# THE APPLICATION OF TENSOR ANALYSIS

# TO

# ELECTRICAL NETWORKS

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# THE INTERPRETATION OF KRON'S METHOD OF ORGANIZING ENGINEERING PROBLEMS

As an introduction to the method of approach to engineering problems which has been developed by Kron,[1] it would be well to review the definition of "engineering", and to compare engineering with physics. Work in the field of physics consists primarily of correlating observed phenomena. New happenings which have not been previously correlated may require the statement of new "laws of nature" or the re-interpreting or extending of old ones. One objective is to create the simplest but most comprehensive correlations. The ultimate goal is a unified field theory or the like that will cover all branches of physics. Since the laws of physics are intended to be the simplest and the most general, they deal with simple units, that is, they may describe a particle, a single conductor, or two unlike charges. Assuming no uncorrelated phenomena are introduced, the laws will still be valid regardless of how the one particle happens to be interrelated with a group of particles, or of how the single conductor is built into a motor.

Unlike physics, engineering deals with specific problems and usually with specific answers. Its problems in most cases consist of finding ways of combining entities already covered by the laws of physics in such manners as to serve useful purposes. New phenomena may and usually do arise out of the combination. A radio, for instance, certainly possesses properties that its independent parts do not possess. If the physicist has done his work well, these new phenomena are due only to the interactions of the components and are included in the existing correlations. Kron's thesis is that since the behaviour of the parts interconnected to make engineering devices is known, it should be possible to develope routine methods of finding the characteristics of these

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1 Gabriel Kron, Tensor Analysis of Networks, pp. xiii-xxiii.

devices. The organized method which he proposes accomplishes this to some extent. In the process of developing an organized mode of approach, several advantages are obtained. Two of these advantages are: The amount of thought necessary to solve a problem is reduced, and shortcuts in numerical calculations may be found. However, these are not the greatest benefits derived from the method of analysis introduced by Kron. The type of organization he uses allows the introduction of tensor analysis, combined with several other mathematical concepts not now of common use in engineering. The application of these, in turn, tends to unify the separate fields in electrical engineering which have grown up haphazardly, and have drifted apart in theories and nomenclature. This, plus certain geometric concepts, allows the engineer to assume a somewhat wider and different viewpoint. Furthermore, tensor analysis is a potent tool for the discovery of essential properties of the entities being studied, since only properties independent of the particular coordinate system from which they are observed may be represented by tensors.

When put in the most basic or the most elegant form, especially in terms of tensors, equations pertaining to seemingly unrelated branches of physics or engineering often show a remarkable similarity. Geometry is a kind of "universal language" for the expression of these formulas. Differential geometry and topology, or analysis situs, are two branches of mathematics employing geometrical reasoning that are beginning to become useful in engineering. Tensor analysis is used to some degree in both subjects. Kron states that differential geometry and topology do not carry the study of geometry far enough to satisfy the demands of engineering problems. For example, the solution may require transformations which tear apart spaces or systems of spaces, while topology only includes the group of transformations which allows stretching and bending.

The analytical tools used by Kron in his organization of engineering problems are matrix analysis and tensor analysis coupled with differential geometry and topology. The specific approach is through three "generalization postulates", the first two of which are applied in Tensor Analysis of Networks.[2] The third postulate is described in Tensor Analysis.[3]

The "First Generalization Postulate" replaces single quantities by n-way matrices. This generalizes from one degree of freedom to n degrees of freedom. No new properties are introduced, because the components of the matrices could still be handled separately. Better organization is the only result.

The "Second Generalization Postulate" uses the organization created by the first postulate to endow the n-way matrices with new content. The matrix with fixed indices becomes a set of components of a geometric object along some reference frame. Invariant equations replace ordinary equations. Unlike ordinary equations, these equations are valid for a large number of coordinate systems of the same type.

Application of the "Third Generalization Postulate" results in still higher organization. Geometric objects are replaced by tensors. The invariant equations valid for one type of reference axis become tensor equations valid for several different types of reference frames.

The "generalization postulates" may be crudely summarized as follows: The "First Generalization Postulate" generalizes from 1 dimension to n dimensions by replacing single quantities by n-way matrices. The "Second Generalization Postulate" changes n-way matrices to geometric objects. The "Third Generalization Postulate" changes geometric objects to tensors.

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[2] Ibid., pp. 47-91.

[3] Gabriel Kron, Tensor Analysis, pp. 242-245.

One implication of the type of analysis just outlined is that whenever it is successfully applied, many problems hitherto considered separate and original become merely routine transformations of some basic tensor. The mathematical analysis has partially paralleled geometrically what the engineer has accomplished physically. The future should see an extension of the parallelism which should result in many benefits to engineering.

# SOME ASPECTS OF THE ELEMENTARY APPLICATION OF TENSOR ANALYSIS TO NETWORK THEORY

Although not a necessity, the application of matrices in tensor analysis of networks is useful, for many of the manipulations in tensor analysis can be expressed very conveniently by the use of matrices.

A matrix is a set of quantities, that is, any aggregation of constants or variables arranged in a row, square, cube, or some other orderly manner, which is manipulated in accordance with certain fixed rules. A summary of the rules will be given shortly.

There are three kinds of notation in fairly common use,[1] direct notation, index notation, and matrix notation. The latter type will not be described since it is seldom used in electrical engineering problems. In direct notation, matrices are represented by bold face letters when printed or by letters with bars for script applications, for example, $\bar{A}$. In index notation, a matrix is written as a base letter with indices attached, for example, $A_{\alpha\beta}$. Each index represents a group of 1-matrices such that when each index attached to the matrix is given a particular value, a specific element in the matrix is defined. In the case of the matrix above, $\alpha$ could represent rows and $\beta$ columns. Thus, $A_{a\beta}$ would be row a, or $A_{\alpha c}$ column c, and A would be the element common to row a and column b. The rules of multiplication of matrices can be remembered easily if the rule for the multiplication of two 2-matrices is remembered. If $\bar{A}$ and $\bar{B}$ are 2-matrices, then $\bar{A} \cdot \bar{B} = \bar{C}$ equals the elements of each row of $\bar{A}$ times the elements of each column of $\bar{B}$, the sum of the products of the corresponding elements of row m and column n forming the element $C_{mn}$ in the 2-matrix $\bar{C}$. In index notation, the product is expressed as $A_{\alpha\beta} B_{\beta\gamma} = C_{\alpha\gamma}$, thus implying the use of the so-called Einstein convention, or summation

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[1] Gabriel Kron, Tensor Analysis of Networks, pp. 12-13.

convention that an index is summed on when it appears twice in the term. Examples of the multiplication of 2-matrices will be given later. Kron gives what he calls the "arrow rule"[2] which, in effect, states that in a matrix product, the matrices are split into 1-matrices along the repeated index and then corresponding 1-matrices are multiplied. Two 1-matrices are multiplied by multiplying corresponding elements and adding the products, thus giving a 0-matrix, or scalar. The product $\bar{A}\cdot\bar{b}$ where $\bar{A}$ is a 2-matrix and $\bar{b}$ a 1-matrix is obtained by multiplying each row of $\bar{A}$ times the row or column which is $\bar{b}$. It is immaterial whether a 1-matrix is expressed as a row or column. Using the arrow rule, $\bar{b}\cdot\bar{A}$ means elements of row (or column) of $\bar{b}$ times columns of $\bar{A}$. The result is a 1-matrix. Matrices are added by adding corresponding elements. Only matrices having the same number of free indices can be added. It is to be noted that free indices are indices such as $\alpha$ and $\beta$ in $A_{\alpha\beta}$, each of which represents several fixed indices.

The transpose of a matrix is a matrix with rows and columns interchanged. It is written in direct notation with the original base letter and a "t" subscript. The index notation of a matrix is not changed by interchanging rows and columns. The use of the transpose of a matrix is to enable products such as $A_{\alpha\beta}B_{\gamma\beta}$ to be represented in direct notation as $\bar{A}\cdot\bar{B}_t$. The inverse of a matrix, $\bar{A}$, $\bar{A}^{-1}$ is defined such that $\bar{A}\cdot\bar{A}^{-1} = \bar{I}$ where $\bar{I}$ is the unit matrix or Kronecker delta. Only 2-matrices have inverses. The inverse of a matrix is found as follows: Interchange rows and columns. Replace each element by its cofactor, Divide each cofactor by the value of the determinant of the matrix.

Because of the organizing power of "The First Generalization Postulate" which brought into view the concepts of transformation, invariance, and group,

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[2] Ibid., pp. 18-21.

the "Second Generalization Postulate" could be applied, geometric objects replacing n-matrices. Actually, a geometric object is represented only by an infinite number of n-matrices, each matrix representing a picture of the object from one reference frame. However, a geometric object may be considered completely represented by an n-matrix the components of which are along a certain reference frame, a set of fixed indices attached to the matrix to identify the reference frame, and a formula of transformation for finding the components along any other reference frame. To avoid clumsy statements, it will be stated that the matrix $\overline{A}$ represents the geometric object A, instead of postulating that it represents the components of A along some axes identified by the indices alongside the matrix.

To show how geometric objects are represented, examples of objects of valence 1, 2, and 3 will be given.

Two examples of vectors, or 1-matrices are:

$\overline{A} = A_\alpha =$

| α | a | b | c | d |
|---|---|---|---|---|
| | f | g | h | k |

$\overline{B} = B_\beta =$

| β | a | b | c | d | f | g | h |
|---|---|---|---|---|---|---|---|
| | 0 | 10 | 0 | 0 | 4 | 7 | 1 |

An example of a 2-matrix is:

$\overline{Z} = Z_{\alpha\beta} =$

| α \ β | a | b | c |
|---|---|---|---|
| a | 11 | 15 | 23 |
| b | 4 | 7 | 2 |
| c | 1 | 5 | 12 |

Particular rows or columns may be indicated as

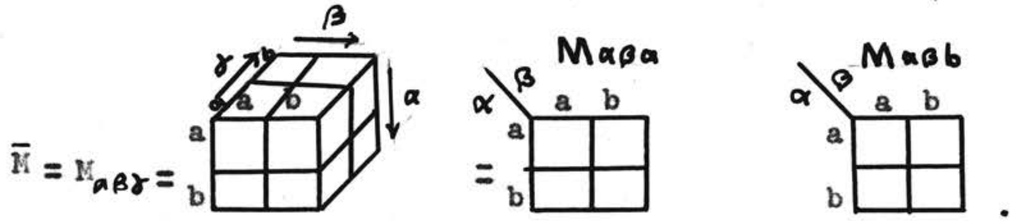
$Z_{\alpha c} =$

| | c |
|---|---|
| a | 23 |
| b | 2 |
| c | 12 |

$Z_{b\beta} =$

| a | b | c |
|---|---|---|
| 4 | 7 | 2 |

Two ways in which a 3-matrix may be represented are shown on the next page.




In the last representation, $M_{\alpha\beta\gamma}$ is split into layers along the index $\gamma$.

The basis for deciding whether or not some geometric object is a tensor is its transformation formula. The transformation formula for a tensor is:

$$T^{\alpha\beta\cdots}_{\gamma\delta\cdots} = T^{\alpha'\beta'\cdots}_{\gamma'\delta'\cdots}C^{\alpha}_{\alpha'}C^{\beta}_{\beta'}C^{\gamma'}_{\gamma}C^{\delta'}_{\delta}\cdots\cdots$$

Whenever an index is repeated in a product it indicates a summation. This is called the summation convention. That is, $a_\alpha x^\alpha = a_1 x^1 + a_2 x^2 + a_3 x^3 + \cdots a_n x^n$. The number of independent variables, $x^1$, $x^2$, $x^3, \ldots x^n$, is n, and is also the range of the free index $\alpha$. The transformation given above changes the components of $\bar{T}$ from the primed set of axes to the unprimed set.

When one set of variables can be expressed as a set of functions of another set of variables, a more definite expression for $C^{\alpha}_{\alpha'}$, the transformation tensor, can be obtained. For instance, if $x^\alpha = f(x^{\alpha'})$, or

$$x^1 = f^1(x^{1'}, x^{2'}, \cdots x^{n'})$$

$$x^2 = f^2(x^{1'}, x^{2'}, \cdots x^{n'})$$

$$x^3 = f^3(x^{1'}, x^{2'}, \cdots x^{n'})$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$x^n = f^n(x^{1'}, x^{2'}, \cdots, x^{n'})$$

then

$$dx^\alpha = \frac{\partial f(x^{\alpha'})}{\partial x^{\alpha'}}dx^{\alpha'} = \frac{\partial x^\alpha}{\partial x^{\alpha'}}dx^{\alpha'}.$$

If it is assumed that

$$dx^\alpha = C^{\alpha}_{\alpha'}dx^{\alpha'}, \quad C^{\alpha}_{\alpha'} = \frac{\partial x^\alpha}{\partial x^{\alpha'}}.$$

The transformation formula for a tensor then becomes

$$T^{\alpha\beta\cdots}_{\gamma\delta\cdots} = T^{\alpha'\beta'\cdots}_{\gamma'\delta'\cdots}\frac{\partial x^\alpha}{\partial x^{\alpha'}}\frac{\partial x^\beta}{\partial x^{\beta'}}\frac{\partial x^{\gamma'}}{\partial x^{\gamma}}\frac{\partial x^{\delta'}}{\partial x^{\delta}}.$$

A transformation for which $x^{\alpha}$ can be expressed as a set of functions of $x^{\alpha'}$ and for which $\frac{\partial x^{\alpha}}{\partial x^{\alpha'}}$ can be determined is called a holonomic transformation. When one set of variables cannot be expressed as a function of the other set, and $\frac{\partial x^{\alpha}}{\partial x^{\alpha'}}$ cannot be determined, the transformation is called non-holonomic. For a non-holonomic transformation, the $C^{\alpha}_{\alpha'}$'s must be determined some other way than by differentiation, perhaps by inspection. In electrical machinery problems which involve spatial motion, most of the transformations performed are non-holonomic.

If $C^{\alpha}_{\alpha'}$ is given, there is a simple way of finding whether or not the transformation is holonomic.[3] Consider two particular components of $C^{\alpha}_{\alpha'}$. If the transformation is holonomic, and

$$C^{a}_{b'} = \frac{\partial x^{a}}{\partial x^{b'}} \qquad C^{a}_{c'} = \frac{\partial x^{a}}{\partial x^{c'}}$$

then

$$\frac{\partial C^{a}_{b'}}{\partial x^{c'}} = \frac{\partial^2 x^{a}}{\partial x^{b'}\partial x^{c'}} = \frac{\partial C^{a}_{c'}}{\partial x^{b'}} = \frac{\partial^2 x^{a}}{\partial x^{c'}\partial x^{b'}}.$$

This is the test for determining what kind of transformation $C^{\alpha}_{\alpha'}$ represents. For covariant variables which are transformed by $C^{\alpha'}_{\alpha}$, there is a similar test. If the transformation is holonomic,

$$\frac{\partial C^{a'}_{a}}{\partial x_{b'}} = \frac{\partial C^{b'}_{a}}{\partial x_{a'}} = \frac{\partial^2 x_{a}}{\partial x_{a'}\partial x_{b'}}.$$

To obtain the geometric objects associated with a given network, the building blocks and the analytical units of the network must be considered. A network may be visualized as several lumped coils interconnected in some manner. It will be assumed that the propagation of superimposed quantities

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[3] Banesh Hoffmann, "What Is Tensor Analysis?", Electrical Engineering, LVII (February, 1938), 61-66.

through the network is instantaneous. The network may either be isolated or be a detached portion of some larger network. A component network not connected, or connected only by mutual inductance to another part of the network is called a sub-network. The two ends of a coil where it is joined to other coils in the network are called junctions. Coils and junctions are the two building blocks of networks. The characteristics of the coils are assumed not to vary with the superimposed electromagnetic quantities. The junctions are assumed to be fixed at the instant under consideration. The two analytical units of networks are the mesh and the junction pair. A mesh is any closed circuit traced through the network. To find the minimum number of meshes in a network, each coil must be traced through at least once. Any two junctions on the same sub-network are called a junction-pair. To avoid confusion in finding the minimum number of junction-pairs, it is usually most convenient to select one junction and pair it with all the other junctions on the same sub-network. This process is repeated for each sub-network. Meshes and junction-pairs will be assumed to include the concept of orientation. For instance, one mesh is the negative of another if they are traced over the same path in opposite directions.

There are two important relations between the analytical units and building blocks of networks.[4] The number of junctions minus the number of sub-networks equals the number of junction-pairs. The number of coils in the network equals the sum of the number of meshes and the number of junction-pairs.[5] These relations are useful in determining the number of meshes or the number of junction-pairs in a complex network.

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[4] O. Veblen, Analysis Situs, pp. 15 and 18.

[5] Kron, op. cit. p. 75.

Two analagous concepts to those of mesh and junction-pair are the branch and open mesh. A branch is a part of the network in which the same current flows. An open mesh is any circuit through the network that joins the two junctions forming a junction-pair. In network analysis, any mesh quantity can be replaced by a corresponding branch quantity. Similarly, any junction-pair quantity can be replaced by a corresponding open mesh quantity.

Electromagnetic quantities superimposed on a network may be divided into two types, impressed quantities and response quantities. These quantities may be either currents or voltages or both.

In setting up the equation of performance of a network, the variables may be either the mesh currents, $\bar{i}$, or the voltages, $\bar{E}$, across the junction-pairs. In some cases, both $\bar{E}$ and $\bar{i}$ must be assumed as variables. Such a network is called an orthogonal network. The equations of performance for mesh and junction networks are

$$\bar{e} = \bar{z}\cdot\bar{i}$$

for mesh networks and

$$\bar{I} = \bar{Y}\cdot\bar{E}$$

for junction networks.

For orthogonal networks either the equation of voltage or the equation of current may be used. The equations of performance are

$$\bar{E} + \bar{e} = \bar{z}\cdot(\bar{i} + \bar{I})$$
$$\bar{i} + \bar{I} = \bar{Y}\cdot(\bar{E} + \bar{e}).$$[6]

The two simplest collections of coils are called primitive networks. The primitive mesh network consists of n coils and n meshes, each coil being short-circuited upon itself. The primitive junction network consists of n

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6 *Ibid.*, pp. 82-84.

coils and n junction pairs, being merely a collection of open-circuited coils.

The solution of networks having n coils and less than n meshes follows from the solution of networks having n coils and n meshes, the only difference being that the opening of impedance-less branches which changes the all mesh network into a network having both meshes and junction-pairs introduces constraints due to the fact that some currents are not allowed to flow. The effect of this is to reduce the number of variables, since only as many branch or mesh currents as there are meshes need to be assumed. The constraints result in a singular transformation tensor which has more rows than columns, for the transformation tensor, $\bar{C}$, is determined by using the assumption $\bar{i} = \bar{C}\cdot\bar{i}'$ where $\bar{i}$ are coil currents in the primitive network and $\bar{i}'$ are branch or mesh currents in the network built out of the coils in the primitive network. A singular transformation matrix has no inverse, so the transformations which can be made with it are restricted.

The transformation from the primitive network to some other network leaves power an invariant. That is, $\bar{e}\cdot\bar{i} = \bar{e}'\cdot\bar{i}'$. From this, and the assumption that $\bar{i} = \bar{C}\cdot\bar{i}'$, the transformation formulas of all the quantities associated with a mesh network can be determined.[7] The most important of these are summarized below, both in direct and index notation.

$$\bar{i} = \bar{C}\cdot\bar{i}' \qquad\qquad i^{\alpha} = C^{\alpha}_{\alpha'}\, i^{\alpha'}$$

$$\bar{e}' = \bar{C}_t\cdot\bar{e} \qquad\qquad e_{\alpha'} = C^{\alpha}_{\alpha'}\, e_{\alpha}$$

$$\bar{z}' = \bar{C}_t\cdot\bar{z}\cdot\bar{C} \qquad\qquad z_{\alpha'\beta'} = z_{\alpha\beta}\, C^{\alpha}_{\alpha'}\, C^{\beta}_{\beta'}$$

$\bar{e}$ is the impressed voltage vector of the primitive network, $\bar{e}'$ the corresponding quantity for the given network, $\bar{z}$ the impedance tensor of the primitive network, and $\bar{z}'$ the impedance tensor of the given network.

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[7] *Ibid.*, pp. 102-104.

The first thing to do in solving a mesh network is to set up the primitive network and its voltage vector and impedance tensor. Next, arbitrarily assume as many independent branch or mesh currents in the given network as there are meshes. By using Kirchhoff's current law, determine all coil currents in the network in terms of the assumed branch currents. Since $\bar{i} = \bar{C}\cdot\bar{i}'$, the transformation tensor, $\bar{C}$, may be determined from the above process. Next, find $\bar{z}'$. Then invert $\bar{z}'$ to get $\bar{y}'$, the admittance tensor of the given network. Find $\bar{i}'$ by $\bar{i}' = \bar{y}'\cdot\bar{e}'$. If desired, find $\bar{e}_c$, the coil voltages, by $\bar{e}_c = \bar{z}\cdot\bar{C}\cdot\bar{i}'$. An example showing how $\bar{C}$ is determined and $\bar{z}'$ is found will be given later.

Besides merely solving for currents, various other transformations may be made by using a transformation tensor.[8] A few of these are: An n-coil all mesh network may be used instead of the primitive network to analyze an n-coil, less than n-mesh network. One set of branch currents may be replaced either by another set of branch currents flowing in the same network, or by mesh currents flowing in the network. Magnetizing currents may be neglected, the number of turns in a coil may be changed, and meshes may be opened. All of these processes may be expressed in terms of a transformation tensor, $\bar{C}$.

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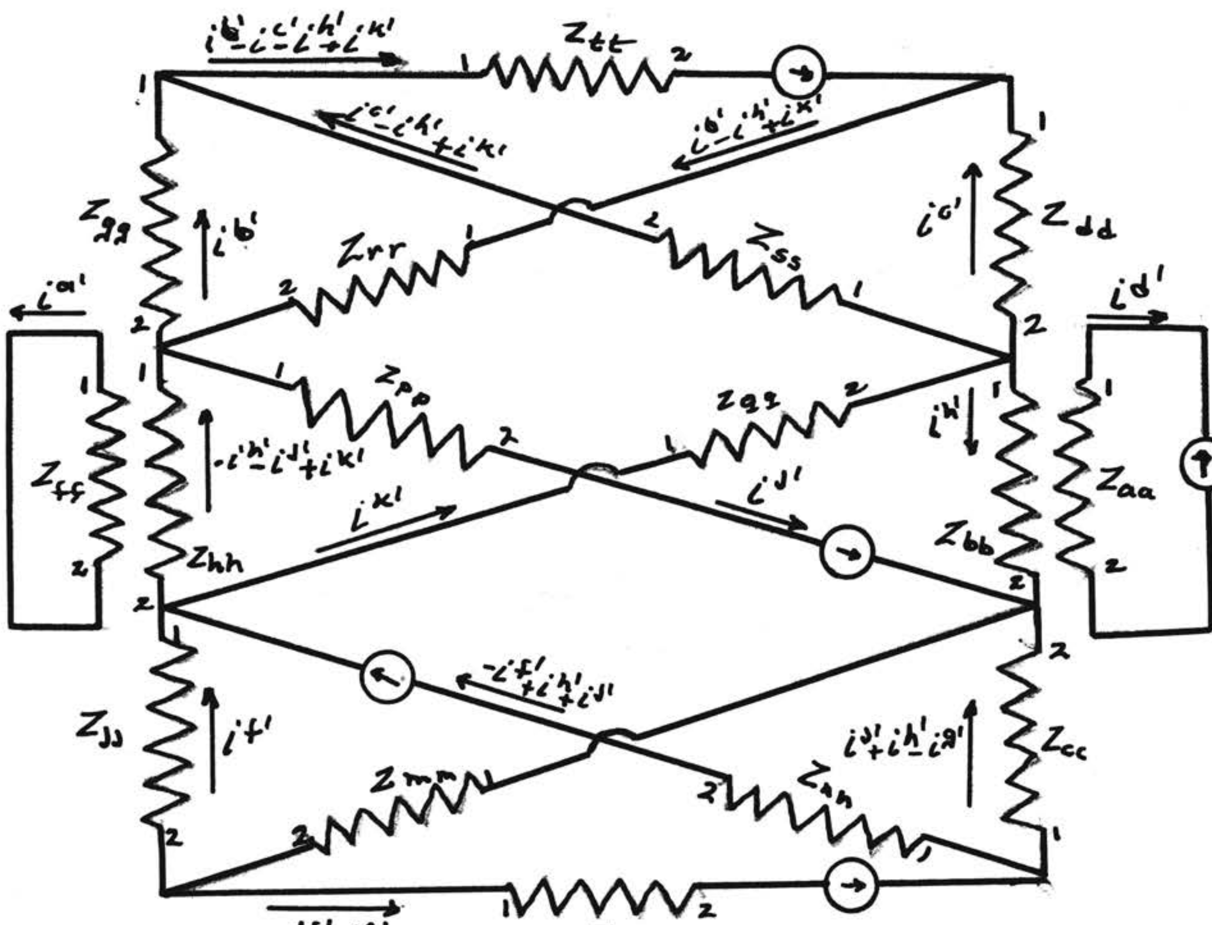
[8] Ibid., pp. 141-173.

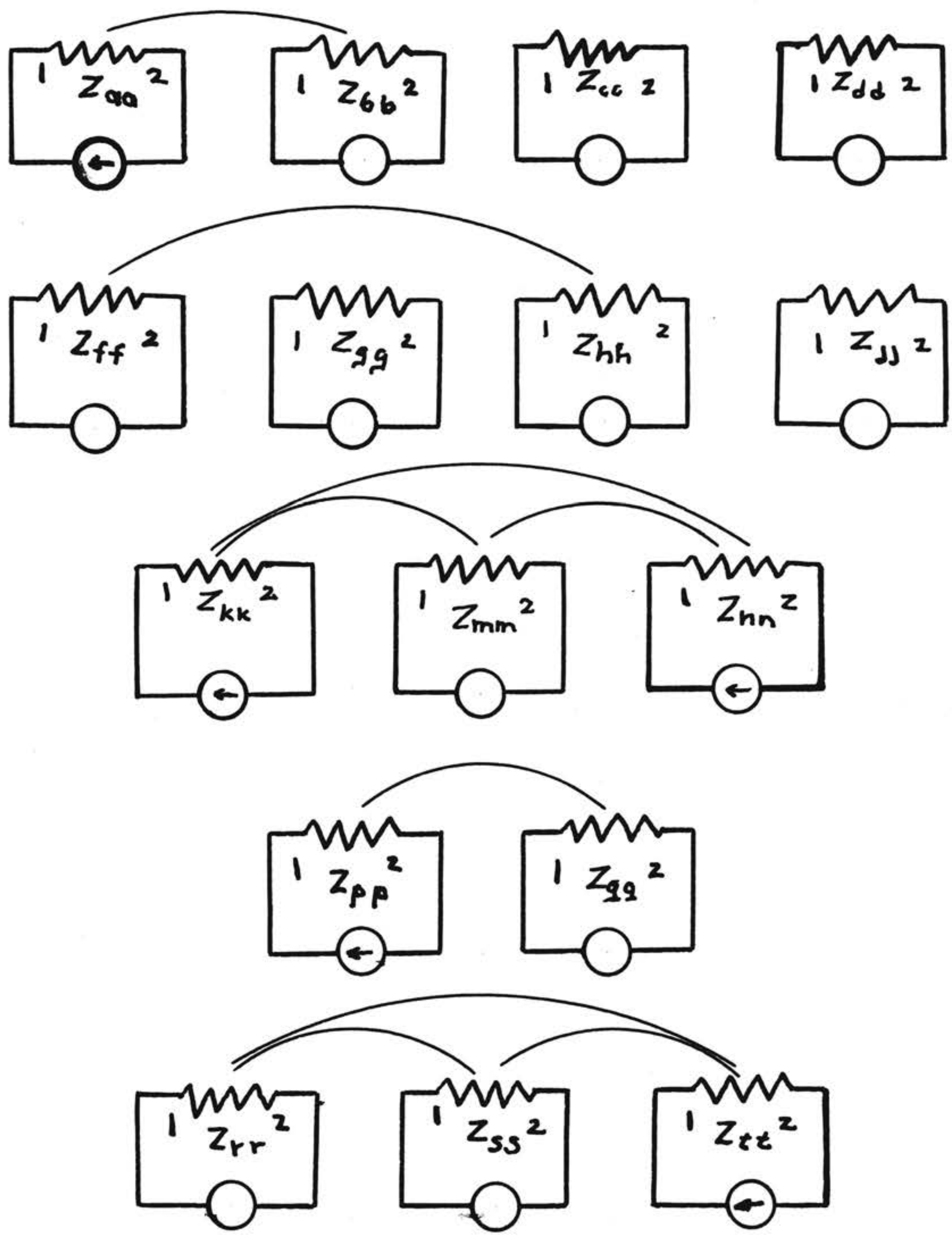
THE APPLICATION OF TENSOR ANALYSIS TO THE SOLUTION OF A MESH NETWORK

In order to show how tensor analysis is used to solve a network, the transformation tensor, impedance tensor, and voltage vector of a given network will be found. The impedance tensor will first be found by means of the primitive network and then by using three other networks as a primitive system from whose interconnection the given network is obtained.

The given network is shown below. There are sixteen coils, ten junctions, and three sub-networks. The number of junctions minus the number of sub-networks gives seven as the number of junction-pairs. The number of coils minus the number of junction-pairs equals nine, the number of meshes. Therefore, nine mesh currents must be assumed. These currents and the coil currents determined from them are also shown.



The primitive network is shown below. The lines connecting the coils indicate mutual inductance



The voltage vector of the primitive network is:

$\bar{e} =$

| a | b | c | d | f | g | h | j | k | l | m | n | p | q | r | s | t |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $e_a$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $e_k$ | 0 | 0 | $e_n$ | $e_p$ | 0 | 0 | 0 | $e_t$ |

Equating old and new currents:

$$i^a = -i^{d'}$$
$$i^b = i^{h'}$$
$$i^c = -i^{g'} + i^{h'} + i^{j'}$$
$$i^d = -i^{c'}$$
$$i^f = -i^{a'}$$
$$i^g = -i^{b'}$$
$$i^h = -i^{h'} - i^{j'} + i^{k'}$$
$$i^j = -i^{f'}$$
$$i^k = -i^{f'} + i^{g'}$$
$$i^m = i^{g'}$$
$$i^n = -i^{f'} + i^{h'} + i^{j'}$$
$$i^p = i^{j'}$$
$$i^q = i^{k'}$$
$$i^r = i^{b'} - i^{h'} + i^{k'}$$
$$i^s = -i^{c'} - i^{h'} + i^{k'}$$
$$i^t = i^{b'} - i^{c'} + i^{k'}$$

Since $\bar{i} = \bar{C} \cdot \bar{i}'$, the components of the transformation tensor, $\bar{C}$, are the coefficients of the i' terms in the above set of equations which express the currents in the primitive network in terms of those in the given network.[1] The transformation tensor is shown in the product $\bar{z} \cdot \bar{C}$. To find the impedance tensor of the given network, the transformation formula $\bar{z}' = \bar{C}_t \cdot \bar{z} \cdot \bar{C}$

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[1] Kron, <u>Tensor Analysis of Networks</u>, pp. 98-102.

will be used. The product $\bar{z}\cdot\bar{C}$ will be calculated first. Whether $\bar{z}\cdot\bar{C}$ or $\bar{C}_t\cdot\bar{z}$ is found is largely a matter of personal choice except when $\bar{e}_c = \bar{z}\cdot\bar{C}\cdot\bar{i}'$ is to be found. Then it is convenient to have the product $\bar{z}\cdot\bar{C}$.[2] The matrix product below gives the intermediary geometric object $z_{\alpha\beta'}$, which is expressed along two different reference frames.[3]

| | a | b | c | d | f | g | h | j | k | m | n | p | q | r | s | t |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| a | $z_{aa}$ | $x_{ab}$ | | | | | | | | | | | | | | |
| b | $x_{ab}$ | $z_{bb}$ | | | | | | | | | | | | | | |
| c | | | $z_{cc}$ | | | | | | | | | | | | | |
| d | | | | $z_{dd}$ | | | | | | | | | | | | |
| f | | | | | $z_{ff}$ | | $x_{fh}$ | | | | | | | | | |
| g | | | | | | $z_{gg}$ | | | | | | | | | | |
| h | | | | | $x_{fh}$ | | $z_{hh}$ | | | | | | | | | |
| j | | | | | | | | $z_{jj}$ | | | | | | | | |
| k | | | | | | | | | $z_{kk}$ | $x_{km}$ | $x_{kn}$ | | | | | |
| m | | | | | | | | | $x_{km}$ | $z_{mm}$ | $x_{mn}$ | | | | | |
| n | | | | | | | | | $x_{kn}$ | $x_{mn}$ | $z_{nn}$ | | | | | |
| p | | | | | | | | | | | | $z_{pp}$ | $x_{pq}$ | | | |
| q | | | | | | | | | | | | $x_{pq}$ | $z_{qq}$ | | | |
| r | | | | | | | | | | | | | | $z_{rr}$ | $x_{rs}$ | $x_{rt}$ |
| s | | | | | | | | | | | | | | $x_{rs}$ | $z_{ss}$ | $x_{st}$ |
| t | | | | | | | | | | | | | | $x_{rt}$ | $x_{st}$ | $z_{tt}$ |

The Impedance Tensor of the Primitive Network •

| | a' | b' | c' | d' | f' | g' | h' | j' | k' |
|---|---|---|---|---|---|---|---|---|---|
| a | | | | -1 | | | | | |
| b | | | | | | | 1 | | |
| c | | | | | | -1 | 1 | 1 | |
| d | | | -1 | | | | | | |
| f | -1 | | | | | | | | |
| g | | -1 | | | | | | | |
| h | | | | | | | -1 | -1 | 1 |
| j | | | | | -1 | | | | |
| k | | | | | -1 | 1 | | | |
| m | | | | | | 1 | | | |
| n | | | | | -1 | | 1 | 1 | |
| p | | | | | | | | 1 | |
| q | | | | | | | | | 1 |
| r | | 1 | | | | | -1 | | 1 |
| s | | | -1 | | | | -1 | | 1 |
| t | | 1 | -1 | | | | -1 | | 1 |

The Transformation Tensor

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2 Ibid., p. 110.

3 Ibid., pp. 178-179.

This is the product of the two tensors on the preceding page, $z_{\alpha\beta}'$.

| | a' | b' | c' | d' | f' | g' | h' | j' | k' |
|---|---|---|---|---|---|---|---|---|---|
| a | | | | $-z_{aa}$ | | | $x_{ab}$ | | |
| b | | | | $-x_{ab}$ | | | $z_{bb}$ | | |
| c | | | | | | $-z_{cc}$ | $z_{cc}$ | $z_{cc}$ | |
| d | | | $-z_{dd}$ | | | | | | |
| f | $-z_{ff}$ | | | | | | $-x_{fh}$ | $-x_{fh}$ | $x_{fh}$ |
| g | | $-z_{gg}$ | | | | | | | |
| h | $-x_{fh}$ | | | | | | $-z_{hh}$ | $-z_{hh}$ | $z_{hh}$ |
| j | | | | | $-z_{jj}$ | | | | |
| k | | | | | $-z_{kk} - x_{km}$ | $z_{kk} + x_{km}$ | $x_{kn}$ | $x_{kn}$ | |
| m | | | | | $-x_{km} - x_{mn}$ | $x_{km} + z_{mm}$ | $x_{mn}$ | $x_{mn}$ | |
| n | | | | | $-x_{kn} - z_{nn}$ | $x_{kn} + x_{mn}$ | $z_{nn}$ | $z_{nn}$ | |
| p | | | | | | | | $z_{pp}$ | $x_{pq}$ |
| q | | | | | | | | $x_{pq}$ | $z_{qq}$ |
| r | | $z_{rr} + x_{rt}$ | $-x_{rs} - x_{rt}$ | | | | $-z_{rr} - x_{rs} - x_{rt}$ | | $z_{rr} + x_{rs} + x_{rt}$ |
| s | | $x_{rs} + x_{st}$ | $-z_{ss} - x_{st}$ | | | | $-x_{rs} - z_{ss} - x_{st}$ | | $x_{rs} + z_{ss} + x_{st}$ |
| t | | $x_{rt} + z_{tt}$ | $-x_{st} - z_{tt}$ | | | | $-x_{rt} - x_{st} - z_{tt}$ | | $x_{rt} + x_{st} + z_{tt}$ |

$\bar{C}_t \cdot (\bar{z} \cdot \bar{C}) = \bar{z}'$, the impedance tensor of the given network which is:

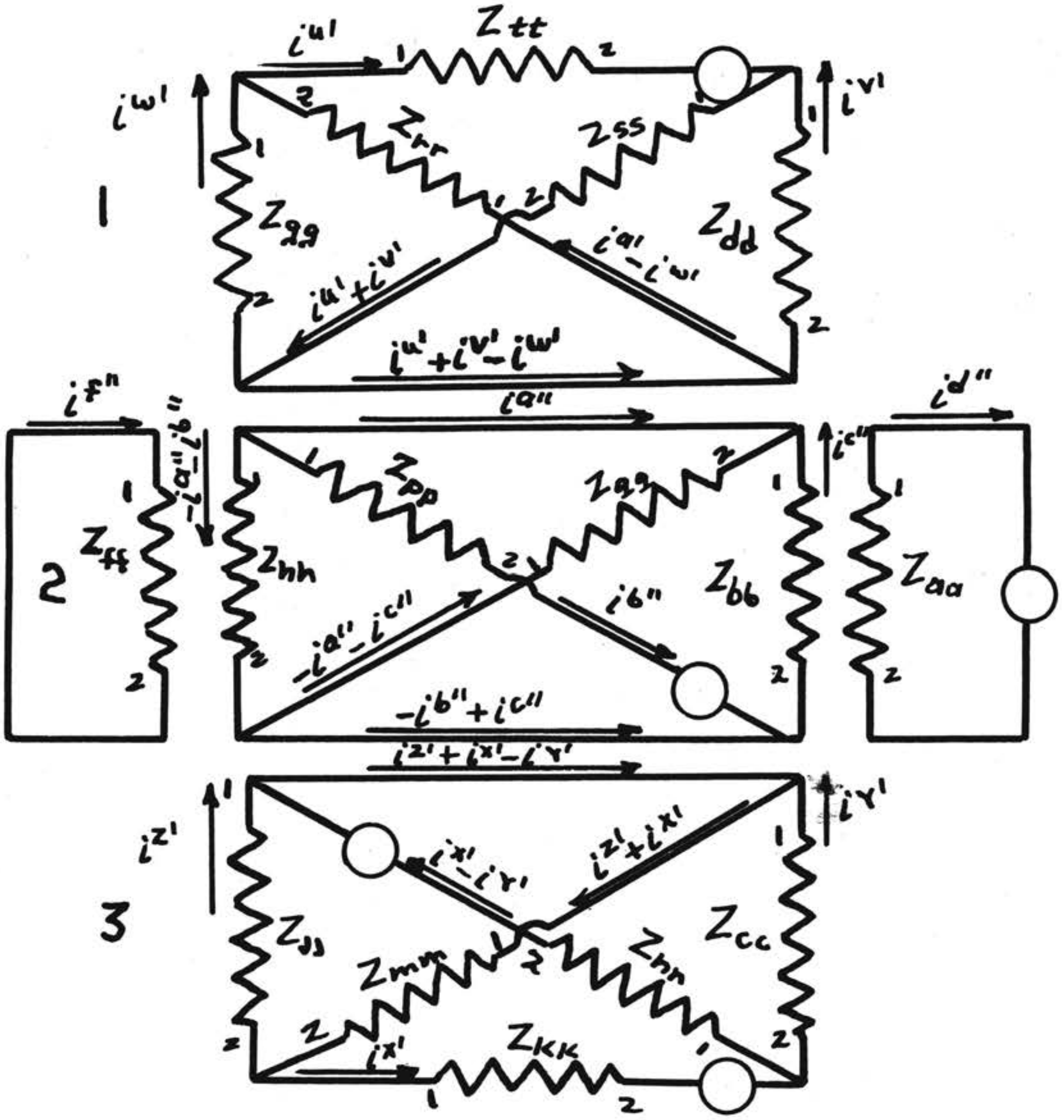
| | a' | b' | c' | d' | f' | g' | h' | j' | k' |
|---|---|---|---|---|---|---|---|---|---|
| a' | $z_{ff}$ | | | | | | $x_{fh}$ | $x_{fh}$ | $-x_{fh}$ |
| b' | | $z_{rr}+z_{tt}+z_{gg}+2x_{rt}$ | $-z_{tt}-x_{rs}-x_{rt}-x_{st}$ | | | | $-z_{rr}-z_{tt}-x_{rs}-2x_{rt}-x_{st}$ | | $z_{rr}+x_{rs}+2x_{rt}+x_{st}+z_{tt}$ |
| c' | | $-x_{rs}-x_{st}-z_{tt}-x_{rt}$ | $z_{ss}+z_{tt}+z_{dd}+2x_{st}$ | | | | $z_{ss}+z_{tt}+x_{rs}+x_{rt}+2x_{st}$ | | $-z_{ss}-x_{rs}-x_{rt}-2x_{st}-z_{tt}$ |
| d' | | | | $z_{aa}$ | | | $-x_{ab}$ | | |
| f' | | | | | $z_{kk}+z_{kn}+z_{nn}+2x_{kn}$ | $-z_{kk}-x_{km}-x_{kn}-x_{mn}$ | $-z_{mn}-x_{kn}$ | $-z_{mn}-x_{kn}$ | |
| g' | | | | | $-z_{kn}-x_{kn}-x_{km}-x_{mn}$ | $z_{kk}+z_{cc}+z_{mm}+2x_{km}$ | $-z_{cc}+x_{kn}+x_{mn}$ | $-z_{cc}+x_{kn}+x_{mn}$ | |
| h' | $x_{fh}$ | $-z_{pr}-x_{rs}-x_{st}-2x_{rt}-z_{tt}$ | $x_{rs}+x_{rt}+z_{ss}+2x_{st}+z_{tt}$ | $-x_{ab}$ | $-z_{mn}-x_{kn}$ | $-z_{cc}+x_{kn}+x_{mn}$ | $z_{rr}+z_{ss}+z_{tt}+z_{cc}+z_{hh}+z_{mm}+z_{bb}+2x_{rs}+2x_{rt}+2x_{st}$ | $z_{cc}+z_{hh}+z_{nn}$ | $-z_{rr}-z_{ss}-z_{tt}-2x_{rt}-2x_{st}-2x_{rs}-z_{hh}$ |
| j' | $x_{fh}$ | | | | $-x_{kn}-z_{nn}$ | $-z_{cc}+x_{kn}+x_{mn}$ | $z_{cc}+z_{hh}+z_{nn}$ | $z_{cc}+z_{hh}+z_{nn}+z_{pp}$ | $x_{pq}-z_{hh}$ |
| k' | $-x_{fh}$ | $z_{rr}+z_{tt}+2x_{rt}+x_{rs}+x_{st}$ | $-x_{rs}-x_{rt}-z_{ss}-2x_{st}-z_{tt}$ | | | | $-z_{hh}-z_{rr}-z_{ss}-z_{tt}-2x_{rs}-2x_{rt}-2x_{st}$ | $x_{pq}-z_{hh}$ | $z_{hh}+z_{rr}+z_{ss}+z_{tt}+2x_{rs}+z_{gg}+2x_{rt}+2x_{st}$ |

The voltage vector of the given network, $C_t\ e$, equals

| a' | b' | c' | d' | f' | g' | h' | j' | k' |
|---|---|---|---|---|---|---|---|---|
| 0 | $e_t$ | $-e_t$ | $-e_a$ | $-e_k - e_n$ | $e_k$ | $e_n - e_t$ | $e_p + e_n$ | $e_t$ |

.

The given network may be considered as having been built out of the interconnection of several component networks. If the impedance tensors of these networks were already known, it might be desirable to work the problem from this standpoint. To illustrate this method, the impedance tensors of three separate networks, 1, 2, and 3, will be combined and transformed to obtain the impedance tensor of the given network. The three networks are called the primitive system.




The Primitive System

The impedance tensor of the primitive system is shown below. The heavy lines border the impedance tensors of networks 1, 2, and 3.

| | u' | v' | w' | a" | b" | c" | d" | f" | x' | y' | z' |
|---|---|---|---|---|---|---|---|---|---|---|---|
| u' | $z_{rr}+z_{ss}+z_{tt}+2x_{rs}+2x_{rt}+2x_{st}$ | $z_{rr}+x_{rs}+x_{rt}$ | $-x_{rs}-z_{ss}-x_{st}$ | | | | | | | | |
| v' | $z_{rr}+x_{rs}+x_{rt}$ | $z_{dd}+z_{rr}$ | $-x_{rs}$ | | | | | | | | |
| w' | $-x_{rs}-z_{ss}-x_{st}$ | $-x_{rs}$ | $z_{gg}+z_{ss}$ | | | | | | | | |
| a" | | | | $z_{hh}+z_{qq}$ | $z_{hh}-x_{pq}$ | $z_{qq}$ | | $-x_{fh}$ | | | |
| b" | | | | $z_{hh}-x_{pq}$ | $z_{hh}+z_{pp}$ | $-x_{pq}$ | | $-x_{fh}$ | | | |
| c" | | | | $z_{qq}$ | $-x_{pq}$ | $z_{bb}+z_{qq}$ | $x_{ab}$ | | | | |
| d" | | | | | | $x_{ab}$ | $z_{aa}$ | | | | |
| f" | | | | $-x_{fh}$ | $-x_{fh}$ | | | $z_{ff}$ | | | |
| x' | | | | | | | | | $z_{kk}+z_{mn}+z_{nn}+2x_{km}+2x_{mn}+2x_{kn}$ | $x_{km}+x_{mn}+z_{nn}$ | $-x_{km}-z_{mm}-x_{mn}$ |
| y' | | | | | | | | | $x_{kn}+x_{mn}+z_{nn}$ | $z_{cc}+z_{nn}$ | $-x_{mn}$ |
| z' | | | | | | | | | $-x_{km}-x_{mn}-z_{mn}$ | $-x_{mn}$ | $z_{jj}+z_{mm}$ |

The transformation tensor, $\bar{C}'$, which changes the primitive system to the given network must now be found by setting corresponding currents in the primitive system and the given network equal to each other and by again using the property

$$\bar{i} = \bar{C}' \cdot \bar{i}' .$$

$$i^{u'} = i^{b'} - i^{c'} - i^{h'} + i^{k'}$$

$$i^{v'} = i^{c'}$$

$$i^{w'} = i^{b'}$$

$$i^{a''} = -i^{u'} - i^{v'} + i^{w'} = -i^{h'} - i^{k'}$$

$$i^{b''} = i^{j'}$$

$$i^{c''} = -i^{h'}$$

$$i^{d''} = i^{d'}$$

$$i^{f''} = -i^{a''}$$

$$i^{x'} = -i^{f'} + i^{g'}$$

$$i^{y'} = -i^{g'} + i^{h'} + i^{j'}$$

$$i^{z'} = -i^{f'}$$

$\bar{C}' =$

| | a' | b' | c' | d' | f' | g' | h' | j' | k' |
|---|---|---|---|---|---|---|---|---|---|
| u' | | 1 | -1 | | | | -1 | | 1 |
| v' | | | 1 | | | | | | |
| w' | | 1 | | | | | | | |
| a" | | | | | | | 1 | | -1 |
| b" | | | | | | | | 1 | |
| c" | | | | | | | -1 | | |
| d" | | | | 1 | | | | | |
| f" | -1 | | | | | | | | |
| x' | | | | | -1 | 1 | | | |
| y' | | | | | | -1 | 1 | 1 | |
| z' | | | | | -1 | | | | |

The impedance tensor of the given network, $\bar{z}'$, is found by the formula

$$\bar{z}' = \bar{C}_t' \cdot \bar{z}_p \cdot \bar{C}'$$

where $\bar{z}_p$ is the impedance tensor of the primitive system. If the multiplications are performed, the result will be found to be equal to the one obtained by using the primitive network.

## COMPOUND TENSORS, MULTIPLE TENSORS, AND COMPOUND COILS

Oftentimes in a set of equations such as

$$\bar{e} = \bar{\bar{z}} \cdot \bar{i},$$

there is a natural subdivision of the set into two or more groups. One group may represent those meshes having impressed voltages, the other group those having no impressed voltages, or in the equation

$$\bar{i} = \bar{\bar{y}} \cdot \bar{e},$$

all the currents may not be required, so there are two groups of currents, one group wanted and one unwanted. In a junction network, there are often three or four different types of junction-pairs,[1] ie., those across which there are currents impressed, those which supply currents to outside loads, and those that are inactive. The inactive ones may be further subdivided as to those whose voltages are wanted and those whose voltages are not wanted. This subdivision of the network quantities is one reason for the use of compound tensors. In the example to follow, the equation

$$\bar{e}' = \bar{\bar{z}}' \cdot \bar{i}'$$

will be divided into two invariant equations in two different ways. The method of accomplishing this is shown below. Originally,

$$\begin{array}{c|c|} & \\ \hline 1 & e'_1 \\ \hline 2 & e'_2 \\ \hline 3 & e'_3 \\ \hline \end{array} = \begin{array}{c|c|c|c|} & 1 & 2 & 3 \\ \hline 1 & z'_{11} & z'_{12} & z'_{13} \\ \hline 2 & z'_{21} & z'_{22} & z'_{23} \\ \hline 3 & z'_{31} & z'_{32} & z'_{33} \\ \hline \end{array} \cdot \begin{array}{c|c|} & \\ \hline 1 & i^{1'} \\ \hline 2 & i^{2'} \\ \hline 3 & i^{3'} \\ \hline \end{array}$$

$$\bar{e}' = \bar{\bar{z}}' \cdot \bar{i}'$$

$\bar{e}'$, $\bar{\bar{z}}'$, and $\bar{i}'$ are subdivided along the heavy lines giving

$$\bar{e}'_1 = \begin{array}{|c|c|} 1 & 2 \\ \hline e'_1 & e'_2 \\ \hline \end{array}$$

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[1] Gabriel Kron, <u>Tensor Analysis of Networks</u>, pp. 484-485.

$$\bar{e}_2 = \begin{bmatrix} e_3' \end{bmatrix}$$

$$\bar{i}^1 = \begin{bmatrix} i^{1'} & i^{2'} \end{bmatrix}$$

$$\bar{i}^2 = \begin{bmatrix} i^{3'} \end{bmatrix}$$

$$\bar{z}_1 = \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & z_{11}' & z_{12}' \\ 2 & z_{21}' & z_{22}' \end{array}$$

$$\bar{z}_2 = \begin{array}{c|c} 1 & z_{13}' \\ 2 & z_{23}' \end{array}$$

$$\bar{z}_3 = \begin{array}{cc} 1 & 2 \\ \hline z_{31}' & z_{32}' \end{array}$$

$$\bar{z}_4 = \begin{bmatrix} z_{33}' \end{bmatrix}$$

The two invariant equations are

$$\begin{array}{c|c} 1 & e_1' \\ 2 & e_2' \end{array} = \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & z_{11}' & z_{12}' \\ 2 & z_{21}' & z_{22}' \end{array} \cdot \begin{array}{c|c} 1 & i^{1'} \\ 2 & i^{2'} \end{array} + \begin{array}{c|c} 1 & z_{13}' \\ 2 & z_{23}' \end{array} \cdot \begin{bmatrix} i^{3'} \end{bmatrix}$$

and

$$\begin{bmatrix} e_3' \end{bmatrix} = \begin{bmatrix} z_{31}' & z_{32}' \end{bmatrix} \cdot \begin{bmatrix} i^{1'} & i^{2'} \end{bmatrix} + \begin{bmatrix} z_{33} \end{bmatrix} \cdot \begin{bmatrix} i^{3'} \end{bmatrix}$$

These invariant equations can be written

$$\bar{e}_1 = \bar{z}_1 \cdot \bar{i}^1 + \bar{z}_2 \cdot \bar{i}^2$$

$$\bar{e}_2 = \bar{z}_2 \cdot \bar{i}^1 + \bar{z}_2 \cdot \bar{i}^2 \quad .$$

The original equation may now be expressed in terms of compound tensors as

$$\begin{array}{cc} \bar{1} & \bar{2} \\ \hline \bar{e}_1 & \bar{e}_2 \end{array} = \begin{array}{c|cc} & \bar{1} & \bar{2} \\ \hline \bar{1} & \bar{z}_1 & \bar{z}_2 \\ \bar{2} & \bar{z}_3 & \bar{z}_4 \end{array} \cdot \begin{array}{c|c} \bar{1} & \bar{i}^1 \\ \bar{2} & \bar{i}^2 \end{array}$$

$$\bar{e}' = \bar{z}' \cdot \bar{i}'$$

The index, $\bar{1}$, represents indices 1 and 2 and is called a compound index. Index $\bar{2}$ in this particular case only represents index 3, but it is a compound index and, in general, would represent more than one index. The subdivision above will be applied in another example to show how a reduction formula may be used to eliminate a mesh from a network.

There are other ways of subdividing the above equation. It might have been divided so that

$$\begin{array}{c|c} 1 & e_1' \\ \hline 2 & e_2' \\ 3 & e_3' \end{array} = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & z_{11}' & z_{12}' & z_{13}' \\ \hline 2 & z_{21}' & z_{22}' & z_{23}' \\ 3 & z_{31}' & z_{32}' & z_{33}' \end{array} \cdot \begin{array}{c|c} 1 & i^{1'} \\ 2 & i^{2'} \\ 3 & i^{3'} \end{array}$$

which is equivalent to

$$\begin{array}{c|c} \bar{1} & \bar{e}_1' \\ \bar{2} & \bar{e}_2' \end{array} \cdot \begin{array}{c|c} \bar{1} & \bar{z}_1' \\ \bar{2} & \bar{z}_2' \end{array} \cdot \boxed{\bar{i}'}$$

This represents the equations

$$\bar{e}_1' = \bar{z}_1' \cdot \bar{i}'$$
$$\bar{e}_2' = \bar{z}_2' \cdot \bar{i}' \ .$$

By following this same line of reasoning, geometric objects of higher valence may be subdivided in many ways. Compound tensors are manipulated analogously to ordinary tensors,[2] but a few precautions must be observed.

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[2] Ibid., pp. 222-227.

Operations may be performed only when the compound indices involved each represent the same fixed indices. In multiplying, the order of multiplication of the geometric objects which compose the compound object must not be disturbed, since matrix algebra is non-commutative. To illustrate this, the multiplication of two compound vectors will be performed.

$$\begin{bmatrix} \bar{A} & \bar{B} & \bar{C} \end{bmatrix} \begin{bmatrix} \bar{D} & \bar{E} & \bar{F} \end{bmatrix} = \begin{bmatrix} \bar{A}\cdot\bar{D} + \bar{B}\cdot\bar{E} + \bar{C}\cdot\bar{F} \end{bmatrix}$$

and not $\bar{D}\cdot\bar{A} + \bar{E}\cdot\bar{B} + \bar{F}\cdot\bar{C}$ .

In taking the transpose of a compound tensor, the transpose of each element is taken in addition to interchanging rows and columns. For instance, if

$$\bar{C} = \begin{bmatrix} \bar{A} & \bar{D} \\ \bar{B} & \bar{E} \\ \bar{H} & \bar{F} \end{bmatrix}$$

then

$$\bar{C}_t = \begin{bmatrix} \bar{A}_t & \bar{B}_t & \bar{H}_t \\ \bar{D}_t & \bar{E}_t & \bar{F}_t \end{bmatrix}$$

In index notation, compound tensors may be represented by some scheme such as letting Greek letters represent all indices, letters from a to f represent one group of fixed indices, g to k another group, and other letters in the remainder of the alphabet the compound indices, each of which represents a group of the fixed indices.[3]

The compounding of tensors may be carried several degrees further. A compound tensor may be subdivided to form a doubly compound tensor which may again be subdivided to form a triply compound tensor and so on. The compound

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[3] Ibid., pp. 227-229.

tensor

$$\bar{z} = \begin{array}{c|c|c|} & \bar{1} & \bar{2} \\ \hline \bar{1} & \bar{z}_1 & \bar{z}_2 \\ \hline \bar{2} & \bar{z}_3 & \bar{z}_4 \\ \hline \end{array}$$

might have been divided along the heavy line to form the doubly compound tensor

$$\bar{z} = \begin{array}{c|c|} \hline \bar{a} & \bar{z}_a \\ \hline \bar{b} & \bar{z}_b \\ \hline \end{array} \; .$$

This is a very simple example, but all subdivisions follow the same method regardless of the number of elements.[5]

The compound tensors which have been described have the common property that all the components are expressed along the same reference frame. Sometimes there are problems where it is convenient or necessary to use tensors having several sets of variable indices each set of which belongs to a different reference frame and may transform under a different group of transformation tensors.[5] The tensor of valence four, $A_{\alpha\beta mn}$ might conceivably be a tensor whose indices $\alpha\beta$ transform under the group of transformations $C^{\alpha}_{\beta'}$, while the indices mn transform under an entirely different group of transformations, $C^{m'}_{n}$, the two sets of indices being expressed along different reference axes. This condition occurs in tube circuits where several currents of different frequencies may flow simultaneously, thus requiring the use of multiple tensors, as these entities are called.[6] Multiple tensors may be subdivided just as ordinary tensors are to form compound multiple tensors.

When the equation of performance of a network is subdivided into several

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[5] <u>Ibid</u>., pp. 538-543.

[6] <u>Ibid</u>., pp. 547-549.

equations, a fictitious network called a compound network is sometimes used to give a physical picture of the equations.[7] The voltage and current in each compound coil of the network are vectors while the impedance is a tensor of valence two. Each coil represents a whole network and its reference frames may be changed by its "individual impedance tensor" without affecting the remainder of the network.[8] To distinguish them from ordinary networks, the coils in a compound network are drawn with heavy lines. These networks may be considered a generalized concept of the single line diagram used in ordinary three-phase circuit analysis. A single line diagram and a compound network could be made to correspond for a given three-phase network if the compound coils were properly chosen.

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[7] Ibid., pp. 480-496.

[8] Ibid., pp. 501-509.

# REDUCTION FORMULAS

When dealing with a network, it is sometimes advantageous to reduce the number of meshes which must be considered. This is done either to reduce the labor of computation, or because some of the mesh currents are not wanted. The eliminated meshes may or may not have impressed voltages in them. The analysis for each case differs slightly.

An example of the reduction by one of the number of meshes in a simple network will be given and a formula will be developed which can be used for any similar type of reduction. There will be no impressed voltage in the eliminated mesh. It will also be shown how the elimination of one mesh is equivalent to a delta-wye conversion.[1] The elimination of more than one mesh is equivalent to several delta-wye conversions.

Consider the equation

$$\bar{e} = \bar{z} \cdot \bar{i}$$

for the performance of a mesh network. The same process applies to networks having different numbers of meshes than the one used for the illustration below.

$$\begin{array}{|c|}\hline e_1 \\ \hline e_2 \\ \hline e_3 \\ \hline 0 \\ \hline \end{array} = \begin{array}{|c|c|c|c|}\hline a & b & c & d \\ \hline e & f & g & h \\ \hline i & j & k & l \\ \hline m & n & o & p \\ \hline \end{array} \cdot \begin{array}{|c|}\hline i^1 \\ \hline i^2 \\ \hline i^3 \\ \hline i^4 \\ \hline \end{array}$$

In terms of compound tensors, this equation may be expressed as

$$\begin{array}{|c|}\hline \bar{e}_1 \\ \hline \bar{0} \\ \hline \end{array} = \begin{array}{|c|c|}\hline \bar{z}_1 & \bar{z}_2 \\ \hline \bar{z}_3 & \bar{z}_4 \\ \hline \end{array} \cdot \begin{array}{|c|}\hline \bar{i}^1 \\ \hline \bar{i}^2 \\ \hline \end{array}$$

which is equivalent to the two equations which follow on the next page.

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[1] Ibid., pp. 261-264.

$$\bar{e}_1 = \bar{z}_1 \cdot \bar{i}^1 + \bar{z}_2 \cdot \bar{i}^2$$
$$\bar{0} = \bar{z}_3 \cdot \bar{i}^1 + \bar{z}_4 \cdot \bar{i}^2$$

It is desired to eliminate $\bar{i}^2$ from these equations. Solving for $\bar{i}^2$ in the second equation,

$$\bar{i}^2 = - \bar{z}_4^{-1} \cdot \bar{z}_3 \cdot \bar{i}^1$$

Substituting this expression for $\bar{i}^2$ in the first equation,

$$\bar{e}_1 = \bar{z}_1 \cdot \bar{i}^1 - \bar{z}_2 \cdot \bar{z}_4^{-1} \cdot \bar{z}_3 \cdot \bar{i}^1$$

or,

$$\bar{e}_1 = (\bar{z}_1 - \bar{z}_2 \cdot \bar{z}_4^{-1} \cdot \bar{z}_3) \cdot \bar{i}^1 .$$

This last equation can be written

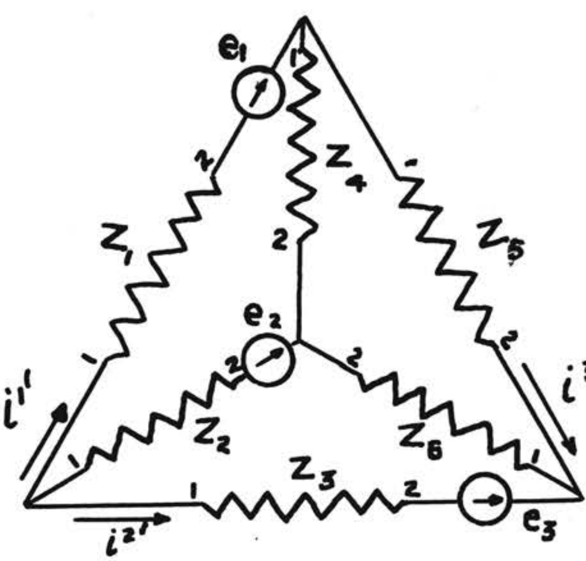
$$\bar{e}_1 = \bar{z}_1' \cdot \bar{i}^1$$

where

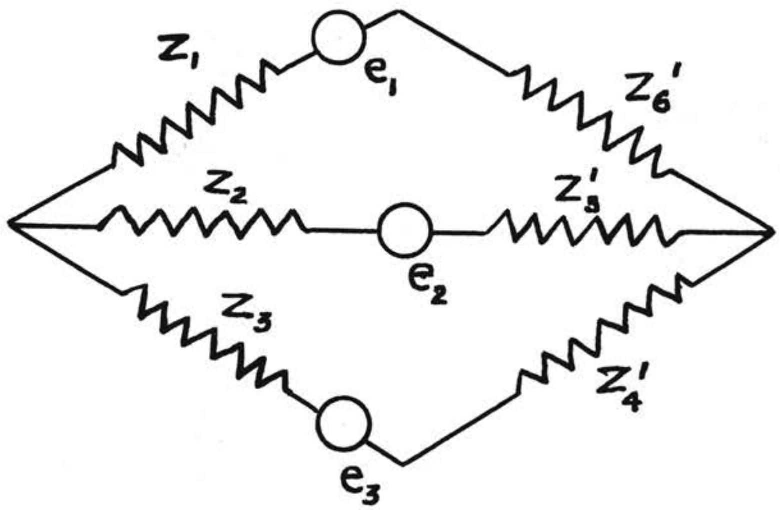
$$\bar{z}_1' = \bar{z}_1 - \bar{z}_2 \cdot \bar{z}_4^{-1} \cdot \bar{z}_3,$$

which is the reduction formula for this case, $\bar{z}_1'$ being the impedance tensor of the reduced network.

The network below will be used as an example for the above discussion. The mesh containing $z_4$, $z_5$, and $z_6$ will be eliminated.

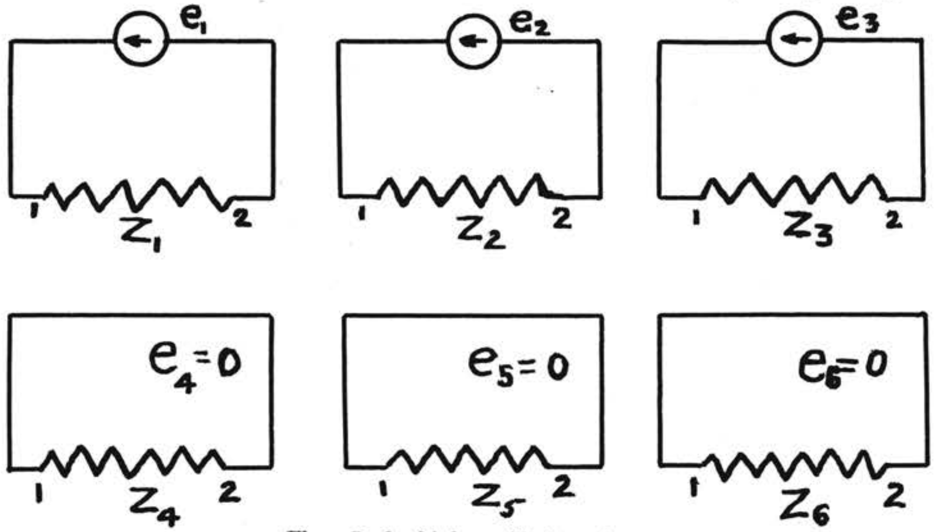



The Original Network and the Assumed Reference Axes




The Reduced Network

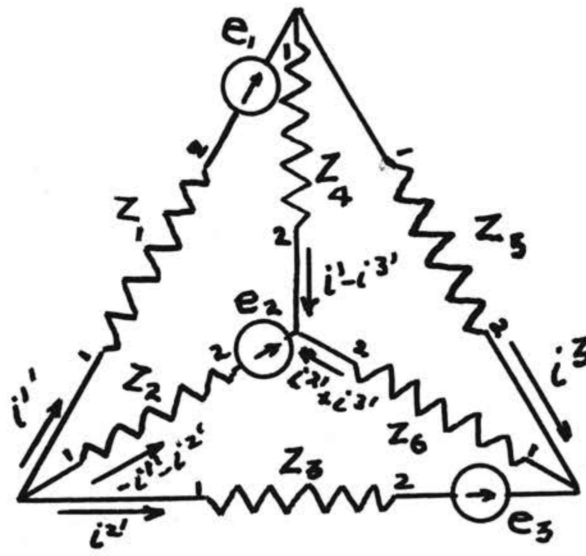
The Delta, $z_4$, $z_5$, $z_6$, Replaces the Wye, $z_4'$, $z_5'$, $z_z'$.




The Primitive Network

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} z_1 & & & & & \\ & z_2 & & & & \\ & & z_3 & & & \\ & & & z_4 & & \\ & & & & z_5 & \\ & & & & & z_6 \end{bmatrix} \cdot \begin{bmatrix} i^1 \\ i^2 \\ i^3 \\ i^4 \\ i^5 \\ i^6 \end{bmatrix}$$

The Equation of Performance of The Primitive Network




Coil Currents in Terms of $i^{1'}$, $i^{2'}$, and $i^{3'}$

Old and new currents are equated to find the components of the transformation tensor, $\bar{C}$.

$$i^1 = i^{1'}$$
$$i^2 = -i^{1'} - i^{2'}$$
$$i^3 = i^{2'}$$
$$i^4 = i^{1'} - i^{3'}$$
$$i^5 = i^{3'}$$
$$i^6 = i^{2'} + i^{3'}$$

Since

$$\bar{i} = \bar{C}\,\bar{i}',$$

$\bar{C} =$

| | 1' | 2' | 3' |
|---|---|---|---|
| 1 | 1 | | |
| 2 | -1 | -1 | |
| 3 | | 1 | |
| 4 | 1 | | -1 |
| 5 | | | 1 |
| 6 | | 1 | 1 |

$\bar{z}' = \bar{C}_t \cdot \bar{z} \cdot \bar{C}$

$= $

| | 1' | 2' | 3' |
|---|---|---|---|
| 1' | $z_1 + z_2 + z_4$ | $z_2$ | $-z_4$ |
| 2' | $z_2$ | $z_2 + z_3 + z_6$ | $z_6$ |
| 3' | $-z_4$ | $z_6$ | $z_4 + z_5 + z_6$ |

In this instance, since there are no mutual inductances to complicate matters, it can easily be seen how $\bar{z}'$ corresponds to the network. The expressions along the main diagonal of $\bar{z}'$ are the impedances around the meshes while those off the main diagonal are the ones common to two meshes, as,

$$z_{32}' = z_6$$

which is common to the meshes in which $i^{3'}$ and $i^{2'}$ flow. The heavy lines on $\bar{z}'$ show that it is to be subdivided so that

$\bar{z}_1 =$

| | 1' | 2' |
|---|---|---|
| 1' | $z_1 + z_2 + z_4$ | $z_2$ |
| 2' | $z_2$ | $z_2 + z_3 + z_6$ |

$\bar{z}_2 =$

| | 3' |
|---|---|
| 1' | $-z_4$ |
| 2' | $z_6$ |

$\bar{z}_4 =$

| | 3' |
|---|---|
| 3' | $z_4 + z_5 + z_6$ |

$\bar{z}_3 =$

| | 1' | 2' |
|---|---|---|
| 3' | $-z_4$ | $z_6$ |

The reduction formula,

$$\bar{z}_1' = \bar{z}_1 - \bar{z}_2 \cdot z_4^{-1} \cdot \bar{z}_3$$

is used to eliminate the third mesh.

$$\bar{z}_4^{-1} = \begin{array}{c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \hline 3' & \dfrac{1}{z_4 + z_5 + z_6} \\ \hline \end{array}$$

$$\bar{z}_4^{-1} \cdot \bar{z}_3 = \begin{array}{c|c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{1'} & \multicolumn{1}{c}{2'} \\ \hline 3' & \dfrac{-z_4}{z_4 + z_5 + z_6} & \dfrac{z_6}{z_4 + z_5 + z_6} \\ \hline \end{array}$$

$$\bar{z}_2 \cdot \bar{z}_4^{-1} \cdot \bar{z}_3 = \begin{array}{c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{3'} \\ \hline 1' & -z_4 \\ \hline 2' & z_6 \\ \hline \end{array} \cdot \begin{array}{c|c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{1'} & \multicolumn{1}{c}{2'} \\ \hline 3' & \dfrac{-z_4}{z_4 + z_5 + z_6} & \dfrac{z_6}{z_4 + z_5 + z_6} \\ \hline \end{array}$$

$$= \begin{array}{c|c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{1'} & \multicolumn{1}{c}{2'} \\ \hline 1' & \dfrac{z_4^2}{z_4 + z_5 + z_6} & \dfrac{-z_4 z_6}{z_4 + z_5 + z_6} \\ \hline 2' & \dfrac{-z_4 z_6}{z_4 + z_5 + z_6} & \dfrac{z_6^2}{z_4 + z_5 + z_6} \\ \hline \end{array}$$

$$\bar{z}_1' = \bar{z}_1 - \bar{z}_2 \cdot \bar{z}_4^{-1} \cdot z_3 = \begin{array}{c|c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{1'} & \multicolumn{1}{c}{2'} \\ \hline 1' & z_1 + z_2 + z_4 - \dfrac{z_4^2}{z_4 + z_5 + z_6} & z_2 + \dfrac{z_4 z_6}{z_4 + z_5 + z_6} \\ \hline 2' & z_2 + \dfrac{z_4 z_6}{z_4 + z_5 + z_6} & z_2 + z_3 + z_6 - \dfrac{z_6^2}{z_4 + z_5 + z_6} \\ \hline \end{array}$$

$$= \begin{array}{c|c|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{1'} & \multicolumn{1}{c}{2'} \\ \hline 1' & z_1 + z_2 + \dfrac{z_4 z_5 + z_4 z_6}{z_4 + z_5 + z_6} & z_2 + \dfrac{z_4 z_6}{z_4 + z_5 + z_6} \\ \hline 2' & z_2 + \dfrac{z_4 z_6}{z_4 + z_5 + z_6} & z_2 + z_3 + \dfrac{z_4 z_6 + z_5 z_6}{z_4 + z_5 + z_6} \\ \hline \end{array}$$

The tensor at the bottom of the preceding page is the impedance tensor of the reduced network. It is also equal to

| | 1' | 2' |
|---|---|---|
| 1' | $z_1 + z_2 + z_6' + z_5'$ | $z_2 + z_5'$ |
| 2' | $z_2 + z_5'$ | $z_2 + z_3 + z_5' + z_4'$ |

,

where

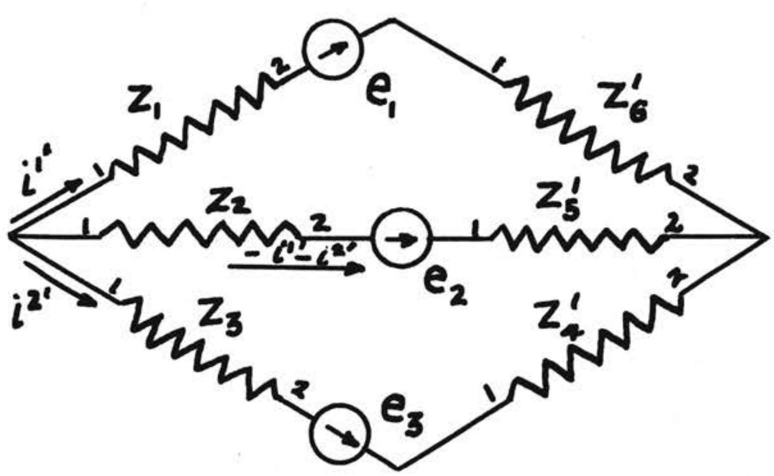
$$z_4' = \frac{z_5\, z_6}{z_4 + z_5 + z_6}$$

$$z_5' = \frac{z_4\, z_6}{z_4 + z_5 + z_6}$$

$$z_6' = \frac{z_4\, z_6}{z_4 + z_5 + z_6}.$$

These are the equations which convert the delta, $z_4$, $z_5$, $z_6$, to the wye, $z_4'$, $z_5'$, $z_6'$.

As an additional check, the impedance tensor of the reduced network will be determined directly from the primitive network in order to see if it equals the one found by using the reduction formula.




The Reduced Network and Its Coil Currents

$\bar{z} =$

| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | $z_1$ | | | | | |
| 2 | | $z_2$ | | | | |
| 3 | | | $z_3$ | | | |
| 4 | | | | $z_4'$ | | |
| 5 | | | | | $z_5'$ | |
| 6 | | | | | | $z_6'$ |

The Impedance Tensor of the Primitive Reduced Network

Equating old and new currents to determine the transformation tensor,

$$i^1 = i^{1'}$$
$$i^2 = -i^{1'} - i^{2'}$$
$$i^3 = -i^{2'}$$
$$i^4 = i^{2'}$$
$$i^5 = -i^{1'} - i^{2'}$$
$$i^6 = i^{1'}$$

$\bar{z}' = \bar{C}_t \cdot \bar{z} \cdot \bar{C} =$

| | $1'$ | $2'$ |
|---|---|---|
| $1'$ | $z_1 + z_2 + z_5' + z_6'$ | $z_2 + z_5'$ |
| $2'$ | $z_2 + z_5'$ | $z_2 + z_3 + z_4' + z_5'$ |

The above tensor equals the one found by using the reduction formula.

Since the illustration just given was a simple one, it could have been solved more easily by conventional methods. However, if there were perhaps twelve or more meshes, the use of the reduction formula would prove quite a labor saver. In such cases, it is usually most convenient to eliminate three rows and columns of the impedance matrix in one operation because the determinant of a three rowed matrix can be easily found. Successive eliminations can be performed until the desired number of meshes have been eliminated.

The use of tensor methods for determining the impedance between two points of a network involves the elimination of several meshes. An additional mesh is introduced by assuming a voltage across the two points between which the impedance is to be measured, and assuming an additional current in this extra branch. If all the rows and columns of z except the additional one are eliminated, the scalar that remains is the impedance between the two points.

## USE OF THE SEQUENCE TENSOR

The impedance tensors of three-phase networks and machines sometimes assume simpler forms when expressed along other reference frames than the phase axes. The other reference frames most used are the so-called sequence axes. To transform from the phase axes to the sequence axes, a tensor, which is really a spinor, or Hermitian tensor since some of its components are complex, called the sequence tensor, $\bar{C}_s$, is used. It equals

$$\frac{1}{\sqrt{3}} \cdot \begin{array}{c|c|c|c|} & 0 & 1 & 2 \\ \hline a & 1 & 1 & 1 \\ \hline b & 1 & a^2 & a \\ \hline c & 1 & a & a^2 \\ \hline \end{array} = \bar{C}_{st} = \bar{C}^{*\,-1}_{st}.$$

$$a = -\frac{1}{2} + j.866 = e^{j120^\circ}$$

$$a^2 = -\frac{1}{2} - j.866 = e^{j120^\circ}$$

The three rows of the sequence tensor are the sequence operators used in the theory of symmetrical components. It should be noted that

$$1 + a + a^2 = 0$$

$$a^3 = 1$$

$$a^4 = a\ .$$

The symbol, $\bar{C}^*$, means, "the conjugate of $\bar{C}$", which is $\bar{C}$ with each element replaced by its conjugate.

$$\bar{C}^*_s = \frac{1}{\sqrt{3}} \cdot \begin{array}{c|c|c|c|} & 0 & 1 & 2 \\ \hline a & 1 & 1 & 1 \\ \hline b & 1 & a & a^2 \\ \hline c & 1 & a^2 & a \\ \hline \end{array} = \bar{C}_{st}^{\ -1}$$

The letters a, b, and c represent the phase axes and the numbers 0, 1, and 2 are the sequence axes. The phase currents in terms of the sequence currents are

$$i^a = \frac{1}{3}(i^0 + i^1 + i^2)$$

$$i^b = \frac{1}{3}(i^0 + a^2\, i^1 + a\, i^2)$$

$$i^c = \frac{1}{3}(i^0 + a\, i^1 + a^2\, i^2)\,.$$ [1]

The transformation formulas of spinors differ slightly from those of tensors. Usually the only difference, when there is any, is that wherever $C_t$ occurs it is replaced by $\bar{C}_t^*$. In index notation, indices to be transformed by some form of $\bar{C}^*$ are written with a bar over them, as $z_{\bar{\alpha}\beta}$ and are called barred indices.[2] The indices of a spinor are called spin indices, those of a tensor, tensor indices.[3]

To demonstrate the use of the sequence tensor and the use of compound coils, the network shown in Fig. 1 consisting of a star, or wye, connected to a delta will be used.

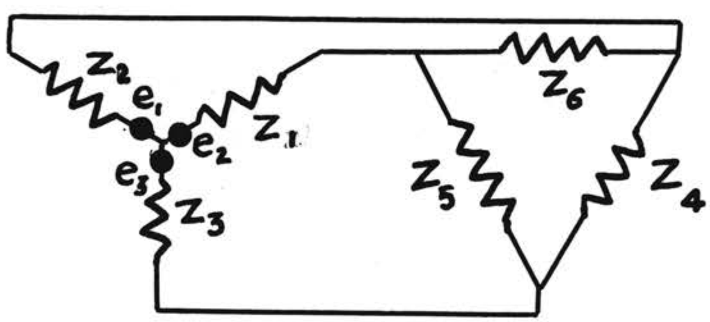



Fig. 1

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1 Gabriel Kron, Tensor Analysis of Networks, p. 328.

2 Ibid., pp. 345-349.

3 Ibid., pp. 349-353.

The compound network is shown in Fig. 2.

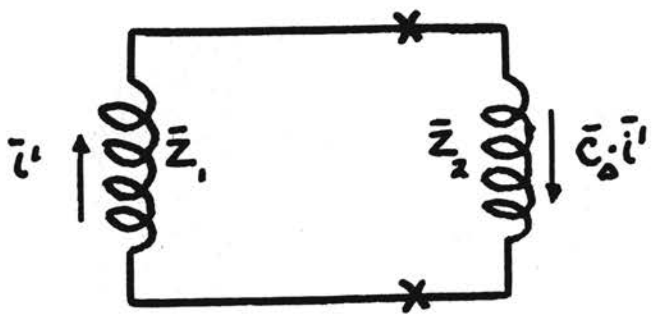


Fig. 2

In using compound networks, a transformation tensor, the junction tensor must be used when some three-phase apparatus are connected to the line. For this network, the junction tensor $\bar{C}_{\Delta}$ is used to connect the delta to the star. Whenever the current undergoes a transformation as it enters a compound coil, crosses are put on the leads to the coil and the value of the current in the coil is indicated.

The junction tensor of the delta expressed along the phase axes is obtained in a manner similar to that of finding the transformation tensor of any other network.

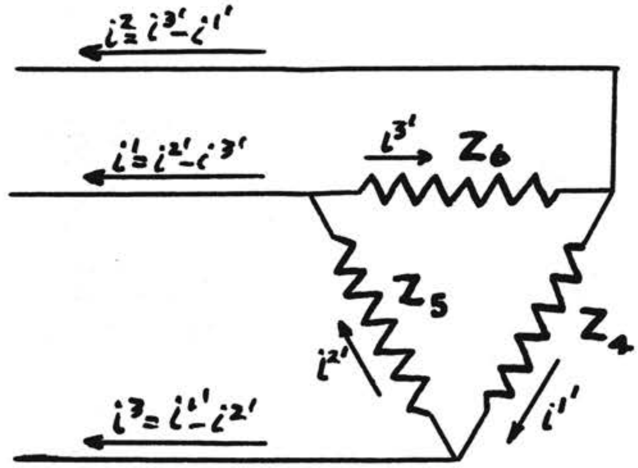


Fig. 3

From Fig. 3,

$$i^1 = i^{2'} - i^{3'}$$
$$i^2 = - i^{1'} + i^{3'}$$
$$i^3 = i^{1'} - i^{2'}$$

The coefficients of primed currents form the components of the junction tensor.

$$\bar{C}_\Delta = \begin{array}{c|c|c|c|} & 1' & 2' & 3' \\ \hline 1 & & 1 & -1 \\ \hline 2 & -1 & & 1 \\ \hline 3 & 1 & -1 & \\ \hline \end{array}$$

The Junction Tensor

The components of $\bar{C}_\Delta$ along the sequence axes are found by the transformation formula

$$\bar{C}_{\Delta s} = \bar{C}_s^{-1} \cdot \bar{C}_\Delta \cdot \bar{C}_s .$$

$$\bar{C}_s^{-1} \cdot \bar{C}_\Delta = \frac{1}{\sqrt{3}} \begin{array}{c|c|c|c|} & 1 & 2 & 3 \\ \hline 0 & 1 & 1 & 1 \\ \hline 1 & 1 & a & a^2 \\ \hline 2 & 1 & a^2 & a \\ \hline \end{array} \begin{array}{c|c|c|c|} & 1' & 2' & 3' \\ \hline 1 & & 1 & -1 \\ \hline 2 & -1 & & 1 \\ \hline 3 & 1 & -1 & \\ \hline \end{array} = \frac{1}{\sqrt{3}} \begin{array}{c|c|c|c|} & 1' & 2' & 3' \\ \hline 0 & 0 & 0 & 0 \\ \hline 1 & a^2 - a & 1 - a^2 & a - 1 \\ \hline 2 & a - a^2 & 1 - a & a^2 - 1 \\ \hline \end{array}$$

$$\bar{C}_s^{-1} \cdot \bar{C}_\Delta \cdot \bar{C}_s = \frac{1}{\sqrt{3}} \begin{array}{c|c|c|c|} & 1' & 2' & 3' \\ \hline 0 & 0 & 0 & 0 \\ \hline 1 & a^2 - a & 1 - a^2 & a - 1 \\ \hline 2 & a - a^2 & 1 - a & a^2 - 1 \\ \hline \end{array} \frac{1}{\sqrt{3}} \begin{array}{c|c|c|c|} & 0 & 1 & 2 \\ \hline 1' & 1 & 1 & 1 \\ \hline 2' & 1 & a^2 & a \\ \hline 3' & 1 & a & a^2 \\ \hline \end{array}$$

$$= \frac{1}{3} \begin{array}{c|c|c|c|} & 0 & 1 & 2 \\ \hline 0 & 0 & 0 & 0 \\ \hline 1 & a^2 - a + 1 - a^2 + a - 1 & a^2 - a + a^2 - a + a^2 - a & a^2 - a + a - 1 + 1 \\ \hline 2 & a - a^2 + 1 - a + a^2 - 1 & a - a^2 + a^2 - 1 + 1 - a & a - a^2 + a - a^2 + a - a^2 \\ \hline \end{array} = \bar{C}_{\Delta s}$$

After cancelling and collecting terms,

$$\bar{C}_{\Delta s} = \begin{array}{c|c|c|c|} & 0 & 1 & 2 \\ \hline 0 & 0 & 0 & 0 \\ \hline 1 & 0 & a^2 - a & 0 \\ \hline 2 & 0 & 0 & a - a^2 \\ \hline \end{array}$$

In the compound network of Fig. 2,

$$\bar{i}^{\bar{1}} = \bar{i}^{1'}$$

$$\bar{i}^{\bar{2}} = \bar{C}_{\Delta} \cdot \bar{i}^{1'} .$$

The transformation tensor of the network is, from the preceding two equations,

$$\bar{C} = \begin{array}{c|c|} \hline \bar{1} & \bar{I} \\ \hline \bar{2} & \bar{C}_{\Delta} \\ \hline \end{array}$$

The impedance tensor of the network is

$$\bar{z} = \begin{array}{c|c|c|} & \bar{1} & \bar{2} \\ \hline \bar{1} & \bar{z}_1 & \\ \hline \bar{2} & & \bar{z}_2 \\ \hline \end{array}$$

where

$$\bar{z}_1 = \begin{array}{c|c|c|c|} & 1 & 2 & 3 \\ \hline 1 & z_1 & & \\ \hline 2 & & z_2 & \\ \hline 3 & & & z_3 \\ \hline \end{array}$$

$$\bar{z}_2 = \begin{array}{c|c|c|c|} & 4 & 5 & 6 \\ \hline 4 & z_4 & & \\ \hline 5 & & z_5 & \\ \hline 6 & & & z_6 \\ \hline \end{array} .$$

$$\bar{z}' = \bar{C}_t \cdot \bar{z} \cdot \bar{C} = \begin{array}{|c|c|} \bar{1} & \bar{2} \\ \hline \bar{z}_1 & \bar{C}_{\Delta t} \cdot \bar{z}_2 \\ \hline \end{array} \; \begin{array}{c|c|} \hline \bar{1} & \bar{I} \\ \hline \bar{2} & \bar{C}_{\Delta} \\ \hline \end{array} = \bar{z}_1 + \bar{C}_{\Delta t} \cdot \bar{z}_2 \cdot \bar{C}_{\Delta}$$

Along the phase axes,

$$\bar{C}_{\Delta t} \cdot \bar{z}_2 \cdot \bar{C}_{\Delta} = \begin{array}{c|ccc|} & 4 & 5 & 6 \\ \hline 1' & & -z_5 & z_6 \\ 2' & z_4 & & -z_6 \\ 3' & -z_4 & z_5 & \\ \hline \end{array} \quad \begin{array}{c|ccc|} & 1' & 2' & 3' \\ \hline 4 & & 1 & -1 \\ 5 & -1 & & 1 \\ 6 & 1 & -1 & \\ \hline \end{array}$$

$$= \begin{array}{c|ccc|} & 1' & 2' & 3' \\ \hline 1' & z_5 + z_6 & -z_6 & -z_5 \\ 2' & -z_6 & z_4 + z_6 & -z_4 \\ 3' & -z_5 & -z_4 & z_4 + z_5 \\ \hline \end{array}$$

$$\bar{z}' = \bar{z}_1 + \bar{C}_{\Delta t} \cdot \bar{z}_2 \cdot \bar{C}_{\Delta} = \begin{array}{c|ccc|} & 1' & 2' & 3' \\ \hline 1' & z_1 + z_5 + z_6 & -z_6 & -z_5 \\ 2' & -z_6 & z_2 + z_4 + z_6 & -z_4 \\ 3' & -z_5 & -z_4 & z_3 + z_4 + z_5 \\ \hline \end{array}$$

Since the networks are similar, this tensor is quite similar to that for the network in the previous example.

To find the impedance tensor along the sequence axes, the procedure is repeated. Expressing phase currents in terms of sequence currents,

$$\bar{i}_p^{1'} = \bar{C}_s \cdot \bar{i}^{1'}$$

$$\bar{i}_p^{2'} = \bar{C}_{\Delta s} \cdot \bar{i}^{1'}$$

$$\bar{C} = \begin{array}{c|c|} \bar{1} & \bar{C}_s \\ \hline \bar{2} & \bar{C}_{\Delta s} \\ \end{array}$$

The sequence impedance tensor is calculated next.

$$\bar{z}'_s = \bar{C}_t \cdot \bar{z} \cdot \bar{C} = \begin{array}{|c|c|} \bar{1} & \bar{2} \\ \hline \bar{C}^*_{st} \cdot \bar{z}_1 & \bar{C}^*_{\Delta st} \cdot \bar{z}_2 \\ \hline \end{array} \quad \begin{array}{c|c|} \bar{1} & \bar{C}_s \\ \hline \bar{2} & \bar{C}_{\Delta s} \\ \end{array} = \bar{C}^*_{st} \cdot \bar{z}_1 \cdot \bar{C}_s + \bar{C}^*_{\Delta st} \cdot \bar{z}_2 \cdot \bar{C}_{\Delta s}$$

$$\bar{C}^*_{\Delta st}\cdot\bar{z}_2\cdot\bar{C}_{\Delta s} = \begin{array}{c|c|c|c|}
 & 4 & 5 & 6 \\ \hline
0 & & & \\ \hline
1 & & (a-a^2)\,z_5 & \\ \hline
2 & & & (a^2-a)\,z_6 \\ \hline
\end{array}
\quad
\begin{array}{c|c|c|c|}
 & 0 & 1 & 2 \\ \hline
4 & & & \\ \hline
5 & & a^2-a & \\ \hline
6 & & & a-a^2 \\ \hline
\end{array}$$

$$= \begin{array}{c|c|c|c|}
 & 0 & 1 & 2 \\ \hline
0 & & & \\ \hline
1 & & (a-a^2)(a^2-a)\,z_5 & \\ \hline
2 & & & (a^2-a)(a-a^2)\,z_6 \\ \hline
\end{array}
= \begin{array}{c|c|c|c|}
 & 0 & 1 & 2 \\ \hline
0 & & & \\ \hline
1 & & 3z_5 & \\ \hline
2 & & & 3z_6 \\ \hline
\end{array}$$

$$\bar{C}^*_{st}\cdot\bar{z}_1\cdot\bar{C}_s = \frac{1}{\sqrt{3}}\begin{array}{c|c|c|c|}
 & 1 & 2 & 3 \\ \hline
0 & z_1 & z_2 & z_3 \\ \hline
1 & z_1 & az_2 & a^2z_3 \\ \hline
2 & z_1 & a^2z_2 & az_3 \\ \hline
\end{array}
\cdot\frac{1}{\sqrt{3}}\begin{array}{c|c|c|c|}
 & 0 & & \\ \hline
1 & 1 & 1 & 1 \\ \hline
2 & 1 & a^2 & a \\ \hline
3 & 1 & a & a^2 \\ \hline
\end{array}$$

$$= \begin{array}{c|c|c|c|}
 & 0 & 1 & 2 \\ \hline
0 & z_1+z_2+z_3 & z_1+a^2z_2+az_3 & z_1+az_2+a^2z_3 \\ \hline
1 & z_1+az_2+a^2z_3 & z_1+z_2+z_3 & z_1+a^2z_2+az_3 \\ \hline
2 & z_1+a^2z_2+az_3 & z_1+az_2+a^2z_3 & z_1+z_2+z_3 \\ \hline
\end{array}$$

$$\bar{z}_s' = \frac{1}{3}\begin{array}{c|c|c|c|}
 & 0 & 1 & 2 \\ \hline
0 & z_1+z_2+z_3 & z_1+a^2z_2+az_3 & z_1+az_2+a^2z_3 \\ \hline
1 & z_1+az_2+a^2z_3 & z_1+z_2+z_3+9z_5 & z_1+a^2z_2+az_3 \\ \hline
2 & z_1+a^2z_2+az_3 & z_1+az_2+a^2z_3 & z_1+z_2+z_3+9z_6 \\ \hline
\end{array}$$

If the star and delta are balanced, that is, if

$$z_1 = z_2 = z_3 = z_A$$

and

$$z_4 = z_5 = z_6 = z_B$$

then $\bar{z}_s'$ reduces to the diagonal matrix

| | 0 | 1 | 2 |
|---|---|---|---|
| 0 | $z_A$ | | |
| 1 | | $z_A + 3z_B$ | |
| 2 | | | $z_A + 3z_B$ |

which is much easier to manipulate than the corresponding tensor expressed along the phase axes. This is one of the main reasons for the use of the sequence tensor.

THE APPLICATION OF TENSOR ANALYSIS TO TUBE CIRCUITS

When, as is often the case, there are only small variations of voltage in a tube circuit, the tube may be considered a linear circuit element and the method of network analysis previously described may be applied. The condition of small voltage variation occurs mainly in amplifiers and oscillators. Other types of circuits such as modulators and rectifiers make use of the non-linear properties of the tube.

Since most tube circuits have fewer junction pairs than meshes, they are usually treated as junction networks. The filament or cathode serves as a common lead from which the various "coils" of the tube, which are the electron paths from the emitter to the grids and plate, branch. Using this scheme, a tetrode would be represented as shown in Fig. 4.[1]

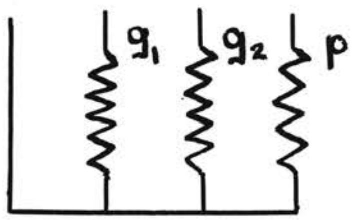


Fig. 4

Analytically, no distinction is made between the grids and the plate.

Suppose a change of voltage,

$$\Delta E_u = \begin{array}{c|c|c|c|} u & a & b & c & p \\ \hline & \Delta E_a & \Delta E_b & \Delta E_c & \Delta E_p \\ \hline \end{array}$$

occurs on the grids, a, b, and c, and the plate, p, of a pentode. Neglecting the curvature of the static characteristic curve, the changes in current are,[2]

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[1] Gabriel Kron, Tensor Analysis of Networks, p. 379.

[2] Ibid., p. 381.

$$\Delta I^u = \begin{array}{|c|c|c|c|} \multicolumn{1}{c}{a} & \multicolumn{1}{c}{b} & \multicolumn{1}{c}{c} & \multicolumn{1}{c}{p} \\ \hline \Delta I^a & \Delta I^b & \Delta I^c & \Delta I^p \\ \hline \end{array}$$

where

$$\Delta I^a = \frac{\partial I^a}{\partial E_a}\Delta E_a + \frac{\partial I^a}{\partial E_b}\Delta E_b + \frac{\partial I^a}{\partial E_c}\Delta E_c + \frac{\partial I^a}{\partial E_p}\Delta E_p$$

$$\Delta I^b = \frac{\partial I^b}{\partial E_a}\Delta E_a + \frac{\partial I^b}{\partial E_b}\Delta E_b + \frac{\partial I^b}{\partial E_c}\Delta E_c + \frac{\partial I^b}{\partial E_p}\Delta E_p$$

$$\Delta I^c = \frac{\partial I^c}{\partial E_a}\Delta E_a + \frac{\partial I^c}{\partial E_b}\Delta E_b + \frac{\partial I^c}{\partial E_c}\Delta E_c + \frac{\partial I^c}{\partial E_p}\Delta E_p$$

$$\Delta I^p = \frac{\partial I^p}{\partial E_a}\Delta E_a + \frac{\partial I^p}{\partial E_b}\Delta E_b + \frac{\partial I^p}{\partial E_c}\Delta E_c + \frac{\partial I^p}{\partial E_p}\Delta E_p$$

Note that the vector $I^v$ has been differentiated with respect to the vector $E_u$. The question may be asked whether $\frac{\partial I^v}{\partial E_u}$ is a tensor. The answer is in the affirmative so long as the components of $C^{\alpha}_{\alpha'}$, the transformation tensor, are constants. Otherwise, the concept of absolute, or covariant, differentiation must be introduced. To illustrate, let

$$I^{\alpha} = C^{\alpha}_{\alpha'} I^{\alpha'},$$
$$E_u = C^{u'}_{u} E_{u'},$$

and it follows that,

$$\frac{\partial I^{\alpha}}{\partial E_u} = \frac{\partial (C^{\alpha}_{\alpha'} I^{\alpha'})}{\partial (C^{u'}_{u} E_{u'})} = \frac{\partial C^{\alpha}_{\alpha'}}{\partial (C^{u'}_{u} E_{u'})} I^{\alpha'} + C^{\alpha}_{\alpha'} \frac{\partial I^{\alpha'}}{\partial (C^{u'}_{u} E_{u'})}.$$

Since there is an extra term in the last expression, $\frac{\partial I^{\alpha}}{\partial E_u}$ does not transform as a tensor. However, if the components of $C^{\alpha}_{\alpha'}$ are constants, then

$$\frac{\partial C^{\alpha}_{\alpha'}}{\partial E_u} = 0,$$

$$\frac{\partial}{\partial (C^{u'}_{u} E_{u'})} = \frac{\partial}{C^{u'}_{u}\partial E_{u'}},$$

and

$$\frac{\partial I^{\alpha}}{\partial E_u} = \frac{\partial I^{\alpha'}}{\partial E_{u'}} C^{\alpha}_{\alpha'} C^{u}_{u'}$$

which agrees with the characteristic transformation formula of a tensor.

The four equations for current change can be written

$$\Delta I^u = Y^{uv} \Delta E_v$$

where

$$Y^{uv} = \frac{\partial I^v}{\partial E_u} .$$

$Y^{uv}$ is called the admittance tensor of the tube. Defining the various amplification factors and resistances of the tube in the conventional manner, the admittance tensor of a pentode is[3]

$Y^{uv}$ =

| | a | b | c | p |
|---|---|---|---|---|
| a | $\frac{1}{r_{aa}}$ | $\frac{\mu_a^b}{r_{aa}}$ | $\frac{\mu_a^c}{r_{aa}}$ | $\frac{\mu_a^p}{r_{aa}}$ |
| b | $\frac{\mu_b^a}{r_{bb}}$ | $\frac{1}{r_{bb}}$ | $\frac{\mu_b^c}{r_{bb}}$ | $\frac{\mu_b^p}{r_{bb}}$ |
| c | $\frac{\mu_c^a}{r_{cc}}$ | $\frac{\mu_c^b}{r_{cc}}$ | $\frac{1}{r_{cc}}$ | $\frac{\mu_c^p}{r_{cc}}$ |
| p | $\frac{\mu_p^a}{r_{pp}}$ | $\frac{\mu_p^b}{r_{pp}}$ | $\frac{\mu_p^c}{r_{pp}}$ | $\frac{1}{r_{pp}}$ |

Omitting the row and column, c, the admittance tensor of a tetrode is[4]

$Y^{uv}$ =

| | a | b | p |
|---|---|---|---|
| a | $\frac{1}{r_a}$ | $\frac{\eta_a}{r_a}$ | $\frac{\nu_a}{r_a}$ |
| b | $\frac{\eta_b}{r_b}$ | $\frac{1}{r_b}$ | $\frac{\nu_b}{r_b}$ |
| p | $\frac{\mu_a}{r_p}$ | $\frac{\mu_b}{r_p}$ | $\frac{1}{r_p}$ |

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[3] *Ibid.*, p. 386.

[4] *Ibid.*, p. 387.

where $\mu$ is the plate amplification factor, $\nu$ the grid amplification factor, $\eta$ the cross amplification factor. The admittance tensor of a triode is found by omitting rows and columns b and c from the admittance tensor of a pentode. It is[5]

$$Y^{uv} = \begin{array}{c|c|c|} & g & p \\ \hline g & \frac{1}{r_g} & \frac{\mu g}{r_g} \\ \hline p & \frac{\mu p}{r_p} & \frac{1}{r_p} \\ \hline \end{array} = \begin{array}{c|c|c|} & g & p \\ \hline g & G^{gg} & G^{gp} \\ \hline p & G^{pg} & G^{pp} \\ \hline \end{array}$$

If the grid current of a triode is zero, $r_g$ is infinite and

$$Y^{uv} = \begin{array}{c|c|c|} & g & p \\ \hline g & 0 & 0 \\ \hline p & \frac{\mu p}{r_g} & \frac{1}{r_p} \\ \hline \end{array}$$

Similarly, when no grid current flows in a tetrode,

$$Y^{uv} = \begin{array}{c|c|c|c|} & a & b & p \\ \hline a & 0 & 0 & 0 \\ \hline b & 0 & 0 & 0 \\ \hline p & \frac{\mu a}{r_p} & \frac{\mu b}{r_p} & \frac{1}{r_p} \\ \hline \end{array}$$

The impedance tensor and transformation tensor of a circuit containing tubes can be set up in the manner of any junction network, the tube adding two or more axes to the tensors. Also it may be added that the tube network and the remainder of the network may be separated, analyzed and recombined.

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[5] *Ibid.*, p. 387.

To find the amplification of an amplifier, the ratio of the change in output voltage to the corresponding change in input voltage is required. To find this, the reduction formulas are used to eliminate all but the input and output axes of $Y'$. This leaves two equations

$$I^{*} = Y^{aa}\Delta E_a + Y^{ab}\Delta E_b$$
$$0 = Y^{ba}\Delta E_a + Y^{bb}\Delta E_b \,.$$

From the last equation,

$$\frac{\Delta E_b}{\Delta E_a} = -\frac{Y^{ba}}{Y^{bb}}$$

which is the desired ratio.

One method for determining the conditions necessary for a junction network to be oscillatory is to consider the equation

$$\bar{I} = \bar{Y}\cdot\bar{E}$$

or

$$\bar{E} = \bar{Y}^{-1}\cdot\bar{I}\,.$$ [6]

If there are no impressed currents, then

$$\bar{I} = \bar{0}\,.$$

The inverse of $\bar{Y}$ is a matrix whose elements are cofactors of $\bar{Y}$ divided by the determinant of $\bar{Y}$, that is,

$$\bar{Y}^{-1} = \frac{\bar{Y}_c}{D}\,.$$

In order for $\bar{E}$ not to be zero when $\bar{I}$ equals zero,

$$\frac{\bar{Y}_c}{D}\cdot\bar{0} \neq \bar{0}$$

which can be true only if

$$D = 0\,.$$

Therefore, if the determinant of the admittance tensor is zero, the network is

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6 *Ibid.*, pp. 399-400.

oscillatory. By using the equation,

$$D = 0,$$

the necessary relations between circuit components for oscillation may be determined.

When it happens that a tube circuit has fewer meshes than junction pairs, it is advantageous to treat it as a mesh network. The tube's impedance tensor is found by taking the inverse of its admittance tensor. The impedance tensor of a tetrode is

$z_{mn} =$

| | a | b | p |
|---|---|---|---|
| a | $\frac{1 - \mu_b \nu_b}{r_b r_p D}$ | $\frac{\mu_b \nu_a - \eta_a}{r_a r_p D}$ | $\frac{\eta_a \nu_b - \nu_a}{r_a r_b D}$ |
| b | $\frac{\mu_a \nu_b - \eta_b}{r_b r_p D}$ | $\frac{1 - \mu_a \nu_a}{r_a r_p D}$ | $\frac{\eta_b \nu_a - \nu_b}{r_a r_p D}$ |
| p | $\frac{\eta_b \mu_b - \mu_a}{r_b r_p D}$ | $\frac{\mu_a \eta_a - \mu_b}{r_a r_p D}$ | $\frac{1 - \eta_a \eta_b}{r_a r_p D}$ |

where

$$D = \frac{1 + \mu_a(\eta_a \nu_b - \nu_a) + \mu_b(\eta_b \nu_a - \nu_b) - \eta_a \eta_b}{r_a r_b r_p}$$

The impedance tensor of a triode is

$z_{mn} =$

| | g | p |
|---|---|---|
| g | $\frac{r_g}{1 - \mu_g \mu_p}$ | $\frac{\mu_g r_p}{1 - \mu_p \mu_g}$ |
| p | $\frac{-\mu_p \mu_g}{1 - \mu_g \mu_p}$ | $\frac{r_p}{1 - \mu_g \mu_p}$ |

The value of the grid current in a triode is usually not required and is often equal to zero, so the axes g in the impedance tensor of the triode may be eliminated by using the reduction formula

$$\bar{z}' = \bar{z}_1 - \bar{z}_2 \cdot \bar{z}_4^{-1} \cdot \bar{z}_3 .$$

Applying this reduction formula,

$$\bar{z}' = r_p' = \frac{r_p}{1-\mu p \mu g} - \frac{\mu p \mu g}{1-\mu p \mu g} \frac{(1-\mu p \mu g)}{r_g} \left( \frac{\mu g \, r_p}{1-\mu p \mu g} \right) = r_p \frac{(1-\mu p \mu g)}{(1-\mu p \mu g)} = r_p$$

The grid coil of the triode may be omitted by changing the self-impedance of the plate coil from $\dfrac{r_p}{1-\mu p \mu g}$ to $r_p$.

If there is a voltage, $e_g$, impressed in series with the grid coil, or if there is a difference of potential, $e_g$, across it, then the equivalent impressed voltage on the plate coil is found by the reduction formula

$$\bar{e}' = \bar{e}_1 - \bar{z}_2 \cdot \bar{z}_4^{-1} \cdot \bar{e}_2$$

and it follows that

$$e_p' = e_p - \left( \frac{-\mu p \mu g}{1-\mu p \mu g} \right) \left( \frac{1-\mu p \mu g}{r_g} \right) e_g = e_p + \mu p e_g \ .$$

If a more exact analysis of a vacuum tube circuit is required, one or two more terms in the series expansion of $\Delta I^u$ may be used,[7] giving

$$\Delta I^u = Y^{uv} \Delta E_v + M^{uvw} \Delta E_v \Delta E_w + D^{uvwz} \Delta E_v \Delta E_w \Delta E_z \ .$$

$Y^{uv}$, the admittance tensor, is also called the amplification tensor.

$$M^{uvw} = \frac{1}{2!} \frac{\partial Y^{uv}}{\partial E_w} = \frac{1}{2!} \frac{\partial^2 I^u}{\partial E_v \partial E_w}$$

is called the modulation tensor, and is a tensor of valence three. For a tetrode it consists of three two-matrices, one of which is

$M^{avw} = \frac{1}{2}$

| | a | b | p |
|---|---|---|---|
| a | $-\frac{1}{2} \frac{\partial r_a}{r_a \partial E_a}$ | $-\frac{1}{2} \frac{\partial r_a}{r_a \partial E_b}$ | $-\frac{1}{2} \frac{\partial r_a}{r_a \partial E_p}$ |
| b | $\frac{\partial G^{ab}}{\partial E_a}$ | $\frac{\partial G^{ab}}{\partial E_b}$ | $\frac{\partial G^{ab}}{\partial E_p}$ |
| p | $\frac{\partial G^{ap}}{\partial E_a}$ | $\frac{\partial G^{ap}}{\partial E_b}$ | $\frac{\partial G^{ap}}{\partial E_p}$ |

.

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7 Ibid., pp. 547-564.

The other two matrices, $M^{bvw}$, and $M^{pvw}$ are similar in form to $M^{avw}$. When no grid currents flow, $M^{avw}$ and $M^{bvw}$ are zero. The tensor $D^{uvwz}$ is of valence four and is called the distortion tensor.

$$D^{uvwz} = \frac{1}{3!}\frac{\partial M^{uvw}}{\partial E_z} = \frac{1}{3!}\frac{\partial^2 Y^{uv}}{\partial E_w \partial E_v} = \frac{1}{3!}\frac{\partial^3 I^u}{\partial E_v \partial E_w \partial E_z} \;.$$

For a pentode, the distortion tensor consists of sixteen two-matrices, each having four rows and four columns. In general, where n is the range of one of its indices, it can be represented as $n^2$ matrices, each matrix having $n^2$ components.

## PRACTICAL EXAMPLES

As an example of how the reduced form of the triode impedance tensor may be applied, the basic circuit of the cathode-follower type of amplifier will be analyzed. Its circuit is shown in Fig. 5

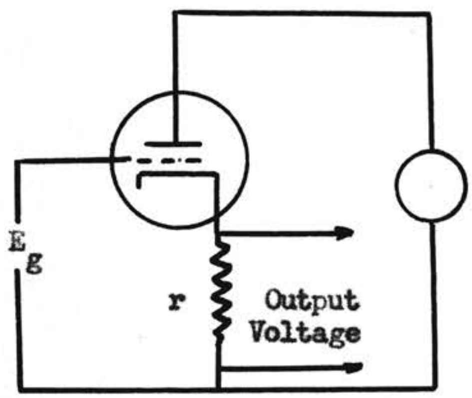


Fig. 5

The reduced network of this circuit is shown in Fig. 6. The tube is replaced by $r_p$ in series with a voltage $\mu E_g$.

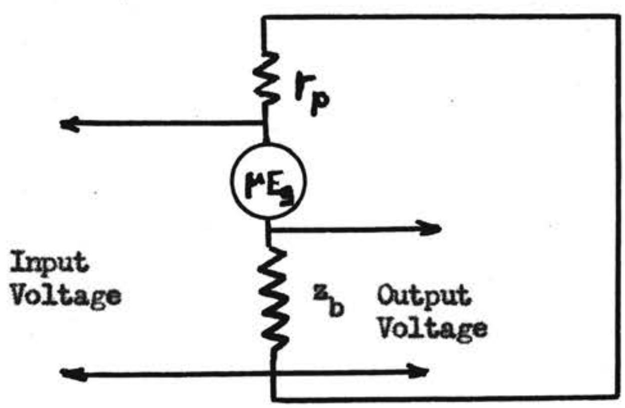


Fig. 6

$z_b$ includes r and any outside load. By inspection, the input voltage equals

$$E_g + I_p z_b .$$

Since

$$I_p = \frac{\mu E_g}{r_p + z_b},$$

the input voltage also equals

$$E_g + \frac{\mu E_g z_b}{r_p + z_b} \, .$$

The output voltage equals

$$I_p z_b = \frac{\mu E_g z_b}{r_p + z_b} \, .$$

Amplification equals output voltage divided by input voltage

$$= \frac{\dfrac{\mu E_g z_b}{r_p + z_b}}{\dfrac{r_p E_g + E_g z_b + \mu E_g z_b}{r_p + z_b}}$$

$$= \frac{\mu z_b}{r_p + (1 + \mu) z_b} \, .$$ [1]

The circuit of the Hartley oscillator shown in Fig. 7 will be analyzed to determine its frequency of oscillation from the formula for its criterion of oscillation.

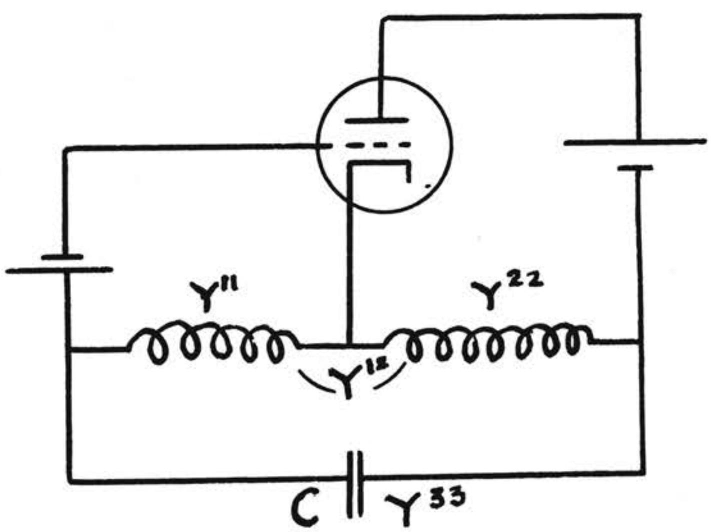


Fig. 7

[1] Herbert J. Reich, Theory and Applications of Electron Tubes, p. 166.

The values of the various admittances in terms of impedances are given below.

$$Y^{11} = \frac{z_{11}}{z_{11}z_{22} - (x_{12})^2} = \frac{j\omega L_1}{-\omega^2(L_1 L_2 - M^2)} = \frac{-jL_1}{\omega(L_1 L_2 - M^2)}$$

$$Y^{22} = \frac{z_{22}}{z_{11}z_{22} - (x_{12})^2} = \frac{j\omega L_2}{-\omega^2(L_1 L_2 - M^2)} = \frac{-jL_2}{\omega(L_1 L_2 - M^2)}$$

$$Y^{12} = \frac{-x_{12}}{z_{11}z_{22} - (x_{12})^2} = \frac{j\omega M}{-\omega^2(L_1 L_2 - M^2)} = \frac{-jM}{\omega(L_1 L_2 - M^2)}$$

M is the mutual inductance between $L_1$ and $L_2$.

$$Y^{33} = j\omega C$$

$$G^{pg} = \frac{\mu}{r_p}$$

$$G^{pp} = \frac{1}{r_p}$$

In Fig. 8, the network is redrawn, the tube being replaced by a grid coil and a plate coil. Assumed junction-pair voltages and coil voltages are shown.

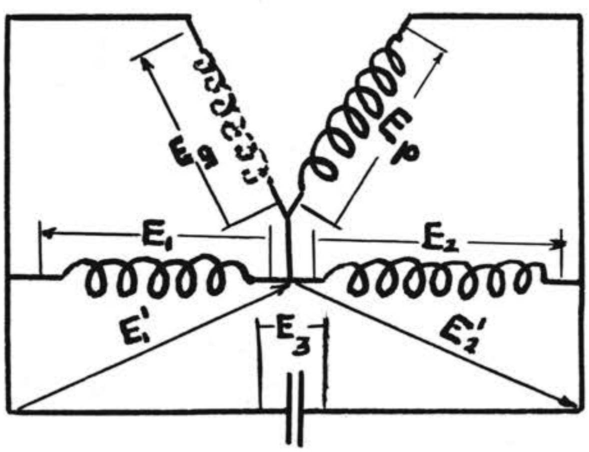


Fig. 8

There are three junctions and one sub-network so there are two junction pairs. Assumed junction pairs are shown in Fig. 9

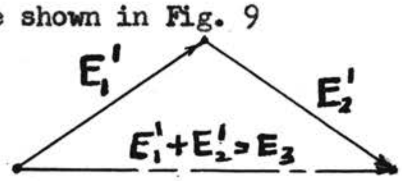


Fig. 9

Old and new voltages across each coil are equated in order to find the transformation tensor.[2]

$$E_1 = - E_1'$$
$$E_2 = E_2'$$
$$E_3 = E_1' + E_2'$$
$$E_g = - E_1'$$
$$E_p = E_2'$$

The components of $\bar{A}$, the transformation tensor, are obtained from the coefficients of the new voltages.

$\bar{A} =$

| | 1' | 2' |
|---|---|---|
| 1 | -1 | |
| 2 | | 1 |
| 3 | 1 | 1 |
| g | -1 | |
| p | | 1 |

Assuming that no grid current flows, the admittance tensor of the primitive network is

$\bar{Y} =$

| | 1 | 2 | 3 | g | p |
|---|---|---|---|---|---|
| 1 | $Y^{11}$ | $Y^{12}$ | | | |
| 2 | $Y^{12}$ | $Y^{22}$ | | | |
| 3 | | | $Y^{33}$ | | |
| g | | | | 0 | 0 |
| p | | | | $G^{pg}$ | $G^{pp}$ |

The transformation formula for the admittance tensor, where $\bar{A}$ equals $\bar{C}_t^{*-1}$, is

$$\bar{Y}' = \bar{A}_t \cdot \bar{Y} \cdot \bar{A} \ .$$

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[2] Gabriel Kron, Tensor Analysis of Networks, pp. 357-374.

$$
\bar{Y}\cdot\bar{A} = \begin{array}{c|c|c|c|c|c|}
 & 1 & 2 & 3 & g & p \\ \hline
1 & Y^{11} & Y^{12} & & & \\ \hline
2 & Y^{12} & Y^{22} & & & \\ \hline
3 & & & Y^{33} & & \\ \hline
g & & & & 0 & 0 \\ \hline
p & & & & G^{pq} & G^{pp} \\ \hline
\end{array}
\cdot
\begin{array}{c|c|c|}
 & 1' & 2' \\ \hline
1 & -1 & \\ \hline
2 & & 1 \\ \hline
3 & 1 & 1 \\ \hline
g & -1 & \\ \hline
p & & 1 \\ \hline
\end{array}
=
\begin{array}{c|c|c|}
 & 1' & 2' \\ \hline
1 & -Y^{11} & Y^{12} \\ \hline
2 & -Y^{12} & -Y^{22} \\ \hline
3 & Y^{33} & Y^{33} \\ \hline
g & 0 & 0 \\ \hline
p & -G^{pq} & -G^{pp} \\ \hline
\end{array}
$$

$$
A_t\cdot\bar{Y}\cdot\bar{A} = \begin{array}{c|c|c|c|c|c|}
 & 1 & 2 & 3 & g & p \\ \hline
1' & -1 & & 1 & -1 & \\ \hline
2' & & 1 & 1 & & 1 \\ \hline
\end{array}
\cdot
\begin{array}{c|c|c|}
 & 1' & 2' \\ \hline
1 & -Y^{11} & Y^{12} \\ \hline
2 & -Y^{12} & Y^{22} \\ \hline
3 & Y^{33} & Y^{33} \\ \hline
g & 0 & 0 \\ \hline
p & -G^{pq} & G^{pp} \\ \hline
\end{array}
=
\begin{array}{c|c|c|}
 & 1' & 2' \\ \hline
1' & Y^{11} + Y^{33} & -Y^{12} + Y^{33} \\ \hline
2' & -Y^{12} + Y^{33} - G^{pq} & Y^{22} + Y^{33} + G^{pp} \\ \hline
\end{array}
$$

This product is the admittance tensor of the oscillator. If the circuit is to oscillate, the determinant of $\bar{Y}'$ must be equal to zero. Therefore,

$$(Y^{11} + Y^{33})(Y^{22} + Y^{33} + G^{pp}) - (-Y^{12} + Y^{33})(-Y^{12} + Y^{33} - G^{pq}) = 0$$

Expanding,

$$Y^{11}Y^{22} + Y^{11}Y^{33} + Y^{11}G^{pq} + Y^{33}Y^{22} + (Y^{33})^2 + Y^{33}G^{pp} - (Y^{12})^2 + Y^{12}Y^{33}$$
$$-Y^{12}G^{pq} + Y^{33}Y^{12} - (Y^{33})^2 + Y^{33}G^{pq} = 0$$

Substituting the admittance values into this equation,

$$\frac{-L_1L_2}{\omega^2(L_1L_2 - M^2)} + \frac{L_1C}{L_1L_2 - M^2} - \frac{jL_1}{r_p\omega(L_1L_2 - M^2)} + \frac{L_2C}{L_1L_2 - M^2} + \frac{j\omega C}{r_p} + \frac{M^2}{\omega^2(L_1L_2 - M^2)}$$
$$+ \frac{MC}{L_1L_2 - M^2} + \frac{jM}{r_p\omega(L_1L_2 - M^2)} + \frac{MC}{L_1L_2 - M^2} + \frac{j\omega C}{r_p} = 0$$

Since the sum of the real and imaginary parts is equal to zero, both real and imaginary parts must be equal to zero. Equating the real parts to zero,

$$\frac{-L_1L_2}{\omega^2(L_1L_2 - M^2)} + \frac{L_1C}{L_1L_2 - M^2} + \frac{L_2C}{L_1L_2 - M^2} + \frac{M^2}{\omega^2(L_1L_2 - M^2)} + \frac{MC}{L_1L_2 - M^2} + \frac{MC}{L_1L_2 - M^2} = 0$$

Multiplying through by $\omega^2(L_1L_2 - M^2)$,

$$-L_1L_2 + \omega^2(L_1 + L_2 + 2M)C + M^2 = 0$$

$$\omega^2 = \frac{L_1L_2 - M^2}{(L_1 + L_2 + 2M)C}$$

$$L_1 + L_2 + 2M = L,$$

which is the total inductance of the coil, so,

$$\omega = \sqrt{\frac{L_1L_2 - M^2}{LC}}$$

For a given $L_1$, $L_2$, $M$, and $C$, this is the theoretical frequency of oscillation.

# CONCLUSIONS

Tensor analysis is a comparitively new engineering tool. Only in the last fifteen years has it been applied to electrical engineering. In that time, there have been applications to such diverse electrical engineering subjects as electrical rotating machinery, linear networks, vacuum tube circuits, transformers, and gaseous rectifiers.

The point of view introduced by tensor analysis is helpful to the engineer in allowing him to solve his problems with a minimum of analysis for each individual problem. Its organized methods sometimes save labor on complicated problems or allow the solution of problems which are so complex that ordinary modes of solution fail.

This thesis has covered a very limited portion of the field in which tensor analysis is applicable. However, it may help to give some idea of the nature of tensor analysis and some aspects of its application to electrical networks to those who do not wish to do a great deal of reading on the subject. Enough material is included to give the reader a working knowledge of the solution of mesh networks by the tensorial method. The thesis may also dispel some of the fears of those who have the erroneous belief that tensor analysis is too esoteric for the average engineer.

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