\mathcal{L}(u,u)} . \tag{7}$$

When the minimizing solutions in $\bar{\mathcal{K}}$ already belong to $\mathcal{K}$, as in the usual cases considered, then both variational problem (6) and (7) are equivalent to the differential problem (1) -- otherwise the differential problem needs some clarification as to the required regularity of its solutions.

The disappearance of the unstable boundary conditions in the complete space $\bar{\mathcal{K}}$ can be explained by considering the variational problem (7). When integrating the first variation by parts to derive Euler's equation (which will be (1a)) boundary integrals will arise. Using only the stable boundary conditions, which are satisfied by functions in $\bar{\mathcal{K}}$, we will then obtain the corresponding 'natural' boundary conditions which will in fact be our original unstable boundary conditions. Thus when deriving Euler's equation for (6) all boundary conditions are present throughout, and for (7) we start with the stable boundary conditions and the unstable boundary conditions appear automatically.

We should remark that although the forms given by (2) and (3) and by (4) and (5) are equivalent in the space $\mathcal{K}$ they are not equivalent in $\bar{\mathcal{K}}$ (even for functions in $C^{(2t)}$ so that (2) will have meaning). For functions in $\bar{\mathcal{K}}$ which belong to $C^{(2t)}$ we could try to transform (4) (or (5)) back into the form (2) (or (3)) but when making this transformation the boundary integrals which vanished in the space $\mathcal{K}$ will no longer vanish unless the function u also satisfies the unstable boundary conditions. (In the bilinear form u must satisfy the unstable boundary conditions but v need not in order to perform this transformation.) Hereafter when we refer to $\mathcal{A}(u,u)$ and $\mathcal{L}(u,u)$ we shall mean the expressions given by (4) and (5).

Our final remark concerns a third space $\tilde{\mathcal{K}}$ consisiting of all functions $u \varepsilon \tilde{C}^{(t)}$ 4. in $\bar{D}$ and satisfying the stable boundary conditions. This space $\tilde{\mathcal{K}}$ will be very useful at times because the complete space $\bar{\mathcal{K}}$ is often quite difficult to find explicitly and

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4. By $\tilde{C}^{(t)}$ we mean the class of all functions belonging to $C^{(t-1)}$ whose $t-1^{th}$ derivatives are absolutely continuous in each variable separately and whose t-th derivatives belong to $\mathcal{L}^{(2)}$.

$\tilde{\mathcal{K}}$ approximates $\overline{\mathcal{K}}$ closely enough for most purposes of analysis. The actual relation is this: $\mathcal{K} \subset \tilde{\mathcal{K}} \subset \overline{\mathcal{K}}$.

2. Approximation methods. At the present time there are many approximation methods at our disposal for this type of problem (see [6]). Two of these which have been extensively analyzed and are important in many applications are the Rayleigh-Ritz and the Weinstein methods (see [1], [2], [3], [4], [5]). In the 'generalized' Rayleigh-Ritz method a subspace $\mathcal{K}^{(0)}$ of $\overline{\mathcal{K}}$ is considered in which the variational problem is explicitly solvable. A sequence $\mu_n^{(m)}$ of approximations to the eigenvalue μ_n is then obtained by successively adding to $\mathcal{K}^{(m-1)}$ a one-dimensional subspace and then solving the problem again in this new subspace $\mathcal{K}^{(m)}$. The approximations $\mu_n^{(m)}$ form a decreasing sequence of upper bounds for the eigenvalues μ_n. In the Weinstein method a similar procedure is used, starting with a larger space containing $\overline{\mathcal{K}}$ and producing an increasing sequence of lower bounds for the eigenvalues μ_n.

Both the Rayleigh-Ritz and Weinstein methods are based on the same fundamental principle -- the Monotony Theorem. This theorem is based on the simple property that the minimum over a smaller class of functions will be an upper bound for the minimum over the original class. The new approximation method to be discussed here gives upper bounds for the desired eigenvalues as does the 'generalized' Rayleigh-Ritz method, but beyond the first step the new method is essentially different from the Rayleigh-Ritz method. In both methods an increasing sequence of subspaces is utilized and the variational problem must be solved in each of these subspaces. However, in the new method the difference between each of these subspaces and the preceding subspace will be of infinite dimension; thus the new method gives rise to a sequence of approximations which should converge more rapidly than the sequence obtained from the Rayleigh-Ritz method.

3. The new approximation method. As mentioned in the introduction we shall actually apply the approximation method to the equivalent variational problem (7). Our first step in this process is to choose a fixed sequence of functions $\phi_k(x_1,x_2,\ldots,x_\nu)$, $k = 1,2,\ldots$, which must be restricted to some extent by the boundary conditions as we shall see. We next form the subspaces $\mathcal{K}^{(m)}$ each composed of all functions of the form

$$u = \sum_{1}^{m} \phi_k(x_1,\ldots,x_\nu) f_k(x_1) , \tag{8}$$

where the functions f_k are allowed to vary through an appropriate class. The only restriction which must be imposed upon the functions ϕ_k and f_k is that the product $\phi_k f_k$ must satisfy the stable boundary conditions (and of course some regularity conditions) so that the spaces $\mathcal{K}^{(m)}$ will actually be subspaces of $\overline{\mathcal{K}}$. [5]

There are many ways in which this can be achieved; for illustration we shall mention two of these.

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5. It is often very important that we construct subspaces of $\overline{\mathcal{K}}$ rather than subspaces of $\mathcal{K} \subset \overline{\mathcal{K}}$ (i.e. that we consider the variational problem (7) rather than (6)) so that we may discard the unstable boundary conditions. Otherwise in many cases we could not apply the method in so simple and convenient a manner.

I) If the stable boundary operators form a Dirichlet system (i.e. $\frac{\partial^i u}{\partial n^i} = 0$, $i = 0,1,\ldots,r$ for some $r \leq t-1$) then we can choose the functions ϕ_k to satisfy these boundary conditions and restrict the functions f_k only to be sufficiently regular.

II) If D is a cylindrical domain with axis in the x_1 direction and bases given by $x_1 = a$ and $x_1 = b$, if the stable boundary operators on the lateral surface are independent of x_1, and if the stable boundary operators on the bases depend only upon x_1 (i.e. they are independent of the particular point of the base), then we can choose the functions ϕ_k to be independent of x_1 and to satisfy these boundary conditions on the lateral surface while restricting the functions $f_k(x_1)$ to satisfy the boundary conditions on the bases.

We now turn to the solution of the variational problem (7) in the subspace $\mathcal{K}^{(m)}$ as given by (8) (or in its completion $\overline{\mathcal{K}^{(m)}}$ if $\mathcal{K}^{(m)}$ is not already complete). We shall let $u = \sum_1^m \phi_k f_k$ represent the minimizing solution and consider a variation function $v = \sum_1^m \phi_k g_k$ for an arbitrary system of functions $g_k(x_1)$ (in the appropriate class). By equating the first variation to zero (and using the corresponding bilinear forms) we obtain

$$\mathfrak{A}(u,v) - \mu \mathfrak{L}(u,v) = 0 ,$$

which can also be written as

$$(9) \qquad \sum_{k=1}^{m} \left[\mathfrak{A}(\phi_k f_k, \phi_\ell g_\ell) - \mu \mathfrak{L}(\phi_k f_k, \phi_\ell g_\ell) \right] = 0, \quad \ell = 1,2,\ldots,m .$$

The forms $\mathfrak{A}$ and $\mathfrak{L}$ as represented in (4) and (5) can now be integrated with respect to all of the variables except x_1 which will give us expressions of the type

$$\sum_{k,p,q} \int_{\alpha}^{\beta} A_{k,\ell,p,q}(x_1) f_k^{(p)}(x_1) \overline{g_\ell^{(q)}(x_1)}\, dx_1 = 0, \quad \ell = 1,\dots,m ,$$

where the superscripts p and q refer to derivatives of those orders. While keeping in mind that the functions $A_{k,\ell,p,q}(x_1)$ (and their derivatives) may have discontinuities at some points (caused by the shape of the boundary, cf. Example 1), we next integrate each of the above terms by parts q times so as to transfer all derivatives from the g_ℓ to the functions $A_{k,\ell,p,q}$ and f_k. By the usual procedure in each interval where $A_{k,\ell,p,q} \in \tilde{C}^{(t)}$ we then obtain a system of ordinary differential equations, at the endpoints α and β we obtain unstable boundary conditions, and at the points of irregularity of one of the functions $A_{k,\ell,p,q}$ (i.e. a discontinuity of some derivative or order less than or equal t-1) we obtain unstable linear differential conditions relating the solution in two adjoining intervals.

In each interval we now solve the sytem of m linear differential equations of order 2t in the m functions $f_1,\dots,f_m$. The general solutions will each depend linearly upon 2t parameters so that the total number of parameters will be 2tm in each interval or 2tmn all together, where n is the number of intervals. To determine these parameters we must now apply the boundary conditions. At each of the endpoints α and β we have t boundary conditions for each of the m functions giving us 2tm linear homogeneous equations in the parameters. At each of the n-1 interior points of irregularity we have for each function t conditions of continuity (since $f_k \in C^{(t-1)}$) and for all m functions we have tm unstable matching conditions for the derivatives. All of the above conditions together give us 2tmn linear homogeneous equations in the 2tmn parameters. In general the coefficients of these equations will be transcendental functions of the variable μ representing the eigenvalue of the variational problem in the class $\mathcal{K}^{(m)}$. Since this is an eigenvalue problem and the function $\sum_1^m \phi_k f_k$ is the eigenfunction, we are only interested in the case when the functions f_k do not all vanish identically, i.e. when the 2tmn parameters do not all vanish. Thus we know that the determinant (of order 2tmn) of the above equations must vanish and this gives us a transcendental equation which determines the eigenvalues $\mu_n^{(m)}$, which are the desired upper bounds for the eigenvalues μ_n.

By an appropriate choice of the fixed sequence $\{\phi_k\}$ the subspaces $\overline{\mathcal{K}^{(m)}}$ will converge to the space $\overline{\mathcal{K}}$ and the corresponding bounds $\mu_n^{(m)}$ will converge to the eigenvalue μ_n, for each n.

We should also remark at this time that in forming the subspaces $\mathcal{K}^{(m)}$ we might have chosen a more general representation, with arbitrary functions of different variables, such as

$$u = \sum_{k=1}^{\nu} \sum_{\ell=1}^{m_k} \phi_{k\ell}(x_1,\ldots,x_\nu) f_{k\ell}(x_k) ,$$

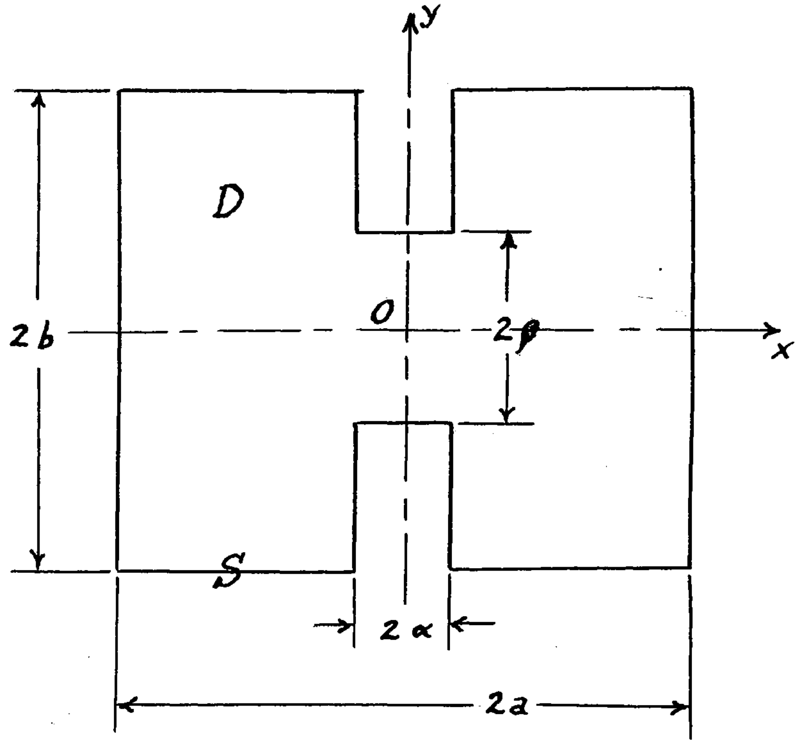
but in this generalized case we would obtain integro-differential equations with integro-differential boundary conditions which would be much more difficult to handle.

4. <u>Example 1</u>. <u>The notched rectangle</u>. As the first example to which we shall apply the new approximation method we shall consider the differential problem

(10a) $$\Delta u + \mu u = 0 \quad \text{in} \quad D ,$$

(10b) $$\frac{\partial u}{\partial n} = 0 \quad \text{on} \quad S ,$$

where D is the notched rectangular domain shown in the figure, with origin taken at the center. In this problem we have a second order equation (t=1) and the one and only boundary operator is unstable. Thus when we apply the new method we may choose the functions ϕ_k to be arbitrary functions only of the variable x and belonging to $\tilde{C}^{(1)}$ but



subject to no boundary conditions, and similarly for the functions $f_k(y)$ depending on the other variable. This would not have been the case if our boundary conditions had been u = 0 on S (as for the membrane problem) and in this other problem a slightly more complicated choice of the functions $\phi_k(x,y)$ would be necessary, making the computations much more involved.

Using our previous notation the class $\mathcal{K}$ will be composed of functions $u \in C^{(2)}$ in $\bar{D}$ and satisfying the boundary condition $\frac{\partial u}{\partial n} = 0$ on S. In the class $\mathcal{K}$ the bilinear forms will be defined as

$$\mathcal{O}\!\!\mathcal{l}(u,v) = - \int_D \Delta u \, \bar{v} \, dxdy ,$$

$$\mathcal{L}(u,v) = \int_D u \, \bar{v} \, dxdy ,$$

where the positive operator A has been replaced by $-\Delta$ and the operator B is the identity operator. Before extending these forms or considering the complete space we first transform them as mentioned in the introduction to their equivalent representations

(11) $$\mathcal{A}(u,v) = \int_D (u_x \overline{v}_x + u_y \overline{v}_y)\, dxdy ,$$

(12) $$\mathcal{B}(u,v) = \int_D u \overline{v}\, dxdy ,$$

where formula (11) follows immediately from Green's identity

$$-\int_D \Delta u\, \overline{v}\, dxdy = \int_D (u_x \overline{v}_x + u_y \overline{v}_y)\, dxdy - \int_S \frac{\partial u}{\partial n} \overline{v}\, ds ,$$

with subscripts referring to derivatives and the exterior normal derivative being used. The corresponding variational problem in the incomplete space $\mathcal{K}$ would then be

(13) $$\mu = \inf_{\mathcal{K}} \frac{\int_D (|u_x|^2 + |u_y|^2)\, dxdy}{\int_D |u|^2\, dxdy} .$$

Before we can consider the complete space $\overline{\mathcal{K}}$ and thus put the variational problem and methods on a more rigorous footing we must clarify a point that has thus far been neglected. In order to form this completion of $\mathcal{K}$ we should have a proper norm, i.e. the quadratic form $\mathcal{A}(u,u)$ should be positive <u>definite</u>. However we see immediately from (11) that this form is not definite since it vanishes for a function which is constant. Thus our original analysis of the problem breaks down at this point and we must go back and reinterpret the problem in a slightly different manner.

Instead of defining the bilinear forms in terms of the operators $-\Delta$ and I (the identity) we shall rewrite equation (10a) in the form

(10a') $$-\Delta u + \varkappa u = \mu' u \quad \text{in} \quad D ,$$

where $\varkappa$ is a fixed positive number and $\mu' = \mu + \varkappa$. We can now define the new bilinear forms corresponding to equation (10a')

(11') $$\mathcal{H}'(u,v) = \int_D (u_x \bar{v}_x + u_y \bar{v}_y + \varkappa u\bar{v}) \, dxdy,$$

(12') $$\mathcal{L}'(u,v) = \mathcal{L}(u,v) .$$

The new quadratic form $\mathcal{H}'(u,u)$ will now be positive definite and all our methods can be applied to the new variational problem ,

(13') $$\mu' = \inf_{\mathcal{K}} \frac{\int_D (|u_x|^2 + |u_y|^2 + \varkappa |u|^2) \, dxdy}{\int_D |u|^2 \, dxdy}$$

and the corresponding problem in the complete space

(14') $$\mu' = \min_{\bar{\mathcal{K}}} \frac{\int_D (|u_x|^2 + |u_y|^2 + \varkappa |u|^2) \, dxdy}{\int_D |u|^2 \, dxdy}$$

where the completion is now taken with respect to the norm $\|u\|^2 = \mathcal{H}'(u,u)$. When dealing with the completion $\bar{\mathcal{K}}$ of the space $\mathcal{K}$ we obviously need to consider this auxiliary form $\mathcal{H}'(u,u)$ but in analyzing the variational problem itself we may use either formula (14') or the corresponding formula

(14) $$\mu = \min_{\bar{\mathcal{K}}} \frac{\int_D (|u_x|^2 + |u_y|^2) \, dxdy}{\int_D |u|^2 \, dxdy} ,$$

where $\bar{\mathcal{K}}$ again represents the completion with respect to $\mathcal{H}'(u,u)$.

This freedom to use either (14) or (14') is due to the fact that $\mu' = \mu + \kappa$ and

$$\frac{\int_D (|u_x|^2 + |u_y|^2 + \kappa |u|^2)\, dxdy}{\int_D |u|^2\, dxdy} = \frac{\int_D (|u_x|^2 + |u_y|^2)\, dxdy}{\int_D |u|^2\, dxdy} + \kappa .$$

Thus we may consider the form $\mathfrak{A}'(u,u)$ as merely an auxiliary form that we use to complete the space -- actually analyzing the variational problem (14).

Having discussed the variational problem corresponding to the differential problem (10) we shall now apply the new approximation method to this variational problem (14). Due to the symmetry of the domain we can divide the problem into four part problems -- considering functions that are even in x and even in y, even in x and odd in y, odd in x and even in y, and odd in x and odd in y. We shall now analyze the four part problems together and separate them at the end.

We first choose our fixed sequence of functions $\phi_k(x) \not\equiv 0$ [6] which belong to $\tilde{C}^{(1)}$ in the closed interval $-a \leq x \leq a$ but subject to no boundary conditions. The subspace $\mathcal{K}^{(m)}$ is composed of all functions $u(x,y) = \sum_1^m \phi_k(x) f_k(y)$, where the functions $f_k(y)$ belong to $\tilde{C}^{(1)}$ in $-b \leq y \leq b$. In order to follow the method to the end and not overcomplicate matters with notation we shall solve the variational problem only in the first subspace $\mathcal{K}^{(1)}$. Consequently we may also drop the subscripts and refer simply to functions of the form $\phi(x)f(y)$.

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6. More accurately we shall require that $\phi_k(x)$ be not identically zero in the subintervals $-a \leq x \leq -\alpha$ and $\alpha \leq x \leq a$.

To solve the variational problem (14) in the space $\mathcal{K}^{(1)}$ we set the first variation equal to zero and obtain

$$\int_D (|\phi'|^2 fg + |\phi|^2 f'g' - \mu|\phi|^2 fg)\; dxdy = 0\ ,$$

where g(y) is an arbitrary function (belonging to $\tilde{C}^{(1)}$. Using the property that the solution f(y) belongs to $C^{(2)}$ in each of the intervals $-b < y < -\beta$, $-\beta < y < \beta$, and $\beta < y < b$, [7] we can now integrate by parts with respect to y in the middle term. Doing this for the integral over one corner of the domain (by symmetry this integral must also vanish) and using the notation

$$R_o = \frac{\int_o^a |\phi'|^2 dx}{\int_o^a |\phi|^2 dx}\ , \qquad R_1 = \frac{\int_\alpha^a |\phi'|^2 dx}{\int_\alpha^a |\phi|^2 dx}\ , \qquad K = \frac{\int_\alpha^a |\phi|^2 dx}{\int_o^a |\phi|^2 dx}\ ,$$

(recalling that we are dealing with one of the part problems) we obtain

$$\int_0^\beta \Big[\Big(\int_0^A |\phi'|^2 dx\Big)f - \Big(\int_0^a |\phi|^2 dx\Big)f' - \mu\Big(\int_0^a |\phi|^2 dx\Big)f\Big]\, g\; dy$$

$$+ \int_\beta^b \Big[\Big(\int_\alpha^a |\phi'|^2 dx\Big)f - \Big(\int_\alpha^a |\phi|^2 dx\Big)f'' - \mu\Big(\int_\alpha^a |\phi|^2 dx\Big)f\Big]\, g\; dy$$

$$- \Big(\int_0^a |\phi|^2 dx\Big)f'(0)g(0) + \Big[\Big(\int_0^a |\phi|^2 dx\Big)f'(\beta-0) - \Big(\int_\alpha^a |\phi|^2 dx\Big)f'(\beta+0)\Big]g(\beta)$$

$$+ \Big(\int_\alpha^a |\phi|^2 dx\Big)f'(b)g(b) = 0\ ,$$

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7. This property is obtained in the usual manner of integrating by parts in the other direction to obtain g' as a common factor. Then we obtain f' as a constant times the indefinite integral of

which by the classical procedure gives us the system

$$(15)\quad \begin{cases} f''(y) + (\mu - R_o) f(y) = 0, & 0 \leqq y < \beta , \\ f''(y) + (\mu - R_1) f(y) = 0, & \beta < y < b , \\ f'(b) = 0, \quad f'(\beta - 0) - Kf'(\beta + 0) = 0 . \end{cases}$$

(Note that the conditions $f'(0)g(0) = 0$ is automatically satisfied due to the restriction to part problems.) The ordinary differential system (15) may now be solved easily. Using the boundary condition $f'(b) = 0$ and the restriction of the particular part problem the solutions must be of the form

$$f(y) = C_1 \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} \left[\sqrt{\mu - R_o}\; y \right] , \qquad 0 \leqq y < \beta ,$$

$$f(y) = C_2 \cos \left[\sqrt{\mu - R_1}\; (b-y) \right], \qquad \beta < y < b ,$$

where $\begin{Bmatrix} \cos \\ \sin \end{Bmatrix}$ stands for cosine in the part problems with even functions of y and sine in the part problems with odd functions of y. Using the remaining boundary condition of (15) and the continuity of the function at $y = \beta$ we now obtain a system of two linear homogeneous equations in the parameters C_1 and C_2. Since ϕf is an eigenfunction in $\mathcal{K}^{(1)}$ we know that $f \not\equiv 0$ and C_1 and C_2 cannot both vanish, which meqns that the determinant

$$\begin{vmatrix} \sqrt{\mu - R_o} \begin{Bmatrix} -\sin \\ \cos \end{Bmatrix} \left[\beta \sqrt{\mu - R_o} \right] & K \sqrt{\mu - R_1} \sin \left[(b-\beta) \sqrt{\mu - R_1} \right] \\ \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} \left[\beta \sqrt{\mu - R_o} \right] & \cos \left[(b-\beta) \sqrt{\mu - R_1} \right] \end{vmatrix}$$

must vanish. Dividing by the expression $\begin{Bmatrix} \cos \\ \sin \end{Bmatrix} \left[\beta \sqrt{\mu - R_o} \right] \cos \left[(b-\beta) \sqrt{\mu - R_1} \right]$ (assuming the constants are such that it does not vanish with the

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f in each interval. By induction this implies that f belongs to $C^{(2)}$ and in fact $C^{(\infty)}$.

determinant) we are left only with the equation

$$(16) \qquad \sqrt{\mu-R_o}\left\{\begin{matrix}\tan\\-\cot\end{matrix}\right\}\left[\beta\sqrt{\mu-R_o}\right]+K\sqrt{\mu-R_1}\tan\left[(b-\beta)\sqrt{\mu-R_1}\right]=0 \ .$$

The values $\mu_n^{(1)}$ which are roots of this equation for each part problem are the eigenvalues for the subspace $\mathcal{K}^{(1)}$ and are thus the first in the sequence of upper bounds for the eigenvalues μ_n of the original problem.

5. Example 2. The clamped rectangular plate. As our second example we consider the differential eigenvalue problem for the clamped plate

$$(17a) \qquad \Delta^2 u = \mu u \quad \text{in} \quad D \ ,$$

$$(17b) \qquad u = \frac{\partial u}{\partial n} = 0 \text{ on } S \ ,$$

where D is the rectangular domain $-a < x < a$, $-b < y < b$. For this problem the class $\mathcal{K}$ is the class of functions $u \in C^{(4)}$ in $\overline{D}$ which satisfy the two boundary conditions $u = \frac{\partial u}{\partial n} = 0$ on S. The bilinear forms corresponding to the operators in (17) are then given by

$$(18) \qquad \mathcal{O}\!\!\mathcal{l}(u,v) = \int_D \Delta^2 u\, \overline{v}\, dxdy = \int_D \Delta u\, \overline{\Delta v}\, dxdy \ , \qquad u,v \in \mathcal{K} \ ,$$

$$(19) \qquad \mathcal{L}(u,v) = \int_D u\, \overline{v}\, dxdy \ , \qquad u,v \in \mathcal{K} \ ,$$

where in (18) we have made use of a form of Green's identity

$$\int_D \Delta^2 u\, \overline{v}\, dxdy = \int_D \Delta u\, \overline{\Delta v}\, dxdy + \int_S \left(\frac{\partial \Delta u}{\partial n}\, \overline{v} - \Delta u\, \overline{\frac{\partial v}{\partial n}}\right) ds \ ,$$

using exterior normal derivatives. In connection with completing $\mathcal{K}$ to form the complete Hilbert space $\overline{\mathcal{K}}$ we do not have the difficulty here that we encountered in the first example. The quadratic form $\mathfrak{N}(u,u)$ is positive definite and the completion may be carried out as usual, leading to the variational problem

$$\mu = \min_{\tilde{\mathcal{K}}} \frac{\int_D |\Delta u|^2 dxdy}{\int_D |u|^2 \, dxdy} \quad . \tag{20}$$

In applying the new method to this problem we now choose a fixed sequence of functions $\phi_k(x)$ belonging to $\tilde{C}^{(2)}$ (or even $C^{(4)}$) in the interval $-a \leqq x \leqq a$ and satisfying the conditions $\phi_k'(\pm a) = 0$. Again, as we did in Example 1, we shall split the original problem into four part problems by symmetry and in each of these part problems we shall solve the variational problem (20) in the first subspace $\mathcal{K}^{(1)}$, which is composed of functions of the form $u(x,y) = \phi(x)f(y)$ (where we have dropped the subscript 1) with $f(y) \, \varepsilon \, \tilde{C}^{(2)}$ in $-b \leqq y \leqq b$ and $f(\pm b) = 0$.

Now upon equating the first variation to zero we obtain

$$\int_D (|\phi''|^2 fg + \phi\overline{\phi''} f'' g + \overline{\phi}\phi'' f g'' + |\phi|^2 f'' g'' - \mu |\phi|^2 fg) dxdy = 0,$$

where $g(y)$ is an arbitrary function belonging to $\tilde{C}^{(2)}$ in $-b \leqq y \leqq b$ (and of course an even or odd function depending on the part problem). By using the fact that the solutions $f(y)$ belong to $C^{(4)}$ in the interval $-b \leqq y \leqq b$ (see footnote 7, page 15) we can integrate by parts the terms containing derivatives of g, considering the integral only over one corner of the domain where it will also vanish because of the symmetry. The equation then takes the form

$$\int_0^b \Big[\Big(\int_0^a |\phi''|^2 dx \Big) f + \Big(\int_0^a \phi \overline{\phi''} \, dx \Big) f'' + \Big(\int_0^a \overline{\phi} \phi'' dx \Big) f'' + \Big(\int_0^a |\phi|^2 dx \Big) f^{(4)} - \mu \Big(\int_0^a |\phi|^2 dx \Big) f \Big] g \, dy$$

$$+ \Big(\int_0^a \overline{\phi} \phi'' \, dx \Big) f(y) g'(y) \Big|_0^b - \Big(\int_0^a \overline{\phi} \phi'' \, dx \Big) f'(y) g(y) \Big|_0^b$$

$$+ \Big(\int_0^a |\phi|^2 dx \Big) f''(y) g'(y) \Big|_0^b - \Big(\int_0^a |\phi|^2 dx \Big) f^{(3)}(y) g(y) \Big|_0^b = 0 .$$

The above equation can be simplified by using the relation

$$\overline{\int_0^a \overline{\phi} \phi'' \, dx} = \int_0^a \phi \, \overline{\phi''} \, dx = \phi \, \overline{\phi'} \Big|_0^a - \int_0^a \phi'^{\,2} \, dx = - \int_0^a \phi'^{\,2} \, dx .$$

Using the classical procedure at this point and introducing the notation

$$A = \frac{\int_0^a |\phi'|^2 dx}{\int_0^a |\phi|^2 dx} , \qquad B = \frac{\int_0^a |\phi''|^2 dx}{\int_0^a |\phi|^2 \, dx} ,$$

we obtain the ordinary differential equation

$$f^{(4)}(y) - 2Af^{(2)}(y) + (B-\mu) f(y) = 0 . \tag{21}$$

In this case the terms that were integrated give us no additional boundary conditions since for each part problem they vanish automatically at both of the values 0 and b. The only boundary conditions that we have are those that were originally imposed either by the symmetry of the part problem or the conditions $f(\pm b) = f'(\pm b) = 0$.

The solutions of equation (21) are immediately found to be

$$f(y) = C_1 \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} \lambda_1 y + C_2 \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} \lambda_2 y , \tag{22}$$

where the $\begin{Bmatrix} \cos \\ \sin \end{Bmatrix}$ again refers to the two cases of part problems with

$f(y)$ even and with $f(y)$ odd and the parameters λ_1 and λ_2 are the two values given by

$$\lambda_j = i\sqrt{A \pm \sqrt{A^2 - B+\mu}} = \frac{1}{\sqrt{2}}\left[\sqrt{\sqrt{B-\mu}-A} \pm i\sqrt{\sqrt{B-\mu}+A}\right].$$

Using the two boundary conditions $f(b) = f'(b) = 0$ we again obtain two linear homogeneous equations in the parameters C_1 and C_2, and since they cannot both vanish this means that the determinant

$$D = \begin{vmatrix} \lambda_1 \begin{Bmatrix} -\sin \\ \cos \end{Bmatrix} \lambda_1 b & \lambda_2 \begin{Bmatrix} -\sin \\ \cos \end{Bmatrix} \lambda_2 b \\ \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} \lambda_1 b & \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} \lambda_2 b \end{vmatrix}$$

must vanish. This determinant D however can be transformed into a more convenient form, $D = 0$, then becoming

$$(23) \qquad \mp \sqrt{\sqrt{B-\mu}+A}\,\sin\left[\sqrt{2}\,b\sqrt{\sqrt{B-\mu}-A}\right] + \sqrt{\sqrt{B-\mu}-A}\,\sinh\left[\sqrt{2}\,b\sqrt{\sqrt{B-\mu}+A}\right] = 0,$$

where the minus sign is used when $f(y)$ is even, the plus sign when $f(y)$ is odd. The solutions $\mu_n^{(1)}$ of equation (23) thus give us the eigenvalues (for each part problem) of the auxiliary problem in the subspace $\mathcal{K}^{(1)}$, these eigenvalues $\mu_n^{(1)}$ being the first in the sequence of upper bounds for the eigenvalues μ_n of the original problem.

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TYPIST PAGE

THESIS TITLE: SOME DEVELOPMENTS AND APPLICATIONS OF A NEW APPROXIMATION METHOD FOR PARTIAL DIFFERENTIAL EIGENVALUE PROBLEMS+

NAME OF AUTHOR: A. K. Jennings

THESIS ADVISOR: N. Aronszajn

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NAME OF TYPIST: Sylvia L. Aronszajn