THE FOUR-TERMINAL NETWORK METHOD OF PREDICTING SINGLE-PHASE INDUCTION MOTOR PERFORMANCE CHARACTERISTICS FROM TEST DATA

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PREFACE

In this country there are primarily two types of transmission systems, the three phase system, and the single phase system. One third more copper is required for single-phase transmission as compared to three-phase transmission for a given amount of power to be transmitted a fixed distance with the same line loss. Or stated in different terms: for a given line-to-line voltage and a given amount of copper, the three-phase, three-wire system is more efficient than the single-phase system (or any other type of transmission system). Industrial requirements call for three-phase power almost exclusively. Single-phase motors displays less satisfactory characteristics than three-phase motors, but, nevertheless, single-phase motors have become useful in a wide field of applications. Part of this demand for single-phase motors is because of the rapid growth of single-phase supply to large rural areas which formerly had no electric service of any kind. Part of the demand may also be attributed to builders of post-war homes, who are demanding an ever increasing application of electric motor power to new and better home appliances. This growing demand has established the importance of single-phase motors despite its handicaps.

The Department of Commerce Census of Manufactures for 1939 reveals that sales for the year in question were as follows:

Single-phase motors, 1/20 hp and over, 5,870,722 units, valued at $44,166,285.

Polyphase induction motors, 1 hp and over, 366,581 units, valued at $33,688,725.

These figures do not include synchronous motors, nor railway and vehicle motors. In 1939, a pre-war year, single-phase motor sales composed over 50% of the total sales of alternating-current motors.

Single-phase motor theory is much more difficult than that of the three-phase motor, and lacks the beauty of symmetry enjoyed in three-phase motors. Because of its increasing importance and the lack of any unified theory, the single-phase motor has been approached from practically every angle in the field of predicting performance characteristics.

In any method of predicting single-phase motor performance characteristics, it is of prime importance that an equivalent circuit be drawn, which will give an electric representation of the single-phase motor under all load conditions. The choice of an equivalent circuit, and its associated solution, is the basis from which all prediction characteristics must be taken.

Since the single-phase motor may be represented by an electric network, it is then a rather simple matter to analyze it as a four-terminal network and solve for the general network constants A, B, C, and D. Transmission lines have long been considered as four-terminal networks and analyzed in terms of the general constants: however, the application of the four-terminal network method to electric motors has been made only recently. It is a method which attacks the problem of motor analysis by an entirely different approach and is an improvement over many of the so-called "conventional methods."

Mr. Tarboux gives an excellent treatment of the four-terminal network method as it applies to induction motors in general, but he points out the fact that there is considerable work to be done, especially concerning circle diagrams for single-phase motors.[1] Since his work is so generalized, and is developed from the standpoint of design data, there is room for improvement in the development of more specific single-phase motor cases. It is the purpose of this thesis to improve on the four-terminal network method of predicting single-phase motor performance characteristics, especially from the standpoint of using the data from blocked-rotor and no-load tests rather than from design data.

It is the hope of the writer that the material presented in this thesis may be of some value to those interested in predicting performance characteristics of single-phase induction motors, and that it may prove beneficial to others interested in research along these lines.

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[1] J. G. Tarboux, Alternating-Current Machinery.

ACKNOWLEDGEMENT

The writer wishes to express his sincere appreciation to Professor C. F. Cameron for his detailed reading of this material and for his helpful criticisms and suggestions concerning the subject matter.

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CHAPTER I

In the prediction of single-phase motor characteristics it is of prime importance to set up an equivalent circuit or network to represent the actual motor as nearly as possible. Since all performance characteristics are based on the choice of an equivalent circuit and the solution of that circuit it is important to review a few of the basic network theorems.

THEVENIN'S THEOREM

Thevenin's theorem is primarily used when the network reactions are to be analyzed at a particular pair of terminals and thus is frequently referred to as a "two-terminal network". The theorem can be stated in several different ways of which the following is typical:

> At any particular frequency any linear network of bilateral elements viewed from any two terminals of the network can be replaced by a generated voltage E_o and an impedance Z_i in series, where E_o is the open circuit voltage measured across the terminals in question and Z_i is the impedance of the network viewed from these same terminals with all generators replaced by their internal impedances.[1]

RECIPROCITY THEOREM

The reciprocity theorem may be stated as follows: In a network composed of linear bilateral circuit elements, if any source of emf, E, located in the i th mesh, produces a current, I, in the k th mesh, the same source of emf, E, if placed in the k th mesh will produce the same current, I, in the i th mesh.

The ratio of the emf, E, in the i th mesh to the current, I,

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[1] Myril b. Reed, Alternating-Current Circuit Theory, p. 358.

in the k th mesh is called the transfer impedance, Z_{ik}. Thus, if the transfer impedance Z_{ik} is equal to the transfer impedance Z_{ki}, the network is said to be fully reciprocal. This also implies that the circuit is absolutely linear and bilateral.

FOUR-TERMINAL NETWORKS

In general, a four-terminal network may be defined as any electric circuit composed of resistances, inductances, and capacitances where there is involved only one pair of energy input terminals and only one pair of energy output terminals. Any such network connecting two pairs of terminals can have its electrical characteristics expressed in terms of four network constants, A, B, C, and D.

It has been shown by various ways[2] that the voltages and currents in such a network, where power flows from the sending end to the receiving end, can be expressed in terms of the network constants by the equations

$$V_s = AV_r + BI_r \qquad (1)$$

$$I_s = CV_r + DI_r \qquad (2)$$

where

V_s is the voltage at the sending end

I_s is the current at the sending end

V_r is the voltage at the receiving end

I_r is the current at the receiving end.

Only three of the four network constants are independent however, since by the reciprocity theorem, the relation AD-BC=1

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[2] J. G. Tarboux, Alternating-Current Machinery, p.1-2.

exists for all reversible bilateral networks. Mr. Tarboux gives a proof of this relationship in his book on alternating-current machinery.[3]

In a network where current and voltage measurements can be obtained at both the sending and receiving ends it is a relatively simple matter to obtain the network constants. From equation (1) the value of the constant A is obtained by opening the circuit at the receiving end, i.e., when $I_r = 0$. Under this condition equation (1) gives

$$A = \frac{V_1}{V_2} \quad ^{4}$$

and equation (2) gives

$$C = \frac{I_1}{V_2}$$

The constant B can also be obtained from equation (1) if the receiving end is short circuited, i.e., when $V_2 = 0$. This gives

$$B = \frac{V_1}{I_2}$$

and equation (2) gives

$$D = \frac{I_1}{I_2}$$

From the above solutions for the network constants in terms of the currents and voltages, the network constants obviously have the following significance.

"A is the voltage impressed at the sending end per volt at the open-circuited receiver. It is a dimensionless voltage ratio.

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[3]Ibid, p.6.

[4]The subscripts "s" and "r" used in equations (1) and (2) will now be changed to 1 and 2 respectively and will be used throughout the rest of this paper.

"B is the voltage impressed at the sending end per ampere in the short-circuited receiver. It is the transfer impedance used in network theory. It is also equal to the voltage impressed at the receiving end per ampere in the short-circuited sending terminals.

"C is the current in amperes into the sending end per volt on the open-circuited receiver. It has the dimensions of admittance.

"D is the current in amperes into the sending end per ampere in the short-circuited receiver. It is a dimensionless current ratio."[5]

Although the constants A and D are dimensionless ratios, they should not be considered as pure numbers. Since all voltages and currents are complex quantities, all of the constants will also be of a complex nature.

It is obvious from the above discussions that certain modifications and assumptions will have to be made before the four-terminal network analysis can be made to apply to single-phase motors. Despite the fact that the output of a motor is mechanical energy instead of electric energy, motors still qualify as having only one point of energy input and only one point of energy output. The single-phase induction motor may be represented electrically by an equivalent circuit in which a variable resistance represents varying load conditions. In such an equivalent circuit the receiving end represents the rotor circuit. Since neither current

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[5]Westinghouse Electric and Manufacturing Company (Central Station Engineers), Electrical Transmission and Distribution Reference Book, p. 101.

nor voltage measurements can be made on the rotor circuit, the network constants must be evaluated from the measurable values of input current and voltage.

Due to the presence of iron cores in most motors, the difficulty of non-linearity is encountered because of the saturation effects of the iron. However, if special care is used in applying the general equations for the determination of the network constants, this factor will be eliminated. Fortunately electric motors generally operate at practically constant voltage, the range of any voltage variation being, usually, rather small, so that the constants may be obtained for average operating conditions. In setting up an equivalent circuit certain assumptions and modifications necessarily have to be made and these usually are of such a nature that the effects of non-linearity are negligible.

In the ordinary four-terminal network the looking-in and looking-back impedances may be found by the use of a bridge circuit or by the ordinary voltmeter-ammeter method and the network constants evaluated from the resulting readings. However, these methods obviously are eliminated in motors by the fact that the receiving-end circuit (the rotor) is inaccessible. Consequently it is necessary to use the voltmeter, the ammeter and the wattmeter in the input circuit only. Measurements, whenever possible, should be made at the normal operating voltage of the motor and not at the rated current. Short-circuit tests at rated voltage will approximate more nearly the normal operating conditions than tests taken with rated current. Mr. Tarboux discusses this point in the following manner:

> It is incorrect to make short-circuit tests at only rated current, because at rated current there is no assurance that the magnetic circuits are operating at rated flux densities. In fact, there is no absolute assurance that such is the case at rated applied voltage. For special accuracy, the induced emf's of the circuit should be computed and a short-circuit test should be made at the rated induced emf, since induced emf's are generally proportional to flux density and thus serve as a measure of the degree of magnetic saturation.[6]

For small single-phase motors, the difference between the induced emf and the rated applied voltage will be small, especially in the difference in magnitude of the induced emf and the rated voltage. Blocked-rotor tests at rated voltage will not produce excessive and damaging currents for single-phase motors, and thus may be safely used.

SOLUTION OF THE CIRCLE DIAGRAM

In order to determine the necessary information for the construction of a circle diagram for any motor, an equivalent circuit must be set up which will approximate the motor behavior. In the four-terminal network method of analysis it is very important to make the proper choice of an equivalent circuit, since the network constants are obtained from circuit parameters. If design data is to be used to obtain the circuit parameters, the exact equivalent circuit may be used. Figure 1 shows the exact equivalent circuit of the single-phase induction motor. However, if test data is to be used, the exact equivalent circuit will have to be modified and an approximate equivalent circuit used, such as the equivalent circuit shownin Figure 2. This may be done so that the

[6]Tarboux, *op. cit.*, p. 12.

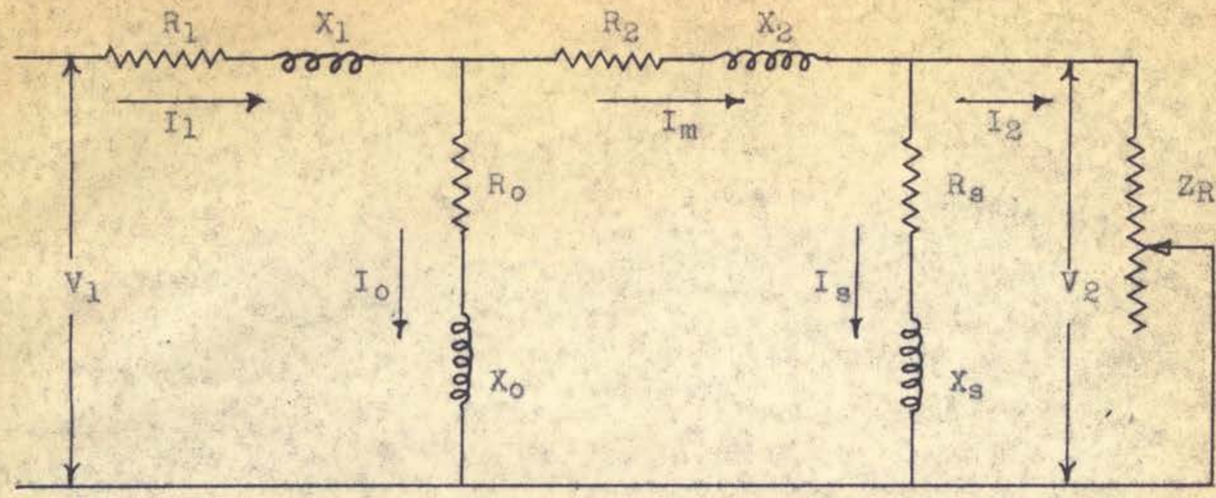

Figure 1. The exact equivalent circuit for the single-phase induction motor.

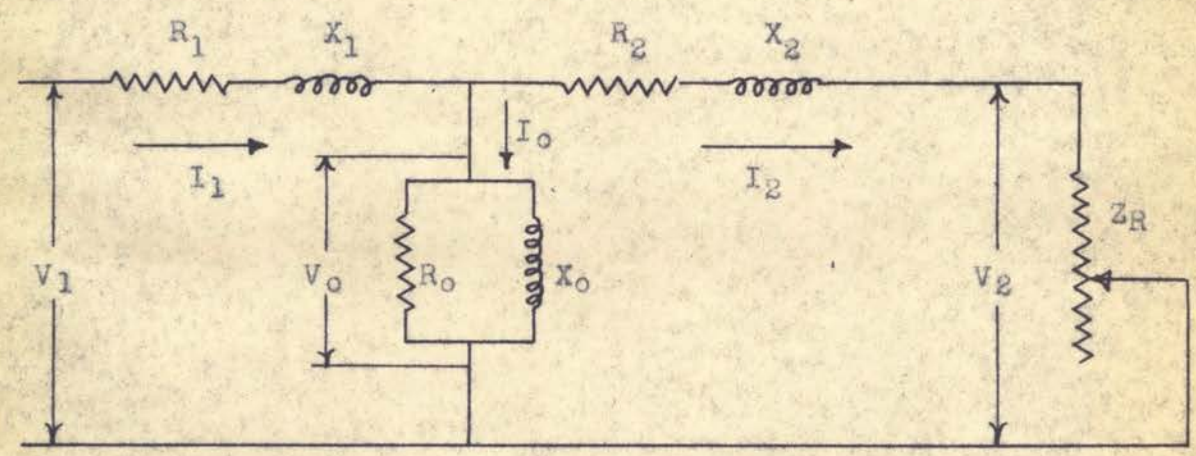

Figure 2. The approximate equivalent circuit for the single-phase induction motor.

the error involved will be small enough to give results of a desired accuracy, even when the performance characteristics are to be predicted. Whenever the exact equivalent circuit is modified to the extent that all shunt windings (Z_0 and Z_s in Figure 1) are moved so as to be directly across the input voltage terminals, and the solution by the use of the network constants will be little improvement over conventional solutions. When more complicated circuits (such as Figure 1 or Figure 2) are to be solved, the four-terminal network method gives a straightforward and much shorter way of obtaining the equation for the input current and, subsequently, the circle diagram.

Tarboux, in his treatment of the equivalent network for the single-phase induction motor, determines the network constants in terms of design data rather than the data obtained from the ordinary blocked-rotor and no-load tests.[7] Since there has been no appreciable work done along these lines by the four-terminal network method, it is the purpose of this thesis to attack the problem of the four-terminal network solution of single-phase induction motors from the standpoint of laboratory tests which may be performed with a minimum of equipment and time. A circle diagram will then be constructed in terms of the network constants, which can be obtained from the test data, and from this circle diagram a calculation sheet will be set up so that the complete performance of a single-phase induction motor may be obtained with a minimum expenditure of time and effort.

The equations expressing the relationship of the voltages and

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[7]Tarboux, <u>op. cit.</u>

currents in four-terminal networks were previously given as

$$V_1 = AV_2 + BI_2 \qquad (1)$$

$$I_1 = CV_2 + DI_2 \qquad (2)$$

From these basic equations an equation of the input (stator) current can be obtained in terms of the general constants, the applied voltage, and the load impedance. In all of the equations used in this derivation the quantities are understood to be complex.

From equation (1)

$$V_2 = \frac{V_1 - BI_2}{A} \qquad (3)$$

From Figure 2 $$V_2 = I_2Z_R \qquad (4)$$

Giving $$Z_RI_2 = \frac{V_1 - BI_2}{A} \qquad (5)$$

Or $$I_2 = \frac{V_1}{AZ_R + B} \qquad (6)$$

Substituting this value of I_2 into equation (4) gives

$$V_2 = \frac{V_1Z_R}{AZ_R + B} \qquad (7)$$

Using equations (6) and (7) in equation (2) gives

$$I_1 = C\,\frac{(V_1Z_R)}{(AZ_R + B)} + D\,\frac{(V_1)}{(AZ_R + B)}$$

$$I_1 = \frac{V_1(CZ_R + D)}{AZ_R + B} \qquad (8)$$

If both the numerator and the denominator of the right hand member are multiplied by A, equation (8) becomes

$$I_1 = \frac{V_1(ACZ_R + AD)}{A(AZ_R + B)}$$

Adding and subtracting BCV_1 in the numerator of the right hand side, and rearranging

$$I_1 = \frac{V_1C}{A} + \frac{V_1(AD - BC)}{A(AZ_R + B)} \quad (9)$$

From the reciprocity theorem the relationship AD - BC = 1 holds true if the circuit is bilateral. The application of this relationship to single-phase motors has been discussed previously. Using this relationship in equation (9) gives the input current as

$$I_1 = V_1\frac{C}{A} + \frac{V_1}{A(AZ_R + B)}$$

$$= V_1\frac{C}{A} + \frac{\frac{V_1}{A^2}}{ZR + \frac{B}{A}} \quad (10)$$

Another method of approach frequently used to obtain the same results will be presented so that it may be compared with the writer's method given above.[8-9]

The input admittance, Y_1, of a four-terminal network is defined by $Y_1 = I_1/V_1$. By substitution from equations (1) and (2) the input admittance may be written

$$Y_1 = \frac{CV_2 + DI_2}{AV_2 + BI_2} \quad (11)$$

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[8]Tarboux, Op. cit.

[9]Electrical Engineering Staff of Massachusetts Institute of Technology, Electric Circuits, pp. 492-3.

If both the numerator and denominator of the right hand side are divided by I_2, equation (11) becomes

$$Y_1 = \frac{C\frac{V_2}{I_2} + D}{A\frac{V_2}{I_2} + B}$$

But

$$\frac{V_2}{I_2} = Z_R$$

Then

$$Y_1 = \frac{CZ_R + D}{AZ_R + B} \tag{12}$$

Carrying out the division indicated in equation (12) gives

$$Y_1 = \frac{C}{A} + \frac{D - \frac{BC}{A}}{AZ_R + B} = \frac{C}{A} + \frac{\frac{AD - BC}{A}}{AZ_R + B}$$

But since AD - BC = 1, then

$$Y_1 = \frac{C}{A} + \frac{\frac{1}{A^2}}{Z_R + \frac{B}{A}} \tag{13}$$

The input current can then be obtained by

$$I_1 = V_1 Y_1$$

And substituting equation (13) for Y_1

$$I_1 = \frac{V_1 C}{A} + \frac{\frac{V_1}{A^2}}{Z_R + \frac{B}{A}}$$

which is identical to equation (10). This is probably a more elegant development, but not any more rigorous than the first method.

It might be well to mention again that the network constants A, B, C, and D are all complex quantities and must be used as such in equation (10).

Equation (10) gives the input current, in terms of the network constants, as

$$I_1 = \frac{V_1 C}{A} + \frac{\frac{V_1}{A_2}}{Z_R + \frac{B}{A}}$$

Let

$$\left.\begin{array}{lll} K' = \overline{C}/\overline{A} & \text{(an admittance)} & \text{(a)} \\ K'' = 1/\overline{A}_2 & \text{(a complex ratio)} & \text{(b)} \\ K = K_1 + jK_2 = \overline{B}/\overline{A} & \text{(an impedance)} & \text{(c)} \end{array}\right\} \text{-(14)}$$

Then equation (10) becomes

$$I_1 = K'V_1 + \frac{K''V_1}{Z_R + K} = K'V_1 + \frac{K''V_1}{Z_R + (K_1 + jK_2)}$$

Since $K_1 + jK_2$ gives an equivalent impedance (evaluated from B/A), then K_1 must be an equivalent resistance and K_2 an equivalent reactance.

The load impedance, Z_R, is considered to be a varying resistance whose value depends upon the speed of the rotor. The grouping of the resistive components and the reactive components into two separate parts in the denominator gives

$$I_1 = K'V_1 + \frac{K''V_1}{(Z_R + K_1) + jK_2} \qquad (15)$$

From equation (15) it can be seen that the input (stator) current is composed of a constant component, $K'V_1$, and a variable component,

$$\frac{K'V_1}{(Z_R + K_1) + jK_2}$$

If a vector quantity, $\overline{R}$, is given as $\overline{R} = \frac{\overline{K}}{M + jN}$, where $\overline{K}$ is a vector constant, and either M or N is varied while the other (N or M) remains constant, then the locus of the vector quantity, $\overline{R}$, is a circle.

The value of K_2, the equivalent reactance, is considered to be constant as are K" and V_1.

Thus if only the variable portion of equation (15) is considered, its locus is a circle whose center is not at the origin. The analytical equation of a circle whose center is not at the origin is given by the equation

$$r^2 = (x - h)^2 + (y - k)^2$$

where

r is the radius of the circle
x and y are the coordinates of any point on the circle
h and k are the coordinates of the center of the circle

Consider the variable portion of equation (15)

$$I' = \frac{\overline{K}''\overline{V}_1}{(Z_R + K_1) + jK_2} \qquad (15\text{-a})$$

Where I' is a vector quantity which represents the rotor current.

If $Z_R = \infty$, which can be obtained when the rotor is rotating at synchronous speed (the only way this can be accomplished in an induction motor is by driving the rotor with another machine), equation (15-a) reduces to zero. If $(Z_R + K_1)$ is equal to zero, equation (15-a) becomes

$$I' = \frac{\overline{K}''\overline{V}_1}{jK_2}$$

These two conditions represent the minimum and maximum values of I' respectively. Since the locus of the vector I' is a circle, the diameter of the circle will be

$$\text{diameter} = \frac{\overline{K}''\overline{V}_1}{jK_2}$$

If the voltage reference is chosen as jV_1, then the diameter will become $\frac{V_1}{K_2}(\overline{K}'')$ and the radius will be $\frac{V_1}{2K_2}(\overline{K}'')$. However, the vector $\overline{K}''$ is equal to $1/A^2$ from equation (14-b).

Let $$1/A^2 = 1/A^2\underline{/\phi}$$

The coordinates of the center of the circle (h,k) then can be evaluated as

$$h = \frac{V_1}{2K_2A^2}\text{Cos }\phi$$

$$k = \frac{V_1}{2K_2A^2}\text{Sin }\phi$$

The circle which represents the locus of the rotor current, I', is shown in Figure 3. The diameter of the circle is seen to be tilted by the angle, ϕ, which is easily determined from the network constant, A. There is considerable controversy among different writers about whether or not the circle diagram should be tiled, and, if so, whether it should be tilted upwards or downward. The ease with which this problem is handled by the four-terminal network method adds to the effectiveness and adaptability of this method as compared to other methods of analyses.

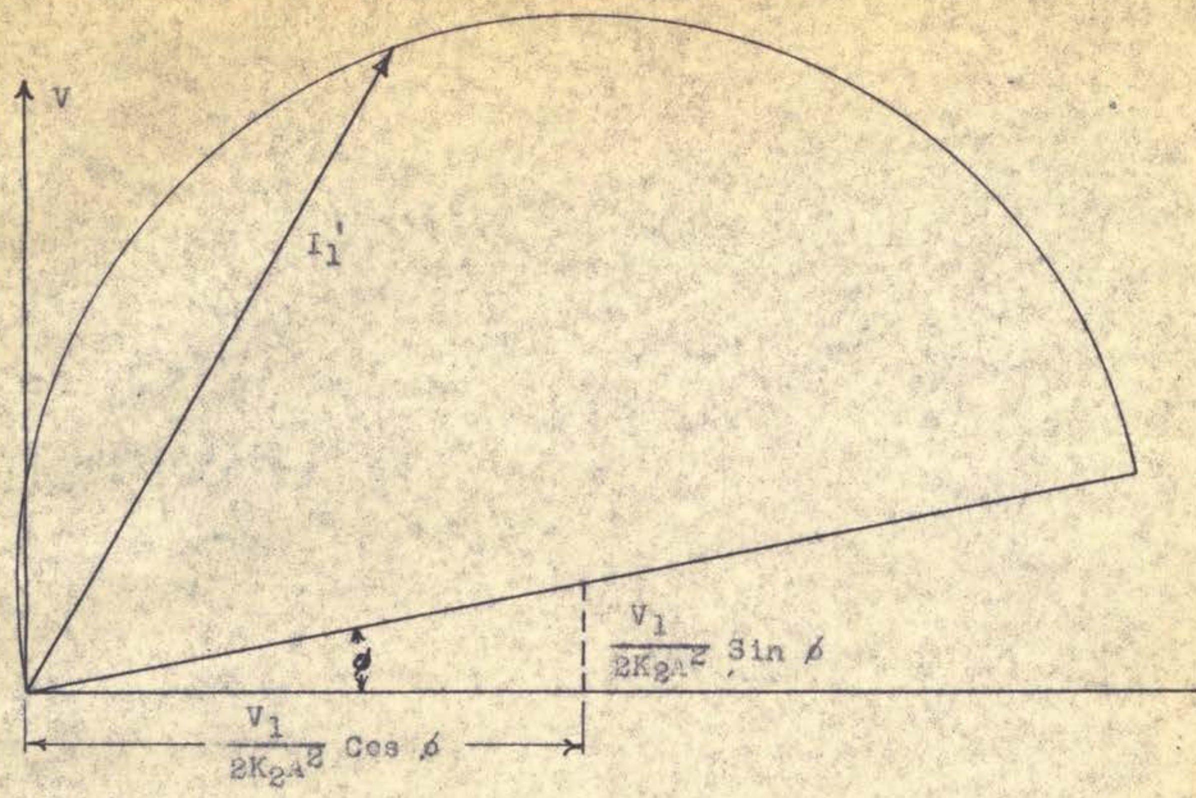

Figure 3. The circle diagram representing the locus of the rotor current of a single-phase induction motor.

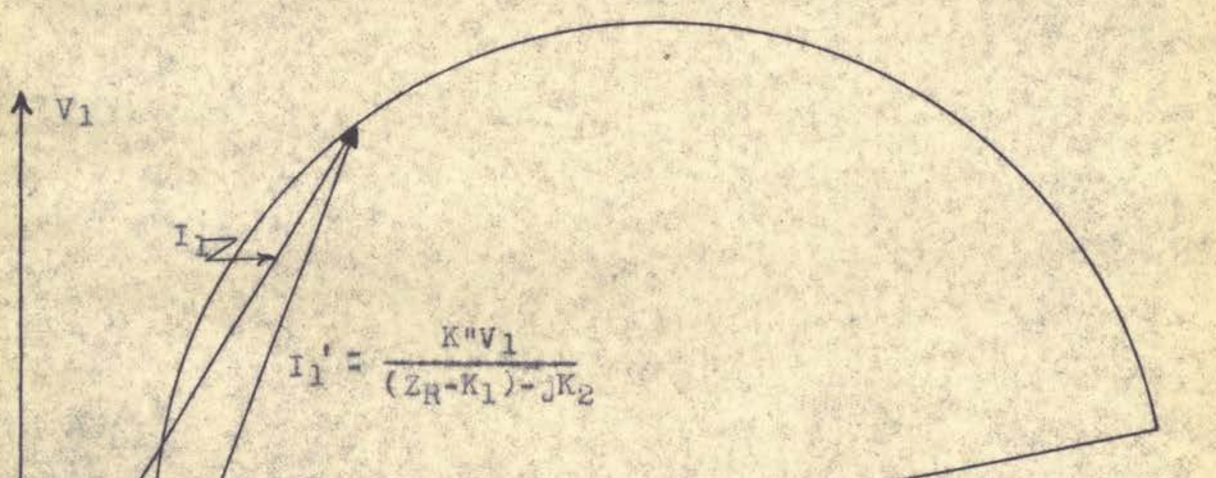

Figure 4. The circle diagram representing the locus of the stator current of a single-phase induction motor.

If the constant vector $K'V_1$ is added to equation (15-a), the center of the circle will be displaced by an amount equal to the magnitude of $K'V_1$ from the original coordinates (h,k) and in the direction of the vector $K'V_1$. The same results can be obtained by moving the original axes in the direction of $-K'V_1$.

The circle which represents the locus of the input current as obtained from equation (15) is shown in Figure 4.

FOUR-TERMINAL NETWORK METHOD - DESIGN DATA

Since the four-terminal network method is just as easily applied to complicated networks as to the more simple networks, it thus is an ideal method of attacking the exact equivalent circuit of the single-phase motor when the design data is known or can be obtained. This approach to the single-phase motor is adequately covered in Mr. Tarboux's book, Alternating-Current Machinery. In order to make a valid comparison of the four-terminal network method put forth in this thesis with the work done by Mr. Tarboux, his work on the single-phase induction motor will be given in brief form.[10]

The equivalent circuit obtained from the theory portrayed by the double-revolving-field theory is given in Figure 5.[11] From this equivalent circuit the parameters are given as

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[10]Tarboux, op. cit., pp. 316-336.

[11]The same equivalent circuit is obtained by the cross-field theory bearing out the conclusions of V. Karapetoff, Jr., in his paper "On the Equivalence of the Two Theories of the Single-phase Motor" given in the A.I.E.E., August, 1921, p 640.

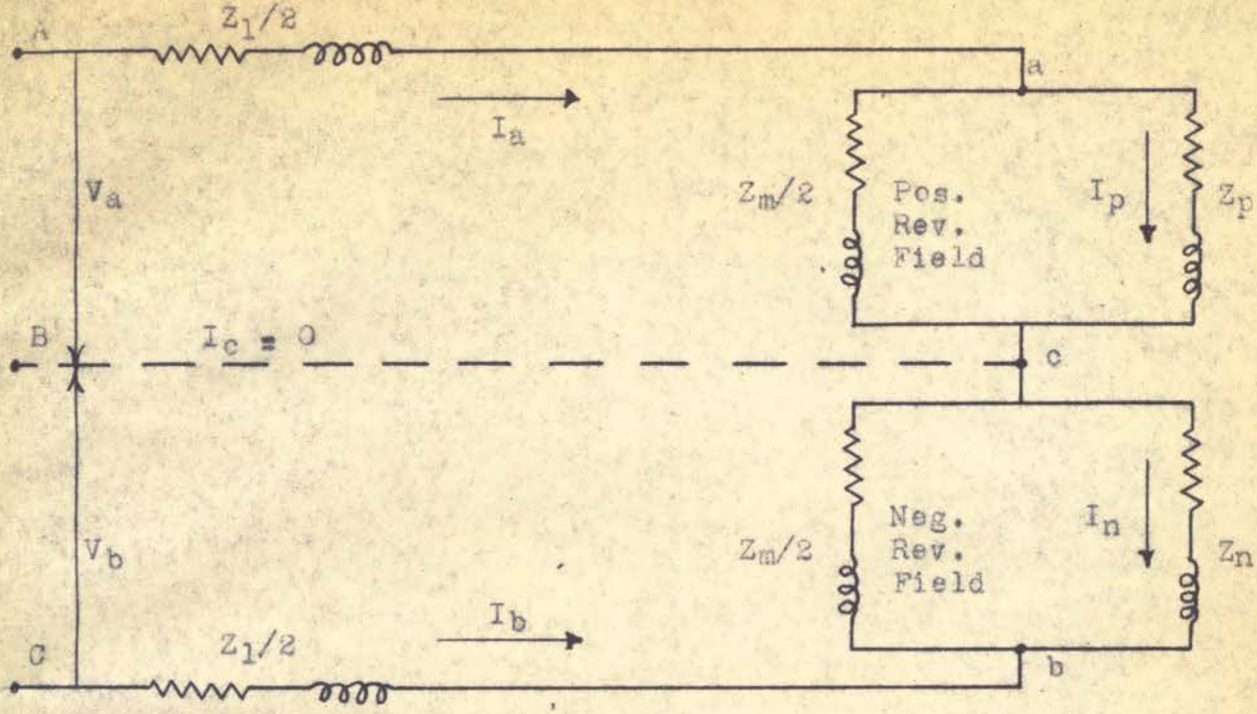

Figure 5. Equivalent-rotating-field circuit diagram of a single-phase motor.

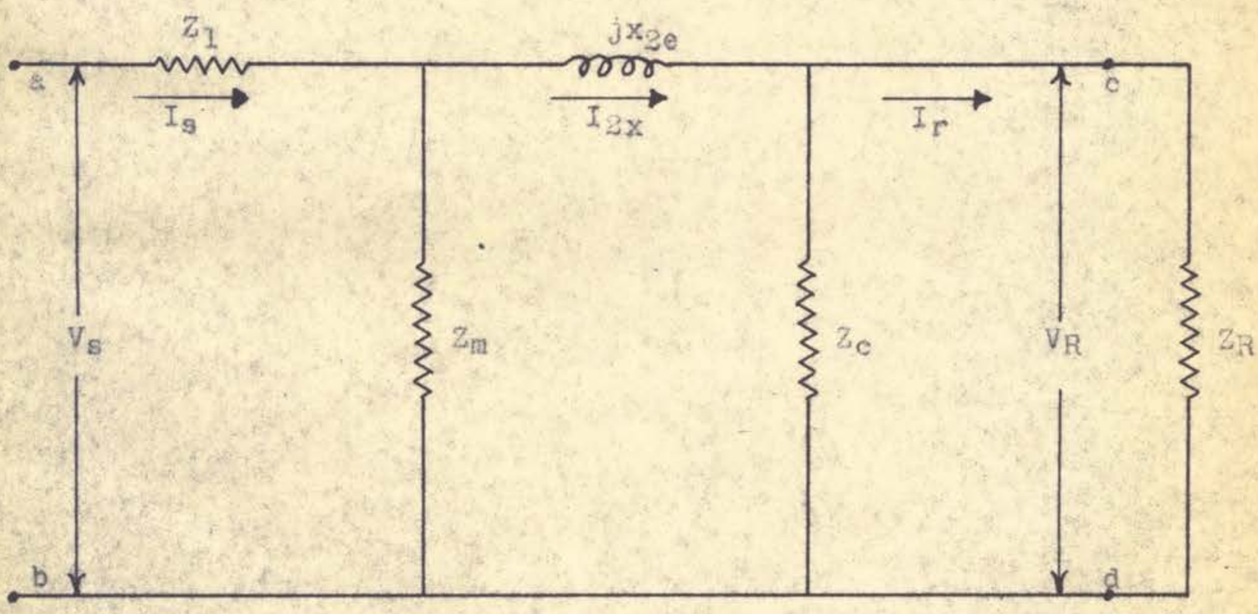

Figure 6. An equivalent circuit for a single-phase motor.

$$Z_p = \frac{1}{Y_p} = \frac{r_{2e}}{2s} + j\frac{x_{2e}}{2} \tag{16}$$

$$Z_n = \frac{1}{Y_n} = \frac{r_{2e}}{2(2-s)} + j\frac{x_{2e}}{2} \tag{17}$$

$Z_m = 1/Y_m$ = the exciting impedance of the main magnetic circuit.

r_1 = primary-winding resistance

x_1 = primary-winding reactance

r_{2e} = secondary-winding resistance

x_{2e} = secondary-winding reactance

$Z_1 = r_1 + jx_1$

$Z_{2e} = r_{2e} + jx_{2e}$

$s = \frac{N_s - N}{N_s}$ = slip

Solving for the impedance between points a and b gives

$$Z_{ab} = \frac{Z_m Z_p}{Z_m + 2Z_p}$$

and similarly, the impedance between points b and c is

$$Z_{bc} = \frac{Z_m Z_n}{Z_m + 2Z_n}$$

adding Z_{ab} and Z_{bc} gives

$$Z_{ac} = \frac{Z_m Z_p}{Z_m + 2Z_p} + \frac{Z_m Z_n}{Z_m + 2Z_n}$$

or

$$Z_{ac} = \frac{Z_m^2 Z_p + 4Z_m Z_n Z_p + Z_m^2 Z_n}{Z_m^2 + 2Z_m(Z_n + Z_p) + 4Z_n Z_p} \tag{18}$$

or in terms of admittance

$$Y_{ac} = \frac{1}{Z_m} + \frac{Z_m+Z_n+Z_p}{Z_m(Z_n+Z_p)+4Z_nZ_p} \tag{19}$$

Equation (19) represents a parallel circuit of Z_m and Z_e in which

$$Z_e = \frac{Z_m(Z_n+Z_p)+4Z_nZ_p}{Z_m+Z_n+Z_p} \tag{20}$$

But, from the initial definitions of the several impedances used in equation (20), the following are obtained:

$$Z_p+Z_n = \frac{r_{2e}+jx_{2e}s(2-s)}{s(2-s)} \tag{21}$$

$$Z_pZ_n = \frac{r_{2e}^2-x_{2e}^2s(2-s)+2jx_{2e}r_{2e}}{4s(2-s)} \tag{22}$$

Using equations (22) and (21) in equation (20) gives

$$Z_e = jx_{2e} + \frac{r_{2e}^2+Z_mr_{2e}+jx_{2e}r_{2e}}{Z_ms(2-s)+r_{2e}+jx_{2e}s(2-s)}$$

or

$$Z_e = jx_{2e} + Z_g \tag{23}$$

where

$$Z_g = \frac{r_{2e}^2+Z_mr_{2e}+jx_{2e}r_{2e}}{Z_ms(2-s)+r_{2e}+jx_{2e}s(2-s)}$$

If the impedance, Z_g, is changed to an admittance, Y_g, and the equation rearranged, the result is

$$Y_g = \frac{s(2-s)}{r_{2e}} + \frac{(1-s)^2}{r_{2e}+jx_{2e}+Z_m} \tag{24}$$

By further manipulation, equation (24) may be given in the form

$$Y_g = \frac{s(2-s)\ (jx_{2e}+Z_m)}{r_{2e}(r_{2e}+jx_{2e}+Z_m)} + \frac{1}{r_{2e}+jx_{2e}+Z_m} \qquad (25)$$

Equation (25) represents two parallel admittances Y_c and Y_d where $Z_c = 1/Y_c = r_{2e} + jx_{2e} + Z_m$

$$= Z_{2e} + Z_m \qquad (26)$$

and

$$Z_r = \frac{1}{Y_d} = \frac{r_{2e}(r_{2e}+jx_{2e}+Z_m)}{s(2-s)\ (jx_{2e}+Z_m)}$$

$$= \frac{r_{2e}(Z_{2e}+Z_m)}{s(2-s)\ (Z_m+jx_{2e})} \qquad (27)$$

By considering equations (19), (26), and (27) along with the circuit given in Figure 5, the equivalent circuit shown by Figure 6 is obtained.

The general network constants may be obtained from Figure 6 as

$$A = \frac{Z_1(Z_{2e}+2Z_m+jx_{2e})+Z_m(Z_m+Z_{2e}+jx_{2e})}{Z_m(Z_{2e}+Z_m)}$$

$$B = \frac{Z_1(Z_m+jx_{2e})+jZ_m x_{2e}}{Z_m}$$

$$C = \frac{Z_{2e}+2Z_m+jx_{2e}}{Z_m(Z_{2e}+Z_m)}$$

$$D = \frac{Z_m+jx_{2e}}{Z_m}$$

By considering Figure 6 under no-load (s=o) and blocked-rotor (s=1) conditions, it is obvious that the parameters Z_1, Z_{2e}, and Z_m cannot be found from the conventional blocked-rotor and no-load tests. However, they may be obtained from the design data provided the values of rotor resistance and inductance are converted into primary terms (r_{2e} and x_{2e}).

FOUR-TERMINAL NETWORK THEORY - TEST DATA

The exact equivalent circuit shown in Figure 1 resembles the equivalent circuit used by Mr. Tarboux, which is shown in Figure 6. The network constants obtained from the solution of the equivalent circuit of Figure 1 are:

$$A = 1 + Y_sZ_1 + Y_oZ_1 + Z_1Z_2Y_oY_s + Y_sZ_2$$

$$B = Z_1 + Z_1Z_2Y_o + Z_2$$

$$C = Y_o + Y_oY_sZ_2 + Y_s$$

$$D = 1 + Y_oZ_2$$

These constants, like the constants resulting from the soluion of Figure 6, cannot be obtained from the no-load and blocked-rotor tests, nor from any practical laboratory tests. Consequently, if the test data is to be used to evaluate the general network constants, an approximate equivalent circuit must be drawn. This equivalent circuit should eliminate all "complicating parameters", and still approximate the actual motor behavior with the desired accuracy.

The parameters of the exact equivalent circuit of Figure 1 are defined as:

$Z_1 = R_1 + jX_1$ = stator winding impedance

$Z_2 = R_2 + jX_2$ = equivalent rotor winding impedance

$Z_o = R_o + jX_o$ = exciting impedance of the stator circuit

$Z_s = R_s + jX_s$ = rotor exciting impedance in the speed axis[12]

$Z_R = \frac{S^2 R_2}{1-S^2}$ = equivalent load impedance

The rotor exciting impedance in the speed axis is the "complicating parameter" of the equivalent circuit. It is usually neglected (or absorbed by the other parameters as constants) which would result in the approximate equivalent circuit given in Figure 2, where $Y_o = 1/Z_o$ = the exciting admittance of the stator circuit. The parameters of Figure 2 may be obtained from the no-load and blocked-rotor tests if certain assumptions are made. Consider the data obtained from these tests as the following:

Blocked-Rotor Test - V_b, I_b, W_b;

No-Load Test - V_n, I_n, W_n.

The resistance of the stator winding, R_1, should also be measured, either by the "IR-drop" method or by the use of a resistance-bridge.

From the blocked-rotor test the equivalent impedance, Z_e, and the equivalent resistance, R_e, may be found by

$$Z_e = \frac{V_b}{I_b} \qquad (7a)$$

and

$$R_e = \frac{W_b}{I_b^2} \qquad (7b)$$

Under blocked-rotor conditions the total resistance, R_e is equal to $R_1 + \frac{R_2 R_o}{R_2 + R_o}$, but since R_o is much larger than R_2, the total resistance is assumed to be given by $R_e = R_1 + R_2$ from which the (8)

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[12] $R_s = S^2 R_r$

$X_s = S^2(X_r + X_2)$

value of R_2 can be obtained.

Since $Z_e = R_e + jX_e$ then the value of the equivalent reactance can be found by

$$X_e = \sqrt{Z_e^2 - R_e^2} .$$

Assuming that the equivalent reactance of the rotor is equal to the reactance of the stator, then $X_1 = X_2 = .5X_e$. The stator and rotor impedances become $\overline{Z}_1 = R_1 + jX_1 = Z_1 \angle\theta_1$

$$\overline{Z}_2 = R_2 + jX_2 = Z_2 \angle\theta_2 .$$

The vector quantity $\overline{I}_n$ will be determined when the phase angle θ_n is found. From the no-load test, $\theta_n = \text{Cos}^{-1} \frac{W_n}{V_n I_n}$. If the voltage vector jV_1 is taken as the reference vector, then the no-load current vector, $\overline{I}_n$, will be equal to $I_n \angle 90-\theta_n$.

By applying Kirchhoff's law, the voltage, V_o, may be determined by

$$\overline{V}_o = \overline{V}_n - \overline{I}_n\overline{Z}_1$$

$$= (V_n \angle 90) - (I_n \angle 90-\theta_n)(Z_1 \angle \theta_1)$$

From which the exciting admittance of the stator circuit, Y_o, may be found as $\overline{Y}_o = \frac{I_n}{V_o}$.

Thus all of the network parameters of the approximate equivalent circuit shown in Figure 2 may be derived from the data of the no-load and blocked-rotor tests. By solving the circuit given in Figure 2 the general network constants may be obtained in terms of the network parameters, $\overline{Z}_1$, $\overline{Z}_2$, and $\overline{Y}_o$.

All of the quantities are vector quantities unless otherwise noted.

By use of Kirchhoff's laws

$$V_1 = I_1Z_1 + I_2Z_2 + V_2 \quad (28)$$

$$I_1 = I_n + I_2 \quad (29)$$

$$V_o = I_2Z_2 + V_2 \quad (30)$$

but

$$I_n = V_oY_o$$

$$= Y_o(I_2Z_2 + V_2) \quad (31)$$

Substituting the value of I_n from equation (31) into equation (29) gives

$$I_1 = Y_o(I_2Z_2 + V_2) + I_2 \quad (32)$$

Substituting equation (32) into equation (28) and expanding, the result is

$$V_1 = Z_1(I_2 + I_2Z_2Y_o + V_2Y_o) + I_2Z_2 + V_2$$

$$= V_2(Z_1Y_o + 1) + I_2(Z_1 + Z_1Z_2Y_o + Z_2) \quad (33)$$

Comparing equation (33) with equation (1) gives

$$A = Z_1Y_o + 1 \quad (34)$$

$$B = Z_1 + Z_1Z_2Y_o + Z_2 \quad (35)$$

Rearranging equation (32) gives

$$I_1 = V_2(Y_o) + I_2(Z_2Y_o + 1) \quad (36)$$

A comparison between equations (36) and (2) shows

$$C = Y_o \quad (37)$$

$$D = Z_2Y_o + 1 \quad (38)$$

An interesting point is brought out about the constant component, $K'V_1$, of equation (15) if it is evaluated in terms of the above test data.

$$K' = \frac{C}{A} = \frac{Y_o}{Z_1Y_o + 1}$$

But

$$Y_o = \frac{I_n}{V_n - I_nZ_1}$$

Giving

$$K' = \frac{\dfrac{I_n}{V_n - I_n Z_1}}{\dfrac{I_n Z_1}{V_n - I_n Z_1} + 1} = \frac{I_n}{V_n}$$

$$K'V_1 = \frac{V_1 I_n}{V_n}$$

But the no-load test is taken at rated voltage giving

$$V_n = V_1$$

Then

$$K'V_1 = I_n$$

There actually is a slight discrepancy between the actual value of $V_1(C/A)$ and the value of no-load current, I_n, since the value of $V_1(C/A)$ should correspond to the no-load current only when S = 1. This could occur in induction motors only by having the rotor driven at synchronous speed by another machine. The no-load current, I_n, would have its tip on the circle slightly above the point where $V_1(C/A)$ would be, but the error due to this discrepancy will be negligible in comparison to the error in obtaining the test data itself. Therefore, the vector $V_1(C/A)$ and the no-load current are assumed to be equal, both in magnitude and phase angle.

The other components, K and $K''V_1$, of equation (15) also may be obtained in terms of the test data.

SOLUTION OF K.

$$K = B/A$$

Substituting the values from equations (34) and (35) into the above gives

$$K = \frac{Z_1 + Z_1 Z_2 Y_o + Z_2}{Z_1 Y_o + 1}$$

But

$$Y_o = \frac{I_n}{V_n - I_n Z_1}$$

$$K = \frac{Z_1 + \dfrac{I_n Z_1 Z_2}{V_n - I_n Z_1} + Z_2}{\dfrac{Z_1 I_n}{V_n - I_n Z_1} + 1}$$

$$= \frac{Z_1(V_n - I_n Z_1) + Z_1 Z_2 I_n + Z_2(V_n - I_n Z_1)}{V_n}$$

$$= \frac{(Z_1 + Z_2)V_n - I_n Z_1^2}{V_n}$$

But $\quad Z_1 + Z_2 = Z_e$

Therefore

$$K = Z_e - \frac{I_n Z_1^2}{V_n}$$

Since $\quad K' = I_n/V_n$

Then $\quad K = Z_e - K' Z_1^2$

SOLUTION OF $K''V_1$.

$$K''V_1 = \frac{V_1}{A^2}$$

Substitution from equation (34) gives

$$K''V_1 = \frac{V_1}{(Z_1 Y_o + 1)^2} = \frac{V_1}{Z_1^2 Y_o^2 + 2Z_1 Y_o + 1}$$

But

$$Y_o = \frac{I_n}{V_n - I_n Z_1}$$

Giving

$$K''V_1 = \frac{V_1}{\dfrac{I_n^2 Z_1^2}{(V_n - I_n Z_1)^2} + \dfrac{2 I_n Z_1}{V_n - I_n Z_1} + 1}$$

$$= \frac{V_1 (V_n - I_n Z_1)^2}{I_n^2 Z_1^2 + 2 Z_1 I_n (V_n - I_n Z_1) + (V_n - I_n Z_1)^2}$$

$$= \frac{V_1 (V_n^2 - 2 V_n I_n Z_1 - I_n^2 Z_1^2)}{V_n^2}$$

$$= V_1 - \frac{2 V_1 I_n Z_1}{V_n} - \frac{V_1 I_n^2 Z_1^2}{V_n^2}$$

However, in single-phase motors the no-load test is practically always made at rated voltage, i.e., $V_1 = V_n$. If this is the case then

$$K''V_1 = V_1 - 2 I_n Z_1 - \frac{I_n^2 Z_1^2}{V_n}$$

but

$$K' = \frac{I_n}{V_n}$$

and

$$V_o = V_1 - I_n Z_1$$

thus

$$K''V_1 = V_o - I_n Z_1 (1 + K' Z_1)$$

CHAPTER II

After the network constants have been evaluated in terms of the data obtained from the no-load and blocked-rotor tests, it is necessary to draw the circle diagram or to use a calculation sheet, which is based on the circle diagram, in order that the performance curves may be plotted from which the motor performance under any load condition may be obtained.

MOTOR PERFORMANCE FROM THE CIRCLE DIAGRAM

A circle diagram based upon equation (15) is given in Figure 7 where

$$PR = \text{diameter of circle} = \frac{K''V_1}{jK_2}$$

$$OP = \text{no-load current} = K'V_1$$

$$Of = \text{blocked-rotor current}$$

In the circle diagram constructed for three-phase and single-phase motors where there is no tilting of the circle diameter, the division of the in-phase component of the blocked-rotor current is handled in the following manner :

$$\frac{\text{rotor copper loss}}{\text{stator copper loss}} = \frac{I_2^2R_2}{(I_b^2 - I_n^2)R_1}$$

When the circle diameter is tilted, there is no length on the conventional circle diagram which represents the stator copper loss. Thus the location of the point L must be done by different methods when then circle diagram is tilted. The method of locating L proposed in this thesis is a relatively simple and straightforward method. The length h'L' can be obtained by the equation

$$h'L' = \frac{(I_b^2 - I_n^2)R_1}{V_1}$$

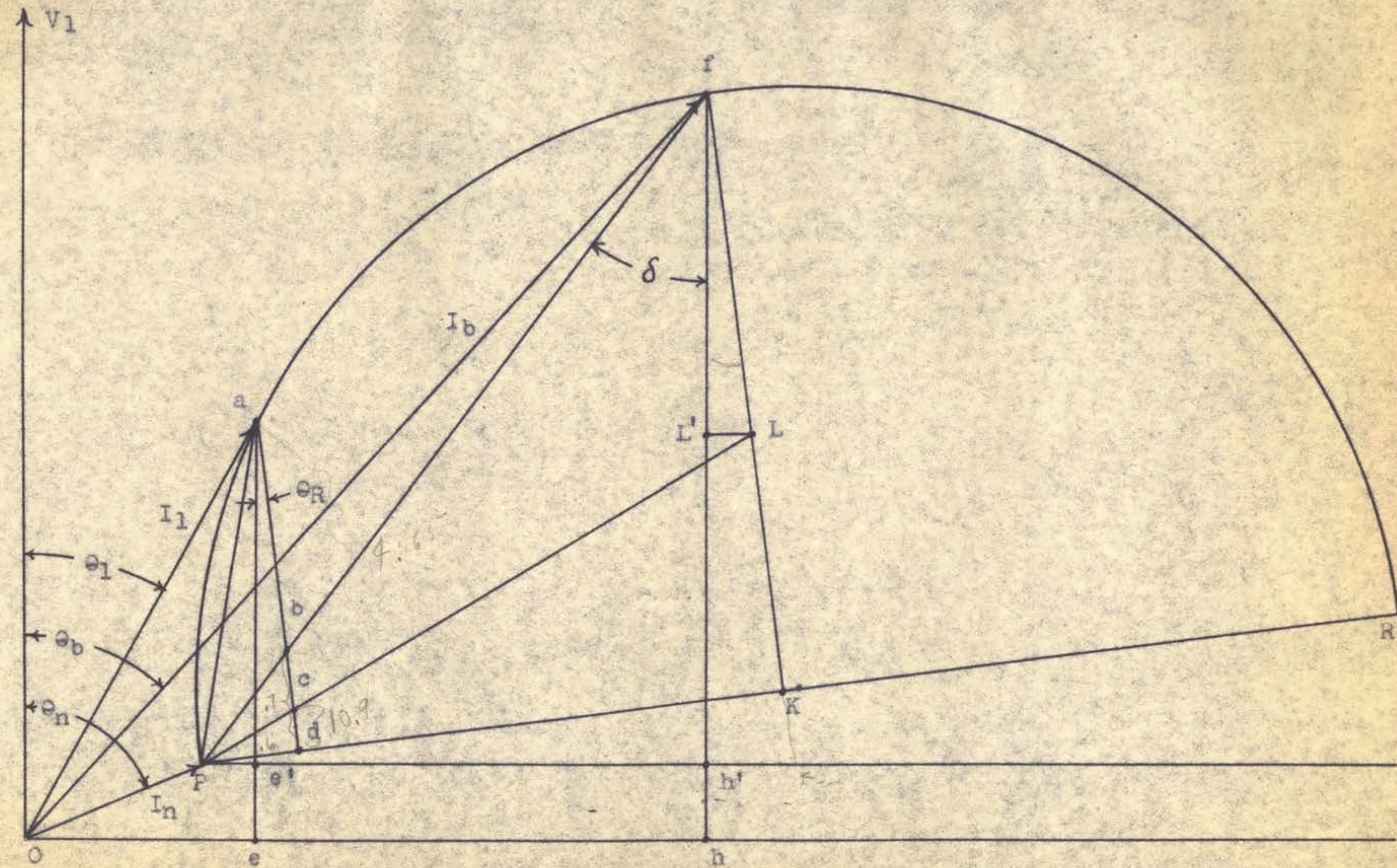

Figure 7. Circle diagram of a single-phase induction motor used to obtain the calculation sheet.

The horizontal projection from point L' to the line representing the in-phase component of the blocked-rotor current locates the point L. The length fL then represents the rotor copper loss during blocked-rotor conditions. The division of fK into the rotor and stator copper losses can be somewhat simplified by two assumptions: (a) that the in-phase components of both the rotor and the stator lie along the same line, and (b) that the quantity $(I_b^2 - I_n^2)$ is equal to I_2^2. Under these two assumptions the location of L is determined by the equation $LK = (R_1/R_e)fK$.

For small single-phase motors (less than 1/2 HP) these assumptions are valid and the errors introduced will be negligible. For larger single-phase motors the errors will be appreciable. Based on the data of a 3-hp motor[1], the errors introduced by using the above-mentioned assumptions exceed 13%. This error could hardly be called negligible. Whenever the term "neglgible" is applied to a situation, it implies that an error is involved, but supposes that the rror does not affect the result to a greater degree than can be tolerated. Thus the question of whether or not certain assumptions can be made or certain parameters neglected is not properly met unless the maximum magnitude of the error involved is specified.

Because of the angle of tilt of the circle diameter, it is also necessary to measure the power output along a line perpendicular to the x-axis while the power input is measured along a line perpendicular to the circle diameter.

For any value of stator current, Oa, the performance charac-

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[1]Puchstein, A. F. and Lloyd, T. C., Alternating-Current Machines, p. 638 (Motor#5).

teristics may be represented by the following quantities on the circle diagram (Figure 7):

Power input to stator = ae x V_1 (Watts)

Torque = ac x V_1 (synchronous Watts)

Useful power output = ab x V_1 (Watts)

Power factor = Cos θ = ae/Oa (as a decimal fraction)

Efficiency = ab/ae (as a decimal fraction)

EXAMPLE

An example will be given of a typical solution of the circle diagram for a single-phase induction motor based upon the equations derived from the approximate equivalent circuit shown in figure 2.

A $\frac{1}{4}$-hp, 115 volt, single-phase, 60 cycle, 4-pole, split-phase motor shows the following test results:

(a) Blocked rotor. Main winding.

$I_b = 14.1$ $V_b = 115$ $W_b = 1100$ $R_1 = 2.42$ ohms

(b) No-load. Main winding.

$I_n = 2.8$ $V_n = 115$ $W_n = 61.6$

The vector $V_1 = 115\underline{/90}$ will be taken as the reference vector.

$$\text{Cos } \theta_b = \frac{W_b}{V_b I_b} = \frac{1100}{(115)(14.1)} = .679$$

$$\theta_b = 47.3$$

$$\text{Cos } \theta_n = \frac{W_n}{I_n V_n} = \frac{61.6}{(2.8)(115)} = .191$$

$$\theta_n = 79.0$$

Thus $I_b = 14.1\underline{/90-\theta_b} = 14.1\underline{/42.7}$

and $I_n = 2.8\underline{/90-\theta_n} = 2.8\underline{/11.0}$

$$Z_e = \frac{V_b}{I_b} = \frac{115\underline{/90}}{14.1\underline{/42.7}} = 8.15\underline{/47.3}$$

$$R_e = \frac{W_b}{I_b^2} = \frac{1100}{(14.1)^2} = 5.52$$

$$R_2 = R_e - R_1 = 3.10$$

$$X_e = \sqrt{Z_e^2 - R_e^2} = \sqrt{8.15^2 - 5.52^2} = 6.0$$

$$X_1 = X_2 = .5X_e = 3.0$$

$$Z_1 = R_1 + jX_1 = 2.42 + j3.0 = 3.86\angle 51.1$$

$$Z_2 = R_2 + jX_2 = 3.10 + j3.0 = 4.31\angle 44$$

$$V_o = V_n - I_n Z_1$$

$$= 115\angle 90 - (2.8\angle 11.0)(3.86\angle 51.1) = 105.56\angle 92.74$$

$$Y_o = \frac{I_n}{V_o} = \frac{2.8\angle 11.0}{105.56\angle 92.74} = .0265\angle -81.74$$

$$A = Z_1 Y_o + 1$$

$$= (3.86\angle 51.1)(.0265\angle -81.74) + 1$$

$$= 1.091\angle -2.76$$

$$B = Z_1 + Z_1 Z_2 Y_o + Z_2$$

$$= (3.86\angle 51.1) + (3.86\angle 51.1)(4.31\angle 44)(.0265\angle -81.74) + 4.31\angle 44$$

$$= (2.42 + j3.0) + (.43 + j.10) + (3.10 + j3.0)$$

$$= 5.95 + j6.10 = 8.53\angle 45.8$$

$$C = Y_o = .0265\angle -81.74$$

$$D = Z_2 Y_o + 1 = (4.31\angle 44)(.0265\angle -81.74) + 1$$

$$= 1.0904 - j.07 = 1.0925\angle -3.68$$

$$K'V_1 = \frac{CV_1}{A} = \frac{(.0265\angle -81.74)(115\angle 90)}{(1.091\angle -2.76)} = 2.8\angle 11.0$$

This gives a good check on the calculations because it has been previously shown that $K'V_1$ is equal to the value of I_n.

$$K''V_1 = \frac{V_1}{A^2} = \frac{115\angle 90}{(1.091\angle -2.76)^2} = 96.8\angle 95.5$$

$$K = K_1 + jK_2 = \frac{B}{A} = \frac{8.53\angle 45.8}{1.091\angle -2.76} = 7.83\angle 48.56$$

$$K_1 = 7.83 \text{ Cos } 48.56 = 5.17$$

$$K_2 = 7.83 \text{ Sin } 48.56 = 5.38$$

$$\text{Diameter of circle} = \frac{K''V_1}{jK_2} = \frac{96.8\angle 95.5}{5.38\angle 90} = 16.6\angle 5.5$$

Now, in order to find the stator current, it is necessary only to assume a value of rotor speed (as a fraction of the synchronous speed) and solve equation (15).

Assume $S = .96$

$$Z_R = \frac{S^2R_2}{1-S^2} = \frac{(.96)^2(3.1)}{1-(.96)^2} = 36.5\angle 0$$

$$I_1 = K'V_1 + \frac{K''V_1}{(Z_R+K_1)+jK_2}$$

$$= 2.8\angle 11.0 + \frac{96.8\angle 95.5}{(36.5+5.17)+j5.38}$$

$$= 2.8\angle 11.0 + 2.3\angle 88.2 = 4.02\angle 45.1$$

The circle diagram for this example is given in Figure 8. From this circle diagram the other performance characteristics may be measured.

P_o = Power output = ab x V_1 = (1.955)(115) = 224.5 watts

P_i = Power input = ae x V_1 = (2.86)(115) = 329.0 watts

T = Torque = ac x V_1 = (2.17)(115) = 250 synchronous watts

PF = Power factor = ae/Oa = 2.86/4.02 = 71.3%

Efficiency = ab/ae = 1.955/2.86 = 68.4%

S = Rotor speed = ab/ac = 1.955/2.17 = .95

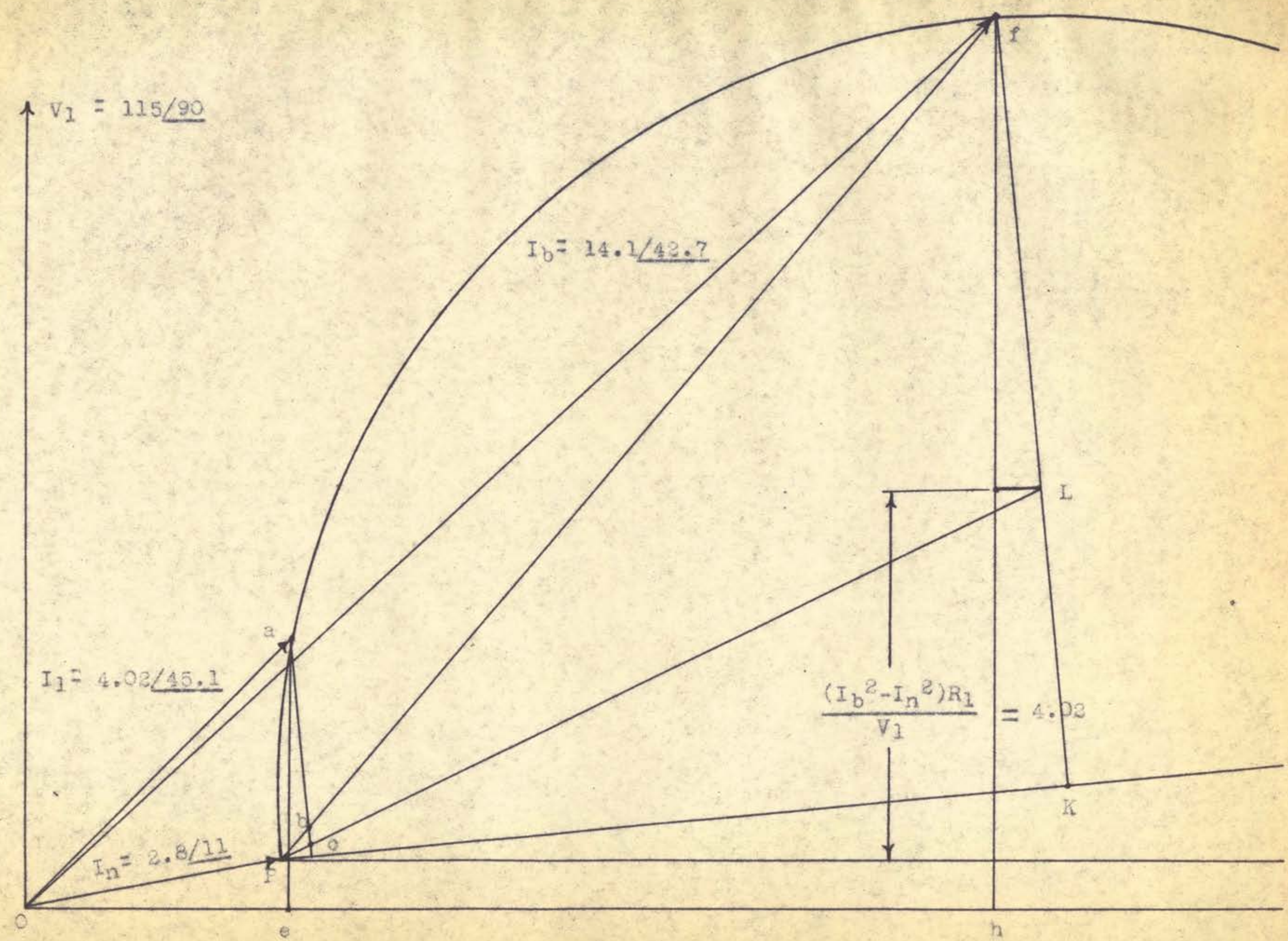

Figure 8. Circle diagram of the ½-hp single-phase induction motor in the Example.

Comparative Values

	P_i	I	%PF	%Eff.	S
From Circle Diagram	329.0	4.02	71.3	68.4	.95
From Test Data	334.0	4.10	71.0	67.0	.96

A comparison of these values to actual test values shows that the maximum error doesn't exceed 2.0%.

CALCULATION SHEET

Since it is the purpose of this thesis to predict the characteristics of single-phase induction motors from test data by the four-terminal network method, the discussion would not be complete without a performance sheet which utilizes the principles of the four-terminal network method.

Many authors abhor the calculation sheet method of determining motor performance because of its purely mechanical nature and also because the significance of the quantities involved are difficult to picture without the aid of a circle diagram. It does not enable them to follow through the calculations with the usual physical interpretation. Nevertheless, the analytical method has the advantages of being a much faster and a more accurate method and is accepted as the method for determination of motor characteristics by many companies.

An ideal calculation sheet contains no vectors; requires only one slide-rule operation per step; contains no large numbers nor any small numbers; and has few steps. It is, of course, impossible to satisfy all of these ideal characteristics. For example, if vectors are resolved into scalars this means that for every vector there will be two scalars thus increasing the number of calculations. The following calculation sheet has been set up with the above ideal characteristics in mind and patterned somewhat after the three-phase induction motor calculation sheets which are frequently used.[2,3]

The calculations upon which a calculation sheet are based must have an assumed starting point. A sheet can be set up with assumed values of power output, current input, or rotor speed and all of the other characteristics may be obtained from this assumed value. It is not even necessary to begin with an assumed performance characteristic. The calculation sheet may begin with assumed values of the "central angle" (the angle bounded by the diameter of the circle and a radius to the input current vector) and the characteristics may be obtained from it.

In deciding which characteristic should be assumed from which a calculation sheet may be constructed, the factor of whether the characteristic is or is not a linear function of the output power should be considered. If the characteristic is a linear function (or nearly so) of the output power, four or five equally spaced values of power output, from no-load to 125% of rated load, may be obtained. This will give a better distribution of points for the plotting of the performance curve.

Since the input current does not vary linearly with the output power, it would be difficult to obtain equally spaced values of power output. This point can be illustrated by considering the

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[2]Calculation sheet by Professor C. F. Cameron - Oklahoma Agricultural and Mechanical College

[3]Wayne J. Morrill, "The Apparent Impedance Method of Calculating single-phase Motor Performance," <u>Electrical Engineering</u>, LX, (December, 1941) 1037 (Transactions)

$\frac{1}{4}$-hp motor used in the previous example. At rated load the motor draws 3.6 amps from the line. If the quarter-load point is desired, it cannot be obtained by taking 25% of 3.6 amps (.9 amps) since the no-load current was found to be 2.8 amps. Neither can the solution be gotten by taking 25% of the difference between the full-load current and the no-load current and adding this to the no-load current. This would give a value of .25(3.6-2.8) - 2.8 = 2.0 amps, which is the current input at about 40% of rated load.

The method of assuming values of power output would be the most logical approach, but the calculations for obtaining the other characteristics in terms of the output power are too long and complicated to be of any practical use. The method of assuming values of the "central angle" has the disadvantage of there being no way of knowing the range of the "central angle" from no-load to 125% of rated load. However, this objection is not too great for single-phase induction motors as the range of the "central angle" is fairly constant for small motors.

Since the rotor speed, S, is practically a linear function of power output, up to about 150% of rated load (and frequently up to 200%), there is no difficulty in obtaining four or five equally spaced values of power output for plotting purposes. The motor name-plate data will furnish the full-load speed, N_{fl}, from which definite values of S can be obtained to give the values of 25%, 50%, 75%, 100%, and 125% of full load. For example; if it is desired to find the characteristics at nearly 75% of full load, the value of S could be found as $S = \frac{N_s - .75(N_s - N_{fl})}{N_s}$, where

N_s = synchronous speed and N_{fl} = full-load speed. The corresponding value of power output for this value of S will be very close to 75% of the full load value.

The following calculations are based, therefore, upon an assumed value of S. The load impedance can be found by

$$Z_R = \frac{S^2 R_2}{1-S^2}$$

Equation (15) gives the stator current in terms of the network constants and Z_R. The network constants are obtained from the no-load and blocked rotor test data, and will have the general values of

$$\overline{V}_1 = V_1\angle 90 \qquad \frac{\overline{C}}{\overline{A}} = \frac{C}{A}\angle\beta$$

$$\frac{1}{\overline{A}^2} = \frac{1}{A^2}\angle\phi \qquad \frac{\overline{B}}{\overline{A}} = \frac{B}{A}\angle\gamma$$

Equation (15) could be re-written, in terms of the magnitudes and angles of the vector quantities, in the following form

$$I_1 = V_1\frac{C}{A}\cos(90+\beta) + jV_1\frac{C}{A}\sin(90+\beta) + \frac{\frac{V_1}{A^2}\cos(90+\phi) + j\frac{V_1}{A^2}\sin(90+\phi)}{(Z_R + \frac{B}{A}\cos\gamma) + j\frac{B}{A}\sin\gamma} \tag{39}$$

or

$$\overline{I}_1 = (m + jn) + \frac{r + js}{x + jy} \tag{40}$$

where

$$m = V_1(C/A)\cos(90+\beta)$$

$$n = V_1(C/A)\sin(90+\beta)$$

$$r = V_1/A^2\cos(90+\phi)$$

$$s = V_1/A^2 \text{ Sin}(90+\phi)$$

$$x = Z_R + (B/A)\text{Cos}\,\gamma$$

$$y = (B/A)\text{Sin}\,\gamma$$

Rationalizing equation (40) gives

$$\bar{I}_1 = (m+jn) + \frac{(rx+sy)+j(xs-ry)}{x^2+y^2}$$

Let $u = x^2+y^2$

thus $$\bar{I}_1 = \left[m + \frac{(rx+sy)}{u}\right] + j\left[n + \frac{(xs-ry)}{u}\right] \qquad (41)$$

The magnitude of $\bar{I}_1$ is

$$I_1 = \sqrt{\left[m + \frac{(rx+sy)}{u}\right]^2 + \left[n + \frac{(xs-ry)}{u}\right]^2}$$

And the phase angle of $\bar{I}_1$ with reference to $\bar{V}_1$ is

$$\theta = \text{Tan}^{-1} \frac{m + \frac{(rx+sy)}{u}}{n + \frac{(xs-ry)}{u}}$$

The power factor is then found as the Cosine of θ.

By referring to Figure 7 the input current, I_1, is made up of vectors OP and Pa. By comparison with equations (40) and (41)

$$Pa = \frac{(rx+sy)}{u} + j\frac{(xs-ry)}{u}$$

or the length of Pa is equal to $\frac{1}{u}\sqrt{(rx+sy)^2 + (xs-ry)^2}$

thus $$Pe' = \frac{(rx+sy)}{u}$$

and $$ae' = \frac{(xs-ry)}{u}$$

$$\theta_R = \mathrm{Tan}^{-1}\frac{Pe'}{ae'} = \mathrm{Tan}^{-1}\frac{(rx+sy)}{(sx-ry)}$$

By trigonometry

$$ad = Pa\ \mathrm{Cos}(\theta_R+\phi)$$

and $$Pd = Pa\ \mathrm{Sin}(\theta_R+\phi)$$

In the triangle Pfh'

$$fh' = I_b\ \mathrm{Cos}\ \theta_b - I_n\ \mathrm{Cos}\ \theta_n$$

$$Ph' = I_b\ \mathrm{Sin}\ \theta_b - I_n\ \mathrm{Sin}\ \theta_n$$

and $$Pf = \sqrt{(Ph')^2+(fh')^2}$$

$$\delta = \mathrm{Tan}^{-1}\frac{Ph'}{fh'}$$

Then $$fK = Pf\ \mathrm{Cos}(\delta+\phi)$$

The length h'L' represents the stator copper loss and can be found by

$$h'L' = \frac{(I_b{}^2-I_n{}^2)R_1}{V_1}$$

Thus $$fL' = fh'-h'L'$$

and $$fL = \frac{fL'}{\mathrm{Cos}\ \phi}$$

This locates point L.

Considering triangles Pbd and PfK, the line bc and cd may be found by the following relations:

$$bd = Pd\ \mathrm{Cot}(\delta+\phi)$$

$$bc = bd\ \frac{(fL)}{(fK)}$$

The power output, P_o, is obtained by multiplying the quantity ab by the applied voltage, V_1, giving

$$P_o = V_1 \times ab = V_1(ad-bd)$$

The torque, in synchronous watts is found as

$$T = V_1 \times ac = V_1(ad-cd)$$

The torque in oz-ft may be obtained by the equation

$$T = \frac{112.8(ad-cd)V_1}{N_s}$$

where N_s = synchronous speed.

The power input, in watts, may be gotten from

$$P_i = ae \times V_1$$

The efficiency is given as the ratio of power output to power input, or

$$Eff. = \frac{ab}{ae}$$

All of the performance characteristics thus obtained are based upon an assumed value of rotor speed. From these calculations the single-phase induction motor calculation sheet shown in Figure 10 is set up so that the performance curves may be plotted from the results. Before this calculation sheet can be used, certain preliminary calculations must be made to obtain the necessary constants which are used in the calculation sheet. Figure 9 shows the preliminary calculation sheet which must be used in conjunction with the calculation sheet of Figure 10. All values given on Figure 9 are constant for a particular motor and are determined from the no-load and blocked-rotor test data. In Figure 9 all vector quantities are indicated either as $\overline{Z}$ or as $Z\underline{/\theta}$.

So that the accuracy of the calculation sheet may be illustrated, the no-load and blocked-rotor test data of the same motor given in the example is used in the calculation sheet of Figure 10. The resulting values are plotted as performance curves in Figure 11 and compared with performance curves from a normal load test. All calculations were made with a slide-rule. If greater accuracy is

V_b = ____ I_b = ____ W_b = ____ R_1 = ____

V_n = ____ I_n = ____ W_n = ____

$\cos\Theta_b = W_b/V_bI_b$ = ____ $\cos\Theta_n = W_n/V_nI_n$ = ____

Θ_b = ____ Θ_n = ____ $90-\Theta_n$ = ____

$Z_e = V_b/I_b$ = ____ $R_e = W_b/I_b^2$ = ____ $R_2 = R_e - R_1$ = ____

$X_e = \sqrt{Z_e^2 - R_e^2}$ = ____ $X_1 = X_2 = .5X_e$ = ____

$Z_1 \underline{/\Theta_1} = R_1 + jX_1$ = ____ $Z_2 \underline{/\Theta_2} = R_2 + jX_2$ = ____

$Z_e \underline{/\Theta_e} = R_e + jX_e$ = ____ $\overline{V}_o = \overline{V}_n - \overline{I}_n\overline{Z}_1$ = ____

$Y_o = \overline{I}_n/\overline{V}_o$ = ____ $V_1(C/A)\underline{/90+\beta} = I_n \underline{/90-\Theta_n}$ = ____

$m = I_n \cos(90-\Theta_n)$ = ____ $n = I_n \sin(90-\Theta_n)$ = ____

$K' = (I_n/V_n)\underline{/-\Theta_n}$ = ____ $K'\overline{Z}_1^2 = K'Z_1^2\underline{/2\Theta_1-\Theta_n}$ = ____

$K\underline{/\gamma} = \overline{Z}_e - K'\overline{Z}_1^2$ = ____ $y = K \sin\gamma$ = ____

$1+K'\overline{Z}_1 = 1+K'Z_1\underline{/\Theta_1-\Theta_n}$ = ____ $\overline{I}_n\overline{Z}_1 = I_nZ_1\underline{/90+(\Theta_1-\Theta_n)}$ = ____

$\overline{I}_n\overline{Z}_1(1+K'\overline{Z}_1)$ = ____ $K''V_1\underline{/90+\phi} = \overline{V}_o - \overline{I}_n\overline{Z}_1(1+K'\overline{Z}_1)$ = ____

$r = K''V_1\cos(90+\phi)$ = ____ $s = K''V_1\sin(90+\phi)$ = ____

$fh' = I_b\cos\Theta_b - I_n\cos\Theta_n$ = ____ $Ph' = I_b\sin\Theta_b - I_n\sin\Theta_n$ = ____

$Pf = \sqrt{(fh')^2+(Ph')^2}$ = ____ $\delta = \tan^{-1}(Ph')/(fh')$ = ____

$\delta+\phi$ = ____ $\cot(\delta+\phi)$ = ____ ry = ____ sy = ____

$fK = Pf\cos(\delta+\phi)$ = ____ $h'L' = R_1(I_b^2 - I_n^2)/V_1$ = ____

$A\,\overline{/\phi/2} = \overline{Z}_1\overline{Y}_o + 1$ = ____ $fL = (fh' - h'L')/\cos\phi$ = ____

Figure 9. Preliminary calculation sheet.

CALCULATION SHEET						
(1)	S					
(2)	$Z_R = (R_2S^2)/(1-S^2)$					
(3)	K Cos γ + (2)					
(4)	y^2 + $(3)^2$					
(5)	r x (3)					
(6)	(5) + sy					
(7)	(6) / (4)					
(8)	m + (7)					
(9)	s x (3)					
(10)	(9) - ry					
(11)	(10) / (4)					
(12)	n + (11)					
(13)	$P_i = V_1$ x (12)					
(14)	I = $[(8)^2 + (12)^2]^{1/2}$					
(15)	$(90-\theta) = \text{Tan}^{-1}$ (12) / (8)					
(16)	P.F. = Sin$(90-\theta)$					
(17)	Pa = $[(7)^2 + (11)^2]^{1/2}$					
(18)	$\lambda = \text{Tan}^{-1}$ (11) / (7)					
(19)	ad = (17) x Sin$(\lambda - \phi)$					
(20)	Pd = (17) x Cos$(\lambda - \phi)$					
(21)	bd = (20) x Cot$(\delta + \phi)$					
(22)	cd = (21) x (fK-fL)/fK					
(23)	$P_o = V_1$((19) - (21))					
(24)	T = $(112.8/N_s)$((19) - (22))					
(25)	Eff. = (23) / (13)					

Figure 10. Calculation sheet for single-phase induction motors.

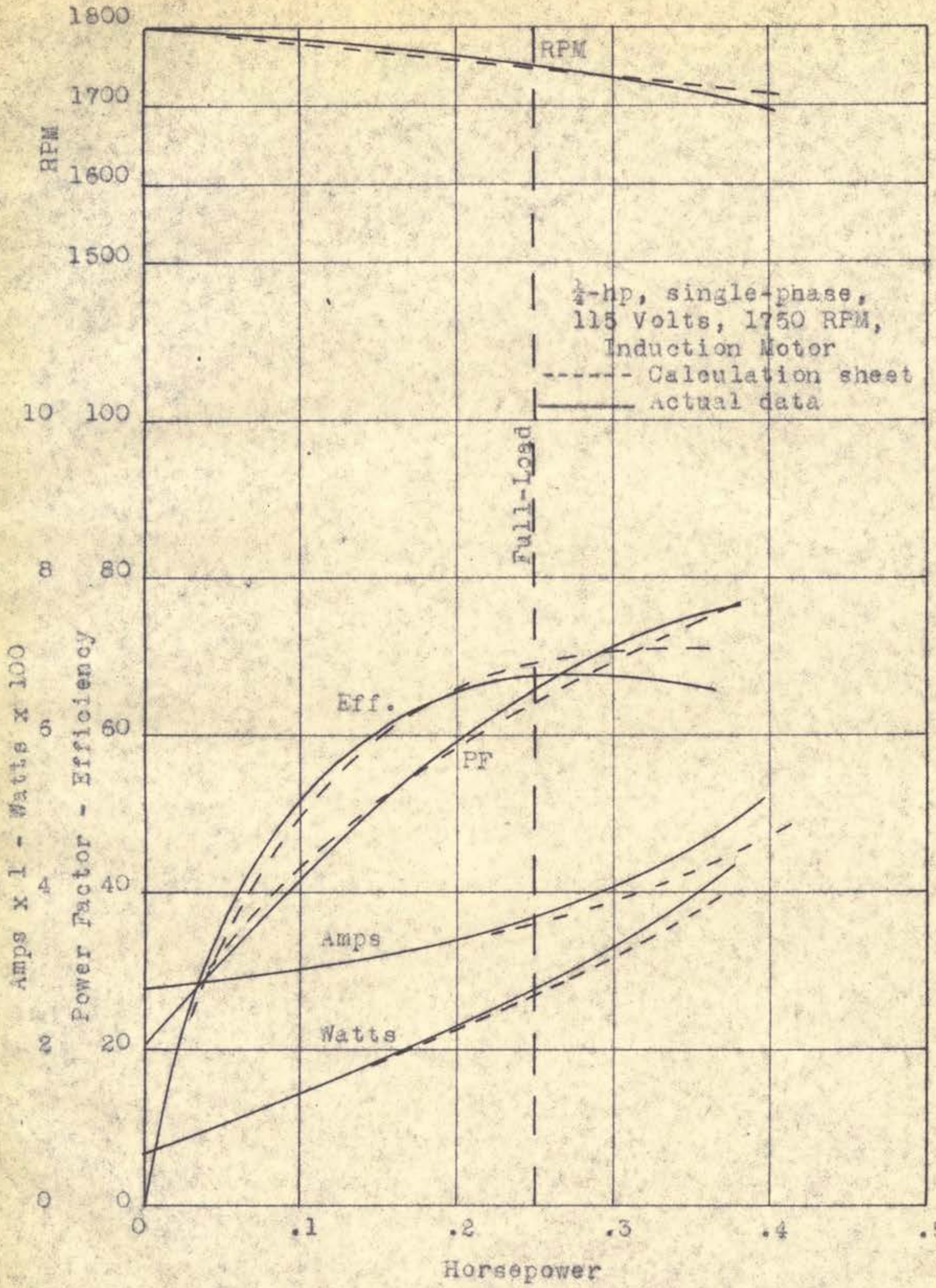

Figure 11. Comparison of the performance curves from the calculation sheet and from actual test data.

desired, tables may be used for trigonometric functions and logarithms used for multiplication and division. The results obtained by the slide-rule, however, are sufficiently accurate for most situations as is shown by the curves of Figure 11.

CONCLUSIONS

The four-terminal network method presented in this thesis is based upon the approximate equivalent circuit of the single-phase motor. Since single-phase motors are relatively small in size, the shunt branch representing the rotor exciting branch in the speed field has partly been combined with the other parameters forming the approximate equivalent circuit while the rest has been neglected entirely. This modification of the exact equivalent circuit was made in order that all of the network constants A, B, C, and D could be evaluated in terms of data obtained from the no-load and blocked-rotor tests.

The four-terminal network method very definitely indicates that the circle diagram should be tilted upwards for more accurate results. Because the circle diameter is tilted, precautions should be taken to remember that the stator "added" copper loss should be in phase with the applied voltage perpendicular to the x-axis) while the rotor copper loss is ϕ degrees ahead of the applied voltage (perpendicular to the diameter of the circle). It has been shown that the error in assuming both the stator and rotor copper losses to be in phase and perpendicular to the circle diameter may be quite large for larger single-phase motors. For smaller motors the errors due to these assumptions may be neglected depending upon the accuracy desired for the prediction characteristics. If greater accuracy is desired, these errors must be corrected for in the calculations.

The application of the four-terminal network method to motors, especially to single-phase motors, is a fairly new field of approach to the solution of the equivalent circuits of motors. The aspects of the application of this method certainly have not been fully explored in the field of predicting motor performance.

The method presented in this thesis for the prediction of single-phase induction motor characteristics from test data is essentially a simple, but accurate solution of the approximate equivalent circuit. From this solution either a circle diagram or a calculation sheet may be used in order to obtain the motor performance characteristics. Since there has been no work done in applying the four-terminal network method to an equivalent circuit of a single-phase motor from the stand point of test data, a calculation sheet is constructed based entirely upon data which can be gotten from the no-load and blocked-rotor tests. This provides a means of determining the performance curves of a single-phase motor from which the behavior of the motor can be predicted. The accuracy of the calculation sheet has been shown to be very good even when using a slide rule for the calculations. Greater accuracy may be had by the use of tables and an ordinary calculating machine. The calculation sheet gives a much faster method of obtaining the performance curves than the circle diagram, and, in addition, is a more accurate method.

As compared to the calculation sheet developed by Mr. Veinott,[4]

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[4]C. G. Veinott, "Performance Calculations on Induction Motors", AIEE Transactions, Volume 51 (September, 1942), 743-755.

the calculation sheet presented herein requires only twenty-five calculations (after the constants are determined) while Mr. Veinott's sheet contains thirty-eight steps.

BIBLIOGRAPHY

Branson, W. J. "Single Phase Induction Motors." Transactions of American Institute of Electrical Engineers, XXXI (1912), 1749-1787.

Guillemin, Ernst A. Communication Networks, Vol.II. New York: John Wiley and Sons, 1935.

Karapetoff, V. and Dennison, Boyd C. Experimental Electrical Engineering, Vol. II. New York: John Wiley and Sons, 1941.

Karapetoff, V. Jr. "On the Equivalence of the Two Theories of the Single-Phase Motor." American Institute of Electrical Engineers, XL (August, 1921), 640.

Langsdorf, Alexander S. Theory of Alternating-Current Machinery. New York: McGraw-Hill Book Co., 1937.

Massachusetts Institute of Technology (Staff Members of the Department of Electrical Engineering) Electric Circuits. New York: John Wiley and Sons, 1943.

Morrill, Wayne J. "The Apparent Impedance Method of Calculating Single-phase Motor Performance." Electrical Engineering LX (December, 1941), 1037 Transactions.

Puchstein, A. F. and Lloyd, T. C. Alternating-Current Machinery. New York: John Wiley and Sons, 1936.

Reed, Myril B. Alternating-Current Circuit Theory. New York: Harper and Brothers, 1948.

Tarboux, J. G. Alternating-Current Machinery. Scranton, Pennsylvania: International Textbook Company, 1947.

Tarboux, J. G. "A Generalized Circle Diagram for a Four-Terminal Network and Its Application to the Capacitor Single-Phase Motor." Electrical Engineering, LXIV (March, 1945), 881-889.

Terman, F. E., Lenzen, T. W., Freedman, C. L., and Rogers, K. A. "General Circle Diagram of Electrical Machinery." Transactions of American Institute of Electrical Engineers, XLIX (January, 1930), 374-380.

United States Department of Commerce. Sixteenth Census of the United States Manufactures 1939, Vol.II, Part 2. Washington: United States Government Printing Office, 1942.

Veinott, C. G. "Performance Calculations on Induction Motors." Transactions of American Institute of Electrical Engineers, LI September, 1932), 743-755.

Westinghouse Electric & Manufacturing Co. (Central Station Engineers). *Electrical Transmission and Distribution Reference Book.* Chicago: The Lakeside Press, 1942.

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