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YUN HWANG BOO

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AN EMPIRICAL COMPARISON OF THEORETICAL MODELS FOR A UTILITY UNDER SEPARABILITY HYPOTHESES: A SYSTEM OF DERIVED DEMAND EQUATIONS OF FUELS FOR U.S. HOUSEHOLD USE, 1937-1970

APPROVED BY

James E. Hibdon

A. J. Kondonassis

Kutler, Stephen

John L. Lovelston

Ed Crim by JEH

DISSERTATION COMMITTEE

AN EMPIRICAL COMPARISON OF THEORETICAL MODELS FOR A UTILITY UNDER SEPARABILITY HYPOTHESES: A SYSTEM OF DERIVED DEMAND EQUATIONS OF FUELS FOR U.S. HOUSEHOLD USE, 1937-1970

BY: YUN HWANG BOO

MAJOR PROFESSOR: JAMES E. HIBDON

This study is concerned with building an econometric model of the American household demand for natural gas, fuel oil, and electricity needed for space-heating, lighting, cooking, and for the operation of other home appliances. In the process of making a choice among alternative econometric specifications for empirical work, the dissertation reviews separability hypotheses, examines four of five chosen econometric specifications for a utility function through application of separability hypotheses, and provides an in-depth comparison of five chosen specifications in terms of both their theoretical properties and results on the demand elasticities and on the Hicks-Allen partial elasticities of substitution.

Five econometric specifications were chosen for this empirical study of demand: the Cobb-Douglas, the CES, the Uzawa CES, the Sato two-level CES, and the translog utility functions. They were approximated by a Taylor series expansion about a fixed point to derive a system of behavioral equations in forms suitable for econometric testing and comparison. Parameters of these approximations were then estimated by Zellner's efficient least-squares method. From these estimates, the demand elasticities, the Hicks-Allen partial elasticities of substitution, and the Slutsky's price elasticities of the compensated demand were computed and evaluated in terms of whether or not the empirical results are in conformity with theoretical results on various kinds of elasticities. Finally, an empirical assessment was made concerning the performance of each of the five utility functions.

The assessment revealed that the translog utility function dominates over the other four utility functions. Therefore, an econometric model of the American household demand for energy fuels should be built from the translog utility function. The choice of this utility function implies that restrictions implied by separability hypotheses on the parameters of the other four utility functions are invalid in the case of the American household demand for energy fuels; hence, the separability hypotheses should be rejected.

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CHAPTER I

INTRODUCTION

If one were looking for a single criterion by which to distinguish modern economic theory from its classical precursors, he would probably decide that this is to be found in the introduction of the so-called subjective theory of value into economic theory.[1] This revolution in thought broke out almost simultaneously along three fronts, and with it are the names of Jevons, Menger, and Walras associated.[2] All three founders of the utility theory, in their pioneering contributions, adopted the cardinal hypothesis with independent utilities. On this assumption, the utility which the consumer derives from each good consumed is a function of the quantity of that good alone. The total utility of the whole collection of goods

[1] P. Samuelson, Foundations of Economic Analysis (Cambridge: Harvard University Press, 1971), p. 90.

[2] Ibid.

is simply the sum of these separate (or independent) utilities, i.e., $U = \sum_{i=1}^{n} U^i(x_i)$ where U is the total utility and U^i is a sub-utility function of the quantity of good x_i consumed.

In fact, the consumer's behavior can be explained just as well in terms of an ordinal utility function as in terms of a cardinal one, i.e., $V = F(U) = F\left(\sum_{i=1}^{n} U^i(x_i)\right)$ where V is the total utility, and F is an arbitrary function of the sum of independent utilities and, hence, an ordinal concept. The consumer's choice is completely determinate if he possesses a ranking of consumer goods according to his preferences. It is not necessary to assume that he possesses a cardinal measure of utility; the much weaker assumption that he possesses a consistent ranking of preferences is sufficient.[3] J. R. Hicks comments:

> It is possible that it might be more convenient to use the cardinal properties as a sort of scaffolding, useful in erecting the building, but to be taken down when the building has been completed. This is in fact what Marshall very largely did, and there is not in principle any objection to it. The objection is merely that in practice it does not seem to help. It is true that the more elementary parts of the theory can be established almost as well by the one method as by the other; but in the more difficult branches cardinal utility becomes a nuisance.[4]

From the point of view of cardinalism, the rejection of the cardinal hypothesis with independent utilities is a serious matter. For if independence were to be maintained, the way would be clear

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[3]J. M. Henderson and R. E. Quandt, Microeconomic Theory (New York: McGraw-Hill Book Company, 1958), p. 8.

[4]J. R. Hicks, A Revision of Demand Theory (London: The Clarendon Press, 1969), p. 9.

for the econometric determination of the main properties of the utility function.[5] Since the cardinal hypothesis was a very severe restriction on the preference field, its rejection led to contemplating the possibility that utilities might be interdependent. That is to say, the marginal utility of any good might depend not only upon the consumption of that good but also upon the consumption of any other good purchased. As a result, the idea of a completely generalized utility function was introduced by F. Y. Edgeworth, i.e., $\phi = f(x_1, x_2, \ldots, x_n)$ where ϕ is the total utility derived from the whole collection of "n" goods, f is an arbitrary function of the quantities of "n" goods consumed, and $f_{ij} = \frac{\partial^2 \phi}{\partial x_i \partial x_j} \gtreqless 0$ for $i \neq j$ $(i,j = 1, 2, \ldots, n)$.[6]

While some of the implications of the cardinal hypothesis with independent utilities led to the rejection of the additive (or cardinal) utility function and its replacement by a completely generalized utility function as, for example, in the works of Edgeworth and Hicks,[7] this in turn generated dissatisfaction because of the relative paucity of its meaningful empirical implications. Consequently, considerably increased attention has been paid in demand analysis to the concept of separability as a theoretical solution of this empirical issue. W. Leontief comments:

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[5] Ibid., pp. 11-12.

[6] F. Y. Edgeworth, Mathematical Psychics (London: Routledge and Kegan Paul, Ltd., 1881), p. 97.

[7] J. R. Hicks, Value and Capital (Oxford: The Clarendon Press, 1968), Chapters I, II and III, and A Revision of Demand Theory (London: The Clarendon Press, 1969).

> The analysis of consumer's choice offers what seems to be a particularly illuminating example of a concrete theoretical issue, the solution of which can be effectively advanced through application of the concept of separable functions. The evolution of theoretical thought on this particular subject followed, as in many other similar instances, a deviously dialectical path of development. It started with the acceptance of conventional and supposedly self-evident notions of the so-called common experience; it went through the antithesis of a rigorous but essentially destructive phase of negative criticism to move finally toward the higher stage of positive synthesis which vindicates again some valuable elements of the original common-sense experience after distilling it in the refining apparatus of exact logical analysis.[8]

It can be admitted that the cardinal hypothesis with independent utilities is the notion that the individual consumer is capable of ordering all conceivable alternatives presented to him--all the positions represented by points on his indifference map. But all that has to be assumed is that he can order those alternatives which he actually does have to compare.[9] In other words, given a collection of consumer goods, a partition of those goods into the subgroups of at least one good--a partition in which the sequence of subgroups is put into an ordered relation, but in which there is no ordering within the subgroups--is desirable in reality because the consumer commonly allocates expenditure among broad groups of goods. If such a commodity-wise partition is permissible, then the consumer will be capable of comparing and ordering the sequence of subgroups. Furthermore, to such a commodity-wise partition there corresponds functional separability:

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[8]W. Leontief, "Introduction to a Theory of the Internal Structure of Functional Relationships," Econometrica, Vol. 15 (1947), p. 371.

[9]J. R. Hicks, A Revision of Demand Theory (london: The Clarendon Press, 1969), pp. 20-24. Also, see Section 2, Chapter III of this dissertation.

A utility function of the quantities consumed of "n" consumer goods will be functionally separable with respect to a commodity-wise partition. Functional separability is essential not only in explaining the consumer's budgetary behavior, but also in making a generalized utility function operationally manageable. The conditions for such functional separability are referred to as the separability hypothesis, which is based on the logical theory of ordering. The concept of separability has enriched the theory of consumer behavior in a number of directions, perhaps the most celebrated of which has been the utility tree. It has been used to analyze the internal structure of utility functions, and its implications have been of primary importance to empirical studies in demand analysis.

The primary purpose of this dissertation is to review the concept of separability, examine the internal structure of utility functions chosen for the present study of demand through application of the separability hypothesis, and make an in-depth empirical comparison among them in connection with U.S. households' demand for energy fuels needed for heating, cooking, lighting, and other home appliances.[10] Then, on an empirical basis of performances of the chosen utility functions, an econometric model of demand for energy fuels will be built. Finally, the demand elasticities and the elasticities of substitution among energy fuels will be examined, both theoretically and empirically.

Chapter II centers on the analysis of historical records of U.S. total energy consumption and of the changing level and pattern of household energy use in the United States during the selected period

[10] The chosen utility functions are the Cobb-Douglas, the CES, the Uzawa CES, the Sato two-level CES, and the transcendental logarithmic utility functions. See Chapter IV.

(1947-1965), so that an explanation of the statistics showing the consumption of energy fuels, a hypothesis which will account for them, may be found.[11] The reason for selecting this particular period is that dramatic shifts in the relative importance of individual energy fuels were revealed during this period; and that U.S. energy total also underwent important changes in its composition.

Chapter III makes an in-depth theoretical comparison of an additive utility function to a completely generalized utility function in order to provide the theoretical background to the development of the concept of separability. It also presents a detailed discussion of the separability hypothesis, and reviews separability theorems. Chapter IV deals with the analysis of the internal structure of chosen utility functions through application of separability theorems, and seeks Taylor approximations to chosen utility functions in order to derive a system of demand equations in forms suitable for econometric testing and comparison.

Chapter V discusses the derivation of a system of demand equations from a Taylor's second order approximation, restrictions on the parameters of demand equations, and the demand elasticities and the elasticities of substitution among energy fuels. It also discusses the estimation method used. Chapter VI presents empirical results, evaluates them in terms of whether or not they are in conformity with the theoretical results derived in Chapter V, and assesses how well each of the chosen utility functions performs. The final chapter synthesizes the conclusions drawn from the empirical analysis of Chapter VI, and the choice among the utility functions will be made on an empirical basis.

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[11] Time-series data for energy fuels used in this demand study range from 1937 to 1970. See Appendix C.

CHAPTER II

BACKGROUND TO THE ANALYSIS OF DEMAND FOR FUELS IN THE U.S.

The best way of approaching the econometric theory of demand is from the point of view of the empirical problem which generates the need for such a theory. The econometrist who seeks to make a demand study contemplates certain factual data showing the consumption of some good (or goods) purchased by a particular group of people during certain periods of time. He seeks an explanation of these statistics, a hypothesis which will account for them. A number of possible explanations may be suggested--hypotheses which cannot be tested directly, but which can be used for the arrangement of empirical data in meaningful ways, and which are accepted or rejected according to their success or failure as instruments of arrangement.[1]

The primary purposes here are: (1) empirical choice of fuel variables which satisfy energy needs within U.S. households for heating, cooking, lighting and other home appliances; and (2) making some assumption about the principles governing the consumer's behavior--the preference hypothesis associated with consumer demand for energy

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[1]J. R. Hicks, A Revision of Demand Theory (London: The Clarendon Press, 1969), p. 17.

fuels.

Total Energy Consumption in the U.S., 1947-1965

The abundant use of energy, mainly from mineral fuels, was fundamental to the economic circumstances of mid-century America. With a population accounting for slightly more than 6% of the world's total in the early 1950's,[2] the amount of energy fuels consumed in the United States was more than one-third of the world's total energy supply, as shown in Table 2-1, and per capita consumption of the U.S. energy fuels was roughly six times the world's average.

Much of the significance of the level of total energy use by an economy, and of changes in that level over time, lies not in the level itself, but in its relationship to such indicators of the development of the economy as population and gross national product. The historical path which the United States followed in reaching its positions in total energy consumption, population, gross national product, and per capita energy consumption is traced in Table 2-1.

Between 1947 and 1965, consumption of energy in the United States rose by an annual average of 2.8% compounded. Although it rose in all but five of these eighteen years, the rate of increase was markedly below the 2.8% average in the first few years of the period and markedly above it in the first half of the 1960's. During the same two decades population rose by 1.7% per year, and gross national pro-

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[2]U.S. population was 157,022,000; world population was 2,550,000,000. See U.S. Bureau of the Census, Statistical Abstract of the United States (Washington, D.C.: U.S. Government Printing Office, 1957).

Table 2-1

WORLD ENERGY PRODUCTION AND U.S. ENERGY CONSUMPTION IN 1953

Energy Source	World Production in BTU Equivalent (trillions)	U.S. Consumption in BTU Equivalent (trillions)
Coal	45,380	11,868
Petroleum	26,272	15,334
Natural Gas	9,212	7,550
Hydropower	1,365	382
Vegetable Fuels	15,695	1,125
Total	97,924	36,259
Per Capita (million BTU)	38.4	230.9

Sources: Department of Economic and Social Affiars, United Nations. "World Energy Requirements in 1975 and 2000," Proceedings of the International Conference on the Peaceful Use of Atomic Energy, Geneva, 1955, Vol. 1; Bureau of Mines, U.S. Department of the Interior, Mineral Yearbook, Vol. 2 (Washington, D.C.: U.S. Government Printing Office, 1956).

duct, in real terms, by 3.9% annually. Thus, energy consumption followed the historical pattern of rising substantially faster than population, but not quite as fast as gross national product.

This continuous long-term growth in total energy consumption was followed by very great changes in the composition of energy supply, due to availability of various sources and forms of energy, their relative prices, advances in technology, changes in the structure of the nation's output of goods and services, and shifts in consumer preferences. Dramatic shifts in the relative importance of the individual energy sources emerged during the period 1947-1965, and the remarkable pace of growth in oil and natural gas was evident, as shown in Table 2-2.

The heavy predominance of oil and natural gas was a relatively new development. Up to a couple of years following World War II, coal accounted for about one-half of the nation's total energy consumption, oil for about one-third, and natural gas for slightly more than one-tenth. The enrgy total had since undergone important changes in its composition. Among them were: (1) shifts among primary energy sources, such as the major shift in relative importance from coal to oil and natural gas; (2) the long-term trend away from the direct consumption of raw energy materials to the use of processed and converted energy products, such as the switch from coal to diesel oil as a railroad fuel and the growth of electric power generation; and (3) in the field of mechanical energy, the replacement of steam power by electricity.[3] These shifts were dependent on and closely interconnected with changes

[3]Schurr and Netschert, Energy in the American Economy, 1850-1975 (Baltimore: The John Hopkins Press, 1960), p. 174.

Table 2-2

DISTRIBUTION OF U.S. ENERGY CONSUMPTION BY PRIMARY FUELS, SELECTED YEARS, 1947-1965

Year	Bituminous Coal	Anthracite	Natural Gas	Natural Gas Liquids	Hydro-Electric Power	Crude Oil
1947	43.5%	3.7%	13.8%	1.7%	4.4%	32.9%
1950	34.8	3.0	18.0	2.3	4.7	37.2
1955	27.8	1.5	23.1	3.0	3.8	40.8
1960	22.0	1.0	28.4	3.2	3.6	41.6
1961	21.5	0.9	29.0	3.3	3.7	41.6
1962	21.3	0.8	29.4	3.4	3.8	41.3
1963	21.6	0.7	29.9	3.4	3.6	40.8
1964	21.6	0.7	30.2	3.5	3.7	40.0
1965	22.4	0.6	30.0	3.5	3.9	39.6

Sources: Department of Statistics, American Gas Association, Gas Facts: 1971 Data, 1972 issue; American Petroleum Institute, 1970 Petroleum Facts and Figures, 1971 issue; Bureau of Mines, U.S. Department of the Interior, Mineral Yearbook, various issues; Department of Commerce, Historical Statistics of the United States: Colonial Times to 1957, (Washington, D.C.: U.S. Governemnt Printing Office); U.S. Department of Commerce, Statistical Abstract of the United States, 1950-1970 issues, (Washington, D.C.: U.S. Government Printing Office).

in the equipment in which the various sources and forms of energy were utilized.

Coal ceased to be the dominant source of energy, being surpassed at the beginning of the 1950's by oil and, less than a decade later, by natural gas. By the mid-1960's these changes seemed to be leveling off, and an approximate pattern had emerged. The changes were due mainly to coal's loss of railroad and space-heating and, to a lesser extent, industrial markets for technological, economic, or performance reason.[4] While both oil and natural gas moved heavily into the space-heating market, natural gas made rapid gains as a boiler fuel especially in electric power generation and simultaneously made heavy inroads on oil especially in the residential heating market, due mainly to the non-price attributes of the fuel such as cleanliness, convenience and dependability.[5]

Energy consumption in large amounts is typical of many different aspects of American life. As would be expected of the world's highly-industrialized and energy-intensive nation, the United States used much of its energy consumption to provide heat and power for mills and factories. Indeed, the industrial sector was foremost among the energy-consuming sectors, accounting for 41.6% of all energy consumed in 1965, as shown in Table 2-3. The transportation sector accounted for 30.4% of all energy consumed, or about three-fourths as much as the amounts consumed by the industrial sector. The household sector with

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[4]Texas Eastern Transmission Co., Competition and Growth in American Energy Market, 1947-1985. (1968), p. 12.

[5]Ibid., p. 20.

Table 2-3

DISTRIBUTION OF U.S. ENERGY CONSUMPTION BY SECTORS, SELECTED YEARS, 1947-1965

Year	Industrial	Transportation	Residential	Commercial
1947	41.8%	32.1%	19.2%	6.9%
1955	44.2	29.4	20.5	5.9
1960	42.0	29.8	21.9	6.3
1965	41.6	30.4	21.0	7.0

Sources: See sources in Table 2-2.

Table 2-4

DISTRIBUTION OF INDUSTRIAL ENERGY CONSUMPTION, SELECTED YEARS, 1947-1965

Year	Natural Gas	Coal	Oil	Electricity
1947	20.8%	56.8%	18.9%	3.5%
1955	31.6	38.7	23.6	6.1
1960	39.7	30.3	22.1	7.9
1965	42.9	28.3	20.5	8.3

Sources: See sources in Table 2-2.

its energy requirements for heating, cooking, lighting, and numerous other household tasks consumed 21.0% of the nation's energy total. Industry, transportation and household together used almost nine-tenths of all energy consumed, with the remainder accounted for mainly by commercial establishments.

Changes in the composition of energy consumed within the sectors are more pronounced than changes in sectoral shares of total energy consumption. As shown in Table 2-4, the industrial energy picture was characterized by a marked shift in the relative importance of coal and natural gas in direct fuel use between 1947 and 1965: coal declined and natural gas rose. Each energy fuel, however, retained a significant share in industrial consumption largely because of coal's firm roots in the metal industry and a few other large industries.[6]

In the transportation sector, oil almost preempted coal, as shown in Table 2-5. Coal's loss to oil of its rail market and its disappearance from the transportation scene was virtually completed by mid-1950's. The rapid expansion in road and air transport markets favored oil, not coal; what little demand coal provided was for nonmotive purposes and, through its indirect use as a fuel source for the electricity consumed by railroads.[7] There was no cushion in the transportation market that softened coal's decline. In both household and commercial sectors, oil and natural gas virtually eliminated direct burning of coal, as shown in Tables 2-6 and 2-7. Coal's maintenance of its relative position in the face of losses to oil and natural gas in

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[6]Schurr and Netschert, op. cit., pp. 224-225.

[7]Ibid., pp. 284-285.

Table 2-5

DISTRIBUTION OF TRANSPORTATION ENERGY CONSUMPTION, SELECTED YEARS, 1947-1965

Year	Natural Gas	Coal	Oil	Electricity
1947	- %	31.6%	68.2%	0.2%
1955	-	3.6	96.3	0.1
1960	-	0.3	99.6	0.1
1965	-	0.2	99.7	0.1

Sources: See sources in Table 2-2.

Table 2-6

DISTRIBUTION OF HOUSEHOLD ENERGY CONSUMPTION, SELECTED YEARS, 1947-1965

Year	Natural Gas	Coal	Oil	Electricity
1947	19.3%	47.5%	30.1%	3.1%
1955	33.5	18.7	41.4	6.4
1960	41.1	9.0	41.4	8.5
1965	45.4	4.2	39.8	10.6

Sources: See sources in Table 2-2.

Table 2-7

DISTRIBUTION OF COMMERCIAL ENERGY CONSUMPTION, SELECTED YEARS, 1947-1965

Year	Natural Gas	Coal	Oil	Electricity
1947	18.2%	59.4%	13.4%	9.0%
1955	34.0	30.0	18.7	16.7
1960	47.7	14.7	18.0	20.6
1965	50.1	6.0	17.2	26.7

Sources: See sources in Table 2-2.

Table 2-8

DISTRIBUTION OF AVERAGE ANNUAL RATE OF CHANGE IN RESIDENTIAL ENERGY CONSUMPTION, SELECTED YEARS, 1947-1965

Year	Natural Gas	Coal	Oil	Electricity
1947-65	8.2%	-9.8%	4.8%	10.5%
1955-65	6.3	-11.1	2.7	8.4
1960-65	4.9	-11.6	2.0	7.5

Sources: See sources in Table 2-2.

direct fuel consumption was thus tied to the growth of electricity and to coal's role in providing fuel for the power plants.[8]

In summary, natural gas, electricity and fuel oil retained the significant shares in household and commercial consumption of energy and virtually eliminated the direct burning of coal from both household and commercial sectors during the period 1947-1965. Coal, nevertheless, maintained a significant share in industrial consumption of energy. The extent to which particular forms of energy were applied to particular uses depended in part upon changing supply conditions and prices of various energy sources and in part upon changing technologies which established preferential efficiencies in various uses. In some cases a single source of energy entirely displaced another. More commonly, however, two or three of energy sources were in use at the same time for the same purposes, as for space-heating and industrial boiler fuel.

The Changing Level and Pattern of Household Energy Use, 1947-1965

The most significant supply change in the residential energy market during the period 1947-1965 was the replacement of coal by natural gas, electricity and fuel oil (see Table 2-6).[9] The average annual rates of growth in consumption of natural gas, electricity and fuel oil are shown in Table 2-8. The negative rates of growth for

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[8]Ibid., pp. 279-281.

[9]The residential energy market represents the sum of energy needs within individual households for heating, cooking, lighting, and other home appliances. This market does not include the transportation energy needs connected with household operations.

coal reveals a decline in the relative importance of coal in direct fuel use. Coal no longer plays a significant role as a supplier of household energy.

Close examination of considerably decreased rates of growth since the 1940's reveals the difficulty of any given energy source maintaining an accelerated growth rate as high levels of market penetration are realized and that here must be an element of competition among fuel oil, natural gas and electricity. Closeness in the magnitudes of the long-term growth rates for natural gas and electricity (i.e., 8.2% and 10.5% during the period 1947-1965, respectively) suggests further that they are very close substitutes. However, it is not possible at this stage to explain which fuel U.S. households most prefer, or rank most highly, over any other alternative open to them.

The Hypothesis about the Preferences of U.S. Households

Common sense suggests a number of possible explanations of the statistics showing the quantities of energy fuels consumed within U.S. households during the period 1947-1965: nonprice-explanations and price-explanations. But what the demand theory, considered from the econometric point of view, has to do is to find a hypothesis which will account for the ways in which U.S. households would be likely to react if variations in prices and incomes were the only causes of changes in consumption. It proceeds by making some assumption about the principles governing their behavior. The assumption of behavior according to a scale of preferences comes in here as the simplest, although not necessarily the only possible, hypothesis, and therefore the one which, initially at least, seems to be the most sensible one to try.[10]

There are two forms of the preference hypothesis in the theory of demand: a strong ordering hypothesis and a weak ordering hypothesis. If a collection of consumer goods is strongly ordered, it is such that each good has a place of its own in the order; it is, in principle, given a number (or a utility), and to each number there corresponds one good, and only one good. Accordingly, the preferences of the consumer will exhibit consistency and transitivity. It is not necessary that there should be any indifferent positions. If the whole order is a strong one, it is sufficient to say that he always chooses the most preferred position open to him, and his choice is explained; preference is always sufficient to explain choice.[11]

Weak ordering, on the other hand, allows for the possibility that some consumer goods may be incapable of being arranged in front of one another and put into an ordered relation with the ordered goods, that is, the possibility of indifferent positions exists. A weak ordering consists of a partition of a collection of goods into the subgroups of at least one good, in which the sequence of subgroups is strongly ordered, but in which there is no ordering within the subgroups. If the consumer's ordering is weak, it is possible that there may be two (or more) positions which stand together at the top of his list. His choice between two such positions remains unexplained purely

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[10]J. R. Hicks, op. cit., pp. 16-17.

[11]Ibid., p. 20. Also see Section 2, Chapter III of this dissertation.

on the basis of preference.[12]

A problem arises as to which kind of preference hypothesis the present study of demand ought to be based on to be the most useful. To deny the preference hypothesis in its weak form is to accept the other extreme--the preference hypothesis in its strong form. In fact, the consumers do sometimes find themselves confronted with alternatives between which they are indifferent. As seen in Table 2-8, the tempting hypothesis is that energy fuels between which choice is actually made are strongly ordered, due to the fact that the long-term growth rates for electricity, natural gas, and fuel oil for 1947-1965 were 10.5%, 8.2%, and 4.8% in that order. But there is no reason to assume a priori strong ordering to be the case for U.S. households. Thus, the present empirical study of demand will adopt the preference hypothesis in both strong and weak forms, and investigate which form of the preference hypothesis provides a substantially realistic picture of U.S. households' choice among electricity, natural gas, and fuel oil.[13] Coal will be eliminated from the present study of demand because of its insignificant role as a supplier of household energy.

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[12]Ibid., p. 21. Also see Section 2, Chapter III of this dissertation.

[13]See Chapters III and IV.

CHAPTER III

SEPARABILITY OF UTILITY FUNCTIONS

To describe a choice process in a manner faithful to reality, a utility function must include a large number of consumer goods as its arguments, while a partition of the collection of those consumer goods into the subgroups of at least one good--a commodity-wise partition in which the sequence of subgroups is strongly ordered, but in which there is no ordering within the subgroups--is desirable. For such a commodity-wise partition is essential not only in making that utility function operationally manageable, but also in adequately explaining the consumer's budgetary behavior in allocating expenditure among broad groups of consumer goods. Any utility function for which commodity-wise partitioning is permissible will be functionally separable with respect to that partition. The conditions for such functional separability will be referred to as the separability hypothesis, which is based on the ordering (or preference) hypothesis.

The primary purposes of this chapter are to present a detailed discussion of the preference hypothesis in both strong and weak forms and to review separability theorems, so that the assumptions underlying different utility functions chosen for the present study of demand and the internal structure of those utility functions can be thoroughly investigated. (See Chapter IV for different utility functions chosen for this empirical study of demand.) In addition, an in-depth comparison of an additive utility function to a completely generalized utility function will be made in order to provide the theoretical background to the development of the concept of separability and reveal the significance of the concept of separability, both theoretical and empirical.

An additive utility function assumes that utility is cardinal and additive. The cardinal hypothesis with independent utilities places very severe restrictions on the preference field and the empirical data and, hence, limits the field of its applicability. Its strong implications, both theoretical and empirical, may possibly lead to the replacement of an additive utility function by a completely generalized utility function. The great increase in generality, however, generates dissatisfaction because of the relative paucity of its meaningful empirical implications. The theoretical solution of this empirical issue has been sought through the application of the concept of separability. Therefore, a comparison of an additive utility function to a completely generalized utility function is essential to a discussion of the concept of separability and will aid the understanding of the significance of the concept of separability.

Theoretical Background to the Development of the Concept of Separability

As observed in the introductory chapter, early contributions to the theory of consumer behavior were characterized by the assumption that utility was measurable and a cardinal concept. Such a utility function could be written as

$$(3.1) \qquad U = \sum_{i=1}^{n} U^i(x_i)$$

where U^i is a sub-utility function of the quantity of good x_i and U is the total utility of the whole collection of goods, x_i's, and the sum of separate utilities, U^i's. In the function (3.1), the preferences of the consumer exhibit consistency and transitivity, because its formulation employs "strong ordering" as the maintained preference hypothesis (i.e., a set of goods, x_i's, is strongly ordered).[1]

The indifference differential equation of the utility function (3.1) under the hypothesis of independent utilities may be written as

$$(3.2) \qquad dU = \sum_{i=1}^{n} U_i(x_i)dx_i = 0$$

where U_i is the first order partial derivative of the utility function U with respect to good x_i and a function of good x_i alone. Equation (3.2) is always integrable in the effective region of a given commodity space because the utility function (3.1) employs strong ordering hypothesis and what corresponds to transitivity, in the mathematical theory, is integrability.[2] The general integral of the equation (3.2) will be of the form

$$(3.3) \qquad V = \int_R \sum_{i=1}^{n} U_i(x_i)dx_i$$
$$= \int_R U_1(x_1)dx_1 + \int_R U_2(x_2)dx_2 + \dots + \int_R U_n(x_n)dx_n$$

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[1]See Section 3 in Chapter II and Sections 2 and 3 in Chapter III.

[2]J. R. Hicks, op. cit., p. 23; W. Rudin, Principles of Mathematical Analysis, (New York: McGraw-Hill Book Co., 1964), Chapter 6 entitled "The Riemann-Stieljes Integral".

$$= U^1(x_1) + U^2(x_2) + \ldots + U^n(x_n) + C$$

$$= F\left(\sum_{i=1}^{n} U^i(x_i)\right)$$

$$= F(U)$$

where V is the total utility, a function of the utility function (3.1) (or, alternatively, a function of the sum of "n" sub-utility functions, U^i's, of good x_i alone), and F is an additive function and R represents the effective region of a given commodity space.

The function (3.3) states that even if the utility function exists at all, it is by no means unique and any other function F(U) can equally well be taken as the utility function. The fact that the utility function is indeterminate to this extent shows that it is a function index of utility, and not a measure of utility.[3] However, even under the assumption that utility is an ordinal concept, the additive utility function (3.1) can be justified if it is interpreted as the normalized utility index of the function (3.3), which is obtained only if the marginal rate of substitution between any two independent goods depends on the quantities of those goods alone.[4]

Since the cardinal hypothesis with independent utilities was a very severe restriction on the preference field, it generated dissatisfaction and led to contemplating the possibility that utilities might be interdependent. That is to say, the marginal utility of any good might depend not only upon the consumption of that good but

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[3]R. G. D. Allen, "The Nature of Indifference Curves," The Review of Economic Studies, Vol. 1 (1933-1934), pp. 110-121.

[4]See Section 3 in Chapter III.

also upon the consumption of any other good purchased. As a result, the idea of a generalized utility function was introduced by F. Y. Edgeworth:[5]

$$(3.4) \qquad \phi = f(x_1, x_2, \ldots, x_n)$$

where ϕ is the total utility and f is an arbitrary function of the quantities of "n" goods, x_i's.

The difference between functions (3.1) and (3.4) is that the utility function (3.4) concedes the interdependence between any pair of goods, x_i and x_j $(i \neq j)$, and the nonadditivity of utility functions. According to the Edgeworth-Pareto definition associated with the utility function (3.4), a pair of goods are complementary, independent, or substitutive, depending upon the sign of the second order partial derivative of the utility function (3.4):

$$(3.5) \qquad f_{ij} = \frac{\partial^2 \phi}{\partial x_i \partial x_j} \gtreqless 0 \text{ for } i \neq j \ (i,j = 1, 2, \ldots, n).$$

In case of the additive utility function (3.1),

$U_{ij} = \frac{\partial^2 U}{\partial x_i \partial x_j} = 0$ for $i \neq j$ because the consumer goods, x_i's, in (3.1) are independent goods. In other words, any pair of goods are neither complementary nor substitutive. Thus, the definition (3.5) appears substantially realistic, and the generalized utility function (3.4) seems to be of the greatly improved form in comparison with the additive utility function (3.1). The theoretical and empirical implications of the definition (3.5) are investigated below.

First, the definition (3.5) depends on the notion of utility as

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[5] F. Y. Edgeworth, op. cit., p. 97.

a determinate function. Even if the definition assumes the existence of the utility function, the function ϕ is not to be taken, in general, as unique, that is, $F(\phi)$ can equally well be taken as the utility function.

Second, the form and sign of f_{ij} in the definition (3.5) are not determinate; in other words,

$$\frac{\partial^2 F(\phi)}{\partial x_i \partial x_j} = F'(\phi)\phi_{ij} + F''(\phi)\phi_i\phi_j \tag{3.6}$$

does not, in general, have the same sign as that of ϕ_{ij} in (3.5), even though $F'(\phi)$, ϕ_i and ϕ_j are assumed to be positive. For $F''(\phi)$ can be either positive or negative, depending entirely upon the functional form of F. Thus, the only case in which the second order partial derivative in (3.6) is invariant in sign is when either ϕ_i or ϕ_j is zero, i.e., when the individual consumer is saturated with one of the goods, x_i and x_j $(i \neq j)$.

Third, even if ϕ_{ij} in (3.5) can be made determinate, its value and sign vary according to the position of the individual consumer, i.e., according to the amount of the various consumer goods he happens to possess.[6] It would seem either that Edgeworth and Pareto intended their definition to apply only in the special cases when ϕ_{ij} preserves a uniform sign in all situations, or that they allowed a pair of goods for a given individual to be complementary in one set of circumstances and substitutive in another.[7] In the former case the definition loses in

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[6] R. G. D. Allen, "A Comparison Between Different Definitions of Complementary and Competitive Goods," Econometrica, Vol. 2 (1934), pp. 168-169.

[7] Ibid.

generality, while in the latter case it does not fit in with the everyday notion of the meanings of the terms "complementary" and "substitutive" when applied to goods.[8]

It was from equations (3.4), (3.5) and (3.6) that the works of Slutsky, Johnson, Hicks, and Allen[9] proceeded to design the criterion of complementary and sbustitutive goods, which are independent both of the existence of a utility function and of indeterminateness in a utility function, if it can be assumed to exist.

While some of the implications of the cardinal hypothesis with independent utilities led to the rejection of the additive utility function (3.1) and its replacement by a completely generalized utility function (3.4), this in turn generated dissatisfaction because of the relative paucity of its meaningful empirical implications. The solution of this theoretical issue has been sought through the application of the concept of separability, which is based on the strong and weak forms of the preference hypothesis.

The Preference Hypothesis in Strong and Weak Forms

The demand theory, which is based on the preference hypothesis, turns out to be nothing else but an economic application of the logical theory of ordering.[10] There are two forms of preference hypothesis--the

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[8] Ibid.

[9] E. E. Slutsky, "On the Theory of Budget of the Consumer." In: Stigler and Boulding, Reading in Price Theory (Chicago: Richard D. Irwin, Inc., 1952), Vol. 6, pp. 27-56; W. E. Johnson, "The Pure Theory of Utility Curves," The Economic Journal (December 1913), pp. 483-513; J. R. Hicks, Value and Capital (Oxford: The Clarendon Press, 1968), Chapters 1-3; R. G. D. Allen and J. R. Hicks, "A Reconsideration of the Theory of Value," Econometrica, Vol. 1 (May 1934), pp. 196-221.

[10] J. R. Hicks, A Revision of Demand Theory, p. 19.

assumption of consumer behavior according to a scale of preference. One is the strong ordering hypothesis, and the other the weak ordering hypothesis.

Given the collection of consumer goods, which is sought to be put into an order, the first necessity is that any good X should be selected as the basis, and, according to the relation which exists between X and the remaining goods, all goods other than X should be arranged with respect to the basis X. It is at this point that the distinction between strong and weak ordering should be drawn. If the ordering is to be strong, all goods other than X must be placed either on the left of X, implying that X is superior to all other goods, or on the right of X, implying that X is inferior to all other goods. As a result, those goods are partitioned into two mutually exclusive commodity groups, one consisting of goods having a sort of relation to the basis X, the other of goods having a different sort of relation. That is to say, the two commodity groups must fulfill the following preliminary condition of strong ordering:[11]

(3.7)
(1) Two commodity groups must include all goods other than X;
(2) two commodity groups must not overlap, so that some goods are in both groups.

Once the preliminary conditions of strong ordering are fulfilled, it must be established that partitions with respect to different bases are consistent with one another; in other words, two-term consistency conditions and the transitivity condition must be fulfilled

[11]Ibid., p. 25.

in order to achieve a final strong ordering. Two-term consistency conditions[12] are such that

(3.8)
(1) if Y is on the left of X, Y being a basis different from the basis X, then X must be on the right of Y;
(2) if Y is on the right of X, then X must be on the left of Y.

Even if two-term consistency conditions are fulfilled, for every possible pair of bases, the whole set of goods are not necessarily capable of being put into an order in a straightforward unidirectional manner, because there may exist the possibility of circular ordering. Hence, in addition to the preliminary conditions and two-term consistency conditions, the transitivity condition must be fulfilled:

(3.9) If Y is on the left of X, and Z is on the left of Y, Z being a basis different from X and Y, then Z is on the left of X.

An alternative interpretation of the transitivity condition in terms of two-term consistency conditions is that if X is on the right of Y, and Y is on the right of Z, then Y is on the left of X, and Z is on the left of Y (second consistency condition), then Z is on the left of X (transitivity condition), then X is on the right of Z (first consistency condition). As a result of the transitivity condition, there are three nonoverlapping commodity groups. The same process can be continued by introducing additional bases, until the whole set of goods are put into an ordered relation. Thus, strong ordering depends upon the prelimi-

[12]Ibid., p. 26.

nary conditions, two-term consistency conditions, and the transitivity condition.

If the ordering is to be weak, there may be goods other than the basis X, which will be placed neither on the left of X nor on the right of X in the ordering. This situation does not fulfill one or the other of the preliminary conditions of strong ordering in (3.7); there is only one preliminary condition of weak ordering, as against the two preliminary conditions of strong ordering. With weak ordering, there is a further important deduction to be drawn from two-term consistency. It is possible that X may be neither on the left of Y nor on the right of Y, which is called "neutral to Y" for brevity. The neutrality of transitivity is reversible, and it occurs with weak ordering, because, with respect to any basis, the remaining goods can be partitioned into two possibly overlapping commodity groups. Thus, in addition to the transitivity condition (3.9), the neutrality of transitivity can be deduced:[13]

(3.10) If X is neutral to Y, and Y is neutral to Z, then X is neutral to Z.

However, wholly unordered goods, which belong to the intersection of two overlapping commodity groups, cannot occur, because if X is neutral to Y, and Y is on the left of Z, then X is on the left of Z. Hence, any good X which is not ordered with respect to Y is nevertheless ordered with respect to such goods as are ordered with respect to Y.

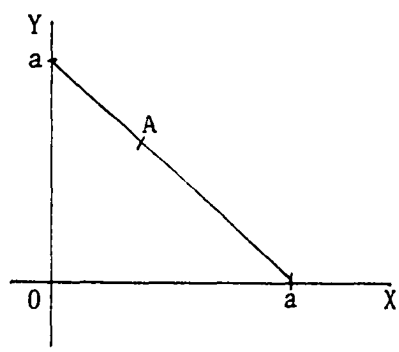
The situation of strong ordering is described in Figure 1, in which the quantities of two goods, X and Y, are measured along the axes. With given prices and income, the quantities available to the consumer

[13]Ibid., p. 28.

are limited by a budget line "aa", and the available alternatives are represented by points within the triangle "a0a" and on the boundary of the triangle. Suppose that the consumer is not affected by anything else than current market conditions, and the choices he makes always express the same ordering. With strong ordering, the assumption of indivisibility (or discontinuity) of the goods is required, i.e., the goods are available in discrete units.

Figure 1

STRONG ORDERING



Source: J. R. Hicks, A Revision of Demand Theory (London: The Clarendon Press, 1969), p. 39.

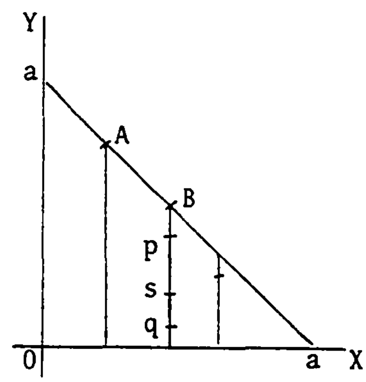
If the available alternatives are strongly ordered, then the consumer reveals his preference for the position A over any other position within the triangle a0a or on the boundary of the triangle. Thus, under strong ordering the chosen position is shown to be preferred to all other positions within and on the triangle only by assuming indivisibility of the goods.

However, the strong form of the preference hypothesis cannot be maintained if divisibility of the goods is assumed. With weak ordering, one more assumption is needed in addition to divisibility:

a positive marginal utility of the good. Suppose that good Y is finely divisible, has a positive marginal utility, and the consumer prefers a larger amount of Y to a smaller amount of Y, provided that the amount of X at his disposal is unchanged. As seen in Figure 2, any point on one of the vertical lines is an effective alternative.

Figure 2

WEAK ORDERING



Source: J. R. Hicks, A Revision of Demand Theory (London: The Clarendon Press, 1969), p. 41.

But such alternatives cannot be strongly ordered, unless the whole set of alternatives on one vertical line is preferred to the whole set of alternatives on the next vertical line, and so on. For if there are two alternatives, p and q, on the same vertical line, which are such that p is preferred to r on the next vertical line, while r is preferred to q , then an alternative, s, between p and q which is indifferent to r can be found, so that strong ordering must be abandoned. Moreover, it cannot be shown that the chosen position on the line "aa" is preferred over any other position which lies on the same line, i.e., A is preferred over B, or B is preferred over A.

As observed above, the difference between the consequences of strong and weak forms of the preference hypothesis amounts to no more than this: that under strong ordering the chosen position is shown to be preferred over any other positions open to the consumer and rejected, which lie within and on the triangle a0a, while under weak ordering the chosen position is preferred over all positions within the triangle, but may be indifferent to other positions on the boundary of the same triangle.

A question arises as to which ordering of the preference hypothesis the demand theory ought to remain based. But it must be noted that the weak ordering is the less restrictive assumption. The weak form of preference hypothesis implies most of the results of demand theory, but does not imply the integrability conditions that the matrix of substitution effects is symmetric, conditions needed to construct a utility function.[14] These conditions are, however, implied by the strong form of the preference hypothesis; the strong form of the preference hypothesis implies a consistent set of preferences, so that the integrability conditions needed to construct a utility function are met, if continuity (or divisibility) is assumed (i.e., the strong ordering approach commits itself to discontinuity or indivisibility).[15]

Separability

According to the separability hypothesis, there corresponds to a commodity-wise partition, achieved by either strong or weak order-

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[14] M. D. Intriligator, Mathematical Optimization and Economic Theory (Englewood Cliffs, N.J.: Prentice-Hall, Inc., 1971), p. 165.

[15] Ibid., p. 166.

ing, a functional separability. A continuously twice differentiable utility function is functionally (i.e., strongly or weakly) separable with respect to a commodity-wise partition, and, hence, can be written with two or more of its independent variables (i.e., consumer goods) grouped in an aggregate. The separability hypothesis was first advanced by W. Leontief[16] and M. Sono.[17] Leontief showed that if $F(x_1, x_2, x_3)$ is continuously twice differentiable, then there exists a function $\phi(x_1, x_2)$ and a function $G(\phi, x_3)$ such that

$$F(x_1, x_2, x_3) = G\{\phi(x_1, x_2), x_3\}$$

if, and only if,

$$\frac{\partial(F_1/F_2)}{\partial x_3} = 0 \tag{3.11}$$

where F_1 and F_2 are first order partial derivatives with respect to x_1 and x_2, respectively.

Sono also derived the same results as Leontief's. But Leontief's work was presented in the context of the theory of production, while Sono's work in the context of the theory of utility. As observed in (3.11), Leontief's functional separability is valid "locally" in the neighborhood of a particular point; that is to say, x_3 in (3.11) is excluded from the preference field, and only two goods, x_1 and x_2, are left to choice, and, hence, the group of x_1 and x_2 is said to be locally separable from the whole set of x_1, x_2 and x_3. Thus, S. M. Goldman and

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[16] W. Leontief, "A Note on the Interrelation of Subsets of Independent Variables of a Continuous First Derivatives," Bulletin of the American Mathematical Society, Vol. 53 (1947), pp. 343-350.

[17] M. Sono, "The Effect of Price Change on the Demand and Supply of Separable Goods," International Economic Review, Vol. 2 (1961), pp. 239-269.

H. Uzawa examined Leontief's necessary and sufficient conditions for functional separability and, as a result, introduced separability theorems which are proved "globally".[18]

The following assumptions and notations are required for the separability theorems introduced below.

(1) The utility function $U(x)$ is a continuous mapping from the set of all nonnegative commodity bundles onto the set of nonnegative utility level with $U(0) = 0$; that is to say, the utility function $U(x)$ assumes "one-to-one and onto" mapping, so that the utility function has an inverse function.

(2) The utility function $U(x)$ is continuously twice differentiable and its symmetric Hessian matrix is negative definite, implying that the utility function is strictly concave.

(3) The set $N = \{x_1, x_2, \ldots, x_n\}$ is the collection of "n" consumer goods; the set $N = \{N_1, N_2, \ldots, N_r\}$ with $r < n$ is the class of "r" commodity groups, each consisting of at least one consumer good, x_i $(i = 1, 2, \ldots, n)$, from the set N.

Under the assumptions (1) and (2), the indifference surfaces are convex toward the origin, and the demand functions for the consumer goods are uniquely determined and stable within the effective region of a given commodity space.[19] The following theorems for weak and strong separability were introduced and proved by Goldman and Uzawa.[20]

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[18] S. M. Goldman and H. Uzawa, "A Note on Separability in Demand Analysis," Econometrica, Vol. 32 (1964), pp. 387-398.

[19] R. G. D. Allen, Mathematical Analysis for Economists (London: Macmillan, 1938), pp. 509-513.

[20] S. M. Goldman and H. Uzawa, op. cit.

Definition 1 (weak separability): A utility function $U(x)$ is weakly separable with respect to a partition $\{N_1, N_2, \ldots, N_r\}$ if the utility function $U(x)$ has the property

$$(3.12) \quad \frac{\partial}{\partial x_k} \cdot \left(\frac{U_i(x)}{U_j(x)}\right) = 0 \text{ for all } x_i,\ x_j \in N_h \text{ and } x_k \notin N_h$$

$$(i,j,k = 1, 2, \ldots, n;$$

$$h = 1, 2, \ldots, r)$$

where U_i and U_j are the first order partial derivatives of $U(x)$ with respect to x_i and x_j, respectively; N_h is any one of "r" commodity groups.

Theorem 1 (weak separability): A utility function $U(x)$ is weakly separable with respect to a partition $\{N_1, N_2, \ldots, N_r\}$ if, and only if, $U(x)$ is of the form

$$(3.13) \quad U(x) = F\ U^1(x^1),\ U^2(x^2),\ \ldots,\ U^r(x^r)$$

where $U^i(x^i)$ $(i = 1, 2, \ldots, r)$ is a sub-utility function of subvector x^i consisting of at least one x_i $(i = 1, 2, \ldots, n)$; F is a monotonically increasing function of "r" sub-utility functions, U^i's.

Definition 2 (strong separability): A utility function $U(x)$ is strongly separable with respect to a partition $\{N_1, N_2, \ldots, N_r\}$ if the utility function $U(x)$ has the property

$$(3.14) \quad \frac{\partial}{\partial x_k} \cdot \left(\frac{U_i(x)}{U_j(x)}\right) = 0 \text{ for } x_i \in N_h,\ x_j \in N_t, \text{ and } x_k \notin N_h \cup N_t$$

$$(h \neq t;\ h,t = 1, 2, \ldots, r).$$

Theorem 2 (strong separability): A utility function $U(x)$ is strongly separable with respect to a partition $\{N_1, N_2, \ldots, N_r\}$ with $r>2$ if, and only if, $U(x)$ is of the form

$$(3.15) \quad U(x) = F\left(U^1(x^1) + U^2(x^2) + \ldots + U^r(x^r)\right)$$

where F is a monotonically increasing function of the sum of "r" sub-utility functions, each U^i being a function of subvector x^i.

As observed in Definition 2 for strong separability, the condition (3.14) reduces to the condition (3.12) for weak separability when the whole set of "n" consumer goods is partitioned into two subgroups, N_1 and N_2 with $r=2$. That is to say, if $r=2$, then $x_i \in N_1$ and $x_k \in N_2$, and hence, $x_j \in N_1$. Therefore, x_i and x_j must belong to N_1, and x_k must belong to N_2. This means that strong separability implies weak separability.

In Chapter IV, the internal structure of the chosen utility functions will be analyzed through application of separability theorems introduced above, while the implications of separability restrictions on the parameters of utility functions will be examined in Chapter V, in relation to demand elasticities and elasticities of substitution derived from the chosen utility functions.

CHAPTER IV

DIFFERENT THEORETICAL MODELS UNDER SEPARABILITY

In empirical studies of demand functions under utility assumption, a problem arises as to which of a number of alternative specifications of the theoretical model for a utility is to be regarded as a correct one. On theoretical grounds, none of the models dominates its competitors. The choice of specification must then be made on an empirical basis: Which model performs best?[1]

Two issues involved are: (1) the choice of specification of the theoretical model for a utility; and (2) the empirical verification of that particular model in terms of its usefulness. These two issues are equally important because if it is true that the chosen model ought to be theoretically sound, then it is also true that its usefulness can be measured in terms of its ability to explain facts. However, no microeconomic data ever give an exact fit to linear or nonlinear forms of the utility function since they are only an approximation to possibly complex but unknown forms. Thus, the following functional forms for a utility are selected to determine the most useful one which will yield a substantially realistic picture of U.S. households' choices among electricity,

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[1]R. S. Parks, "Systems of Demand Equations: An Empirical Comparison of Alternative Functional Forms," Econometrica, Vol. 37 (October 1969), pp. 629-650.

natural gas and fuel oil.

(1) Cobb-Douglas (CD for short) utility function,[2]

(2) CES utility function,[3]

(3) Uzawa's CES (Uzawa for short) utility function,[4]

(4) Sato's two-level CES (Sato for short) utility function,[5]

(5) Transcendental logarithmic (translog for short) utility function.[6]

These utility functions are selected for three reasons. First, the CD and CES are strongly separable utility functions, while the Uzawa and Sato are weakly separable utility functions. The translog utility function does not employ separability as part of the maintained hypothesis, and it is the unrestricted (or generalized) functional form for utility. The choice of these specifications will enable this demand study to cover three possible cases: the case of strong separability, the case of weak separability, and the case of neither strong nor weak separability. Second, while there may be reasons to suspect the impli-

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[2] P. H. Douglas, "Are There Laws of Production?," The American Economic Review, Vol. 28 (1948), pp. 1-41.

[3] K. J. Arrow, H. B. Chernery, B. S. Minhas, and R. M. Solow, "Capital-Labor Substitution and Economic Efficiency," The Review of Economics and Statistics, Vol. 63 (1961), pp. 225-249.

[4] H. Uzawa, "Production Functions with Constant Elasticities of Substitution," The Review of Economic Studies, Vol. 29 (1962), pp. 291-299.

[5] K. Sato, "Two-Level Constant-Elasticity-of-Substitution Production Function," The Review of Economic Studies, Vol. 34 (1967), pp. 201-217.

[6] L. R. Christensen, D. W. Jorgenson, and L. J. Lau, "Transcendental Logarithmic Utility Function," The American Economic Review, (June 1975), pp. 367-383.

cations of the properties of any particular utility function, there is no reason to assume a priori that this particular function is applicable to the case for individual households. Third, nothing can be said about the quality of the estimates of parameters if one, and only one, function is selected a priori.

The primary purpose here is to derive the chosen utility functions from the generalized utility function through application of separability theorems, so that the internal structure of each of the chosen utility functions will be investigated.

Strongly Separable Utility Functions: CD and CES Utility Functions

Suppose that an arbitrary utility function of the quantities demanded of three consumer goods is given:

$$U = F(x_1, x_2, x_3) \tag{4.1}$$

where U is the total utility, and F is an arbitrary function which is assumed to be continuously twice differentiable. Let x_1, x_2, and x_3 represent the quantities demanded of fuel oil, natural gas, and electricity, respectively. To derive the three-good CD and CES utility functions, assume that a set of three goods has ordering among themselves, and is capable of being put into an ordered relation with the ordered goods; that is to say, a set of three goods is strongly ordered. Then in principle, there is given an ordinal utility measure (or a number), and to each utility measure (or each number) there corresponds one and only one good.

Because of the strong ordering hypothesis, utilities are independent of one another and, hence, additive. Thus, by Theorem 2 (strong separability), function (4.1) can be written as

$$(4.2) \quad U = F(x_1, x_2, x_3) = G\left(U^1(x_1) + U^2(x_2) + U^3(x_3)\right)$$

where G is a monotonically increasing function of the sum of sub-utility functions U^i (independent utilities), and each U^i is a monotonically increasing function of the quantity demanded of one good x_i.

The three-good CD utility function can be derived from function (4.2) Since sub-utility functions U^i in (4.2) are monotonically increasing, define U^i's as logarithmic functions which are monotonically increasing:

$$(4.3) \quad \begin{aligned} U^1(x_1) &= \ln \theta_1 x_1^{b_1} \\ U^2(x_2) &= \ln \theta_2 x_2^{b_2} \\ U^3(x_3) &= \ln \theta_3 x_3^{b_3} \end{aligned}$$

where θ_i's and b_i's are constants. Substitution of (4.3) into (4.2) yields

$$(4.4) \quad \begin{aligned} U &= G\ U^1(x_1) + U^2(x_2) + U^3(x_3) \\ &= G(\ln \theta_1 x_1^{b_1} + \ln \theta_2 x_2^{b_2} + \ln \theta_3 x_3^{b_3}) \\ &= G\left(\ln(\theta_1 \theta_2 \theta_3 \cdot x_1^{b_1} \cdot x_2^{b_2} \cdot x_3^{b_3})\right) \\ &= G\left(\ln(\theta \cdot x_1^{b_1} \cdot x_2^{b_2} \cdot x_3^{b_3})\right) \end{aligned}$$

where $\theta = \theta_1\theta_2\theta_3$. Since G is a monotonically increasing function, define G as an exponential function which is monotonically increasing. Then, function (4.4) becomes

$$(4.5) \quad U = \mathrm{EXP}\left[\ln(\theta \cdot x_1^{b_1} \cdot x_2^{b_2} \cdot x_3^{b_3})\right] = \theta \cdot x_1^{b_1} \cdot x_2^{b_2} \cdot x_3^{b_3} .$$

Function (4.5) is the three-good CD utility function. Since it assumes strong separability, the CD utility function is the strongly separable utility function. Furthermore, function (4.5) is linearly homogeneous when $b_1 + b_2 + b_3 = 1$.[7]

The three-good CES utility function can also be derived from function (4.2). Since sub-utility functions U^i and function G are monotonically increasing, define U^i and G as power functions which are monotonically increasing:

$$(4.6) \quad \begin{aligned} U^1(x_1) &= \delta_1 x_1^{-p} \\ U^2(x_2) &= \delta_2 x_2^{-p} \\ U^3(x_3) &= \delta_3 x_3^{-p} \\ G &= (a \cdot U)^{-p} \end{aligned}$$

where δ_i's and p are constants, and a represents any level of utility U. Substitution of (4.6) into (4.2) yields

$$(a \cdot U)^{-p} = \delta_1 x_1^{-p} + \delta_2 x_2^{-p} + \delta_3 x_3^{-p}$$

(4.7) or

$$U = \theta \cdot (\delta_1 x_1^{-p} + \delta_2 x_2^{-p} + \delta_3 x_3^{-p})^{-\frac{1}{p}} , \text{ where } \theta = a^{-1} .$$

Function (4.7) is the three-good CES utility function. Since it assumes strong separability, the CES utility function is the strongly separable

[7]See the latter part of Section 3 in Chapter IV for the validity of homogeneity restrictions on the form of the utility function.

utility function. Furthermore, this function is linearly homogeneous.

Weakly Separable Utility Functions: Uzawa and Sato Utility Functions

H. Uzawa, in his 1962 paper,[8] proposed a generalization of the n-good CES utility function. The characteristic of this function is a hybrid of the CD and CES utility functions; that is to say, sub-utility functions possess CES properties, and they are combined with an overall CD utility function. K. Sato, in his 1967 paper,[9] proposed a function which generalizes Uzawa's n-good CES utility function; that is to say, sub-utility functions possessing CES properties are combined with an overall CES utility function. Thus, this function is called the two-level CES utility function.

To derive the three-good Uzawa and Sato utility functions, assume a partition of a set of three goods into two subgroups, in which the subgroups are strongly ordered, but in which there is no ordering within the subgroups. Thus, the set of three goods is weakly ordered. Assume further that the one subgroup consists of x_1 alone, and the other subgroup consists of x_2 and x_3.[10] Then, there is given a utility, and to each utility there corresponds one and only one subgroup.

Because of the weak ordering hypothesis, utilities corresponding to the subgroups are independent of each other and, hence, additive. Thus, by Theorem 1 (weak separability) and Theorem 2 (strong separabi-

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[8] H. Uzawa, op. cit.

[9] K. Sato, op. cit.

[10] This type of commodity grouping is one of the possible cases discussed in Chapter II.

lity), function (4.1) can be written as

$$(4.8) \quad U = F(x_1, x_2, x_3) = G\left(U^1(x_1) + U^2(x_2, x_3)\right)$$

where G is a monotonically increasing function of the sum of sub-utility functions U^i (independent utilities), U^1 is a monotonically increasing function of the quantity demanded of good x_1, and U^2 is a monotonically increasing function of the quantities demanded of goods x_2 and x_3.

The three-good Uzawa utility function can be derived from function (4.8). By Uzawa's definition, sub-utility functions U^i in (4.8) possess CES properties. This implies that the subgroup of x_2 and x_3 is strongly ordered; and that there exist such utility functions as U^{21} and U^{22}, and they are additive by Theorem 2 (strong separability). Thus, function (4.8) becomes

$$(4.9) \quad \begin{aligned} U &= G\left(U^1(x_1) + U^2(x_2, x_3)\right) \\ &= G\left[U^1(x_1) + U^2\left(U^{21}(x_2) + U^{22}(x_3)\right)\right]. \end{aligned}$$

Since U^1 and U^2 are, by Uzawa's definition, CES sub-utility functions and combined with an overall CD function, define U^1 and U^2 in (4.9) as the logarithm of CES utility function:

$$(4.10) \quad \begin{aligned} U^1(x_1) &= b_1 \cdot \ln\left(\theta_1 \cdot (\delta_1 x_1{}^{-P})^{-\frac{1}{P}}\right), \\ U^2(x_2, x_3) &= b_2 \cdot \ln\left(\theta_2 \cdot (\delta_2 x_2{}^{-P} + \delta_3 x_3{}^{-P})^{-\frac{1}{P}}\right) \end{aligned}$$

where the definitions of U^2 and U^3 in (4.6) are substituted into U^{21} and U^{22} in (4.9). Since G in (4.9) is a monotonically increasing function, define G as an exponential function which is monotonically increasing. Then, function (4.9) becomes, by substituting (4.10) into

(4.9),

$$U = EXP\left[b_1 \cdot \ln\left(\theta_1 \cdot (\delta_1 x_1^{-p})^{-\frac{1}{p}}\right) + b_2 \cdot \ln\left(\theta_2 \cdot (\delta_2 x_2^{-p} + \delta_3 x_3^{-p})^{-\frac{1}{p}}\right)\right]$$

$$(4.11) \qquad = \left(\theta_1 \cdot (\delta_1 x_1^{-p})^{-\frac{1}{p}}\right)^{b_1} \cdot \left(\theta_2 \cdot (\delta_2 x_2^{-p} + \delta_3 x_3^{-p})^{-\frac{1}{p}}\right)^{b_2}$$

$$= \theta \cdot x_1^{b_1} \cdot (\delta_2 x_2^{-p} + \delta_3 x_3^{-p})^{-\frac{b_2}{p}}$$

where $\theta = \theta_1^{b_1} \cdot \theta_2^{b_2} \cdot \delta_1^{-\frac{1}{p}}$. Function (4.11) is the three-good Uzawa utility function, which is a hybrid of the CD and CES functions. Since it assumes weak separability (see (4.8)), the Uzawa utility function is the weakly separable utility function. Furthermore, this function is linearly homogeneous.

The three-good Sato utility function can be derived from function (4.9), which is weakly separable. By Sato's definition, sub-utility functions U^i in (4.9) possess CES properties. Thus, define U^1 and U^2 in (4.9) as

$$U^1(x_1) = \theta_1 \cdot (\delta_1 x_1^{-p})^{-\frac{1}{p}},$$

(4.12)

$$U^2(x_2, x_3) = \theta_2 \cdot (\delta_2 x_2^{-p} + \delta_3 x_3^{-p})^{-\frac{1}{p}}$$

where the definitions of U^2 and U^3 in (4.6) are substitutde into U^{21} and U^{22} in (4.9). Since U^1 and U^2 are, by Sato's definition, combined with an overall CES function, define G in (4.9) as a CES function of CES sub-utility functions, U^1 and U^2. Then, function (4.9) becomes, by substituting (4.12) into (4.9),

$$U = G\left(\theta_1\cdot(\delta_1 x_1^{-p})^{-\frac{1}{p}} + \theta_2\cdot(\delta_2 x_2^{-p} + \delta_3 x_3^{-p})^{-\frac{1}{p}}\right)$$

$$(4.13) \qquad = \theta\cdot\left[a_1\left(\theta_1(\delta_1 x_1^{-p})^{-\frac{1}{p}}\right)^{-w} + a_2\left(\theta_2(\delta_2 x_2^{-p} + \delta_3 x_3^{-p})^{-\frac{1}{p}}\right)^{-w}\right]^{-\frac{1}{w}}$$

$$= \theta\cdot\left(b_1 x_1^{-w} + b_2\cdot(\delta_2 x_2^{-p} + \delta_3 x_3^{-p})^{\frac{w}{p}}\right)^{-\frac{1}{w}}$$

where $b_1 = a_1\theta_1^{-w}\delta_i^{-\frac{w}{p}}$ and $b_2 = a_2\theta_2^{-w}$.

Function (4.13) is the three-good Sato utility function, which is weakly separable (see (4.8)). Furthermore, this function is linearly homogeneous.

Transcendental Logarithmic Utility Function

The transcendental logarithmic (translog for short) utility function proposed by Christensen, Jorgenson, and Lau[11] is nothing but an approximation by a Taylor series expansion about a fixed point different from zero to a generalized utility function in logarithmic form of n variables. That is to say, it is a generalized Taylor series expansion in n variables, truncated after the second order term for an arbitrary function of n variables.

To see this, suppose that an arbitrary utility function of the quantities demanded of n consumer goods, x_i's (i = 1, 2, ..., n) is given:

$$(4.14) \qquad U(x) = F(x_1, x_2, \ldots, x_n)$$

where x is a commodity vector consisting of n goods. Applying logarithmic transformation to (4.14) yields

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[11]L. R. Christensen, D. W. Jorgenson, and L. J. Lau, op. cit.

(4.15) $\quad \ln U(x) = G(\ln x_1, \ln x_2, \ldots, \ln x_n)$

where G is a logarithmic transformation of F in (4.14).

Assume that the utility function U(x) is continuously twice differentiable in the effective region of a given commodity space. Then, the logarithmic utility function (4.15) can be approximated by a Taylor series expansion about a fixed point different from zero, i.e., $(\ln \bar{x}_1, \ln \bar{x}_2, \ldots, \ln \bar{x}_n)$:

$$\ln U(x) = G(\ln \bar{x}_1, \ln \bar{x}_2, \ldots, \ln \bar{x}_n)$$

(4.16)
$$+ \sum_{i=1}^{n} \frac{G}{\ln x_i} (\ln x_i - \ln \bar{x}_i)$$

$$+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{{}^2G}{\ln x_i \ln x_j} (\ln x_i - \ln \bar{x}_i)\cdot(\ln x_j - \ln \bar{x}_j)$$

+ higher order terms.

Truncating (4.16) after the second order term and evaluating it at $\ln \bar{x} \neq 0$ (i.e., $\ln \bar{x} = \ln \bar{x}_1 = \ln \bar{x}_2 = \ldots = \ln \bar{x}_n$) yields

$$\ln U(x) = \ln a_0 + \sum_{i=1}^{n} a_i\cdot(\ln x_i - \ln \bar{x})$$

$$+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} B_{ij}\cdot(\ln x_i - \ln \bar{x})\cdot(\ln x_j - \ln \bar{x})$$

$$= \ln a_0 + \sum_{i=1}^{n} a_i\cdot\ln x_i + \ln \bar{x} \cdot \sum_{i=1}^{n} (-a_i)$$

(4.17)
$$+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} B_{ij}\cdot\ln x_i\cdot\ln x_j$$

$$+ \sum_{i=1}^{n} \ln x_i \cdot \ln \bar{x} \left(\frac{1}{2} \sum_{j=1}^{n} B_{ij} + \frac{1}{2} \sum_{k=1}^{n} B_{ki} \right)$$

$$+ \frac{1}{2} \cdot (\ln \bar{x})^2 \cdot \sum_{i=1}^{n} \sum_{j=1}^{n} B_{ij}$$

$$= \ln a_0 + \sum_{i=1}^{n} a_i \cdot \ln x_i + a_A \cdot \ln A$$

$$+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} B_{ij} \cdot \ln x_i \cdot \ln x_j + \sum_{i=1}^{n} C_{iA} \cdot \ln x_i \cdot \ln A$$

$$+ \frac{1}{2} \cdot C_{AA} \cdot (\ln A)^2$$

where $\ln a_0 = G(\ln \bar{x}_1, \ln \bar{x}_2, \ldots, \ln \bar{x}_n)$,

$$a_i = \frac{\partial G}{\partial \ln x_i} \quad (i = 1, 2, \ldots, n) ,$$

$$B_{ij} = \frac{\partial^2 G}{\partial \ln x_i \partial \ln x_j} \quad (i,j = 1, 2, \ldots, n) ,$$

(4.18)

$$a_A = \sum_{i=1}^{n} (-a_i), \ \ln A = \ln \bar{x}, \ C_{AA} = \sum_{i=1}^{n} \sum_{j=1}^{n} B_{ij}, \text{ and}$$

$$C_{iA} = \frac{1}{2} \sum_{j=1}^{n} B_{ij} + \frac{1}{2} \sum_{k=1}^{n} B_{ki} = \sum_{j=1}^{n} B_{ij} \text{ for } i = 1, 2, \ldots, n$$

(since B_{ij}'s are the second order partial derivatives which are symmetric).

Function (4.17) is the translog utility function, which is just a generalized Taylor series expansion in n variables, truncated after the second order term for an arbitrary utility function of n variables, and evalauted at a fixed point $\ln \bar{x}$ different from zero. However, it is both possible and valid to evaluate the truncated Taylor expansion (4.16) at $\ln \bar{x} = 0$, because a set of data can always be scaled such that the actual data points include any points of expansion. If function (4.16) is truncated after the second order term and evaluated at $\ln \bar{x} = 0$, then the translog utility function (4.17) will be of the form

$$(4.19) \qquad \ln U(x) = \ln a_0 + \sum_{i=1}^{n} a_i \cdot \ln x_i + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} B_{ij} \cdot \ln x_i \cdot \ln x_j$$

which is quadratic in the logarithms of the quantities demanded of n consumer goods x_i's. Function (4.19) is the simplified version of the translog utility function (4.17); both functions in (4.17) and (4.19) are the translog utility functions. Therefore, this empirical demand study will employ function (4.19) as the translog utility function, instead of function (4.17).

The three-good translog utility function will be of the form

$$(4.20) \quad \ln U(x) = \ln a_0 + \sum_{i=1}^{3} a_i \cdot \ln x_i + \frac{1}{2} \sum_{i=1}^{3} \sum_{j=1}^{3} B_{ij} \cdot \ln x_i \cdot \ln x_j \ .$$

As observed in (4.17) and (4.19), restrictions implied by homogeneity and separability are not imposed on the form of translog utility function, while the CD, the CES, the Uzawa, and the Sato utility functions employ homogeneity and separability as part of the maintained hypothesis. Since homogeneous utility functions are selected for this demand study, a question arises as to the validity of homogeneity restrictions, which are a very severe restriction on the preference field and on the form of the utility function.

In the traditional approach to demand analysis, the additive and homothetic utility functions have played an important role in formulating the following tests of the theory of demand. If the utility function is homothetic, expenditure proportions are independent of total expenditure.[12] If the utility function is additive and homothetic, elasticities of substitution among all pairs of goods are constant and equal.[13] An example is a linear logarithmic utility function which is both additive and homothetic and employed in the demand studies by H. Wold and R. Stone.[14]

In their 1975 paper,[15] L. R. Christensen, D. W. Jorgenson, and L. J. Lau have developed tests of the theory of demand that do not employ additivity or homotheticity as part of the maintained hypothesis. For this purpose they introduce new representations of the utility function: the "direct" and "indirect" translog utility functions. The direct translog utility function is quadratic in the logarithms of the quantities demanded of n goods, and, hence, exactly identical to the translog utility function in (4.19). Employing parallel treatment, the indirect translog utility function is defined as quadratic in the logarithms of ratios of prices to total expenditure:

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[12]If the utility function is homothetic, it can be written as $\ln U = H\{G(\ln x_1, \ln x_2, \ldots, \ln x_n)\}$, where G is homogeneous of degree one and H is a monotonically increasing function.

[13]A. Bergson, "Real Income, Expenditure Proportionality, and Frisch's New Methods," The Review of Economic Studies, Vol. 4 (1936), pp. 33-52.

[14]H. Wold, Demand Analysis: A Study in Econometrics (New York: McGraw-Hill Book Company, 1953); J. R. N. Stone, Measurement of Consumers' Expenditures and Behavior in the United Kingdom, 1920-1938, (London: Oxford University Press, 1954), Vol. 1.

[15]L. R. Christensen, D. W. Jorgenson, and L. J. Lau, op. cit.

$$(4.21)\quad \ln V = \ln a_0 + \sum_{i=1}^{n} a_i \cdot \ln \frac{P_i}{M} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} B_{ij} \cdot \ln \frac{P_i}{M} \cdot \ln \frac{P_j}{M}$$

where M is total expenditure, and p_i's are prices of consumer goods. Furthermore, they have exploited the duality between the direct and indirect translog utility functions, and presented statistical tests of restrictions on the form of the utility function implied by additivity or homotheticity.

A statistical test is made of the validity of restrictions on the direct and indirect translog utility functions (i.e., functions (4.19) and (4.21)) implied by linear homogeneity, given that they are homothetic. The test statistics are computed on the basis of the likelihood ratio--the ratio of the maximum value of the likelihood function with restriction to the maximum value of the likelihood function without restriction. The computed values of test statistics for the direct and indirect translog utility functions are 1.47 and 4.73, respectively. At a level of significance of 0.01 with one degree of freedom the critical value is 6.63. Since the critical value is greater than the computed values of test statistics, the null hypothesis that linear homogeneity is valid is accepted. Furthermore, the result from the test leads to the conclusion that the direct translog utility function is homothetic (or linearly homogeneous) if, and only if, the indirect translog utility function is homothetic (or linearly homogeneous), and, hence, the duality between them is established. Thus, both functions, direct and indirect, represent the same prefer-

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[16] The empirical results are based on time-series data (1929-1972) which include prices and quantities of the services of consumers' durables, nondurable goods, and other services.

ences.[17]

Throughout this dissertation, linear homogeneity is thus employed as part of the maintained hypothesis, and restrictions implied by linear homogeneity are imposed on the translog utility function (4.19). And, due to the duality established between the direct and indirect translog utility functions, a system of demand equations will be derived from the former, instead of the latter. (See Chapter V.)

Taylor Approximations to the Chosen Utility Functions

A major advantage of using the translog utility function in (4.19) is that a system of demand equations can be derived in forms suitable for econometric testing. Since the translog utility function is a generalized Taylor series expansion truncated after the second order term, Taylor approximations to other utility functions will be of the same form as the translog utility function in (4.19). The only difference between them is that the coefficients of the Taylor approximation to one utility function are different from those of Taylor approximations to other utility functions. In other words, Taylor approximations to chosen utility functions other than the translog utility function are nothing but the constrained translog utility function. Thus, a system of demand equations derived from the translog utility function can be treated as systems of demand equations derived from other utility functions, provided that appropriate restrictions are imposed on parameters of the translog utility

[17] This was theoretically proved by L. J. Lau. See L. J. Lau, "Duality and the Structure of Utility Functions," Journal of Economic Theory, Vol. 1 (1970), pp. 374-396.

function. (See the latter part of this section, Section 1 in Chapter V, and Table 6-2 in Chapter VI.) Accordingly, econometric testing for utility functions other than the translog utility function can also be performed, using estimates of parameters of the translog utility function and imposing appropriate parameter restrictions on them. (See Section 1 in Chapter V and Table 6-2 in Chapter VI.)

The chosen utility functions will be approximated by a Taylor series expansion about $\ln x_i = 0$ $(i = 1, 2, 3)$, paralleling the treatment of the translog utility function in (4.19).[18]

(a) Taylor approximation to the CD utility function (4.5):[19]

$$\ln U(x) = \ln \theta + \sum_{i=1}^{3} b_i \cdot \ln x_i \, . \tag{4.22}$$

(b) Taylor approximation to the CES utility function (4.7):

$$\begin{aligned} \ln U(x) = {} & \ln \theta + \sum_{i=1}^{3} \delta_i \cdot \ln x_i + \frac{1}{2} \sum_{i=1}^{3} p \cdot (\delta_i^2 - \delta_i) \cdot (\ln x_i)^2 \\ & + \frac{1}{2} \sum_{\substack{i=1 \\ i \neq j}}^{3} \sum_{j=1}^{3} p \cdot \delta_i \cdot \delta_j \cdot \ln x_i \cdot \ln x_j \, . \end{aligned} \tag{4.23}$$

(c) Taylor approximation to the Uzawa utility function (4.11):

$$\begin{aligned} \ln U(x) = {} & \ln \theta + b_1 \cdot \ln x_1 + b_2 \cdot \sum_{i=2}^{3} \delta_i \cdot \ln x_i \\ & - \frac{1}{2} \cdot (p \cdot b_2 \cdot \delta_2 \cdot \delta_3) \cdot \sum_{i=2}^{3} (\ln x_i)^2 \end{aligned} \tag{4.24}$$

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[18] Logarithmic transformation is applied to the chosen utility functions prior to Taylor approximations.

[19] As seen in (4.22), no appeal to approximation by a Taylor series expansion is required.

$$+ \frac{1}{2} \cdot (p \cdot b_2 \cdot \delta_2 \cdot \delta_3) \cdot \sum_{\substack{i=2 \\ i \neq j}}^{3} \sum_{j=2}^{3} \ln x_i \cdot \ln x_j \ .$$

(d) Taylor approximation to the Sato utility function (4.13):

$$\begin{aligned} \ln U(x) = {} & \ln \theta + b_1 \cdot \ln x_1 + b_2 \cdot \sum_{i=2}^{3} \delta_i \cdot \ln x_i \\ & - \frac{1}{2} \cdot b_2 \cdot [w \cdot b_1 \cdot (\ln x_1)^2 + \delta_2 \cdot (p \cdot \delta_3 + w \cdot b_1 \cdot \delta_2) \\ & \cdot (\ln x_2)^2 + \delta_3 \cdot (p \cdot \delta_2 + w \cdot b_1 \cdot \delta_3) \cdot (\ln x_3)^2] \\ & + w \cdot b_1 \cdot b_2 \cdot \delta_2 \cdot \ln x_1 \cdot \ln x_2 \\ & + w \cdot b_1 \cdot b_2 \cdot \delta_3 \cdot \ln x_1 \cdot \ln x_3 \\ & + b_2 \cdot \delta_2 \cdot \delta_3 \cdot (p - w \cdot b_1) \cdot \ln x_2 \cdot \ln x_3 \ . \end{aligned} \tag{4.25}$$

The derivation of Taylor approximations to the chosen utility functions are contained in Appendix A. Functions (4.22), (4.23), (4.24), (4.25), and (4.20) are the alternative functional forms for the CD, the CES, the Uzawa, the Sato, and the translog utility functions, and they will be used to derive a system of demand equations. It must be noted that the alternative functional forms for the CD, the CES, the Uzawa, and the Sato utility function are nothing else but the constrained translog utility function. In other words, they are derived from the translog utility function (4.20) by imposing restrictions implied by linear homogeneity and separability on the translog parameters and by identifying the restricted translog parameters with the original parameters of other chosen utility functions.

CHAPTER V

DEMAND UNDER HOMOGENEITY AND SEPARABILITY

To build an econometric model of U.S. households' demand for energy fuels, the (direct) translog utility function employs linear homogeneity but not separability as part of the maintained hypothesis, while other chosen utility functions embody both linear homogeneity and separability in their systems of preferences.[1] As discussed in Chapter IV, the alternative functional forms for utility--Taylor approximations to the chosen utility functions--will be used to derive the system of demand equations. However, it is not necessary to derive the demand equations separately from each of alternative forms. What is required is to derive the system of demand equations from the translog utility function (4.19) and utilize it as the system of demand equations for other chosen utility functions, taking into account parameter restrictions implied by separability.[2] This is the very

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[1]For this demand study, the CD utility function is assumed to be linearly homogeneous.

[2]The translog utility function (4.19) is treated as an alter-functional form, due to the fact that it is a Taylor approximation to a generalized utility function. This estimating method was suggested by E. R. Berndt and L. R. Christensen. See E. R. Berndt and L. R. Christensen, "The Translog Production Function and Factor Substitution in the U.S. Manufacturing, 1929-1968," Journal of Econometrics, Vol. 1 (1973), pp. 81-113.

reason that the Taylor Approximation is preferred, particularly in this demand study.

A System of Equations for Budget Shares[3]

The neoclassical problem of the household is that of choosing a commodity bundle, given the utility function in (4.20) and given the budget constraint:

$$(5.1) \qquad \max_{x} \ln U(x) \text{ subject to } px \leqq M \text{ and } x \geqq 0$$

where

$$\ln U(x) = \ln a_0 + \sum_{i=1}^{3} a_i \cdot \ln x_i + \frac{1}{2} \sum_{i=1}^{3} \sum_{j=1}^{3} B_{ij} \cdot \ln x_i \cdot \ln x_j \ ;$$

$$(5.2) \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \ ; \quad p = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} \ ; \quad M = p_1 x_1 + p_2 x_2 + p_3 x_3 \ ;$$

x_1, x_2, and x_3 are the quantities demanded of fuel oil, natural gas, electricity, respectively; p_1, p_2, and p_3 are prices of fuel oil, natural gas, and electricity, respectively; and M is total expenditure.

Differentiating ln U(x) in (5.1) with respect to ln x_i and rearranging gives

$$(5.3) \qquad \frac{\partial \ln U}{\partial \ln x_i} = a_i + \sum_{j=1}^{3} Q_{ij} \cdot \ln x_j \ , \quad i = 1, 2, 3$$

where $Q_{ij} = \frac{1}{2} \cdot (B_{ij} + B_{ji})$ for $i \neq j$, and $Q_{ij} = Q_{ji}$ for $i \neq j$.[4]

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[3] Dr. C. K. Liew, Associate Professor of Economics, the University of Oklahoma, gave valuable assistance to the author in the completion of this section.

[4] It is assumed in Chapter IV that the translog utility function is continuously twice differentiable, so that the Hessian matrix of this utility function is symmetric. See Taylor approximations.

By the rule of differentiating the logarithmic function with respect to variables in the logarithms, function (5.3) can be written as

$$(5.4) \qquad \frac{\partial \ln U}{\partial \ln x_i} = \frac{\frac{\partial U}{U}}{\frac{\partial x_i}{x_i}} = \frac{\partial U}{\partial x_i} \cdot \frac{x_i}{U}, \quad i = 1, 2, 3.$$

When utility U is maximized subject to the income constraint, the consumer will spend his income so that

$$(5.5) \qquad \frac{\partial U}{\partial x_i} = \lambda p_i , \quad i = 1, 2, 3$$

where λ is the Lagrangian multiplier. Substituting (5.5) into (5.4) yields

$$(5.6) \qquad \frac{\partial \ln U}{\partial \ln x_i} = \lambda p_i \cdot \frac{x_i}{U}, \quad i = 1, 2, 3 .$$

Since the translog utility function employs linear homogeneity as part of the maintained hypothesis, Euler's theorem holds for any values of x_1, x_2 and x_3 on a linearly homogeneous surface:

$$(5.7) \qquad U(x) = x_1 \cdot \frac{\partial U}{\partial x_1} + x_2 \cdot \frac{\partial U}{\partial x_2} + x_3 \cdot \frac{\partial U}{\partial x_3} .$$

Substituting (5.5) into (5.7) yields

$$(5.8) \qquad U(x) = \lambda x_1 \cdot p_1 + \lambda x_2 \cdot p_2 + \lambda x_3 \cdot p_3 = \lambda \sum_{i=1}^{3} x_i \cdot p_i = \lambda M .$$

Then, by substituting (5.8) into (5.6), function (5.6) can be written as

$$(5.9) \qquad \frac{\partial \ln U}{\partial \ln x_i} = \lambda p_i \cdot \frac{x_i}{\lambda M} = \frac{p_i x_i}{M} , \text{ since } U(x) = M .$$

Accordingly, substituting (5.9) into (5.3) yields

$$\frac{\partial \ln U}{\partial \ln x_i} = \frac{p_i x_i}{M} = a_i + \sum_{j=1}^{3} Q_{ij} \cdot \ln x_j \;, \quad i = 1, 2, 3 \;. \tag{5.10}$$

Function (5.10) is the system of equations for budget shares such that

$$\begin{aligned} \frac{p_1 x_1}{M} &= a_1 + Q_{11} \cdot \ln x_1 + Q_{12} \cdot \ln x_2 + Q_{13} \cdot \ln x_3 \\ \frac{p_2 x_2}{M} &= a_2 + Q_{21} \cdot \ln x_1 + Q_{22} \cdot \ln x_2 + Q_{23} \cdot \ln x_3 \\ \frac{p_3 x_3}{M} &= a_3 + Q_{31} \cdot \ln x_1 + Q_{32} \cdot \ln x_2 + Q_{33} \cdot \ln x_3 \end{aligned} \tag{5.11}$$

where $\frac{p_i x_i}{M}$ is recognized as the budget share spent on good x_i. Hence, a complete econometric model for the (direct) translog utility function is provided by three equations for the budget shares, as seen in (5.11).

Restrictions on the parameters of equations in (5.11) implied by linear homogeneity are:

$$\begin{aligned} &\sum_{i=1}^{3} a_i = 1 \;; \quad \sum_{i=1}^{3} Q_{ij} = 0 \;, \quad j = 1, 2, 3 \;; \\ &\sum_{j=1}^{3} Q_{ij} = 0 \;, \quad i = 1, 2, 3 \;. \end{aligned} \tag{5.12}$$

The logarithm of the (direct) translog utility function is continuously twice differentiable in the logarithms of the quantities demanded, so that the Hessian of this function is symmetric. Thus, the parameters of equations in (5.11) satisfy equality and symmetry restrictions, in addition to restrictions in (5.12):

$$Q_{ij} = Q_{ji} \text{ for } i \neq j \; (i,j = 1, 2, 3). \tag{5.13}$$

Hence, the system of equations in (5.11) for the (direct) translog utility function requires parameter restrictions in (5.12) and (5.13) implied by linear homogeneity, equality, and symmetry. However, the system of equations in (5.11) for the CD, the CES, the Uzawa, and the Sato utility functions requires at least one restriction implied by either strong separability or weak separability, in addition to restrictions in (5.12) and (5.13).

One additional restriction is required for the Sato utility function:

$$a_2 \cdot Q_{13} = a_3 \cdot Q_{12} . \tag{5.14}$$

Two additional restrictions are required for the Uzawa utility function:

$$\begin{aligned} &a_2 \cdot Q_{13} = a_3 \cdot Q_{12} , \\ &a_2 \cdot Q_{13} = a_3 \cdot Q_{12} = 0 \quad \text{implies} \quad Q_{12} = Q_{13} = 0 . \end{aligned} \tag{5.15}$$

Two additional restrictions are required for the CES utility function:

$$\begin{aligned} &a_2 \cdot Q_{13} = a_3 \cdot Q_{12} , \\ &a_3 \cdot Q_{12} = a_1 \cdot Q_{23} . \end{aligned} \tag{5.16}$$

Three additional restrictions are required for the CD utility function:

$$\begin{aligned} &a_2 \cdot Q_{13} = a_3 \cdot Q_{12} , \\ &a_3 \cdot Q_{12} = a_1 \cdot Q_{23} , \\ &Q_{12} = Q_{13} = Q_{23} = 0 . \end{aligned} \tag{5.17}$$

The parameters of the behavioral equations (5.11) for each of the chosen utility functions will be estimated, taking into account the parameter restrictions outlined in (5.14), (5.15), (5.16), and (5.17). A summary of the parameters which are to be estimated is presented in Table 6-2 in Chapter VI.

Demand Elasticities and Partial Elasticities of Substitution

The purposes here are: (1) to obtain the theoretical results on the Hicks-Allen partial elasticities of substitution for each of the chosen utility functions, using the definition of the Hicks-Allen partial elasticity of substitution and the system of the behavioral equations for budget shares in (5.11); (2) to obtain the theoretical results on price-, cross-, and income-elasticities of demand for each of the chosen utility functions and express those results in terms of the Hicks-Allen partial elasticities of substitution and the budget shares; and (3) to obtain the theoretical results on the price elasticities of the compensated demand--Slutsky's price elasticities of demand--for the translog utility function.

The Hicks-Allen partial elasticity of substitution between two goods, x_i and x_j $(i \neq j)$, is defined as[5]

(5.18)
$$\sigma_{ij} = \frac{x_1 U_1 + x_2 U_2 + \dots + x_n U_n}{x_i x_j} \cdot \frac{D_{ij}}{D},$$
$$\sigma_{ij} = \sigma_{ji} \text{ for } i \neq j \ (i,j = 1, 2, \dots, n) \quad .$$

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[5]R. G. D. Allen, Mathematical Analysis for Economists (London: Macmillan, 1938), p. 512.

where U_i is the first order partial derivative of the (direct) translog utility function with respect to x_i, D_{ij} is the determinant of the cofactor matrix of the element U_{ij} of the negative definite bordered Hessian matrix of U(x), and D is the bordered Hessian determinant such that

$$D = \begin{vmatrix} 0 & U_1 & U_2 & \cdots & U_n \\ U_1 & U_{11} & U_{12} & \cdots & U_{1n} \\ U_2 & U_{21} & U_{22} & \cdots & U_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ U_n & U_{n1} & U_{n2} & & U_{nn} \end{vmatrix} .$$

The chosen utility functions discussed in Chapter IV are strictly quasi-concave homogeneous utility functions.[6] Since the duality between the direct and indirect forms of the utility function is established (see p. 51), it follows that a strictly quasi-concave homogeneous direct utility function is strongly (weakly) separable with respect to a commodity-wise partition if, and only if, the indirect utility function is strongly (weakly) separable price-wise.[7] Berndt and Christensen investigated the relationships between the Hicks-Allen partial elasticities of substitution and separability,

[6]Strict quasi-concavity is a condition which is equivalent to the utility function having convex indifference surface. In fact, for any member of the translog family there exist configurations of goods such that neither monotonicity nor convexity is satisfied. This follows simply from the quadratic nature of the translog function (i.e., a Taylor approximation). On the other hand, there are regions in a commodity space where these conditions are satisfied. See E. R. Berndt and L. R. Christensen, op. cit.

[7]This was proved by L. J. Lau. See L. J. Lau, op. cit.

and introduced the following three theorems:[8]

Theorem 3: A strictly quasi-concave homogeneous direct utility function and its (dual) indirect utility function are weakly separable with respect to a partition $\{N_1, N_2, \ldots, N_r\}$ if, and only if, $\sigma_{ik} = \sigma_{jk}$ for $i \neq j \neq k$ and for x_i, $x_j \in N_h$, $x_k \in N_h$ ($h = 1, 2, \ldots, r$).

Theorem 4: A strictly quasi-concave homogeneous direct utility function and its (dual) indirect utility function are strongly separable with respect to a partition $\{N_1, N_2, \ldots, N_r\}$ if, and only if, $\sigma_{ik} = \sigma_{jk}$ for $i \neq j \neq k$ and for $x_i \in N_h$, $x_j \in N_t$, and $x_k \notin N_h \cup N_t$ ($h \neq t$; $h,t = 1, 2, \ldots, r$).

Corollary to Theorem 4: For any strictly quasi-concave homogeneous utility function and its (dual) indirect utility function with each good or price forming its own subset, strong separability with respect to a partition $\{N_1, N_2 \ldots, N_r\}$ is necessary and sufficient for all Hicks-Allen elasticities of substitution σ_{ij} for $i \neq j$ to be equal.

The CD utility function requires that the Hicks-Allen partial elasticities of substitution are all equal to unity by Theorem 4 and its Corollary and because of the assumption of linear homogeneity:

(5.19) $$\sigma_{12} = \sigma_{13} = \sigma_{23} = 1 .$$

The CES utility function requires that the Hicks-Allen partial

[8] E. R. Berndt and L. R. Christensen, op. cit.

elasticities of substitution are all equal by Theorem 4 and its Corollary:

(5.20) $\sigma_{12} = \sigma_{13} = \sigma_{23}$.

By Theorem 3, the Uzawa utility function requires that

(5.21) $\sigma_{12} = \sigma_{13} = 1$,

since the Uzawa utility function is a hybrid of the CD and CES functions.

By Theorem 3, the Sato utility function requires that

(5.22) $\sigma_{12} = \sigma_{13}$.

The income-elasticity of demand for a good x_i is defined as[9]

$$\frac{M}{x_i} \cdot \frac{\partial x_i}{\partial M} = \frac{x_1 U_1 + x_2 U_2 + \ldots + x_n U_n}{x_i} \cdot \frac{D_i}{D} \tag{5.23}$$

$$(i = 1, 2, \ldots, n)$$

where D_i is the determinant of the cofactor matrix of the element U_i of the negative definite bordered Hessian matrix of $U(x)$, and D is the bordered Hessian determinant (see (5.18)). Since the chosen utility functions are linearly homogeneous, the income-elasticity of demand for any good x_i ($i = 1, 2, 3$) is unity, i.e., the consumption of each good increases in the same proportion as income.

The price- and cross-elasticities of demand for a good x_j can be defined in terms of the Hicks-Allen elasticity of substitution in (5.18) and the income-elasticity of demand in (5.23):[10]

[9] R. G. D. Allen, op. cit., p. 520.

[10] Ibid., pp. 510-513.

$$\frac{p_i}{x_j} \cdot \frac{\partial x_j}{\partial p_i} = k_i \cdot (\sigma_{ij} - \frac{M}{x_j} \cdot \frac{\partial x_j}{\partial M}) = k_i \cdot \sigma_{ij} - \frac{M}{x_j} \cdot \frac{\partial x_j}{\partial M} \tag{5.24}$$

$$(i,j = 1, 2, \ldots, n)$$

where $k_i = \frac{p_i x_i}{M}$ is the budget share spent on a good x_i.

The price- and cross-elasticities of demand implied by separability are as follows:[11]

(1) The CD utility function

$$\frac{p_i}{x_j} \cdot \frac{\partial x_j}{\partial p_i} = \begin{cases} -1 \text{ for } i = j \\ 0 \text{ for } i \neq j \end{cases} \quad (i,j = 1, 2, 3). \tag{5.25}$$

(2) The CES utility function

$$\frac{p_i}{x_j} \cdot \frac{\partial x_j}{\partial p_i} = \begin{cases} -\sigma^* \cdot (1 - k_i) = k_i \text{ for } i = j \text{ ,} \\ k_i \cdot (\sigma^* - 1) \text{ for } i \neq j \end{cases} \tag{5.26}$$

$$(i,j = 1, 2, 3)$$

where $\sigma^* = \sigma_{12} = \sigma_{13} = \sigma_{23}$ (see (5.20)).

(3) The Uzawa utility function

$$\frac{p_i}{x_j} \cdot \frac{\partial x_j}{\partial p_i} = \begin{cases} -1 \text{ for } i = j = 1 \text{ ,} \\ -(k_1 + k_2 + k_3 \cdot \sigma_{23}) \text{ for } i = j = 2 \text{ ,} \\ -(k_1 + k_2 \cdot \sigma_{23} + k_3) \text{ for } i = j = 3 \text{ ,} \\ 0 \text{ for } i,j = 1, 2 \text{ ,} \\ 0 \text{ for } i,j = 1, 3 \text{ ,} \\ k_i \cdot (\sigma_{ij} - 1) \text{ for } i \neq j \text{ and } i,j = 2, 3 \text{ .} \end{cases} \tag{5.27}$$

[11] See Appendix B for the derivation of the price- and the cross-elasticities of demand implied by separability.

(4) The Sato utility function

$$(5.28)\qquad \frac{p_i}{x_j} \cdot \frac{\partial x_j}{\partial p_i} = \begin{cases} -\sigma^{**} \cdot (1 = k_1) - k_1 \text{ for } i = j = 1 , \\ -(k_1 \cdot \sigma_{12} + k_3 \cdot \sigma_{23}) - k_2 \text{ for } i = j = 2 , \\ -(k_1 \cdot \sigma_{13} + k_2 \cdot \sigma_{23}) - k_3 \text{ for } i = j = 3 , \\ k_i \cdot (\sigma_{ij} - 1) \text{ for } i \neq j \text{ and } i,j = 1,\ 2,\ 3 \end{cases}$$

where $\sigma^{**} = \sigma_{12} = \sigma_{13}$ (see (5.22)).

(5) The translog utility function

$$(5.29)\qquad \frac{p_i}{x_j} \cdot \frac{\partial x_j}{\partial p_i} = k_i \cdot (\sigma_{ij} = 1) \text{ for } i \neq j \text{ and } i,j = 1,\ 2,\ 3$$

which is obtained by imposing linear homogeneity, but not separability, on the definition (5.24).

Using price-, cross-, and income elasticities of demand, the price elasticities of the compensated demand--Slutsky's price elasticities of demand--can be computed:

$$(5.30)\qquad \frac{\partial x_j}{\partial p_i} = \left(\frac{\partial x_j}{\partial p_i}\right)_{comp} - x_i \cdot \left(\frac{\partial x_j}{\partial M}\right) \qquad \text{(the Slutsky equation)} \quad (i,j = 1,\ 2,\ 3)$$

where $\frac{\partial x_j}{\partial p_i}$ is the total effect of a change in price on demand, $\left(\frac{\partial x_j}{\partial p_i}\right)_{comp}$ is the substitution effect of a compensated change in price on demand, and $x_i \cdot \left(\frac{\partial x_j}{\partial M}\right)$ is the income effect of a change in income on demand. Multiplying (5.30) through by $\frac{p_i}{x_j}$ and multiplying the last term on the right by $\frac{M}{M}$,

$$\frac{p_i}{x_j} \cdot \frac{\partial x_j}{\partial p_i} = \frac{p_i}{x_j} \cdot \left(\frac{\partial x_j}{\partial p_i}\right)_{comp} - x_i \cdot \frac{p_i}{x_j} \cdot \frac{M}{M} \cdot \left(\frac{\partial x_j}{\partial M}\right)$$

$$(5.31) \qquad = \left(\frac{p_i}{x_j} \cdot \frac{\partial x_j}{\partial p_i}\right)_{comp} - \frac{p_i x_i}{M} \cdot \left(\frac{M}{x_j} \cdot \frac{\partial x_j}{\partial M}\right) .$$

This is the Slutsky equation which is expressed in terms of the price and income elasticities, and it states that the price elasticity of demand equals the price elasticity of the compensated demand less the corresponding income elasticity of demand multiplied by the proportion of the total expenditure spent on x_i. From (5.31):

$$\left(\frac{p_i}{x_j} \cdot \frac{\partial x_j}{\partial p_i}\right)_{comp} = \frac{p_i}{x_j} \cdot \frac{\partial x_j}{\partial p_i} + \frac{p_i x_i}{M} \cdot \left(\frac{M}{x_j} \cdot \frac{\partial x_j}{\partial M}\right)$$

$$(5.32) \qquad = \frac{p_i}{x_j} \cdot \frac{\partial x_j}{\partial p_i} + k_i \quad (i = 1, 2, 3) ,$$

since the income elasticity of demand is unity. Equation (5.32) is the price elasticity of the compensated demand (or Slutsky's price elasticity of demand).

Estimation

The behavioral equations in (5.11) for budget shares generated by the (direct) translog utility function are estimated by the method of Zellner's efficient least-squares (ZELS for short), using restrictions in (5.12) and (5.13) implied by linear homogeneity, equality, and symmetry.[12] Then, on the basis of the estimates of the parameters of the behavioral equations for the translog utility function, the parameter estimates of the behavioral equations for other chosen utility functions are derived, taking into account the parameter restrictions

[12] A. Zellner, "An Efficient Method of Estimating Seemingly Unrelated Regressions and Tests for Aggregate Bias," Journal of the American Statistical Association, Vol. 57 (June 1962), pp. 348-368.

outlined in (5.14), (5.15), (5.16), and (5.17). The estimation is based on time-series data which show prices and quantities of fuel oil, natural gas, and electricity for 1937-1970, and there are thirty-four observations for each behavioral equation.[13]

The ZELS method is an application of the generalized least-squares estimation, which occurs in the estimation of a group of euqations. To apply the ZELS method, two conditions must be fulfilled: (1) the equations do not have the same list of regressors: and (2) there must be nonzero correlations between disturbance terms in two or more equations.[14] If these two conditions are fulfilled, then the ZELS estimators will be asymptotically more efficient than single-equation least-squares estimators. According to Zellner,[15] even if the correlation in the second condition is unknown, an estimate of the correlation from an equation-by-equation application of the ordinary least-squares is quite likely to improve the efficiency of estimation. On the other hand, if the first condition is not fulfilled, the ZELS estimators will collapse to yeild single-equation least-squares estimators (OLSQ estimators) even when the second condition is fulfilled. As seen in (5.11), the system of the behavioral equations does not satisfy the first condition, because the equations have the same set of explanatory variables (i.e., in x_i, i = 1, 2, 3).

An exceptional case to which the ZELS estimators are not

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[13]The data are contained in Appendix C.

[14]J. Johnston, Econometric Methods, 2nd ed. (New York: McGraw-Hill Book Company, 1972), p. 238.

[15]A. Zellner, op. cit.

applicable is, however, the one where the set of equations have the same list of regressors and there are no restrictions on the regression coefficients.[16] That is to say, if there are restrictions on the regression coefficients, the ZELS estimators will be applicable even when the first condition is not fulfilled. Now that the linear homogeneity and symmetry conditions in (5.12) and (5.13) have been imposed on the coefficients of the behavioral equations in (5.11), the ZELS estimators are applicable to the system in (5.11) and can realize a gain in efficiency by taking into account the correlation between the disturbances. Hence, the ZELS estimators are preferred over the ordinary least-squares estimators (OLSQ).

Suppose that the i^{th} behavioral equation in the system (5.11) is

$$Y_i = X_i \cdot Q_i + u_i, \quad i = 1, 2, 3 . \tag{5.33}$$

Then, the system of the behavioral equations can be set out in matrix notation: Letting M_i be total expenditure for the i^{th} year, and $n = 33$,[17]

$$\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} = \begin{pmatrix} X_1 & 0 & 0 \\ 0 & X_2 & 0 \\ 0 & 0 & X_3 \end{pmatrix} \cdot \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \tag{5.34}$$

where

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[16] E. R. Berndt and L. R. Christensen, op. cit.

[17] There are thirty-four observations, but they are reduced to thirty-three observations, due to the data transformation. This is explained in the next section.

$$Y_i = \begin{pmatrix} \frac{p_{i1}x_{i1}}{M_1} \\ \frac{p_{i2}x_{i2}}{M_2} \\ \cdot \\ \cdot \\ \cdot \\ \frac{p_{in}x_{in}}{M_n} \end{pmatrix} ; \quad X_i = \begin{pmatrix} 1 & \ln x_{11} & \ln x_{21} & \ln x_{31} \\ 1 & \ln x_{12} & \ln x_{22} & \ln x_{32} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \ln x_{1n} & \ln x_{2n} & \ln x_{3n} \end{pmatrix} ;$$

$$Q_i = \begin{pmatrix} a_i \\ Q_{i1} \\ Q_{i2} \\ Q_{i3} \end{pmatrix} ; \quad u_i = \begin{pmatrix} u_{i1} \\ u_{i2} \\ \cdot \\ \cdot \\ \cdot \\ u_{in} \end{pmatrix} \quad (i = 1, 2, 3).$$

Using the estimates of the variance-covariance for the disturbances, u_i's, in (5.34) obtained from the single-equation least-squares residuals and, also, using restrictions in (5.12) and (5.13), the constrained ZELS estimators, Q*, can be obtained:[18]

$$(5.35) \quad \begin{aligned} Q^* &= Q + (X'\textstyle\sum_*^{-1}X)^{-1}R'\left(R(X'\textstyle\sum_*^{-1}X)^{-1}R'\right)^{-1}(r - RQ) , \\ V(Q^*) &= {}^2A(X'\textstyle\sum_*^{-1}X)^{-1} \doteq \frac{e'e}{n-k} \cdot A(X'\textstyle\sum_*^{-1}X)^{-1} \end{aligned}$$

where r is a known column vector of order 7x1 (the number of restrictions) and R is a known matrix of order 7x12 such that

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[18]The method of Lagrangian multipliers is applied. See J. Johnston, op. cit., pp. 157-158.

$$r = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \end{pmatrix};$$

(5.36)

$$X = \begin{pmatrix} X_1 & 0 & 0 \\ 0 & X_2 & 0 \\ 0 & 0 & X_3 \end{pmatrix} \quad \text{(see (5.34))};$$

$$\Sigma_*^{-1} = \begin{pmatrix} s^{11}\cdot I & s^{12}\cdot I & s^{13}\cdot I \\ s^{21}\cdot I & s^{22}\cdot I & s^{23}\cdot I \\ s^{31}\cdot I & s^{32}\cdot I & s^{33}\cdot I \end{pmatrix}$$, which is the inverse matrix

of the estimates of the variance-covariance for the disturbance terms, u_i's, with the identity matrix of order 4x4; X' is the transpose of X matrix; $A = I - (X'X)^{-1}R'\left(R(X'X)^{-1}R'\right)^{-1}R$; Q is the unconstrained ZELS estimators such that

$$Q = \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix} = \begin{pmatrix} s^{11}X_1'X_1 & s^{12}X_1'X_2 & s^{13}X_1'X_3 \\ s^{21}X_2'X_1 & s^{22}X_2'X_2 & s^{23}X_2'X_3 \\ s^{31}X_3'X_1 & s^{32}X_3'X_2 & s^{33}X_3'X_3 \end{pmatrix}^{-1} \cdot \begin{pmatrix} \sum_{j=1}^{3} s^{1j}X_1'Y_j \\ \sum_{j=1}^{3} s^{2j}X_2'Y_j \\ \sum_{j=1}^{3} s^{3j}X_3'Y_j \end{pmatrix}$$

and Q_i (i = 1, 2, 3) is the column vector of order 12x1 such that

$$Q_1 = \begin{pmatrix} Q_{11} \\ Q_{12} \\ Q_{13} \end{pmatrix}, \quad Q_2 = \begin{pmatrix} Q_{21} \\ Q_{22} \\ Q_{23} \end{pmatrix}, \quad Q_3 = \begin{pmatrix} Q_{31} \\ Q_{32} \\ Q_{33} \end{pmatrix}$$

(see (5.34) for X_i and Y_i); and $V(Q^*)$ is the variance-covariance matrix for Q^*.

F-statistics for testing overall homogeneity (i.e., $H_0 = Q_1^* = Q_2^* = Q_3^*$ where H_0 is the null hypothesis) is given by

$$(5.37)\quad F_{q,n-m} = \frac{T \cdot V(Q^*) \cdot C' \left(C \cdot V(Q^*) \cdot C'\right)^{-1} C \cdot V(Q^*) \cdot T'}{Y'Y - T \cdot V(Q^*) \cdot T'} \cdot \frac{(n-m)}{q}$$

where m is the number of independent variables; q is the number of restrictions; $T = Y'\Sigma_*^{-1}X$ and T' is the transpose of T; C is such that $C = \begin{bmatrix} I & -I & 0 \\ 0 & I & -I \end{bmatrix}$, I and 0 being the identity and null matrices of order 4x4, respectively. F-statistics for testing the hypothesis of linear homogeneity can be obtained by replacing C matrix in (5.37) with R matrix in (5.36).[19]

Data Description

The consumption of different types of energy fuels consumed in households, to a large extent, is governed by the stock of home appliances in existence. To the extent that these appliances are not replaced, there is a "committed demand" for a particular fuel. But some appliances will be replaced and new ones will be added. Those new home appliances can institute a shift in the demand for fuels, creating a "new demand" for one or the other.

The concept of "new demand" refers to the demand for fuels arising from both "replacement demand" due to the retirement (and the replacement) of old home appliances and "incremental demand" due to

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[19]See A. Zellner, op. cit.

net increases in the stock of home appliances and, hence, purges the "committed demand" of the total demand for fuels.[20] This concept describes the behavior of a consumer not committed by past contracts to any form of technique or any type of service. To generate the new demand by the total demand, the concept of new demand incorporates a stock effect and permits some assumptions about the adjustment of the stock of home appliances over time, that is, the rate of utilization of home appliances and the rate of depreciation.

This empirical study employs the concept of new demand. Time-series data for the quantities consumed of fuel oil, natural gas, and electricity for 1937-1970--total demand--are converted into "new demand" data (the sum of replacement demand and incremental demand). The method of conversion is as follows:[21]

$$F_t^N = F_t^T - (1 - r) \cdot F_{t-1}^T$$

$$G_t^N = G_t^T - (1 - r) \cdot G_{t-1}^T \qquad (5.38)$$

$$E_t^N = E_t^T - (1 - r) \cdot E_{t-1}^T$$

where F_t^N, G_t^N, and E_t^N are new demand (N) for fuel oil (F), natural gas (G), and electricity (E) in period t, respectively; F_t^T, G_t^T, and E_t^T are total demand (T) for fuel oil, natural gas, and electricity in period t, respectively; F_{t-1}^T, G_{t-1}^T, and E_{t-1}^T are total demand for fuel oil,

[20] As for the concept of new demand, see P. Balestra and M. Nerlove, "Pooling Cross Section and Time Series Data in the Estimation of a Dynamic Model: The Demand for Natural Gas," Econometrica, Vol. 34 (July 1966), pp. 585-612.

[21] As for the algebraic derivation of (5.38), see P. Balestra and M. Nerlove, op. cit.

natural gas, and electricity in period (t-1), respectively; and r is the depreciation rate. Since the lagged variables are involved in (5.38), the original thirty-four observations (1937-1970) for each behavioral equation in (5.11) are reduced to thirty-three observations (i.e., n = 33).

As for the rate of depreciation, Balestra and Nerlove argue, on an empirical basis, that 11% depreciation rate for all fuel-consuming appliances is not unreasonable.[22] According to M. L. Bernstein, 10% depreciation rate for household refrigeration in the United States is preferred, even though such depreciation rates as 10%, 20%, and 25% have produced the similar estimates of regression coefficients.[23] Because of these similar results in two different empirical studies, the 10% depreciation rate is chosen for present purposes.[24]

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[22] Ibid.

[23] M. L. Berstein, "The Demand for Household Refrigeration in the United States." In: A. C. Harberger, The Demand for Durable Goods, (Chicago: The University of Chicago Press, 1960).

[24] The cases which involve depreciation rates ranging from 5% to 20% were tested in this empirical study by the author. But only those cases with such depreciation rates as 5%, 7%, 8% and 10% produced similar and reasonably satisfactory results. This led to the choice of 10% depreciation rate.

CHAPTER VI

EMPIRICAL RESULTS

A summary of the empirical results obtained by the ZELS method with 10% depreciation rate is presented in Table 6-1. The values of the a_i's and Q_{ij}'s are the restricted estimates of the parameters of hehavioral equations (5.11) for the (direct) translog utility function. As seen in Table 6-1, the standard errors of regression coefficients are much smaller in magnitude than the values of a_i's and Q_{ij}'s and the F-value for testing the hypothesis of regression-coefficient vector equality (i.e., H_0: $Q_1 = Q_2 = Q_3$) is 92.9101 and much greater than the critical value at any level of significance. These two results lead to the conclusion that there does exist a functional relationship between the variables--a relationship between budget shares ($\frac{p_i x_i}{M}$) and the quantities demanded of fuel oil (x_1), natural gas (x_2), and electricity (x_3). R^2's for three behavioral equations are greater than 0.8, and the standard errors of estimates are small in magnitude. These imply that more than 80% of the variation of budget shares is explained by the quantities demanded of three energy fuels. In addition, the F-value for testing the hypothesis of linear homogeneity is 0.000403. This value is smaller than the critical value

Table 6-1

SUMMARY OF RESULTS OF CONSTRAINED ZELS ESTIMATION OF EQUATIONS (5.11) FOR THE TRANSLOG FUNCTION WITH 10% DEPRECIATION RATE, SELECTED YEARS, 1938-1970

			a_i	Q_{i1}	Q_{i2}	Q_{i3}
	Q_1		0.567702	0.059862	-0.014707	-0.045154
	SER		0.001884	0.000294	0.000186	0.000325
Equation I	R^2	0.861046				
	Ad. R^2	0.846671				
	SEE	0.027062				
	Q_2		0.307138	-0.014707	0.083437	-0.068729
	SER		0.000087	0.000013	0.000026	0.000024
Equation II	R^2	0.809923				
	Ad. R^2	0.790259				
	SEE	0.053558				
	Q_3		0.125160	-0.045154	-0.068729	0.113884
	SER		0.000023	0.000003	0.000003	0.000005
Equation III	R^2	0.837169				
	Ad. R^2	0.820324				
	SEE	0.049165				

	F-Value	Number of Restrictions
Overall Homogeneity	92.910100	8
Linear Homogeneity	0.000403	7

Note: Q_i is the column vector 4x1; SER is the standard error of regression coefficients; Ad. R^2 is the adjusted R^2; SEE is the standard error of the estimates (see (5.11)) for the notation).

at any level of significance, and, hence, the parameter estimates satisfy the hypothesis of linear homogeneity. Therefore, the parameter estimates can be regarded as qualitatively satisfactory. Furthermore, these parameter estimates can be used to determine the values of the parameters of the (direct) translog utility function itself, because $Q_{ij} = \frac{1}{2} \cdot (B_{ij} + B_{ji})$ and $B_{ij} = B_{ji}$ for $i \neq j$ (see (4.20) and (53.)). This is an advantage of using a Taylor approximation to a utility function.

The restricted estimates of the parameters for other chosen utility functions are presented in Table 6-3. They are derived from the parameter estimates for the translog utility function. First, alternative functional forms (Taylor approximations) for the CD, the CES, the Uzawa, and the Sato utility functions are differentiated with respect to $\ln x_i (i = 1, 2, 3)$. The parameters of the derivative equations are summarized in Table 6-2. Equating the unknown parameters of the derivative equations with the known parameters for the translog utility function in the last column of Table 6-2 and solving them simultaneously yield the solutions to the parameters for other chosen utility function.[1] For example, the alternative functional form (a Taylor approximation) for the CES utility function is treated as a special case of the translog utility function (a Taylor approximation to an arbitrary utility function). Then, the following restrictions on the translog parameters are obtained:

$$\delta_i = a_i \quad (i = 1, 2, 3)$$

[1]This estimating method was introduced by Berndt and Christensen. See E. R. Berndt and L. R. Christensen, op. cit.

Table 6-2

PARAMETER RESTRICTIONS ON A SYSTEM OF EQUATIONS
(5.11) UNDER HOMOGENEITY AND SEPARABILITY

CD	CES	Uzawa	Sato	Translog
b_1	δ_1	b_1	b_1	a_1
b_2	δ_2	$b_2\delta_2$	$b_2\delta_2$	a_2
b_3	δ_3	$b_2\delta_3$	$b_2\delta_3$	a_3
0	$p(\delta_1^2 - \delta_1)$	0	$-wb_1b_2$	Q_{11}
0	$p(\delta_2^2 - \delta_2)$	$-pb_2\delta_2\delta_3$	$-b_2\delta_2(p\delta_3 + wb_1\delta_2)$	Q_{22}
0	$p(\delta_3^2 - \delta_3)$	$-pb_2\delta_2\delta_3$	$-b_2\delta_3(p\delta_2 + wb_1\delta_3)$	Q_{33}
0	$p\delta_1\delta_2$	0	$wb_1b_2\delta_2$	$Q_{12} = Q_{21}$
0	$p\delta_1\delta_3$	0	$wb_1b_2\delta_3$	$Q_{13} = Q_{31}$
0	$p\delta_2\delta_3$	$pb_2\delta_2\delta_3$	$b_2\delta_2\delta_3(p - wb_1)$	$Q_{23} = Q_{32}$

Note: The translog utility function does not employ separability; the parameters of the CD, the CES, the Uzawa, and the Sato utility functions are obtained by imposing restrictions in (5.14), (5.15), (5.16), and (5.17) on the behavioral equations in (5.11), in addition to restrictions implied by linear homogeneity, equality, and symmetry ((5.12) and (5.13)), and are identified with the parameters of the alternative functional forms (Taylor approximations) in (4.22), (4.23), (4.24), and (4.25).

Table 6-3

PARAMETER ESTIMATES OF CONSTRAINED ZELS ESTIMATION OF EQUATIONS (5.11) FOR ALL CHOSEN UTILITY FUNCTIONS WITH 10% DEPRECIATION RATE, SELECTED YEARS, 1938-1970

PARAMETER	CD	CES p=-0.24392	UZAWA p=-0.77291	SATO p=-0.911384 w=-0.084352	TRANSLOG
a_1	0.567702	0.567702	0.567702	0.567702	0.567702
Q_{11}	0.0	0.059862	0.0	0.020701	0.059862
Q_{12}	0.0	-0.042531	0.0	-0.014707	-0.014707
Q_{13}	0.0	-0.017331	0.0	-0.005994	-0.045154
a_2	0.307138	0.307138	0.307138	0.307138	0.307138
Q_{21}	0.0	-0.042531	0.0	-0.014707	-0.014707
Q_{22}	0.0	0.051907	0.068730	0.091493	0.083437
Q_{23}	0.0	-0.009377	-0.068730	-0.076786	-0.068729
a_3	0.125160	0.125160	0.125160	0.125160	0.125160
Q_{31}	0.0	-0.017331	0.0	-0.005994	-0.045154
Q_{32}	0.0	-0.009377	-0.068730	-0.076786	-0.068729
Q_{33}	0.0	0.026708	0.068730	0.082778	0.113884

Note: See Appendix D for the computation of a_i's, p , and w.

$$(6.1) \qquad Q_{ij} = p\cdot(\delta_i^2 - \delta_i) \qquad (i = 1, 2, 3)$$

$$Q_{ij} = p\cdot\delta_i\cdot\delta_j \qquad (i = 1, 2, 3)$$

In (6.1), there are nine equations in total, each equating the unknown parameters of the derivative equation with the known parameters in the last column of Table 6-2. The solution can be obtained by solving nine equations simultaneously for p. This is another advantage of using Taylor approximations to utility functions. Inspection of Table 6-2 indicates that the CES, the Uzawa, and the Sato utility functions have multiple solutions to such unknown parameters as p and w. Particular solutions to them must be found and the manner in which that was accomplished is discussed in Appendix D. The particular values determined are: $p = -0.24392$ for the CES utility function; $p = -0.77291$ for the Uzawa utility function; $p = -0.084352$ and $w = -0.911384$ for the Sato utility function. Using these values, the parameters in the first, the second, the third and the fourth columns of Table 6-2 were determined (see Appendix D). These parameter estimates satisfy restrictions implied by linear homogeneity, equality, symmetry, and separability outlined in (5.12), (5.13), (5.14), (5.15), (5.16), and (5.17).

The demand elasticities and the elasticities of substitution for the chosen utility functions are presented in Tables 6-4, 6-5, 6-6, 6-7, and 6-8. As expected, the CD utility function exhibits unitary price-elasticities, zero cross-elasticities, and unitary elasticities of substitution. These empirical results are in conformity with the theoretical results outlined in (5.19) and (5.25). The former two results imply that the relative changes in quantity of any fuel, x_i $(i = 1, 2, 3)$, and its price are equal, and, hence, budget share on

Table 6-4

ELASTICITIES OF DEMAND AND PARTIAL ELASTICITIES OF SUBSTITUTION OBTAINED FROM THE CD FUNCTION, SELECTED YEARS, 1938-1970

p \ q	ED in 1938 FO	NG	E
FO	-1.0000	0.0000	0.0000
NG	0.0000	-1.0000	0.0000
E	0.0000	0.0000	-1.0000

q \ q	ES in 1938 FO	NG	E
FO	--	1.0000	1.0000
NG	1.0000	--	1.0000
E	1.0000	1.0000	--

p \ q	ED in 1940 FO	NG	E
FO	-1.0000	0.0000	0.0000
NG	0.0000	-1.0000	0.0000
E	0.0000	0.0000	-1.0000

q \ q	ES in 1940 FO	NG	E
FO	--	1.0000	1.0000
NG	1.0000	--	1.0000
E	1.0000	1.0000	--

p \ q	ED in 1945 FO	NG	E
FO	-1.0000	0.0000	0.0000
NG	0.0000	-1.0000	0.0000
E	0.0000	0.0000	-1.0000

q \ q	ES in 1945 FO	NG	E
FO	--	1.0000	1.0000
NG	1.0000	--	1.0000
E	1.0000	1.0000	--

p \ q	ED in 1950 FO	NG	E
FO	-1.0000	0.0000	0.0000
NG	0.0000	-1.0000	0.0000
E	0.0000	0.0000	-1.0000

q \ q	ES in 1950 FO	NE	E
FO	--	1.0000	1.0000
NG	1.0000	--	1.0000
E	1.0000	1.0000	--

Note: (1) ED represents elasticities of demand, on-diagonal terms in ED represent price-elasticities of demand, and off-diagonal terms in ED represent cross-elasticites of demand; (2) p and q represent price and quantity, respectively; (3) ES repr^sents the Hicks-Allen partial elasticities of substitution; and (4) FO, NG, and E represent fuel oil, natural gas, and electricity, respectively.

Table 6-4 (Continued)

q \ p	ED in 1955 FO	NG	E
FO	-1.0000	0.0000	0.0000
NG	0.0000	-1.0000	0.0000
E	0.0000	0.0000	-1.0000

q \ q	ES in 1955 FO	NG	E
FO	--	1.0000	1.0000
NG	1.0000	--	1.0000
E	1.0000	1.0000	--

q \ p	ED in 1960 FO	NG	E
FO	-1.0000	0.0000	0.0000
NG	0.0000	-1.0000	0.0000
E	0.0000	0.0000	-1.0000

q \ q	ES in 1960 FO	NG	E
FO	--	1.0000	1.0000
NG	1.0000	--	1.0000
E	1.0000	1.0000	--

q \ p	ED in 1965 FO	NG	E
FO	-1.0000	0.0000	0.0000
NG	0.0000	-1.0000	0.0000
E	0.0000	0.0000	-1.0000

q \ q	ES in 1965 FO	NG	E
FO	--	1.0000	1.0000
NG	1.0000	--	1.0000
E	1.0000	1.0000	--

q \ p	ED in 1970 FO	NG	E
FO	-1.0000	0.0000	0.0000
NG	0.0000	-1.0000	0.0000
E	0.0000	0.0000	-1.0000

q \ q	ES in 1970 FO	NG	E
FO	--	1.0000	1.0000
NG	1.0000	--	1.0000
E	1.0000	1.0000	--

that fuel is unaffected by changes in its price and the demand is unitary; and that there exists no measurable interdependence between energy fuels, X_i's. Unitary elasticities of substitution indicate that there are substitution possibilities among energy fuels.

For the CES utility function, the magnitudes of price elasticities of demand for any fuel range from -1.0439 to -1.9255, and the magnitudes of cross elasticities of demand of any fuel range from 0.0106 to 1.1896, except those for the year 1945. However, since the 1950's, the ranges have narrowed. Price elasticities are between -1.0509 and -1.5688, and cross elasticities between 0.0106 and 0.4479. Moreover, the magnitudes are very close to one another. These results imply that the percent change in quantity of any fuel, $x_i (i = 1, 2, 3)$, exceeds the percent change in its price. Hence, the budget share of that fuel would increase for a price decline and the demand would be elastic. In 1945, the price elasticity of demand for fuel oil, x_1, was positive, and some cross elasticities of demand were negative. These are inconsistent with the price- and cross-elasticities for other years, as observed in Table 6-5. As for elasticities of substitution, the magnitudes are not identical and, hence, do not satisfy restrictions implied by strong separability (see (5.20)). Therefore, it is concluded that nothing can be said either about elasticities of substitution or about price- and cross-elasticities of demand, because price- and cross-elasticities are not independent indices, i.e., they are a function of elasticity of substitution and income elasticity of demand (see (5.24) and (5.26)). Accordingly, the CES utility function will be ruled out of consideration for use in the present demand study.

Table 6-5

ELASTICITIES OF DEMAND AND PARTIAL ELASTICITIES OF SUBSTITUTION OBTAINED FROM THE CES FUNCTION, SELECTED YEARS, 1938-1970

p \ q	ED in 1938 FO	NG	E
FO	-1.9255	1.1896	0.0330
NG	0.7971	-2.2527	0.0109
E	0.1285	0.0631	-1.0439

q \ q	ES in 1938 FO	NG	E
FO	--	7.6155	1.1836
NG	7.6155	--	1.0901
E	1.1836	1.0901	--

p \ q	ED in 1940 FO	NG	E
FO	-1.3678	0.5372	0.0354
NG	0.2993	-1.6019	0.0186
E	0.0685	0.0647	-1.0541

q \ q	ES in 1940 FO	NG	E
FO	--	2.8757	1.1236
NG	2.8757	--	1.1169
E	1.1236	1.1169	--

p \ q	ED in 1945 FO	NG	E
FO	1.9882	-0.5699	-0.0401
NG	-2.3616	-0.6756	0.0651
E	-0.6266	0.2455	-1.0250

q \ q	ES in 1945 FO	NG	E
FO	--	-10.8306	0.1669
NG	-10.8306	--	1.3264
E	0.1669	1.3264	--

p \ q	ED in 1950 FO	NG	E
FO	-1.4483	0.3207	0.0437
NG	0.3451	-1.3497	0.0132
E	0.1031	0.0289	-1.0569

q \ q	ES in 1950 FO	NG	E
FO	--	2.4220	1.1939
NG	2.4220	--	1.0544
E	1.1939	1.0544	--

p \ q	ED in 1955 FO	NG	E
FO	-1.3897	0.3244	0.0429
NG	0.3012	-1.3584	0.0154
E	0.0885	0.0340	-1.0583

q \ q	ES in 1955 FO	NG	E
FO	--	2.2934	1.1714
NG	2.2934	--	1.0659
E	1.1714	1.0659	--

Table 6-5 (Continued)

p \ q	ED in 1960 FO	NG	E
FO	-1.4525	0.3921	0.0450
NG	0.3466	-1.3288	0.0130
E	0.1059	0.0267	-1.0581

q \ q	ES in 1960 FO	NG	E
FO	--	2.3591	1.2026
NG	2.3591	--	1.0510
E	1.2026	1.0510	--

p \ q	ED in 1965 FO	NG	E
FO	-1.3945	0.2863	0.0456
NG	0.3016	-1.3154	0.0151
E	0.0928	0.0292	-1.0607

q \ q	ES in 1965 FO	NG	E
FO	--	2.1709	1.1865
NG	2.1709	--	1.0585
E	1.1865	1.0585	--

p \ q	ED in 1970 FO	NG	E
FO	-1.5688	0.4434	0.0402
NG	0.4479	-1.4749	0.0106
E	0.1208	0.0315	-1.0509

q \ q	ES in 1970 FO	NG	E
FO	--	3.2226	1.2018
NG	3.2226	--	1.0526
E	1.2018	1.0526	--

For the Uzawa CES utility function, the price elasticities of demand for fuel oil and cross elasticities of demand (on- and off-diagonal terms in the first row and the first column of Table 6-6) are unity and zero, respectively. These results are in conformity with theoretical results outlined in (5.27), and they are expected from the Uzawa function, since the relation between the partitioned subgroups in the Uzawa CES utility function is of the Cobb-Douglas form, i.e., two sub-groups of goods are combined with an overall CD function (see (4.11)). On the other hand, the magnitudes of price elasticities of demand for natural gas (or electricity) range from -1.1619 to -2.7215, and the magnitudes of cross elasticities of demand of natural gas (or electricity) for electricity (or natural gas) range from 0.1619 to 1.7215. However, since the 1950's, the ranges have narrowed. Price elasticities are between -1.2108 and -1.6266, and cross elasticities between 0.2108 and 0.6266. These results imply that the demand for natural gas (or electricity) is elastic and that there does exist measurable interdependence between natural gas and electricity, i.e., they are substitutes.

Elasticities of substitution between fuel oil and natural gas and between fuel oil and electricity were unity for all years. These results are in conformity with theoretical results outlined in (5.21), and they are expected from the Uzawa utility function for the same reason mentioned above. But elasticities of substitution between natural gas and electricity range from 1.8111 to 3.4604. Since the 1950's, the range has narrowed. They are between 1.8608 and 2.0462. The magnitude of σ is an indication of the ease with which natural

Table 6-6

ELASTICITIES OF DEMAND AND PARTIAL ELASTICITIES OF SUBSTITUTION OBTAINED FROM THE UZAWA FUNCTION, SELECTED YEARS, 1938-1970

p \ q	ED in 1938 FO	NG	E	q \ q	ES in 1938 FO	NG	E
FO	-1.0000	0.0000	0.0000	FO	--	1.0000	1.0000
NG	0.0000	-2.7215	0.2965	NG	1.0000	--	3.4604
E	0.0000	1.7215	-1.2965	E	1.0000	3.4604	--

p \ q	ED in 1940 FO	NG	E	q \ q	ES in 1940 FO	NG	E
FO	-1.0000	0.0000	0.0000	FO	--	1.0000	1.0000
NG	0.0000	-1.9674	0.2786	NG	1.0000	--	2.7461
E	0.0000	0.9674	-1.2786	E	1.0000	2.7461	--

p \ q	ED in 1945 FO	NG	E	q \ q	ES in 1945 FO	NG	E
FO	-1.0000	0.0000	0.0000	FO	--	1.0000	1.0000
NG	0.0000	-1.6101	0.1619	NG	1.0000	--	1.8111
E	0.0000	0.6101	-1.1619	E	1.0000	1.8111	--

p \ q	ED in 1950 FO	NG	E	q \ q	ES in 1950 FO	NG	E
FO	-1.0000	0.0000	0.0000	FO	--	1.0000	1.0000
NG	0.0000	-1.4819	0.2199	NG	1.0000	--	1.9064
E	0.0000	0.4819	-1.2199	E	1.0000	1.9064	--

p \ q	ED in 1955 FO	NG	E	q \ q	ES in 1955 FO	NG	E
FO	-1.0000	0.0000	0.0000	FO	--	1.0000	1.0000
NG	0.0000	-1.5162	0.2328	NG	1.0000	--	1.9997
E	0.0000	0.5161	-1.2328	E	1.0000	1.9997	--

Table 6-6 (continued)

p \ q	ED in 1960 FO	NG	E
FO	-1.0000	0.0000	0.0000
NG	0.0000	-1.4499	0.2195
E	0.0000	0.4499	-1.2195

q \ q	ES in 1960 FO	NG	E
FO	--	1.0000	1.0000
NG	1.0000	--	1.8608
E	1.0000	1.8608	--

p \ q	ED in 1965 FO	NG	E
FO	-1.0000	0.0000	0.0000
NG	0.0000	-1.4483	0.2319
E	0.0000	0.4483	-1.2319

q \ q	ES in 1965 FO	NG	E
FO	--	1.0000	1.0000
NG	1.0000	--	1.9005
E	1.0000	1.9005	--

p \ q	ED in 1970 FO	NG	E
FO	-1.0000	0.0000	0.0000
NG	0.0000	-1.6266	0.2108
E	0.0000	0.6266	-1.2108

q \ q	ES in 1970 FO	NG	E
FO	--	1.0000	1.0000
NG	1.0000	--	2.0462
E	1.0000	2.0462	--

gas and electricity can be substituted in consumption to maintain a given level of indifference.[2] There are two limiting cases. If they are perfect substitutes, then σ is infinite. If they are perfect complements, then σ is zero. Accordingly, it is concluded that substitution possibilities between natural gas and electricity do exist, but substitution is not easy.

The evaluation of performances of Sato's two-level CES utility function is similar to that of the CES utility function. As seen in Table 6-7, elasticities of substitution between fuel oil (x_1) and natural gas (x_2) and between fuel oil and electricity (x_3) are not equal, and, hence, they do not satisfy restrictions implied by weak separability (see (5.22)). Accordingly, nothing can be said about elasticities of substitution and, also, about the quality of the parameter estimates for the Sato CES utility function. Moreover, since price- and cross-elasticities are a function of elasticity of substitution and income elasticity of demand, nothing can be said about price- and cross-elasticities. Therefore, Sato's two-level CES utility function will be ruled out of consideration for use in the present demand study. As far as the present study of energy demand is concerned, the CES family cannot be used to explain U.S. households' demand for energy fuels.

For the translog utility function, the magnitudes of price elasticities of demand for any fuel range from -1.2959 to -4.2684, and the magnitudes of cross elasticities of demand (off-diagonal terms

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[2] J. R. Hicks and R. G. D. Allen, op. cit.

Table 6-7

ELASTICITIES OF DEMAND AND PARTIAL ELASTICITIES OF SUBSTITUTION OBTAINED FROM THE SATO FUNCTION, SELECTED YEARS, 1938-1970

p \ q	ED in 1938		
	FO	NG	E
FO	-1.2066	0.8666	-0.0961
NG	0.5807	-6.6152	0.8177
E	-0.3741	0.4749	-1.7216

q \ q	ES in 1938		
	FO	NG	E
FO	--	5.8192	0.4653
NG	5.8192	--	7.7867
E	0.4654	7.7867	--

p \ q	ED in 1940		
	FO	NG	E
FO	-1.0921	0.2699	-0.0301
NG	0.1504	-2.9088	0.4721
E	-0.0582	1.6389	-1.4419

q \ q	ES in 1940		
	FO	NG	E
FO	-	1.9422	0.8949
NG	1.9422	--	3.9583
E	0.8949	3.9583	--

p \ q	ED in 1945		
	FO	NG	E
FO	-1.8936	0.2673	-0.137
NG	1.1077	-2.1662	0.2386
E	-0.2140	0.8989	-1.2249

q \ q	ES in 1945		
	FO	NG	E
FO	--	6.5491	0.7154
NG	6.5491	--	2.1951
E	0.7154	2.1951	--

p \ q	ED in 1950		
	FO	NG	E
FO	-1.1088	0.1099	-0.0039
NG	0.1183	-1.7695	0.3010
E	-0.0094	0.6596	-1.2970

q \ q	ES in 1950		
	FO	NG	E
FO	--	1.4873	0.9823
NG	1.4873	--	2.2403
E	0.9823	2.2403	--

p \ q	ED in 1955		
	FO	NG	E
FO	-1.0972	0.1172	-0.0056
NG	0.1088	-1.8339	0.3233
E	-0.115	0.7168	-1.3177

q \ q	ES in 1955		
	FO	NG	E
FO	--	1.4672	0.9777
NG	1.4672	--	2.3882
E	0.9777	2.3882	--

Table 6-7 (continued)

p \ q	ED in 1960 FO	NG	E
FO	-1.1100	0.1010	-0.0025
NG	0.1159	-1.7093	0.2968
E	-0.0059	0.6083	-1.2943

q \ q	ES in 1960 FO	NG	E
FO	--	1.4545	0.9887
NG	1.4545	--	2.1638
E	0.9887	2.1638	--

p \ q	ED in 1965 FO	NG	E
FO	-1.0989	0.0984	-0.0023
NG	0.1037	-1.7052	0.3139
E	-0.0048	0.6068	-1.3116

q \ q	ES in 1965 FO	NG	E
FO	--	1.4025	0.9905
NG	1.4025	--	2.2187
E	0.9905	2.2187	--

p \ q	ED in 1970 FO	NG	E
FO	-1.1284	0.1581	-0.0104
NG	0.1597	-2.0659	0.3054
E	-0.0313	0.9078	-1.2950

q \ q	ES in 1970 FO	NG	E
FO	--	1.7924	0.9478
NG	1.7924	--	2.5156
E	0.9478	2.5156	--

in Table 6-8) range from 0.0589 to 2.7539, except those for the year 1945. But, since the 1950's, the ranges have narrowed. Price elasticities are between -1.3519 and -1.8686, and cross elasticities between 0.0589 and 0.7551. These empirical results imply that the demand for any fuel is elastic, and there does exist measurable interdependence between energy fuels, i.e., they are substitutes in some degree.
In 1945, the price elasticity of demand for fuel oil and some of cross elasticities of demand show positive and negative signs, respectively. These are inconsistent with other elasticity coefficients for other years, as observed in Table 6-8.

Elasticities of substitution between fuel oil and natural gas, between fuel oil and electricity, and between natural gas and electricity, which are derived from the translog utility function, range from 1.2287 to 3.8620, from 1.3438 to 1.6554, and from 2.0215 to 4.9357, respectively. However, since the 1950's, the ranges have narrowed. Elasticities of substitution between fuel oil and natural gas, between fuel oil and electricity, and between natural gas and electricity are between 1.2287 and 1.5689, between 1.5581 and 1.6554, and between 2.0215 and 2.2607, respectively. Accordingly, it is concluded that substitution possibilities among energy fuels do exist, but substitution is not easy.

Throughout this chapter, performances of the chosen utility functions have been investigated in terms of demand elasticities and elasticities of substitution. Some conclusions can be drawn about these functions. First, the functions which possess the CES properties will be ruled out of consideration for use in the present study,

Table 6-8

ELASTICITIES OF DEMAND AND PARTIAL ELASTICITIES OF SUBSTITUTION OBTAINED FROM THE TRANSLOG FUNCTION, SELECTED YEARS, 1938-1970

p \ q	ED in 1938		
	FO	NG	E
FO	-1.5854	0.5147	0.0618
NG	0.3449	-4.2684	0.4742
E	0.2405	2.7537	-1.5360

q \ q	ES in 1938		
	FO	NG	E
FO	--	3.8620	1.3438
NG	3.8620	--	4.9357
E	1.3438	4.9357	--

p \ q	ED in 1940		
	FO	NG	E
FO	-1.2959	0.1517	0.1093
NG	0.0845	-2.4497	0.3739
E	0.2113	1.2980	-1.4831

q \ q	ES in 1940		
	FO	NG	E
FO	--	1.5297	1.3815
NG	1.5297	--	3.3429
E	1.3815	3.3429	--

p \ q	ED in 1945		
	FO	NG	E
FO	3.0540	-0.2764	-0.1863
NG	-1.1455	-1.7314	0.2675
E	-2.9086	1.0078	-1.0811

q \ q	ES in 1945		
	FO	NG	E
FO	--	-4.7384	-2.8667
NG	-4.7384	--	2.3398
E	-2.8667	2.3398	--

p \ q	ED in 1950		
	FO	NG	E
FO	-1.4058	0.0692	0.1405
NG	0.0745	-1.6440	0.2623
E	0.3313	0.5748	-1.4029

q \ q	ES in 1950		
	FO	NG	E
FO	--	1.3068	1.6231
NG	1.3068	--	2.0809
E	1.6231	2.0809	--

p \ q	ED in 1955		
	FO	NG	E
FO	-1.3519	0.0687	0.1399
NG	0.0638	-1.6944	0.2822
E	0.2881	0.6257	-1.4222

q \ q	ES in 1955		
	FO	NG	E
FO	--	1.2739	1.5581
NG	1.2739	--	2.2118
E	1.5581	2.2118	--

Table 6-8 (Continued)

p \ q	ED in 1960 FO	NG	E
FO	-1.4148	0.0629	0.1457
NG	0.0722	-1.5969	0.2605
E	0.3426	0.5339	-1.4062

q \ q	ES in 1960 FO	NG	E
FO	--	1.2831	1.6554
NG	1.2831	--	2.0215
E	1.6554	2.0215	--

p \ q	ED in 1965 FO	NG	E
FO	-1.3655	0.0559	0.1505
NG	0.0589	-1.5939	0.2783
E	0.3066	0.5381	-1.4289

q \ q	ES in 1965 FO	NG	E
FO	--	1.2287	1.6157
NG	1.2287	--	2.0806
E	1.6157	2.0806	--

p \ q	ED in 1970 FO	NG	E
FO	-1.4800	0.1135	0.1217
NG	0.1147	-1.8686	0.2541
E	0.3654	0.7551	-1.3758

q \ q	ES in 1970 FO	NG	E
FO	--	1.5689	1.6100
NG	1.5689	--	2.2607
E	1.6100	2.2607	--

because estimates of elasticities of substitution derived from both CES and Sato utility functions do not satisfy the restrictions implied by separability. Consequently, nothing can be said either about estimates of price- and cross-elasticities of demand or about the quality of the parameter estimates for both utility functions.

Second, as discussed in Chapter IV, the CD utility function is a special case of the CES utility function and a limiting case of the Uzawa CES utility function which is a hybrid of the CD and CES utility functions. As observed above, estimates of elasticities of substitution derived from both CD and Uzawa utility functions satisfy the restrictions implied by separability. It is noted that since no appeal to approximation by a Taylor series expansion is required in case of the CD utility function there is no approximation error for the CD utility function and, consequently, the restrictions on the Hicks-Allen partial elasticities of substitution implied by separability are exactly satisfied. (See Section 4 in Chapter IV.) In fact all of the Hicks-Allen partial elasticities of substitution for the CD utility function are exactly equal to unity for all years, as seen in Table 6-4. Since it is not necessary to use both CD and Uzawa CES utility functions in order to build an econometric model of demand, the Uzawa CES utility function will be chosen over the CD utility function. A major reason is that the former has fewer restrictions and is therefore more general than the latter. The latter will be ruled out of consideration for use in this empirical study of demand.

Third, only two utility functions remain--the Uzawa CES and translog utility functions. Table 6-9 presents the averages of estimates

Table 6-9

AVERAGES OF DEMAND ELASTICITIES AND ELASTICITIES OF SUBSTITUTION FOR THE UZAWA AND TRANSLOG UTILITY FUNCTIONS, 1950-1970

Elasticity	Uzawa (1)	Translog (2)	Difference* (2)-(1)	$\frac{\text{Translog}}{\text{Uzawa}}$ (2)/(1)
$\frac{p_1}{x_1} \cdot \frac{\partial x_1}{\partial p_1}$	-1.0	-1.4036	0.4036	1.4036
$\frac{p_2}{x_2} \cdot \frac{\partial x_2}{\partial p_2}$	-1.5046	-1.6795	0.1749	1.1162
$\frac{p_3}{x_3} \cdot \frac{\partial x_3}{\partial p_3}$	-1.2249	-1.4072	0.1823	1.1488
$\frac{p_1}{x_2} \cdot \frac{\partial x_2}{\partial p_1}$	0.0	0.0740	0.0740	∞
$\frac{p_1}{x_3} \cdot \frac{\partial x_3}{\partial p_1}$	0.0	0.1396	0.1396	∞
$\frac{p_2}{x_1} \cdot \frac{\partial x_1}{\partial p_2}$	0.0	0.0768	0.0768	∞
$\frac{p_3}{x_1} \cdot \frac{\partial x_1}{p_3}$	0.0	0.3268	0.3268	∞
$\frac{p_2}{x_3} \cdot \frac{\partial x_3}{\partial p_2}$	0.2229	0.2654	0.0425	1.1906
$\frac{p_3}{x_2} \cdot \frac{\partial x_2}{\partial p_3}$	0.5045	0.6055	0.1010	1.2001
σ_{12}	1.0	1.3322	0.3322	1.3322
σ_{13}	1.0	1.6124	0.6124	1.6124
σ_{23}	1.9427	2.1311	0.1884	1.0969

*The differences are computed in absolute value.

of both demand elasticities and elasticities of substitution derived from the Uzawa CES and translog utility functions, the differences between the averages, and the ratios of the averages for the translog utility function to those for the Uzawa CES utility function. The purpose of Table 6-9 is to empirically determine which function performs better. Since the empirical results for the 1945 (Table 6-8) are qualitatively unsatisfactory in case of the translog utility function, Table 6-9 covers the estimates ranging from 1950 to 1970. The unsatisfactory results may possibly be attributed to errors in measurement of quantities consumed of energy fuels and their prices or some other unknown factors. If the rows with zero values in Table 6-9 are ignored, there exist insignificant differences between the averages. It is very difficult to judge which function performs better. But the translog utility function is preferred over the Uzawa CES utility function. A major reason is that the former has fewer restrictions and is therefore more general than the latter. On that basis alone, the translog utility function is superior to all the other utility function.

Finally, Table 6-10 presents estimates of Slutsky's price elasticities of the compensated demand for the translog utility function. Estimates of Slutsky's price elasticities of the compensated demand is smaller in absolute value than estimates of price elasticities of the ordinary demand, because income elasticities of demand are positive and equal to unity due to the hypothesis of linear homogeneity (see (5.32)). Hence, the ordinary demand curve has greater price elasticities of demand in absolute value than the compensated demand

Table 6-10

PRICE ELASTICITIES OF THE COMPENSATED DEMAND FOR THE TRANSLOG UTILITY FUNCTION, 1950-1970

Year	p \ q	FO	NG	E
1950	FO	-1.1657	0.3093	0.3806
	NG	0.2885	-1.4300	0.4763
	E	0.8772	1.1207	-0.8570
1955	FO	-1.0759	0.3447	0.4069
	NG	0.3048	-1.4534	0.5232
	E	0.7711	1.1087	-0.9392
1960	FO	-1.1364	0.3413	0.4241
	NG	0.3444	-1.3247	0.5327
	E	0.7920	0.9833	-0.9568
1965	FO	-1.1112	0.3102	0.4048
	NG	0.3450	-1.3078	0.5644
	E	0.7662	0.9977	-0.9693
1970	FO	-1.2326	0.3609	0.3691
	NG	0.3709	-1.6034	0.5193
	E	0.8528	1.2425	-0.8884

curve.

However, there exists one important difference between Table 6-8 and Table 6-10. According to Table 6-8, the demand for fuel oil, natural gas, or electricity is elastic--the coefficients of price elasticities of demand for each fuel are greater than unity in absolute value. On the other hand, according to Table 6-10, the demand for fuel oil or natural gas is elastic, but the demand for electricity has the coefficients of price elasticities of demand of less than unity in absolute value and is, therefore, inelastic.

CHAPTER VII

CONCLUSION

To build an econometric model of U.S. households' demand for energy fuels, five utility functions have been chosen. As observed in Chapter IV, the CD and CES utility functions are strongly separable, while the Uzawa CES and Sato two-level CES utility functions are weakly separable. These functions place a priori restrictions implied by linear homogeneity and separability on their parameters and, hence, on their various elasticities. The translog utility function places no a priori restrictions on its parameters and, hence, no a priori restrictions on its various elasticities, yet allows various restrictions to be tested parametrically. However, the restrictions implied by linear homogeneity are imposed on the translog parameters in this empirical study of demand, as discussed in Chapter IV.

The chosen utility functions have been applied to the problem of estimating elasticities of both ordinary demand and compensated demand and elasticities of substitution. As observed in Chapter VI, estimates of elasticities of substitution derived from the CES and Sato two-level CES utility functions do not satisfy restrictions implied by separability. Since price- and cross-elasticities of demand are not indices independent of elasticities of substitution (i.e., they are a function of the elasticity of substitution and the income-elasticity of demand), nothing can be said about estimates of various elasticities. Accordingly, the quality of the parameter estimates for the CES and Sato two-level CES utility functions is unsatisfactory beyond dispute. Therefore, these two utility functions have been ruled out of consideration for use in this empirical study of demand.

On the other hand, estimates of elasticities of substitution derived from the CD and Uzawa CES utility functions do satisfy restrictions implied by separability, and, hence, estimates of their parameters and various elasticities can be regarded as qualitatively satisfactory. Since the choice must be made among utility functions, the Uzawa CES utility function is preferred over the CD utility function, even though the empirical performances of both utility functions are satisfactory,in terms of the fulfillment of restrictions. A major reason is that the former has fewer restrictions and is therefore more general than the latter. Hence, the CD utility function has been ruled out of consideration for use in this empirical study of demand.

There remain only two utility functions--the Uzawa CES and translog utility functions. As for a choice between them, the latter is preferred over the former for two reasons. First, as observed in Table 6-9, the empirical performances of both utility functions provide similar results, but some of estimates of cross elasticities of demand for the Uzawa CES utility function are zero due to restrictions implied by separability. These empirical results are unrealistic. Second, the translog utility function has fewer restrictions and therefore more general than the Uzawa CES utility function. Hence, the

Uzawa CES utility function has been ruled out of consideration for use in this empirical study of demand. So far as the performances of the chosen utility functions are concerned, the translog utility function dominates its competitors. Therefore, an econometric model of U.S. households' demand for energy fuels is built from the translog utility function. The choice of the translog utility function implies that restrictions implied by either strong separability or weak separability on the parameters of other chosen utility functions are invalid in case of U.S. households' demand for energy fuels, and, hence, the hypothesis of separability must be rejected.

BIBLIOGRAPHY

BIBLIOGRAPHY

Allen, R. G. D. "A Comparison Between Different Definitions of Complementary and Competitive Goods." Econometrica, Vol. 2 (1934).

______. Macro-Economic Theory. London: Macmillan, 1956.

______. Mathematical Analysis for Economists. London: Macmillan, 1938.

______. "The Nature of Indifference Curves." The Review of Economic Studies, Vol. 1 (1933-1934).

______. "Professor Slutsky's Theory of Consumer's Choice." The Review of Economic Studies (1936).

American Petroleum Institute. 1970 Petroleum Facts and Figures. New York, 1971.

Arrow, K. J., H. G. Chenery, B. S. Minhas, and R. M. Solow. "Capital-Labor Substitution and Economic Efficiency," The Review of Economics and Statistics, Vol. 63 (1961).

Balestra, P., and M. Nerlove. "Pooling Cross Section and Time Series Data in the Estimation of a Dynamic Model: The Demand for Natural Gas." Econometrica, Vol. 34 (July 1966).

Berndt, E. R., and L. R. Christensen. "The Translog Function and Factor Substitution in the U.S. Manufacturing, 1929-1968." Journal of Econometrics, (1973).

Bureau of Mines, U.S. Department of the Interior. Mineral Yearbook. Washington, D.C.: U.S. Government Printing Office, 1940-1970.

Christensen, L. R., D. W. Jorgensen, and L. J. Lau. "Transcendental Logarithmic Production Frontiers." The Review of Economics and Statistics (1973).

______. "Transcendental Logarithmic Utility Functions." The American Economic Review, Vol. 65 (June 1975).

Department of Economic and Social Affairs, United Nations. "World Energy Requirements in 1975 and 2000." Proceedings of the International Conference on the Peaceful Use of Atomic Energy, Vol. 1 Geneva, 1955.

Department of Statistics, American Gas Association. Gas Facts: 1971 Data. Arlington, Virginia, 1972.

Douglas, P. H. "Are There Laws of Production?" The American Economic Review, Vol. 28 (1948).

Edgeworth, F. Y. Mathematical Psychics. London: Routledge and Kegan Paul, Ltd., 1881.

Goldberger, A. S. Econometric Theory. New York: John Wiley and Sons, Inc., 1964.

Goldman, S. M., and H. Uzawa. "A Note on Separability in Demand Analysis." Econometrica, Vol. 32 (1964).

Gorman, W. N. "Capital Aggregation in Vintage Models." Value, Capital and Growth: Essays in Honor of Sir John Hicks, ed., J. N. Wolfe, 1969.

______. "Separable Utility and Aggregation." Econometrica, Vol. 27 (1959).

Henderson, J. M., and R. E. Quandt. Microeconomic Theory. New York: McGraw-Hill Book Company, 1958.

Hibdon, J. E. Price and Welfare Theory. New York: McGraw-Hill Book Company, 1969.

Hicks, J. R. "Notes and Comments: Direct and Indirect Additivity." Econometrica, Vol. 37 (1969).

______. A Revision of Demand Theory. Oxford: The Clarendon Press, 1969.

______. Value and Capital. Oxford: The Clarendon Press, 1968.

Hicks, J. R., and R. G. D. Allen. "A Reconsideration of the Theory of Value." Econometrica, Vol. 1 (1934).

Houthakker, H. S. "Additive Preferences." Econometrica (1960).

Intriligator, M. D. Mathematical Optimization and Economic Theory. Englewood Cliffs, N.J.: Prentice-Hall, 1971.

Johnson, W. E. "The Pure Theory of Utility Curves." The Economic Journal (December 1913).

Johnston, J. Econometric Method. 2nd ed. New York: McGraw-Hill Book Company, 1963.

Lau, L. J. "Duality and the Structure of Utility Function." Journal of Economic Theory (1970).

Leontief, W. "Introduction to a Theory of the Internal Structure of Functional Relationships." Econometrica, Vol. 15 (1947).

______. "A Note on the Interrelation of Subsets of Independent Variables of a Continuous Function with Continuous First Derivatives." Bulletin of the American Mathematical Society, Vol. 5 (1947).

Liew, C. K. "The Price Structure and Market Share in the Case of Energy Industries." Working Paper No. 30. Norman, Oklahoma: Center for Economic and Management Research, The University of Oklahoma, November 1972.

______. "Measuring the Substitutibility of Energy Consumption." Working Paper. Norman, Oklahoma: Center for Economic and Management Research, The University of Oklahoma, December 1974.

Morishima, M. Theory of Demand: Real and Monetary. Oxford: The Clarendon Press, 1973.

Moursund, D. G., and C. S. Duris. Elementary Theory and Application of Numerical Analysis. New York: McGraw-Hill Book Company, 1967.

Parks, R. W. "Systems of Demand Equations: An Empirical Comparison of Alternative Functional Forms." Econometrica, Vol. 37 (October 1969).

Pearce, I. F. "An Exact Method of Consumer Demand Analysis." Econometrica, Vol. 29 (1961).

Rudin, W. Principles of Mathematical Analysis. New York: McGraw-Hill Book Company, 1964.

Samuelson, P. "Corrected Formulation of Direct and Indirect Additivity." Econometrica, Vol. 37 (April 1969).

______. Foundations of Economic Analysis. Cambridge, Massachusetts: Harvard University Press, 1971.

______. "Using Full Duality to Show that Simultaneously Additive Direct and Indirect Utilities Implies Unitary Elasticity of Demand." Econometrica, Vol. 33 (October 1965).

______. "Consumption Theory in Terms of Revealed Preference." Economica, Vol. 15 (1948).

Sato, K. "Two-Level Constant-Elasticity-of-Substitution Production Function." The Review of Economic Studies, Vol. 34 (1967).

Schurr and Netschert. Energy in the American Economy, 1850-1975. Baltimore: The John Hopkins Press, 1960.

Slutsky, E. E. "On the Theory of Budget of the Consumer." In: Stigler and Boulding, Readings in Price Theory. Chicago: Richard D. Irwin, Inc., 1952.

Sono, M. "The Effect of Price Change on the Demand and Supply of Separable Goods." International Economic Review, Vol. 2 (1961).

Strotz, R. H. "The Empirical Implications of a Utility Tree." Econometrica, Vol. 25 (1957).

______. "The Utility Tree--A Correction and Further Appraisal." Econometrica, Vol. 27 (1959).

Texas Eastern Transmission Company. Competition and Growth in American Energy Market, 1947-1985 (1968).

U.S. Department of Commerce. Statistical Abstract of the United States. Washington, D.C.: U.S. Government Printing Office, 1940-1973.

______. Historical Statistics of the United States: Colonial Times to 1957. Washington, D.C.: U.S. Government Printing Office.

Uzawa, H. "Production Function with Constant Elasticities of Substitution." The Review of Economic Studies, Vol. 29 (1962).

Zellner, A. "An Efficient Method of Estimating Seemingly Unrelated Regressions and Tests for Aggregation Bias." Journal of the American Statistical Association, Vol. 57 (June 1962).

APPENDICES

APPENDIX A

APPROXIMATIONS BY A TAYLOR SERIES EXPANSION

(1) The CES Utility Function

$$F(x) = \theta \cdot \left(\sum_{i=1}^{3} \delta_i x_i^{-p}\right)^{-\frac{1}{p}} \quad \text{from (4.3).}$$

$$\ln F(x) = \ln \theta - \frac{1}{p} \cdot \ln\left(\sum_{i=1}^{3} \delta_i x_i^{-p}\right) \quad \text{from (4.21).}$$

(a) $\ln F(x) \Big|_{x_i=1} = \ln \theta$, since $\delta_1 + \delta_2 + \delta_3 = 1.$

(b) $$\frac{\partial \ln F(x)}{\partial \ln x_i} = \frac{\partial F(x)}{\partial x_i} \frac{x_i}{F(x)} = \frac{\delta_i x_i^{-p}}{(\delta_1 x_1^{-p} + \delta_2 x_2^{-p} + \delta_3 x_3^{-p})}\Bigg|_{x_i=1} = \delta_i .$$

(c) $$\frac{\partial^2 \ln F(x)}{\partial (\ln x_i)^2} = \frac{\partial}{\partial x_i} \cdot \left(\frac{\partial F(x)}{\partial x_i} \cdot \frac{x_i}{F(x)}\right) \cdot x_i = -p\delta_i x_i^{-p} \cdot \left(\sum_{j=1}^{3} \delta_j x_j^{-p}\right)^{-1}$$

$$+ p\delta_i^2 x_i^{-2p} \cdot \left(\sum_{j=1}^{3} \delta_j x_j^{-p}\right)^{-2}\Bigg|_{x_i=1} = p \cdot (\delta_i^2 - \delta_i).$$

(d) $$\frac{\partial}{\partial \ln x_j} \cdot \left(\frac{\partial \ln F(x)}{\partial \ln x_i}\right)_{i \neq j} = \frac{\partial}{\partial x_j} \cdot \left(\frac{\partial F(x)}{\partial x_i} \cdot \frac{x_i}{F(x)}\right) \cdot x_j$$

$$= p\delta_i \delta_j x_i^{-p} x_j^{-p} \cdot \left(\sum_{k=1}^{3} \delta_k x_k^{-p}\right)^{-2}\Bigg|_{x_i=1, x_j=1} = p\delta_i \delta_j .$$

Hence,

$$\ln F(x) = \ln \theta + \sum_{i=1}^{3} \delta_i \cdot \ln x_i + \frac{1}{2} \cdot p \sum_{i=1}^{3} (\delta_i^{\,2} - \delta_i) \cdot (\ln x_i)^2$$

$$+ \frac{1}{2} \cdot p \sum_{\substack{i=1 \\ i \neq j}}^{3} \sum_{j=1}^{3} \delta_i \delta_j \cdot \ln x_i \cdot \ln x_j .$$

(2) The Uzawa CES Utility Function

$$F(x) = \theta \cdot x_1^{\,b_1} \cdot \Big(\sum_{i=2}^{3} \delta_i x_i^{-p}\Big)^{-\frac{b_2}{p}} \quad \text{from (4.10).}$$

$$\ln F(x) = \ln \theta + b_1 \cdot \ln x_1 - \frac{b_2}{p} \cdot \ln\Big(\sum_{i=2}^{3} \delta_i x_i^{-p}\Big) \quad \text{from (4.23).}$$

(a) $\ln F(x)\Big|_{x_i=1} = \ln \theta$, since $\delta_2 + \delta_3 = 1$.

(b) $$\frac{\partial \ln F(x)}{\partial \ln x_1} = \frac{\partial F(x)}{\partial x_1} \cdot \frac{x_1}{F(x)}$$

$$= \theta \cdot b_1 x_1^{\,b_1 - 1} \cdot \Big(\sum_{i=2}^{3} \delta_i x_i^{-p}\Big)^{-\frac{b_2}{p}} \cdot \frac{x_1}{F(x)}\Bigg|_{x_i=1} = b_1 .$$

$$\frac{\partial \ln F(x)}{\partial \ln x_2} = \frac{\partial F(x)}{\partial x_2} \cdot \frac{x_2}{F(x)}$$

$$= b_2 \delta_2 x_2^{-p} \cdot \Big(\sum_{i=2}^{3} \delta_i x_i^{-p}\Big)^{-1}\Bigg|_{x_i=1} = b_2 \delta_2 .$$

$$\frac{\partial \ln F(x)}{\partial \ln x_3} = \frac{\partial F(x)}{\partial x_3} \cdot \frac{x_3}{F(x)}$$

$$= b_2 \delta_3 x_3^{-p} \cdot \Big(\sum_{i=2}^{3} \delta_i x_i^{-p}\Big)^{-1}\Bigg|_{x_i=1} = b_2 \delta_3 .$$

(c) $$\frac{\partial^2 \ln F(x)}{\partial (\ln x_1)^2} = \frac{\partial}{\partial x_1} \cdot \Big(\frac{\partial F(x)}{\partial x_1} \cdot \frac{x_1}{F(x)}\Big) \cdot x_1 = x_1 \cdot \frac{\partial b_1}{\partial x_1} = 0 \Big|_{x_i=1} = 0 .$$

$$\frac{\partial^2 \ln F(x)}{\partial(\ln x_2)^2} = \frac{\partial}{\partial x_2} \cdot \left(\frac{\partial F(x)}{\partial x_2} \cdot \frac{x_2}{F(x)}\right) \cdot x_2$$

$$= -pb_2\delta_2 x_2^{-p} \cdot \left(\sum_{i=2}^{3} \delta_i x_i^{-p}\right)^{-1}$$

$$+pb_2\delta_2^{2} x_2^{-2p} \cdot \left(\sum_{i=2}^{3} \delta_i x_i^{-p}\right)^{-2}\Bigg|_{x_i=1}$$

$$= -pb_2\delta_2 \cdot (1 - \delta_2) = -pb_2\delta_2\delta_3.$$

$$\frac{\partial^2 \ln F(x)}{\partial(\ln x_3)^2} = \frac{\partial}{\partial x_3} \cdot \left(\frac{\partial F(x)}{\partial x_3} \cdot \frac{x_3}{F(x)}\right) \cdot x_3$$

$$= -pb_2\delta_3 x_3^{-p} \cdot \left(\sum_{i=2}^{3} \delta_i x_i^{-p}\right)^{-1}$$

$$+pb_2\delta_3 x_3^{-2p} \cdot \left(\sum_{i=2}^{3} \delta_i x_i^{-p}\right)^{-2}\Bigg|_{x_i=1}$$

$$= -pb_2\delta_3 \cdot (1 - \delta_3) = -pb_2\delta_2\delta_3.$$

(d) $$\frac{\partial}{\partial \ln x_2} \cdot \left(\frac{\partial \ln F(x)}{\partial \ln x_1}\right) = \frac{\partial}{\partial x_2} \cdot \left(\frac{\partial F(x)}{\partial x_1} \cdot \frac{x_1}{F(x)}\right) \cdot x_2$$

$$= x_2 \cdot \frac{\partial b_1}{\partial x_2} = 0\Big|_{x_i=1} = 0.$$

$$\frac{\partial}{\partial \ln x_3} \cdot \left(\frac{\partial \ln F(x)}{\partial \ln x_1}\right) = \frac{\partial}{\partial x_3} \cdot \left(\frac{\partial F(x)}{\partial x_1} \cdot \frac{x_1}{F(x)}\right) \cdot x_3$$

$$= x_3 \cdot \frac{\partial b_1}{\partial x_3} = 0\Big|_{x_i=1} = 0.$$

$$\frac{\partial}{\partial \ln x_3} \cdot \left(\frac{\partial \ln F(x)}{\partial \ln x_2}\right) = \frac{\partial}{\partial x_3} \cdot \left(\frac{\partial F(x)}{\partial x_2} \cdot \frac{x_2}{F(x)}\right) \cdot x_3$$

$$= pb_2\delta_2\delta_3 x_2^{-p} x_3^{-p} \cdot \left(\sum_{i=2}^{3} \delta_i x_i^{-p}\right)^{-2} \Big|_{x_i=1}$$

$$= pb_2\delta_2\delta_3.$$

Hence,

$$\ln F(x) = \ln \theta + b_1 \cdot \ln x_1 + b_2 \sum_{i=2}^{3} \delta_i \cdot \ln x_i$$

$$- \frac{1}{2} \cdot pb_2\delta_2\delta_3 \cdot \sum_{i=2}^{3} (\ln x_i)^2$$

$$+ \frac{1}{2} \cdot pb_2\delta_2\delta_3 \cdot \sum_{\substack{i=2 \\ i \neq j}}^{3} \sum_{j=2}^{3} \ln x_i \cdot \ln x_j.$$

(3) The Sato Two-Level CES Utility Function

$$F(x) = \theta \cdot \left(b_1 x_1^{-w} + b_2 \cdot \left(\sum_{i=2}^{3} \delta_i x_i^{-p}\right)^{\frac{w}{p}}\right)^{-\frac{1}{w}} \text{ from (4.11).}$$

$$\ln F(x) = \ln \theta - \frac{1}{w} \cdot \ln\left(b_1 x_1^{-w} + b_2 \cdot \left(\sum_{i=2}^{3} \delta_i x_i^{-p}\right)^{\frac{w}{p}}\right) \text{ from (4.25).}$$

(a) $\ln F(x)\big|_{x_i=1} = \ln \theta$, since $b_1 + b_2 = \delta_2 + \delta_3 = 1$.

(b) $$\frac{\partial \ln F(x)}{\partial \ln x_1} = \frac{\partial F(x)}{\partial x_1} \cdot \frac{x_1}{F(x)} = b_1 x_1^{-w} \cdot \left(b_1 x_1^{-w} + b_2 \cdot \left(\sum_{i=2}^{3} \delta_i x_i^{-p}\right)^{\frac{w}{p}}\right)^{-1} \Big|_{x_i=1}$$

$$= b_1.$$

$$\frac{\partial \ln F(x)}{\partial \ln x_2} = \frac{\partial F(x)}{\partial x_2} \cdot \frac{x_2}{F(x)}$$

$$= b_2\delta_2 x_2^{-p} \cdot \left(b_1 x_1^{-w} + b_2 \cdot \left(\sum_{i=2}^{3} \delta_i x_i^{-p}\right)^{\frac{w}{p}}\right)^{-1} \cdot \left(\sum_{i=2}^{3} \delta_i x_i^{-p}\right)^{\frac{w}{p}-1} \Big|_{x_i=1}$$

$$= b_2\delta_2.$$

$$\frac{\partial \ln F(x)}{\partial \ln x_3} = \frac{\partial F(x)}{\partial x_3} \cdot \frac{x_3}{F(x)}$$

$$= b_2\delta_3 x_3^{-p}\cdot(b_1x_1^{-w}+b_2\cdot(\sum_{i=2}^{3} \delta_i x_i^{-p})^{\frac{w}{p}})^{-1} \cdot (\sum_{i=2}^{3} \delta_i x_i^{-p})^{\frac{w}{p}-1}\Big|_{x_i=1}$$

$$= b_2\delta_3.$$

(c) $$\frac{\partial^2 \ln F(x)}{\partial(\ln x_1)^2} = \frac{\partial}{\partial x_1} \cdot (\frac{\partial F(x)}{\partial x_1} \cdot \frac{x_1}{F(x)}) \cdot x_1$$

$$= -b_1 w x_1^{-w}\cdot(b_1x_1^{-w} + b_2\cdot(\sum_{i=2}^{3} \delta_i x_i^{-p})^{\frac{w}{p}})^{-1}$$

$$+ b_1^2 w x_1^{-2w}\cdot(b_1x_1^{-w} + b_2\cdot(\sum_{i=2}^{3} \delta_i x_i^{-p})^{\frac{w}{p}})^{-2}\Big|_{x_i=1}$$

$$= -b_1 w\cdot(1 - b_1) = -wb_1b_2.$$

$$\frac{\partial^2 \ln F(x)}{\partial(\ln x_2)^2} = \frac{\partial}{\partial x_2} \cdot (\frac{\partial F(x)}{\partial x_2} \cdot \frac{x_2}{F(x)}) \cdot x_2$$

$$= (b_1x_1^{-w} + b_2\cdot(\sum_{i=2}^{3} \delta_i x_i^{-p})^{\frac{w}{p}})^{-1} \cdot (\sum_{i=2}^{3} \delta_i x_i^{-p})^{\frac{w}{p}-1}$$

$$\cdot \Big\{-pb_2\delta_2x_2^{-p} + b_2^2\delta_2^2 w x_2^{-2p}\cdot(b_1x_1^{-w}+b_2\cdot(\sum_{i=2}^{3} \delta_i x_i^{-p})^{\frac{w}{p}})$$

$$- b_2\delta_2^2\cdot(w-p)\cdot x_2^{-2p}\cdot(\sum_{i=2}^{3} \delta_i x_i^{-p})^{-1}\Big\}\Big|_{x_i=1}$$

$$= -b_2\delta_2\cdot(p\delta_3 + wb_1\delta_2).$$

Similarly,

$$\frac{\partial^2 \ln F(x)}{\partial(\ln x_3)^2} = \frac{\partial}{\partial x_3}\cdot(\frac{\partial F(x)}{\partial x_3} \cdot \frac{x_3}{F(x)})\cdot x_3 = -b_2\delta_3\cdot(p\delta_2 + wb_1\delta_3) \text{ at } x_i=1.$$

(d) $$\frac{\partial}{\partial \ln x_2}\cdot\left(\frac{\partial \ln F(x)}{\partial \ln x_1}\right) = \frac{\partial}{\partial x_2}\cdot\left(\frac{\partial F(x)}{\partial x_1}\cdot\frac{x_1}{F(x)}\right)\cdot x_2$$

$$= wb_1b_2\delta_2x_1^{-w}x_2^{-p}\cdot(b_1x_1^{-w}+b_2\cdot(\sum_{i=2}^{3}\delta_ix_i^{-p})^{\frac{w}{p}})^{-2}\cdot(\sum_{i=2}^{3}\delta_ix_i^{-p})^{\frac{w}{p}-1}\Big|_{x_i=1}$$

$$= wb_1b_2\delta_2.$$

$$\frac{\partial}{\partial \ln x_3}\cdot\left(\frac{\partial \ln F(x)}{\partial \ln x_1}\right) = \frac{\partial}{\partial x_3}\cdot\left(\frac{\partial F(x)}{\partial x_1}\cdot\frac{x_1}{F(x)}\right)\cdot x_3$$

$$= wb_1b_2\delta_3x_1^{-w}x_3^{-p}\cdot(b_1x_1^{-w}+b_2\cdot(\sum_{i=2}^{3}\delta_ix_i^{-p})^{\frac{w}{p}})^{-2}\cdot(\sum_{i=2}^{3}\delta_ix_i^{-p})^{\frac{w}{p}-1}\Big|_{x_i=1}$$

$$= wb_1b_2\delta_3.$$

$$\frac{\partial}{\partial \ln x_3}\cdot\left(\frac{\partial \ln F(x)}{\partial \ln x_2}\right) = \frac{\partial}{\partial x_3}\cdot\left(\frac{\partial F(x)}{\partial x_2}\cdot\frac{x_2}{F(x)}\right)\cdot x_3$$

$$= b_2\delta_2x_2^{-p}\cdot\Big\{wb_2\delta_3x_3^{-p}\cdot(b_1x_1^{-w}+b_2\cdot(\sum_{i=2}^{3}\delta_ix_i^{-p})^{\frac{w}{p}})^{-2}\cdot(\sum_{i=2}^{3}\delta_ix_i^{-p})^{2(\frac{w}{p}-1)}$$

$$-p\cdot(\frac{w}{p}-1)\cdot\delta_3x_3^{-p}\cdot(b_1x_1^{-w}+b_2\cdot(\sum_{i=2}^{3}\delta_ix_i^{-p})^{\frac{w}{p}})^{-1}\cdot(\sum_{i=2}^{3}\delta_ix_i^{-p})^{\frac{w}{p}-2}\Big\}$$

$$= b_2\delta_2\delta_3\cdot(p - wb_1)\quad\text{at } x_i = 1.$$

Hence,

$$\ln F(x) = \ln\theta + b_1\cdot\ln x_1 + b_2\cdot\sum_{i=2}^{3}\delta_i\cdot\ln x_i$$

$$-\frac{1}{2}\cdot b_2\cdot\Big\{wb_1\cdot(\ln x_1)^2 + {}_2\cdot(p\delta_3+wb_1\delta_2)\cdot(\ln x_2)^2$$

$$+\delta_3\cdot(p\delta_2+wb_1\delta_3)\cdot(\ln x_3)^2\Big\} + wb_1b_2\delta_2\cdot\ln x_1\cdot\ln x_2$$

$$+ wb_1b_2\delta_3\cdot\ln x_1\cdot\ln x_3 + b_2\delta_2\delta_3\cdot(p - wb_1)\cdot\ln x_2\cdot\ln x_3.$$

APPENDIX B

DERIVATION OF DEMAND ELASTICITIES

From (5.24):

$$\text{(A.1)} \quad \frac{p_i}{x_j} \cdot \frac{\partial x_j}{\partial p_i} = k_i \cdot (\sigma_{ij} - \frac{M}{x_j} \cdot \frac{\partial x_j}{\partial M}) \quad (i,j = 1, 2, 3)$$

Among partial elasticities of substitution defined in (5.18) there exists a linear dependence:[1]

$$\text{(A.2)} \quad k_1 \cdot \sigma_{i1} + k_2 \cdot \sigma_{i2} + k_3 \cdot \sigma_{i3} = 0 \ , \quad i = 1, 2, 3.$$

From (A.2), the following relations can be obtained:

$$k_1 \cdot \sigma_{11} = -k_{12} \cdot \sigma_{12} - k_3 \cdot \sigma_{13} \quad \text{for } i=1,$$

$$\text{(A.3)} \quad k_2 \cdot \sigma_{22} = -k_1 \cdot \sigma_{21} - k_3 \cdot \sigma_{23} = -k_1 \cdot \sigma_{12} - k_3 \cdot \sigma_{23} \quad \text{for } i=2,$$

$$k_3 \cdot \sigma_{33} = -k_1 \cdot \sigma_{31} - k_2 \cdot \sigma_{32} = -k_1 \cdot \sigma_{13} - k_2 \cdot \sigma_{23} \quad \text{for } i=3$$

since $\sigma_{ij} = \sigma_{ji}$ for i,j = 1, 2, 3 (see (5.18)).

The price- and cross-elasticities of demand for a good x_j (j = 1, 2, 3) for the chosen utility functions can be obtained by using (A.1) and (A.3):

(1) The CD utility function

$$\frac{p_1}{x_1} \cdot \frac{\partial x_1}{\partial p_1} = (-k_2 \cdot \sigma_{12} - k_3 \cdot \sigma_{13}) - k_1 \cdot 1 = -k_2 - k_3 - k_1 = -1 \text{ for } i=1,$$

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[1] R. G. D. Allen, Mathematical Analysis for Economists, pp. 503-505.

$$\text{(A.4)} \quad \begin{aligned} &\frac{p_2}{x_2} \cdot \frac{\partial x_2}{\partial p_2} = (-k_1 \cdot \sigma_{12} - k_3 \cdot \sigma_{23}) - k_2 \cdot 1 = -k_1 \cdot k_3 - k_2 = -1 \quad \text{for } i=2, \\ &\frac{p_3}{x_3} \cdot \frac{\partial x_3}{\partial p_3} = (-k_1 \cdot \sigma_{13} - k_2 \cdot \sigma_{23}) - k_3 \cdot 1 = -k_1 - k_2 - k_3 = -1 \quad \text{for } i=3 \end{aligned}$$

$$\frac{p_i}{x_j} \cdot \frac{\partial x_j}{\partial p_i} = k_i \cdot (1-1) = 0 \quad \text{for } i \neq j \; (i,j = 1, 2, 3)$$

where $\sigma_{12} = \sigma_{13} = \sigma_{23} = 1$ (see (5.19)), and the income-elasticity of demand is unity, and $k_1 + k_2 + k_3 = 1$ due to linear homogeneity.

(2) The CES utility function

$$\begin{aligned} \frac{p_1}{x_1} \cdot \frac{\partial x_1}{\partial p_1} &= (-k_2 \cdot \sigma_{12} - k_3 \cdot \sigma_{13}) - k_1 \cdot 1 \\ &= -\sigma^*(k_2 + k_3) - k_1 \\ &= -\sigma^*(1 - k_1) - k_1 \quad \text{for } i=1, \end{aligned}$$

$$\text{(A.5)} \quad \begin{aligned} \frac{p_2}{x_2} \cdot \frac{\partial x_2}{\partial p_2} &= (-k_1 \cdot \sigma_{12} - k_3 \cdot \sigma_{23}) - k_2 \cdot 1 \\ &= -\sigma^*(k_1 + k_3) - k_2 \\ &= -\sigma^*(1 - k_2) - k_2 \quad \text{for } i=2, \end{aligned}$$

$$\begin{aligned} \frac{p_3}{x_3} \cdot \frac{\partial x_3}{\partial p_3} &= (-k_1 \cdot \sigma_{13} - k_2 \cdot \sigma_{23}) - k_3 \cdot 1 \\ &= -\sigma^* \cdot (k_1 + k_2) - k_3 \\ &= -\sigma^* \cdot (1 - k_3) - k_3 \quad \text{for } i=3, \end{aligned}$$

$$\frac{p_i}{x_j} \cdot \frac{\partial x_j}{\partial p_i} = k_i \cdot (\sigma^* - 1) \quad \text{for } i \neq j \; (i,j = 1, 2, 3)$$

where $\sigma^* = \sigma_{12} = \sigma_{13} = \sigma_{23}$ (see (5.20)).

(3) The Uzawa utility function

$$\frac{p_1}{x_1} \cdot \frac{\partial x_1}{\partial p_1} = (-k_2 \cdot \sigma_{12} - k_3 \cdot \sigma_{13}) - k_1 \cdot 1 = (-k_2 - k_3) - k_1 = -1 \quad \text{for } i=1,$$

$$\frac{p_2}{x_2} \cdot \frac{\partial x_2}{\partial p_2} = (-k_1 \cdot \sigma_{12} - k_3 \cdot \sigma_{23}) - k_2 \cdot 1 = -(k_1 + k_2 + k_3 \cdot \sigma_{23}) \quad \text{for } i=2,$$

(A.6)
$$\frac{p_3}{x_3} \cdot \frac{\partial x_3}{\partial p_3} = (-k_1 \cdot \sigma_{13} - k_2 \cdot \sigma_{23}) - k_3 \cdot 1 = -(k_1 + k_2 \cdot \sigma_{23} + k_3) \quad \text{for } i=3,$$

$$\frac{p_1}{x_2} \cdot \frac{\partial x_2}{\partial p_1} = \frac{p_2}{x_1} \cdot \frac{\partial x_1}{\partial p_2} = \frac{p_3}{x_1} \cdot \frac{\partial x_1}{\partial p_3} = \frac{p_1}{x_3} \cdot \frac{\partial x_3}{\partial p_1} = 0 \ ,$$

$$\frac{p_i}{x_j} \cdot \frac{\partial x_j}{\partial p_i} = k_i \cdot (\sigma_{ij} - 1) \quad \text{for } i \neq j \ (i,j \ (i,j = 1,\ 2,\ 3)$$

where $\sigma_{12} = \sigma_{13} = 1$ and $\sigma_{ij} = \sigma_{ji}$ for $i \neq j$ (see (5.21)).

(4) The Sato utility function

$$\frac{p_1}{x_1} \cdot \frac{\partial x_1}{\partial p_1} = -\sigma^{**} \cdot (k_2 + k_3) - k_1 = -\sigma^{**} \cdot (1 - k_1) - k_1 \quad \text{for } i=1,$$

$$\frac{p_2}{x_2} \cdot \frac{\partial x_2}{\partial p_2} = -(k_1 \cdot \sigma_{12} + k_3 \cdot \sigma_{23}) - k_2 \quad \text{for } i=2,$$

(A.7)
$$\frac{p_3}{x_3} \cdot \frac{\partial x_3}{\partial p_3} = -(k_1 \cdot \sigma_{13} + k_2 \cdot \sigma_{23}) - k_3 \quad \text{for } i=3,$$

$$\frac{p_i}{x_j} \cdot \frac{\partial x_j}{\partial p_i} = k_i \cdot (\sigma_{ij} - 1) \quad \text{for } i \neq j \ (i,j = 1,\ 2,\ 3)$$

where $\sigma^{**} = \sigma_{12} = \sigma_{13}$ and $\sigma_{ij} = \sigma_{ji}$ for $i \neq j$ (see (5.22)).

APPENDIX C

DATA FOR ENERGY FUELS CONSUMED IN U.S. HOUSEHOLDS, 1937-1970*

Year	Fuel Oil		Natural Gas		Electricity	
	Quantity (Mil Bbl)	Price (¢/Bbl)	Quantity (Mil Therm)	Price (¢/Therm)	Quantity (Mil KWH)	Price (¢/KWH)
1937	64.355	415.0	3353.4	7.0569	17691.0	4.3529
1938	67.016	390.0	3356.3	7.0279	19371.0	4.0022
1939	81.740	385.0	3646.0	6.8376	21433.0	3.9100
1940	98.436	381.0	4064.3	6.7431	24068.0	3.8357
1941	103.763	378.0	4112.8	6.6969	26574.0	3.7520
1942	103.991	375.0	6913.6	6.4075	27000.0	3.6167
1943	107.437	370.0	6662.5	6.3850	29000.0	3.4983
1944	111.225	366.0	5290.0	6.2582	31000.0	3.3819
1945	103.634	362.0	5600.9	6.3071	34000.0	3.2724
1946	118.058	357.0	6162.7	6.1615	39000.0	3.1521
1947	150.695	38910	7513.6	6.0605	44000.0	3.0845
1948	167.623	437.0	8227.6	6.0676	51000.0	3.0059
1949	152.872	465.0	9541.9	6.1991	58000.0	2.9403
1950	174.821	481.0	11561.6	6.4813	67000.0	2.8536
1951	204.977	498.0	14008.7	6.6463	77000.0	2.7817
1952	219.746	499.0	15253.9	7.1811	87000.0	2.7598
1953	221.543	529.0	16013.2	7.7436	97000.0	2.7402
1954	254.994	553.0	17830.3	8.0624	108000.0	2.7037
1955	282.991	671.0	20085.7	8.2541	125000.0	2.6584
1956	301.497	713.0	22444.8	8.5160	134000.0	2.5948
1957	301.916	882.0	24277.7	8.7285	147000.0	2.5571
1958	343.700	829.0	26320.0	9.0729	159000.0	2.5314
1959	347.490	841.0	28026.9	9.3544	173000.0	2.5040
1960	364.142	826.0	30231.2	9.7282	196000.0	2.4776
1961	378.057	859.0	31575.0	9.9857	209000.0	2.4478
1962	392.519	859.0	33861.5	10.0191	226000.0	2.4124
1963	390.403	878.0	35309.9	10.0443	242000.0	2.3649
1964	378.156	864.0	37699.1	10.0114	262000.0	2.3057
1965	397.016	882.0	39164.2	10.0596	281000.0	2.2523
1966	396.392	904.0	40932.8	10.0415	307000.0	2.1935
1967	418.014	930.0	42811.0	10.0373	332000.0	2.1639
1968	430.608	959.0	44682.0	10.0367	368000.0	2.1201
1969	456.608	980.0	47374.5	10.1394	402000.0	2.1224
1970	470.337	1016.0	48394.4	10.5884	448000.0	2.1018

Source: See sources in Tables 2-2.

"The quantity of fuel oil is the quantity of distillate heating oil (grade 2), and prices are obtained by dividing total revenues by total quantities.

APPENDIX D

FINDING THE SOLUTION FOR THE PARAMETERS*

(1) The solutions for the parameters of the CES utility function are:

$\delta_i = a_i$ for 1, 2, 3 ,

$$(D.1) \quad p = \begin{cases} \dfrac{a_2 Q_{13}}{a_1 a_3} \\[2ex] \dfrac{Q_{11}}{a_1^2 - a_1} \\[2ex] \dfrac{Q_{22}}{a_2^2 - a_2} \\[2ex] \dfrac{Q_{33}}{a_3^2 - a_3} \end{cases}$$

(2) The solutions for the parameters of the Uzawa CES utility function are:

$b_1 = a_1$ and $b_2 = 1 - a_1$ since $b_1 + b_2 = 1$,

*This method was used by Berndt and Christensen. See Berndt and Christensen, op. cit.

$$\text{(D.2)} \quad p = \begin{cases} \dfrac{-Q_{22}\cdot(1 - a_1)}{a_2 a_3} \\[2ex] \dfrac{-Q_{33}\cdot(1 - a_1)}{a_2 a_3} \\[2ex] \dfrac{Q_{23}\cdot(1 - a_1)}{a_2 a_3} \end{cases} .$$

(3) The solutions for the parameters of the Sato two-level CES utility function are:

$b_1 = a_1$ and $b_2 = 1 - a_1$ since $b_1 + b_2 = 1$,

$\delta_1 = \dfrac{a_i}{1 - a_1}$ for $i = 2, 3$ since $\delta_2 + \delta_3 = 1$,

$$w = \begin{cases} \dfrac{-Q_{11}}{a_1\cdot(1-a_1)} \\[2ex] \dfrac{Q_{31}}{a_1 a_3} \\[2ex] \dfrac{Q_{21}}{a_1 a_2} \end{cases}$$

(D.3)

$$p = \begin{cases} \dfrac{-Q_{22}\cdot(1-a_1)-wa_1a_2}{a_2 a_3} \\[2ex] \dfrac{Q_{23}\cdot(1-a_1)+wa_1a_2a_3}{a_2 a_3} \\[2ex] \dfrac{-Q_{33}\cdot(1-a_1)-w\cdot a_1\cdot a_3^2}{a_2 a_3} \end{cases}$$

(4) The solutions for the parameters of the CD utility function are:

(D.4) $b_i = a_i$ for $i = 1, 2, 3$.

As seen in (D.1), (D.2), and (D.3), the CES, the Uzawa, and the Sato utility functions have multiple solutions to their substitution parameters, p or w. For particular parameter values, the value of each true function is compared with the value of its second order approximation for various points in the commodity space. This procedure is interpreted as measuring how close together are the true and approximate indifference surfaces. The comparisons are based on the differences between their values:

(1) For a particular value of p of the CES utility function, the true and approximate CES utility function in (4.7) and (4.23) are used:

$$\ln U(x) - \ln U(x)^* = \ln \theta - \frac{1}{p} \cdot \ln(\sum_{i=1}^{3} \delta_i x_i^{-p}) - [\ln \theta$$

$$+ \sum_{i=1}^{3} \delta_i \ln x_i + \frac{1}{2} \sum_{i=1}^{3} p(\delta_i^2 - \delta_i)\cdot(\ln x_i)^2 \tag{D.5}$$

$$+ \frac{1}{2} \sum_{\substack{i=1 \\ i\neq j}}^{3} \sum_{j=1}^{3} p\cdot\delta_i\cdot\delta_j\cdot\ln x_i\cdot\ln x_j]$$

where ln U* is the Taylor approximation to the CES utility function, and ln U is the logarithm of the true CES utility function in (4.7).

(2) For a particular value of p of the Uzawa utility function, the true and approximate Uzawa utility function in (4.11) and (4.24) are used:

$$\ln U(x) - \ln U(x)^* = \ln \theta + b_1\cdot\ln x_1 - \frac{b_2}{p} \cdot \ln(\sum_{i=2}^{3} \delta_i x_i^{-p})$$

$$- [\ln \theta + b_1\cdot\ln x_1 + b_2 \sum_{i=2}^{3} \delta_i \ln x_i \tag{D.6}$$

$$- \frac{1}{2}\cdot(pb_2\delta_2\delta_3) \sum_{i=2}^{3} (\ln x_i)^2$$

$$+ \frac{1}{2}\cdot(pb_2\delta_2\delta_3) \sum_{i=2}^{3} \sum_{\substack{j=2 \\ i \neq j}}^{3} \ln x_i \cdot \ln x_j]$$

where ln U* is the Taylor approximation to the Uzawa utility function, ln U is the logarithm of the true Uzawa utility function in (4.11).

(3) For a particular value of p and w of the Sato utility function, the true and approximate Sato utility functions in (4.13) and (4.25) are used:

$$\begin{aligned} \ln U(x) - \ln U(x)^* &= \ln \theta - \frac{1}{w}\cdot \ln(b_1 x_1^{-w} + b_2 \cdot (\sum_{i=2}^{3} \delta_i x_i^{-p})^{\frac{w}{p}}) \\ &- [\ln \theta + b_1 \cdot \ln x_1 + b_2 \sum_{i=2}^{3} \delta_i \ln x_i \\ &- \frac{1}{2} b_2(wb_1 \cdot (\ln x_1)^2 + \delta_2(p\delta_3 + wb_1\delta_2) \\ &\cdot (\ln x_2)^2 + \delta_3(p\delta_2 + wb_1\delta_3)\cdot(\ln x_3)^2)] \end{aligned} \tag{D.7}$$

where ln U* is the Taylor approximation to the Sato utility function, and ln U is the logarithm of the true Sato utility function in (4.13).

Using (D.5), (D.6), and (D.7), the differences between the true and approximate functions, which correspond to multiple parameter values, p or w, are computed and presented in Tables D-1, D-2, and D-3. To choose a particular value for p or w, the percentage distribution of differences is considered: The smaller differences the p or w gives, the better the p or w. In other words, the distribution of smaller differences means the relative closeness of the true and approximate indifference surfaces. The percentage distribution of differences for the CES, the Uzawa, and the Sato utility functions are presented in Tables D-4, D-5, and D-6, respectively.

For a particular value of p of the CES utility function, p = -0.243920 is chosen among four different values of p, because all the values of differences lie between 0.0 and 20.0, as shown in Table D-4. Similarly, for the Uzawa utility function, p = -0.772910 is chosen among three different values of p, as shown in Table D-5. For particular values of p and w of the Sato utility function, a careful comparison suggests that the combination of p = -0.911384 and w = -0.084352 is the best choice among nine different combinations of p and w, as shown in Table D-6.

Table D-1

CLOSENESS (OR DIFFERENCE) BETWEEN THE TRUE CES UTILITY FUNCTION AND ITS ALTERNATIVE FUNCTIONAL FORM WITH 10% DEPRECIATION RATE

(Magnitudes in Absolute Value)

Year	p=-0.635495	p=-0.243920	p=-0.392086	p=-1.040084
1938	52.430900	8.271940	13.868800	700.232000
1939	67.450100	9.599270	17.290300	916.417000
1940	76.891500	10.105900	18.880600	1117.830000
1941	66.370000	9.151020	16.387600	1005.210000
1942	120.169000	11.297600	24.456700	2075.830000
1943	43.113700	6.386720	9.829960	796.590000
1944	44.347500	6.439640	10.006700	332.116000
1945	80.142000	8.961730	17.383100	1359.190000
1946	103.469000	10.992700	22.318500	1898.670000
1947	126.485000	12.414800	26.501300	2349.570000
1948	135.285000	12.295400	26.843100	2793.160000
1949	125.036000	10.062300	22.577100	2783.790000
1950	167.188000	13.377800	30.893500	3841.860000
1951	185.673000	14.120500	33.401600	4466.330000
1952	171.911000	13.319500	30.977600	4237.450000
1953	167.066000	12.744200	29.597700	4263.470000
1954	197.717000	14.455700	34.629900	5135.450000
1955	229.140000	15.069300	37.550700	6730.180000
1956	213.293000	14.741000	36.192000	5684.510000
1957	222.006000	14.383200	35.967100	6467.560000
1958	234.440000	15.587400	38.872300	6720.680000
1959	237.448000	14.899200	37.746300	7245.290000
1960	277.330000	15.926700	41.749100	9521.660000
1961	247.384000	15.347900	39.064700	7777.190000
1962	276.974000	16.028700	42.015900	9252.450000
1963	268.371000	15.415300	40.347300	9112.390000
1964	295.491000	15.651400	42.356900	10648.800000
1965	293.661000	16.300200	43.270100	10564.100000
1966	318.614000	16.283600	44.517000	12465.200000
1967	329.202000	16.979600	46.347000	12914.900000
1968	360.686000	17.232700	48.263900	15618.900000
1969	378.431000	17.906500	50.531300	16382.800000
1970	391.333000	17.574900	50.115300	18791.700000

Note: p is the substitution parameter in the CES function.

Table D-2

CLOSENESS (OR DIFFERENCE) BETWEEN THE TRUE UZAWA UTILITY FUNCTION AND ITS ALTERNATIVE FUNCTIONAL FORM WITH 10% DEPRECIATION RATE

(Magnitudes in Absolute Value)

Year	p=-0.938309	p=-0.772910	p=-1.280698
1938	0.030090	0.025910	0.027645
1939	0.020066	0.015852	0.024765
1940	0.019028	0.014924	0.024006
1941	0.031179	0.027190	0.027232
1942	0.000053	0.000036	0.000100
1943	0.956296	0.551171	1.892350
1944	0.972984	0.562711	1.919780
1945	0.022903	0.018475	0.026480
1946	0.023579	0.019121	0.026802
1947	0.012678	0.009541	0.018036
1948	0.017906	0.013937	0.023112
1949	0.022624	0.018211	0.026336
1950	0.014620	0.011139	0.020088
1951	0.013161	0.009934	0.018563
1952	0.021068	0.016764	0.025434
1953	0.025536	0.021042	0.027523
1954	0.018854	0.014770	0.023872
1955	0.021709	0.017355	0.025826
1956	0.014393	0.010950	0.019858
1957	0.020342	0.016101	0.024956
1958	0.018551	0.014503	0.023635
1959	0.021732	0.017377	0.025840
1960	0.025170	0.020677	0.027414
1961	0.023514	0.019058	0.026773
1962	0.021960	0.017588	0.025972
1963	0.024994	0.020502	0.027357
1964	0.023516	0.019060	0.026774
1965	0.026686	0.022213	0.027783
1966	0.028715	0.024369	0.027886
1967	0.028306	0.023923	0.027906
1968	0.031920	0.028097	0.026814
1969	0.029484	0.025221	0.027786
1970	0.036141	0.034267	0.020700

Note: p is the substitution parameter in the Uzawa CES utility function.

Table D-3

CLOSENESS (OR DIFFERENCE) BETWEEN THE TRUE SATO UTILITY FUNCTION AND ITS ALTERNATIVE FUNCTIONAL FORM WITH 10% DEPRECIATION RATE

(Magnitudes in Absolute Value)

Year	w=-0.243920		
	p=0.168066	p=-0.555706	p=1.944176
1938	0.018387	0.109116	0.132710
1939	0.009349	0.064261	0.060134
1940	0.009144	0.061438	0.052339
1941	0.017483	0.112005	0.151621
1942	0.017734	0.016669	0.020784
1943	0.151943	0.281368	5.369460
1944	0.156246	0.274240	5.431580
1945	0.042950	0.074657	0.011095
1946	0.014844	0.081129	0.071821
1947	0.007248	0.043155	0.015057
1948	0.013512	0.061335	0.030720
1949	0.092638	0.038781	0.057777
1950	0.013831	0.050079	0.009808
1951	0.012084	0.045264	0.005691
1952	0.018278	0.073551	0.041959
1953	0.025983	0.093007	0.062281
1954	0.014233	0.064735	0.035465
1955	0.018896	0.076109	0.045458
1956	0.015570	0.048937	0.005032
1957	0.026629	0.069467	0.020074
1958	0.014109	0.063669	0.033656
1959	0.025088	0.075932	0.032760
1960	0.025837	0.091299	0.059268
1961	0.022425	0.083817	0.052156
1962	0.023824	0.077116	0.036924
1963	0.030951	0.089909	0.047706
1964	0.041663	0.078496	0.017621
1965	0.026508	0.098521	0.072107
1966	0.037711	0.108665	0.090756
1967	0.029176	0.106943	0.083173
1968	0.036607	0.129081	0.112517
1969	0.031675	0.113594	0.091227
1970	0.040995	0.166194	0.183965

Note: p and w are the substitution parameters in the Sato CES utility function.

Table D-3 (Continued)

Year	w=-0.084352		
	p=-0.911384	p=-0.820797	p=-1.133682
1938	0.086827	0.079009	0.100934
1939	0.052695	0.047067	0.064842
1940	0.050540	0.045089	0.062478
1941	0.088081	0.080359	0.101450
1942	0.004767	0.004728	0.004857
1943	0.678795	0.468699	1.244930
1944	0.695426	0.482490	1.268250
1945	0.082042	0.073956	0.099285
1946	0.067098	0.060322	0.081096
1947	0.036260	0.032184	0.045864
1948	0.052993	0.047328	0.065648
1949	0.093838	0.084751	0.113496
1950	0.045550	0.040604	0.057010
1951	0.041029	0.036530	0.051599
1952	0.064288	0.057665	0.078560
1953	0.081820	0.073891	0.097916
1954	0.055766	0.049857	0.068821
1955	0.066368	0.059581	0.080874
1956	0.046001	0.041028	0.057557
1957	0.067364	0.060470	0.082440
1958	0.054965	0.049126	0.067913
1959	0.070457	0.063320	0.085778
1960	0.080664	0.072809	0.096707
1961	0.073676	0.066325	0.089033
1962	0.070348	0.063226	0.085579
1963	0.083274	0.075183	0.099917
1964	0.083557	0.075365	0.100876
1965	0.085557	0.077404	0.101785
1966	0.099184	0.090093	0.116790
1967	0.092413	0.083848	0.108983
1968	0.110146	0.100686	0.127043
1969	0.098004	0.089123	0.114802
1970	0.132616	0.122802	0.147067

Table D-3 (Continued)

Year	w=-0.635495		
	p=-1.224270	p=-1.261184	p=-1.133682
1938	0.215878	0.224355	0.193284
1939	0.131837	0.138751	0.114110
1940	0.111864	0.118610	0.094624
1941	0.260780	0.269204	0.238194
1942	0.605051	0.604972	0.605246
1943	0.226933	0.316359	0.014998
1944	0.257142	0.347734	0.042346
1945	0.364484	0.356303	0.385644
1946	0.106699	0.114482	0.086509
1947	0.000036	0.005383	0.013377
1948	0.008688	0.001880	0.026026
1949	0.979950	0.971662	1.001370
1950	0.097175	0.091089	0.112525
1951	0.098086	0.092419	0.112321
1952	0.021253	0.013707	0.040657
1953	0.038143	0.029764	0.060026
1954	0.000418	0.007437	0.017507
1955	0.016290	0.008621	0.036054
1956	0.137130	0.131061	0.152427
1957	0.183899	0.176327	0.203324
1958	0.004644	0.002312	0.022394
1959	0.123181	0.115379	0.143287
1960	0.045608	0.037277	0.067335
1961	0.032116	0.024082	0.052948
1962	0.096147	0.088328	0.116314
1963	0.132196	0.123808	0.154056
1964	0.330444	0.322172	0.351886
1965	0.015179	0.006658	0.037532
1966	0.129534	0.120654	0.153024
1967	0.011893	0.003167	0.034938
1968	0.011439	0.002397	0.035741
1969	0.015761	0.006902	0.039277
1970	0.101385	0.110262	0.076725

Table D-4

DISTRIBUTION OF DIFFERENCES BETWEEN THE TRUE CES FUNCTION AND ITS ALTERNATIVE FUNCTIONAL FORM WITH 10% DEPRECIATION RATE

Class	p=-0.635495	p=-0.243920	p=-0.392086	p=-1.04008
0.00 - 10.00	-	18.2 %	6.1 %	-
10.01 - 20.00	-	81.8 %	15.2 %	-
20.01 - 30.00	-	-	18.2 %	-
30.01 - 40.00	-	-	30.3 %	-
40.01 - 50.00	6.0 %	-	24.2 %	-
50.01 - 100.00	15.2 %	-	6.0 %	-
100.01 - 200.00	30.3 %	-	-	-
201.00 - 300.00	33.3 %	-	-	-
301.00 and over	15.2 %	-	-	100.0 %
The Smallest Number	43.1137	6.38672	9.82996	332.116
The Largest Number	391.333	17.9065	50.5313	18791.7

Table D-5

DISTRIBUTION OF DIFFERENCES BETWEEN THE TRUE UZAWA FUNCTION AND ITS ALTERNATIVE FUNCTIONAL FORM WITH 10% DEPRECIATION RATE

Class	p=-0.938309	p=-0.772910	p=-1.280698
0.0000 - 0.0200	30.3 %	66.7 %	15.2 %
0.0201 - 0.0300	51.5 %	24.2 %	78.7 %
0.0301 - 0.0400	12.1 %	3.0 %	-
0.0401 and over	6.1 %	6.1 %	6.1 %
The Smallest Number	0.000053	0.000035	0.000100
The Largest Number	0.972984	0.562711	1.919780

Table D-6

DISTRIBUTION OF DIFFERENCES BETWEEN THE TRUE SATO FUNCTION AND ITS ALTERNATIVE FUNCTIONAL FORM WITH 10% DEPRECIATION RATE

Class	w=-0.243920		
	p=0.168066	p=-0.555706	p=-1.944176
0.0000 - 0.0200	45.5 %	3.0 %	21.2 %
0.0201 - 0.0300	27.3 %	-	3.0 %
0.0301 - 0.0400	12.1 %	3.0 %	9.1 %
0.0401 and over	15.1 %	94.0 %	66.7 %
The Smallest Number	0.007248	0.016669	0.005691
The Largest Number	0.156246	0.281368	5.431580

Class	w = -0.084352		
	p=-0.911384	p=-0.820797	p=-1.133682
0.0000 - 0.0200	3.0 %	3.0 %	3.0 %
0.0201 - 0.0300	-	-	-
0.0301 - 0.0400	3.0 %	6.0 %	-
0.0401 and over	94.0 %	91.0 %	97.0 %
The Smallest Number	0.004767	0.004728	0.004857
The Largest Number	0.695426	0.482490	1.268560

Class	w=-0.635495		
	p=-1.224270	p=-1.261184	p=-1.133682
0.0000 - 0.0200	27.0 %	30.0 %	12.0 %
0.0201 - 0.0300	3.0 %	6.0 %	6.0 %
0.0301 - 0.0400	6.0 %	3.0 %	18.0 %
0.0401 and over	64.0 %	61.0 %	64.0 %
The Smallest Number	0.000036	0.001880	0.001370
The Largest Number	0.605051	0.604972	0.605246