AN IMPROVED METHOD FOR PREDICTING SINGLE-PHASE INDUCTION MOTOR CHARACTERISTICS

AN IMPROVED METHOD

FOR PREDICTING

SINGLE-PHASE

INDUCTION MOTOR CHARACTERISTICS

By

JESSE ROSS

Bachelor of Science

Missouri School of Mines and Metallurgy

Rolla, Missouri

1948

Submitted to the Department of Electrical Engineering

Oklahoma Agricultural and Mechanical College

In Partial Fulfillment of the Requirements

for the Degree of

MASTER OF SCIENCE

1949

APPROVED BY:

Chas. F. Cameron

Chairman, Thesis Committee

Herbert L. Jones

Member of the Thesis Committee

A. Naeter

Head of the Department

D. C. McIntosh

Dean of the Graduate School

PREFACE

The single-phase motor is one of the most useful machines that has been developed. Although its use by industry is of secondary importance, the single-phase motor is of great importance in the home. There have been millions of these motors manufactured, the bulk of them being utilized for domestic purposes. They are used for everything from electric clocks to various types of shop power equipment. The Department of Commerce Census of Manufacturers for 1937 shows that for that year there were more than sixteen million fractional-horsepower motors sold.

Aside from its commercial importance the single-phase motor presents a rather complex but interesting collection of theoretical analysis. The two most prominent theories are the cross field and the revolving field theories.

The single-phase induction motor is one of the earliest types of a-c motor developed for use on power distribution circuits. With the more recent development of three-phase power, the single-phase motor is limited to use as small and fractional-horsepower sizes. They are of particular use when the three-phase power is not available. However, for larger power applications (above 5 to 7.5 horsepower) where polyphase power is available, the polyphase induction motor is used. The polyphase motor is cheaper, has better starting torque, is simpler to maintain, has a better power factor, and does not throw an unbalanced voltage on the line.

Because of the lack of starting torque, the single-phase motor must be provided with some sort of starting device. This may be accomplished by use of an auxiliary winding, commutator, or a simple twist of the rotor with the hand.

In this discussion, the starting characteristics and the theory of operation of the motor will not be considered. The purpose of this thesis is to present a simple and workable method of calculating the characteristics of the single-phase induction motor from the usual no-load and blocked rotor tests. This is accomplished by first defining the circle of the primary current locus and then by developing an analytical solution of that circle.

Since the purpose of this thesis is only to present simple straightforward methods of calculating the performance of the single-phase motor and not to introduce any new performance theory, the paper itself deals only with the methods themselves, the assumptions upon which they are based, and how to make practical use of them.

The methods can be applied directly after the data for the no-load and short-circuit tests have been obtained.

All of the characteristics will be found in terms of an independently varying angle to be called alpha.[1] This angle is the angle subtended by the diameter of the circle and the radius to the end of the input current. The selection of the angle alpha depends upon the end to be accomplished. If an ordinary curve sheet of characteristics is to be plotted, the angle may be arbitrarily selected; if the performance is to be found for a definite value of input current, the angle is found for that current; or if the performance is to be found in terms of the output load, a simple substitution into an equation, which is developed in Chap. 2. will give the angle alpha to be used.

The methods of solution will be demonstrated by use of an example

---

[1] Paper by Professor C. F. Cameron - Oklahoma Agricultural and Mechanical College.

solution. For this example a quarter-horsepower split-phase induction motor will be used.

A method will be presented which shows how to calculate the performance characteristics of single-phase motors from data obtained by laboratory tests. For single-phase motors, tilting the circle diagram improves the accuracy of the results. A method is given which defines the angle of tilt in terms of easily calculated data. An analytical solution of the circle diagram of the single-phase induction motor is developed in this thesis. It is believed that the analytical solution of the circle diagram of the single-phase motor may be used to a greater extent when this information becomes available.

This thesis is presented with the hope that the material will be of some value especially to those engaged in the manufacture of induction motors, and also to anyone who has reason to determine induction motor characteristics.

ACKNOWLEDGEMENT

The writer wishes to express his sincere appreciation to Professor C. F. Cameron for his many helpful suggestions and careful scrutiny of this material.

TABLE OF CONTENTS

CHAPTER I

Characteristics from the Non-tilted Circle

To find the characteristics of a single-phase motor, the method that is usually considered for three-phase motors is used. With very little variation, this method is applied to the single-phase motor. The diameter of the circle is considered to be horizontal, thus for clarity the method will be called the non-tilt method.

Figure 1 is the circuit of a single-phase motor in which g_t and b_t represent the exciting path of the transformer field. The terms, g_c and b_c, represent the exciting path of the speed field.[1] This circuit, as it stands, is complex and the standard polyphase equation for the circle diagram does not hold. In the actual motor, the exciting current for the speed field flux flows in the secondary in addition to the load current. The exciting current for the transformer field flux flows in the primary in addition to the current necessary to overcome the M. M. F. of the secondary current.

If figure 1 were assumed to be approximately the same as figure 2, the circuit could be treated as is done in the case of the polyphase induction motor. Both the speed and the transformer exciting paths are represented as being across the voltage source. This change of the circuit will cause considerable error in the results that will be obtained; but the method is applicable and will give a solution.

In figures 1 and 2, the following notations are used:

---

[1] J. M. Bryant and E. W. Johnson, Alternating Current Machinery. p. 627.

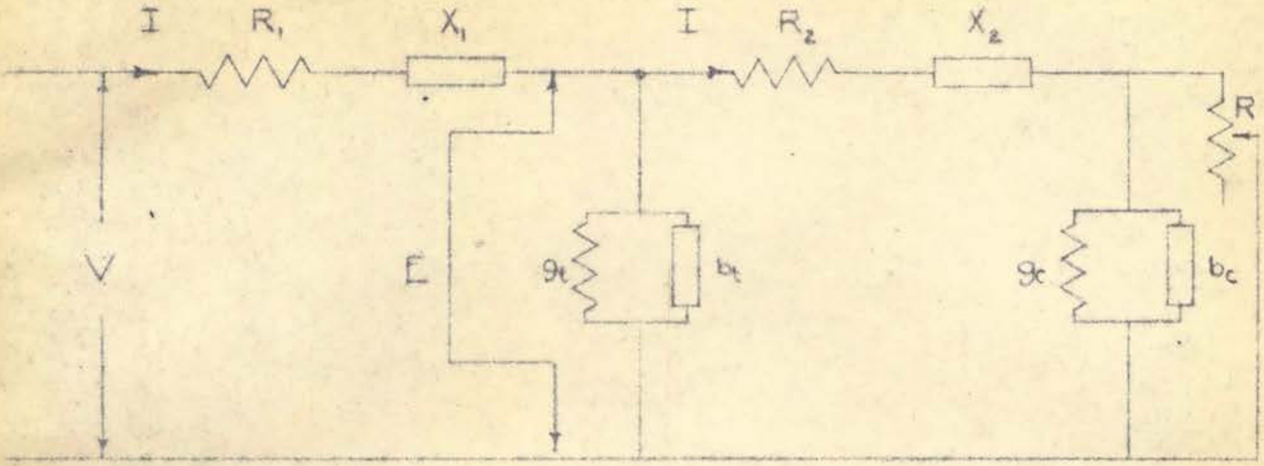

Figure 1. Equivalent circuit for a single-phase induction motor.

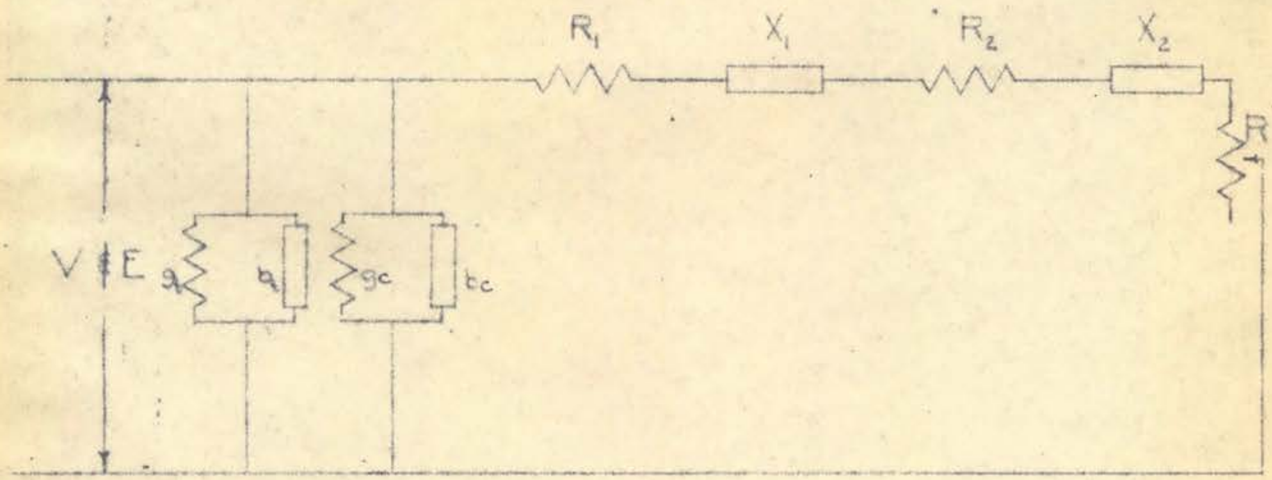

Figure 2. Modified circuit diagram for a single-phase induction motor.

R_1 is the resistance of the primary or stator coils;

R_2 is the equivalent one-to-one resistance of the rotor coils in either path;

R is the resistance accredited to the load applied to the motor;

X_1 is the leakage reactance of the primary or stator coils;

X_2 is the equivalent one-to-one leakage reactance of the rotor coils;

V is the impressed voltage on the primary;

g_t is the exciting conductance to supply primary core loss;

b_t is the exciting susceptance for the primary flux;

g_c is the exciting conductance to supply the secondary losses;

b_c is the exciting susceptance for the rotor to supply the flux ϕ_b;

E is the secondary emf induced by ϕ

In figure 2 the equation for the secondary current may be written as:

$$I_{2s} = \frac{V}{\sqrt{(R_1+R_2+R)^2+(X_1+X_2)^2}}$$

Since there are both primary and secondary leakage reactances I_{2s} must lag the voltage by some angle ϕ. It can be seen from the figure that:

$$\sin \theta = \frac{X_1 + X_2}{\sqrt{(R_1+R_2+R)^2+(X_1+X_2)^2}}$$

All the terms on the right hand side of the equation except R may be considered as constant. R varies as the load varies.

Solving for the value of the square root of each equation gives:

$$\sqrt{(R_1+R_2+R)^2+(X_1+X_2)^2} = \frac{V}{I_{2s}}$$

and

$$\sqrt{(R_1+R_2+R)^2+(X_1+X_2)^2} = \frac{X_1+X_3}{\sin \phi}$$

equating the equals

$$\frac{V}{I_{2s}} = \frac{X_1+X_2}{\sin \theta}$$

from which

$$I_{2s} = \frac{V \sin \theta}{X_1 + X_2}$$

This is the equation of a circle in polar form and is shown graphically in figure 3. The maximum value of I_{2s} is when $\theta = 90^{\circ}$ or the sin of $\theta = 1$. At this point:

$$I_{2s} = \frac{V}{X_1 + X_2}$$

From this equation and by the application of analytical geometry the diameter of the circle may be written as:

$$D = \frac{V}{X_1 + X_2}$$

The condition for the secondary current to be the diameter of the circle can not be reached since this would mean that the value of $R_1 + R_2 + R$ would have to equal zero.

The primary current is:

$$I_p = I_{2s} + I_{et} + I_{es}$$

where I_{et} is the exciting current for the transformer field;

I_{es} is the exciting current for the speed field;

It can be seen in figure 2 that at no-load conditions the secondary current (I_{2s}) is equal to zero. The primary current is equal to the sum of the two exciting currents and may be written as:

$$I_p = I_{et} + I_{es}$$

Where the symbols are the same as used before. The no-load vector diagram is shown in figure 4. In that figure the symbols are as follows:

MO is the sum of $I_{et} + I_{es}$;

θ_{nl} is the no-load power factor angle;

No is the total magnitizing component of the exciting currents.

It should be noted in this figure that the flux vector lags the induced voltage, or in this case the terminal voltage, by 90 degrees.

In figure 2 the primary current is the vector sum of two currents, the secondary current, and the constant exciting currents. If figure 3 were placed such that point "O" coincides with the point "O" of figure 4 the diagram would be as shown in figure 5.

where

MO is the total exciting current;

P is any point on the circular locus between O and A;

MP is the primary current (I_p);

OP is the secondary equivalent current (I_{2s});

ϕ_1 is the angle of lag between the primary current and the impressed voltage;

cos ϕ_1 is the power factor of the motor as a load;

PC is the active component of the primary current;

DC = MN is the portion of PC which supplies the core losses of the motor;

PD is the equivalent active component of the rotor current.

As the motor is loaded, the point P will move along the circle OPA, of figure 5, until at blocked rotor or starting conditions. This at the position MB. The secondary current is OB, and MB is the primary current. The motor

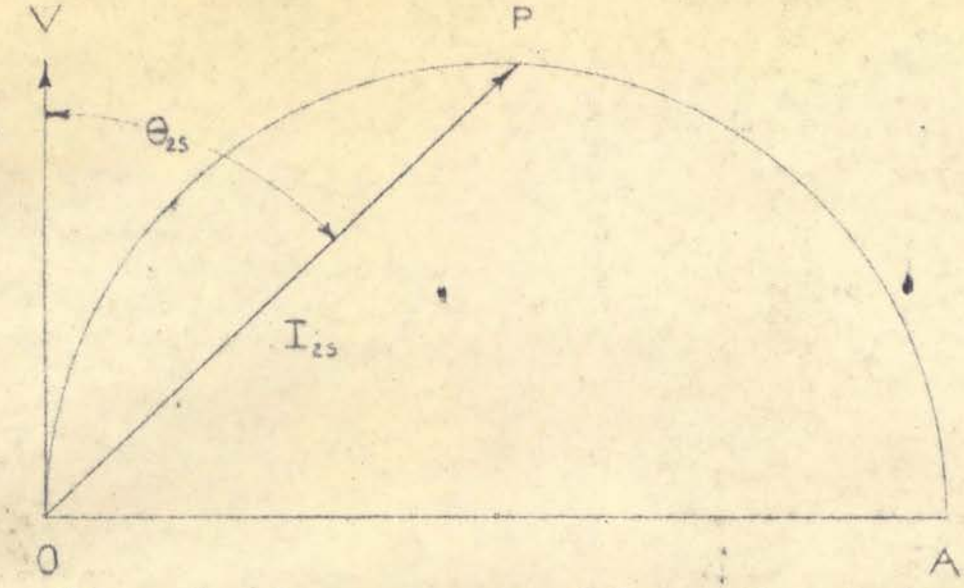

Figure 3. Secondary current locus of a single-phase motor.

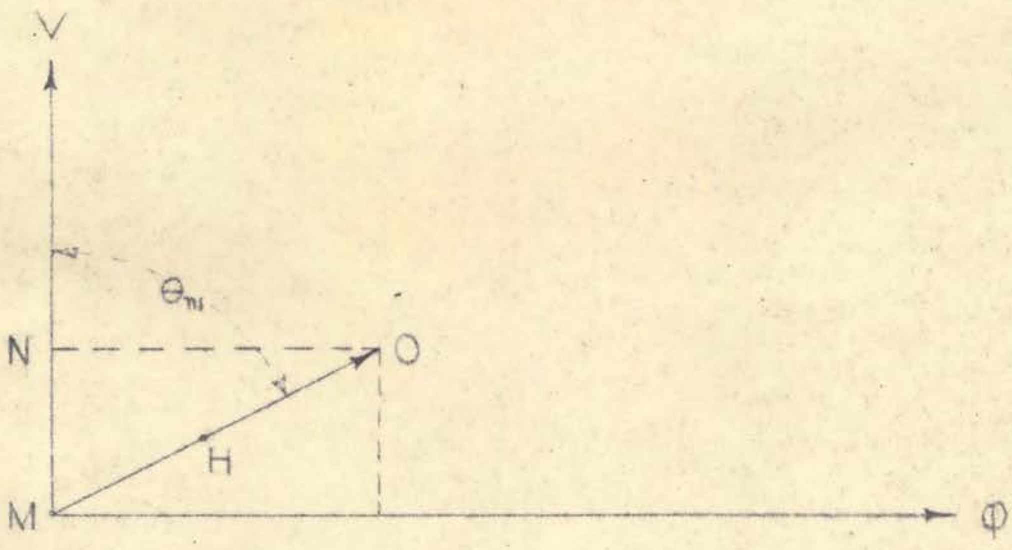

Figure 4. Vector diagram of exciting current.

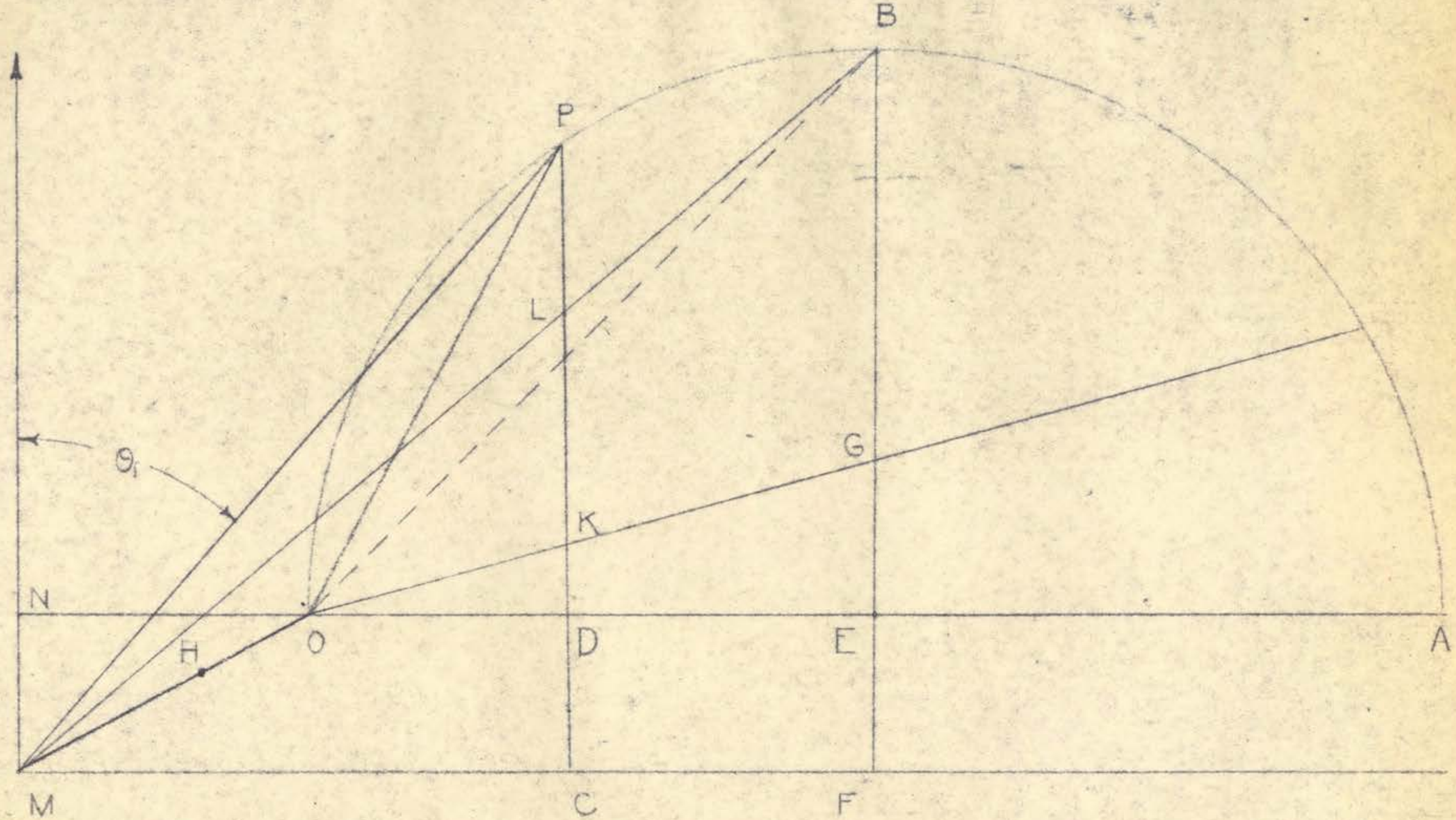

Figure 5. Complete non-tilted circle diagram,single-phase induction motor.

is at rest and there can be no output, thus the active component of the current BF must represent the current supplying the primary and secondary copper loss and the core loss. (Actually the friction and windage loss disappears, but there is an increase in the core loss due to the increase of frequency of the flux passing through the rotor steel. These two changes cause errors that tend to cancel each other). The portion BE of BF will represent the rotor I^2R loss and the primary I^2R loss. The portion EF was already accounted for as core loss. By measuring the primary resistance the primary I^2R loss may be calculated. Thus the portion of the line shown as EG is proportional to the loss, $(I_B^2 - I_{nl}^2)R_1$. Since the lines of the diagram are lines that represent currents it follows that the proportionality constant must be $\frac{1}{V}$ so that GE will have units that are the same as the other lines of the diagram (amperes).

Hence

$$EG = \frac{(I_B^2 - I_{NL}^2)R_1}{V}$$

From this discussion the following features of the diagram shown in figure 5 can be recognized[1].

MN is the active component of the no-load primary current;

No is the reactive component of the no-load primary current;

BF is the active component of the primary current at standstill;

HO is the secondary speed field current at synchronous speed;

MH is the transformer field current at synchronous speed;

GE is the component of I_p at standstill to supply the added I^2R loss of the primary;

---

[1] *Ibid.*, p. 629.

BG is the component of I_p at standstill to supply the rotor I^2R loss;

HP is the total rotor current at any speed;

$\angle$VMP is the angle of lag of the primary current behind V at any load and speed;

PC/PM is the power factor of the motor at any load;

KD is the added primary I^2R loss to scale;

LK is the added secondary I^2R loss to scale;

LC is the total loss of the motor to scale;

PL is the total motor output to scale;

PC is the total motor input to scale;

PL/PC is the motor efficiency;

PK is the total secondary power input exclusive of speed field I^2R loss and rotational loss:

$\sqrt{PL/PK}$ is the speed;

$\sqrt{(PL)X(PK)}$ is the torque in synchronous watts.

The preceding method of finding the characteristics of the single-phase motor is widely used. Although the accuracy of this method is poor, it is used since other methods are not so easily applied to laboratory data. The larger part of the error that is encountered when using this method is caused by the assumption that the exciting branches of the circuit are across the terminal voltage. When the circuit is changed in this manner (although not brought out in the previous discussion) the induced voltage of the secondary was assumed to be in phase with the primary voltage. From figure 1 it can be seen that this is not the case since the flow of current through the primary resistance and reactance would cause both a phase shift and a drop from the terminal voltage to the secondary or induced voltage. This phase shift actually causes the circle to tilt upward. The tilt phenomena will be elaborated on in the following discussions.

The error that was caused by assuming the core and speed field losses across the terminal voltages and thus not tilting the circle will increase as the load is increased. This is obvious since the tilted circle will be the same as the non-tilted circle at no-load and will then start to deviate as the primary current proceeds around the circle.

Veinott Method of Performance Calculations[1]

The method of calculating the performance of the single-phase induction motor that was developed by Mr. Veinott and presented in <u>Electrical Engineering</u> in 1932 makes extensive use of the West theory.[2] For the paper, "Performance Calculations on Inductions Motors", Mr. Veinott uses the same assumptions that were used to develop the West theory. The angle of hysteretic lag between flux and the mmf is neglected, and the magnetomotive force or current required to produce a certain flux is assumed to be proportional to the flux, just as in the West paper. The general basis of this method is that the voltage equations of the circuits in figure 6 are set up according to Kirchhoff's law The solution of these three simultaneous equations give the current in each branch. The circuits are: the primary circuit, the transformer field in the secondary, and the cross field circuit in the secondary. From these currents and the known reactances the performance data is calculated. The theory of this method will not be gone into in detail but the method of calculation will be shown. Before proceeding with the method of calculation the terms and notations to be used must be defined. They are as follows:

E is the impressed voltage;

r_1 is the primary resistance;

r_2 is the secondary resistance referred to the primary;

r_g is the resistance representing core loss when connected in parallel with X_m;

---

1 C. G. Veinott, "Performance Calculation on Induction Motors", <u>Electrical Engineering</u>, September 1932, p. 743.

2 H. R. West, "The Cross Field Theory of A. C. Machinery", <u>Electrical Engineering</u>, February 1926, p. 160.

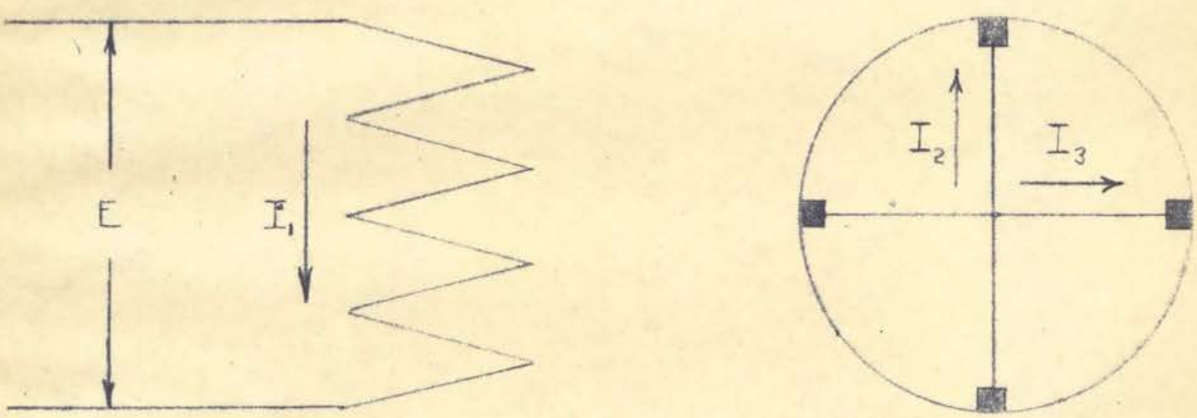

Figure 6. The single-phase induction motor viewed from the cross field theory.

r_m is the resistance representing core loss when connected in series with X_m;

x_1 is the primary leakage reactance;

x_2 is the secondary leakage reactance referred to the primary;

X_m is the magnetizing reactance;

X is the ideal short-circuit reactance of motor ($r_2 = 0$) $= x_1 + x_2 \frac{X_m}{X_m + x_2}$. This is the reactance the motor would have with short-circuited secondary if the iron loss and secondary resistance were zero. This expression, however, does take into account the shunting of the secondary leakage reactance by the magnetizing reactance;

$Xo = Xm + x_1$ = reactance of the primary winding with the secondary open-circuited;

$Kp = \frac{Xm}{Xo} = \sqrt{\frac{Xo - X}{Xo}}$ (used only when $x_1 = x_2$);

I_1 = primary current;

I_2 = secondary current in transformer axis for single-phase motors;

I_3 = secondary current in cross field axis;

Io is the no-load current

$I_m = \frac{E}{Xo}$ = primary current which would flow if secondary were open-circuited;

$S = \frac{\text{actual speed}}{\text{synchronous speed}}$

The next step in the solution is to find the motor constants. They are defined in Mr. Veinott's[3] article as:

---

[3] C. G. Veinott, op. cit., p. 744.

$K_p = \sqrt{(X_o - X) / X_o}$

$r_2/X = n\, r_2/X_o$

n is the turns ratio

$i_m = \frac{E}{X_o}$

$F_1 = (2 - K_p^2) r_2$

$F_2 = (2\, r_1 + r_2)\, (r_2/X_o)$

$F_3 = (i_m\, r_2)\, (r_2/X_o)$

$F_4 = (i_m\, r_2)\, 2$

$F_5 = (i_m\, r_2)\, K_p$

$F_6 = (i_m\, r_2)\, K_p^{\,2} r_2$

$F_7 = EK_p$

$F_8 = (EK_p)^2 r_2$

$F_9 = \frac{\text{Fe loss (m)}}{E}$

From this the actual solution can be performed. This will be done as it was done in the original article. That is, each step will be numbered and as one of the secured values is used over, its number will be used. This simplifies the procedure and makes the tabulation easier.

1 $S = \frac{\text{r.p.m.}}{\text{syn.}}$

2 S^2

3 $(1-S^2)$

4 $(1-S^2) r_1$

5 F_1

6 $U = (4) + (5)$

7 $(1-S^2)X$

8 F_2

9 $W = (7) - (8)$

10 $\sqrt{U^2+W^2}$

11 $(1-S^2)E$

12 F_3

13 $M = (11) - (12)$

14 F_9U

15 $N = (13) + (14)$

16 $\sqrt{N^2+F_4{}^2}$

17 $I_1 = (16) /(10)$

18 $(1-S^2)F_7$

19 $\sqrt{(18)^2 + F_5{}^2}$

20 $I_2 = (19) /(10)$

21 SF_5

22 $I_3 = (21) /(10)$

23 $(1-S^2)F_8$

24 F_6

25 $(23) - (24)$

26 Prim. cu. loss = $I_1{}^2 r_1$

27 Sec. cu loss (m) = $I_2{}^2 r_2$

28 Sec. cu. loss (c) = $I_3{}^2 r_2$

29 Fe loss (m)

30 $(25) \times (2) /(10)^2$

31 Input = (26) + (27) + (28) + (29) + (30)

32 Fe loss (c) + (F and W)

33 Output = (30) - (32)

34 R.p.m. = S x syn.

35 Torque = $\frac{112.6 \times (33)}{(34)}$

36 Eff. = (33) /(31)

37 p.F. = (31) /EI_1

38 App. eff. = (36) x (37)

This method is very complicated even when making the assumption that the reactance of the primary equals the reactance of the secondary. The calculation of the motor constants is laborious and subject to error due to its many operations.

To calculate the motor constants by this method many physical facts are required that are not ordinarily available or obtainable by laboratory methods. These calculations would require the use of the design data of the motor.

The Veinott method looks as though it would be long but very accurate when used with design data, but it would be of little use for finding the performance from laboratory data.

Langsdorf Downward Tilt

The explanation of the method of finding the characteristics of the single-phase motor as stated in the text by A. S. Langsdorf[4] will not greatly further the work that is being done here. However, a few comments on some of the work presented in Mr. Langsdorf's book in the chapter on single-phase induction motors are in order.

Mr. Langsdorf develops a solution of the performance characteristics of the single-phase induction motor in terms of a circle diagram of the input current. To locate the locus of the input current, Mr. Langsdorf developes an equation of the input current from the exact equivalent circuit. The exact equivalent circuit is shown in Figure 1. The final equation of the input current is a complicated equation that is the equation of a circle in polar form. The locus of the input current, according to the equation of the input current, is a circle that has a diameter inclined slightly downward from the tip of the no-load current vector. The downward inclination of the diameter is an angle of very small magnitude. Since the angle of downward tilt is so small, Mr. Langsdorf comes to the final conclusion that it is just as accurate to use the method that has been previously described in this paper as the non-tilt method.

A study of the actual plot of the primary current of several single-phase motors has led to the conclusion that there is some mistake in the downward tilt theory. Also this downward tilt phenomena has not been recognized by any other reference that was consulted.

In view of what has been stated above and since nothing else of significance on this subject was in that reference it will not be further discussed.

---

[4] A. S. Langsdorf, Theory of Alternating Current Machinery, pg. 678.

Other Solutions

One of the most prominent methods that has been developed that has not been discussed, is the four terminal network solution.[5] This method is also a method that has the disadvantages of the inherent elusive motor constants. Its solution is also useful when the design data of the motor is known, but it fails for laboratory calculations since some of the values that are involved are reactances that are extremely difficult to evaluate. This method shows that the primary current locus, of the single-phase induction motor, follows a circle which has its diameter from the tip of the no-load current vector tilted upward.

Performance characteristics of the single phase inductionmotor can also be found by use of symmetrical components.[6] This method has the same disadvantages that were discussed for the other solutions. That is, the motor constants that are used are not ordinarily evaluated in the laboratory when running an induction motor test. According to Mr. Lawrence, this method is simpler than the other methods developed.

Another development that is well worth discussion is the amortisseur action[7] and its resulting method of finding the performance characteristics of single-phase induction motors.[8]

This proposition (amortisseur action) is presented as a coordination

---

[5] J. G. Tarboux, Alternating Current Machinery, pp. 335.

[6] R. R. Lawrence, Principles of Alternating Current Machinery, pp. 567.

[7] E. Bretch, "Amortisseur Action of the Squirrel Cage," Electrical Engineer, June, 1946.

[8] E. Bretch, "Letters to the Editor," Electrical Engineering, January, 1947.

between the two single-phase motor theories, the rotary field and cross field theories, and corrects some of the inconsistency of these theories. The proposition is that the quadrature flux of the single-phase squirrel cage motor is developed by the amortisseur action of the squirrel cage.

This theory and its application to motors is and has for years been applied to the repulsion start induction run motor even though the method of application is not in text books.

For solution of this method, as presented by Mr. Brotch, the following data must be evaluated:

1 Amperes, zero speed open rotor.

2 Watts, zero speed open rotor.

3 Amperes, synchronous speed short-circuited rotor.

4 Watts, synchronous speed short-circuited rotor.

5 Amperes, locked short-circuited rotor.

6 Watts, locked short-circuited rotor.

7 Primary resistance.

From the above, it seems that the solution of performance characteristics is applicable only to repulsion start induction motors. This follows from the fact that the only type of motor that items 1 and 2 can be evaluated for is the repulsion start type of induction motor.

Method of Tilt from Laboratory Data

The discussion of the previously described methods for handling the circle diagram of the single-phase motor shows that there has been no really usable solution devised as yet. This, at least, is true for all the references that it was possible to consult. In all the references that were consulted, some of which were discussed, the method of finding the circle diagram was in each case faulty in one of two ways. Either the results were inaccurate (this was the case for the discussed non-tilt method[9]) or the method was so complicated that it could not be applied directly to the usual induction motor tests. In some cases the method of solution involved reactances that could not be evaluated if the design data were not at hand.

From this, the problem resolved itself into finding a method that is accurate and involves only the usual induction motor test data. The test data is secured from the no-load test, the blocked rotor test, and the determination of the primary resistance.

At first it was thought that the answer was to be found within the equation that was developed for the approximate circuit in figure 2 of this chapter. The no-load test and the blocked-rotor test each establish a point on the circle. One of three things remains to be found to define the circle. The angle that the diameter makes with the horizontal from the no-load point, the length of the diameter, or some other point on the circle must be found for the circle to be completed.

The equation for the secondary current[10] using the approximate equivalent

---

[9] J. M. Bryant and E. W. Johnson, _op. cit._, p. 627.

[10] _Ibid._, p. 628.

circuit was found to be:

$$I_s = \frac{V}{X_1 + X_2} \sin \theta$$

Where $\frac{V}{X_1 + X_2}$ is the diameter of the circle.

From this it can be seen that the circle can be completed if the value of X_1 and X_2 or $X_1 + X_2$ can be found. This line of attack was tried for a split-phase motor. Specifically, the motor tested was a single-phase, one-quarter horsepower Century motor. The test showed that the value of $X_1 + X_2$ varied over a considerable range from no-load to blocked rotor. By plotting the circle from actual load test, it was found that the correct value of reactance was at some point a little less than full load. From this it was thought that there might possibly be some saturation constant that would give the proper reactance. This was proven not to be the case by a similar run of tests on a different motor. The second motor used was a single-phase, one-half horsepower Century motor. The correct value of reactance for this motor was found to be at a point a little over one-half load.

The main error involved in the use of the approximate circuit is that the excitation current that passes through primary impedance is neglected. This is the error introduced when the core and speed field branches of the circuit were moved out to a point at which they were directly across the terminal voltage as in figure 2. This, as is seen in its use with three-phase motors, is permissible when the primary impedance is small with respect to the other values of the circuit. The smaller the motor, usually, the greater the error that is encountered when the core loss component is moved. As a rule single-phase motors would have relatively large impedance drops in the primary circuit and because of the impedance drop the core loss and speed field loss cannot be moved about when the single-phase motor is under consideration.

If the core loss and speed field branches of the exact circuit are left in place as in figure 1, it can be shown that the difference between this exact circuit and the approximate circuit is that the core loss and speed field exciting currents now flow through the primary impedance.

This circuit (figure 1) is the exact circuit since it was derived directly from either the cross field or the revolving field theory.[11]

The same method of attack can be applied to the exact circuit that was applied to the equivalent circuit.

To eliminate some of the mathematics that would be involved, the first step is to lump the speed field exciting branch in with the load resistance and reduce this to a single series impedance. This is done without changing the circuit materially. As can be seen, this is shown in figure 7.

Now, an expression for I_{2s} can be written as:

$$I_{2s} = \frac{E}{\sqrt{(R_2+R_{sl})^2 + (X_2+X_3)^2}}$$

Where E is the induced voltage in the rotor

R_2 is the secondary resistance

R_{sl} is the speed field and load resistance

X_2 is the secondary reactance

X_3 is the speed field series reactance

---

11 J. G. Tarboux, op. cit., pp. 332.

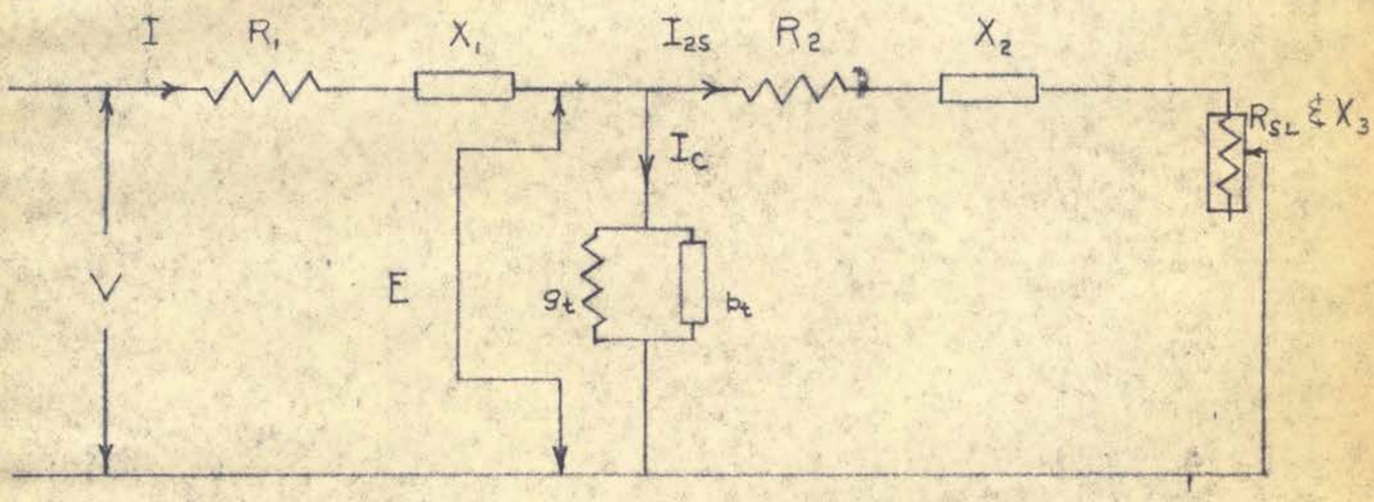

Figure 7. Equivalent circuit for a single-phase induction motor.

There is both secondary and speed field reactance so I_{2s} must lag the induced voltage by some angle θ where

$$\sin\theta = \frac{X_2 + X_3}{\sqrt{(R_2+R_{s1})^2 + (X_2+X_3)^2}}$$

solving for the term under the radical sign

$$\sqrt{(R_2+R_{s1})^2+(X_2+X_3)^2} = \frac{X_2 + X_3}{\sin\theta}$$

and from the secondary current equation

$$\sqrt{(R_2+R_{s1})^2+(X_2+X_3)^2} = \frac{E}{I_{2s}}$$

Equating the equals

$$\frac{E}{I_{2s}} = \frac{X_2 + X_3}{\sin\theta}$$

from which

$$I_{2s} = \frac{E\sin\theta}{(X_2 + X_3)}$$

This is an equation of the same apparent form as the secondary current equation that was previously developed. However, there is one difference of form. The term E is the rotor induced voltage instead of the terminal voltage. The induced voltage is a variable that is dependent upon the current through primary impedance.

The new secondary current equation requires some manipulation before it will be of any value. This is not a polar equation of a circle since

$\frac{E}{X_1 + X_3}$ is not a constant.

$$I_{2s} = \frac{E \sin \theta}{X_2 + X_3}$$

E in this equation is the terminal voltage minus the primary impedance drop, thus:

$$E = V - I Z_1$$

and

$$I_{2s} = \frac{(V-IZ_1) \sin \theta}{X_2 + X_3}$$

The current, I, in this equation, is the sum of the core loss and secondary branch currents.

$$I = I_{2s} + I_c$$

now

$$I_{2s} = \frac{\left[V-(I_{2s}+I_c)Z_1\right] \sin \theta}{X_2 + X_3}$$

or

$$I_{2s} = \frac{(V-I_{2s}Z_1-I_cZ_1) \sin \theta}{X_2 + X_3}$$

expanding

$$= \frac{(V-I_cZ_1) \sin \theta}{X_2 + X_3} - \frac{I_{2s}Z_1 \sin \theta}{X_2 + X_3}$$

then collecting terms

$$I_{2s} + \frac{I_{2s}(Z_1 \sin \theta)}{X_2 + X_3} = \frac{(V-I_cZ_1) \sin \theta}{X_2 + X_3}$$

factoring

$$I_{2s}\left(\frac{1+Z_1 \sin \theta}{X_2 + X_3} \right) = \frac{(V-I_cZ_1) \sin \theta}{X_2 + X_3}$$

dividing through by

$$1 + \frac{Z_1 \sin \theta}{X_2 + X_3}$$

then the result is an equation in terms of I_{2s}

$$I_{2s} = \frac{(V - I_c Z_1) \sin \theta}{(X_2 + X_3)\ (1 + \dfrac{I_c Z_1 \sin \theta}{X_2 + X_3})}$$

and simplifying

$$I_{2s} = \frac{(V - I_c Z_1) \sin \theta}{X_2 + X_3 + Z_1 \sin \theta}$$

the induced voltage $E = V - IZ_1$ and $I = I_c + I_{2s}$ but at no-load $I_{2s} = 0$

thus, $E_{nl} = V - I_c Z_1$

Now by substitution

$$I_{2s} = \frac{E_{nl} \sin \theta}{X_2 + X_3 + Z_1 \sin \theta}$$

This is the equation of the secondary current in terms of the constants and the varying angle θ. The angle θ is the angle between the induced voltage at no load and the secondary current. When θ is zero the secondary current is zero, and when θ is 90° the secondary current is maximum. From this it follows that the diameter of the circle (assuming this equation to be an approximate circle) must lag the induced voltage by 90°.

According to this equation the secondary current does not follow the locus of a circle but tends to flatten slightly at the top. All the references that were consulted and the tests that were run agreed that I_{2s} follows a circle. Thus the variation of the above equation from a circle probably is due to the assumption of constant core loss which is not true.[12]

[12] Bryant and Johnson, op. cit., p. 614

Throughout the possible load range of a motor this equation will conform very closely with the experimental data in that both can be considered as circles.

This discrepancy (the difference between the equation for I_{2s} and the equation for a circle) does not hinder this solution since the equation was derived to show that the diameter of the circle is perpendicular to the no-load induced voltage and this is adequately demonstrated. The induced voltage at no-load is defined as V the terminal voltage minus the I_nZ_p drop. This primary impedance drop shows that the induced voltage has a phase shift as well as a decrease in magnitude.

The construction involved, when drawing the circle diagram, will be such that the diameter will be perpendicular to the induced voltage and the horizontal will be perpendicular to the terminal voltage. Then since the sides of one of the angles are perpendicular to the sides of the other angle the two angles are equal. Thus, the angle that the diameter makes with the horizontal is equal to the angle that the induced voltage at no-load makes with the terminal voltage.

The primary impedance (Z_p) is made up of the primary resistance and reactance. The primary reactance is an even more obscure value than the sum of the primary and the secondary reactances. Since the reactances could not be found for the previously discussed method, it would seem that the use of the primary reactance would cause this method to be of no value. This is not the case. The primary impedance drop is shown in its proper place in figure 8. The resistance drop is in phase with the no-load current and the reactance is perpendicular to it. The power-factor angle at no-load is usually large thus making the reactance drop lie such that it is approximately parallel to the

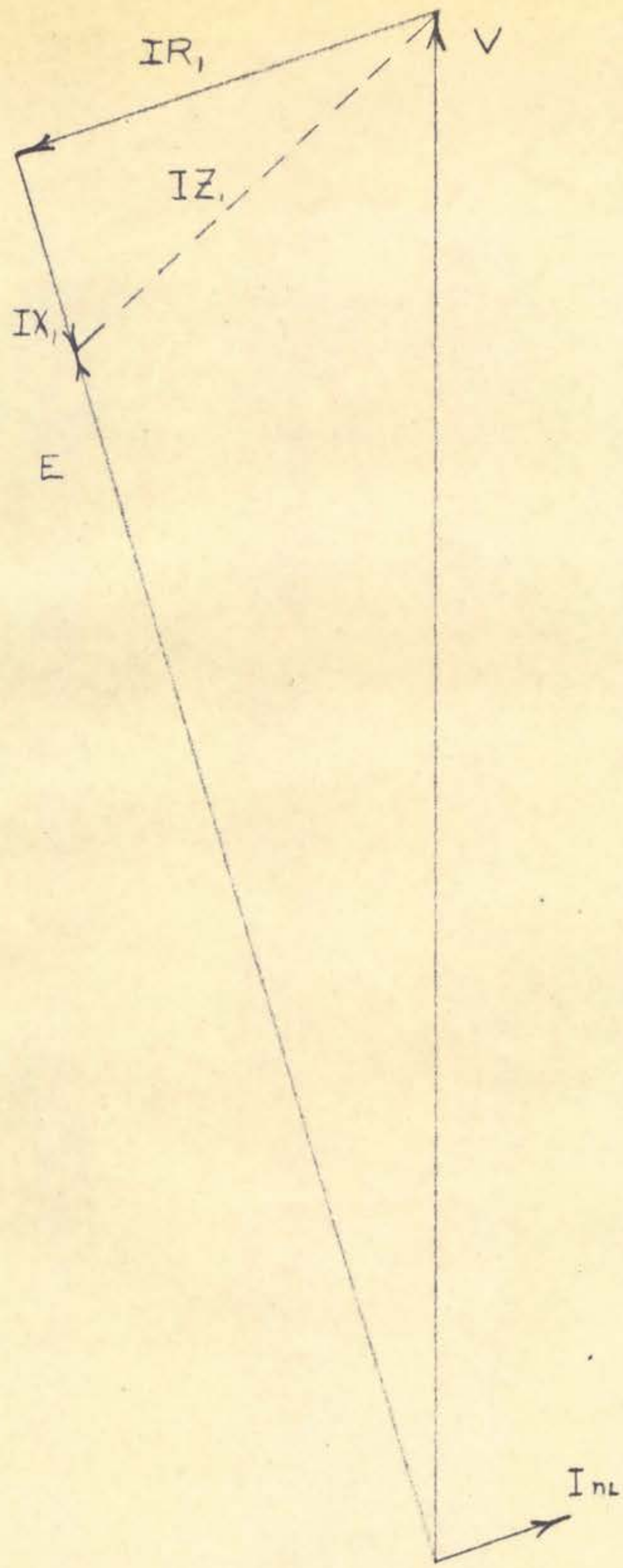

Figure 8. No-load vector diagram of the voltages of a single-phase induction motor.

line that will represent the induced voltage. The reactance drop then principally changes only the magnitude and has very little effect upon the phase shift of the induced voltage.

As was stated before, the purpose of finding the induced voltage is to facilitate the construction of the diameter of the circle. The induced voltage is used only in finding the angle that the diameter makes with the horizontal. The diameter of the circle is constructed perpendicular to the induced voltage at the no-load current point. From this it follows that the magnitude of the induced voltage has no bearing on the circle, thus, the reactance component of the primary drop may be neglected.

- The above method was applied to the test data as before and the results were in keeping with the above assumptions. The errors that were found were negligible. The check was made by calculating the angle of tilt from the no-load test and the primary resistance and then comparing this calculated angle with the actual angle. The actual angle was found by plotting the circle from the brake test and then constructing the true diameter. The brake test circles were found to be no better than the constructed circles since in finding the center of the test circle it was necessary to construct several perpendicular bisectors to the chords of the circle and then to find the average point of intersection of all these bisectors.

The next step was to try the tilt method using the primary reactance. As has been explained, this reactance is very difficult to find and could not be used if only the usual test data was at hand. The reactance of the primary was found by taking the rotor out of the motor and measuring the reactance direct. Several other methods were tried but the best results were achieved by the direct method.

The angle of tilt of several motors was found from the induced voltage as before, but this time the measured reactance was used instead of being neglected. The results were compared to the results that were achieved when the reactance was neglected. It was found that the method that neglected the reactance was in error about five percent for the motor of maximum error. The error is negligible in magnitude since the reactance that was used was in error an indeterminate amount.

The circle locus of the input current is now completely defined. The no-load and blocked rotor rests each define a point that lies upon the circle. The angle of tilt of a diameter from the no-load point can be determined by use of the no-load current and the primary resistance. With the above information, the circle can be completed by conventional methods. The chord is drawn between no-load and blocked rotor points. The perpendicular bisector of the chord intersects the tilted diameter at the center of the circle. The center of the circle and two points on the circle are then known and the circle can be drawn.

As shown above, the circle locus of the input current can be completely constructed from the conventional laboratory test data. When the circle is drawn for the smaller single-phase motors, it will involve very small lines and angles that will add to the usual graphical errors. It will follow then, that an analytical solution using this circle will enhance considerably the work that is being done here.

CHAPTER II

Introduction to Chapter II

The following is a method for predicting the performance characteristics of an induction motor which embodies the advantages of the graphical method but increases the accuracy to that of a mathematical solution. The derivation is developed from the standpoint of a single-phase motor.

The circle diagram is solved by an analytical method. It has been found that tilting the circle diagram increased the accuracy, and this scheme is likewise solved by the analytical method.

Since the value of the no-load reactance drop has been found to be negligible, it has been omitted and the angle of tilt has been defined using only the primary no-load resistance drop. This has been discussed in Chapter I.

Study of the following development will lead to the conclusion that for a high degree of accuracy the small angles involved will necessitate the uses of trigonometric tables in place of the slide rule; although ordinary slide rule accuracy gives better results than the usual method of solving by the graphical solution of a circle diagram. The use of the tables as suggested requires a little discretion. The number of significant figures to which the data is carried out depends on the known accuracy of the given design or test data. For all the information available it was found that the circle could be considered as correct for the calculation purposes.

The derivation of the analytical solution is in four parts. Each of the last three parts is of value for a different set of boundry conditions. If an ordinary set of characteristic curves is to be plotted,it is necessary only to pick convenient values of the angle alpha and solve. When the input current is the known boundry condition then the proper values of the current are substituted, and from this the proper angle alpha is found for each current used. The last but probably the most useful method, is the method where by the angle alpha can be found for any output load for which the characteristics are to be found.

PART 1

Defining the Circle

The conventional tests for the characteristics of an induction motor are to be used. For the single-phase motor the no-load and blocked rotor tests are usually run at rated voltage. From these tests the no-load current and power-factor as well as the blocked rotor current and power-factor can be calculated. The primary resistance is measured directly.

The following (Part 1) uses the information obtained from the above discussed tests for actually placing the circle, thus defining it.

The circle as defined here, differs from the conventional circle only in the angle of tilt. To construct the circle, the same procedure is followed that would be normally used except that the no-load induced voltage would be found by construction or in the same way as it is calculated and the diameter would, as usual, be constructed perpendicular to the no-load induced voltage. There would be some question as to the value of tilting the circle when the inaccurate graphical method is to be applied to the small angles that are involved in the single-phase circle diagram. It would be improbable that a circle could be enlarged enough to facilitate the measuring of the lines that are involved in the small angles. It was found that the tilt angle is usually around three degrees while the quarter load alpha angle is sometimes less than two degrees and usually about two and one-half degrees.

Given data (either design or experimental):

I_n at an angle θ_n ---- from no load tests

I_b at an angle θ_b ---- from blocked rotor tests

Applied voltage ---- V

Primary resistance---- R_p

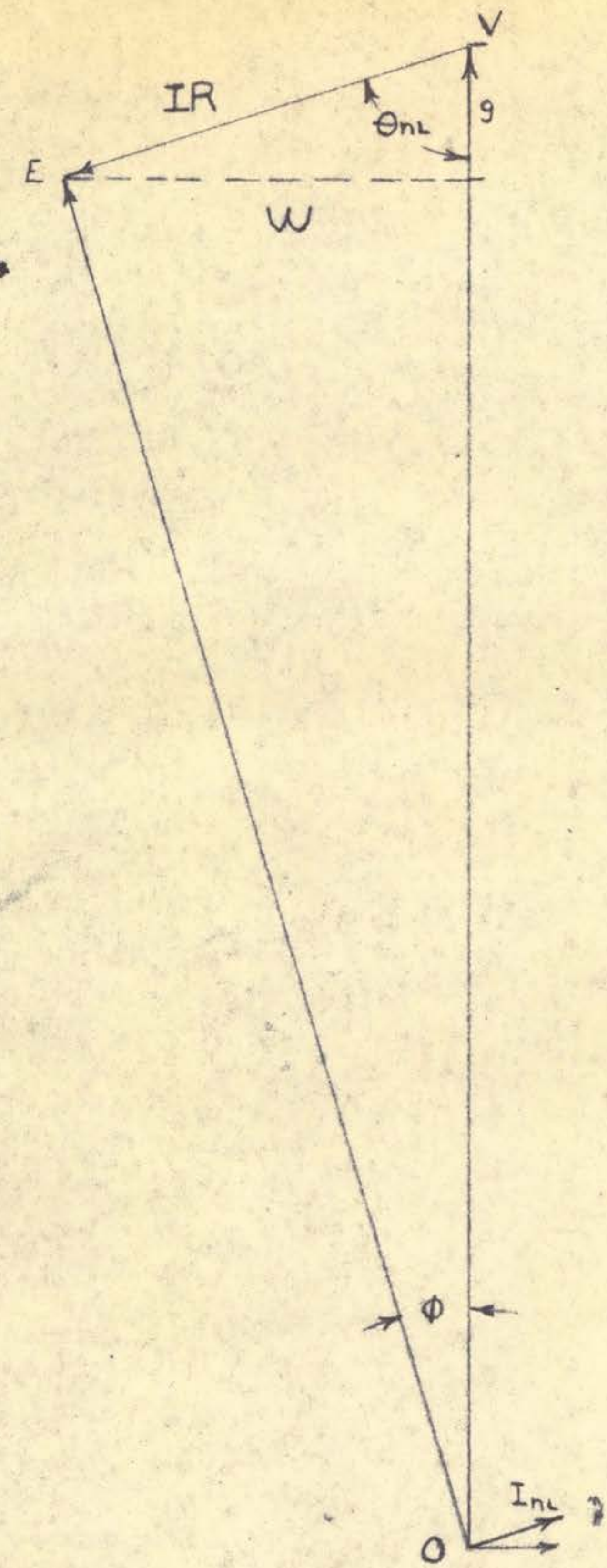

Figure 9. Approximate vector diagram that defines w and g.

$I_n R_p$ = no load current times stator resistance

From figure 9 it is seen that:

$$W = I_n R_p \sin \theta_n$$

and

$$g = I_n R_p \cos \theta_n$$

The angle ϕ will be found in figure 9

$$\phi = \tan^{-1} \frac{W}{V-g}$$

If the sides of an angle are perpendicular to each other, the angles are equal; hence, the angles ϕ of figure 11 are equal or $\phi_1 = \phi_2 = \phi_3$. All three of these angles will be referred to as ϕ for the rest of the analytical solution.

The diameter of the circle of figure 10 was constructed perpendicular to the induced voltage E. OA is no-load current. OP′ is blocked rotor current.

Then from figure 10 it can be seen that:

$$X_n = I_n \sin \theta_n$$

$$Y_n = I_n \cos \theta_n$$

$$X_b = I_b \sin \theta_b$$

$$Y_b = I_b \cos \theta_b$$

from this

$$\rho + \phi = \tan^{-1} \frac{Y_b - Y_n}{X_b - X_n}$$

then collecting terms

$$\beta = \tan^{-1} \frac{Y_b - Y_n}{X_b - X_n} - \phi$$

Angle M = 90^o (angles inscribed in a semicircle are equal to 90^o)

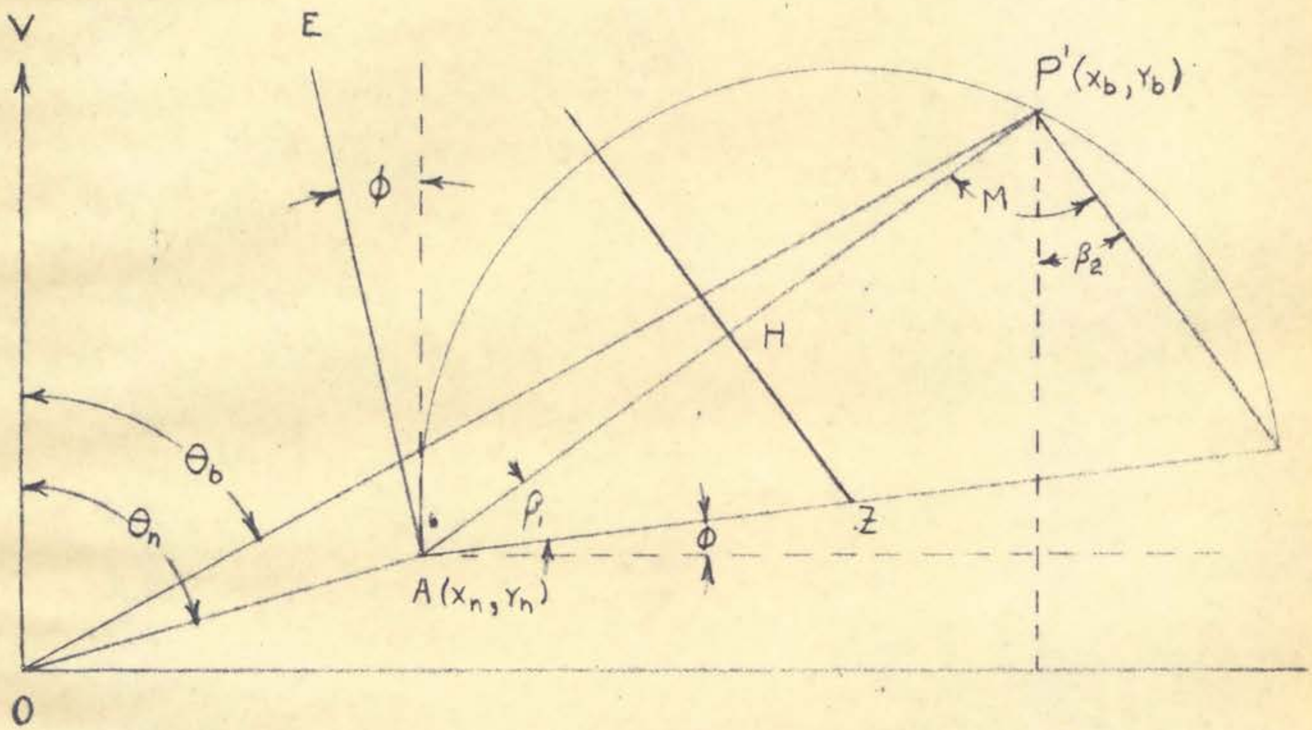

Figure 10. Primary current locus of a single-phase induction motor.

then since $(90^{\circ} - \beta_1) = (90^{\circ} - \beta_2)$

$$\beta_1 = \beta_2$$

The two angles will be referred to as β. In figure 10 AP' is a chord of the circle. Then since the perpendicular bisector of a chord of a circle passes through the center of the circle, it follows that AH = 1/2 AP' and the angle at H is constructed as a right angle.

From this it can be seen that

$$R = \frac{AH}{\cos\beta}$$

$$AH = \sqrt{\left(\frac{X_b - X_n}{2}\right)^2 - \left(\frac{Y_b - Y_n}{2}\right)^2}$$

and by substitution

$$R = \sqrt{\left(\frac{X_b - X_n}{2}\right)^2 - \left(\frac{Y_b - Y_n}{2}\right)^2} \div \cos\beta$$

The circle is now completely defined. The angle $\emptyset$ gives the tilt of the circle, the value R gives the radius of the circle, and the angle β which will be useful later is now known.

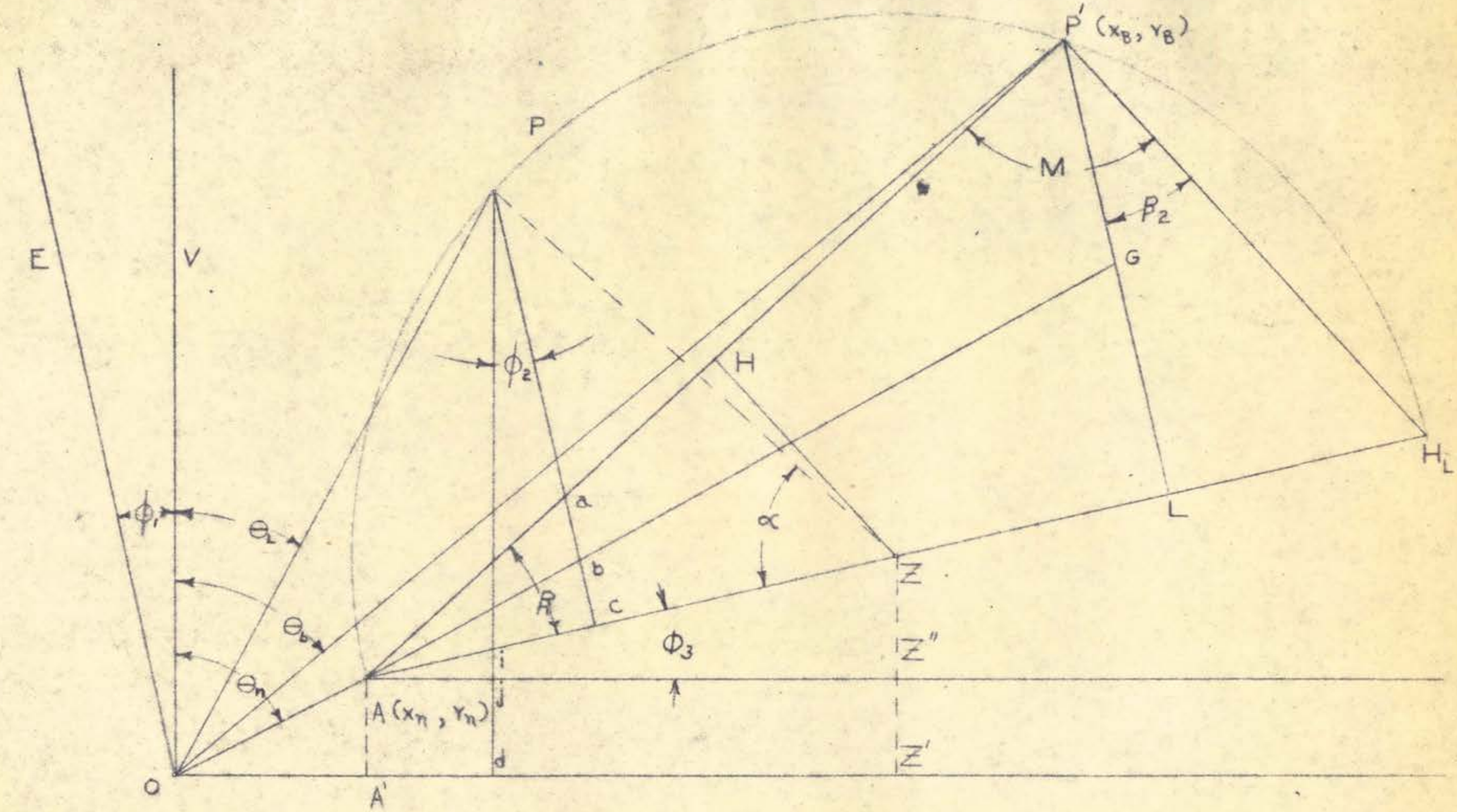

Figure 11. Complete tilted circle diagram, single-phase induction motor.

PART 2

Characteristics from Arbitrary Angles Alpha

The circle diagram of figure 11 is somewhat different from the conventional diagram. The characteristics will be determined in terms of the variable angle alpha instead of in terms of the input current and its power-factor. The conventional circle diagram of the three-phase motor is determined graphically from the points of blocked rotor and no-load currents and the horizontal diameter. The characteristics of the current, power-factor, watts output, watts input, slip, torque, and efficiency are found by measuring graphically short distances on certain parts of the circle diagram. Because of graphical construction errors this method has been found inaccurate. To get rid of these graphical errors the analytical solution of the circle diagram for the three-phase induction motor was developed by Professor C. F. Cameron.[1] Since the circle diagrams for three-phase and single-phase motors are quite similar, it was thought that the same type of analytical solution could be made for the single-phase motor.

Another factor which is of considerable importance in predicting the characteristic of the single-phase motor is to obtain the characteristics from the circle that has been tilted. The necessity for tilting the circle diagram has been discussed in Chapter I. The magnitude of the angle of tilt is calculated by use of the equation derived in Part 1 of Chapter II, and the characteristics from the tilted circle will be derived here.

In figure 11 it is seen that the power output and the power input lines do not coincide as they do in the simple three-phase circle diagram. This is

---

[1] C. F. Cameron, op. cit., Passim.

because of the angle to which the circle has been tilted. The power input is in phase with the input voltage while the power output is in phase with the voltage that is induced in the rotor at no-load. It also can be seen in figure 11 that the angle between the power input and power output is equal to the angle between the induced and the terminal voltage. To accomplish this difference between the input and output power is the main argument in favor of the tilting process.

The central angle alpha was seen to be a convenient method of attack so it will be used here as the variable, and all the characteristics will be found from it. To use the following development, the circle is first defined as in Part 1. The next step is to assume values of alpha such that a desirable range of values of outputs are found when the solution is carried through for several loads. The characteristics as found for these arbitrary values of alpha can then be plotted. The characteristics can then be read off of the curves in the usual fashion.

Referring to figure 11, P is any point on the circle. The input current is equal to OP at a power-factor angle θ_1. Now by trigonometric manipulations the following lines may be defined.

$$Pc = R \sin a,$$

$$Zc = R \cos a,$$

$$Pi = \frac{Pc}{\cos \phi}$$

and

$$ic = Pc \tan \phi$$

by subtraction

$$Ai = R - Zc - ic$$

and

$$ij = Ai \sin \phi$$

since

$$dj = Y_n$$

then

$$Pd = pi + ij + Y_n$$

and

$$A'd = Ai \cos \phi$$

$$Od = X_n + A'd$$

From figure 11 it can be seen that Pd is the inphase component of the current and OD is the reactive component of the current.

Then

$$\phi L = \tan^{-1} \frac{Od}{Pd} \text{ input power factor}$$

From this

$$OP = \frac{Pd}{\cos \theta_L} \text{ input current}$$

Now

$$LG = \frac{(I_b^2 - I_n^2)\, R_P}{V}$$

The length of LG places AG. AG is known as the torque line. From the known radius and the defined length, LG, the following may be written.

$$P'H_1 = 2R \sin\beta$$

$$P'L = P'H_1 \cos\beta$$

$$LH_1 = P'H_1 \sin\beta$$

by subtraction

$$AL = 2R - LH_1$$

and

$$Ac = R - Zc$$

By use of similar triangles it follows that

$$\frac{ac}{P'L} = \frac{Ac}{AL}$$

therefore

$$ac = \frac{P'L \times Ac}{AL}$$

By the same method, it is seen that

$$\frac{bc}{LG} = \frac{Ac}{AL}$$

Therefore

$$bc = \frac{Ac \times LG}{AL}$$

By subtraction the following lines may be defined

$$Pa = Pc - ac$$

$$Pb = Pc - bc$$

$$ab = ac - bc$$

The lines that are to be used to calculate the performance characteristics are now known and the characteristics may be defined as:

$$P\ \text{input} = Pd \times V$$

$$P\ \text{output} = Pa \times V$$

$$\text{torque} = \sqrt{Pa \times Pb} \times V$$

$$\text{efficiency} = \frac{P\ \text{output}}{P\ \text{input}}$$

$$RPM = \sqrt{\frac{ab}{Pb}} \times \text{Synchronous speed}$$

PART 3

Characteristics from Known Power Factor or Known Line Current

The limitations of finding the characteristics of the motor from the arbitrary angle alpha is easily recognized. The arbitrary selection of the angle allows the characteristics to be found, but until the solution is almost complete it is not known at what output or input the characteristics will be. Thus, if the output is to be found for full load, it is necessary to follow the solution through for several values of alpha and the results either interpolated or plotted to find the correct value. From this it was decided that the analytical solution would be considerably improved if the correct angle alpha could be found for a given input condition. The following derivation is such a method.

Refer to figure 12.

The equation of a circle in polar form is

$$P^2 - P_1^2 - 2PP_1 \cos(\theta_L - \theta_1) = R^2$$

Where R is the radius defined in Part 1

P_1 constant of the circle

θ_1 constant of the circle

P is length of the line from origin to a point on the circle and is equal to the input current

θ_L is the angle P makes with the horizontal axis and is $(90^\circ - \theta_1)$

To define the constants of the circle the following method is used

$ZZ'' = R \sin \phi$

$AZ'' = R \cos \phi$

from which

$ZZ' = ZZ'' + Z''Z' = ZZ'' + Y_n$

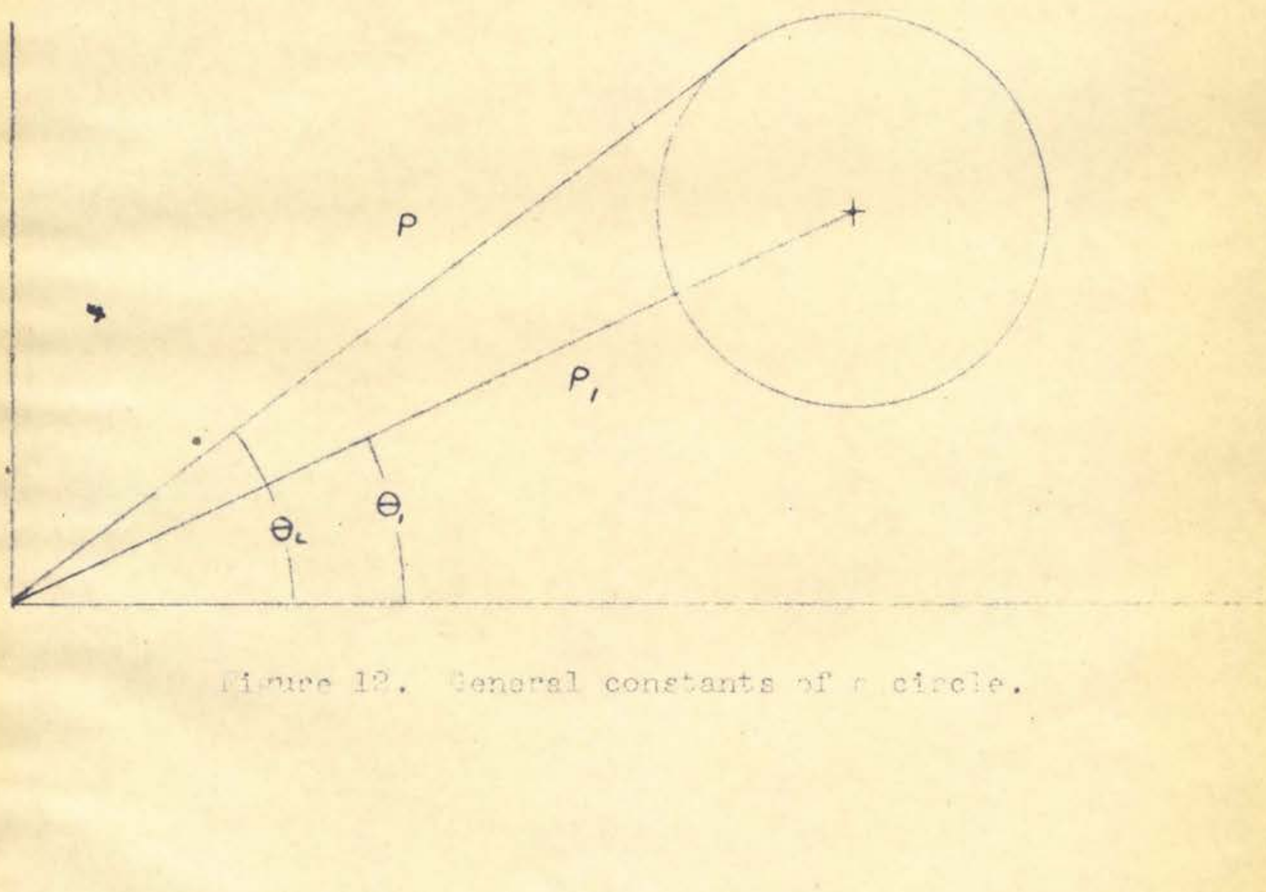

Figure 12. General constants of a circle.

$$OZ' = OA' + A'Z' = A'Z' + X_n = AZ'' + X_n$$

Then from figure 12 it can be seen that

$$\theta_1 = \tan^{-1} \frac{ZZ'}{OZ'}$$

By uses of the Pythagorean theorem

$$P_1 = \sqrt{(ZZ')^2 - (OZ')^2}$$

The constants of the circle are now known, and the above polar equation of a circle is an equation with two variable (the input current and its power-factor) one of which will be the assumed value.

The following lines of figure 11 can be defined by the use of trigonometry.

$$Pd = I \sin \theta$$

$$Od = I \cos \theta$$

$$A'd = Od - OA' = Od - X_n$$

ϕ was found in Part 1

$$ij = A'd \tan \phi$$

$$Pi = Pd - ij - jd = Pd - ij - Y_n$$

$$Pc = Pi \cos \phi$$

$$a = \sin^{-1} \frac{Pc}{R}$$

Substituting the circle constants P_1 and θ_1 into the given polar equation leaves two unknowns. From this it follows that if either of the two unknowns, namely the input current or its power-factor, are known the other can be found. Hence, if the motor is to be operated at a certain power-factor or at a certain current, the characteristic of the motor can be found for that power-factor or for that current in terms of the angle alpha. The method of finding the characteristics in terms of the angle alpha was developed in Part 2 of this chapter.

PART 4

Characteristics from Arbitrary Output Load

For ordinary purposes a method of finding the input of the motor in terms of the output load is the most desirable condition. The load that the motor will carry is usually the known. As developed thus far, the analytical solution would have to be used as discussed in Part 2 to find the characteristics in terms of some output load. This type of solution was found to be clumsy and rigorous if just a few values of load are to be solved. It follows that a solution would do away with the solving for a multitude of values which when just a few are needed would be very desirable. Such a method is given in this section. Though not complicated, this method is a powerful tool in solving for the characteristics of the single-phase motor. This is especially true due to the ease of the calculation.

The angle alpha, from which the motor characteristics can be found as developed in Part 2, is found in terms of the angle β, the radius R, and the length of the output load line Pa. The term Pa, is directly dependent on the output load as shown in the following. The constants R and β of the circle have been evaluated in Part 1.

Refer to figure 11

$$P \text{ output} = Pa \times V$$

$$Pa = \frac{P \text{ output}}{V} \quad \text{where P output is in watts}$$

$$\sin a = \frac{Pc}{R}$$

$$\cos a = \frac{Zc}{R} \quad \text{therefore } Zc = R \cos a$$

$$Pc = Pa + ac$$

$$\sin a = \frac{Pa + ac}{R}$$

$$ac = Ac \tan \beta$$

$$Ac = R - Zc = R - R \cos a$$

By substitution of Ac

$$ac = (R - R \cos a) \tan \beta$$

again substituting

$$\sin a = \frac{(R-R \cos a) \tan \beta}{R}$$

Expanding

$$\sin a = \frac{Pa}{R} + \frac{(R-R \cos a) \tan \beta}{R}$$

Or

$$\sin a = \frac{Pa}{R} + (1 - \cos a) \tan \beta$$

Also

$$\sin a = \frac{Pa}{R} + \tan \beta - \cos a \tan \beta$$

Now collecting terms

$$\sin a + \cos a \tan \beta = \frac{Pa}{R} + \tan \beta$$

Multiplying by $\cos \beta$

then $\sin a \cos \beta + \cos a \frac{\sin \beta}{\cos \beta} \times \cos \beta = \frac{Pa \cos \beta}{R} + \frac{\sin \beta}{\cos \beta} \times \cos \beta$

From which

$$\sin a \cos \beta + \cos a \sin \beta = \frac{Pa \cos \beta}{R} - \sin \beta$$

Then by application of the proper trigonometric identity the above equation may be written as

$$\sin (a + \beta) = \frac{Pa \cos \beta}{R} + \sin \beta$$

or

$$a + \beta = \sin^{-1} \left[\frac{Pa \cos \beta}{R} + \sin \beta \right]$$

then

$$a = \left(\sin^{-1} \left[\frac{Pa \cos\beta}{R} + \sin\beta \right] \right) - \beta$$

From this the angle alpha can be found for any output load. The completely defined circle is first found by use of Part 1. This gives the values of R and the angle β. The power output to be used is placed in the first of the solution and the length of the output load line will be found. With this information, the finding of alpha is a simple substitution problem. The known values β, Po and R are substituted into the above equations and alpha is evaluated.

The simplicity of this method also makes it useful for purposes of plotting the characteristics. This was discussed in Part 2. The angle alpha can be solved for the most desirable points such as 25%, 50%, 75%, full load, and 125% load values and the curves plotted from these points. This gives the more accurate data at the most frequently used points.

Example Solution

Motor Description

Split phase induction motor

Manufactured by Century Motor Company

Mod. Sp - 65L - CKK4 - 230 F   3.7 Amper   Single-phase

$\frac{1}{4}$ H. P. 1750 RPM 115 V 60 cycles

Data from Laboratory Tests

No Load Test

I_n = 2.78 ampers 60.6 watts 115 volts

Blocked Rotor Test (running winding only)

I_B = 15 ampers 1,285 watts 115 volts

Resistance of the primary running winding. R = 2.42

$$\cos \theta_n = \frac{P_n}{E \times I_n} = \frac{60.6}{115 \times 2.78} = .19$$

or

$$\theta_n = 79.05^o$$

$$\cos \theta_B = \frac{P_B}{E \times I_B} = \frac{1,285}{115 \times 14.7} = .745$$

or

$$\theta = 48.2^o$$

The primary resistance voltage drop at no-load is $I_nR_p = 2.78 \times 2.42$

$= 6.727$

then W by definition

$$W = I_nR_p \sin \theta_n = 6.727 \sin 79.05 = 6.6$$

and

$$g = I_nR_p \cos \theta_n = 6.727 \cos 79.05 = 1.27$$

from which

$$\tan \phi = \frac{W}{V - g} = \frac{6.6}{115 - 1.27} = .05805$$

and

$$\phi = 3.33^\circ$$

then

$$X_n = I_n \sin \theta_n = 2.78 \times \sin 79.1 = 2.73$$

$$Y_n = I_n \cos \theta_n = 2.78 \times \cos 79.1 = .528$$

$$Y_B = I_B \sin \theta_B = 15 \times \sin 48.2 = 11.16$$

$$Y_B = I_B \cos \theta_B = 15 \times \cos 48.2 = 10.03$$

Then substituting the values obtained above

$$\beta = (\tan^{-1} \frac{Y_B - Y_n}{X_B - X_n} - \phi = \tan^{-1} \frac{9.502}{8.43} - 3.33^\circ = 45.15^\circ$$

and

$$AH = \sqrt{\frac{Y_B - Y_n}{2}^2 + \frac{X_B - X_n}{2}^2} = \sqrt{(17.7) + (22.6)} = 6.35$$

Then the radius of the circle can be found

$$R = \frac{AH}{\cos \beta} = \frac{6.35}{\cos 45.15} = 9.004$$

This completes the defining of the circle. The performance characteristics at rated load will be found for the motor in this example. The characteristics are found in terms of the central angle alpha, therefore the next step is to find the value of alpha at rated load. The angle alpha will, for explanation purposes, be found from rated input current and from rated output load. Both methods should give the same results.

Alpha will first be found from the rated input current. The rated input current is found on the name plate of the motor and is 3.7 amperes for the motor of this example.

First the constants of the polar equation of the circle must be found, thus following the previously determined solution.

$$ZZ'' = R \sin \phi = 9.004 \times .0616 = .555$$

and

$$AZ'' = R \cos \phi = 9.004 \times .9981 = 8.986$$

Also

$$ZZ' = ZZ'' + Z''Z' = .555 + .528 = 1.083$$

and

$$OZ' = A'Z' + X_n = 8.986 + 2.73 = 11.716$$

Using the above determined values

$$\theta_1 = \tan^{-1} \frac{ZZ'}{OZ'} = \frac{1.083}{11.716} = 5.28^\circ$$

and

$$P_1 = \frac{ZZ'}{\sin \phi_1} = \frac{1.083}{\sin 5.28} = 11.765$$

By direct substitution into the polar equation of the circle the complementary angle of the power factor angle is found. The polar equation of the circle is:

$$P^2 + P_1^2 - 2PP_1 \cos (\theta - \theta_1) = R^2$$

$$13.839 + 138.37 - 87.5 \cos (\theta - \theta_1) = 81.063$$

$$- 86.9 \cos (\theta - \theta_1) = -71.143$$

$$\cos (\theta - \theta_1) = .821$$

$$\theta - \theta_1 = 35^\circ$$

$$\theta = 40.28^\circ$$

Therefore the power factor angle is equal to (90 - 40.28) which equals 49.72°

By definition

$$PD = I \sin \theta = 3.7 \sin 40.28 = 2.43$$

$$OD = I \cos \theta = 3.7 \cos 40.28 = 2.8125$$

and

$$A'D = OD - OA' = 2.8125 - 2.73 = .0831$$

$$ij = .0831 \tan \phi = .0048$$

hence

$$Pi = PD - ij - jd = 2.43 - .0048 - .528 = 1.9019$$

and

$$PC = Pi \cos \phi = 1.90 \times .998 = 1.8983$$

By definition

$$\sin a = \frac{1.898}{9.004} = .21$$

$$a = 12.1^{\circ}$$

This is the value of alpha for rated load found from the rated input current.

Alpha will now be found for the rated output load. The motor is rated one-quarter H. P.

Therefore

$$746 \times .25 = 186.5 \text{ watts}$$

and

$$P_a = \frac{\text{watts}}{\text{volts}} = \frac{186.5}{115} = 1.621$$

By substituting into the equation

$$a = (\sin^{-1} \frac{Pa \cos}{R} + \sin \beta) - \beta$$

Alpha is found

$$a = \sin^{-1} \frac{1.621 \times .705}{9.004} + .709 \ -45.15$$

$= (\sin^{-1} .836) - 45.15$

$= 11.75^{\circ}$

This is the value of alpha for rated load found from the rated output current. The difference between the value of alpha from input current and from output load is $.35^{\circ}$. This error is negligible and is an arithmetical error that is probably due to the use of the slide rule.

Using the value of alpha determined by either of the above methods the solution of the performance characteristics can be completed. Since there was a slight difference in the result of the two different solutions and it is not known which value of alpha is the more accurate, the value of alpha will be chosen at 12° in order to conform closely with the value obtained by either of the two above solutions.

Now, substituting into the equations of Part 2 of Chapter II of this paper, the solution of the characteristics will be completed.

$$Pc = R \sin a = 1.308$$

$$Zc = R \cos a = 3.549$$

$$Pi = \frac{Pc}{\cos \phi} = 1.81$$

$$ic = Pc \tan \phi = .105$$

$$Ai = R - Zc - ic = .036$$

$$ij = Ai \sin \phi = .005$$

$$PD = Pi + ij + Y_n = 2.340$$

then

$$\text{Power input} = PD \times V = 272 \text{ watts}$$

now

$$A'D = AI \cos \phi = .0359$$

$$OD = X_n + A'D = 2.816$$

$$\theta_L = \tan^{-1} \frac{GD}{PD} \quad 50.15^\circ$$

Power factor = .645

$$CP = \frac{PD}{\cos \theta_L} = 3.66$$

$$LG = \frac{(I_b^2 - I_n^2) R_P}{V} = 4.009$$

$$P'h = 2R \sin = 12.9$$

$$P'L = P'h \cos = 8.69$$

$$Lh = P'h \sin = 9.55$$

$$AL = 2R - Lh = 7.91$$

$$Ac = R - Zc = .181$$

$$ac = \frac{P'L \times Ac}{AL} = .1985$$

$$bc = \frac{Ac \times LG}{AL} = .0915$$

$$Pa = Pc - ac = 1.6095$$

$$Pb = Pc - bc = 1.7165$$

$$ab = ac - bc = .1070$$

$$\text{Speed in RPM} = \sqrt{\frac{Pa}{Pb}} \times 1800 = 1740$$

$$\text{Power output} = Pa \times V = 1.6095 \times 115 = 185$$

$$\text{Efficiency} = \frac{P\ out}{P\ in} \times 100 = \frac{185}{2.72} = 67.95\%$$

The solution of the rated load characteristics of the quarter horsepower motor is now complete. The performance characteristics found by the solution of the example compare very well with brake test values. The following is a comparison of calculated and brake test characteristics.

Load	1/4	1/2	3/4	4/4	5/4	Found From
Current	2.9	3.12	3.32	3.66	4.17	Calculation
	2.9	3.1	3.3	3.7	4.2	Brake Test
Pin Watts	110	172	214.5	272	344	Calculation
	115	165	215	275	340	Brake Test
Efficiency %	45.1	54.1	64.1	67.95	69	Calculation
	42.5	56	64	66.9	67	Brake Test
P. F.	.33	.48	.562	.645	.716	Calculation
	.34	.47	.57	.65	.72	Brake Test
Speed RPM	1770	1769	1760	1740	1725	Calculation
	1786	1775	1760	1720	1730	Brake Test

The above comparison shows that the analytical solution is not perfect but the error between the brake test and calculated results is negligible.

CONCLUSIONS

The method presented here for the prediction of induction motor characteristics from laboratory data is essentially simple, yet the results are better than any method devised to this date. It is far better than the graphical solution because the results are more accurate and the extensive use of drawing instruments is not required.

An added advantage is in the fact that the characteristic of a single-phase induction motor may be determined directly for any particular value of load or input current. That is, it is not necessary to assume several values for the angle alpha and follow the entire solution through for each angle. Such a process usually requires several attempts in order to determine the characteristics at some particular value of load.

The accuracy of this method depends upon the primary reactance that is neglected, or the accuracy depends upon whether or not the primary reactance is of such a value that it can be neglected. If the primary reactance of the motor is such that it can not be neglected, this will probably be a special type motor, the tilt of the circle will be in error. This tilt error will cause an error in the calculated performance characteristics of the motor. As was stated, for all the motors that were available for testing, the method of tilt proved to be very accurate with the largest error being in the region of three to five percent. This error of tilt is not directly proportional to the error that it causes in the calculated performance characteristics but is considerably larger. This is because the inphase power component is a sine function of the angle, and the angle is small in magnitude. From this it follows that the method developed in this paper is of most value for motors that have relatively large angle of tilt, or large phase shifts of the induced voltage at no-load.

If the circle diagram is to be plotted from design data, greater accuracy would be reached by using one of the other methods that have been developed; but if time or labor is of consequence, the method developed in this paper would be satisfactory. This applies to solutions from design data only.

The analytical solution is definitely an improvement over the conventional graphical solution of a circle diagram. The analytical solution as developed here can be applied to any circle diagram that is of the form of the induction motor circle. For the usual induction motor circle the angle of tilt would be zero, and the rest of the solution could be applied directly. The analytical solution that is developed here is a better solution than the previously developed analytical solutions, because it does not require several assumptions of the angle alpha and then a solution for each assumption plus interpolation to find the characteristics that are desired.

The solution developed in this paper is useful for any type of induction run single-phase motor. This is brought out since there are many types of starting devices in use. The type of starting circuit does not affect the running characteristics since the starting circuit is switched out of the circuit when the proper speed is reached. This is usually done by some sort of centrifugal device.

The non-tilt method of calculating the performance of a single-phase induction motor is the only solution that has been developed to date which can be compared to the solution of the characteristics of the single-phase induction motor that is presented in this paper. This is true since the non-tilt solution is the only solution that can be directly applied from the usual laboratory test data. The analytical solution from the tilted circle, as presented here, is a definite improvement when compared to the non-tilted circle. There are two reasons for this improvement; the method of tilting the circle increases the accuracy of the circle and the analytical solution is a mathematical solution that does away with the use of inaccurate graphical methods.

If the method presented by Mr. Veinott[1] or the method presented by Mr. Tarboux[2] were to be applied as solutions of the characteristics of single-phase induction motor calculated from laboratory data, then the analytical solution as developed in this paper would be a better solution. The solutions that were developed by Mr. Veinott and Mr. Tarboux require that certain motor constants be known that are extremely difficult to evaluate by laboratory procedure. As previously stated, this thesis presents a solution of the single-phase induction motor performance characteristics that do not require the evaluation of any motor circuit constant other than the primary resistance. The measurement of the primary resistance is a criterion for the complete solution of any induction motor circle diagram and is readily determined from laboratory tests. The comparison of the solution in this paper to that presented by Mr. Veinott or Mr. Tarboux, however, is not a fair comparison since it is thought that the method of solutions that were presented by those men were methods that were to be applied when design data was to be used.

Therefore, it may be concluded that the performance characteristics

---

[1] C. G. Veinott, op. cit., p. 745.

[2] J. G. Tarboux, op. cit., p. 335.

for the single-phase induction motor may be calculated by the method developed in this paper from the usual induction motor tests. The tests are the no-load test, the blocked rotor test, and the measurement of the primary resistance. The characteristics of the motor can be calculated without the use of any type of graph or curve sheet.

BIBLIOGRAPHY

Bretch, E. "Amortisseur Action of the Squirrel Cage." Electrical Engineering, (June, 1946).

Bretch, E. "Letters to the Editor." Electrical Engineering, (January, 1947).

Bryant, J. M., and Johnson, E. W. Alternating-Current Machinery. New York: McGraw-Hill Book Company, 1935.

Cameron, C. F., and Webking, H. "Induction Motor Characteristics." Oklahoma Engineering Experiment Station Publication, No. 63 (May, 1948).

Karapetoff, V. Experimental Electrical Engineering, Vol. II. New York: John Wiley and Sons, 1927.

Langsdorf, A. S. Theory of Alternating-Current Machinery. New York: McGraw-Hill Book Company, 1937.

Lawrence, R. R. Principles of Alternating-Current Machinery. New York: McGraw-Hill Book Company, 1940.

Mc Allister, A. S. Alternating-Current Motors. New York: McGraw-Hill Book Company, 1909.

Puchstein, A. F., and Lloyd, T. C. Alternating-Current Machines. New York: John Wiley and Sons, 1942.

Ricker, C. W., and Tucker, C. E. Electrical Engineering Laboratory Experiments. New York: McGraw-Hill Book Company, 1940.

Tarboux, J. G. Alternating-Current Machinery. Scranton, Pa.: International Textbook Company, 1947.

Veinott, C. G. "Performance Calculation on Induction Motors." Transactions of American Institute of Electrical Engineers, (September, 1932).

Typist: Grace Peebles