A CLOSED ORTHONORMAL SYSTEM OF BIHARMONIC FUNCTIONS DEFINED ON THE UNIT CIRCLE

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Stillwater, Oklahoma

1943

Submitted to the Department of Mathematics

Oklahoma Agricultural and Mechanical College

In Partial Fulfillment of the Requirements

for the Degree of

MASTER OF SCIENCE

1949

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PREFACE

I deeply appreciate the supervision and guidance given me by Professor Josephine Mitchell and Dr. Ainsley H. Diamond in the writing of this paper.

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INTRODUCTION

The purpose of this paper is to construct an orthonormal set of biharmonic functions defined on the unit circle C using an inner product which arises as the result of a certain variational problem, discussed in Section II.

In Section III, following Bergman's method,[1] an orthonormal system of biharmonic functions is constructed for the unit circle: the kernel function is defined and certain of its properties proved. The orthogonal series is defined and certain of its properties proved. Then, assuming that the system is closed, an arbitrary biharmonic function is represented by means of this orthogonal series. Finally the value of this function at any interior point is expressed in terms of its value and the value of its radial derivative on the boundary of the unit circle.

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[1] Stefan Bergman, Partial Differential Equations, Advanced Topics, Chapter VI, pp. 38-50.

SECTION I

This section contains fundamental definitions and theorems which are needed in order to prove later results. Reference will be made to them as used.

The biharmonic equation[1] is the partial differential equation of fourth order

(1.1) $$\Delta \Delta f = 0,$$

where in rectangular coordinates the operator

(1.2) $$\begin{aligned}\Delta\Delta &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \\ &= \frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2}\end{aligned}$$

or in polar coordinates

(1.3) $$\Delta\Delta = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)$$

Any function which satisfies the biharmonic equation is called a biharmonic function. Any complex solution of the biharmonic equation may be expressed in the form

(1.4) $$U(z,\bar{z}) = \bar{z} f(z) + g(z)$$

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[1] Philipp Frank and Richard v. Mises, Die Differentialgleichungen der Mechanik and Physik, pp. 845-846.

where $z = x+iy$, $\bar{z} = x-iy$, and $f(z)$ and $g(z)$ are analytic functions of z.[2] Since any complex function may be separated into its real and imaginary parts, a real biharmonic function[3] F (x,y) may be written in the form

$$(1.5) \quad \begin{aligned} F(x,y) &= \operatorname{Re} U(z,\bar{z}) = \operatorname{Re}\left[\bar{z} f(z) + g(z)\right] \\ \text{or} \quad &= \operatorname{Im} U(z,\bar{z}) = \operatorname{Im}\left[\bar{z} f(z) + g(z)\right]. \end{aligned}$$

The <u>class</u> <u>L^2</u> is defined to be the subset $\{\operatorname{Re} \bar{z}f(z), \operatorname{Im} \bar{z}f(z)\}$ of the set of biharmonic functions with the additional property

$$(1.6) \qquad \iint_D \left[\Delta F(x,y)\right]^2 dx\, dy < \infty,$$

where D is a simply connected domain.

A system of functions $\{\varphi_n(x,y)\}$, $(n = 1, 2, 3, \cdots)$ belonging to L^2, is called an orthogonal set[4] if

$$(1.7) \qquad \iint_D \varphi_n(x,y)\, \varphi_m(x,y)\, dx\, dy = \begin{cases} 0 & \text{for } n \neq m \\ a \text{ constant} & \text{for } n = m. \end{cases}$$

---

[2] If $f(z)$ is an analytic function of z, then Re $[f(z)]$ and Im $[f(z)]$ satisfy the harmonic equation, $\Delta \operatorname{Re}[f(z)] = 0$ and $\Delta \operatorname{Im}[f(z)] = 0$. $\operatorname{Re}[f(z)]$ and $\operatorname{Im}[f(z)]$ are called harmonic functions.

[3] Frank and v. Mises, <u>Op</u>. <u>cit</u>., p. 848.

[4] Stefan Bergman, <u>Op</u>. <u>cit</u>., p. 39.

If

$$(1.8) \qquad \iint_D \varphi_n(x,y)\,\varphi_m(x,y)\,dx\,dy = 1$$

for m = n, the system is called normal. A system is called an orthonormal set[5] if it is both orthogonal and normal.

An orthogonal set may be normalized by dividing each $\varphi_n(x,y)$ by

$$(1.9) \qquad \left\{ \iint_D \left[\varphi_n(x,y)\right]^2 dx\,dy \right\}^{\frac{1}{2}}$$

which we call the normalizing factor.[6]

If $\{\psi_n(x,y)\}$ is a sequence of linearly independent functions of class L^2 and if every function $f(x,y)$ of class L^2 can be approximated by linear combinations of the form

$$(1.10) \qquad \sum_{n=1}^{\nu} c_n \psi_n(x,y) \qquad (c_n \text{ being constants})$$

so that the average quadratic error

$$(1.11) \qquad \iint_D \left| f(x,y) - \sum_{n=1}^{\nu} c_n \psi_n \right|^2 dx\,dy$$

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[5] Ibid., p. 40.

[6] E. W. Hobson, The Theory of Functions of Real Variables, Vol II, p. 753.

approaches 0 as $\nu \longrightarrow \infty$, then $\{\psi_n(x,y)\}$ is said to be a <u>closed</u> <u>system</u>.[7]

If $\{\psi_n(x,y)\}$ be a closed sequence of functions, a closed orthonormal system $\{\varphi_n(x,y)\}$ can be determined by the Gramm-Schmidt process so that $\varphi_n(x,y)$ is a linear combination of $\{\psi_n(x,y)\}$.[8]

Let

$$(1.12) \qquad a_n = \iint_D f(x,y)\,\varphi_n(x,y)\,dx\,dy\,, \quad n = 1, 2, 3, \cdots,$$

where $\{\varphi_n(x,y)\}$ is an orthogonal system of class L^2 on D and $f(x,y)$ belongs to L^2 on D, then

$$(1.13) \qquad \sum_{n=1}^{\infty} a_n^2 \leq \iint_D [f(x,y)]^2\,dx\,dy\,.$$

This statement is called <u>Bessel's</u> <u>inequality</u>. If $\{\varphi_n(x,y)\}$ is a closed system, this inequality becomes <u>Parseval's</u> <u>equality</u>.[9]

$$(1.14) \qquad \sum_{n=1}^{\infty} a_n^2 \leq \iint_D [f(x,y)]^2\,dx\,dy\,.$$

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[7] Bergman, <u>Op</u>. <u>cit</u>., p. 41.

[8] Bergman, <u>Op</u>. <u>cit</u>., p. 42, and Hobson, <u>Op</u>. <u>cit</u>., p. 754.

[9] Bergman, <u>Op</u>. <u>cit</u>., p. 41.

As a tool in our proofs we shall use the following two inequalities of Schwartz:

$$(1.15)\quad \left(\sum_{n=1}^{\infty}|a_n b_n|\right)^2 \leq \sum_{n=1}^{\infty}|a_n|^2 \sum_{n=1}^{\infty}|b_n|^2 \text{ [10]}$$

$$(1.16)\quad \left|\iint f(x,y)\, g(x,y)\, dx\,dy\right|^2 \leq \iint |f|^2 dx\,dy \iint |g|^2 dx\,dy \text{ [11]}$$

We need the definitions of absolute convergence and uniform convergence.

A series $\sum_{n=1}^{\infty} a_n$ is said to be <u>absolutely convergent</u> if $\sum_{n=1}^{\infty}|a_n|$ converges.

The series $\sum_{n=1}^{\infty} u_n(x,y)$ is said to be <u>uniformly convergent</u> to the function $S(x,y)$ over the domain D if, given any positive number ϵ, we can find a number n_o, depending on ϵ but not on x or y, such that

$$(1.17)\quad |S(x,y) - S_n(x,y)| < \epsilon$$

for $n > n_o$, and for every value of (x,y) in D.[13]

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10 <u>Ibid</u>., p. 39.

11 E. C. Titchmarsh, <u>The Theory of Functions</u>, p. 281.

12 Lawrence M. Graves, <u>The Theory of Functions of Real Variables</u>, p. 108.

13 Titchmarsh, <u>Op. cit</u>., p. 2.

In order to establish absolute and uniform convergence we use the Weierstrass M-test.[14] If the series $\sum_{n=1}^{\infty} M_n$ converges and if $|u_n(x,y)| \leq M_n$, $(n = 1, 2, 3, \cdots)$ for every (x,y) in a closed domain D, then $\sum_{n=1}^{\infty} u_n(x,y)$ converges absolutely and uniformly.

The Weierstrass convergence theorem[15] states that the sum of a uniformly convergent series of continuous functions is a continuous function.

A series may be differentiated term by term if the differentiated series is a uniformly convergent series of continuous functions and its sum is equal to the derivative of the sum of the original series.[16]

The Lebesque convergence theorem[17] is also used. A uniformly convergent series of continuous functions may be integrated term by term; that is, if $\sum_{n=1}^{\infty} u_n(x,y)$ converges uniformly for all (x,y) in D, then

$$\iint_D \sum_{n=1}^{\infty} u_n(x,y)\,dx\,dy = \sum_{n=1}^{\infty} \iint_D u_n(x,y)\,dx\,dy. \tag{1.18}$$

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14 Graves, Op. cit., p. 11.

15 Titchmarsh, Op. cit., p. 7.

16 Ibid., pp. 37-38.

17 Ibid., p. 36.

SECTION II

The following theorem is proved by applying the classical methods of the calculus of variations to functions involving higher order derivatives.[1]

Theorem 2.1. If the integral

$$(2.1) \qquad I = \iint_D F(x,y,u,p,q,r,s,t) \; dx\, dy$$

is a minimum, then

$$(2.2) \qquad F_u - \frac{\partial F_p}{\partial x} - \frac{\partial F_q}{\partial y} + \frac{\partial^2 F_r}{\partial x^2} + \frac{\partial^2 F_s}{\partial x \partial y} + \frac{\partial^2 F_t}{\partial y^2} = 0,$$

where F is a continuous function on the domain D and has continuous third order partial derivatives, and

$$(2.3) \quad p = \frac{\partial u}{\partial x}, \; q = \frac{\partial u}{\partial y}, \; r = \frac{\partial^2 u}{\partial x^2}, \; s = \frac{\partial^2 u}{\partial x \partial y}, \; t = \frac{\partial^2 u}{\partial y^2}.$$

Proof: Let us assume that $u = u(x,y)$, continuous along with its second order partial derivatives, is a certain function of x and y which makes the integral (2.1) a minimum. We shall consider u(x,y) to be a one-parameter family of functions constructed as follows. Take any arbitrary function

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[1] William F. Osgood, Advanced Calculus, pp. 406-419, and R. Courant, Differential and Integral Calculus, Vol II, pp. 491-521.

$h(x,y)$, which is continuous along with its second order partial derivatives in D, and also vanishes along with its partial derivatives on the boundary B of D. Form the family of functions $u(x,y) + \epsilon h(x,y)$, ϵ being the parameter.

If we consider x and y as fixed, then (2.1) becomes a function of ϵ:

$$(2.4)\quad I(\epsilon) = \iint_D F(x, y, u+\epsilon h, p+\epsilon h_x, q+\epsilon h_y, r+\epsilon h_{xx}, s+\epsilon h_{xy}, t+\epsilon h_{yy})\,dx\,dy.$$

The integral (2.4) is at least as great as the original integral (2.1) and is equal to it when $\epsilon = 0$. By ϵ varying we may vary the integral (2.4). Thus if $I(\epsilon)$ is minimum, then

$$(2.5)\qquad \frac{dI(\epsilon)}{d\epsilon} = 0 \quad \text{when} \quad \epsilon = 0$$

By Leibniz' rule[2] we differentiate (2.4) with respect to ϵ remembering that x and y are independent of ϵ, and get the following equation

$$(2.6)\quad \frac{dI(\epsilon)}{d\epsilon} = \iint_D \Big[\frac{\partial F}{\partial(u+\epsilon h)}\frac{\partial(u+\epsilon h)}{\partial\epsilon} + \frac{\partial F}{\partial(p+\epsilon h_x)}\frac{\partial(p+\epsilon h_x)}{\partial\epsilon} + \frac{\partial F}{\partial(q+\epsilon h_y)}\frac{\partial(q+\epsilon h_y)}{\partial\epsilon} + \frac{\partial F}{\partial(r+\epsilon h_{xx})}\frac{\partial(r+\epsilon h_{xx})}{\partial\epsilon} + \frac{\partial F}{\partial(s+\epsilon h_{xy})}\frac{\partial(s+\epsilon h_{xy})}{\partial\epsilon} + \frac{\partial F}{\partial(t+\epsilon h_{yy})}\frac{\partial(t+\epsilon h_{yy})}{\partial\epsilon}\Big]\,dx\,dy$$

Using conditions (2.5) this differentiation becomes

$$(2.7)\quad \frac{dI(\epsilon)}{d\epsilon} = \iint_D \left[F_u h + F_p h_x + F_q h_y + F_r h_{xx} + F_s h_{xy} + F_t h_{yy}\right] dx\,dy$$

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[2] Osgood, Op. cit., pp. 461-462.

where the subscripts denote partial differentiation.

Examining the second term of (2.7), we know that

$$\frac{\partial}{\partial x}(F_p h) = \frac{\partial F_p}{\partial x} h + F_p h_x , \tag{2.8}$$

$$F_p h_x = \frac{\partial}{\partial x}(F_p h) - \frac{\partial F_p}{\partial x} h .$$

Integrating (2.8) over the domain D, we have

$$\iint_D F_p h_x \, dx\, dy = \iint_D \frac{\partial}{\partial x}(F_p h)\, dx\, dy - \iint_D \frac{\partial F_p}{\partial x} h \, dx\, dy \tag{2.9}$$

$$= \int_B F_p h \, dy - \iint_D \frac{\partial F_p}{\partial x} h \, dx\, dy .$$

But $h(x,y) = 0$ on the boundary B, so that

$$\iint_D F_p h_x \, dx\, dy = -\iint_D \frac{\partial F_p}{\partial x} h \, dx\, dy . \tag{2.10}$$

Similarly the third term of (2.7) becomes

$$\iint_D F_q h_y \, dx\, dy = -\iint_D \frac{\partial F_q}{\partial y} h \, dx\, dy . \tag{2.11}$$

Using the rule for differention of a product, the fourth term of (2.7) is

$$F_r h_{xx} = \frac{\partial}{\partial x}(F_r h_x) - \frac{\partial F_r}{\partial x} h_x . \tag{2.12}$$

Integrating this over the domain D

$$\iint_D F_r h_{xx} \, dx\, dy = \iint \frac{\partial}{\partial x}(F_r h_x)\, dx\, dy - \iint \frac{\partial F_r}{\partial x} h_x \, dx\, dy \tag{2.13}$$

$$= \int_B F_r h_x \, dy - \iint_D \frac{\partial F_r}{\partial x} h_x \, dx\, dy .$$

Again $h_x(x,y) = 0$ on the boundary B, thus

$$(2.14)\quad \iint_D F_r h_{xx}\,dx\,dy = -\iint_D \frac{\partial F_r}{\partial x} h_x\,dx\,dy.$$

Integrating the right hand member of (2.14) by parts again, and recalling that $h(x,y) = 0$ on B, (2.14) becomes

$$(2.15)\quad \iint F_r h_{xx}\,dx\,dy = -\left[\iint_D \frac{\partial}{\partial x}\left(\frac{\partial F_r}{\partial x} h\right) dx\,dy - \iint_D \frac{\partial^2 F_r}{\partial x^2} h\,dx\,dy\right]$$

$$= -\left[\int_B \frac{\partial F_r}{\partial x} h\,dy - \iint_D \frac{\partial^2 F_r}{\partial x^2} h\,dx\,dy\right]$$

$$= \iint_D \frac{\partial^2 F_r}{\partial x^2} h\,dx\,dy$$

Similarly it can be shown that the last two terms of (2.7) becomes

$$(2.16)\quad \iint_D F_s h_{xy}\,dx\,dy = \iint_D \frac{\partial^2 F_s}{\partial x \partial y} h\,dx\,dy$$

and

$$(2.17)\quad \iint_D F_t h_{yy}\,dx\,dy = \iint_D \frac{\partial^2 F_t}{\partial y^2} h\,dx\,dy.$$

Substituting (2.10), (2.11), (2.15), (2.16) and (2.17) into (2.7) and factoring out h, we have

$$(2.18)\quad \iint_D h\left[F_u - \frac{\partial F_p}{\partial x} - \frac{\partial F_q}{\partial y} + \frac{\partial^2 F_r}{\partial x^2} + \frac{\partial^2 F_s}{\partial x \partial y} + \frac{\partial^2 F_t}{\partial y^2}\right] dx\,dy = 0$$

Since $h(x,y)$ is an arbitrary function, the second factor of (2.18) must vanish at every point on the domain D. Therefore (2.2) is a necessary condition for (2.1) to be a minimum.

An immediate consequence of this theorem is the following statement, which concerns our inner product in Section III.

Corollary 2.1. If the integral

$$\iint_D \left[\Delta f(x,y)\right]^2 dx\,dy \tag{2.19}$$

is a minimum, then $f(x,y)$ is a biharmonic function on D, ie,

$$\Delta\Delta f(x,y) = 0. \tag{2.20}$$

Proof: We know that

$$\iint_D \left[\Delta f(x,y)\right]^2 dx\,dy = \iint_D \left[f_{xx} + f_{yy}\right]^2 dx\,dy \tag{2.21}$$

and that $f_{xx} = r$, and $f_{yy} = t$. Let $F = \left[f_{xx} + f_{yy}\right]^2$ and apply equation (2.2). We immediately find that

$$F_u = 0,\ \frac{\partial F_p}{\partial x} = 0,\ \frac{\partial F_q}{\partial y} = 0,\ \frac{\partial^2 F_s}{\partial x \partial y} = 0, \tag{2.22}$$

$$\frac{\partial^2 F_r}{\partial x^2} = 2\left(f_{xxxx} + f_{yyxx}\right),\ \frac{\partial^2 F_t}{\partial y^2} = 2\left(f_{xxyy} + f_{yyyy}\right).$$

Equation (2.2) becomes

$$2\left(f_{xxxx} + f_{yyyy} + 2 f_{xxyy}\right) = 0 \tag{2.23}$$

$$2\left(\frac{\partial^4 f}{\partial x^4} + \frac{\partial^4 f}{\partial y^4} + 2\frac{\partial^4 f}{\partial x^2 \partial y^2}\right) = 0$$

which is the biharmonic equation (2.20). Therefore $f(x,y)$ is a biharmonic function on D.

SECTION III

We first consider the following system of linearly independent biharmonic functions.

(3.1)
$$\psi_0 = \mathrm{Re}\,\bar{z}z\,,\ \psi_1 = \mathrm{Re}\,\bar{z}z^2,\ \psi_2 = \mathrm{Im}\,\bar{z}z^2,\ \cdots$$
$$\psi_{2n-1} = \mathrm{Re}\,\bar{z}z^{n+1},\ \psi_{2n} = \mathrm{Im}\,\bar{z}z^{n+1},\ \cdots$$

In polar coordinates $z = re^{i\theta}$ and $\bar{z} = re^{-i\theta}$, this system would be

(3.2)
$$\psi_0 = r^2,\ \psi_1 = r^3\cos\theta,\ \psi_2 = r^3\sin\theta,\ \cdots$$
$$\psi_{2n-1} = r^{n+2}\cos n\theta,\ \psi_{2n} = r^{n+2}\sin n\theta,\ \cdots$$

By means of the Schmidt Gram[1] process we form from the set (3.2) a biharmonic orthonormal system $\{\varphi_n(r,\theta)\}$ with respect to the inner product

(3.3)
$$\iint \Delta\varphi_n(r,\theta)\,\Delta\varphi_m(r,\theta)\,dA,$$

where the domain C is the unit circle and $dA = r\,dr\,d\theta$. If $n = m$, the inner product becomes

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1 Bergman, Op. cit., p. 42.

$$(3.4) \qquad \iint_C \left[\Delta \varphi_n(r,\theta) \right]^2 dA$$

which has the desired property of Corollary 2.1.

Theorem 3.1. The system of real biharmonic functions.

$$(3.5) \qquad \{\varphi_n\} = \begin{Bmatrix} \varphi_0 = \dfrac{r^2}{4\sqrt{\pi}} \\ \varphi_{2n-1} = \dfrac{r^{n+2}\cos n\theta}{2\sqrt{2\pi(n+1)}} \\ \varphi_{2n} = \dfrac{r^{n+2}\sin n\theta}{2\sqrt{2\pi(n+1)}} \end{Bmatrix} \quad (n = 1, 2, 3, \cdots)$$

forms an orthonormal set (for functions of class L^2) on the unit circle C with respect to the inner product (3.3).

Proof: The set of functions (3.2) form an orthogonal set with respect to the inner product (3.3). Proof:

$$(3.6) \qquad \begin{aligned} &\iint_C \left[\Delta \psi_n(r,\theta)\, \Delta \psi_m(r,\theta) \right] dA \qquad (n \neq m) \\ &= \iint_C \left[(4n+4) r^n \cos n\theta\, (4m+4) r^m \cos m\theta \right] r\, dr\, d\theta \\ &= 16(n+1)(m+1) \int_0^{2\pi}\int_0^1 r^{n+m+1} \cos n\theta \cos m\theta\, dr\, d\theta \\ &= \frac{16(n+1)(m+1)}{n+m+2} \int_0^{2\pi} \cos n\theta \cos m\theta\, d\theta = 0 \end{aligned}$$

(This last integral may also contain products such as $\left[\sin n\theta \cos m\theta\right]$ or $\left[\sin n\theta \sin m\theta\right]$, but these integrals also vanish). Thus the set (3.2) is orthogonal.

To normalize our orthogonal set we find the normalizing factor (cf. Section I)

$$(3.7) \qquad \begin{aligned} \iint_C \left[\Delta \psi_n \right]^2 dA &= \int_0^{2\pi}\int_0^1 \left[(4n+4) r^n \cos n\theta \right]^2 r\, dr\, d\theta \quad (n \neq 0) \\ &= (4n+4)^2 \int_0^{2\pi} \frac{\cos^2 n\theta}{2n+2}\, d\theta \\ &= 8\pi(n+1) \end{aligned}$$

For the case $n = o$, the normalizing factor would be

$$(3.8) \qquad \iint_C [\Delta\psi_0]^2 dA = \int_0^{2\pi}\int_0^1 [4]^2 r\,dr\,d\theta$$
$$= 8\int_0^{2\pi} d\theta = 16\pi .$$

Dividing ψ_0 by $4\sqrt{\pi}$ and ψ_n by $2\sqrt{2\pi(n+1)}$ we have the orthonormal set (3.5).

Theorem 3.2. If $f(r,\theta)$ is of class L^2 on the unit circle C, the constants

$$(3.9) \qquad a_n = \iint_C \Delta f(r,\theta)\,\Delta\varphi_n(r,\theta)\,dA$$

are called the Fourier coefficients and possess the following minimum properties. Of all the linear combinations of the form

$$(3.10) \qquad \sum_{n=0}^{\infty} c_n \varphi_n(r,\theta)$$

the one which gives the best approximation to $f(r,\theta)$ with respect to the average quadratic error is $c_n = a_n$ ie, the integral

$$(3.11) \qquad \iint \left[\Delta\left(f(r,\theta) - \sum_{n=0}^{\nu} c_n\varphi_n\right)\right] dA$$

attains its minimum for $c_n = a_n$.[2]

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[2] Ibid., pp. 40-41.

Proof: We have

$$(3.12)\quad 0 \le \iint_C \left[\Delta\left(f - \sum_{n=0}^{\nu} c_n \varphi_n\right)\right]^2 dA$$

$$= \iint_C [\Delta f]^2 dA - 2\sum_{n=0}^{\nu} c_n \iint_C \Delta f \Delta\varphi_n \, dA + \sum_{n=0}^{\nu}\sum_{m=0}^{\nu} c_n c_m \iint_C \Delta\varphi_n \Delta\varphi_m \, dA$$

But the second integral is the Fourier coefficient (3.9) and the last integral is the inner product (3.3), hence

$$(3.13)\quad 0 \le \iint_C [\Delta f]^2 dA - \sum_{n=0}^{\nu} a_n^2 + \sum_{n=0}^{\nu} (a_n - c_n)^2$$

The minimum is obviously obtained for $c_n = a_n$.
This minimum is then

$$(3.14)\quad 0 \le \iint_C [\Delta f]^2 dA - \sum_{n=0}^{\nu} a_n^2 ,$$

or we may write

$$(3.15)\quad \sum_{n=0}^{\nu} a_n^2 \le \iint_C [\Delta f]^2 dA$$

Finally, since the right member is independent of ν, we have

$$(3.16)\quad \sum_{n=0}^{\infty} a_n^2 \le \iint_C [\Delta f]^2 dA$$

which is Bessel's inequality. Assuming that the system is closed, Bessel's inequality becomes $\sum_{n=0}^{\infty} a_n^2 = \iint_C [\Delta f]^2 dA$ which is Parseval's equality. The series $\sum_{n=0}^{\infty} a_n^2$ is absolutely convergent since it is a convergent series of positive terms.

The kernel function[3] for the orthogonal set $\{\varphi_n\}$, for any points (s,α) and (t,β) in C, is defined by

$$(3.17)\qquad K(s,\alpha,t,\beta) = \sum_{n=0}^{\infty} \varphi_n(s,\alpha)\,\varphi_n(t,\beta)$$

Theorem 3.3. The series $\sum_{n=0}^{\infty} \varphi_n(s,\alpha)\,\varphi_n(t,\beta)$ is an absolutely and uniformly convergent series for all (s,α) and (t,β) belonging to any closed subdomain C' contained in C. The kernel function (3.17) is a continuous function of 4 real variables, (s, α, t, β), may be operated on term by term by Δ, and is a biharmonic function.

Proof: Using Schwartz' inequality, series (3.17) becomes

$$(3.18)\qquad \sum_{n=0}^{\infty} \left|\varphi_n(s,\alpha)\,\varphi_n(t,\beta)\right| = \sum_{n=0}^{\infty}\left|\frac{s^{n+2}\cos n\alpha}{2\sqrt{2\pi(n+1)}}\;\frac{t^{n+2}\cos n\beta}{2\sqrt{2\pi(n+1)}}\right|$$
$$\leq \frac{1}{8\pi}\sqrt{\sum_{n=0}^{\infty}\frac{s^{2n+4}\cos^2 n\alpha}{n+1}\sum_{n=0}^{\infty}\frac{t^{2n+4}\cos^2 n\beta}{n+1}}$$

If we prove that both series under the radical in (3.18) are uniformly and absolutely convergent, then (3.17) will be uniformly and absolutely convergent. Let us examine one of the series under the radical

$$(3.19)\qquad \sum_{n=0}^{\infty}\frac{s^{2n+4}\cos^2 n\alpha}{n+1} \leq \sum_{n=0}^{\infty}\frac{s^{2n+4}}{n+1}$$

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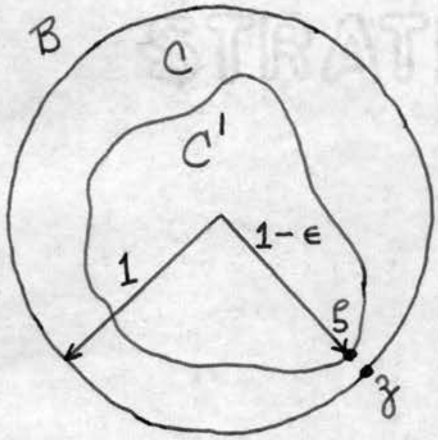
[3] Ibid., pp. 42-47.

follows because the $\cos^2 n\alpha \leq 1$. Also

$$(3.20) \quad \sum_{n=0}^{\infty} \frac{s^{2n+4}}{n+1} \leq \sum_{n=0}^{\infty} s^{2n+4} = \frac{s^4}{1-s^2}$$



The point (s,α) is an arbitrary point in C', and the maximum value of s would be $s = 1 - \epsilon$, where $\epsilon = \min|z-\zeta|$, z belongs to the boundary B of C and ζ belongs to C'. The number ϵ is positive because the closed set C' does not intersect the boundary B.

Replacing s by its maximum value in (3.20), we have

$$(3.21) \quad \sum_{n=0}^{\infty} \frac{s^{2n+4}\cos^2 n\alpha}{n+1} \leq \frac{(1-\epsilon)^4}{2\epsilon-\epsilon^2}, \quad 0 < \epsilon < 1,$$

and the series is absolutely and uniformly convergent. Similarly the other series in (3.18) can be shown to be absolutely and uniformly convergent. Therefore by the Weierstrass M-test, the series $\sum_{n=0}^{\infty} \varphi_n(s,\alpha)\,\varphi_n(t,\beta)$ is absolutely and uniformly convergent in any closed domain C' contained in C. The kernel function is a continuous function since it is the sum of a uniformly convergent series of continuous functions.

To prove that the series (3.17) may be operated on term by term, we must show that the series whose terms are $\sum_{n=0}^{\infty} \Delta\varphi_n(s,\alpha)\,\varphi_n(t,\beta)$ is a uniformly convergent series of continuous functions. By (3.18) this involves showing that the series $\sum_{n=0}^{\infty} \frac{\Delta s^{n+2}\cos n\alpha}{2\sqrt{2\pi(n+1)}}$ converges uniformly for $(s, \alpha) \subset C'$. We find that $\left(\Delta = \frac{\partial^2}{\partial s^2} + \frac{1}{s}\frac{\partial}{\partial s} + \frac{1}{s^2}\frac{\partial^2}{\partial \alpha^2}\right)$

$$(3.22)\quad \sum_{n=0}^{\infty} \left| \frac{\Delta s^{n+2}\cos n\alpha}{2\sqrt{2\pi(n+1)}} \right| = \sum_{n=0}^{\infty} \left| \frac{4(n+1)\, s^n \cos n\alpha}{2\sqrt{2\pi(n+1)}} \right|$$

and since

$$(3.23)\quad \sum_{n=0}^{\infty} \left| \frac{4(n+1) s^n \cos n\alpha}{2\sqrt{2\pi(n+1)}} \right| \le \sum_{n=0}^{\infty} \left| \frac{2(n+1)}{\sqrt{2\pi(n+1)}}\, s^n \right|$$
$$\le \sum_{n=0}^{\infty} \left| \frac{2}{\sqrt{2\pi}} (n+1)\, s^n \right|$$
$$= \sqrt{\frac{2}{\pi}} \sum_{n=0}^{\infty} \left| (n+1)\, s^n \right|$$

But we see that

$$(3.24)\quad \sum_{n=0}^{\infty} \left|(n+1)\, s^n\right| = \sum_{n=0}^{\infty} \left| \frac{d}{ds}(s^{n+1}) \right|,$$

and since s^{n+1} $(s < 1)$ is a convergent power series, the limit and the differentiation may be interchanged,[4] thus

$$(3.25)\quad \sum_{n=0}^{\infty} \frac{d(s^{n+1})}{ds} = \frac{d}{ds} \sum_{n=0}^{\infty} s^{n+1} = \frac{d}{ds}\left(\frac{s}{1-s}\right)$$
$$= \frac{1}{(1-s)^2}$$

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[4] Titchmarsh, Op. cit., p. 38.

Since $(s,\alpha) \subset C'$, we replace s by its maximum $1-\epsilon$, then

$$(3.26) \quad \sum_{n=0}^{\infty} \left| \frac{\Delta s^{n+2} \cos n\alpha}{2\sqrt{2\pi}(n+1)} \right| \leq \sqrt{\frac{2}{\pi}} \, \frac{1}{\epsilon^2}, \; (0 < \epsilon < 1).$$

Consequently (3.22) is an absolutely and uniformly convergent series of continuous functions. Therefore the series (3.17) may be operated on by Δ term by term. Similarly it may easily be verified that the kernel function is a biharmonic function in (s,α) and (t,β).

Theorem 3.4.[5] If $\{\varphi_n(r,\theta)\}$ is a system of orthonormal functions of class L^2 on the unit circle C, and if the sequence of numbers $\{a_n\}$ is such that

$$(3.27) \quad \sum_{n=0}^{\infty} a_n^2 < \infty,$$

then the series

$$(3.28) \quad g(r,\theta) = \sum_{n=0}^{\infty} a_n \varphi_n(r,\theta)$$

converges absolutely and uniformly in every closed subdomain $C' \subset C$ and represents a function $g(r,\theta)$ of class L^2 whose Fourier coefficients are equal to a_n, ie,

$$(3.29) \quad \iint_C \Delta g(r,\theta) \, \Delta \varphi_n(r,\theta) \, dA = a_n .$$

[5] Bergman, Op. cit., pp. 47-49.

If $\{\varphi_n(r,\theta)\}$ is a closed system with respect to functions of class L^2, then every function $f(r,\theta)$ of class L^2 may be represented in the form (3.28). The coefficients of this series are given by (3.29) and the series converges absolutely and uniformly in every closed subdomain $C' \subset C$.

Proof: We shall first show that the series (3.28) converges absolutely and uniformly in C'. We shall show that

$$(3.30) \quad \lim_{p\to\infty} \sum_{n=p}^{p+q} |a_n \varphi_n(r,\theta)| = 0, \quad (q = 1, 2, 3, \cdots)$$

independent of (r,θ) belonging to C'. By Schwartz inequality

$$(3.31) \quad \sum_{n=p}^{p+q} |a_n \varphi_n| \leq \sqrt{\sum_{n=p}^{p+q} |a_n|^2 \sum_{n=p}^{p+q} |\varphi_n|^2}$$

$$\leq \sqrt{\sum_{n=p}^{\infty} |a_n|^2 \sum_{n=0}^{\infty} |\varphi_n|^2}$$

and from (3.21), for any point (r,θ) in C',

$$(3.32) \quad \sqrt{\sum_{n=p}^{\infty} |a_n|^2 \sum_{n=0}^{\infty} |\varphi_n|^2} = \sqrt{\sum_{n=p}^{\infty} |a_n|^2 \frac{(1-\epsilon)^4}{2\epsilon - \epsilon^2}}$$

Letting $p \to \infty$

$$(3.33) \quad \lim_{p\to\infty} \sum_{n=p}^{p+q} |a_n \varphi_n| \leq \lim_{p\to\infty} \sqrt{\sum_{n=p}^{\infty} |a_n|^2 \frac{(1-\epsilon)^4}{2\epsilon - \epsilon^2}}$$

But by (3.27)

$$(3.34) \quad \lim_{p\to\infty} \sum_{n=p}^{\infty} a_n^2 = 0,$$

therefore (3.30) follows. By the Weierstrass convergence theorem the function $g(r,\theta)$ given by (3.28) is continuous in every closed subdomain C' of C.

Since the set $\{\varphi_n\}$ is biharmonic in C, the function $g(r,\theta)$ may be proved to be biharmonic in the same way as this result was proved for the kernel function (cf proof of Theorem 3.3).

To complete the proof that $g(r,\theta)$ is of class L^2 we must show that $\iint_C [\Delta g(r,\theta)]^2 dA < \infty$. Let

$$g_\nu(r,\theta) = \sum_{n=0}^{\nu} a_n \varphi_n(r,\theta) \tag{3.35}$$

then

$$\iint_{C'} [\Delta g]^2 dA = \iint_{C'} \lim_{\nu\to\infty} [\Delta g_\nu]^2 dA \tag{3.36}$$

and since the right integrand is a uniformly and absolutely convergent sequence of continuous functions, by the Lebesque convergence theorem equation (3.36) becomes

$$\begin{aligned} \iint_{C'} [\Delta g]^2 dA &= \lim_{\nu\to\infty} \iint_{C'} [\Delta g_\nu]^2 dA \\ &\leq \lim_{\nu\to\infty} \iint_C [\Delta g_\nu]^2 dA \quad (C' \to C) \\ &= \lim_{\nu\to\infty} \iint_C \Big[\sum_{n=0}^{\nu} a_n \Delta\varphi_n\Big] dA \\ &= \lim_{\nu\to\infty} \iint_C \Big[\sum_{n=0}^{\nu} a_n \Delta\varphi_n\Big]\Big[\sum_{m=0}^{\nu} a_m \Delta\varphi_m\Big] dA, \end{aligned} \tag{3.37}$$

which becomes by our orthogonality properties

$$(3.38)\qquad \iint_{C'} [\Delta g]^2 dA \leq \sum_{n=0}^{\infty} a_n^2 .$$

Let the domain C' $\longrightarrow$ C, then

$$(3.39)\qquad \iint_{C} [\Delta g]^2 dA = \lim_{C' \to C} \iint_{C'} [\Delta g]^2 dA \leq \sum_{n=0}^{\infty} a_n^2 .$$

Therefore $g(r,\theta)$ is of class L^2. We know if the system is closed, then equation (3.39) becomes $\iint_C [\Delta g]^2 dA = \sum_{n=0}^{\infty} a_n^2$, which is Parceval's equality.

In order to prove (3.29) we use Schwartz inequality and the orthogonal properties of the system (3.5)

$$(3.40)\qquad \iint_C \Delta g \Delta \varphi_m dA - a_m = \iint_C \left[\left(\Delta g - \sum_{n=0}^{\nu} a_n \Delta\varphi_n\right)(\Delta\varphi_m)\right] dA, \quad (\nu > m)$$
$$\leq \sqrt{\iint_C \left[\Delta g - \sum_{n=0}^{\nu} a_n \Delta\varphi_n\right]^2 dA \iint_C [\Delta\varphi_m]^2 dA}$$
$$= \sqrt{\iint_C \left[\Delta g - \sum_{n=0}^{\nu} a_n \Delta\varphi_n\right]^2 dA}$$

For any closed set C' C

$$(3.41)\qquad \iint_{C'} \left[\Delta g - \sum_{n=0}^{\nu} a_n \Delta\varphi_n\right]^2 dA = \lim_{\mu\to\infty} \iint_{C'} \left[\sum_{n=0}^{\mu} a_n \Delta\varphi_n - \sum_{n=0}^{\nu} a_n \Delta\varphi_n\right]^2 dA$$
$$\leq \lim_{\mu\to\infty} \iint_{C} \left[\sum_{n=0}^{\mu} a_n \Delta\varphi_n - \sum_{n=0}^{\nu} a_n \Delta\varphi_n\right]^2 dA, \quad (C' \to C)$$
$$= \lim_{\mu\to\infty} \iint_{C} \left[\sum_{n=\nu+1}^{\mu} a_n \Delta\varphi_n\right]^2 dA$$
$$= \lim_{\mu\to\infty} \iint_{C} \left[\sum_{n=\nu+1}^{\mu} a_n \Delta\varphi_n\right]\left[\sum_{n=\nu+1}^{\mu} a_m \Delta\varphi_m\right] dA$$
$$= \lim_{\mu\to\infty} \sum_{n=\nu+1}^{\mu} a_n^2 = \sum_{n=\nu+1} a_n^2$$

Letting $C' \longrightarrow C$, we have

$$(3.42) \quad \sqrt{\iint_C \Big[\Delta g - \sum_{n=0}^{\nu} a_n \Delta \varphi_n\Big]^2 dA} \leq \sqrt{\sum_{n=\nu+1}^{\infty} a_n^2}$$

and if $\nu \longrightarrow \infty$, then

$$(3.43) \quad \lim_{\nu \to \infty} \sqrt{\iint_C \Big[\Delta g - \sum_{n=0}^{\nu} a_n \Delta \varphi_n\Big]^2 dA} \leq \lim_{\nu \to \infty} \sqrt{\sum_{n=\nu+1}^{\infty} a_n^2} = 0$$

Since ν is arbitrary in (3.40) it follows that

$$(3.44) \quad a_m = \iint_C \Delta g \, \Delta \varphi_m \, dA.$$

Hence the numbers a_m are the Fourier coefficients of $g(r,\theta)$, which completes the proof of the first part of our theorem.

Let us now consider a given function $f(r,\theta)$ belonging to L^2. We assume that the system $\{\varphi_n\}$ is closed. Then it follows from Theorem (3.2) and the first part of this theorem that

$$(3.45) \quad \iint [\Delta f]^2 dA = \sum_{n=0}^{\infty} a_n^2, \quad \text{where} \quad a_n = \iint_C \Delta f \, \Delta \varphi_n \, dA$$

As we have just proved the series $\sum_{n=0}^{\infty} a_n \varphi_n(r,\theta)$ converges uniformly and absolutely for all (r,θ) belonging to any closed subset C' of C, and represents a function $g(r,\theta)$ of class L^2. We must show that this function $g(r,\theta)$ is identical to $f(r,\theta)$. It will suffice to show that

$$(3.46)\quad \lim_{\nu\to\infty}\iint_C \left\{\Delta\left[f(r,\theta)-\sum_{n=0}^{\nu} a_n\varphi_n(r,\theta)\right]\right\}^2 dA = 0 .$$

But (3.46) is an immediate consequence of the definition of a closed system (cf. Section I) and the minimum property of the Fourier coefficients.

Theorem 3.5. If $\{\varphi_n\}$ is a closed orthonormal system on the unit circle, any function $f(r,\theta)$ of class L^2 may be represented at any interior point (s,α) of the unit circle C by the series

$$(3.47)\quad f(s,\alpha)=\sum_{m=0}^{\infty}\varphi_m(s,\alpha)\int_0^{2\pi}\left[\Delta\varphi_m(1,\theta)\,\frac{\partial f(1,\theta)}{\partial r}-f(1,\theta)\,\frac{\partial\Delta\varphi_m(1,\theta)}{\partial r}\right]d\theta .$$

This series expresses the value of the function f at any interior point (s,α) of the circle C in terms of its value and the value of $\frac{\partial f}{\partial r}$ on the circumference of the circle.

Proof: The series in equation (3.46) may be written in the following form

$$(3.48)\quad f(s,\alpha) = \sum_{n=0}^{\infty}\varphi_m(s,\alpha)\iint_C \Delta f\,\Delta\varphi_m\, dA .$$

We transform this equation to rectangular coordinates by letting $F(x,y) = f(r,\theta)$ and $\Phi_m(x,y) = \varphi_m(r,\theta)$ and $dA = dxdy$, then

$$(3.49)\quad \iint_C \Delta f\,\Delta\varphi_m\, dA = \iint_C \Delta F\,\Delta\Phi_m\, dx\, dy$$

We now use the following form of Green's Theorem[6]

$$(3.50)\quad \iint_C (u\,\Delta v - v\,\Delta u)\,dx\,dy = \int_{+B}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)ds,$$

where n is the outer normal, +B is the positive direction along the boundary B of C, and ds is the element of arc length. Let $\Delta\Phi_m(x,y) = u$ and $F(x,y) = v$, then (3.50) becomes

$$(3.51)\quad \iint_C \left[\Delta\Phi_m\,\Delta F - F\,\Delta\Delta\Phi_m\right]dx\,dy = \int_{+B}\left[\Delta\Phi_m\frac{\partial F}{\partial n} - F\frac{\partial\Delta\Phi_m}{\partial n}\right]ds.$$

But $\Delta\Delta\Phi_m = 0$, so that

$$(3.52)\quad \iint \left[\Delta F\,\Delta\Phi_m\right]dx\,dy = \int_{+B}\left[\Delta\Phi_m\frac{\partial F}{\partial n} - F\frac{\partial\Delta\Phi_m}{\partial n}\right]ds.$$

If we transform back to polar coordinates, then

$$(3.53)\quad \iint_C \Delta f\,\Delta\varphi_m\,dA = \int_{+B}\left[\Delta\varphi_m\frac{\partial f}{\partial n} - f\frac{\partial\Delta\varphi_m}{\partial n}\right]ds.$$

On the unit circle C, the outer normal coincides with the variable r, ie., $\frac{\partial}{\partial n} = \frac{\partial}{\partial r}$, and ds = $d\theta$, thus (3.53) becomes

$$(3.54)\quad \iint_C \Delta f\,\Delta\varphi_m\,dA = \int_0^{2\pi}\left[\Delta\varphi_m(1,\theta)\frac{\partial f(1,\theta)}{\partial r} - f(1,\theta)\frac{\partial\Delta\varphi_m(1,\theta)}{\partial r}\right]d\theta$$

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[6] Courant, Op. cit., Vol II, p. 367.

Substituting (3.54) into (3.48), we have

$$(3.55)\quad f(s,\alpha) = \sum_{m=0}^{\infty} \varphi_m(s,\alpha) \int_0^{2\pi} \left[\Delta\varphi_m(1,\theta) \frac{\partial f(1,\theta)}{\partial r} - f(1,\theta) \frac{\partial \Delta\varphi_m(1,\theta)}{\partial r} \right] d\theta$$

which was to be proved.

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