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SHERVIN PAHLAVAN

Norman, Oklahoma

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NONLINEAR STABILITY ANALYSIS OF THIN-WALLED SECTIONS

USING THE FINITE ELEMENT METHOD

APPROVED BY

Thomas M. Murray

J. Narasimha Reddy

Fulton K. Fears

Lloyd A. Iverson

DISSERTATION COMMITTEE

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LIST OF SYMBOLS

A	= differential operator
a	= length of plate
B	= displacement to strain transformation matrix
B_o	= displacement to strain transformation matrix for small deflection
B_c	= incremental displacement to strain transformation matrix
B_u	= displacement to strain transformation matrix, in-plane action
B_{uw}	= displacement to strain transformation matrix for interaction part of stiffness matrix
B_L	= displacement to strain transformation matrix for nonlinear terms
b	= width of the plate
C	= differential operator
D	= rigidity matrix
E	= modulus of elasticity
F	= generalized force
$\{F\}^e$	= generalized nodal forces for one element
G	= intermediate equation for computation of geometric matrix
J	= Jacobian matrix
J_2	= second invariant of stress deviator tensor

$\{K\}$ = stiffness matrix for the structure

K^e = element stiffness matrix

K_c = conventional stiffness matrix

K_g = geometric stiffness matrix

K_g^m = membrane geometric stiffness matrix

K_g^b = bending geometric stiffness matrix

K_L = nonlinear stiffness matrix

M = stress matrix

N = shape function for in-plane action

P = load vector

q = displacement vector

q_p = vector of in-plane displacements

q_b = vector of out-of-plane displacements

$\{q\}^e$ = vector of nodal displacements for one element

R = residual

T_x, T_y, T_{xy} = stress resultants per unit length of the boundary of the element

t = thickness

U = approximation assumed for the unknown function, internal work or strain energy, displacements in the x and y direction.

u_i = displacement in the x direction at the node i

u = displacement in the x direction at any point, unknown function

v = displacement in the y direction at any point

v_i = displacement in the x direction at the node i

W = external work

w = deflection in the z direction

$x, y,$ = coordinates an any point

x_i, y_i = coordinates at node i

α = coefficients of polynomial for bending element

$\theta_{x_i}, \theta_{y_i}$ = rotations at node i, about x and y axes

θ_x, θ_y = rotations at any point, about x and y axes

ξ, η = natural coordinates for isoparametric element

ϕ = shape functions for out-of-plane action, angle of twist

χ = curvature

δ = incremental value

σ = stress

ρ = weight function

ψ = coefficients of the terms in the assumed function, coordinate matrix

λ = proportionality constant

ν = Poisson's ratio

ε_p = axial strain due to in-plane loading

ε_b = axial strain due to bending

ABSTRACT

A finite element procedure was developed for analysis of thin plates and thin-walled sections. The procedure is based on the large deflection theory and geometrical nonlinearities were considered. The displacement approach of finite element method was followed. The incremental method was used for solution of nonlinear equations and the effect of membrane stresses was included by means of geometric stiffness matrices. Most of the stiffness matrices were computed by numerical integration. The procedure can be applied to the problems of in-plane and out-of-plane actions and also to the problems of demonstrating combined behavior, thus it is applicable to bending, buckling and post-buckling problems. The formulation was specialized to two types of elements selected from the literature. A computer program was developed and using the program, example problems were solved. Load-deflection behavior is shown for the solved problems and buckling load is determined from the load-deflection diagram. Also the failure load was roughly estimated considering the failure to be first yield. The results of the estimation are very close to failure loads predicted by other methods. Results obtained for the buckling load and post-buckling behavior are in good agreement with the existing solutions and for most of the problems very few elements were required to obtain adequate results.

NONLINEAR STABILITY ANALYSIS OF THIN-WALLED SECTIONS USING THE FINITE ELEMENT METHOD

CHAPTER I

INTRODUCTION

1.1.1 Conventional Analysis

For many years practicing engineers conducted analyses on the basis of the linear theory. In the linear theory it is assumed that deflections are very small, thus the geometry of the structure does not change significantly during the loading process, and linear stress approximation is applicable. The linear theory also assumes that materials are linearly elastic, hence the constitutive matrix remains constant and independent of the load level. Utilizing these assumptions, simplified procedures were formulated to obtain solutions to engineering problems.

In the conventional method of analysis, it is also assumed that different failure modes are independent and each one can be studied separately. Yielding of the cross section due to bending, for instance, is assumed to be independent of local buckling. Analysis of local buckling is based on simple plate buckling theory and in the study of member buckling, distortion of the cross section is neglected.

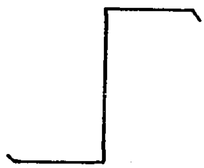
Although behavior of virtually every structure is nonlinear, the linear theory yields sufficient accuracy in many problems. The linear theory is applicable when deflections of structures at working loads are small and the material behaves linearly elastic. The uncertainty and approximation of linear theory can be tolerated with use of high factors of safety, thus the stresses and deflections are much less than the allowable limits.

1.1.2 Present Study

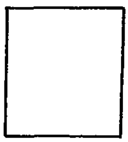
There are many cases which require nonlinear analysis; instances where strains are small but deflections are relatively large and the deformation of the structure is affected by interaction of load and deflection. For such cases the determination of stresses requires consideration of nonlinear behavior. Post-buckling analysis of thin-walled sections fall into this category. Another example is when different phenomena occur simultaneously, i.e., where effects are coupled and must not be considered separately. For example, in the bending of a beam of thin-walled cross section, it may not be realistic to assume that flexure is independent of local buckling.

In the present work the approach is based on the finite element method. Large deflections are accounted for in the derivation of the element stiffness matrix; it is possible to consider different boundary conditions and complicated loading. Of course the method needs an extensive amount of numerical computation and can be handled only with computer facilities. The method can be applied to thin plates and sections composed of thin plates (see Figure 1-1).

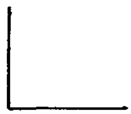
a. channels
simple or lipped



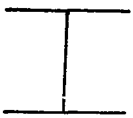
b. Z section



c. box section



d. angle



e. I section

Fig. 1-1. Examples of the Thin-Walled Cross Sections

Loading may be in-plane or transverse. Failure modes may be bending or buckling or, depending on the shape of the cross section, a combination of bending and local buckling.

1.2 Literature Survey

1.2.1 Classical Methods

Exact solutions are available for linear problems related to bending and buckling of bars. In the case of buckling, Euler was the first to develop a theoretical solution for prismatic bars [52]. The linear plate problems of bending and buckling, have been solved assuming an infinite series representation for the deflection of the plate [53, 54]. However, analysis of a structural member by a linear classical method suffers from a number of limitations. It is usually assumed that the cross sections do not distort and buckling occurs from the initial configuration. These assumptions result from separate investigation of different modes. Even with the above simplification the closed form solution of the plate differential equation is mathematically complex and it is only available for a limited number of simple problems and boundary conditions.

Using classical methods attention has also been given to some geometrically nonlinear problems. Several types of simple plate bending problems have been solved in this category [53], but for most cases only approximate solutions exist. Post-buckling behavior of some simple plate problems have also been solved using the large deflection method [9] where out-of-plate deflection is assumed to be zero before the bifurcation load is attained. In this manner, it is determined that, for thin plates, the bifurcation load may be much smaller than the failure load [54].

For computation of post-buckling strength, an exact solution of the governing equation is not available; hence a semi-empirical method called the "effective width" concept [20, 58, 60] has been developed. In many other cases difficulties of obtaining exact solutions have led the investigators and designers to consider approximate methods of analysis.

1.2.2 Numerical Methods Other Than The Finite Element Method

For bending and buckling problems of even moderate complexity, a numerical method must be adopted. Also when the problem involves complex geometry, material properties and boundary conditions, solution is only possible with the aid of a numerical method. The best known of these may be the finite difference method, in which the differential equation is approximated by discrete values of the variable at selected points [17]. The discretization results in a system of algebraic equations whose solution yields the approximate values of the unknowns at the base points. When the differential equation is nonlinear the system of finite difference equations is also nonlinear.

Other well-known numerical procedures are commonly grouped as "weighted residual methods," such as Ritz, Galerkin and Least Square methods. Suppose that the governing differential equation can be written in the operator form $Au = F$, where u is the unknown function, A is a differential operator, and F is a generalized force. If an approximation for u is assumed, say U, then the governing equation becomes $AU - F = R$, where R is the residual of the approximation. Since the assumed function is not exact, in general, the residual will not be equal to zero. The weighted residual methods seek the solution by requiring that some weighted integral of the residual over the domain under consideration be zero: $\int_R R\rho dR = 0$, where ρ is a weight function.

In the Galerkin method the weight is taken to be the trial function used to represent $U = \Sigma\psi_i U_i$. Then $\int_R R\,(\Sigma\psi_j U_j)\,\psi_i\,dR = 0$ for every $i = 1, \ldots, N$.

The least squares method is based on minimizing the integral of the square of the residual or $\frac{\partial}{\partial U_k} \int_R R^2\,(\Sigma U_j \psi_j)\,dR = 0$; here the weight function is $\rho = 2R \frac{\partial}{\partial U_k} (R\,(U_j \psi_j))$.

In the Ritz method, it is assumed that the solution can be represented by a linear combination of simple functions, each function has to satisfy the given boundary conditions. First, the problem is formulated as definite integrals, then the desired unknown function is substituted as a linear combination. Finally, the functional is minimized with respect to the arbitrary coefficients in the linear combination.

The above numerical methods may be used in the analysis of different types of structures. They were originally developed for hand computation and recently adopted to modern digital computers

[19, 11]. The finite element method is a product of computer era and it utilizes variational methods to construct approximate solutions at the element level.

1.3 The Finite Element Method

1.3.1 General

The finite element method can be programmed in a systematic way and it can be adjusted to incorporate nonlinearities, complex geometries and boundary conditions, which are more difficult to accomodate in other numerical methods. The basic concepts of the finite element method was discussed in a very important paper [55] in 1956. Since then there has been much effort toward development and application of the method. The basic concept of finite element method is that the structure can be modeled as an assemblage of a number of subregions, called finite elements. The solution over each element is described by a set of assumed functions. The assumed functions are chosen in such a form to insure certain properties like continuity of the behavior of the structure, inclusion of rigid body modes (displacements), constant strain and curvature state. However, satisfactory solutions have been obtained [63] from elements which do not meet all the aforementioned requirements. Stiffness matrices have been formulated for different types of problems and behavior [4, 14, 24, 33, 36, 47]. In some cases such as buckling problems or nonlinear analysis, the stiffness matrix has to be modified. The modification takes place by adding a corrective matrix which is called the geometric stiffness or initial stress matrix. It is based on the physical consideration that the presence of in-plane loads (stresses) influence subsequent deflection of an already deflected structure. Depending on the nature of stresses and deflections, the initial stress matrix may increase or decrease the stiffness of the structure.

The concept of geometric matrix was first introduced in reference [56]. There the derivation was based on a strain energy formulation. Later on a purely geometrical consideration was used to derive the matrix [4]. Although buckling and nonlinear problems are completely different in theory, some similarity exists in utilizing the geometric matrix concept.

1.3.2 Buckling and Post-Buckling Problems

Considerable literature is available on the use of the finite element method for eigenvalue buckling problems [7, 24, 28, 30, 43]. For this class of problems it is normally assumed that the member is perfectly straight, it has a plane of symmetry and is loaded in that plane. It is also assumed that there is no lateral or torsional displacement until the critical load is reached. In lateral-torsional buckling of beams, deflection about the major axis is neglected. The procedure is called linearized stability [24] for which the matrix formulation may be expressed as

$$P = (K_c + \lambda K_g)\ q$$

where K_c = conventional stiffness matrix, K_g = geometric stiffness matrix, q = displacement vector, p = load vector. At bifurcation load neutral stability must exist or

$$(K_c + \lambda K_g)\ q = 0$$

with the solution

$$\det (K_c + \lambda K_g) = 0$$

The above formulation requires small deflection assumptions. The analysis must be conducted in two steps: a prebuckling analysis in which a small portion of the load to be carried by the structure is applied and "initial stress" computed, then the matrix K_g is formed. The conventional stiffness matrix K_c is assumed to remain constant during the loading process and the geometric matrix at each increment is directly proportional to the applied load with λ being the proportionality factor. Solution yields the buckled shape and the buckling load is equal to the lowest eigenvalue multiplied by the applied load in the prebuckling stage.

The above scaling procedure may be applied successfully to problems which exhibit linear behavior up to the point of failure. It is equivalent to classical Euler buckling formulation in which prebuckling deflections are neglected and buckling is assumed to occur from initial configuration. Nevertheless, the method has some advantages over classical methods such as treating load and geometric irregularities and nonisotropic materials. As will be seen later, nonlinear analysis is also based on the use of the geometric matrix, hence the eigenvalue buckling analysis gives an insight into nonlinear analysis by a matrix method.

In most of the buckling problems some deflection exist from the very beginning, hence the bifurcation load is not meaningful. Also in the problems where deflections are relatively large the change in the geometry cannot be neglected.

Several problems in elastic stability have been solved using a linear theory. In reference [7], using the displacement finite element method, stiffness matrices are formulated for torsional and lateral stability of structural members. The elements are beam segments with two nodes, every node having seven degrees of freedom, [u, w, ψ, v, θ, ϕ, x] where u, v, w are deflections in the x, y, z direction, ϕ = angle of twist, $\theta = \frac{dv}{dx}$, $\psi = \frac{dw}{dx}$ and $x = \frac{d\phi}{dx}$. Then 14 by 14 geometric and conventional stiffness matrices are derived and an eigenvalue problem is formulated whose solution yields the buckling load. The following examples are studied in the mentioned reference: torsional buckling of an axially loaded uniform member where linear displacements are constrained and angular displacements are free; lateral buckling of a narrow rectangular beam subjected to equal end moments; lateral buckling of a cantilever beam subjected to a concentrated load at the shear center; buckling of a simply supported beam with different loading conditions such as distributed or concentrated load at the top flange, at the bottom flange and at the shear center; stability of a circular shaft under conservative torque.

Results for the above problems converged to the classical solutions, whenever available, as the mesh was refined. Better results were obtained for problems governed by flexural and lateral instability relative to those governed by torsion.

Reference [43] gives solutions for lateral buckling of steel beams. The method is the same eigenvalue procedure as mentioned previously. The method uses beam segments with two nodes and lateral displacement, torsional rotation, lateral bending and warping are selected as degrees of freedom at each node. Analyses are performed for a number of examples including simply supported beams under end moments, concentrated load, distributed load. Also, analyses for a series of two span aluminum beams have been carried out with the results given in the form of an interaction diagram. A solution is also presented for a continuous beam with different cross sections in different sapans. The results are in close agreement with those given by classical or experimental methods.

Stability of plates by finite element method is considered in reference [30] where linearized buckling analysis is performed for square and rectangular plates under compressive loads in one or two directions; plates under combined bending and compression and under pure shear; orthotropic plate under uniform compression in one direction. With fine meshes good agreement is obtained compared to other approximate solutions such as Raleigh-Ritz and finite difference method.

In reference [28] using the finite element method, a solution is presented for stability problems of beams. The beams are divided into plate elements. Double symmetric sections (rectangular, wide flange, I) are considered. Numerical examples include: buckling of axially loaded column; lateral buckling of a simpl. supported I beam with a concentrated load at the centroid at the midspan and the same beam loaded at the top flange; cantilever beam loaded at the centroid at the free end; continuous beams with stiffeners and bracing at the top and bottom flange at midspan also with braces only at the compression flange.

Linearized stability discussed in the above references, where applicable to many engineering problems is not sufficient for the cases where the stability of a critical equilibrium configuration or post-buckling behavior must be considered. In such cases the need for a more accurate theory has resulted in considerable effort for the solution of nonlinear problems.

To date much progress has been made in the field of nonlinear analysis. In reference [49], a procedure is formulated for the solution of geometric and material nonlinearity but complete derivation of stiffness matrices and numerical examples are not given. Reference [33] formulates the general nonlinear problems by potential energy, direct and incremental method, two corrective matrices are derived which are denoted by N_1 and N_2 and are called the first order and second order stiffness matrices. The mentioned paper also gives a useful explanation of nonlinear analysis, but numerical results are not presented; in reference [47], the finite element procedure is formulated for the solution of inelastic beam and beam column problems by using one dimensional elements and replacing elastic modulus by the tangent modulus. The formulation is accompanied by numerical results for a cantilever beam-column subjected to doubly eccentric axial load and a beam column with residual stress. Reference [36] gives a general discussion and formulation of geometrically nonlinear problems using one dimensional elements for beam-columns and plane stress triangular elements for plates. Elasto plastic solution of plane stress and plane strain problems by finite element method are considered in reference [42] where results are given for a rectangular panel under tensile load. Plane stress, plane strain and axisymmetrically loaded body of revolution in the nonlinear range (geometric and material) are discussed. In reference [34] numerical results are given for a thick cylinder under internal pressure, a plate under tension with central hole and a notched tension specimen. Formulation of geometrically nonlinear problems under uniform heating with large temperature changes are presented in reference [56]. Stiffness matrices for truss and plane stress problems are given with no numerical results. The plane stress relationship for elastic-plastic material is developed in reference [62] and the procedure is used to solve plane stress problems such as a perforated tension strip and cantilever beam, the method is extended [64] to axisymmetric problems of large deflection and plasticity. Plastic bending problems of plates, assuming small deflection, are solved in reference [8]. Solution to elastic-plastic problems of axially compressed cylinders and columns are presented in reference [35].

Although the above references have contributed greatly to nonlinear analysis, still it is not possible to include geometric and material nonlinearity in a routine manner. To date, solved problems involving elastic-plastic material properties have been limited to one dimensional elements, plane stress or plane strain, axisymmetric members using isoparametric elements, or plastic bending of plates with small deflections. Solutions for complex problems with combined geometric and material nonlinearity and considering buckling have not been presented. The available rigidity matrices required for these problems are not efficient and the computational effort required for even relatively simple problems is enormous. For this reason, the present work is concerned with only geometric nonlinearities, however, relatively complex problems are solved as a first step to developing a complete solution. The incremental method is used wherein the nonlinear problem is replaced with a piecewise linear series of solutions. For every increment of load, geometry is assumed to remain constant, and a new tangential stiffness matrix is formed and deflection is obtained for that increment.

1.3.3 Elements

In the early application of finite element method to stability problems, one dimensional elements were used [4, 7, 36, 51]. Although trusses and some beam-columns may be represented adequately by one dimensional elements, the model seems to have some deficiencies for thin-walled structures. One dimensional elements, cannot take into account the complete geometry, loading and local behavior. Two cross sections with the same moment of inertia and cross-sectional area may have different shapes (e.g., one may be symmetrical, the other one nonsymmetrical), hence, different load-deflection behavior. A finite element mesh in the case of one dimensional element is obtained by dividing the structural member through the length into a number of elements. Every element has two nodes at the two ends. Displacements, rotations, angle of twist and warping are taken as degrees of freedom.

Dividing the member into two dimensional elements may result in a more accurate model [28, 30], since distortion of the cross section and consideration of local and member behavior can be handled simultaneously. When using two dimensional elements the geometric matrix can be obtained from the interation of in-plane stresses and out-of-plane deflections [13, 22, 30]. A more precise formulation takes into account both in-plane and out-of-plane deformations [27].

Two dimensional elements while possessing the advantage of better representing the behavior of the structure have also some disadvantages over one dimensional elements. The increased number of nodes increases the size of stiffness matrices and consequently the computational effort increases. The stress-strain and strain-deformation relationships are not as simple as one dimensional cases.

1.3.4 Plasticity

There are some certain limits of stresses beyond which the stress-strain behavior of material is nonlinear. These limits are defined according to plasticity theories. Plasticity problems are also studied by the use of finite element method. In one approach a linear variation of plastic strain is assumed over the element [22]. For some cases, this procedure gets extremely difficult [8] . Besides it is known that plastic strain is not a continuous function over the surface and through the thickness of the element. In reference [6] a method is presented for finite element solution of elasto-plastic material. In that paper the authors assumed a linear variation of plastic strain between the nodes and in addition assumed that the plastic strain varies linearly from its value on the lower or upper surface to some elastic plastic boundary in the cross section. In a later paper [5] the same authors discarded the mentioned method and suggested the use of well-known plasticity theories.

In reference [64] nonlinear material and geometry is discussed and incremental flow theory of plasticity is used to present material behavior. The formulation was applied to two and three dimensional isoparametric elements. In that paper bending and buckling is not discussed. In fact, the constitutive law given for two or generally three dimensional cases may not be simply reduced to the case of bending. Apparently numerical integration was necessary which will spoil the rather simple form of the constitutive law.

Inclusion of nonlinear material behavior in the case of one dimensional element may be possible by means of simple modification using tangent modulus instead of elastic modulus [47]. For two and three dimensional elements confusion still exists about stress distribution and stress-strain relationship [25]. Little information on nonlinear material is available in the literature and finite element solutions are not of uniformly acceptable quality. It is known that for nonlinear material, the principle of superposition is not valid, hence the analysis becomes more complex. The chances of obtaining closed form solution to specific problems are fairly remote.

Numerical solution usually reduces a nonlinear problem to a piecewise linear one. In the case of nonlinear material it means adjusting the rigidity matrix at the end of each increment and keeping it constant during the next increment. Based on the plasticity theories two approaches are available for computation of rigidity matrix. The two widely accepted plasticity theories are incremental flow theory and deformation theory. Deformation theory gives the relationship between total stress and total strain [26] while incremental flow theory gives the relationship between incremental values of stress and strain [35, 38]. The first one has a simpler form while the latter is theoretically more acceptable [37]. As discussed previously the present work is concerned only with geometrical nonlinearity, however, formulation for both plasticity theories is given in Appendix C.

Since the distribution of stresses at a yielded point in a material is completely different from the other points in its neighborhood, a large number of integration points will be necessary on the surface and through the thickness of each element. Computation of stresses at the nodal points follows the same procedure. All these values must be stored and then used in later computations. Hence for large problems the storage location for material nonlinearity may be needed for 50,000-100,000 values. These computations must be repeated in each step. Working with the deformation theory may simplify the required operation to some extent. The only difference being the stress-strain relationship which is given for total values rather than incremental values. But still most of the above mentioned difficulties exist.

At the present time, simultaneous treatment of large deflection and plasticity results in a computational effort which is extremely large and uneconomical. In the present work using large deflection method for thin-walled steel members, in the elastic range a very good approximation is obtained for elastic buckling, post-buckling behavior and even failure load by assuming the failure at the first yield.

1.4 State-of-the-Art Summary

Although much attention has been devoted to linearized buckling theory, which is a numerical equivalent of the classical method, very few practical problems exist which may follow this theory. Examples are a perfectly straight column under axial loads or a plate without imperfection under in-plane loads. In reality, the ideal case of a perfect structure with a perfectly centroidal or in-plane load may not occur very often. Therefore every load will cause some deflection.

For thin-walled sections, the relative magnitude of pre-buckling deflections are significant and cannot be neglected. The same comment applies to bending of thin plates, where transverse deflection and the resulting membrane forces have considerable effect on the overall behavior. Also cross-sectional distortion (local buckling) which will change the geometrical properties and stress distribution may not be treated separately.

Very little work has been done on the theoretical investigation of post-buckling behavior. In fact, complicated problems of post-bifurcation have not been solved theoretically and the "effective width" concept produces reasonable results in only a semi-empirical way. In the very few simple cases where the solution for post-buckling behavior exists, assumption of bifurcation may cause some inaccuracy in the results of the post-buckling analysis.

Most of the solutions in the literature refer to a member with a plane of symmetry which is not the case for thin-walled zee and channel sections.

Solution procedures presented in the literature for buckling, post-buckling and some bending problems are inadequate. Generally for the problems in which the behavior depends on the load level, nonlinear analysis must be followed. So far no unique treatment for the general nonlinear problem has been presented in the literature. In spite of theoretical and experimental investigations the matter is not clear enough. The general nonlinear problem is still under extensive research. Although in references [39, 40], a formulation for nonlinear structural analysis is presented, computation or detailed derivation of any of the matrices is not given. In the following section a method for solution of nonlinear structural problem is proposed.

1.5 Solution Method

Here the finite element method is used to solve problems of geometrical nonlinearity. The method has a very apparent physical interpretation while it is strongly supported by basics of mechanics.

Most important of all it is cast into matrix formulation which makes possible the use of standard matrix structural analysis. The member is idealized by two dimensional elements which makes possible the simultaneous treatment of local and structural action and gives a better representation (relative to one dimensional mesh). The displacement method is adopted in the present work. It is particularly suited to nonlinear analysis because geometric nonlinearities may be incorporated through displacement formulation directly [31, 33]. For displacement models two approaches are available [49, 64]:

1. Eulerian formulation or moving coordinate system,
2. Lagrangian formulation or fixed coordinate system.

The latter is used here for being more straightforward.

Retaining nonlinear strain-displacement terms and assuming large deflection requires a nonlinear stiffness matrix and also a geometric stiffness matrix. Both are variable and depend on the load level. The stiffness matrices contain bending, membrane and counteraction components.

For the solution of nonlinear matrices incremental technique is used. In this manner an increment of load is applied to the system and deflections computed, then the existing geometry is considered in computation of conventional and geometric matrices for the next increment. For buckling and post-buckling analysis some imperfection in the structural member is needed in order to avoid bifurcation. Here the required deflection is imposed by applying a very small concentrated load in the transverse direction at a critical point.

Details of theoretical formulations are presented in Chapter II, selected elements are discussed in Chapter III and results of numerical studies are given in Chapter IV.

CHAPTER II

ANALYSIS PROCEDURE

2.1 Formulation in Compact Form

2.1.1 Governing Equations of Equilibrium

A finite element formulation may be developed assuming displacements, stresses or both displacements and stresses (a mixed procedure). In the displacement formulation, one may start with the virtual work principle which states that for a body in equilibrium, the algebraic sum of all work done during a virtual displacement is equal to zero or mathematically

$$\delta U = \int_{vol.} \delta\varepsilon \ \sigma \ dv = \delta W \tag{1}$$

where δ denotes incremental value, W = external work by the applied load, σ = stress, U = internal work or strain energy.

Writing equation (1) for a typical element and performing standard manipulation leads to the stiffness equation for the element

$$\{K\}^e \{q\}^e = \{F\}^e$$

where $\{K\}^e$ = element stiffness matrix, $\{q\}^e$ = nodal displacements, $\{F\}^e$ = generalized nodal forces for one element.

The stiffness equation may then be assembled using a direct stiffness approach to obtain the structural stiffness equation.

$$\{K\} \ \{q\} = \{F\}$$

where {K}, {q}, {F} are stiffness matrix, generalized nodal displacements and generalized nodal forces respectively, for the entire structure.

For small deflection assumptions {K} is linear e.g. bending of a beam or a plate may be formulated as a linear function of the applied load; for large deflection {K} is nonlinear and depends on the displacement and load levels and, thus, may be expressed as {K} = {K ({U}, {F})}, a typical nonlinear relationship is shown in Figure 2-1.

2.1.2 Displacement Model

As was mentioned in Chapter I, geometric nonlinearities can be incorporated most readily within a displacement formulation, and since only geometric nonlinearities are considered in this study the displacement approach is used here. The strain expression needed for manipulation of equation (1) may be obtained from the definition of Lagrangian strain tensor:

$$\varepsilon_{ij} = \frac{1}{2} (U_{i,j} + U_{j,i} + U_{m,i} U_{m,j}) + \frac{1}{2} W,_{i} W,_{j} + Z \chi_{ij} \qquad (2)$$

where ε = strain tensor, U = in-plane displacements, W = out-of-plane displacements, Z = distance of the point from the middle plane, χ_{ij} = curvature tensor.

This expression is valid for small and large deflection [46, 64] , and also for combination of bending and in-plane action.

The displacement of any point may be represented as a linear combination of nodal displacements. Hence, if the vector of in-plane nodal displacements is designated $\{q_p\}$ then U = {N} $\{q_p\}$ where {N} is the matrix of shape functions. Similarly w = $\{\phi\}$ $\{q_b\}$ where w = out-of-plane displacement at a point, $\{q_b\}$ = vector of out-of-plane nodal displacements and $\{\phi\}$ = matrix of shape functions.

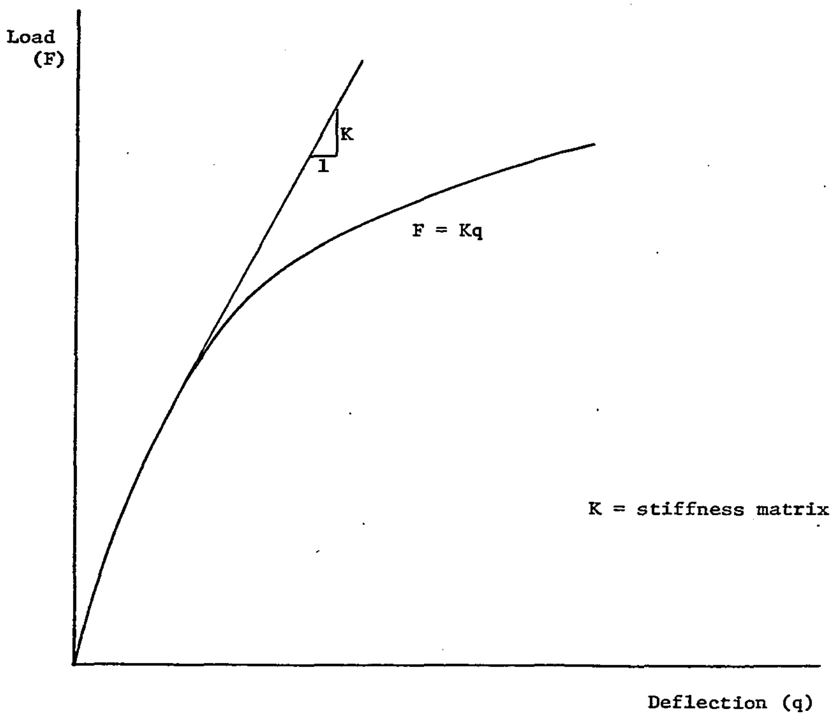


Fig. 2-1. Nonlinear Relationship

Between Load Deflection

Writing $\{\varepsilon\} = \{C\}\ \{U\}$, where $\{C\}$ is a differential operator assuming surface traction per unit area of the element is $\{T\}$, substituting in the virtual work expression and carrying out the finite element manipulation, the following relationship is obtained:

$$\int_V \{B\}^T\ \{\sigma\}dv - \{F\} = 0 \qquad (3)$$

Here $\{F\}$ denotes generalized nodal forces, and with the previously defined nolations $F = \int_A \{\phi\}^T\ \{T\}\ dA$ where the integration is carried over the area of a typical element. $\{B\}$ = displacement to strain transformation matrix for incremental values or

$$\delta\ \{\varepsilon\} = \{B\}\ \delta\ \{q\}$$

For large deflection $\{B\}$ is nonlinear and depends on the displacements or

$$\{B\} = \{B(U)\}$$

Taking the first variation of equation (3) results in

$$\int_V \delta\{B\}^T\ \{\sigma\}\ dv + \int_V \{B\}\ \delta\ \{\sigma\}\ dv = \delta F \qquad (4)$$

$\{\sigma\}$ may be expressed as $\{D\}\ \{\varepsilon\}$ where $\{D\}$ is the constitutive matrix.

The strain-displacement matrix $\{B\}$ is obtained from the variation of $\{B_o\} + \{B_L\}$ where $\{B_o\}$ is related to the linear terms and $\{B_L\}$ is related to the nonlinear terms. Hence $\{B_o\}$ is constant between two displacement configurations and

$$\delta\ (\ \{B_o\} + \{B_L\}\ ) = \delta\ \{B_L\}$$

Since only nonlinear geometry is considered here, the stress vector of equation (4) can be obtained as a linear combination of strains. The two integrals on the left hand side of equation (4) result in two matrices. The first one is independent of material properties and depends only on the stress level it is called the initial stress or geometric stiffness matrix; the second matrix depends on the displacements, hence, it is nonlinear and may be called the large deflection stiffness matrix. Finally, the complete formulation for geometric nonlinearity is given as:

$$(\{K_g\}+\{K_L\})\,\delta\,\{q\} = \delta\,\{F\} \qquad (5)$$

where $\{K_g\}$ = geometric stiffness, K_L = large deflection stiffness and other notations have been defined previously.

The major step in deriving the two stiffness matrices mentioned in the above, consists of obtaining matrix {B} and {B} results from the first variation of $\{B_L\}$. It is seen that for the first integral, first variation of {B} is also needed. The values along with stress-strain expressions when substituted in equation (4) will give the stiffness matrices whose details are presented in the next section.

2.2 Details of Element Stiffness Matrices

To develop the terms of geometric and large deflection matrices, a two-dimensional strain-displacement expression for combined in-plane and bending action is written which is the expanded form of equation (2).

$$\{\varepsilon\} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix} + \frac{1}{2}\begin{bmatrix} (\frac{\partial u}{\partial x})^2 + (\frac{\partial v}{\partial x})^2 + (\frac{\partial w}{\partial x})^2 \\ (\frac{\partial u}{\partial y})^2 + (\frac{\partial v}{\partial y})^2 + (\frac{\partial w}{\partial y})^2 \\ 2\,(\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}) \end{bmatrix} - Z\begin{Bmatrix} \frac{\partial^2 w}{\partial x^2} \\ \frac{\partial^2 w}{\partial y^2} \\ \frac{2\,\partial^2 w}{\partial x\,\partial y} \end{Bmatrix} \qquad (6)$$

The second matrix on the right hand side contains nonlinear terms and is denoted as

$$\{\varepsilon_L\} = \frac{1}{2} \{H\} \{\theta\}$$

where

$$H = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} & \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \end{bmatrix} \qquad (7)$$

$$\text{and } \{\theta\}^T = \left\lfloor \frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial w}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}, \frac{\partial w}{\partial y} \right\rfloor$$

Taking the first variation

$$\delta \{\varepsilon_L\} = \delta \left(\frac{1}{2} \{H\} \{\theta\}\right) = \frac{1}{2} \delta \{H\} \{\theta\} + \frac{1}{2} \{H\} \delta \{\theta\}$$

and writing

$$\theta = \{G\} \{q\}$$

one obtains

$$\delta \{\varepsilon_L\} = \{H\} \{G\} \delta \{q\} \qquad (8)$$

Since $\{B\}$ is related to incremental strain-displacement expression, one may proceed by taking variation of total strain as

$$\delta \{\varepsilon\} = \delta \{\varepsilon_o\} + \delta \{\varepsilon_L\} + \delta \{\varepsilon_b\}$$

in which ε_b is the axial strain due to bending, $\{\varepsilon_o\}$ = strain for small deflection assumptions, $\{\varepsilon_L\}$ = additional values due to large deflection assumptions. Utilizing shape functions and nodal displacements, the above equation is written as:

$$\delta \{\varepsilon\} = \{B_o\} \delta \{q\} + \{B_L\} \delta \{q\} + \{B_b\} \delta \{q\} \qquad (9)$$

Where the displacement to strain transformation matrices for incremental values are defined as: $\{B_o\}$ = Linear transformation matrix, $\{B_1\}$ = non-linear transformation matrix and $\{B_b\}$ = bending transformation matrix.

Comparing (8) and (9), it is concluded that $\{B_1\} = \{H\}\ \{G\}$

where $G = \begin{bmatrix} G_{11} & 0 \\ 0 & G_{22} \end{bmatrix}$

and at node i

$$\{G_{11}\}_i = \begin{bmatrix} \dfrac{\partial N_i}{\partial x} & I_2 \\ \dfrac{\partial N_i}{\partial y} & I_2 \end{bmatrix}$$

where N is the shape function for in-plane action and I_2 is identity matrix of order 2. Also

$$\{G_{22}\}_i = \begin{bmatrix} \dfrac{\partial \phi K_1}{\partial x} & \dfrac{\partial \phi K_2}{\partial x} & \dfrac{\partial \phi K_3}{\partial x} \\ \dfrac{\partial \phi K_1}{\partial y} & \dfrac{\partial \phi K_2}{\partial y} & \dfrac{\partial \phi K_3}{\partial y} \end{bmatrix}$$

where $K_1 = 3i - 2$, $K_2 = 3i-1$, $K_3 = 3i$

Substituting these values in the integral for geometric stiffness and performing required manipulation results in

$$\{K_g\} = \int_V \{G\}^T \{S\} \{G\}\ dv$$

where {S} is a matrix of stress components. Introducing force per unit length of the boundary of the element and carrying the integral over the area.

$$\{K_g\} = \int_A \{G\}^T \{M\} \{G\}\ dA$$

where $\{M\} = \begin{bmatrix} M_{11} & 0 \\ 0 & M_{22} \end{bmatrix}$

and submatrices

$$\{M_{11}\} = \begin{bmatrix} T_x I_2 & T_{xy} I_2 \\ T_{xy} I_2 & T_y I_2 \end{bmatrix}$$

$$\{M_{22}\} = \begin{bmatrix} T_x & T_{xy} \\ T_{xy} & T_y \end{bmatrix}$$

Again I_2 represents an identity matrix or order 2 and T_x, T_y, T_{xy} are stress components per unit length of the boundary of the element. Then

$$\{K_g\} = \begin{bmatrix} K_g^m & 0 \\ 0 & K_g^b \end{bmatrix}$$

where $\{K_g^m\} = \int_A \{G_{11}\}^T \{M_{11}\} \{G_{11}\} \, dA$

and $\{K_g^b\} = \int_A \{G_{22}\}^T \{M_{22}\} \{G_{22}\} \, dA$

considering strain in the middle plane of the element

$$\{\varepsilon\} = \{B\} \{q\}$$

where

$$\{B\} = \{B_u^o + B_u^L \;\vdots\; B_{uw}\} \tag{10}$$

and submatrices are related to the deflections in the following manner:

$\{B_u^o\}$ = transformation matrix for small deflection, $\{B_u^L\}$ = additional terms for in-plane components of large deflection, B_{uw} = additional terms of out-of-plane components of large deflection.

and $\{q\} = \begin{Bmatrix} q_p \\ \hline q_b \end{Bmatrix}$

then stress is given by

$$\{\sigma\} = \{D\} \quad \{\varepsilon\} = \{D\} \quad \{\varepsilon\} = \{D\} \quad \{B_u^o + B_u^L \mid B_{uw}\} \begin{Bmatrix} q_p \\ \hline q_b \end{Bmatrix}$$

Examining matrices $\{B_o\}$ and $\{B_1\}$ of equation (9), it is found that the ith elements are as follows:

$$\{B_o\} = \{B_u^o\}_i = \begin{bmatrix} \frac{\partial N_i}{\partial x} & 0 \\ 0 & \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} \end{bmatrix}$$

$$\{B_L\}\{q\} = \frac{1}{2} \begin{Bmatrix} \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2 \\ \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 \\ 2\left(\frac{\partial u}{\partial x}\right)\left(\frac{\partial u}{\partial y}\right) + 2\left(\frac{\partial v}{\partial x}\right)\left(\frac{\partial v}{\partial y}\right) + 2\left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right) \end{Bmatrix}$$

Now

$$\delta\ (\{B_L\}\{q\}) = \{B_1\}\ \delta\ \{q\}$$

where $\{B_1\} = \{H\}\ \{G\}$

thus

$$\{B_1\}_i = \begin{bmatrix} \frac{\partial u}{\partial x} \frac{\partial N_i}{\partial x} & \frac{\partial v}{\partial x} \frac{\partial N_i}{\partial x} & \frac{\partial w}{\partial x} \frac{\partial \phi_k}{\partial x} \\ \frac{\partial u}{\partial y} \frac{\partial N_i}{\partial y} & \frac{\partial v}{\partial y} \frac{\partial N_i}{\partial y} & \frac{\partial w}{\partial y} \frac{\partial \phi_k}{\partial y} \\ \frac{\partial u}{\partial x} \frac{\partial N_i}{\partial y} & \frac{\partial v}{\partial y} \frac{\partial N_i}{\partial x} & \frac{\partial w}{\partial y} \frac{\partial \phi_k}{\partial x} \\ + & + & + \\ \frac{\partial u}{\partial y} \frac{\partial N_i}{\partial x} & \frac{\partial v}{\partial x} \frac{\partial N_i}{\partial y} & \frac{\partial w}{\partial x} \frac{\partial \phi_k}{\partial y} \end{bmatrix}$$

where $\frac{\partial \phi_k}{\partial x}$ and $\frac{\partial \phi_k}{\partial y}$ are submatrices defined as

$$\frac{\partial \phi_k}{\partial x} = \left\lfloor \frac{\partial \phi_{k_1}}{\partial x}, \frac{\partial \phi_{k_2}}{\partial x}, \frac{\partial \phi_{k_3}}{\partial x} \right\rfloor$$

and

$$\frac{\partial \phi_k}{\partial y} = \left\lfloor \frac{\partial \phi_{k_1}}{\partial y}, \frac{\partial \phi_{k_2}}{\partial y}, \frac{\partial \phi_{k_3}}{\partial y} \right\rfloor$$

$K_1 = 3i - 2,\ K_2 = 3i - 1,\ K_3 = 3i$

Combining (7), (9), (10), and (11)

$$\{B_u^L\} = \begin{bmatrix} \frac{\partial u}{\partial x} \frac{\partial N_i}{\partial x} & \frac{\partial v}{\partial x} \frac{\partial N_i}{\partial x} \\ \frac{\partial u}{\partial y} \frac{\partial N_i}{\partial y} & \frac{\partial v}{\partial y} \frac{\partial N_i}{\partial y} \\ \frac{\partial u}{\partial x} \frac{\partial N_i}{\partial y} & \frac{\partial v}{\partial y} \frac{\partial N_i}{\partial x} \\ + & + \\ \frac{\partial u}{\partial y} \frac{\partial N_i}{\partial x} & \frac{\partial v}{\partial x} \frac{\partial N_i}{\partial y} \end{bmatrix}$$

and

$$\{B_{uw}\} = \frac{1}{2}\begin{bmatrix} \frac{\partial w}{\partial x} \frac{\partial \phi_k}{\partial x} \\ \frac{\partial w}{\partial y} \frac{\partial \phi_k}{\partial y} \\ \frac{\partial w}{\partial y} \frac{\partial \phi_k}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial \phi_k}{\partial y} \end{bmatrix}$$

The stiffness matrix $\{K_L\}$ is obtained from $K_L = \int_V B_C^T DB_C \, dv$

where

$$\{B_C\} = \{B_u \quad B_{uw}\} + Z\{B_b\}$$

and $\{B_u\} = \{B_u^o\} + \{B_u^L\}$

thus

$$\{B_C\}_i = \left[\begin{array}{c|c|c} (1 + 0.5 \frac{\partial u}{\partial x}) \frac{\partial N_i}{\partial x} & 0.5 \frac{\partial v}{\partial x} \frac{\partial N_i}{\partial x} & 0.5 \frac{\partial w}{\partial x} \frac{\partial \phi_k}{\partial x} + Z \frac{\partial^2 \phi_k}{\partial x^2} \\ \hline 0.5 \frac{\partial u}{\partial y} \frac{\partial N_i}{\partial y} & (1 + 0.5 \frac{\partial v}{\partial y}) \frac{\partial N_i}{\partial y} & 0.5 \frac{\partial w}{\partial y} \frac{\partial \phi_k}{\partial y} + Z \frac{\partial^2 \phi_k}{\partial y} \\ \hline 0.5 \frac{\partial u}{\partial y} \frac{\partial N_i}{\partial x} & (1 + 0.5 \frac{\partial v}{\partial y}) \frac{\partial N_i}{\partial x} & 0.5 \frac{\partial w}{\partial y} \frac{\partial \phi_k}{\partial x} + \\ +(1 + 0.5 \frac{\partial u}{\partial x}) \frac{\partial N_i}{\partial y} & +0.5 \frac{\partial v}{\partial x} \frac{\partial N_i}{\partial y} & 0.5 \frac{\partial w}{\partial x} \frac{\partial \phi_k}{\partial y} + 2Z \frac{\partial^2 \phi_k}{\partial x \partial y} \end{array}\right]$$

and

$$q^T = \lfloor U_i, V_i, \ldots \quad W_k, \ldots \rfloor$$

The complete matrix K_L is written as

$$K_L = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$$

and the submatrices are defined as

$$K_{11} = t \int_A B_u^T DB_u \, dA, \quad K_{12} = t \int_A B_u^T DB_{uw} \, dA$$

$$\{K_{21}\} = \{K_{12}\}^T$$

and

$$K_{22} = \frac{t^3}{12} \int_A B_b^T DB_b \, dA + t \int_A B_{uw}^T DB_{uw} \, dA$$

where t is the thickness of the element and t is the constitutive matrix.

The total strain is obtained from

$$\{\varepsilon\} = \{B_u^o + B_u^L\} \{q_p\} + \{B_{uw}\} \{q_b\}$$

and the matrix of surface traction at every point is

$$\begin{Bmatrix} T_x \\ T_y \\ T_{xy} \end{Bmatrix} = t \{D\} \{\varepsilon\}$$

It is noted that the formulation can be reduced very easily to the standard case of bifurcation; however, it is not adopted here since the present work is only restricted to large deflection behavior or geometric nonlinearity, hence solution to equation (5) is sought and the approach for that is discussed in the following section.

2.3 Procedure for Nonlinear Analysis

The nonlinear stiffness equation (5) representing the behavior of the structure may be solved numerically by the incremental method.

Schematically, the nonlinear problem and solution is presented in Figure 2-2. Of course, in the figure the difference between the incremental and the exact solutions is exaggerated. By choosing proper size for the increments the two solutions can be made to converge in most instances. The following summarizes the procedure:

First, a small increment of load is applied and displacements and stresses are computed using the small deflection matrix $\{K_o\}$; these values are then used to compute the large deflection and geometric matrices.

Next the equation

$$(K_L + K_g) \ \delta q = \delta F$$

is solved for the new increment of loads; solution yields δ or increment of displacement. Total displacement at each stage is computed as

$$q_i = q_{i-1} + \delta q_i$$

and then the total strains and stresses may be found, K_L and K_g updated and the process repeated using another increment of load until total load F is reached where

$$F = \Sigma_{i=1}^{n} \ \delta F_i$$

The nonlinear stiffness matrix requires the computation of

$$\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial w}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}, \frac{\partial w}{\partial y}$$

These values are numerically computed as

$$\frac{\partial u}{\partial x} = \Sigma_{i=i}^{n_1} \quad u_i \frac{\partial N_i}{\partial x} \qquad \text{etc.}$$

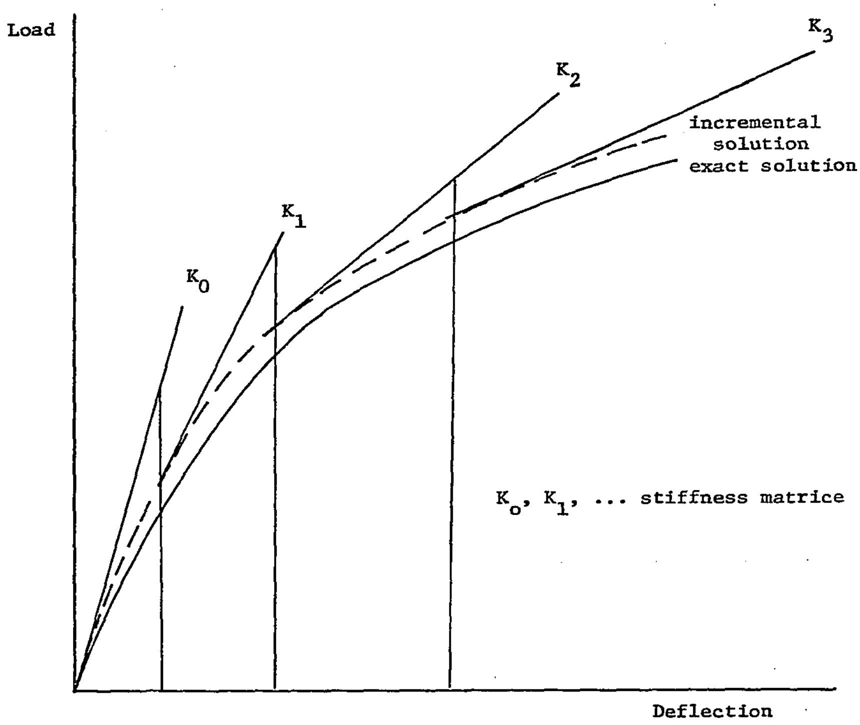


Fig. 2-2. Incremental Solution of Nonlinear Problem

$$\frac{\partial w}{\partial x} = \sum_{i=1}^{n_2} w_i \frac{\partial \phi i}{\partial x} \qquad \text{etc.}$$

where n_1 and n_2 are number of generalized displacements for in-plane and bending action.

CHAPTER III

FORMULATION APPLIED TO THE SELECTED ELEMENTS

3.1 Criteria for Selection of Elements

In Chapter II formulation was developed for combining bending and membrane action, hence for numerical computations two types of elements are needed. It is reasonable to select the two types of elements with the same geometrical shape, otherwise, before assembling the structural stiffness equation the different elements in each region must be assembled to obtain an unique stiffness matrix for that region. This requires an additional amount of computation. Also due to the interaction part of the stiffness matrices, it is necessary to have bending and membrane shape functions which apply to the same region. In fact, the two elements are not acting independently but they are cast into one unique element, thus numerical integrations and other operations for both must be performed in the same region. Comparing the results obtained from different elements for member action, it has been shown that with the same number of nodes complex elements produce better results that simple elements (e.g., results obtained from one rectangle are better than those when the same region is divided in two triangles, the total number of nodes for the region being the same) [63]. Even a rectangle is better than six triangles with additional number of nodes.

For bending, if a single polynomial expansion is assumed over the whole element then, in order to get a complete and compatible function, six degrees of freedom are needed at a non-right-angled corner. Hence, a triangle will have a total of 18 and a quadrilateral will have a total of 24 degrees of freedom [13]. For a triangle at least a quintic polynomial expansion with 21 degrees of freedom must be used. Obviously, this procedure gets very involved and few results are found in the literature for these types of elements.

In another approach, a triangle is divided into three subtriangles and then a polynomial expansion is assumed over each subregion [14]. By imposing compatibility requirements a triangular element with 12 degrees of freedom is obtained. Relating the single degree of freedom at each mid-side node to the degrees of freedom at the corner nodes, triangular elements with eleven, ten and nine degrees of freedom are constructed. In this case every subtriangle has a different set of shape functions. Considering the interaction part of the element stiffness matrix, the incremental procedure necessary for large deflection analysis, and considering the better performance of complex elements (rectangular or quadrilateral) for membrane action, triangular elements are not efficient for the present work and are eliminated from further consideration.

Extending the procedure of dividing an element into triangular sub-regions, a compatible quadrilateral called Q-19 has been developed [13]. The quadrilateral is divided into four triangles and each triangle is subdivided into three subtriangles. Every triangle has eleven degrees of freedom and a set of eleven shape functions is obtained for every subtriangle. The complete quadrilateral has 19 degrees of freedom, seven of them internal which must be condensed out before assembling the structure stiffness matrix. If a membrane element with ten degrees of freedom is to be combined with the Q-19 element, then a 29 x 29 element stiffness matrix is required which must be reduced to a 20 x 20 by condensation and the condensed out terms must be retained for later computation of stress and strain at the internal nodes and integration points which must be used for calculation of geometric and large deflection stiffness. Also considering the 12 subtriangular region, each governed by a different set of shape functions and the required numerical integration for the element stiffness matrix, the tremendous amount of required numerical computations becomes apparent.

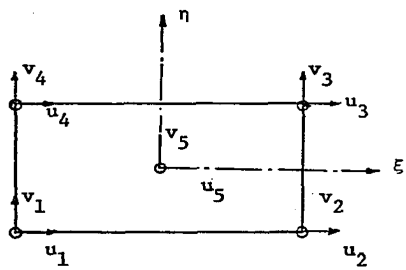
Since the thin walled steel sections which are of interest in this study may be easily divided into rectangular elements and considering economical deficiencies of non-rectangular bending elements for large deflection analysis, attention is here restricted to rectangular bending and membrane elements. The two selected elements are explained in the following sections.

3.2 Membrane Element

It is intended to select an element which does not have a large number of nodes while at the same time produces reasonable results. Hence for membrane action quadrilateral and rectangular elements with a total of four nodes (at the corners) or five nodes (one at the center) have been examined. In fact these are the displacement membrane elements most widely used in the literature.

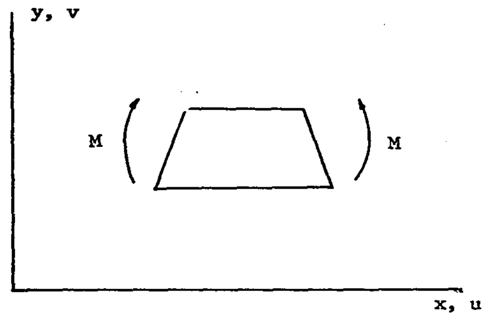
The element called 4CST is composed of 4 constant strain triangles cannot represent a state of pure bending [19] and, as was mentioned in Section 3.1, the more complex elements are superior to this one. The original isoparametric element called Q-4, which has 4 corner nodes [63], produces better results compared to 4CST, but its bending response is not satisfactory. Another isoparametric element with 5 nodes, 4 at the corners and one at the center, produces results which are slightly better than those of Q4, but still its deflection under pure bending is not correct [16]. The incorrect deflection of these elements is shown in Figure 3-2-a. According to references [8, 21, 24, 27] the original isoparametric element (Q4) and the one with a central node have shown improved performance when a constant shear strain is imposed upon the entire element. The resulting elements are called QM4 and QM5. The QM5 element is superior to QM4 and it has given exact results under pure bending. Also, rectangular QM5 element procedures exact results under axial load [16]. Here, element QM5 is selected to model the membrane action.

Element QM5 was first developed in reference [21] and has been extensively used by other authors [8, 27, 28]. This element has 10 degrees of freedom, two at each node. The geometry of the element and degrees of freedom are shown in Figure 3-1. As was mentioned earlier in Section 3.1, the original element with five nodes has been shown to be defective under bending [21], since it is not capable of attaining

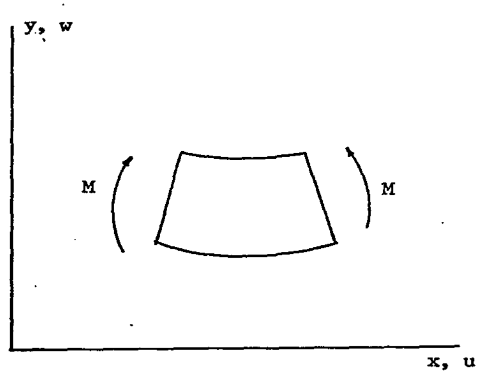


Natural Coordinates and Nodal Degrees of Freedom

Fig. 3-1. Membrane Element



a- Response to pure bending, when actual values of shear strain considered.



b - Response to pure bending when shear strain is set equal to zero everywhere

Fig. 3-2. Membrane Element-Bending Response the correct deflected shape under pure bending. In this case the presence of some shear strain makes the element stiffer than a beam segment by removing some shear energy, the bending performance of the element has been greatly improved [21]. Considering that shear strain is zero at the center of the element, in the integration of the element stiffness matrix the terms which produce shear strain are evaluated at the center regardless of the actual values of the Gauss points [19, 21, 28]. The improvement is illustrated in Figure 3-2-b.

The displacement field for the element is given by:

$$u = \sum_{i=1}^{4} N_i U_i + (1 - \xi^2) \quad (1 - \eta^2) \; U_5$$

$$v = \sum_{i=1}^{4} N_i V_i + (1 - \xi^2) \quad (1 - \eta^2) \; V_5$$

where u and v are displacements at any point of the element in the x and y direction. N_i's are shape functions and ξ and η are natural coordinates. The shape functions are defined as

$$N_1 = \frac{1}{4} (1 - \xi) \; (1 - \eta)$$

$$N_2 = \frac{1}{4} (1 + \xi) \; (1 - \eta)$$

$$N_3 = \frac{1}{4} (1 + \xi) \; (1 + \eta)$$

$$N_4 = \frac{1}{4} (1 - \xi) \; (1 + \eta)$$

where ξ and η are the local coordinates, which take $\pm$ 1 values at the nodes.

Referring to the element stiffness matrix developed in Chapter II, it is seen that derivations of shape functions with respect to cartesian coordinate system are needed. Here shape functions are given in terms of natural coordinates, therefore a relationship is needed for conversion. It is known that for isoparametric element the following relationship exists between cartesian and natural coordinate systems [24].

$$x = \Sigma_{i=1}^{4} N_i x_i \qquad\qquad y = \Sigma_{i=1}^{4} N_i y_i$$

where x_i's and y_i's are nodal coordinates and N_i's are shape functions. Then applying the chain rule and writing in matrix form, the conversion formula is obtained as

$$\begin{Bmatrix} \frac{\partial N}{\partial x} \\ \frac{\partial N}{\partial y} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \frac{\partial N}{\partial \xi} \\ \frac{\partial N}{\partial \eta} \end{Bmatrix}$$

where [J] is the Jacobian matrix

$$[J] = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}$$

Performing the required substitution in the expression of element's large deflection stiffness matrix (Chapter II) and noting that $dA = [J]\, d\xi\, d\eta$ the following expression is obtained:

$$K_{11} = \int_{-1}^{1} \int_{-1}^{1} t \;\{B_u\}^T \;\{D\}\; \{B_u\}\; [J] d\xi \; d\eta$$

where K_{11} = submatrix for membrane action, t = thickness of the element, B_u = displacement to strain transformation matrix and[J]= determinant of Jacobian matrix. The geometric stiffness matrix is developed using the same displacement field as defined for the conventional stiffness (consistent method). If membrane geometric stiffness is denoted by K_g^m then

$$\{K_g^m\} = \begin{bmatrix} K_g^m & 0 \\ 0 & K_g^m \end{bmatrix}$$

the above form results from the following arrangement of in-plane nodal displacements.

$$\{q_p\}^T = \lfloor U_1, \ldots \quad U_5, V_1, \ldots \quad V_5 \rfloor$$

This arrangement simplifies both representation of the terms of the geometric matrix and also numerical computations. K_g^m is a 5 x 5 matrix whose terms are computed as

$$(K_g^m)_{ij} = \int_A \left(\frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} T_x + \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial y} T_{xy} + \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial x} T_{xy} + \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} T_y \right) dA$$

where T_x, T_y, T_{xy} are stress resultants per unit length of the boundary of the element.

Again derivatives are computed in terms of natural coordinates and dA is replaced by J d ξ dη. Hence limits of integration are -1 and +1 for both ξ and η.

3.3 Bending Element

Two types of rectangular bending elements are mostly used in literature: (1) A compatible element [10] which uses Hermitian interpolation functions and has 16 degress of freedom and (2) an incompatible rectangle [14] called ACM, with 12 degrees of freedom. It is seen that the incompatible rectangle [10, 19, 27, 63] gives converging solutions and the obtained results are reasonable and in good agreement with the other existing solutions. This element is used here and the numerical results given in Chapter IV indicate the usefulness of this element.

The element has four corner nodes and three degrees of freedom at each node. Nodal degrees of freedom consist of one transverse displacement and two rotations about the two perpendicular axis in the plane of the element. Figure 3-3 shows geometry and degrees of freedom. The displacement field is expressed by a 12 term polynomial. If displacement at every point is designated by w then $w = \psi\alpha$ where $\{\alpha\}^T = \alpha_1 \ldots \alpha_{12}$ set of coefficients of polynomial and

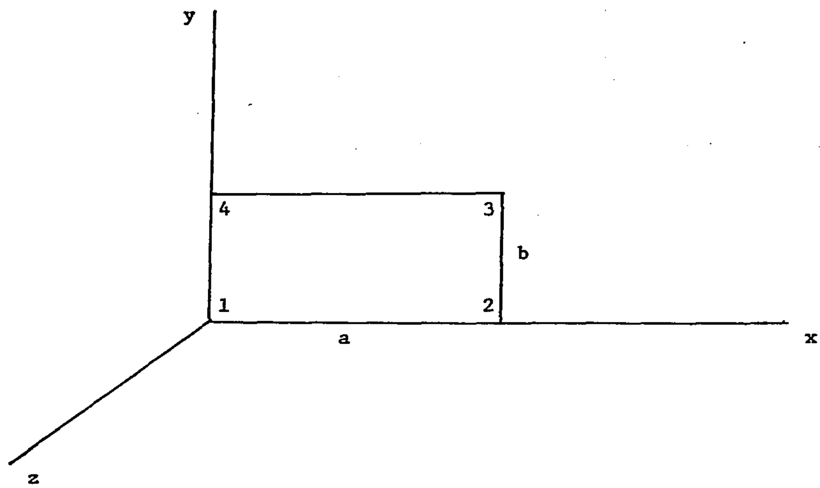
$$\psi = \lfloor 1,\ x,\ y,\ x^2,\ xy,\ y^2,\ x^3,\ x^2y,\ xy^2,\ y^3,\ x^3y,\ xy^3 \rfloor$$

the nodal degrees of freedom are

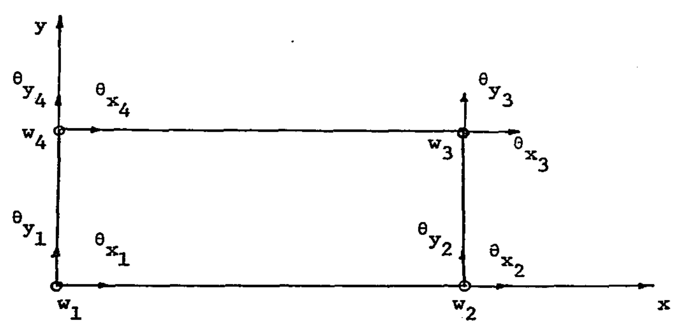
$$\begin{Bmatrix} w \\ \frac{\partial w}{\partial y} \\ -\frac{\partial w}{\partial x} \end{Bmatrix} \quad \text{or} \quad \begin{Bmatrix} w \\ \theta_x \\ \theta_y \end{Bmatrix}$$

To obtain nodal deflections and rotations the above values are evaluated at each node or

$$W_i = W(x_i, y_i), \qquad \theta_{x_i} = \theta_x(x_i, y_i), \qquad \theta_{y_i} = \theta_y(x_i, y_i)$$



Bending Element - Geometry



Bending Element - Nodal Degrees of Freedom

Fig. 3-3.

The vector of 12 nodal displacement for out-of-plane action is written as

$$\{q_b\}^T = \lfloor w_1,\ \theta_{x_1}, \theta_{y_1},\ w_2,\ \theta_{x_2},\ \theta_{y_2},\ w_3,\ \theta_{x_3},\ \theta_{y_3},\ w_4,\ \theta_{x_4},\ \theta_{y_4} \rfloor \tag{1}$$

substituting nodal coordinates in the expression for w and its derivatives, 12 simultaneous equations in terms of α are obtained. Writing equations in matrix form:

$$\{q_b\} = \{C\}\ \{\alpha\}$$

where $\{C\}$ is a 12 x 12 matrix depending on nodal coordinates, and α is the vector of unknown constants. The inverse relationship is written as $\{\alpha\} = \{C\}^{-1}\ \{q_b\}$. It follows that displacement at any point in terms of nodal displacement is given by

$$w = \{\psi\}\ \ \{C\}^{-1}\ \{q_b\}$$

Matrices $\{C\}$ and $\{C\}^{-1}$ are shown in Appendix B.

Considering strain due to bending

$$\varepsilon_x = Z\ \frac{\partial^2 w}{\partial x^2}, \qquad \varepsilon_y = Z\ \frac{\partial^2 w}{\partial y^2}, \qquad \gamma_{xy} = 2\ Z\ \frac{\partial^2 w}{\partial x \partial y}$$

$$\{\varepsilon_b\} = Z\ \ \{B_w\}\ \{q_b\}$$

it can be seen that

$$\{B_w\} = \begin{Bmatrix} \dfrac{\partial^2 w}{\partial x^2} \\ \dfrac{\partial^2 w}{\partial y^2} \\ \dfrac{2\partial^2 w}{\partial x \partial y} \end{Bmatrix} \{C\}^{-1}$$

$$\text{or} \quad \{B_w\} = \{Q\}\{C\}^{-1}$$

The contribution of bending part to conventional stiffness matrix is then

$$\{K_{22}\} = \int_A \{B_w\}^T \{D\} \{B_w\} \quad dA \tag{2}$$

Out-of-plane geometric matrix is designated by $\{K^b_g\}$, it is a 12 x 12 matrix with terms

$$(K^b_{g_{ij}}) = \int_A \left(\frac{\partial \phi_i}{\partial x} \frac{\partial \phi_j}{\partial x} T_x + \frac{\partial \phi_i}{\partial x} \frac{\partial \phi_j}{\partial y} T_{xy} + \frac{\partial \phi_i}{\partial y} \frac{\partial \phi_j}{\partial x} T_{xy} + \frac{\partial \phi_i}{\partial y} \frac{\partial \phi_j}{\partial y} T_y \right) dA$$

where $\{\phi\}$ is a row matrix of shape functions given by $\phi = \{\psi\} \quad \{C\}^{-1}$ and the arrangement of nodal displacement is given in expression (1).

3.4 Coupled Bending - Membrane and Additional Terms Due to Large Deflection

Submatrices developed in the preceding sections are independent of coupling effect. The conventional matrix composed of bending and membrane part is represented as:

$$K = \begin{bmatrix} K_{11} & 0 \\ 0 & K_{22} \end{bmatrix}$$

This matrix is used in the first step of analysis with only a small portion of load applied to the structure. In the succeeding steps, the combined geometric and large deflection matrix is formulated as follows.

The complete form of geometrically nonlinear large deflection matrix may be written as:

$$\{K_L\} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} + K_{22}' \\ & \end{bmatrix}$$

In Chapter II, detailed derivation of submatrices is given. Here their physical interpretation and their structure for numerical computation is briefly overviewed.

Membrane stiffness matrix $\{K_{11}\}$ is similar to $\{K_{11}\}$ of small deflection matrix except for the effect of nonlinear terms in the strain-displacement relationship. Matrix $\{K_{22}'\}$ is the additional bending stiffness due to large deflection and it depends on the first derivatives of the displacement field, matrices K_{12} and K_{21} are submatrices resulting from coupled bending-membrane effect and also depend on the first derivatives of in-plane and out-of-plane displacements in addition to the shape functions.

3.5 Assembling Submatrices and Condensation

In the process of numerical computation, matrices K_{11} and K_{11}' are arranged in the following order of in-plane displacements:

$$\{q_p\}^T = \lfloor u_1, v_1, u_2, v_2, u_3, v_3, u_4, v_4, u_5, v_5 \rfloor$$

The in-plane geometric matrix has a different arrangement (mentioned in Section 3.2), hence before assembling the submatrices into the element stiffness matrix, a rearrangement of $\{K_g^m\}$ is necessary. The complete stiffness matrix for one element, denoted by $(K_L + K_g)^e$, is a 22 x 22 matrix. Before assembling this into structure stiffness matrix, the internal degrees of freedom are condensed out which requires yet another rearrangement conforming with

$$\{q\}^T = \lfloor q_p^T \mid q_b^T \rfloor = \lfloor u_1, v_1, w_1, \theta_{x_1}, \theta_{y_1}, \ldots u_4, v_4, w_4, \theta_{x_4}, \theta_{y_4}, U_5, V_5 \rfloor$$

The condensed elements of load vector and stiffness matrix are saved for later computation of deflections at the condensed nodes which are needed to construct the updated geometric and large deflection matrices.

3.6 Numerical Integration

Except for submatrix K_{22} which may be integrated in the closed form, the submatrices can be evaluated only by numerical integration. In the present work only $\{K_{22}\}$ is integrated in closed form. For numerical integration of $\{K_{11}\}$ a 2 x 2 Gauss rule is used, for all other numerical integrations a 4 x 4 Gauss rule is used. For this reason all expressions for in-plane and out-of-plane actions are obtained in terms of natural coordinates.

Interpolation functions for numerical evaluation of $\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial^2 w}{\partial x^2}, \frac{\partial^2 w}{\partial y^2}, \frac{\partial^2 w}{\partial x \partial y}$ are listed in Appendix B. The functions are listed in terms of x and y again Jacobian matrix and its determinant is used for conversion of ξ and η coordinates.

CHAPTER IV

NUMERICAL STUDIES

The assemblage of two dimensional members is treated as a general three dimensional structure. Thus it is possible for every point of the structure to have rotations and translations in all three directions in the space. Of course, plane structures may be handled by applying proper restraints at the nodes.

The method can cope with relatively complex geometry and boundary conditions. There is no need for the members to have a symmetrical cross section or for the loads to be applied in the plane of symmetry of the member. In contrast to the small deflection procedure which needs different formulations for different classes of problems, such as buckling, bending, membrane, the large deflection formulation can be applied to a wide variety of problems involving combined phenomena.

A computer program was developed for numerical computation of the method discussed in the previous chapters. The program is called "NASM--Nonlinear Analysis of Structural Members." A macro flow chart is shown in Figure 4-1 and the program listing is in Appendix A. Several problems are selected to illustrate the application and accuracy

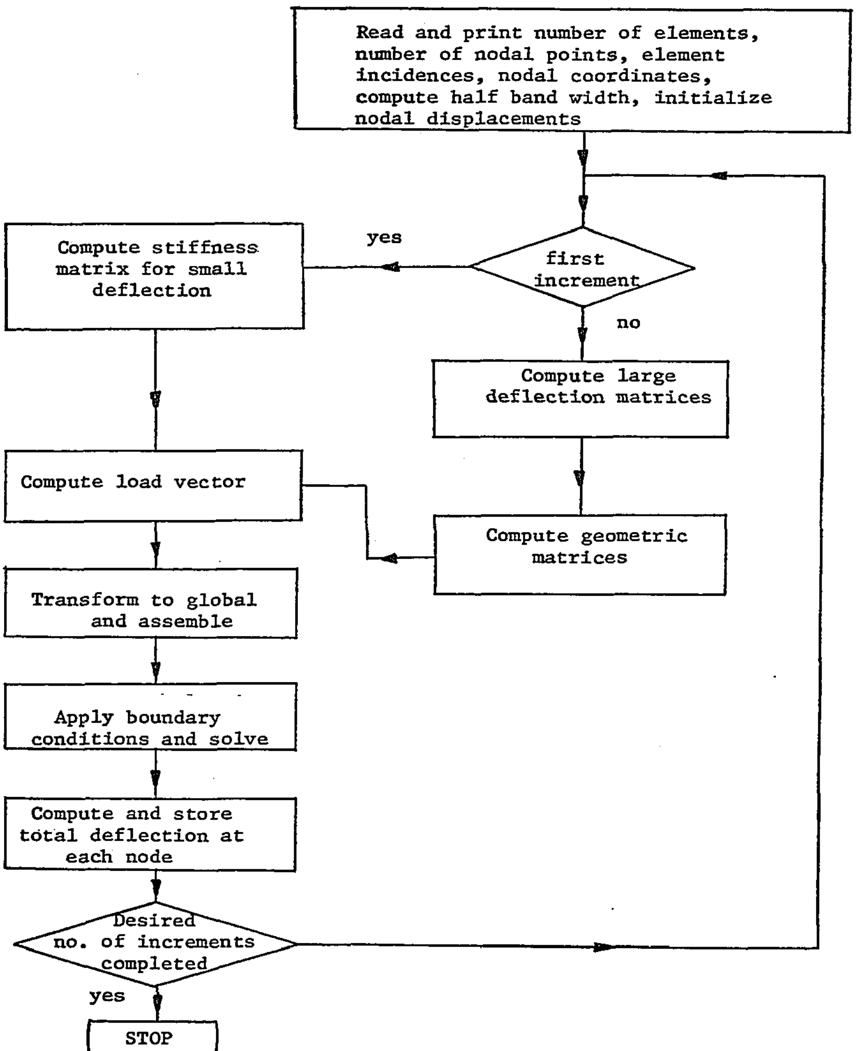


Fig. 4-1. Flow Chart for Computer Program

of the procedure. For some of these problems exact solutions using classical methods are available. For other problems exact solutions do not exist and the approximate solutions are usually based on the empirical formulas. In each case the present solution is in reasonable agreement with the previous ones.

For most of the problems studied herein, symmetry is utilized and boundary conditions for nodes located on the axis of symmetry are handled in the following manner: For plate problems, when single symmetry is used, x and y are taken as the coordinate axes in the plane of the plate and one-half of the plate is analyzed. Zero displacement in the x-direction and zero rotation about the y-axis are imposed along the line of symmetry. For double symmetry, one-quarter of the plate is analyzed and displacements are assumed zero in the x- and y- directions. Rotations about the x and y axes are taken as zero as appropriate.

Example 1. Bending of a Clamped Plate

In this example a square plate, 20 in. by 20 in. by .08 in. thick with all boundaries considered fixed is studied. Utilizing symmetry, only one-quarter of the plate was analyzed. The loading was uniform and applied in the transverse direction in increments of 0.4 lb/in^2.

Since the finite element method is not an exact method, four meshes were used to study convergence, Figures 4-2-b through 4-2-e. For each mesh the plate was loaded to 2.0 lb/in^2 and the resulting maximum deflection compared. Figure 4-3 compares maximum deflection to mesh size and clearly shows that a 16 element mesh (4 elements per quarter) is adequate for this problem.

Using the 4 x 4 mesh, the plate was loaded to 4.82 lb/in^2 which corresponds to first yielding of material having a yield stress of 36000 psi. The resulting deflection and load-stress relationships are plotted in non-dimensionalized form in Figures 4-4 and 4-5.

Classical solutions given for this problem in references [32] and [53] are also plotted in Figure 4.4. Both solutions are approximate and as an example the one presented in reference [53] which is based on the Ritz method will be described here. Applying the virtual work principle, the equation

$$\delta V - \delta \quad q\, w dx\, dy = 0 \tag{1}$$

is obtained, where V = total strain energy for a virtual displacement, q = uniform load per unit area, w = deflection in the transverse direction. The displacements in the middle plane of the plate in the x, y, z directions are denoted by u, v, w and the following functions satisfying boundary conditions are assumed for the mentioned displacements

$$u = (a^2 - x^2)\ (b^2 - y^2)\ x\ (b_{00} + b_{02}\ y^2 + b_{20}\ x^2 + b_{22}\ x^2y^2)$$

$$v = (a^2 - x^2)\ (b^2 - y^2)\ y\ (c_{00} + c_{02}\ y^2 + c_{20}\ x^2 + c_{22}\ x^2y^2)$$

$$w = (a^2 - x^2)^2\ (b^2 - y^2)^2\ (a_{00} + a_{02}\ y^2 + a_{20}\ x^2)$$

where 2a and 2b are length and width of the plate and other coefficients are unknown constants. Substituting the above functions in

Equation (1) and minimizing with respect to $a_{00} \ldots C_{22}$, eleven nonlinear equations were obtained. Numerical solution of the equations resulted in values of the unknowns. The unknowns depend on the shape of the plate and on the value of q. In this manner displacements and their derivatives can be obtained and strains and stresses can be computed using derivatives of displacements. The results of the classical analysis, as presented in reference [53], are plotted in Figure 4-3 and 4-4.

It is realized that deflections using the finite element procedure converge to values slightly larger than those given in [53]. The deviation can be justified considering:

1. Both solutions are approximate, hence there is no reason for getting exactly the same results;
2. According to reference [61] experimental investigations give larger deflections than the values presented in [53];
3. Considering properties of the displacement method of finite element analysis, the assumed function approximate displacements closely but give less accurate values for stresses [24] since in the equation $(K_1 + K_g)\ \delta q = \delta F$, the solution is mainly affected by K_1 which in turn depends on the displacements. The value of $[K_g]$ depends on stresses and has a minor effect on the resulting displacements [13, 27, 28].

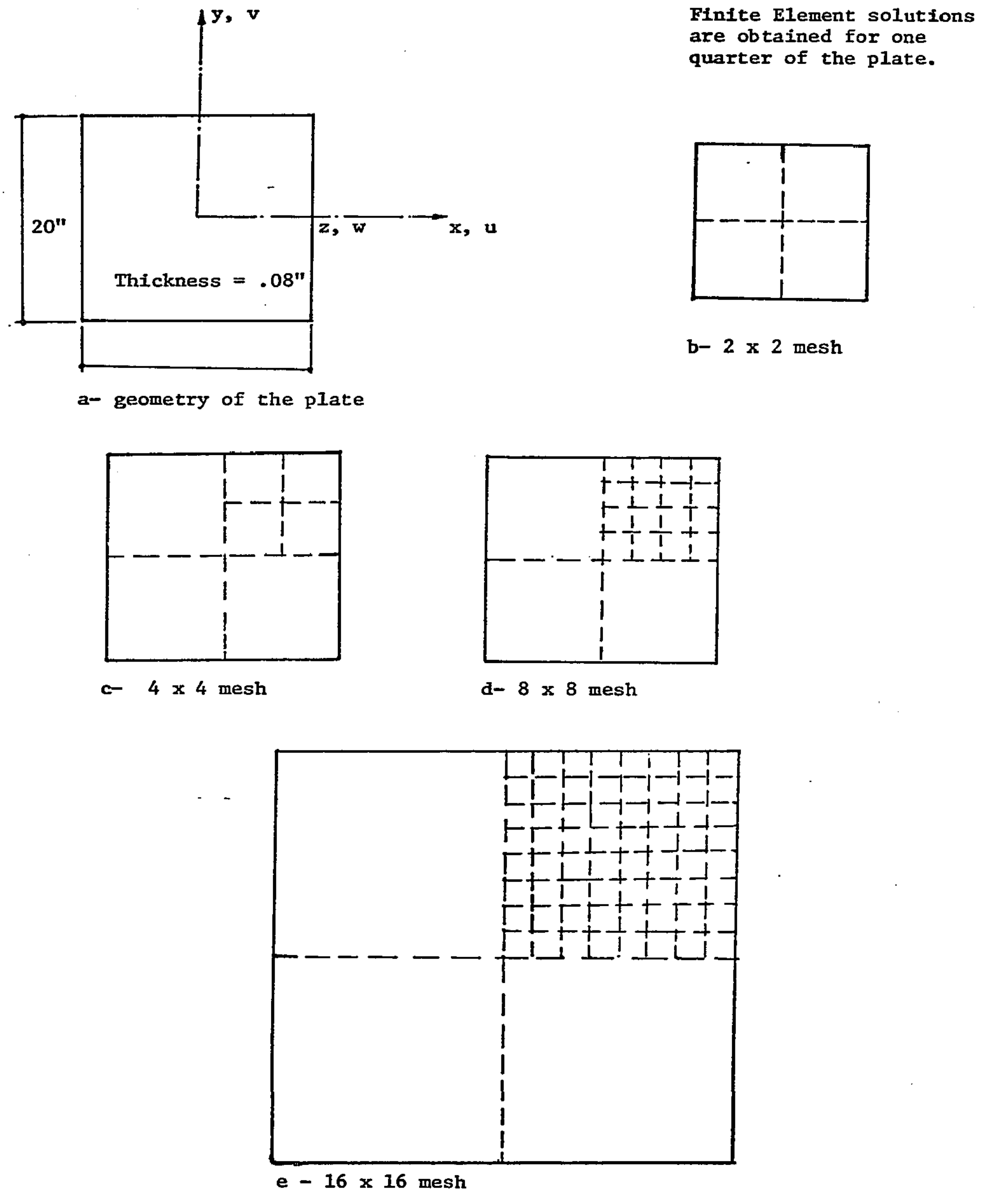


Fig. 4-2. Geometry and Idealization

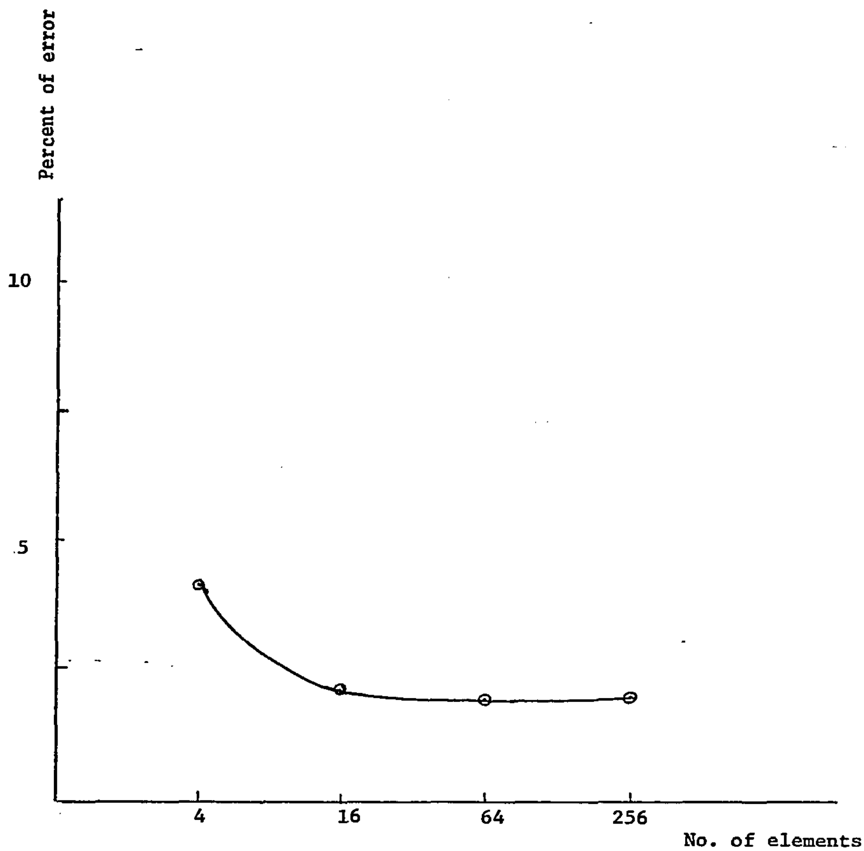
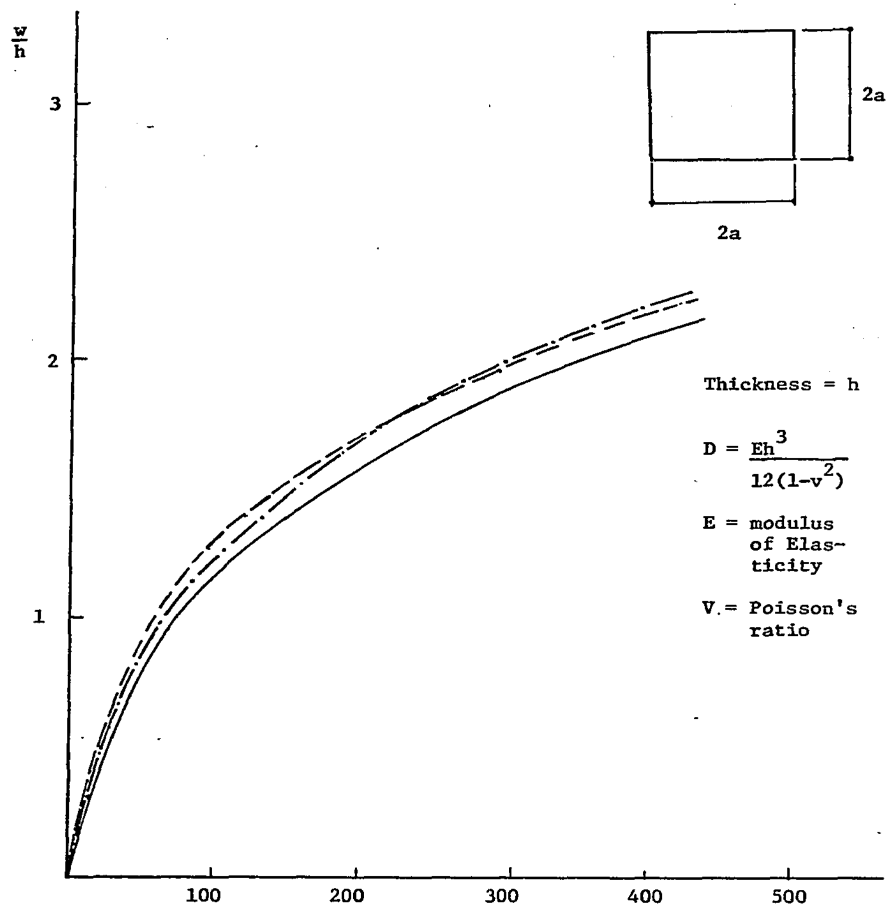


Figure 4-3. Bending of Clamped Plate - Convergence Compared with Reference [32]



Finite Element Solution — — — — — — — q = uniform load

Reference [53] ———————— w = deflection

Reference [32] —·—·—·—·— h = thickness

Figure 4-4. Load-Deflection Diagram for a Clamped Plate (Finite Element Solution is Obtained from 4x4 mesh)

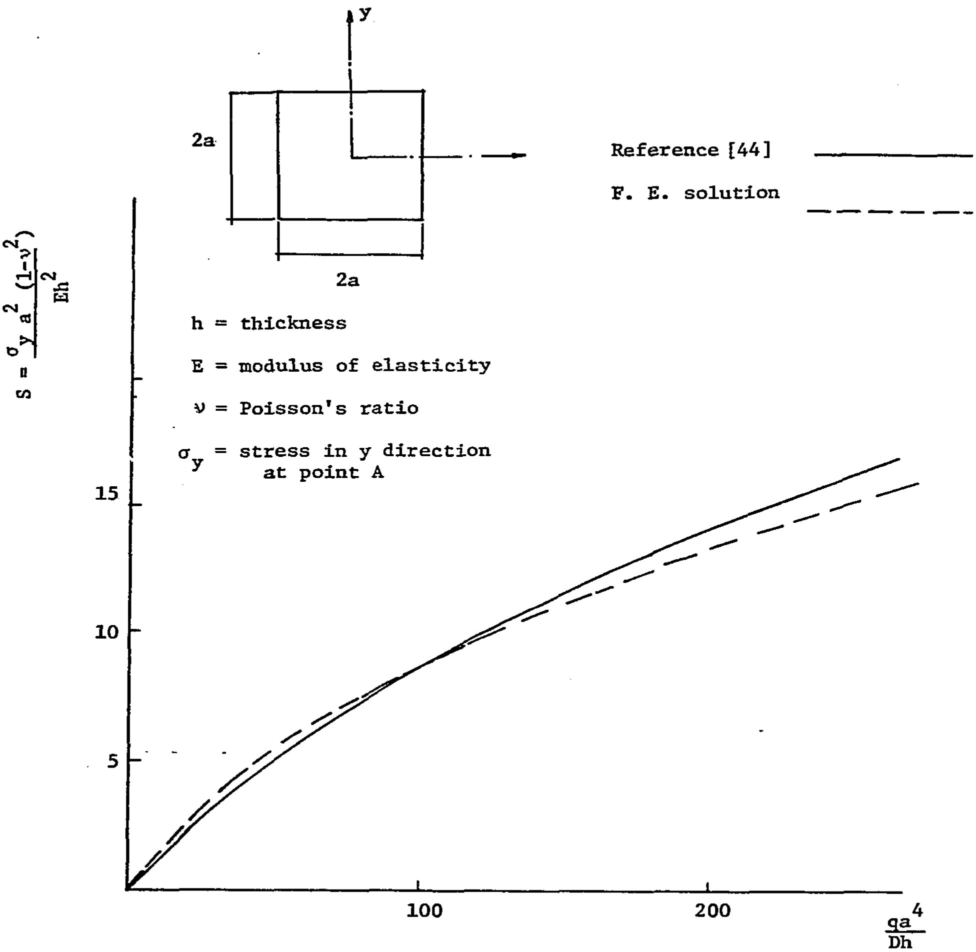


Fig. 4-5. Maximum Stress (at Point A) - Versus Load

Example 2. Buckling of a Simply Supported Plate

A square plate 20 in. by 20 in. by 0.1 in. with all edges considered simply supported is studied. The loading consists of in-plane uniform load acting on the two parallel edges in the x direction in increments of 1900 lb. (total). Because of symmetry only one-quarter of the plate is analyzed. The uniform load is replaced by equivalent concentrated loads at the nodal points. To obtain convergence, solutions were obtained for four different meshes, Figures 4-6-b, 4-6-e. For each mesh the plate was loaded to 9500 lb; maximum deflections are compared in Figure 4-7. It is seen that a 64 element mesh results in a good approximation. Then using an 8 x 8 mesh the plate was loaded to 20900 lb. which is about four times the elastic buckling load. At the last increment the maximum stress was more than 36000 psi and by interpolation the load at first yield (36000 psi) was determined to be 19850 lb.

The load-deflection diagram for maximum deflection is plotted in Figure 4-8. A change in the slope of the curve corresponds to elastic buckling load.

A classical solution is also available for this problem [9]. The solution is based on nonlinear differential equation given by Von Karman

$$\frac{\partial^4 F}{\partial x^4} + 2\frac{\partial^4 F}{\partial x^2 \partial y^2} + \frac{\partial^4 F}{\partial y^4} = E\left[\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right]$$

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{P}{D} + \frac{t}{D}\left(\frac{\partial^2 F}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 F}{\partial y^2}\frac{\partial^2 w}{\partial x^2} - \frac{2\partial^2 F}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y}\right)$$

where P = uniformly distributed load, D = rigidity, E = elastic modulus, F = stress function, w = out-of-plane displacement. Mean values of stresses are defined by P_1 and P_2, where

$$P_2 = \frac{1}{a}\int_{-\frac{a}{2}}^{\frac{a}{2}} \sigma_y \, dy \quad , \qquad\qquad P_1 = \frac{1}{b}\int_{-\frac{b}{2}}^{\frac{b}{2}} \sigma_x \, dy,$$

where a and b are length and width of the plate. The deflection is assumed to be $w = f \cos \frac{\Pi X}{a} \cos \frac{\Pi Y}{b}$ where f = max deflection.

Substituting into the Von Karman's equations and applying boundary conditions an approximate solution is obtained. Writing expression for total potential energy, after some manipulation the equations for a square plate loaded in one direction is obtained as:

$$\frac{\Pi f^2}{8b^2} = \frac{1}{E} (P_1 - \sigma_c)$$

$$\sigma_x = P_1 - (P_1 - \sigma_c) \cos \frac{2\Pi y}{b}$$

$$\sigma_y = (P_1 - \sigma_c) \cos \frac{2\Pi x}{b}$$

where σ_c is critical stress given by

$$\sigma_c = \frac{\Pi^2 E t^2}{3\,(1-v^2) b^2}$$

and σ_x, σ_y = stresses at any point, b = side of the plate.

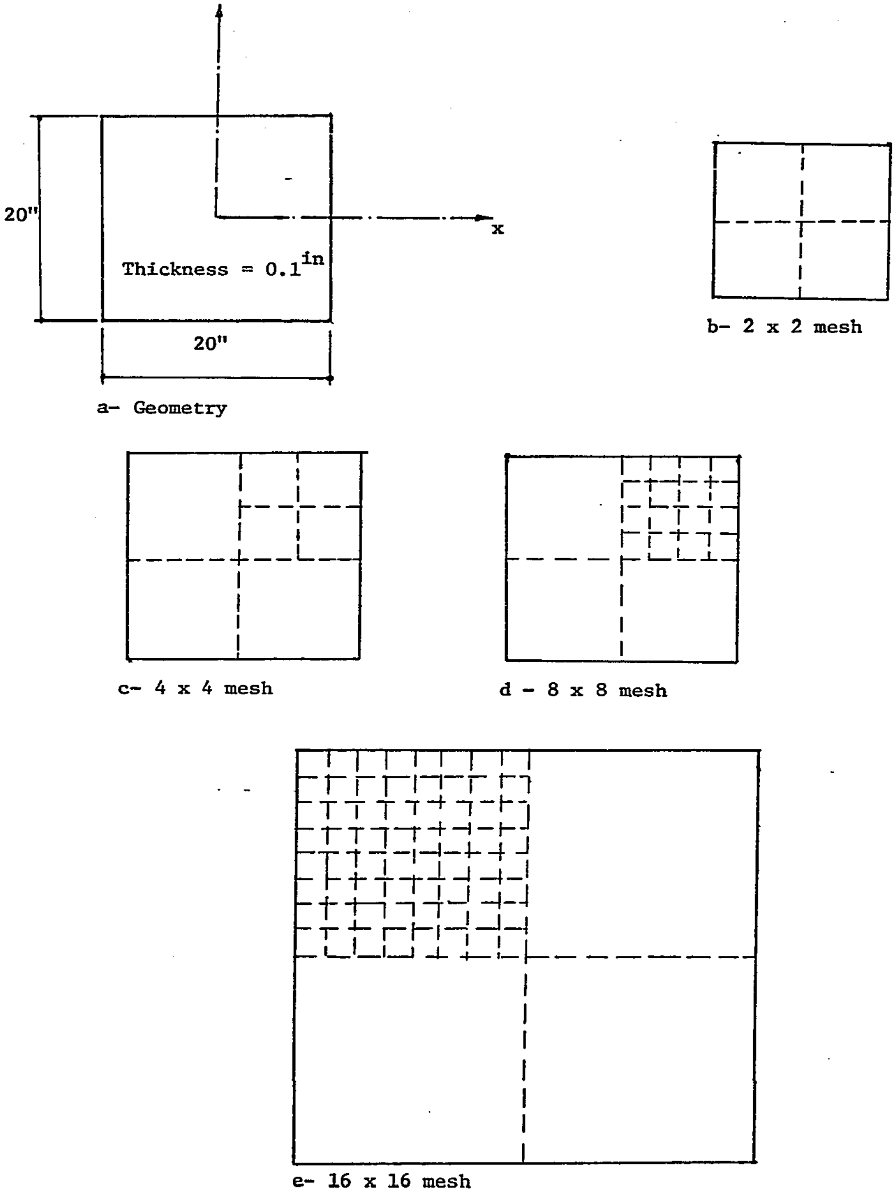


Fig. 4-6. Simply Supported Plate Under
In-Plane Load in the x Direction

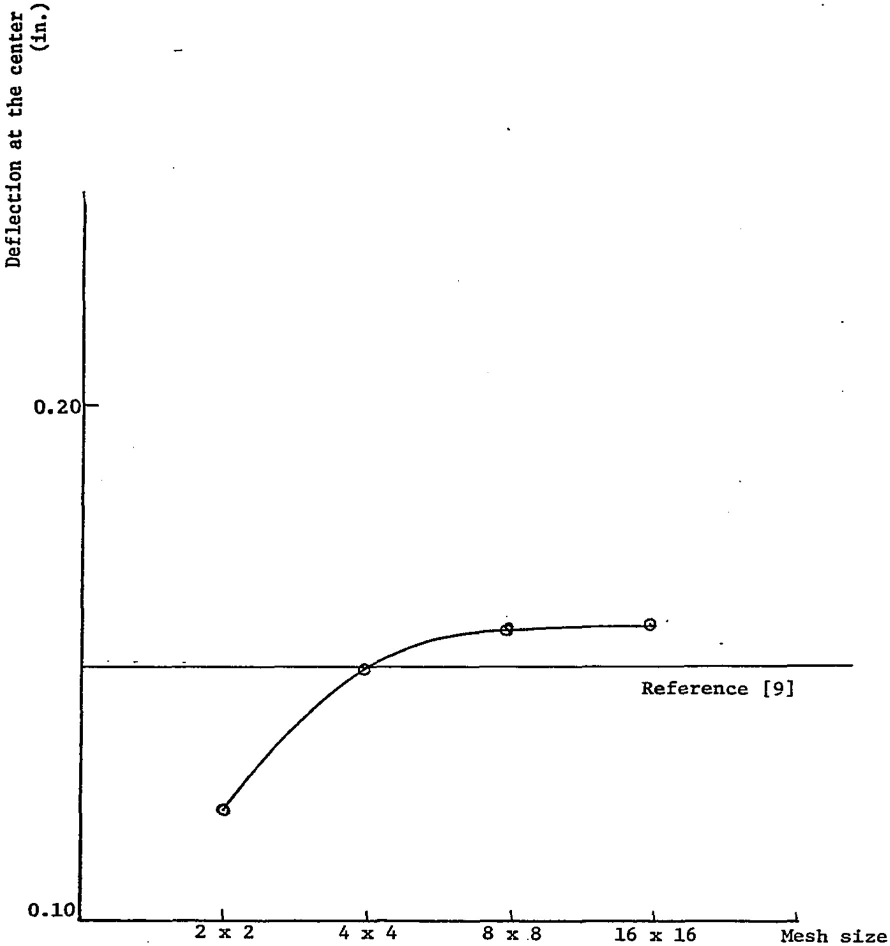


Fig. 4-7. Simply Supported Plate - In-Plane Load in the x Direction - Convergence

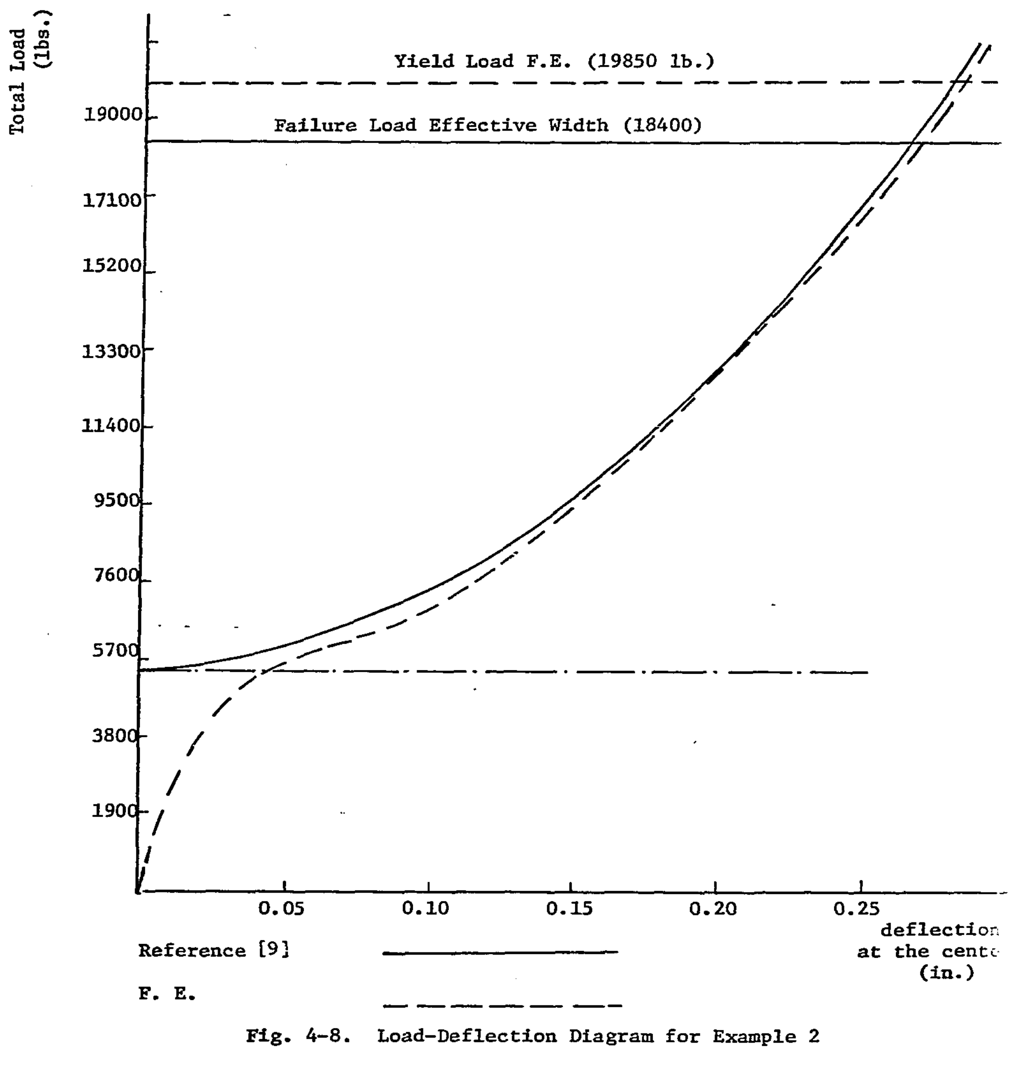


Fig. 4-8. Load-Deflection Diagram for Example 2

Results of the classical analysis are also presented in Figure 4-8. It is seen that finite element analysis yields deflections which are slightly larger than those given by classical method. The difference is explained as: first, both solutions are approximate, second,and more important, the classical method assumes zero deflection at the time of buckling and the plate starts deflection after bifurcation is reached, while in the finite element solution a considerable amount of deflection exists at the time of buckling.

Example 3. Buckling of Plate Under Shear and Bending

A square plate 8 in. by 8 in. by .05 in. thick, with simply supported boundaries was analyzed. The load consist of bending moment and shear force, both in the plane of the plate. The loads were applied in increments of 150 lb. shear force and 1920 in-lb bending moment. The bending moment and shear force were applied as concentrated forces at the nodal points and the entire plate was analyzed. Again to study convergence, solutions were obtained for three different meshes. Loading and geometry of the meshes are shown in Figures 4-9-a - 4-9-c. For each mesh, a 750 lb. shear force and 9600 in-lb bending moment was applied and in order to study out-of-plane behavior, a small out-of-plane deflection was imposed on the plate by applying a concentrated load of 0.5 lb. at each increment. The maximum out-of-plane deflection for each mesh is plotted in Figure 4-10 and a 4 by 4 mesh was found to produce an acceptable approximation. The 4 by 4 mesh was then used for additional studies.

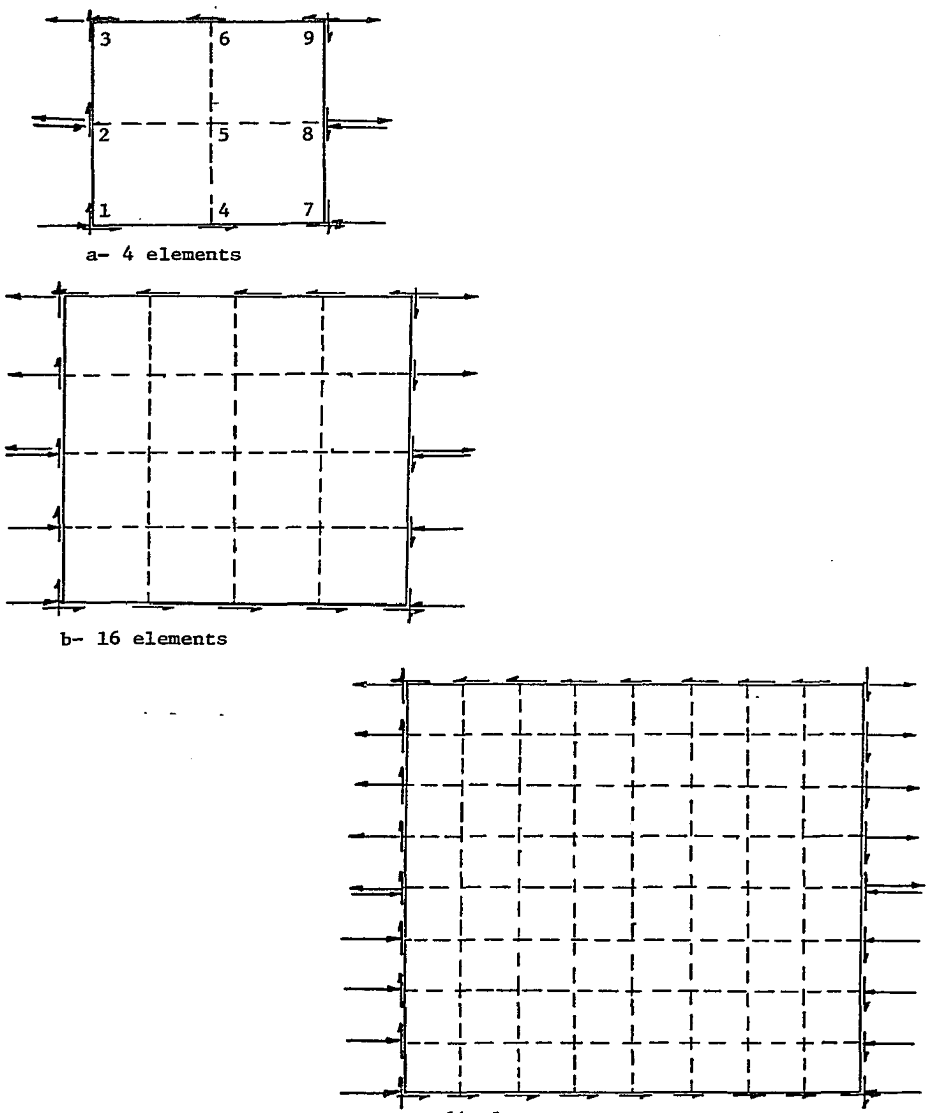


Fig. 4-9. In-Plane Bending and Shear, Three Models
(Loads for each Increment)

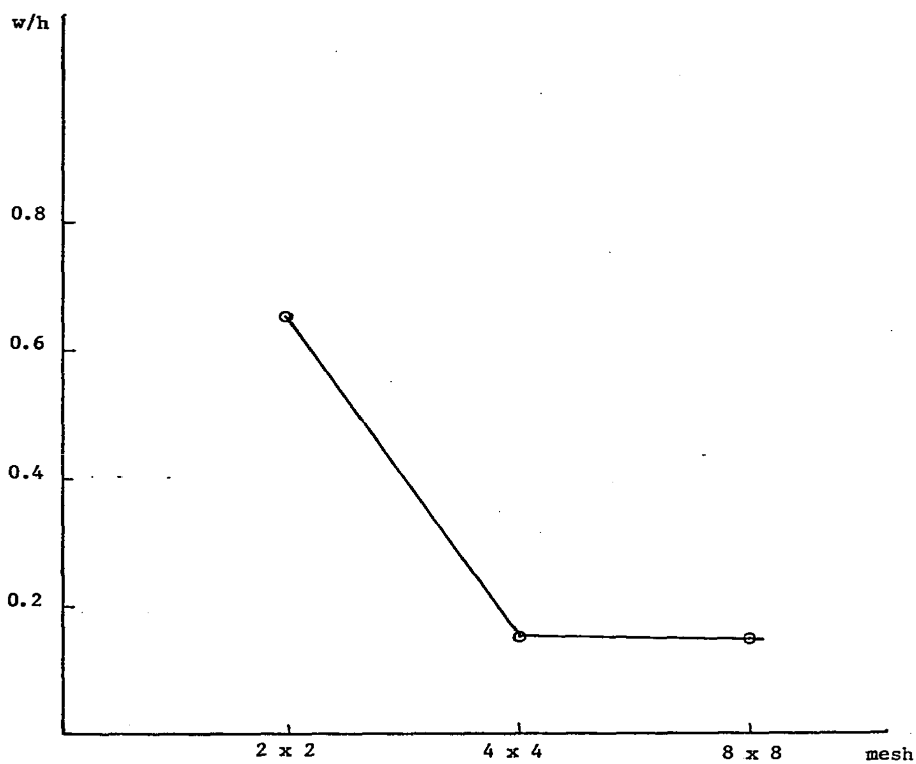


Fig. 4-10. In-Plane Bending and Shear Convergence

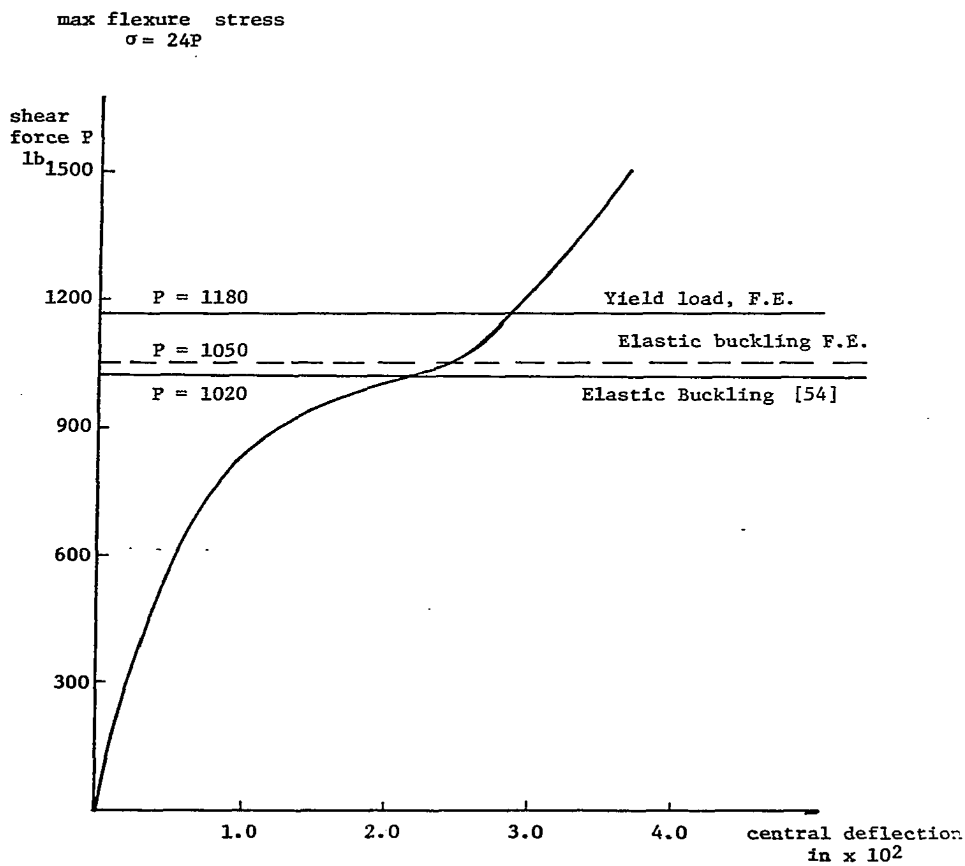


Fig. 4-11. Load-Deflection for In-Plane Bending and Shear

The plate was loaded to 23,040 in-lb bending moment and 1800 lb. shear force. The maximum deflection-versus applied load diagram is plotted in Figure 4-11. A change in the slope of the curve corresponds to elastic buckling load.

A closed form solution to this problem was not found in the literature, however, solution for the elastic buckling load is available in reference [54]. According to that reference the critical shear stress τ_{cr} must first be computed from $\tau_{cr} = K \frac{\Pi^2 D}{b^2 h}$, where h = thickness of the plate, b = width of the plate, K depends on the length to width ratio and D is given by $D = \frac{Eh^3}{12(1 - \nu^2)}$, where E = modulus of elasticity and ν = poisson's ratio. Next the ratio of τ/τ_{cr} is computed where τ = actual shear stress. With this number and using a graph presented in reference [54], a value for K is found which must be substituted in the formula $\sigma_{cr} = K \frac{\Pi^2 D}{b^2 h}$ to find the critical bending stress which causes buckling. This stress was found to be 24,480 psi and is plotted in Figure 4-11. The solution is restricted to elastic buckling load and a solution is not presented for the post-buckling behavior. The finite element method was used to obtain a load-deflection relationship beyond the elastic buckling point.

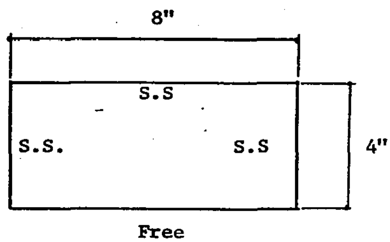
Example 4. Buckling of a Plate with One Free Edge

In this example a rectangular plate 4 in. by 8 in. by .06 in. thick with three sides simply supported and one side free is studied. Geometry of the plate and boundary conditions are illustrated in Figure 4-12-a. In-plane loads in increments of 272 lb. was applied to the plate, an out-of-plane deflection was produced by applying a concentrated load at the center of the plate, this load was applied in increments of 0.272 lb. Utilizing symmetry only one-half of the plate was analyzed.

To study convergence of the procedure, four different meshes were used and the plate loaded to 1360 lb. The maximum deflection for the different meshes is plotted in Figure 4-13. It is seen that a 4 x 8 mesh provides adequate results for this problem.

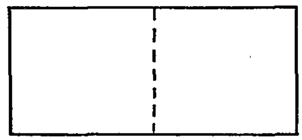
Using the 4 by 8 mesh, the plate was loaded to 3264 lb. (12 increments). Again, the failure load of the plate is approximated considering that it occurs at the first yield of the material which again was assumed to be 36000 psi. The load-deflection diagram is plotted in Figure 4-14. The buckling and failure loads are indicated in the same figure.

For this problem classical solution for elastic buckling load is available in reference [54] where the critical stress may be obtained from $\sigma_{cr} = \frac{K\Pi^2 D}{b^2 h}$, where K is a factor depending on the load and edge conditions and other notations were defined previously. The buckling load computed from this formula and the value obtained from finite element solution are shown in Figure 4-14. Classical solutions for post-buckling behavior are not available, however, an approximate value of the failure load may be obtained using the effective width

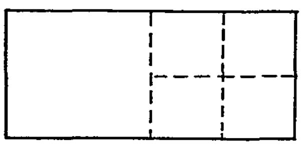


thickness = .06 in.

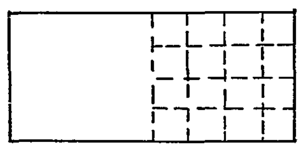
a- geometry and boundary conditions



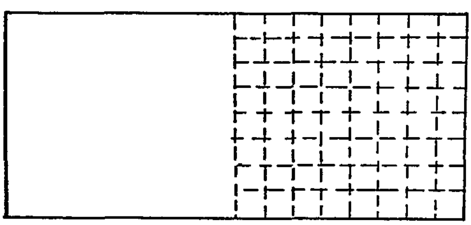
b- 1 by 2 mesh



c- 2 by 4 mesh



d- 4 by 8 mesh



e- 8 by 16 mesh

Fig. 4-12. Geometry, Boundary Conditions, Models

(Example 4)

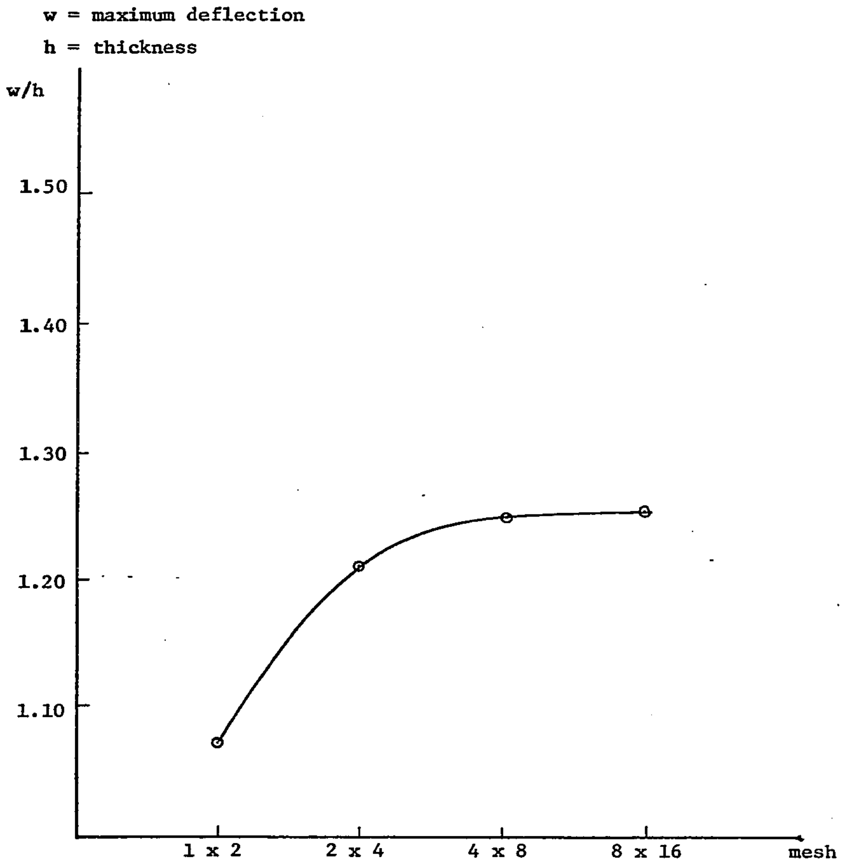


Fig. 4-13. Convergence for Plate Three Sides Simply Supported One Side Free

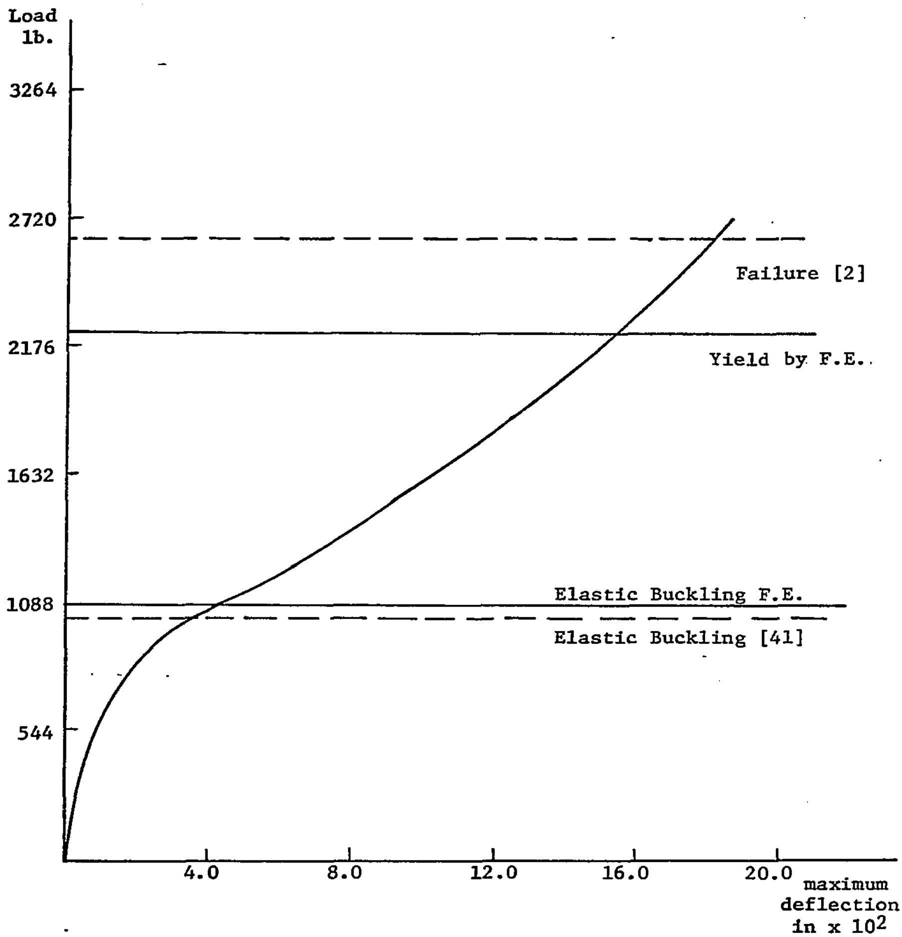


Fig. 4-14. Load Deflection Diagram for a Plate

Three Sides Simply Supported One Side Free concept [20]. Based on the method of reference [20], σ_{cr} is substituted in the formula $\frac{be}{b} = \sqrt{\frac{\sigma_{cr}}{\sigma_{max}}}\left(1.0 - 0.25\sqrt{\frac{\sigma_{cr}}{\sigma_{max}}}\right)$, where σ_{max} = maximum stress (in this case yield stress of the material), be = effective width, b = actual width. Then computing effective width and effective area and multiplying by σ_{max} the failure load is obtained which is indicated in Figure 4-14.

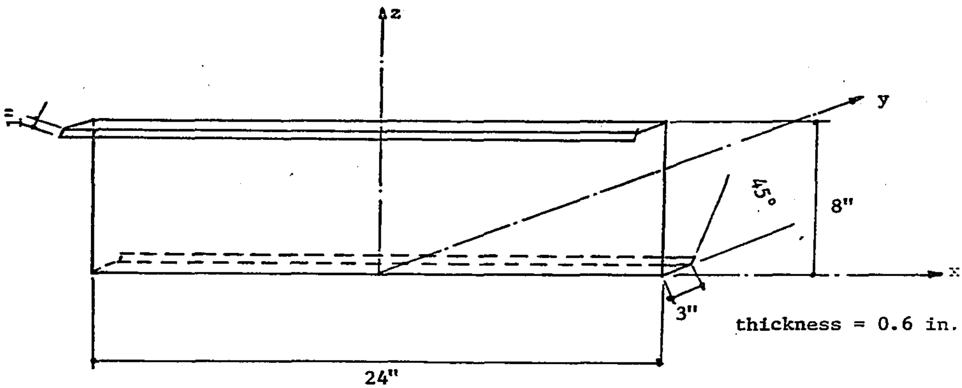
Example 5. Elastic Buckling and Failure of a Thin-Walled Stub Column

For this example a lipped Z section was selected. The dimensions are shown in Figure 4-15-a: the length is three times the maximum dimension of the cross section, the lip angle is 45 degrees, and the thickness is .06 in. Utilizing symmetry only one-half of the column was analyzed. At the nodes located on the center of the column x- displacement and y- and z- rotation were set equal to zero. Lips, flanges and web were divided lengthwise and the web was divided through the width into two equal segments. Thus, between transverse element lines there are 6 elements - two for the lips, two for the flanges and two for the web. The load was applied in the x- direction in increments of 2000 lb., distributed to the nodal points to produce uniform compression. A small load was also applied in the y- direction at the center of the column in increments of 0.5 lb. to study out-of-plane behavior. A convergence study was first performed using the three meshes shown in Figures 4-15-b, 4-15-d. The meshers have 24, 48, and 96 elements, respectively. Maximum deflection versus mesh size for a 1000 lb. load is plotted in Figure 4-16 and it is seen that an adequate result is obtained using 48 elements.

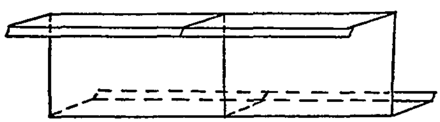
The mesh with 48 elements was then loaded to 26000 lb. and the resulting load-deflection relationship plotted in Figure 4-17. The change in the slope of the curve defines the elastic buckling load. Stress distribution over a cross section near the centerline for two different loads is plotted in Figures 4-18-a and 4-18-b. Figure 4-18-a illustrates the stress before elastic buckling and Figure 4-18-b shows the distribution after elastic buckling. To determine the yield load linear interpolation was made between the increments just before and after the assumed yield stress, 36000 psi, was reached.

The elastic buckling load may be obtained [54] using $\sigma_{cr} = K \frac{\Pi^2 D}{b^2 h}$ where σ_{cr} = buckling stress and K = 4.0 for a plate with all boundaries simply supported (flanges and web), K = 0.456 for a long plate three sides simply supported and one side free (lips). Other notations in the above formula have been defined previously. Applying the above formula to the flange, web, lip it is found that only the web buckles in the elastic range and the buckling load obtained in this manner is shown in Figure 4-17 and it is seen that this load is slightly lower than the load obtained by the finite element solution. The discrepency is caused by the factor K. The value 4.0 is conservative for stiffened elements since it is defined for simple supports and the effect of adjacent elements increases K.

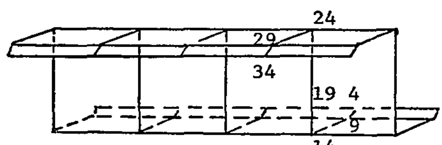
A closed form solution for post-buckling behavior and failure load does not exist, however the effective width method can be used



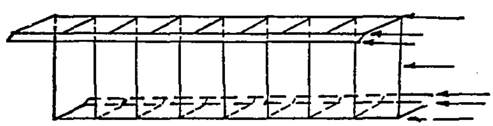
a- geometry and coordinate system for stub column



b- 24 elements (symmetry)



c- 48 elements (symmetry)



d - 96 elements (symmetry)

Fig. 4-15. Geometry and Finite Element Meshes

for Lipped Z Column

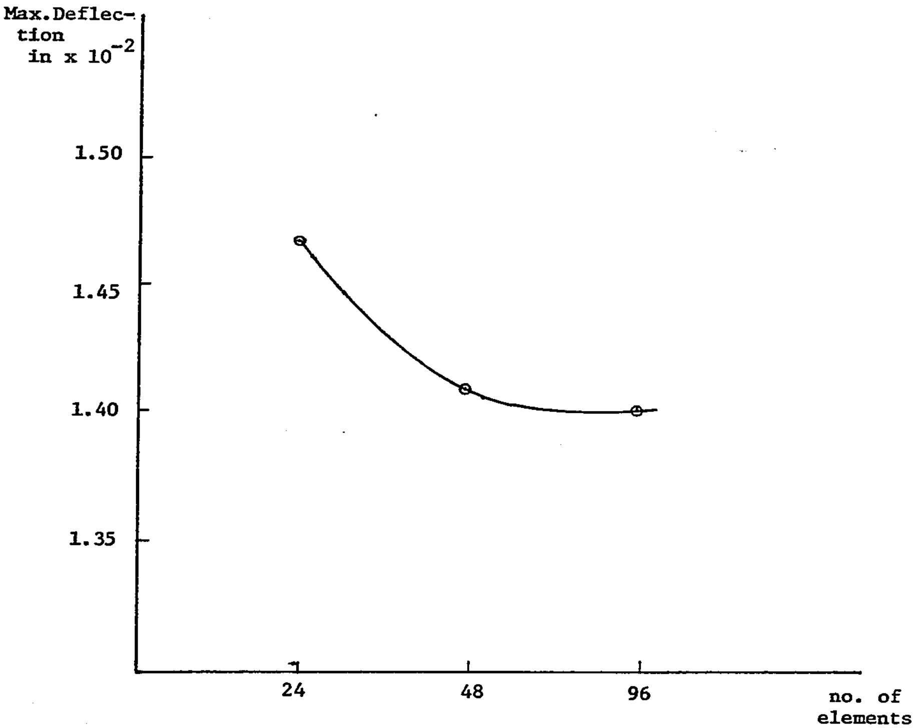


Fig. 4-16. Stub Column Convergence

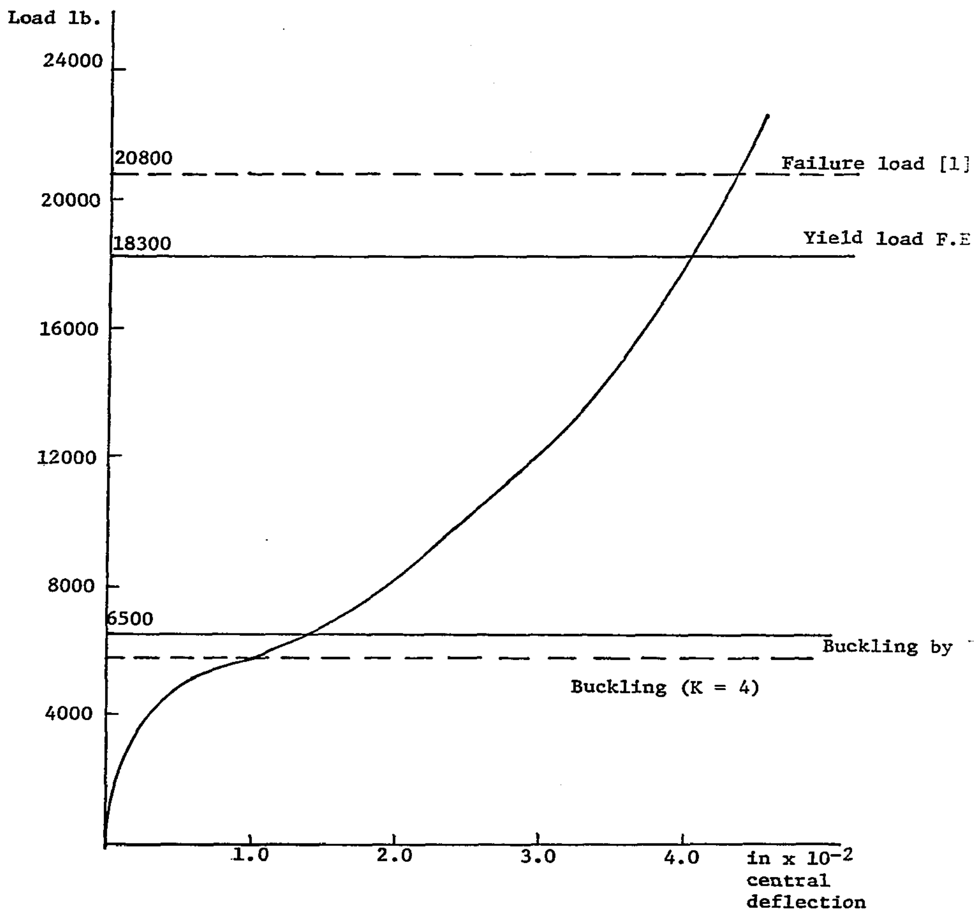
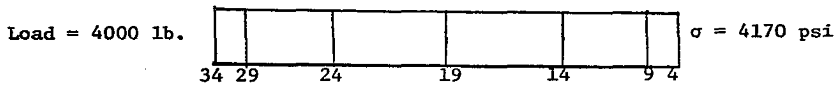
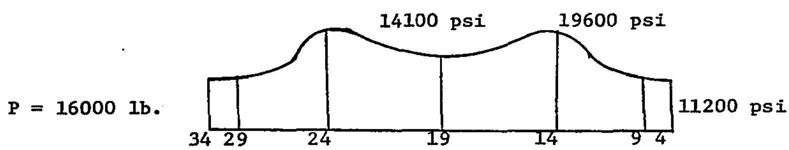


Fig. 4-17. Load-Deflection Diagram for Stub Column



a- stress distribution on the cross section before buckling

For node numbers see Figure 4-14-c.



b- stress after buckling

Fig. 4-18. Stress Distribution on the Cross Section of Stub Column to compute a failure load in the following manner [20]:

For each part of the section (lip, flange, web) the effective width is computed from

$$b_e = 0.95t \sqrt{\frac{KE}{\sigma_{max}}} \left(1.0 - \frac{0.209t}{b} \sqrt{\frac{KE}{\sigma_{max}}}\right)$$

where b_e = effective width, E = elastic modulus, t = thickness, b = actual width, σ_{max} = yield stress (36000 psi), K = 4.0 for stiffened elements and K = 0.456 for unstiffened elements. The sum of the effective width multiplied by σ_{max} results in the failure load. The result of effective width computation is also shown in Figure 4-17. It is seen that failure load by the finite element method is lower than that given by effective width method. The effective width method includes the effect of post yielding resistance which is not included in the finite element method.

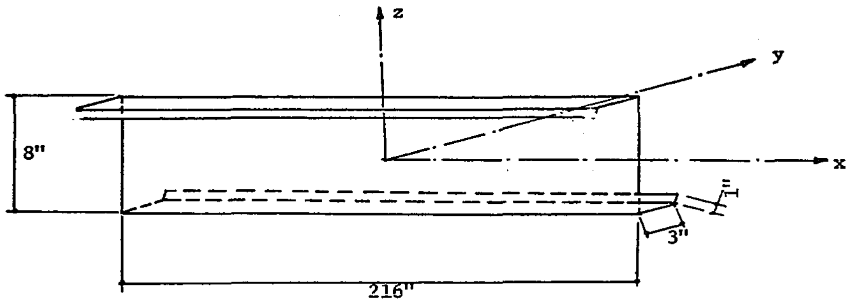
Example 6. Bending of Thin-Walled Beam Having Lipped Z Section

In this example a thin-walled beam having a lipped Z cross section was analyzed. Again symmetry is utilized and boundary conditions at the nodes located on the center of the beam are similar to those mentioned for Example 5. Geometry of the beam is illustrated in Figure 4-19-a. The beam is 18 ft. long simply, supported and subjected to a uniformly distributed load over the length of the top flange. The load was replaced by equivalent concentrated loads which are also shown in Figure 4-19. In this example it was intended to study simple bending, hence to avoid torsion and lateral bending, supports were provided by restraining the displacements of the web in the y-direction and also rotations about the x- and z- axes at the web nodes. The above treatment models the beam as a purlin which is laterally supported and is free to bend in the plate of the web.

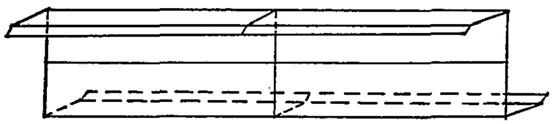
Convergence of the procedure was studied using three different meshes of 24, 48, and 96 elements, Figures 4-19-b, 4-19-d. A load of 6.75 lb/in was applied in increments of 1.35 lb/in and deflections calculated. Comparison of maximum deflection in the z- direction with the mesh size is shown in Figure 4-20.

The mesh with 48 elements was selected and loaded to 16.20 lb/in in 1.35 lb/in increments. The load deflection relationship for the center of the beam is plotted in Figure 4-21 together with the load at the first yield (36,000 psi). Deflection obtained from the beam theory using the formula $\Delta = \frac{5wl^4}{384EI}$ is also shown in Figure 4-21. In this formula w = uniform load, l = length, E = modulus of elasticity, I = moment of inertia. Stress distributions over the cross-section at the centerline for three different loads are plotted in Figure 4-22.

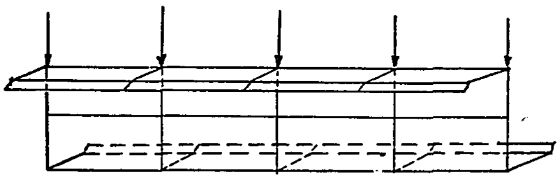
The ultimate bending moment capacity of the beam is found using the method described in reference [2]. In this method the post-buckled strength of unstiffened elements is estimated using a stress reduction factor and post-buckled strength of stiffened elements is estimated using the effective width concept. The predicted failure load 11.28 lb/in agrees quite closely with 12.22 lb/in obtained using the proposed finite element procedure.



a- geometry of the beam

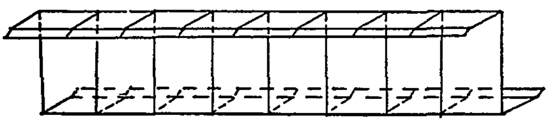


b- 24 elements (symmetry)



loads are shown for 1 increment

c- 48 elements (symmetry)



d- 96 elements (symmetry)

Fig. 4-19. Beam of Example 6

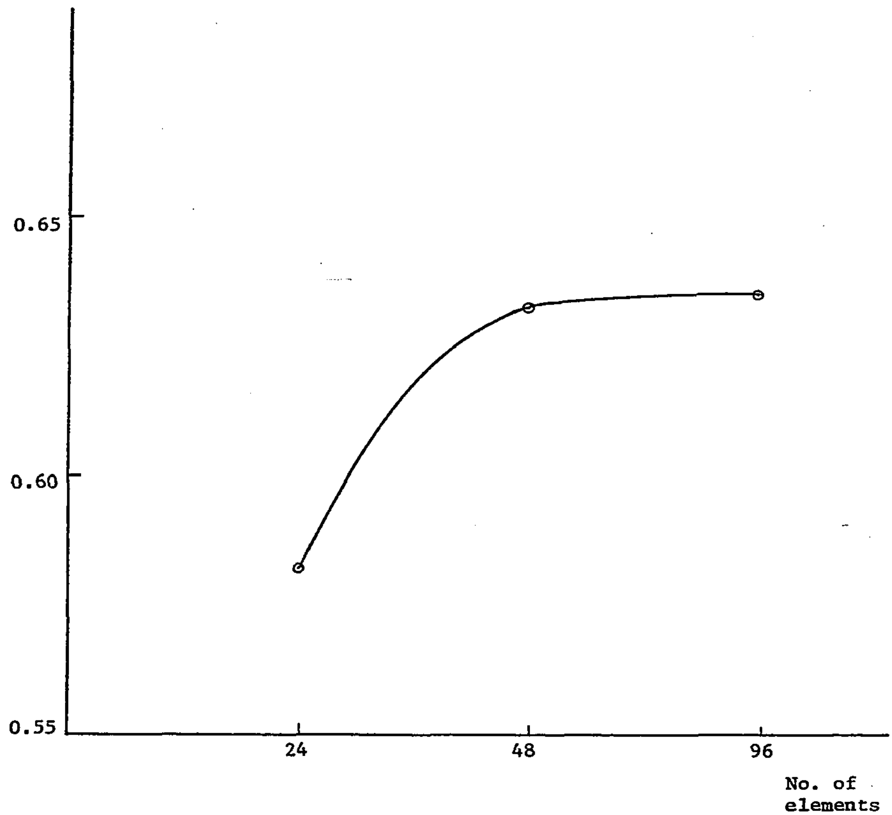


Figure 4-20. Convergence for the Beam of Example 6

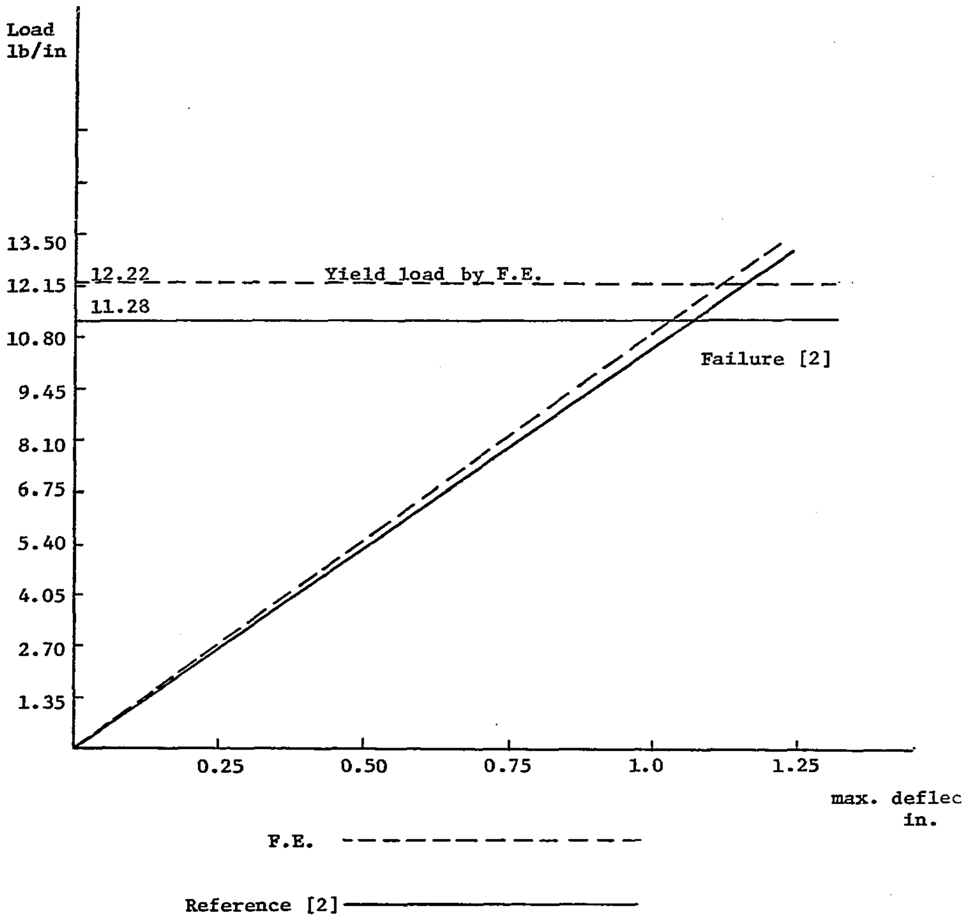


Figure 4-21. Load Deflection Diagram for

the Beam of Example 6

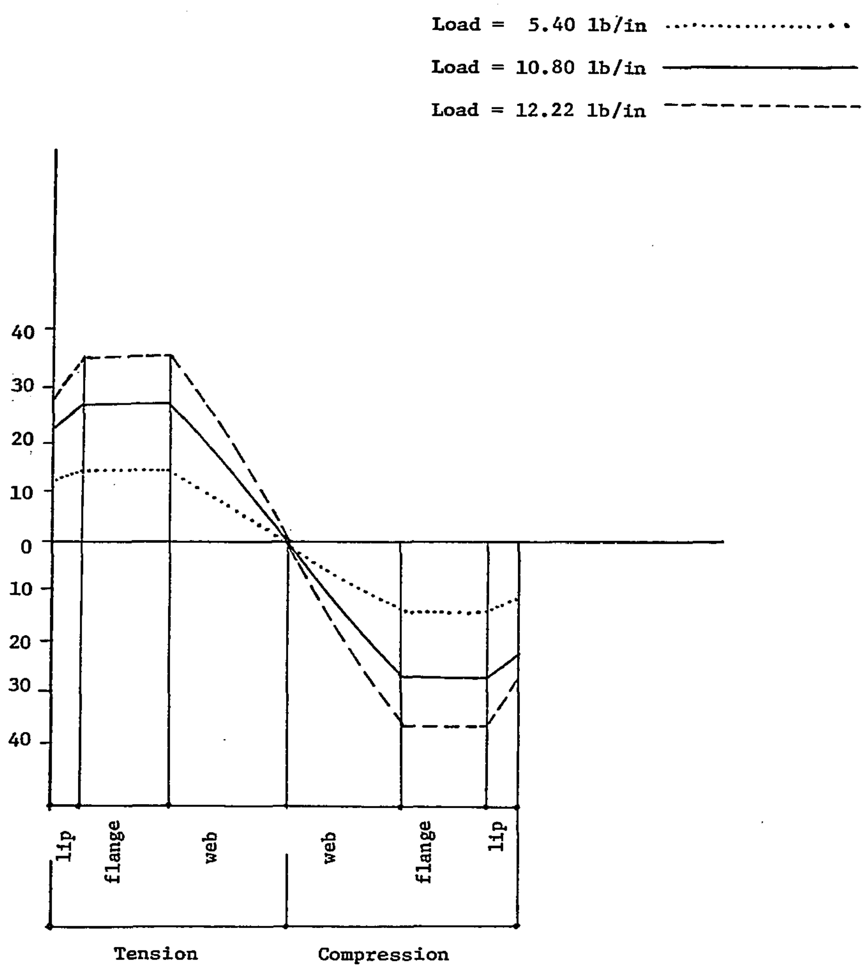


Figure 4-22. Stress Distribution Over the Cross Section at the Centerline (Example 6)

CHAPTER V

SUMMARY AND CONCLUSIONS

The purpose of this study was to investigate post-buckling strength of thin-walled sections. A survey of the literature was performed to determine the state-of-the-art. It is found that classical solutions for post-buckling problems are very limited and are usually based on Von Karman's equation. The Von Karman equation is mathematically complex and a closed form solution cannot be obtained. Approximate solutions are available for a few simple plates with either fixed, simple or free boundary conditions. These solutions were usually obtained using one of the approximate energy methods.

The finite element method has a broad potential for application to post-buckling problems but a survey of the literature indicated that post-buckling analyses using the finite element method have not been extensively studied. Because of the suitability of the finite element method it was selected here for a study of post-buckling strength of thin-walled cross-sections. The displacement approach was selected and nonlinearity was considered using the nonlinear strain-displacement expressions through the Lagrangian definition of the strain tensor. The traditional eigenvalue approach was discarded and both buckling and post-buckling phenomenon was studied using nonlinear analysis. To obtain complete formulation for the interaction of in-plane and out-of-plane behavior in the large deflection range, standard stiffness matrices were modified. The development resulted in the formulation of three submatrices of the element stiffness matrix, which to the knowledge of the author are new. Matrices K_{12} and K_{21}, K_{12} is the transpose of K_{21}, account for the effect of the membrane load on the transverse deflection and the effect of the transverse load on the membrane deflection, respectively. The third matrix is K_{22}' which is the additional stiffness for a bending element due to large deflection behavior. The procedure developed here is equally valid for bending, buckling and post-buckling studies, and unlike the effective width method which uncouples the different modes and considers the plate components of the cross-section separately, the member is treated as a whole.

A computer program was developed for numerical studies and a wide variety of problems were solved. The selected results were presented here and reasonable agreement was obtained with existing solutions either experimental and empirical or theoretical. Several solutions were presented for the problems never before solved in the literature.

Results obtained for a plate bending problem show very good approximation even using very few elements. Results of the post-buckling analysis of thin plates compared to other available solutions showed good correlation. As was mentioned earlier very few solutions exist in the literature for post-buckling problems, but the close agreement obtained indicated that the proposed method is satisfactory. Thin-walled members having lipped Z cross-sections were analyzed and compared to an analysis using the effective width method. These problems were selected because of their complex geometry and also their practical usefulness.

The overall result of the study is indicative of the possibility for further improvement in the field of post-buckling analysis and large deflection behavior for bending and buckling problems using the finite element method. First, it may be possible to develop a more efficient software to increase the practical usefulness of the suggested method; the present work is considered mainly as a research tool. Secondly, for the class of problems considered here, inclusion of combined material and geometric nonlinearity is not efficient with the proposed method. Development of a more efficient constitutive law for the material behavior is needed before the proposed method can be expanded.

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APPENDIX A

COMPUTER PROGRAM

A-1. Description of Computer Program

For the numerical studies described, a computer program was developed. The program consists of a main subprogram and a number of subroutines described as follows:

Main - Reads and prints the input data; computes half-band-width for the structural stiffness matrix.

ASMBL1 - Assembles element stiffness matrices into the structural stiffness matrix; computes displacements at the condensed degrees of freedom; computes total displacements; transforms from local to global coordinates and vice-versa.

STIF - Computes element stiffness matrix for small deflection (first increment).

BNDRY - Imposes boundary conditions.

SOLVE - Solves system of equation taking advantage of the banded matrix.

LARGE - Computes element stiffness matrix for large deflection.

STRESS - Computes stresses at the integration points and at the nodal points.

ARANG - Arranges element stiffness matrices in a proper form for condensation and assemblage.

ARANG2 - Rearranges the displacements which are then used in the computation of the element stiffness matrix for the next increment.

CONDNS - Condenses internal degrees of freedom for each element.

A-2. Input Data

Type 1		**General Parameters**
cols.	1-30	Material constants
	31-35	Number of prescribed boundary conditions
	36-40	Number of concentrated loads
	41-45	Stress printing interval
	46-50	Number of elements
	51-55	Number of nodes
	56-60	Number of increments
Type 2		**Concentrated Loads**
cols.	1-4	Load index
	5-16	Load value; one card for each load.
Type 3		**Prescribed Boundaries**
cols.	1-80	Indices of prescribed boundaries; 20 indices per each card.
Type 4		**Element Information**
cols.	1-20	Nodal numbers in a clockwise sense about the z-axis of the element; five columns for each number.

	21-30	Uniform load per in^2 of the area of the surface of the element. These are the loads uniformly distributed over the surface of the element.
	31-40	Thickness of the element.

<u>Type 5</u> <u>Nodal Coordinate</u>

cols.	1-80	x, y, z coordinates of each node in the order of nodal numbers; 10 columns for each value.

```
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON AK(210,42),QK(22,22),UM(24,22),QG(210),Q(22),X(35),Y(35),
     $Z(35),XO(4),YO(4),PLCAD(24),TH(24),DEL(210),F(210),VBDY(65),E,PR,
     $NOD(24,4),IBDY(65),NEL,NNP,NHEW,NEQ,NBDY,NRMAX,NCMAX,NDF,KTN
      DIMENSION CQK(24,2,22)
C     DEFINITION OF VARIABLES:
C     AK.....GLOBAL STIFFNESS MATRIX FOR STRUCTURE
C     QK.....ELEMENT STIFFNESS MATRIX (LOCAL)
C     UM.....DISPLACEMENTS AT NODAL POINTS (LOCAL)
C     QG.....GLOBAL DISPLACEMENTS AND GLOBAL LOADS FOR STRUCTURE
C     Q.....LOAD VECTOR FOR ELEMENT (LOCAL)
C     X,Y,Z.....GLOBAL COORDINATES OF NODES
C     XO,YO .....LOCAL COORDINATES OF NODES
C     DEL.....DISPLACEMENTS AT NODES
C     NOD.....GLOBAL POSITION OF LOCAL NODES (BOOLEAN MATRIX)
C     NEL.....NUMBER OF ELEMENTS
C     NNP.....NUMBER OF NODAL POINTS
C     NHEW.....HALF-BAND-WIDTH
C     NEQ.....NUMBER OF EQUATIONS
C     CQK.....ARRAY TO STORE PART OF STIFFNESS MATRIX FOR LATER USE IN
C     COMPUTATION OF DISPLACEMENTS AT INTERNAL NODES
C     READ AND PRINT NUMBER OF ELEMENTS NUMBER OF NODES STEPS OF LOADING
      READ 1,E,PR,NBDY,NN,KTN,NEL,NNP,KM
      PRINT 102,NEL,NNP,KM
      DO 55 II=1,210
   55 F(II)=0.0
      DO 5 NI=1,NN
    5 READ 6,N,F(N)
      READ 7,(IBDY(I),I=1,NBDY)
      NRMAX=210
      NCMAX=42
      NDF=6
      DO 20 I=1,NBDY
   20 VBDY(I)=0.0
C     COMPUTE NUMBER OF EQUATIONS
      NEQ=NNP*NDF
C     INITIALIZE LOADING STEPS
      KT=1
C     INITIALIZE NODAL DISPLACEMENTS
      DO 10 M=1,24
      DO 10 II=1,22
   10 UM(M,II)=0.0
      DO 15 M=1,24
      DO 15 I=1,4
   15 NOD(M,I)=0
      PRINT 105
C     INPUT NOD NUMBERS ,LOAD PER UNIT AREA AND ELEMENT THICKNESSES
      READ 3,((NOD(M,I),I=1,4),PLOAD(M),TH(M),M=1,NEL)
      DO 50 M=1,NEL
   50 PRINT 103,M,(NOD(M,I),I=1,4),PLCAD(M),TH(M)
C     COMPUTE HALF-BANDWIDTH
```

```
      MAXDIF=0
      DO 30 N=1,NEL
      DO 30 I=1,4
      DO 30 J=1,4
      L=IABS(NOD(N,I)-NOD(N,J))
      IF(L.GE.MAXDIF)MAXDIF=L
   30 CONTINUE
      NHBW=NDF*(MAXDIF+1)
C.....INPUT NODAL COORDINATES
      READ 4,(X(I),Y(I),Z(I),I=1,NNP)
      PRINT 106
      DO 40 I=1,NNP
   40 PRINT 104,I,X(I),Y(I),Z(I)
C     COMPUTE ELEMENT STIFFNESS MATRICES TRANSFORM TO GLOBAL ASSEMBLE TO
C     STRUCTURAL  STIFFNESS AND SOLVE TO GET DEFLECTIONS
  300 CALL ASMBL1(COK,KT)
      KT=KT+1
C     CHECK TO SEE IF TOTAL NUMBER OF STEPS COMPLETED
      IF(KT.LE.KM) GO TO 300
   60 STOP
    1 FORMAT(2F15.5,6I5)
    3 FORMAT(4I5,2F10.3)
    4 FORMAT(8F10.3)
    6 FORMAT(I4,F12.3)
    7 FORMAT(20I4)
  102 FORMAT(1X,'NUMBER OF ELEMENTS',I5//1X,'NUMBER OF NODS',4X,I5//1X,'
     $NUMBER OF LOADING',2X,I4)
  103 FORMAT(5X,I3,4(5X,I5),2F14.3)
  104 FORMAT(I4,10X,F12.3,2(5X,F12.3))
  105 FORMAT(1H0,'ELEMENT',5X,'NOD-I',5X,'NOD-J',5X,'NOD-K',5X,'NOD-L',
     $10X,'LOAD',5X,'THICKNESS'/56X,'(LB/IN2)',8X,'(IN)')
  106 FORMAT(1H0,'NOD',10X,'X-COORDINATE',5X,'Y-COORDINATE',5X,'Z-COORDI
     $NATE')
      END
      SUBROUTINE ASMBL1 (COK,KT)
C     THIS SUBROUTINE COMPUTES ELEMENT STIFFNESS MATRICES AND LOAD VECTO
C     TRANSFORMS FROM LOCAL TO GLOBAL ASSEMBLES AND SOLVES EQUATIONS
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON AK(210,42),QK(22,22),UM(24,22),QG(210),C(22),X(35),Y(35),
     $Z(35),XQ(4),YQ(4),PLOAD(24),TH(24),DEL(210),F(210),VBDY(65),E,PR,
     $NOD(24,4),IBDY(65),NEL,NNP,NHBW,NEQ,NBDY,NFMAX,NCMAX,NDF,KTN
      DIMENSION AG(24,24),TG(20,24),DUG(24),QR(24),ZG(4),YG(4)
      DIMENSION U(22),DU(22),COK(24,2,22),CQ(20,20),CINV(12,12)
C     DEFINITIONS OF VARIABLES:
C     AG.....GLOBAL STIFFNESS FOR THE ELEMENT
C     TG.....TRANSFORMATION MATRIX (FOR LOCAL GLOBAL TRANSFORMATION)
C     DUG.....INCREMENT OF DISPLACEMENTS(GLOBAL)
C     QR LOAD VECTOR FOR THE ELEMENT (GLOBAL)
C     ZG,YG.....ELEMENT NODAL COORDINATES (GLOBAL)
C     U.....VECTOR OF TOTAL DISPLACEMENTS AT NODES FOR THE ELEMENTS
C     DU.....VECTOR OF INCREMENTAL DISPLACEMENTS AT NODES (ELEMENT LOCAL
```

```
C      CQ.....CONDENSED ELEMENT STIFFNESS MATRIX(LOCAL)
C      CINV.....RELATED TO AXIAL STRAIN DUE TO BENDING
C      IBDY GLOBAL DEGREES OF FREEDOM AT WHICH BOUNDARY CONDITIONS ARE
C      PRESCRIBED
C      VBDY VALUES OF PRESCRIBED BOUNDARY CONDITIONS
C      NRMAX ....MAXIMUM NUMBER OF ROWS
C      NCMAX.....MAXIMUM NUNBER OF COLUMNS
C      NDF ....DGREES OF FREEDOM AT EACH NODE
C      TH(M) THICKNESS OF ELEMENT "M"
C      INITIALIZE TRANSFORMATION MATRIX
       DO 56 I=1,20
       DO 56 J=1,24
   56  TG(I,J)=0.0
C      INITIALIZE STRUCTURE STIFFNESS MATRIX AND LOAD VECTOR
       DO 100 I=1,NRMAX
       QG(I)=0.0
       DO 100 J=1,NCMAX
  100  AK(I,J)=0.0
C      DO LOOP ON ELEMENT
       DO 1 M=1,NEL
       THM=TH(M)
C      INITIALIZE STIFFNESS MATRIX AND LOAD VECTOR FOR THE ELEMENT
       DO 45 I=1,22
       Q(I)=0.0
       DO 45 J=1,22
   45  QK(I,J)=0.0
       DO 2 I=1,4
       II=NCD(M,I)
       XQ(I)=X(II)
       YG(I)=Y(II)
       ZG(I)=Z(II)
    2  CONTINUE
       D14=DSQRT((ZG(4)-ZG(1))**2+(YG(4)-YG(1))**2)
C      FIND DIRECTION COSINE FOR LOCAL GLOBAL Y AND Z COORDINATES
       YPY=(YG(4)-YG(1))/D14
       YPZ=(ZG(4)-ZG(1))/D14
       DO 53 I=1,4
C      FIND LOCAL Y COORDINATE FOR THE ELEMENT
   53  YQ(I)=DABS(YPY*YG(I)+YPZ*ZG(I))
C      FIND DIMENSIONS OF THE ELEMENT IN THE LOCAL X,Y PLANE
       AL=DABS(XQ(2)-XQ(1))
       BL=DABS(YQ(4)-YQ(1))
C      COMPUTE LOAD VECTOR (LOCAL,ELEMENT)
       QL=PLOAD(M)*AL*BL/4.0
       Q(11)=QL
       Q(12)=QL*BL/6.
       Q(13)=-QL*AL/6.
       Q(14)=QL
       Q(15)=Q(12)
       Q(16)=-Q(13)
       Q(17)=QL
```

```
      Q(18)=-Q(12)
      Q(19)=Q(16)
      Q(20)=QL
      Q(21)=Q(18)
      Q(22)=Q(13)
C     COMPUTE TRANSFORMATION MATRIX
      TG(1,1)=1.0
      TG(2,2)=YPY
      TG(2,3)=YPZ
      TG(3,2)=-YPZ
      TG(3,3)=YPY
      TG(4,4)=1.0
      TG(5,5)=YPY
      TG(5,6)=YPZ
      TG(6,7)=1.0
      TG(7,8)=YPY
      TG(7,9)=YPZ
      TG(8,8)=-YPZ
      TG(8,9)=YPY
      TG(9,10)=1.0
      TG(10,11)=YPY
      TG(10,12)=YPZ
      TG(11,13)=1.0
      TG(12,14)=YPY
      TG(12,15)=YPZ
      TG(13,14)=-YPZ
      TG(13,15)=YPY
      TG(14,16)=1.0
      TG(15,17)=YPY
      TG(15,18)=YPZ
      TG(16,19)=1.0
      TG(17,20)=YPY
      TG(17,21)=YPZ
      TG(18,20)=-YPZ
      TG(18,21)=YPY
      TG(19,22)=1.0
      TG(20,23)=YPY
      TG(20,24)=YPZ
C     SELECT GLOBAL DISPLACEMENTS FOR THE ELEMENT
      IF(KT.EQ.1) GO TO 72
      DO 55 I=1,4
      DO 55 IL=1,NDF
      JJ=(NOD(M,I)-1)*NDF+IL
      II=(I-1)*NDF+IL
   55 DUG(II)=DEL(JJ)
C     TRANSFORM ELEMENT NODAL DISPLACEMENTS TO LOCAL COORDINATES
      DO 301 I=1,20
      DU(I)=0.0
      DO 301 J=1,24
  301 DU(I)=DU(I)+TG(I,J)*DUG(J)
```

```
C     RECOVER CONDENSED DISPLACEMENTS AT THE CENTRAL NOD OF THE ELEMENT
      DU(21)=0.0
      DU(22)=0.0
      DO 80 K=1,2
      IK=K+19
      JK=IK+1
      DO 80 L=1,IK
   80 CU(JK)=DU(JK)-CQK(M,K,L)*DU(L)
C     ARRANGE INCFEMENTAL DISPLACEMENTS FIRST IN PLANE ,THEN OUT OF PLANE
      CALL ARANG2(DU,22)
      DO 90 II=1,22
C     COMPUTE TOTAL DISPLACEMENT AT THE NODES
      UM(M,II)=UM(M,II)+DU(II)
   90 U(II)=UM(M,II)
C     CCMPUTE ELEMENT STIFFNESS MATRIX FOR LARGE DEFLECTICN
      CALL LARGE(U,QK,AL,BL,THM,E,PR,NOD,M,KT,KTN)
      GO TO 205
   72 CONTINUE
C     CCMPUTE ELEMENT STIFFNESS MATRIX FOR SMALL DEFLECTICN
      CALL STIF(QK,CINV,AL,BL,THM,E,PR,M,KT)
  205 CCNTINUE
C     TAKE ADVANTAGE OF SYMMETRY TC COMPLETE ELEMENT STIFFNESS MATRIX
      DO 85 II=11,22
      DC 85 JJ=1,10
   85 QK(II,JJ)=QK(JJ,II)
C     ARRANGE ELEMENT STIFFNESS MATRIX IN TERMS OF NCDAL DISPLACEMENT AT
C     EACH NODE
      CALL ARANG(QK,Q,22)
C     CCNDENSE INTERNAL DEGREES CF FREEDOM
      CALL CONDNS (QK,Q,22,2)
C     STORE PARTS OF CONDENSED ELEMENT STIFFNESS MATRIX  FOR LATER USE
C     IN CCMPUTATION OF DISPLACEMENT AT INTERNAL NODES
      DO 95 I=1,2
      DO 95 J=1,22
      II=I+20
   95 CQK(M,I,J)=QK(II,J)
C     TAKE THE CONDENSED PART FOR TRANSFORMATION OF STIFFNESS MATRIX TO GLCBAL
      DO 155 I=1,20
      DO 155 J=1,20
  155 CQ(I,J)=QK(I,J)
C     TRANSFORM STIFFNESS MATRIX TC GLOBAL
      CALL MATMLT(TG,20,24,CQ,TG,24,AG)
C     TRANSFORM LCAD TC GLCBAL
      DO 185 J=1,24
      QR(J)=0.0
      DC 185 I=1,20
  185 QR(J)=QR(J)+TG(I,J)*Q(I)
C.....ASSMBLE TO GLOBAL
      ND=6
      DO 75 I=1,4
      NR=(NOD(M,I)-1)*ND
```

```
      DO 75 II=1,ND
      NR=NR+1
      L=(I-1)*ND+II
      QG(NR)=QG(NR)+QR(L)
      DO 70 J=1,4
      NCL=(NOD(M,J)-1)*ND
      DO 65 JJ=1,ND
      N=(J-1)*ND+JJ
      NC=NCL+JJ+1-NR
      IF(NC)65,65,60
   60 AK(NR,NC)=AK(NR,NC)+AG(L,N)
   65 CONTINUE
   70 CONTINUE
   75 CONTINUE
    1 CONTINUE
      DO 500 II=1,NEQ
  500 QG(II)=QG(II)+F(II)
      IF (KT.GT.1) GO TO 115
      PRINT 160
  160 FORMAT(///,1X,'EQUIVALENT NODAL LOADS FOR EACH STEP'//5X,'NOD',10X
     $,'LOAD-X',10X,'LOAD-Y',10X,'LOAD-Z',8X,'MOMENT-X',8X,'MOMENT-Y',
     $8X,'MOMENT-Z')
      DO 280 II=1,NNP
      JJ=(II-1)*NDF+1
      MJ=JJ+5
  280 PRINT 110,II,(QG(KJ),KJ=JJ,MJ)
  110 FORMAT(5X,I3,6(4X,E12.5))
  115 CONTINUE
C     IMPOSE BOUNDARY CONDITIONS
      DO 118 I=1,NBDY
      IE=IBDY(I)
      VE=VBDY(I)
  118 CALL BNDRY(NRMAX,NCMAX,NEQ,NHBW,AK,QG,IE,VE)
C     SOLVE EQUATIONS
      IRES=0
      CALL SOLVE(NRMAX,NCMAX,NEQ,NHBW,AK,QG,IRES)
C     STORE DISPLACEMENTS
      DO 150 II=1,NEQ
  150 DEL(II)=QG(II)
      PRINT 200,KT
  200 FORMAT(1H1,'STEP',I3)
      PRINT 250
  250 FORMAT(///,5X,'NOD',5X,'DEFLECTION-X',5X,'DEFLECTION-Y',5X,'DEFLEC
     $TION-Z',7X,'ROTATION-X',7X,'ROTATION-Y', 7X,'ROTATION-Z')
      DO 350 II=1,NNP
      JJ=(II-1)*NDF+1
      MJ=JJ+5
  350 PRINT 210,II,(QG(KJ),KJ=JJ,MJ)
  210 FORMAT(5X,I3,6(5X,E12.5))
      RETURN
      END
```

```
      SUBROUTINE MATMLT(A,M,N,C,B,L,ATCB)
      IMPLICIT REAL*8(A-H,O-Z)
C     THIS SUBROUTINE MULTIPLIES TRASPOSE OF MATRIX "A" TIMES MATRIX "C"
C     AND THEN TIMES MAYRIX "B"
      DIMENSION A(M,N),C(M,M),B(M,L),ATC(24,20),ATCB(N,L)
      DO 2 I=1,N
      DO 2 J=1,M
      ATC(I,J)=0.0
      DO 2 K=1,M
    2 ATC(I,J)=ATC(I,J)+A(K,I)*C(K,J)
      DO 1 I=1,N
      DO 1 LL=1,L
      ATCB(I,LL)=0.0
      DO 1 J=1,M
    1 ATCB(I,LL)=ATCB(I,LL)+ATC(I,J)*B(J,LL)
      RETURN
      END
      SUBROUTINE BNDRY(NRMAX,NCMAX,NEQ,NHBW,S,SL,IE,SVAL)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION S(NRMAX,NCMAX),SL(NRMAX)
      IT=NHBW-1
      I=IE-NHBW
      DO 100 II=1,IT
      I=I+1
      IF(I.LT.1) GO TO 100
      J=IE-I+1
      SL(I)=SL(I)-S(I,J)*SVAL
      S(I,J)=0.0
  100 CONTINUE
      S(IE,1)=1.0
      SL(IE)=SVAL
      I=IE
      DO 200 II=2,NHBW
      I=I+1
      IF(I.GT.NEQ)GO TO 200
      SL(I)=SL(I)-S(IE,II)*SVAL
      S(IE,II)=0.0
  200 CONTINUE
      RETURN
      END
      SUBROUTINE SOLVE(NRM,NCM,NEQNS,NBW,BAND,RHS,IRES)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION BAND(NRM,NCM),RHS(NRM)
      MEQNS=NEQNS-1
      IF(IRES.GT.0)GO TO 90
      DO 500 NPIV=1,MEQNS
      NPIVOT=NPIV+1
      LSTSUB=NPIV+NBW-1
      IF(LSTSUB.GT.NEQNS)LSTSUB=NEQNS
      DO 400 NROW=NPIVOT,LSTSUB
      NCOL=NROW-NPIV+1
      IF(DABS(BAND(NPIV,1)).LT.1.0D-4)BAND(NPIV,1)=1.0
```

```
      FACTOR=BAND(NPIV,NCOL)/BAND(NPIV,1)
      DO 200 NCOL=NROW,LSTSUB
      ICOL=NCOL-NROW+1
      JCOL=NCOL-NPIV+1
  200 BAND(NROW,ICOL)=BAND(NROW,ICOL)-FACTOR*BAND(NPIV,JCOL)
  400 RHS(NROW)=RHS(NROW)-FACTOR*RHS(NPIV)
  500 CONTINUE
      GO TO 101
   90 DO 100 NPIV=1,MEQNS
      NPIVOT=NPIV+1
      LSTSUB=NPIV+NBW-1
      IF(LSTSUB.GT.NEQNS)LSTSUB=NEQNS
      DO 110 NROW=NPIVOT,LSTSUB
      NCOL=NROW-NPIV+1
      IF(DABS(BAND(NPIV,1)).LT.1.0D-4)BAND(NPIV,1)=1.0
      FACTOR=BAND(NPIV,NCOL)/BAND(NPIV,1)
  110 RHS(NROW)=RHS(NROW)-FACTOR*RHS(NPIV)
  100 CONTINUE
C.....BACK SUBSTITUTION
  101 DO 800 IJK=2,NEQNS
      NPIV=NEQNS-IJK+2
      IF(DABS(BAND(NPIV,1)).LT.1.0D-4)BAND(NPIV,1)=1.0
      RHS(NPIV)=RHS(NPIV)/BAND(NPIV,1)
      LSTSUB=NPIV-NBW+1
      IF(LSTSUB.LT.1)LSTSUB=1
      NPIVOT=NPIV-1
      DO 700 JKI=LSTSUB,NPIVOT
      NROW=NPIVOT-JKI+LSTSUB
      NCOL=NPIV-NROW+1
      FACTOR=BAND(NROW,NCOL)
  700 RHS(NROW)=RHS(NROW)-FACTOR*RHS(NPIV)
  800 CONTINUE
      IF(DABS(BAND(1,1)).LT.1.0D-4)BAND(1,1)=1.0
      RHS(1)=RHS(1)/BAND(1,1)
      RETURN
      END
      SUBROUTINE STIF(OK,CINV,AL,BL,THM,E,PR,M,KT)
C     THIS SUBROUTINE COMPUTES STIFFNESS MATRIX FOR THE ELEMENT IN LOCAL
C     COORDINATE SYSTEM AND IN THE FIRST STEP OF ANALYSIS WHICH IS THE
C     LINEAR PART
C     DEFINITION OF VARIABLES
C     "B" AND "CINV"  RELATE NODAL DISPLACEMENTS TO STRAINS AT ANY POINT
C     (FOR BENDING)
C     XA,YA NATURAL COORDINATES AT NODES
C     XI,ETA NATURAL COORDINATES AT INTEGRATION POINTS
C     W     VALUE OF WEIGHT FUNCTION AT INTEGRATION POINTS
C     DNDX,DNDY,DNX,DNY.....DERIVATIVES OF SHAPE FUNCTIONS FOR IN PLANE
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION OK(22,22),CINV(12,12),B(12,12)
      DIMENSION XA(4),YA(4),XI(2),ETA(2),W(2),BM(3,10),AK1(10,10)
      DIMENSION DNDX(5),DNDY(5),DNX(5),DNY(5),EM(3,3),AK2(12,12)
      DATA XA,YA,W/-1.D0,1.D0,1.D0,-1.D0,-1.D0,-1.D0,1.DC,1.D0,1.D0,1.D0
```

```
      $/
       DATA XI,ETA/0.57700,-0.57700,0.57700,-0.57700/
       A2=AL*AL
       B2=BL*BL
       AB=AL*BL
       A3=AL*AL*AL
       B3=BL*BL*BL
       A3B3=A3*B3
       A2B=A2*BL
       AB2=AL*B2
       A3B=A2B*AL
       AB3=AB2*BL
       A2B2=A2*B2
       A3B2=A3B*BL
       A2B3=AB3*AL
       A4B=A3B*AL
       AB4=AB3*BL
       A5B=A4B*AL
       AB5=AB4*BL
       IF(KT.GT.1) GO TO 35
       TH3=THM*THM*THM/12.
       TH2=THM/2.
C      ELEMENTS OF RIGIDITY MATRIX FOR BENDING
       D1=TH3*E/(1.0-PR*PR)
       D2=D1*PR
       D3=D1*(1.0-PR)/2.
C      COMPUTE CONSTITUTIVE MATRIX
       EM(1,1)=THM*E/(1.-PR*PR)
       EM(1,2)=EM(1,1)*PR
       EM(3,3)=EM(1,1)*(1.-PR)/2.
       EM(2,1)=EM(1,2)
       EM(2,2)=EM(1,1)
       EM(1,3)=0.0
       EM(2,3)=0.0
       EM(3,1)=0.0
       EM(3,2)=0.0
C      PERFORM NUMERICAL INTEGRATION FOR IN PLANE PART OF STIFFNESS MATRIX
       DO 25 J=1,2
       DO 25 K=1,2
       DO 10 I=1,4
C      COMPUTE DERIVATIVE OF SHAPE FUNCTION AT INTEGRATION POINTS
       DNDX(I)=0.5*XA(I)*(1.+YA(I)*ETA(K))/AL
       DNDY(I)=0.5*YA(I)*(1.+XA(I)*XI(J))/BL
       DNX(I)=0.5*XA(I)/AL
       DNY(I)=0.5*YA(I)/BL
    10 CONTINUE
       DNDX(5)=-4.*XI(J)*(1.-ETA(K)*ETA(K))/AL
       DNDY(5)=-4.*ETA(K)*(1.-XI(J)*XI(J))/BL
       DNX(5)=0.0
       DNY(5)=0.0
       DO 20 I=1,9,2
       K1=I+1
```

```
      K2=K1/2
C     COMPUTE DISPLACEMENT TO STRAIN TRANSFORMATION MATRIX FOR IN PLANE
C     ACTION
      BM(1,I)=DNDX(K2)
      BM(1,K1)=0.0
      BM(2,I)=0.0
      BM(2,K1)=DNDY(K2)
      BM(3,I)=DNY(K2)
   20 BM(3,K1)=DNX(K2)
C     PERFORM MATRIX MULTIPLICATION "
      CALL MATMLT(BM,3,10,EM,BM,10,AK1)
      DO 12 II=1,10
      DO 12 JJ=1,10
   12 QK(II,JJ)=QK(II,JJ)+0.25*AL*BL*AK1(II,JJ)*W(K)*W(J)
   25 CONTINUE
C     END OF NUMERICAL INTEGRATION
C     COMPUTE MATRIX "B" FOR BENDING STIFFNESS
      DO 1 I=1,12
      DO 1 J=1,12
    1 B(I,J)=0.0
      B(4,4)=4.0*D1*AB
      B(4,6)=4.0*D2*AB
      B(4,7)=6.0*D1*A2B
      B(4,8)=2.0*D1*AB2
      B(4,9)=2.0*D2*A2B
      B(4,10)=6.0*D2*AB2
      B(4,11)=3.0*D1*AB22
      B(4,12)=3.0*D2*AB22
      B(5,5)=4.0*D3*AB
      B(5,8)=4.0*D3*A2B
      B(5,9)=4.0*D3*AB2
      B(5,11)=4.0*D3*A3B
      B(5,12)=4.0*D3*AB3
      B(6,4)=4.0*D2*AB
      B(6,6)=4.0*D1*AB
      B(6,7)=6.0*D2*A2B
      B(6,8)=2.0*D2*AB2
      B(6,9)=2.0*A2B*D1
      B(6,10)=6.0*D1*AB2
      B(6,11)=3.0*D2*AB22
      B(6,12)=3.0*D1*AB22
      B(7,4)=6.0*D1*A2B
      B(7,6)=6.0*D2*A2B
      B(7,7)=12.0*A3B*D1
      B(7,8)=3.0*D1*AB22
      B(7,9)=4.0*D2*A3B
      B(7,10)=9.0*D2*AB22
      B(7,11)=6.0*D1*A3B2
      B(7,12)=6.0*D2*A3B2
      B(8,4)=2.0*D1*AB2
      B(8,5)=4.0*D3*A2B
      B(8,6)=2.0*D2*AB2
```

```
      B(8,7)=3.0*D1*AB22
      B(8,8)=(4.0*D1*AB3/3.0)+(16.0*D3*A3B/3.0)
      B(8,9)=D2*AB22+4.0*D3*AB22
      B(8,10)=4.0*D2*AB3
      B(8,11)=2.0*D1*A2B3+6.0*D3*A4B
      B(8,12)=2.0*D2*A2B3+4.0*D3*A2B3
      B(9,4)=2.0*D2*A2B
      B(9,5)=4.0*D3*AB2
      B(9,6)=2.0*D1*A2B
      B(9,7)=4.0*D2*A3B
      B(9,8)=D2*AB22+4.0*D3*AB22
      B(9,9)=(4.0*D1*A3B/3.0)+(16.*D3*AB3/3.)
      B(9,10)=3.0*D1*AB22
      B(9,11)=2.0*D2*A3B2+4.0*D3*A3B2
      B(9,12)=2.0*D1*A3B2+6.0*D3*AB4
      B(10,4)=6.0*D2*AB2
      B(10,6)=6.0*D1*AB2
      B(10,7)=9.0*D2*AB22
      B(10,8)=4.0*D2*AB3
      B(10,9)=3.0*D1*AB22
      B(10,10)=12.0*D1*AB3
      B(10,11)=6.0*D2*A2B3
      B(10,12)=6.0*D1*A2B3
      B(11,4)=3.0*D1*AB22
      B(11,5)=4.0*D3*A3B
      B(11,6)=3.0*D2*AB22
      B(11,7)=6.0*D1*A3B2
      B(11,8)=2.0*D1*A2B3+6.0*D3*A4B
      B(11,9)=2.0*D2*A3B2+4.0*D3*A3B2
      B(11,10)=6.0*D2*A2B3
      B(11,11)=4.0*D1*A3B3+(36.0*D3*A5B/5.0)
      B(11,12)=(4.0*D2*A3B3)+4.0*D3*A3B3
      B(12,4)=3.0*D2*AB22
      B(12,5)=4.0*D3*AB3
      B(12,6)=3.0*D1*AB22
      B(12,7)=6.0*D2*A3B2
      B(12,8)=2.0*D2*A2B3+4.0*D3*A2B3
      B(12,9)=2.0*D1*A3B2+6.0*D3*AB4
      B(12,10)=6.0*D1*A2B3
      B(12,11)=4.0*D2*A3B3+4.0*D3*A3B3
      B(12,12)=4.0*D1*A3B3+(36.0*D3*AB5/5.0)
   35 CONTINUE
C     COMPUTE CINV FOR THE ELEMENT
C     THIS IS INVERSE OF MATRIX "C" , TRANSPOSE OF MATRIX "CINV"
C     MULTIPLIED BY  "B" THEN MULTIPLIED BY CINV GIVES ELEMENT STIFFNESS
C     MATRIX FOR BENDING
      DO 2 I=1,12
      DO 2 J=1,12
    2 CINV(I,J)=0.0
      CINV(1,1)=1.0
      CINV(2,3)=-1.0
      CINV(3,2)=1.0
```

```
CINV(4,1)=-3.0/A2
CINV(4,3)=2./AL
CINV(4,4)=3./A2
CINV(4,6)=1.0/AL
CINV(5,1)=-1.0/AB
CINV(5,2)=-1.0/AL
CINV(5,3)=1.0/BL
CINV(5,4)=1./AB
CINV(5,5)=1.0/AL
CINV(5,7)=-1.0/AB
CINV(5,10)=1.0/AB
CINV(5,12)=-1.0/BL
CINV(6,1)=-3.0/B2
CINV(6,2)=-2.0/BL
CINV(6,10)=3.0/B2
CINV(6,11)=-1.0/BL
CINV(7,1)=2.0/A3
CINV(7,3)=-1.0/A2
CINV(7,4)=-2.0/A3
CINV(7,6)=-1.0/A2
CINV(8,1)=3.0/A2B
CINV(8,3)=-2.0/AB
CINV(8,4)=-3.0/A2B
CINV(8,6)=-1.0/AB
CINV(8,7)=3.0/A2B
CINV(8,9)=1.0/AB
CINV(8,10)=-3.0/A2B
CINV(8,12)=2.0/AB
CINV(9,1)=3.0/AB2
CINV(9,2)=2.0/AB
CINV(9,4)=-3.0/AB2
CINV(9,5)=-2.0/AB
CINV(9,7)=3.0/AB2
CINV(9,8)=-1.0/AB
CINV(9,10)=-3.0/AB2
CINV(9,11)=1.0/AB
CINV(10,1)=2.0/B3
CINV(10,2)=1.0/B2
CINV(10,10)=-2.0/B3
CINV(10,11)=1.0/B2
CINV(11,1)=-2.0/A3B
CINV(11,3)=1.0/A2B
CINV(11,4)=2.0/A3B
CINV(11,6)=1.0/A2B
CINV(11,7)=-2.0/A3B
CINV(11,9)=-1.0/A2B
CINV(11,10)=2.0/A3B
CINV(11,12)=-1.0/A2B
CINV(12,1)=-2.0/AB3
CINV(12,2)=-1.0/AB2
CINV(12,4)=2.0/AB3
CINV(12,5)=1.0/AB2
CINV(12,7)=-2.0/AB3
```

```
      CINV(12,8)=1.0/AB2
      CINV(12,10)=2.0/AB3
      CINV(12,11)=-1.0/AB2
      IF(KT.GT.1) GO TO 50
      CALL MATMLT(CINV,12,12,B,CINV,12,AK2)
      DO 15 II=1,12
      DO 15 JJ=1,12
      I1=II+10
      J1=JJ+10
   15 QK(I1,J1)=QK(I1,J1)+AK2(II,JJ)
   50 RETURN
      END
      SUBROUTINE LARGE(U,QK,AL,BL,THM,E,PR,NOD,M,KT,KTN)
C     THIS SUBROUTINE COMPUTES LARGE DEFLECTION MATRICES FOR THE ELEMENT
C     LOCAL SYSTEM
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION STRMEM(3,10),STRMOP(3,12),BMEM(3,10),BMOP(3,12)
      DIMENSION AK1(10,10),AK2(12,12),AK3(10,12),XA(4),YA(4),XI(6),
     $ETA(6),W(4),DNDX(5),DNDY(5),DNX(5),DNY(5),DFDX(12),DFDY(12)
      DIMENSION U(22),QK(22,22),EM(3,3),GK(12,12),BNDG(3,12),D(3,3)
      DIMENSION CINV(12,12),B(12,12),STOP(3),SBOT(3),NOD(24,4)
      DATA XA,YA/-1.D0,1.D0,1.D0,-1.D0,-1.D0,-1.D0,1.D0,1.D0/
      DATA XI,ETA,W/0.8611D0,0.3399D0,-0.8611D0,-0.3399D0,-1.D0,1.D0,
     $0.8611D0,0.3399D0,-0.8611D0,-0.3399D0,-1.D0,1.D0,0.3478D0,0.6521D0
     $,0.3478D0,0.6521D0/
C     DEFINITION OF VARIABLES
C     STRMEM,STRMOP.....DEFLECTION TO STRAIN TRANSFORMATION MATRICES FOR
C     TOTAL STRAIN
C     BMEM,BMOP ....DEFLECTION TO STRAIN TRANSFORMATION MATRICES FOR
C     INCREMENTAL VALUES
C     AK1,AK2,AK3.....TEMPORARY STORAGE LOCATION FOR SUBMATRICES
C     DFDX,DFDY.....DERIVATIVES OF SHAPE FUNCTIONS FOR CUT OF PLANE
C     DEFLECTIONS
C     BNDG ....DEFLECTION TO STRAIN TRANSFORMATION MATRIX FOR BENDING
C     STOP,SBOT.....STRESSES AT THE TOP AND BOTTOM FIBERS OF THE NODAL POINTS
C     NOD.....ARRAY REPRESENTING BOOLEAN MATRIX
      KT1=KT-1
      KTT=KTN*(KT1/KTN)
      IPRINT=2
      CALL STIF(QK,CINV,AL,BL,THM,E,PR,M,KT)
C     INITIALIZE VALUES FOR DISPLACEMENT TO STRAIN TRANSFORMATION MATRIX
C     FOR BENDING
      DO 50 J=1,12
      DO 50 I=1,3
   50 BNDG(I,J)=0.0
      DO 60 I=1,12
      DO 60 J=1,12
   60 B(I,J)=0.0
C     IF ISTRS=1 COMPUTE STRESSES ; IF ISTRS=2 COMPUTE STIFFNESS MATRIX
      ISTRS=1
      NN=6
      MN=5
```

```
      IF(IPRINT-1)35,135,235
  235 IF(KTT.LT.KT1) GO TO 35
  135 PRINT 122
  122 FORMAT(///)
      PRINT 121,M
  121 FORMAT(1H0,'ELEMENT NO.',I2//25X,'STRESS-X',9 X,'STRESS-Y',8 X,'ST
     $RESS-XY',5X,'EQUIV-STRESS')
      GO TO 35
   25 CONTINUE
      NN=4
      MN=1
   35 CONTINUE
C     DO LOOP FOR NUMERICAL INTEGRATIONS
      DO 10 J=MN,NN
      DO 20 K=MN,NN
      DO 5 I=1,4
      DNDX(I)=0.5*XA(I)*(1.+YA(I)*ETA(K))/AL
      DNDY(I)=0.5*YA(I)*(1.+XA(I)*XI(J))/BL
      DNX(I)=0.5*XA(I)/AL
      DNY(I)=0.5*YA(I)/BL
    5 CONTINUE
      DNDX(5)=-4.*XI(J)*(1.-ETA(K)*ETA(K))/AL
      DNDY(5)=-4.*ETA(K)*(1.-XI(J)*XI(J))/BL
      DNX(5)=0.0
      DNY(5)=0.0
      A2=AL*AL
      AB=AL*BL
      B2=BL*BL
      A3=A2*AL
      B3=B2*BL
      A2B=A2*BL
      AB2=B2*AL
      A3B=A3*BL
      AB3=B3*AL
      X=0.5*AL*(1.+XI(J))
      Y=0.5*BL*(1.+ETA(K))
      X2=X*X
      Y2=Y*Y
      XY=X*Y
      X2Y=XY*X
      XY2=XY*Y
      Y3=Y2*Y
      X3=X2*X
C     COMPUTE VALUES FOR DISPLACEMENT TO STRAIN TRANSFORMATION MATRIX
      BNDG(1,4)=2.0
      BNDG(1,7)=6.*X
      BNDG(1,8)=2.*Y
      BNDG(1,11)=6.*XY
      BNDG(2,6)=2.0
      BNDG(2,9)=2.*X
      BNDG(2,10)=6.*Y
      BNDG(2,12)=6.*XY
      BNDG(3,5)=2.0
```

```
      BNDG(3,8)=4.*X
      BNDG(3,9)=4.*Y
      BNDG(3,11)=6.*X2
      BNDG(3,12)=6.*Y2
C     COMPUTE.DERIVATINE OF SHAPE FUNCTICNS FOR BENCING
      DFDX(1)=-6.*X/A2-Y/AB+6.*X2/A3+6.*XY/A2B+3.*Y2/AB2-6.*X2Y/A3B-2.*Y
     $3/AB3
      DFDX(2)=-Y/AL+2.*Y2/AB-Y3/AB2
      DFDX(3)=-1.+4.*X/AL+Y/BL-3.*X2/A2-4.*XY/AB+3.*X2Y/A2B
      DFDX(4)=6.*X/A2+Y/AB-6.*X2/A3-6.*XY/A2B-3.*Y2/AB2+6.*X2Y/A3B
     $+2.*Y3/AB3
      DFDX(5)=Y/AL-2.*Y2/AB+Y3/AB2
      DFDX(6)=2.*X/AL-3.*X2/A2-2.*XY/AB+3.*X2Y/A2B
      DFDX(7)=-Y/AB+6.*XY/A2B+3.*Y2/AB2-6.*X2Y/A3B-2.*Y3/AB3
      DFDX(8)=-Y2/AB+Y3/AB2
      DFDX(9)=2.*XY/AB-3.*X2Y/A2B
      DFDX(10)=Y/AB-6.*XY/A2B-3.*Y2/AB2+6.*X2Y/A3B+2.*Y3/AB3
      DFDX(11)=Y2/AB-Y3/AB2
      DFDX(12)=-Y/BL+4.*XY/AB-3.*X2Y/A2B
      DFDY(1)=-X/AB-6.*Y/B2+3.*X2/A2B+6.*XY/AB2+6.*Y2/B3-2.*X3/A3B
     $-6.*XY2/AB3
      DFDY(2)=1.-X/AL-4.*Y/BL+4.*XY/AB+3.*Y2/B2-3.*XY2/AB2
      DFDY(3)=X/BL-2.*X2/AB+X3/A2B
      DFDY(4)=X/AB-3.*X2/A2B-6.*XY/AB2+2.*X3/A3B+6.*XY2/AB3
      DFDY(5)=X/AL-4.*XY/AB+3.*XY2/AB2
      DFDY(6)=-X2/AB+X3/A2B
      DFDY(7)=-X/AB+3.*X2/A2B+6.*XY/AB2-2.*X3/A3B-6.*XY2/AB3
      DFDY(8)=-2.*XY/AB+3.*XY2/AB2
      DFDY(9)=X2/AB-X3/A2B
      DFDY(10)=X/AB+6.*Y/B2-3.*X2/A2B-6.*XY/AB2-6.*Y2/B3+2.*X3/A3B
     $+6.*XY2/AB3
      DFDY(11)=-2.*Y/BL+2.*XY/AB+3.*Y2/B2-3.*XY2/AB2
      DFDY(12)=-X/BL+2.*X2/AB-X3/A2B
C     COMPUTE DERIVATIVES OF DISPLACEMENTS AT INTEGRATION POINTS
      DUDX=0.0
      DUDY=0.0
      DVDX=0.0
      DVDY=0.0
      DWDX=0.0
      DWDY=0.0
      DUX=0.0
      DUY=0.0
      DVX=0.0
      DVY=0.0
      DO 1 II=1,9,2
      JJ=II+1
      IJ=JJ/2
      DUDX=DUDX+U(II)*DNDX(IJ)
      DUDY=DUDY+U(II)*DNDY(IJ)
      DVDX=DVDX+U(JJ)*DNDX(IJ)
      DVDY=DVDY+U(JJ)*DNDY(IJ)
      DUX=DUX+U(II)*DNX(IJ)
      DUY=DUY+U(II)*DNY(IJ)
```

```
      DVX=DVX+U(JJ)*DNX(IJ)
      DVY=DVY+U(JJ)*DNY(IJ)
    1 CCNTINUE
      DO 2 IJ=1,9,2
      II=IJ+1
      JJ=II/2
C     COMPUTE DISPLACEMENT TO STRAIN TRANSFCRMATION MATRICES FCR TCTAL
C     ANC INCREMENTAL VALUES
      STRMEM(1,IJ)=(C.5*DUDX+1.)*DNDX(JJ)
      STRMEM(2,IJ)=C.5*DUDY*DNDY(JJ)
      STRMEM(3,IJ)=0.5*DUY*DNX(JJ)+(0.5*DUX+1.)*DNY(JJ)
      STRMEM(1,II)=0.5*DVDX*DNDX(JJ)
      STRMEM(2,II)=(0.5*DVDY+1.)*DNDY(JJ)
      STRMEM(3,II)=(0.5*DVY+1.)*DNX(JJ)+0.5*DVX*DNY(JJ)
      BMEM(1,IJ)=(DUDX+1.)*DNDX(JJ)
      BMEM(2,IJ)=DUDY*DNDY(JJ)
      BMEM(3,IJ)=DUY*DNX(JJ)+(DUX+1.)*DNY(JJ)
      BMEM(1,II)=DVDX*DNDX(JJ)
      BMEM(2,II)=(DVDY+1.)*DNDY(JJ)
      BMEM(3,II)=(DVY+1.)*DNX(JJ)+DVX*DNY(JJ)
    2 CONTINUE
      DO 3 II=1,12
      JJ=II+10
      DWDX=DWDX+U(JJ)*DFDX(II)
      DWDY=DWDY+U(JJ)*DFDY(II)
    3 CCNTINUE
      DO 4 JJ=1,12
      STRMOP(1,JJ)=0.5*DWDX*DFDX(JJ)
      STRMOP(2,JJ)=0.5*DWDY*DFDY(JJ)
      STRMOP(3,JJ)=0.5*DWDY*DFDX(JJ)+0.5*DWDX*DFDY(JJ)
      BMOP(1,JJ)=DWDX*DFDX(JJ)
      BMOP(2,JJ)=DWDY*DFDY(JJ)
      BMOP(3,JJ)=DWDY*DFDX(JJ)+DWDX*DFDY(JJ)
    4 CCNTINUE
C     COMPUTE STRESSES ; IF ISTRS=1 CCMPUTE STRESSES AT THE NODAL POINTS
C     IF ISTRS=2 COMPUTE STRESSES AT THE INTEGRATION POINTS
      CALL STRESS(U,STRMEM,STRMOP,BNDG,EM,D,SBOT,STOP,CINV,TX,TY,TXY,
     $THM,E,PR)
      SET=DSQRT(STOP(1)*STOP(1)+STOP(2)*STOP(2)-STOP(1)*STOP(2)+3.*STOP
     $(3)*STOP(3))
      SEB=DSQRT(SBOT(1)*SBOT(1)+SBOT(2)*SBOT(2)-SBOT(1)*SBOT(2)+3.*SBOT
     $(3)*SBOT(3))
      IF(ISTRS.NE.1)GO TO 65
      IF(IPRINT-1) 20,100,200
  200 IF(KTT.LT.KT1) GO TO 20
  100 CONTINUE
      JN=J/6
      JM=J-K
      IF(JM)85,95,105
   85 KL=4
      GO TO 155
  105 KL=2
      GO TO 155
```

```
   95 IF(JN-1)115,125,125
  115 KL=1
      GO TO 155
  125 KL=3
  155 PRINT 145,NOD(M,KL),(STOP(II),II=1,3),SET,(SBCT(II),II=1,3),SEB
  145 FORMAT(1X,'NOD',I3,6X,'TOP',4(5X,E12.5)/13X,'BCT',4(5X,E12.5))
      GO TO 20
C     COMPUTE SUBMATRICES FOR LARGE DEFLECTION MATRIX
   65 CONTINUE
      CALL MATMLT(BNDG,3,12,D,BNDG,12,AK2)
      DO 55 II=1,12
      DO 55 JJ=1,12
   55 B(II,JJ)=B(II,JJ)+AK2(II,JJ)*W(K)*W(J)*AL*BL/4.
      CALL MATMLT(BMEM,3,10,EM,BMEM,10,AK1)
      DO 51 II=1,10
      DO 51 JJ=1,10
   51 QK(II,JJ)=QK(II,JJ)+AK1(II,JJ)*W(K)*W(J)*AL*BL/4.
      CALL MATMLT(BMEM,3,10,EM,BMOP,12,AK3)
      DO 52 II=1,10
      DO 52 JJ=1,12
      LL=JJ+10
   52 QK(II,LL)=QK(II,LL)+AK3(II,JJ)*W(J)*W(K)*AL*BL/4.
      CALL MATMLT(BMOP,3,12,EM,BMOP,12,AK2)
      DO 53 II=1,12
      DO 53 JJ=1,12
      IL=II+10
      JL=JJ+10
   53 QK(IL,JL)=QK(IL,JL)+AK2(II,JJ)*W(K)*W(J)*AL*BL/4.
C.....COPUTE AND ASSMBLE GEOMETRIC MATRICES
      DO 14 II=1,5
      DO 14 JJ=1,5
      G=DNDX(II)*DNDX(JJ)*TX+DNDX(II)*DNDY(JJ)*TXY+DNDX(JJ)*DNDY(II)*TXY
     $+DNDY(II)*DNDY(JJ)*TY
      I2=2*II
      I1=I2-1
      J2=2*JJ
      J1=J2-1
      QK(I1,J1)=QK(I1,J1)+G*W(K)*W(J)*AL*BL/4.
      QK(I2,J2)=QK(I2,J2)+G*W(K)*W(J)*AL*BL/4.
   14 CONTINUE
      DO 30 II=1,12
      DO 30 JJ=1,12
   30 GK(II,JJ)=DFDX(II)*DFDX(JJ)*TX+DFDX(II)*DFDY(JJ)*TXY+DFDX(JJ)*DFDY
     $(II)*TXY+DFDY(II)*DFDY(JJ)*TY
      DO 40 II=1,12
      DO 40 JJ=1,12
      IL=II+10
      JL=JJ+10
   40 QK(IL,JL)=QK(IL,JL)+GK(II,JJ)*W(J)*W(K)*AL*BL/4.
   20 CONTINUE
   10 CONTINUE
      ISTRS=ISTRS+1
      IF(ISTRS-2) 45,25,45
```

```
   45 CONTINUE
C     COMPUTE BENDING PART OF ELEMENT STIFFNESS MATRIX
      CALL MATMLT(CINV,12,12,B,CINV,12,AK2)
      DO 15 II=1,12
      DO 15 JJ=1,12
      I1=II+10
      J1=JJ+10
   15 QK(I1,J1)=QK(I1,J1)+AK2(II,JJ)
      RETURN
      END
      SUBROUTINE STRESS(U,STRMEM,STRMOP,BNDG,EM,D,SBOT,STOP,CINV,TX,TY,
     $TXY,THM,E,PR)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION U(22),STRMEM(3,10),STRMOP(3,12),EM(3,3),S(3),SBEN(3)
      DIMENSION EONE(3),ETWO(3),D(3,3),EBN(3),BNDG(3,12),SBOT(3),STOP(3)
      DIMENSION CINV(12,12),BBEN(3,12)
C     DEFINITION OF VARIABLES
C     BBEN ....DEFLECTION TO STRAIN TRANSFORMATION FOR BENDING
C     EBN AXIAL STRAIN DUE TO BENDING
C     EONE AXIAL STRAIN DUE TO IN PLANE ACTION
C     ETWO AXIAL STRAIN DUE TO OUT OF PLANE ACTION
C.....OTHER VARIABLES DEFINED PREVIOUSLY
      DO 50 II=1,3
      DO 50 JJ=1,12
      BBEN(II,JJ)=0.0
      DO 50 KK=1,12
   50 BBEN(II,JJ)=BBEN(II,JJ)+BNDG(II,KK)*CINV(KK,JJ)
C     COMPUTE AXIAL STRAIN FOR MEMBRANE ACTION
      DO 5 II=1,3
      EONE(II)=0.0
      DO 5 JJ=1,10
    5 EONE(II)=EONE(II)+STRMEM(II,JJ)*U(JJ)
C     COMPUTE AXIAL STRAIN  FOR LARGE DEFLECTION
      DO 6 II=1,3
      ETWO(II)=0.0
      EBN(II)=0.0
      DO 6 JJ=1,12
      EBN(II)=EBN(II)+0.5*THM*BBEN(II,JJ)*U(JJ+10)
    6 ETWO(II)=ETWO(II)+STRMOP(II,JJ)*U(JJ+10)
C     CONSTITUTIVE MATRIX  FOR PLANE
      EM(1,1)=THM*E/(1.-PR*PR)
      EM(1,2)=PR*EM(1,1)
      EM(2,1)=EM(1,2)
      EM(3,3)=EM(1,1)*(1.-PR)/2.
      EM(2,2)=EM(1,1)
      EM(1,3)=0.0
      EM(2,3)=0.0
      EM(3,1)=0.0
      EM(3,2)=0.0
      THM2=THM*THM/12.
C     CONSTITUTIVE MATRIX FOR BENDING
      DO 10 II=1,3
      DO 10 JJ=1,3
```

```
   10 D(II,JJ)=EM(II,JJ)*THM2
C     COMPUTE STRESSES FOR THE MIDDLE PLANE TOP AND BOTTOM FIBER
      DO 8 II=1,3
      S(II)=0.0
      DO 8 JJ=1,3
    8 S(II)=S(II)+EM(II,JJ)*(EONE(JJ)+ETWO(JJ))
      DO 20 II=1,3
      SBEN(II)=0.0
      DO 20 JJ=1,3
   20 SBEN(II)=SBEN(II)+EM(II,JJ)*EBN(II)
      DO 25 II=1,3
      STOP(II)=(S(II)+SBEN(II))/THM
      SBOT(II)=(S(II)-SBEN(II))/THM
   25 CONTINUE
      TX=S(1)
      TY=S(2)
      TXY=S(3)
      RETURN
      END
      SUBROUTINE ARANG(QK,Q,M)
C     THIS SUBROUTINE REARANGES ELEMENT STIFFNESS MATRIX AND LOAD VECTOR
C     BEFORE CONDENSATION
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION QK(M,M),Q(M)
C     REARRANGEMENT OF ROWS
      L=3
      KL=5
      N=8
      DO 3 LL=1,4
C     DO LOOP ON THE COLUMNS
      DO 1 I=L,KL
      TEMP1=Q(I)
      II=I+N
      Q(I)=Q(II)
C     DO LOOP ON THE ROWS
      DO 8 J=1,M
      TEMP=QK(I,J)
      QK(I,J)=QK(II,J)
      N1=N-1
      DO 2 K=1,N1
      IK=II-K
      IK1=IK+1
      IF(J.LT.M) GO TO 2
      Q(IK1)=Q(IK)
    2 QK(IK1,J)=QK(IK,J)
    8 QK(I+1,J)=TEMP
      Q(I+1)=TEMP1
    1 CONTINUE
      L=L+5
      KL=KL+5
      N=N-2
    3 CONTINUE
```

```
C     REARRANGEMENT OF THE CCLUMNS
      L=3
      KL=5
      N=8
      DO 4 LL=1,4
C     DO LCCP CN THE ROWS
      DO 5 J=L,KL
      JJ=J+N
C     DO LCOP ON THE COLUMNS
      DO 5 I=1,M
      TEMP=CK(I,J)
      QK(I,J)=QK(I,JJ)
      N1=N-1
      DC 6 K=1,N1
      JK=JJ-K
      JK1=JK+1
    6 QK(I,JK1)=QK(I,JK)
    5 QK(I,J+1)=TEMP
      L=L+5
      KL=KL+5
      N=N-2
    4 CONTINUE
      RETURN
      END
      SUBROUTINE ARANG2(U,M)
      IMPLICIT REAL*8(A-H,O-Z)
C     REARRANGEMENT CF INCREMENTAL DEFLECTION TO PUT IN PLANE VALUES FIRST
C     AND OUT OF PLANE VALUES NEXT
      DIMENSION U(M)
      L=3
      N=3
      DO 1 LL=1,4
      L1=L+1
      DO 2 I=L,L1
      TEMP=U(I)
      K=I+N
      U(I)=U(K)
      N1=N-1
      DO 3 J=1,N1
      KJ=K-J
      K1=KJ+1
    3 U(K1)=U(KJ)
      IL=I+1
    2 U(IL)=TEMP
      L=L+2
      N=N+3
    1 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE CONDNS(QK,Q,LI,KK)
C     THIS SUBROUTINE CONDENSES INTERNAL DEGREES OF FREEDOM FOR EACH
C     ELEMENT
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION QK(LI,LI),Q(LI)
      DO 1 J=1,KK
      IJ=LI-J
      IK=IJ+1
      PIVOT=QK(IK,IK)
      DO 2 K=1,IJ
      F=QK(IK,K)/PIVOT
      QK(IK,K)=F
      DO 3 I=1,IJ
      QK(I,K)=QK(I,K)-F*QK(I,IK)
    3 CONTINUE
    2 Q(K)=Q(K)-QK(K,IK)*Q(IK)/PIVOT
    1 Q(IK)=Q(IK)/PIVOT
      RETURN
      END
```

NUMBER OF ELEMENTS 4

NUMBER OF NODS 9

NUMBER OF LOADING 5

ELEMENT	NOD-I	NOD-J	NOD-K	NOD-L	LOAD (LB/IN2)	THICKNESS (IN)
1	1	4	5	2	0.0	0.100
2	2	5	6	3	0.0	0.100
3	4	7	8	5	0.0	0.100
4	5	8	9	6	0.0	0.100

NOD	X-COORDINATE	Y-COORDINATE	Z-COORDINATE
1	0.0	0.0	0.0
2	0.0	5.000	0.0
3	0.0	10.000	0.0
4	5.000	0.0	0.0
5	5.000	5.000	0.0
6	5.000	10.000	0.0
7	10.000	0.0	0.0
8	10.000	5.000	0.0
9	10.000	10.000	0.0

EQUIVALENT NODAL LOADS FOR EACH STEP

NOD	LOAD-X	LOAD-Y	LOAD-Z	MOMENT-X	MOMENT-Y	MOMENT-Z
1	0.0	0.0	-0.95000D 00	0.0	0.0	0.0
2	0.0	0.0	0.0	0.0	0.0	0.0
3	0.0	0.0	0.0	0.0	0.0	0.0
4	0.0	0.0	0.0	0.0	0.0	0.0
5	0.0	0.0	0.0	0.0	0.0	0.0
6	0.0	0.0	0.0	0.0	0.0	0.0
7	-0.23750D 03	0.0	0.0	0.0	0.0	0.0
8	-0.47500D 03	0.0	0.0	0.0	0.0	0.0
9	-0.23750D 03	0.0	0.0	0.0	0.0	0.0

STEP 1

NOD	DEFLECTION-X	DEFLECTION-Y	DEFLECTION-Z	ROTATION-X	ROTATION-Y	ROTATION-Z
1	0.0	0.0	-0.68204D-02	0.0	0.0	0.0
2	0.0	0.47500D-04	-0.41873D-02	0.79388D-03	0.0	0.0
3	0.0	0.95000D-04	0.0	0.86514D-03	0.0	0.0
4	-0.15833D-03	0.0	-0.41873D-02	0.0	-0.79388D-03	0.0
5	-0.15833D-03	0.47500D-04	-0.28254D-02	0.48056D-03	-0.48056D-03	0.0
6	-0.15833D-03	0.95000D-04	0.0	0.60064D-03	0.0	0.0
7	-0.31667D-03	0.0	0.0	0.0	-0.86514D-03	0.0
8	-0.31667D-03	0.47500D-04	0.0	0.0	-0.60064D-03	0.0
9	-0.31667D-03	0.95000D-04	0.0	0.0	0.0	0.0

STEP 2

NOD	DEFLECTION-X	DEFLECTION-Y	DEFLECTION-Z	ROTATION-X	ROTATION-Y	ROTATION-Z
1	0.0	0.0	-0.13930D-01	0.0	0.0	0.0
2	0.0	0.47122D-04	-0.91866D-02	0.15932D-02	0.0	0.0
3	0.0	0.87847D-04	0.0	0.19753D-02	0.0	0.0
4	-0.15886D-03	0.0	-0.90783D-02	0.0	-0.16021D-02	0.0
5	-0.15983D-03	0.46078D-04	-0.62929D-02	0.10223D-02	-0.10462D-02	0.0
6	-0.16138D-03	0.90328D-04	0.0	0.13741D-02	0.0	0.0
7	-0.32394D-03	0.0	0.0	0.0	-0.19313D-02	0.0
8	-0.32147D-03	0.44522D-04	0.0	0.0	-0.13604D-02	0.0
9	-0.31989D-03	0.91909D-04	0.0	0.0	0.0	0.0

STEP 3

NOD	DEFLECTION-X	DEFLECTION-Y	DEFLECTION-Z	ROTATION-X	ROTATION-Y	ROTATION-Z
1	0.0	0.0	-0.41492D-01	0.0	0.0	0.0
2	0.0	0.46256D-04	-0.28702D-01	0.46649D-02	0.0	0.0
3	0.0	0.24624D-04	0.0	0.63238D-02	0.0	0.0
4	-0.16092D-03	0.0	-0.28457D-01	0.0	-0.46862D-02	0.0
5	-0.17091D-03	0.36032D-04	-0.20076D-01	0.31668D-02	-0.32207D-02	0.0
6	-0.18767D-03	0.49389D-04	0.0	0.44652D-02	0.0	0.0
7	-0.38702D-03	0.0	0.0	0.0	-0.62226D-02	0.0
8	-0.36248D-03	0.18455D-04	0.0	0.0	-0.44326D-02	0.0
9	-0.34657D-03	0.65578D-04	0.0	0.0	0.0	0.0

STEP 4

NOD	DEFLECTION-X	DEFLECTION-Y	DEFLECTION-Z	ROTATION-X	ROTATION-Y	ROTATION-Z
1	0.0	0.0	-0.54413D-01	0.0	0.0	0.0
2	0.0	0.45299D-04	-0.38070D-01	0.61158D-02	0.0	0.0
3	0.0	-0.20084D-03	0.0	0.83491D-02	0.0	0.0
4	-0.16395D-03	0.0	-0.38038D-01	0.0	-0.60953D-02	0.0
5	-0.19698D-03	0.91876D-05	-0.27599D-01	0.41344D-02	-0.41120D-02	0.0
6	-0.27915D-03	-0.94691D-04	0.0	0.63531D-02	0.0	0.0
7	-0.61201D-03	0.0	0.0	0.0	-0.83537D-02	0.0
8	-0.50656D-03	-0.72724D-04	0.0	0.0	-0.63411D-02	0.0
9	-0.43803D-03	-0.25379D-04	0.0	0.0	0.0	0.0

ELEMENT NO. 1

		STRESS-X	STRESS-Y	STRESS-XY	EQUIV-STRESS
NOD 1	TOP	0.38215D 04	0.74475D 04	0.51955D 03	0.65130D 04
	BOT	-0.11539D 05	-0.75284D 04	-0.73060D 03	0.10225D 05
NOD 2	TOP	0.11246D 04	0.63318D 04	-0.49487D 03	0.61088D 04
	BOT	-0.77370D 04	-0.10690D 04	0.26378D 03	0.72487D 04
NOD 4	TOP	0.25255D 04	0.47965D 04	-0.47692D 03	0.42371D 04
	BOT	-0.47049D 04	-0.37847D 04	0.26586D 03	0.43434D 04
NOD 5	TOP	0.60989D 03	0.44473D 04	-0.59437D 03	0.43009D 04
	BOT	-0.57461D 04	-0.20133D 04	0.21772D 04	0.63027D 04

ELEMENT NO. 2

		STRESS-X	STRESS-Y	STRESS-XY	EQUIV-STRESS
NOD 2	TOP	0.40119D 03	0.39497D 04	0.45679D 02	0.37660D 04
	BOT	-0.94304D 04	-0.33096D 04	-0.61124D 02	0.73582D 04
NOD 3	TOP	-0.40239D 04	0.24098D 04	-0.29237D 02	0.56361D 04
	BOT	-0.40239D 04	0.22229D 04	-0.62080D 01	0.54844D 04
NOD 5	TOP	0.33778D 03	0.35113D 04	-0.11019D 04	0.38502D 04
	BOT	-0.60193D 04	-0.28913D 04	0.28603D 04	0.71918D 04
NOD 6	TOP	-0.43738D 04	0.13936D 04	-0.20737D 04	0.63299D 04
	BOT	-0.43738D 04	0.90680D 03	0.20383D 04	0.60318D 04

ELEMENT NO. 3

		STRESS-X	STRESS-Y	STRESS-XY	EQUIV-STRESS.
NOD 4	TOP	0.92155D 02	0.40894D 04	0.49887D 02	0.40450D 04
	BOT	-0.67851D 04	-0.44917D 04	-0.76148D 02	0.61327D 04
NOD 5	TOP	-0.36525D 03	0.41766D 04	-0.10799D 04	0.47535D 04
	BOT	-0.65756D 04	-0.22837D 04	0.28436D 04	0.75957D 04
NOD 7	TOP	-0.14762D 04	-0.23356D 03	-0.24973D 02	0.13751D 04
	BOT	-0.15910D 04	-0.23356D 03	-0.42875D 01	0.14981D 04
NOD 8	TOP	-0.24551D 04	-0.57495D 03	-0.20493D 04	0.41894D 04
	BOT	-0.28881D 04	-0.57495D 03	0.20205D 04	0.43885D 04

ELEMENT NO. 4

			STRESS-X	STRESS-Y	STRESS-XY	EQUIV-STRESS
NOD	5	TOP	-0.63736D 03	0.32408D 04	-0.58496D 03	0.37418D 04
		BOT	-0.69477D 04	-0.31620D 04	0.23111D 04	0.71597D 04
NOD	6	TOP	-0.33723D 04	0.16940D 04	-0.24093D 04	0.61131D 04
		BOT	-0.33727D 04	0.12072D 04	0.23416D 04	0.57750D 04
NOD	8	TOP	-0.21552D 04	0.42464D 03	-0.23762D 04	0.47623D 04
		BOT	-0.25882D 04	0.42464D 03	0.23085D 04	0.48954D 04
NOD	9	TOP	-0.38095D 04	-0.67005D 01	-0.33036D 04	0.68723D 04
		BOT	-0.38095D 04	-0.67005D 01	0.32359D 04	0.67750D 04

STEP 5

NOD	DEFLECTION-X	DEFLECTION-Y	DEFLECTION-Z	ROTATION-X	ROTATION-Y	ROTATION-Z
1	0.0	0.0	-0.32801D-01	0.0	0.0	0.0
2	0.0	0.39006D-04	-0.23079D-01	0.37033D-02	0.0	0.0
3	0.0	-0.25961D-03	0.0	0.49656D-02	0.0	0.0
4	-0.16141D-03	0.0	-0.23565D-01	0.0	-0.36266D-02	0.0
5	-0.18930D-03	0.16220D-04	-0.18067D-01	0.23847D-02	-0.22764D-02	0.0
6	-0.29725D-03	-0.12872D-03	0.0	0.44157D-02	0.0	0.0
7	-0.67256D-03	0.0	0.0	0.0	-0.51890D-02	0.0
8	-0.54073D-03	-0.92558D-04	0.0	0.0	-0.44688D-02	0.0
9	-0.45626D-03	-0.46099D-04	0.0	0.0	0.0	0.0

APPENDIX B

MATRICES [c] AND $[c]^{-1}$ AND INTERPOLATION FUNCTIONS FOR BENDING ELEMENT

1											
		1									
	−1										
1	a		a^2			a^3					
		1		a			a^2			a^3	
	−1		$-2a$			$-3a^2$					
1	a	b	a^2	ab	b^2	a^3	a^2b	ab^2	b^3	a^3b	ab^3
		1		a	$2b$		a^2	$2ab$	$3b^2$	a^3	$3ab^2$
	−1		$-2a$	$-b$		$-3a^2$	$-2ab$	$-b^2$		$-3a^2b$	$-b^3$
1		b			b^2				b^3		
		1			$2b$				$3b^2$		
	−1			$-b$				$-b^2$			$-b^3$

[c]

1											
		-1									
	1										
$\frac{-3}{a^2}$		$\frac{-2}{a}$	$\frac{3}{a^2}$		$\frac{1}{a}$						
$\frac{-1}{ab}$	$\frac{-1}{a}$	$\frac{1}{b}$	$\frac{1}{ab}$	$\frac{1}{a}$		$\frac{-1}{ab}$			$\frac{1}{ab}$		$\frac{-1}{b}$
$\frac{-3}{b^2}$	$\frac{-2}{b}$								$\frac{3}{b^2}$	$\frac{-1}{b}$	
$\frac{-2}{a^3}$		$\frac{-1}{a^2}$	$\frac{-2}{a^3}$	$\frac{-1}{a^2}$							
$\frac{3}{a^2b}$		$\frac{-2}{ab}$	$\frac{-3}{a^2b}$		$\frac{-1}{ab}$	$\frac{3}{a^2b}$		$\frac{1}{ab}$	$\frac{-3}{a^2b}$		$\frac{2}{ab}$
$\frac{3}{ab^2}$	$\frac{2}{ab}$		$\frac{-3}{ab^2}$	$\frac{-2}{ab}$		$\frac{3}{ab^2}$	$\frac{-1}{ab}$		$\frac{-3}{ab^2}$	$\frac{1}{ab}$	
$\frac{2}{b^3}$	$\frac{1}{b^2}$								$\frac{-2}{b^3}$	$\frac{1}{b^2}$	
$\frac{-2}{a^3b}$		$\frac{1}{a^2b}$	$\frac{2}{a^3b}$		$\frac{1}{a^2b}$	$\frac{-2}{a^3b}$		$\frac{-1}{a^2b}$	$\frac{2}{a^3b}$		$\frac{-1}{a^2b}$
$\frac{-2}{ab^3}$	$\frac{-1}{ab^2}$		$\frac{2}{ab^3}$	$\frac{1}{ab^2}$		$\frac{-2}{ab^3}$	$\frac{1}{ab^2}$		$\frac{2}{ab^3}$	$\frac{-1}{ab^2}$	

$[c]^{-1}$

Interpolation Functions

$$w = \{\phi\} \{q_b\} = \{\psi\} \{c\}^{-1} \{q_b\}$$

then $\{\phi\} = \{\psi\} \{c\}^{-1}$

$\{\psi\} = \{1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3, x^3y, xy^3\}$

$$\phi_1 = 1 - \frac{3x^2}{a^2} - \frac{xy}{ab} - \frac{3y^2}{b^2} + \frac{2x^3}{a^3} + \frac{3x^2y}{a^2b} + \frac{3xy^2}{ab^2} + \frac{2y^3}{b^3} - \frac{2x^3y}{a^3b} - \frac{2xy^3}{ab^3}$$

$$\phi_2 = y - \frac{xy}{a} - \frac{2y^2}{b} + \frac{2xy^2}{ab} + \frac{y^3}{b^2} - \frac{xy^3}{ab^2}$$

$$\phi_3 = -x + \frac{2x^2}{a} + \frac{xy}{b} - \frac{x^3}{a^2} - \frac{2x^2y}{ab} + \frac{x^3y}{a^2b}$$

$$\phi_4 = \frac{3x^2}{a^2} + \frac{xy}{ab} - \frac{2x^3}{a^3} - \frac{3x^2y}{a^2b} - \frac{3xy^2}{ab^2} + \frac{2x^3y}{a^3b} + \frac{2xy^3}{ab^3}$$

$$\phi_5 = \frac{xy}{a} - \frac{2xy^2}{ab} + \frac{xy^3}{ab^2}$$

$$\phi_6 = \frac{x^2}{a} - \frac{x^3}{a^2} - \frac{x^2y}{ab} + \frac{x^3y}{a^2b}$$

$$\phi_7 = \frac{-xy}{ab} + \frac{3x^2y}{a^2b} + \frac{3xy^2}{ab^2} - \frac{2x^3y}{a^3b} - \frac{2xy^3}{ab^3}$$

$$\phi_8 = \frac{-xy^2}{ab} + \frac{xy^3}{ab^2}$$

$$\phi_9 = \frac{x^2y}{ab} - \frac{x^3y}{a^2b}$$

$$\phi_{10} = \frac{xy}{ab} + \frac{3y^2}{b^2} - \frac{3x^2y}{a^2b} - \frac{3xy^2}{ab^2} - \frac{2y^3}{b^3} + \frac{2x^3y}{a^3b} + \frac{2xy^3}{ab^3}$$

$$\phi_{11} = \frac{-y^2}{b} + \frac{xy^2}{ab} + \frac{y^3}{b^2} - \frac{xy^3}{ab^2}$$

$$\phi_{12} = \frac{-xy}{b} + \frac{2x^2y}{ab} - \frac{x^3y}{a^2b}$$

APPENDIX C

PLASTICITY THEORIES

Deformation Theory of Plasticity

The following relationship is given [26] between total stress and total strain

$$\varepsilon_{ij} = \frac{1+\nu}{E} \sigma_{ij} - \frac{\nu}{E} \sigma_{pp} \delta_{ij} + \frac{g(J_2)}{E} S_{ij}$$

where ε = strain, σ = stress, ν = Poisson's ratio, J_2 = second invariant of stress deviator tensor, E = modulus of elasticity, δ_{ij} = Kronecker delta, and

$$\sigma_{pp} = \sigma_{11} + \sigma_{22} + \sigma_{33}$$

$$S_{ij} = \sigma_{ij} - \frac{1}{3} \sigma_{pp} \delta_{ij}$$

$$g(J_2) = \frac{3}{2} \left(\frac{E}{E_s} - 1\right)$$

$$J_2 = \frac{1}{2} S_{ij} S_{ij}$$

Assuming plane stress, then $\sigma_{33} = \sigma_{13} = \sigma_{23} = 0$, hence

$$\varepsilon_{11} = \frac{1}{E} \left(1 + \frac{2g}{3}\right) \sigma_{11} - \frac{1}{E} \left(\nu + \frac{g}{3}\right) \sigma_{22}$$

$$\varepsilon_{22} = \frac{1}{E} \left(1 + \frac{2g}{3}\right) \sigma_{22} - \frac{1}{E} \left(\nu + \frac{g}{3}\right) \sigma_{11}$$

The inverse relationship may be written as

$$\sigma_{11} = E_{11}\ \varepsilon_{11} + E_{12}\ \varepsilon_{12}$$

$$\sigma_{22} = E_{12}\ \varepsilon_{11} + E_{22}\ \varepsilon_{22}$$

$$\sigma_{12} = \frac{E}{1 + \nu + g}\ \varepsilon_{12}$$

where $\quad E_{22} = E_{11} = \dfrac{a}{a^2 - b^2}, \quad E_{12} = \dfrac{-b}{a^2 - b^2}$

and $\quad a = \dfrac{1}{E}\left(1 + \dfrac{2}{3} g\right), \qquad b = \dfrac{-1}{E}\left(\nu + \dfrac{g}{3}\right)$

Flow Theory of Plasticity

For small increments the strain is decomposed into elastic and plastic parts.

$$\delta\varepsilon = \delta\varepsilon^{e} + \delta\varepsilon^{P}$$

Then the stress increment $\delta\sigma$ is related to elastic strain increment by

$$\delta\sigma = D\ \delta\varepsilon^{e}$$

where D = constitutive matrix. Equation of the yield surface is represented by $f(\sigma, \varepsilon) = 0$, then according to the normality rule

$$\sigma\varepsilon^{P} = \delta\lambda \frac{\partial f}{\partial \sigma}$$

Taking the first variation of $f(\sigma, \varepsilon) = 0$

$$\left\{\frac{\partial f}{\partial \sigma}\right\} \delta\sigma = H' \ \delta\bar{\varepsilon}^{P}$$

where $\delta\lambda$ is a non-negative scalor and H' = slope of equivalent strain curve.

Isotropic material behavior is assumed, hence the yield criterion for subsequent yielding becomes

$$f = \bar{\sigma} - H(\bar{\varepsilon}^P) = 0$$

where $\bar{\sigma}$ = effective stress and H is a function of equivalent plastic strain. The incremental form of equivalent plastic strain is given by

$$\delta\varepsilon^{-P} = \frac{2}{3}\{\delta\varepsilon^P_{ij}\ \delta\varepsilon^P_{ij}\}^{\frac{1}{2}}$$

Writing the elastic strain as

$$\delta\varepsilon^e = \delta\varepsilon - \delta\varepsilon^P$$

Then $\delta\sigma = D\ \delta\varepsilon^e = D(\delta\varepsilon - \delta\varepsilon^P)$

Premultiplying the above equation by $\{\frac{\partial f}{\partial\sigma}\}$

$$\{\frac{\partial f}{\partial\sigma}\}\ \delta\sigma = \{\frac{\partial f}{\partial\sigma}\}\ \{D\}\ \{\delta\varepsilon\} - \{\frac{\partial f}{\partial\sigma}\}\ \{D\}\ \{\delta\varepsilon^P\}$$

$$H\ \delta\varepsilon^{-P} = \{\frac{\partial\bar{\sigma}}{\partial\sigma}\}\ \{D\}\ \{\delta\varepsilon\} - \{\frac{\partial\bar{\sigma}}{\partial\sigma}\}\ \{D\}\ \{\frac{\partial\bar{\sigma}}{\partial\sigma}\}\ \{\delta\varepsilon^{-P}\}$$

Since $\delta\varepsilon^{-P}$ is a scalor

$$\{\delta\varepsilon^{-P}\} = \frac{\{\frac{\partial\sigma}{\partial\sigma}\}\ \{D\}\ \{\delta\varepsilon\}}{H + \{\frac{\partial\bar{\sigma}}{\partial\sigma}\}\ \{D\}\ \{\frac{\partial\bar{\sigma}}{\partial\sigma}\}}$$

Manipulation yields

$$\delta\sigma = D\ \delta\varepsilon - D\ \frac{\{\frac{\partial\bar{\sigma}}{\partial\sigma}\}\ \{D\}\ \{\frac{\partial\bar{\sigma}}{\partial\sigma}\}}{H + \{\frac{\partial\bar{\sigma}}{\partial\sigma}\}\ \{D\}\ \{\frac{\partial\bar{\sigma}}{\partial\sigma}\}}\ \delta\varepsilon$$

or

$$D_{ep} = D - D \frac{\{\frac{\partial \bar{\sigma}}{\partial \sigma}\} \{D\} \{\frac{\partial \bar{\sigma}}{\partial \sigma}\}}{H + \{\frac{\partial \bar{\sigma}}{\partial \sigma}\} \{D\} \{\frac{\partial \bar{\sigma}}{\partial \sigma}\}}$$

where D_{ep} = incremental elasto-plastic constitutive matrix.