SIMPLE SLOPE DEFLECTION EQUATIONS FOR SYMMETRICAL ARCH STRUCTURES

BY

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REPORT AND ABSTRACT APPROVED:

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Report Adviser

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PREFACE

The purpose of this report is to present a new derivation of slope deflection equations for symmetrical, curved members.

The first comprehensive study of structures with curved members by means of slope deflection equations was prepared by K. Beyer[1]. The method of elastic centers was used as a basis of investigation.

In this country the slope deflection equations for curved members were derived by means of column analogy by K. Fowler[2].

The idea of a fictitious fixed end beam was applied in this paper. The curved member is replaced by a straight, fictitious member, passing through the center of the real member.

The slope deflection equation of the curved member was derived in the form

$$M_{AB} \text{ (Curved member)} = M_{AB} \text{ (Straight member)} \pm He,$$

where (H) is the horizontal reaction of the curved member and (e) is the vertical ordinate of its centroid. All investigations are general and are applicable to any structure containing symmetrical, curved or bent members.

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(1) K. Beyer, Die Statik Im Stahlbetonban, Berlin 1933.

(2) K. Fowler, Slope Deflection equations for Curved members, Proc. ASCE, March 1950.

The illustrative examples selected are limited to high parabolic arches of

$$I = I_o \sec \alpha$$

where

I = Moment of inertia of beam at any section,

I_o = Moment of inertia of beam at center line section,

and α = Slope of the tangent line at any point of the beam.

The writer's decision to derive simple slope deflection equations for symmetrical curved members came about as a result of seminar courses taken under Professor Jan Joseph Tuma.

Grateful acknowledgement is due to Professor Tuma for his advice and persistent encouragement as well as for the general procedure laid down in this paper.

Spring, 1955

Stillwater, Oklahoma

R. G. Ungson Jr.

R. G. UNGSON, JR.

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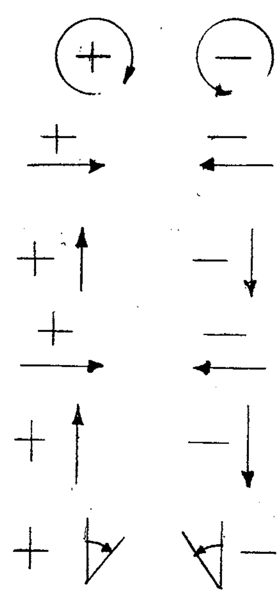


NOMENCLATURE

M_{AB} = Left end moment of arch, AB.

M_{BA} = Right end moment of arch, AB.

FM_{AB} = Fixed end moment of the left end of arch (AB).

FM_{BA} = Fixed end moment of the right end of arch (AB).

GM_{AB} = Left end moment of fictitious beam (A'B').

GM_{BA} = Right end moment of fictitious beam (A'B').

H = Total thrust of the arch (AB).

R_{AY} = Vertical reaction of left end of arch (AB).

R_{AX} = Horizontal reaction of left end of arch (AB).

R_{BY} = Vertical reaction of right end of arch (AB).

R_{BX} = Horizontal reaction of right end of arch (AB).

e = Distance of centroid of arch from its base line (AB).

θ_A = Angular rotation of left end A.

θ_B = Angular rotation of right end B.

Δ_{AY} = Vertical displacement of end A.

Δ_{BY} = Vertical displacement of end B.

Δ_{AX} = Horizontal displacement of end A.

Δ_{BX} = Horizontal displacement of end B.

ψ_Y = Angular rotation of the beam.

$K_{AB}^{(R)}$ = Rotational stiffness factor of end A.

$K_{BA}^{(R)}$ = Rotational stiffness factor of end B.

$K_{AB}^{(C)}$ = Carry over factor from A to B.

$K_{BA}^{(C)}$ = Carry Over factor from B to A.

Δ_Y = Relative vertical displacement between A and B.

Δ_X = Relative horizontal displacement between A and B.

NOMENCLATURE (Cont'd)

K'_{AB} = Modified stiffness factor for a symmetrical beam with a symmetrical load.

K''_{AB} = Modified stiffness factor for a symmetrical beam with an antisymmetrical load.

K'''_{BA} = Modified stiffness factor for a beam with a hinged end.

L = Span of the arch (AB).

BM = Bending moment at any point on a simple beam.

SM = Statical moment of load about any point.

T_{AB} = Moment of the bending moment diagram about the left end A.

T_{BA} = Moment of the bending moment diagram about the right end B.

f = Height of the arch.

I = Moment of inertia at any section of the arch.

α = Angle which the tangent to the arch makes with the horizontal.

CONSTANTS:

$$C_1 = \int_0^L \frac{X^2 ds}{L^2 EI} \qquad C_4 = \int_0^L \frac{Y^2 ds}{EI}$$

$$C_2 = \int_0^L \frac{XX' ds}{L^2 EI} \qquad C_5 = \int_0^L \frac{BM^{(A)}\, Y ds}{EI}$$

$$C_3 = \int_0^L \frac{X'^2 ds}{L^2 EI} \qquad C_6 = \int_0^L \frac{BM^{(B)}\, Y ds}{EI}$$

$$X' = L - X$$

PART I

DERIVATIONS

PART I

DERIVATIONS

1. Fixed End Moments:

A fixed end symmetrical curved bar, loaded by a general system of loads will be considered (Fig. 1). The reactive elements may be related to the axis $\overline{X}$ (R^1_{AX}, R^1_{AY}, GM^1_{AB}, R^1_{BX}, R^1_{BY}, GM^1_{BA}), which passes through the center of the arch, or to the axis $\overline{AB}$ (R^1_{AX}, R^1_{AY}, M^1_{AB}, R^1_{BX}, R^1_{BY}, M^1_{BA}). The first relation has been proven to be more appropriate and will be used.

C. G. is the center of gravity of the arch.

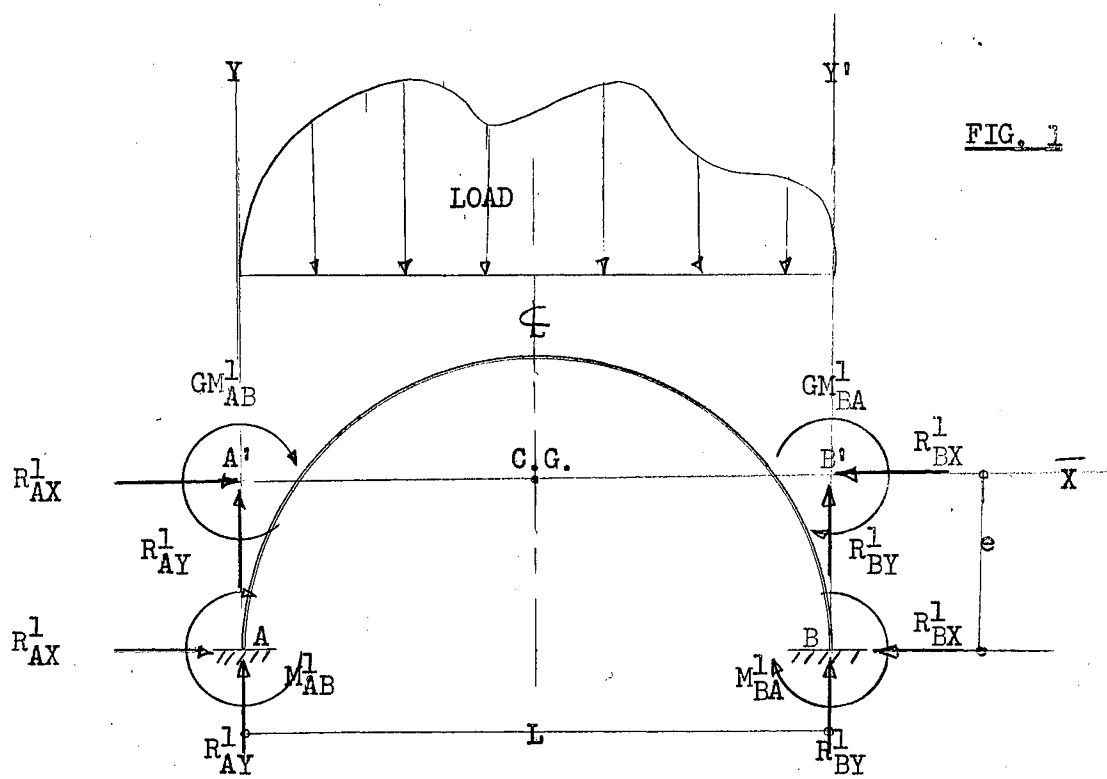


FIG. 1

Thus horizontal reactions are $$R^1_{AX} = R^1_{BX} = H^1 . \qquad (1)$$

The vertical reactions are:

$$R^1_{AY} = BR^1_{AY} - \frac{M^1_{AB} + M^1_{BA}}{L} , \qquad (2)$$

and

$$R^1_{BY} = BR^1_{BY} + \frac{M^1_{AB} + M^1_{BA}}{L} , \qquad (3)$$

where BR^1_{AY} and BR^1_{BY} are the vertical reactions of a simple beam $\overline{AB}$.

And the fixed end moments are

$$GM^1_{AB} = M^1_{AB} + H^1 e , \qquad (4)$$

and

$$GM^1_{BA} = M^1_{BA} - H^1 e , \qquad (5)$$

where e is the ordinate of the centroid of the bar.

The normal force at any right section of the bar is

$$N^{(A)}_{X=0 \longrightarrow L} = + \sum_0^X F_X \operatorname{Cos} \varphi + \sum_0^X F_Y \operatorname{Sin} \varphi \qquad (6)$$

where φ is the angle the tangent line to the bar at any point makes with the horizontal.

The shearing force at any right section of the bar is

$$T^{(A)}_{X=0 \longrightarrow L} = - \sum_0^X F_X \operatorname{Sin} \varphi + \sum_0^X F_Y \operatorname{Cos} \varphi \qquad (7)$$

and the bending moment at any right section of the bar is

$$M^{(A)}_{X=0 \longrightarrow L} = GM^1_{AB} - \frac{GM^1_{AB} + GM^1_{BA}}{L} X + BM^{(A)}_X - H^1 Y \qquad (8)$$

Or, taken from the end B

$$M^{(B)}_{X=0 \longrightarrow L} = -GM^{1}_{BA} + \frac{GM^{1}_{AB} + GM^{1}_{BA}}{L} X' + BM^{(B)}_{X} - H^{1}Y \, , \qquad (9)$$

where $BM^{(A)}_{X}$ and $BM^{(B)}_{X'}$ are the bending moments of a simple beam $\overline{AB}$.

The virtual work equations are

$$\int_0^L \frac{N \, ds}{L \, AE} + \int_0^L \frac{V \, ds}{L \, AG} + \int_0^L \frac{MX' \, ds}{L \, EI} = 0 = \frac{\partial U}{\partial GM^{1}_{AB}} \, , \qquad (10)$$

$$\int_0^L \frac{N \, ds}{L \, AE} + \int_0^L \frac{V \, ds}{J \, AG} + \int_0^L \frac{MX \, ds}{L \, EI} = 0 = \frac{\partial U}{\partial GM^{1}_{BA}} \, , \qquad (11)$$

and

$$\int_0^L \frac{N \, ds}{L \, AE} + \int_0^L \frac{V \, ds}{L \, AG} + \int_0^L \frac{MY \, ds}{L \, EI} = 0 = \frac{\partial U}{\partial H^{1}} \, , \qquad (12)$$

where $X' = (L - X)$.

Considering the normal and the shearing deformations to be small and expressing equations (10, 11, and 12) in terms of equations (8 and 9), the virtual work equations become

$$GM^{1}_{AB} \int_0^L \frac{X'^2 \, ds}{L^2 \, EI} - GM^{1}_{BA} \int_0^L \frac{XX' \, ds}{L^2 \, EI} + \int_0^L \frac{BM^{(A)}_{X} X' \, ds}{L \, EI} = 0, \qquad (13)$$

$$GM^{1}_{AB} \int_0^L \frac{X'X \, ds}{L^2 \, EI} - GM^{1}_{BA} \int_0^L \frac{X^2 \, ds}{L^2 \, EI} + \int_0^L \frac{BM^{(A)}_{X'} X \, ds}{L \, EI} = 0, \qquad (14)$$

and

$$\int_0^L \frac{BM^{(A)}_{X} \, Y \, ds}{EI} - H^{1} \int_0^L \frac{Y^2 \, ds}{EI} = 0 \, . \qquad (15)$$

Denoting

$$\left.\begin{aligned} C_1 &= \int_0^L \frac{X^2\,ds}{L^2\,EI}, & C_3 &= \int_0^L \frac{X'^2 ds}{L^2\,EI}, \\ C_2 &= \int_0^L \frac{X'X\,ds}{L^2\,EI}, & C_4 &= \int_0^L \frac{Y^2\,ds}{EI}, \end{aligned}\right\} \quad (16)$$

$$C_5 = \int_0^L \frac{BM^{(A)}Yds}{EI}, \qquad C_6 = \int_0^L \frac{BM^{(B)}Yds}{EI}, \qquad (17)$$

$$T_{AB} = \int_0^L \frac{BM^{(A)}Xds}{L\,EI}, \qquad T_{BA} = \int_0^L \frac{BM^{(A)}X'ds}{L\,EI}, \qquad (18)$$

the deformation equations become

$$C_3 GM^1_{AB} - C_2 GM^1_{BA} + T_{BA} = 0\,, \qquad (19)$$

$$C_2 GM^1_{AB} - C_1 GM^1_{BA} + T_{AB} = 0\,, \qquad (20)$$

and

$$C_4\,H^1 - C_5 = 0\,. \qquad (21)$$

Solving equations (19, 20, and 21) simultaneously and denoting

$$C_1C_3 - C_2C_2 = N\,, \qquad (22)$$

the reactive elements related to the centroidal axis become

$$GM^1_{AB} = \frac{C_2T_{AB} - C_1T_{BA}}{N}\,, \qquad (23)$$

$$GM^1_{BA} = \frac{C_2T_{BA} - C_3T_{AB}}{N}\,, \qquad (24)$$

and

$$H^1 = \frac{C_5}{C_4} \quad \text{or} \quad \frac{C_6}{C_4}\,. \qquad (25)$$

Finally the reactive elements related to AB become

$$M^1_{AB} = \frac{C_2 T_{AB} - C_1 T_{BA}}{N} + \frac{C_5}{C_4} e \; , \tag{26}$$

$$M^1_{BA} = \frac{C_2 T_{BA} - C_3 T_{AB}}{N} - \frac{C_6}{C_4} e \; , \tag{27}$$

and

$$R^1_{AX} = R^1_{BX} = \frac{C_5}{C_4} = \frac{C_6}{C_4} \; . \tag{28}$$

2. <u>End Moments due to θ_A:</u>

Releasing the fixed end A and producing an angular deformation θ_A (Fig. 2) and using the same procedure as in the previous derivation, the deformation equations similar to equations (19, 20, & 21) become

$$C_3 GM^{(2)}_{AB} - C_2 GM^{(2)}_{BA} = \theta_A \; , \tag{29}$$

$$C_2 GM^{(2)}_{AB} - C_1 GM^{(2)}_{BA} = 0 \; , \tag{30}$$

$$C_4 H^{(2)} - e\theta_A = 0 \; . \tag{31}$$

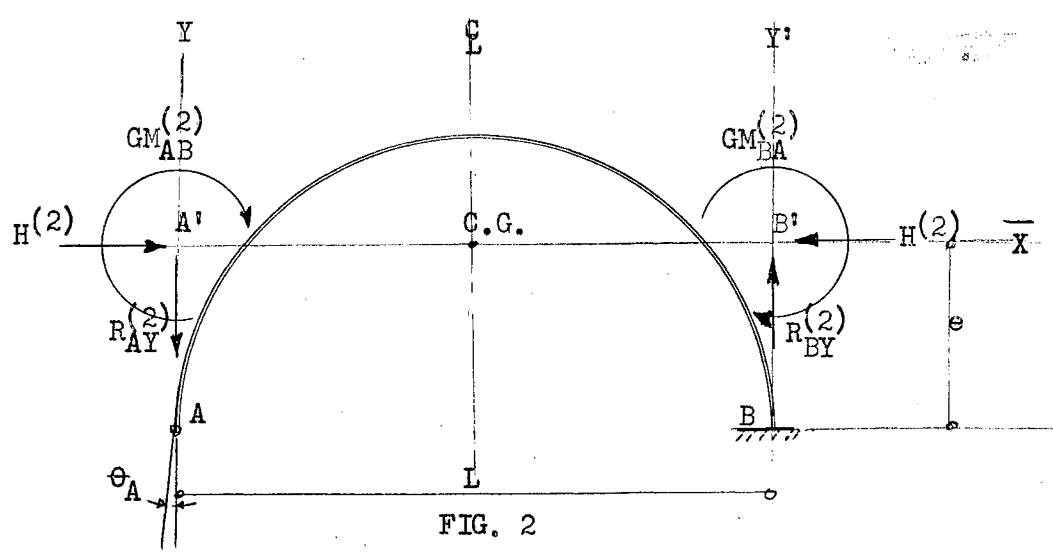


FIG. 2

The reactive elements related to the axis $\overline{X}$ become

$$GM_{AB}^{(2)} = \frac{\theta_A C_1}{N} , \tag{32}$$

$$GM_{BA}^{(2)} = \frac{\theta_A C_2}{N} , \tag{33}$$

and

$$H^{(2)} = \frac{\theta_A e}{C_4} . \tag{34}$$

Finally, the reactive elements related to $\overline{AB}$ are

$$M_{AB}^{(2)} = \frac{\theta_A C_1}{N} + \frac{\theta_A e^2}{C_4} , \tag{35}$$

$$M_{BA}^{(2)} = \frac{\theta_A C_2}{N} - \frac{\theta_A e^2}{C_4} , \tag{36}$$

and

$$R_{AX}^{(2)} = R_{BX}^{(2)} = \frac{\theta_A e}{C_4} . \tag{37}$$

3. End Moments due to θ_B:

Releasing the fixed end B, but holding the end A fixed (Fig. 3) and producing an angular displacement θ_B at B; from cyclosymmetry, the reactive elements related to axis $\overline{X}$ become

$$GM_{AB}^{(3)} = \frac{C_2 \theta_B}{N} , \tag{38}$$

$$GM_{BA}^{(3)} = \frac{C_3 \theta_B}{N} , \tag{39}$$

and $$H^{(3)} = \frac{e\theta_B}{C_4} . \tag{40}$$

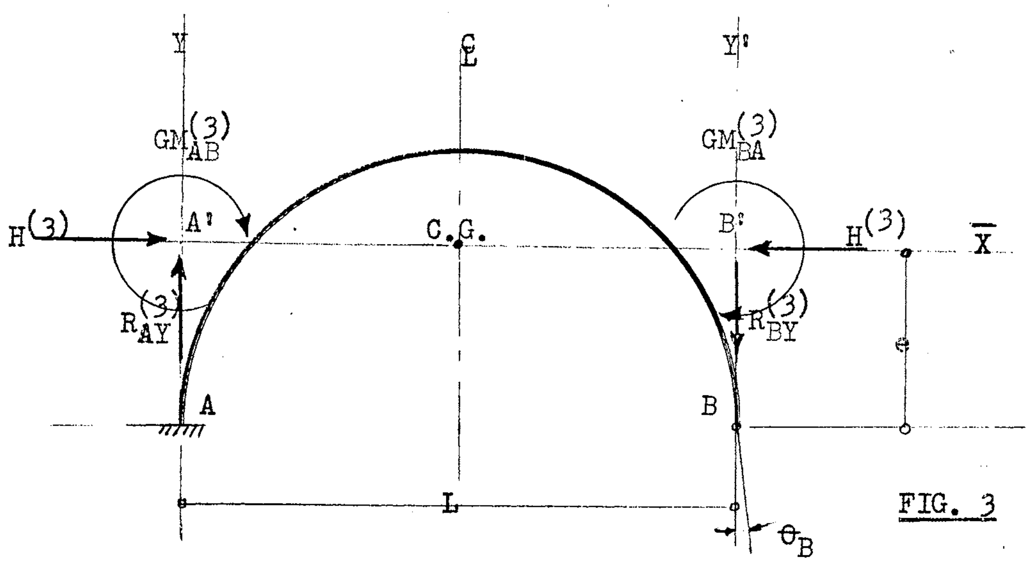


FIG. 3

Finally the reactive elements related to the axis $\overline{AB}$ are

$$M_{AB}^{(3)} = \frac{C_2\theta_B}{N} - \frac{e^2\theta_B}{C_4} , \tag{41}$$

$$M_{BA}^{(3)} = \frac{C_3\theta_B}{N} + \frac{e^2\theta_B}{C_4} , \tag{42}$$

and $$R_{AX}^{(3)} = R_{BX}^{(3)} = -\frac{e\ \theta_B}{C_4} . \tag{43}$$

4. End Moments due to Δ_{AX} and Δ_{BX}:

Locking the ends A and B against rotation, but permitting two independent horizontal translations (Fig. 4) and using the same procedure as in the previous cases (1 to 3), the following deformation equations may be derived

$$C_3 GM_{AB}^{(4)} - C_2 GM_{BA}^{(4)} = 0 \,, \tag{44}$$

$$C_2 GM_{AB}^{(4)} - C_1 GM_{BA}^{(4)} = 0 \,, \tag{45}$$

and

$$C_4 H^{(4)} = \Delta_{AX} - \Delta_{BX} = \Delta_X \,. \tag{46}$$

The reactive elements related to the axis $\overline{X}$ become

$$GM_{AB}^{(4)} = 0 \,, \qquad GM_{BA}^{(4)} = 0 \,, \tag{47)\&(48}$$

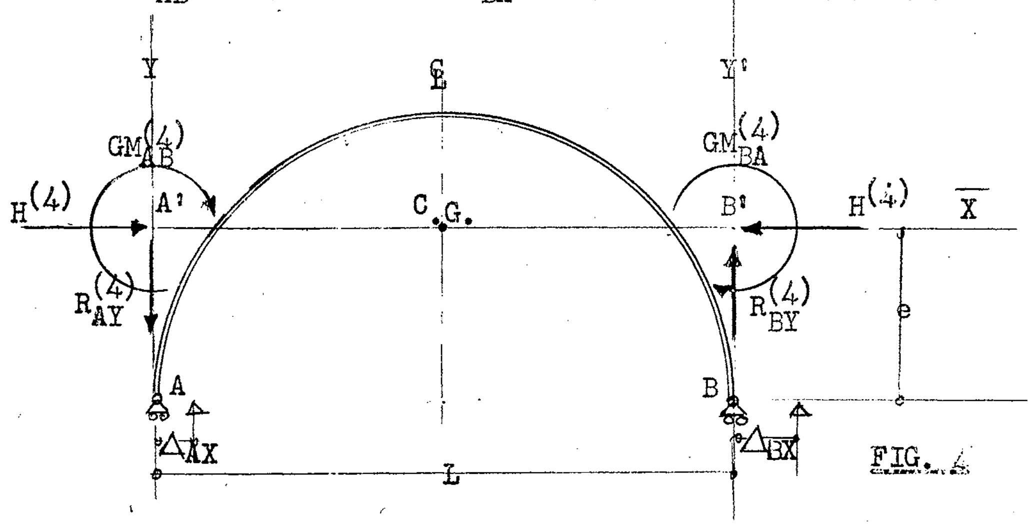


FIG. 4

and

$$H^{(4)} = \frac{\Delta_X}{C_4} \,. \tag{49}$$

Finally the reactive elements related to the axis $\overline{AB}$ are

$$M_{AB}^{(4)} = \frac{\Delta_X e}{C_4} \,, \tag{50}$$

$$M_{BA}^{(4)} = -\frac{\Delta_X e}{C_4} \,, \tag{51}$$

and

$$R_{AX}^{(4)} = R_{BX}^{(4)} = \frac{\Delta_X}{C_4} \,. \tag{52}$$

5. End Moments due to Δ_{AY} and Δ_{BY}:

Locking the ends A and B against rotation and permitting two independent vertical translations (Fig. 5) and using the same procedure as in the previous cases (1 to 4), the following deformation equations may be derived

$$C_3 GM_{AB}^{(5)} - C_2 GM_{BA}^{(5)} = \frac{\Delta_{BY} - \Delta_{AY}}{L} = \frac{\Delta_Y}{L} , \qquad (53)$$

$$C_2 GM_{AB}^{(5)} - C_1 GM_{BA}^{(5)} = - \frac{\Delta_{BY} - \Delta_{AY}}{L} = - \frac{\Delta_Y}{L} , \qquad (54)$$

and

$$- C_4 H^{(5)} = 0 . \qquad (55)$$

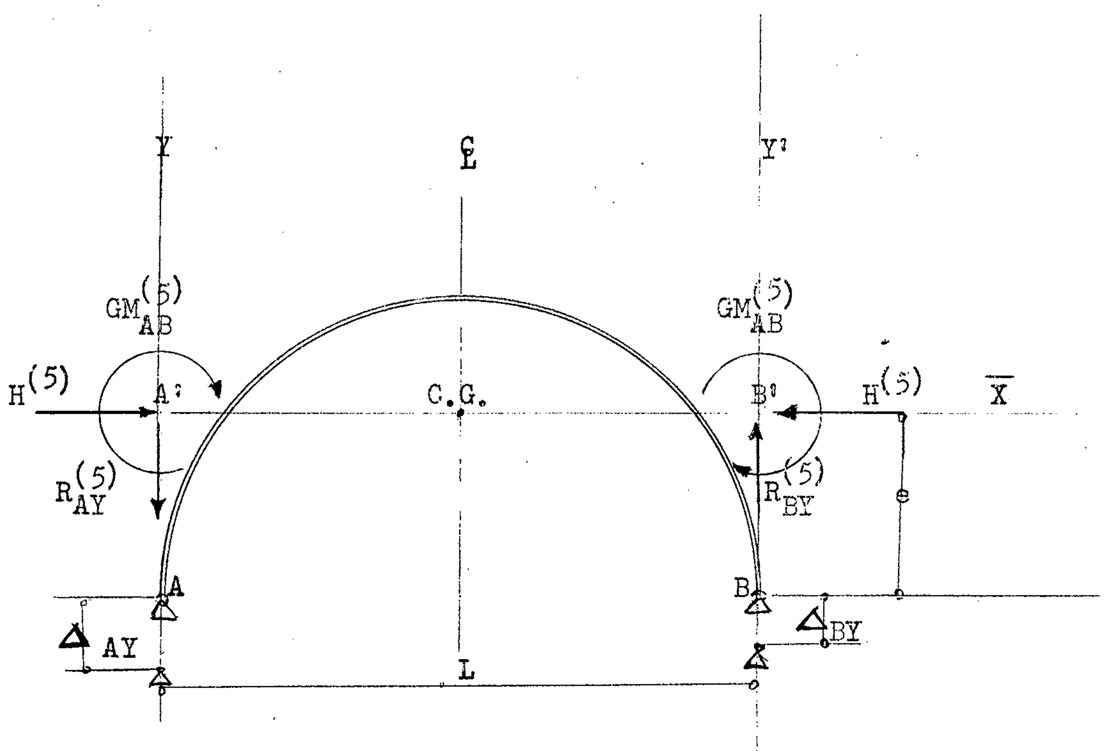


FIG. 5

The reactive elements related to the $\overline{X}$-axis are

$$GM_{AB}^{(5)} = \frac{C_1 + C_2}{N} \frac{\Delta_Y}{L} , \quad (56)$$

$$GM_{BA}^{(5)} = \frac{C_3 + C_2}{N} \frac{\Delta_Y}{L} , \quad (57)$$

and

$$H^{(5)} = 0 . \quad (58)$$

Finally the reactive elements related to the $\overline{AB}$ axis become

$$M_{AB}^{(5)} = \frac{C_1 - C_2}{N} \psi_Y , \quad (59)$$

$$M_{BA}^{(5)} = \frac{C_3 - C_2}{N} \psi_Y , \quad (60)$$

and

$$R_{AX}^{(5)} = R_{BX}^{(5)} = 0 , \quad (61)$$

in which

$$\psi_Y = \frac{\Delta_Y}{L} . \quad (62)$$

6. Slope Deflection Equations:

Superimposing the results of equations (26, 27, 28, 35, 36, 37, 41, 42, 43, 50, 51, 52, 59, 60, and 61), the final slope deflection equations become

$$M_{AB} = \frac{C_1}{N}\theta_A + \frac{C_2}{N}\theta_B + \frac{C_1 + C_2}{N}\psi_Y + GM_{AB} + He, \tag{63}$$

$$M_{BA} = \frac{C_3}{N}\theta_B + \frac{C_2}{N}\theta_A + \frac{C_3 + C_2}{N}\psi_Y + GM_{BA} - He, \tag{64}$$

and

$$H = \frac{e(\theta_A - \theta_B) + \Delta_X + C_5}{C_4}. \tag{65}$$

Considering the curved member to be a high parabolic arch of

$$I = I_o \sec \alpha , \tag{66}$$

then

$$M_{AB} = 4EK\theta_A + 2EK\theta_B + 6EK\psi + GM_{AB} + He, \tag{67}$$

$$M_{BA} = 4EK\theta_B + 2EK\theta_A + 6EK\psi + GM_{BA} - He, \tag{68}$$

and

$$H = \frac{5EK(\theta_A - \theta_B)}{e} + \frac{C_5}{C_4} + \frac{5EK\Delta_X}{e^2}. \tag{69}$$

The similarity of these equations with the regular slope deflection equations for straight beams is apparent.

In the equations above, the relative stiffness (K) of the beam is used where

$$K = \frac{I}{L}. \tag{70}$$

7. Conclusion:

By relating the reactive forces and moments acting on any symmetrical arch to the axis $\overline{X}$ passing through the centroid of the arch, the slope deflection equations can be easily derived.

These equations have two simple parts. The first part is the slope deflection equation of a simple beam (A'B') with a length equal to the span of the arch (L) and loaded with the same load as that of the arch.

The second part is a correction to the end moments given by the first part. This correction (He) is due to the total thrust (H) of the arch multiplied by the vertical distance of the centroid (e) from the base line (AB).

These slope deflection equations make the solution of symmetrical arch structures almost as simple as that of a straight beam.

PART II

TABLES

PART II

TABLES

From the formulas derived in Part I, the following tables are obtained for high parabolic arch structures of

$$I = I_o \sec \alpha$$

where $I =$ the moment of inertia of the arch at any section,

$I_o =$ the moment of inertia at the center line section,

$\alpha =$ the slope of the tangent line to the arch.

8. Table of Constants:

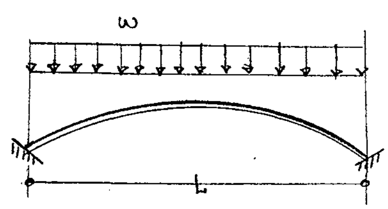
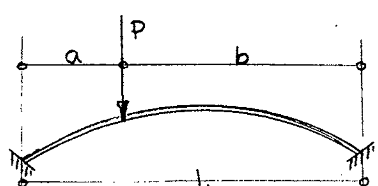
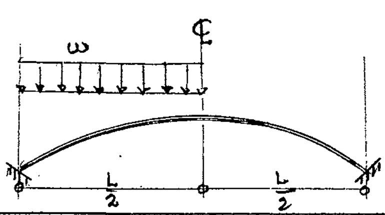
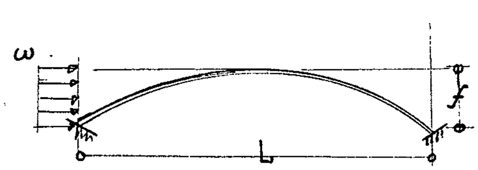
$C_1 = \int_0^L \frac{X^2 ds}{L^2 EI}$	$\frac{L}{3EI}$
$C_2 = \int_0^L \frac{XX' ds}{L^2 EI}$	$\frac{L}{6EI}$
$C_3 = \int_0^L \frac{X'^2 ds}{L^2 EI}$	$\frac{L}{3EI}$
$C_4 = \int_0^L \frac{Y^2 ds}{EI}$	$\frac{4f^2 L}{45EI}$
$N = C_1 C_3 - C_2 C_2$	$\frac{L^2}{12 E^2 I^2}$

In all the tables, the following dimensions are used:

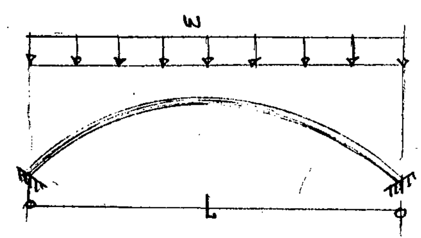
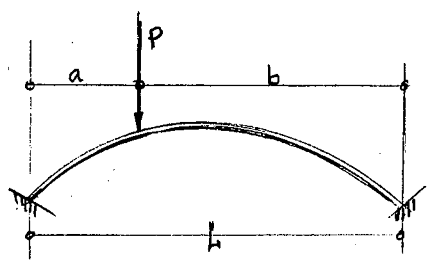
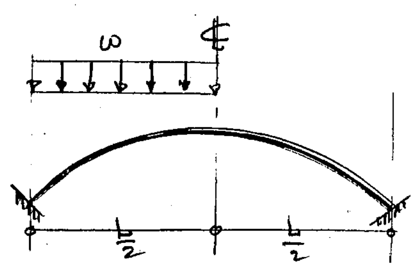
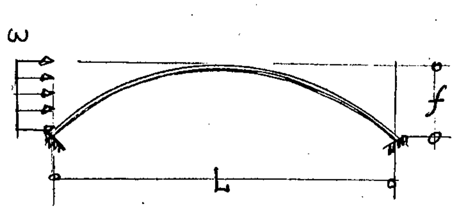
$f =$ the height of the arch,

and $e = \frac{2f}{3}$, is the centroidal height of the arch.

9. Table of Fixed end Moments:

LOADING	GM_{AB} $\frac{C_2 T_{AB} - C_1 T_{BA}}{N}$	GM_{BA} $\frac{C_2 T_{BA} - C_3 T_{AB}}{N}$
	$-\frac{wL^2}{12}$	$+\frac{wL^2}{12}$
	$-\frac{Pab^2}{L^2}$	$+\frac{Pa^2b}{L^2}$
	$-\frac{11wL^2}{192}$	$+\frac{5wL^2}{192}$
	$+\frac{3we^2}{5}$	$+\frac{69we^2}{40}$

10. Table for Thrust Due to the Load:

<table>
<tr><th rowspan="2">L O A D I N G</th><th>$H^{(1)}$</th></tr>
<tr><th>$\frac{C_5}{C_4}$</th></tr>
<tr><td>
</td><td>$+\frac{wL^2}{12e}$</td></tr>
<tr><td>
</td><td>$+\frac{5Pa^2b^2}{2eL^3}$</td></tr>
<tr><td>
</td><td>$+\frac{wL^2}{24e}$</td></tr>
<tr><td>
</td><td>$-\frac{4wf}{7}, -\frac{3wf}{7}$</td></tr>
</table>

11. Table of Thrust Due to Angular Rotations and Lateral Displacements:

$H^{(2)} = \frac{e\theta_A}{C_4} = \frac{5EI\theta_A}{eL}$	$\frac{5EK\theta_A}{e}$
$H^{(3)} = -\frac{e\theta_B}{C_4} = -\frac{5EI\theta_B}{eL}$	$-\frac{5EK\theta_B}{e}$
$H^{(4)} = \frac{\triangle_Y}{C_4} = \frac{0}{C_4}$	0
$H^{(5)} = \frac{\triangle_X}{C_4} = \frac{5EI\triangle_X}{e^2L}$	$\frac{5EK\triangle_X}{e^2}$

PART III

ILLUSTRATIVE EXAMPLES

PART III

ILLUSTRATIVE EXAMPLES

12. Illustrative Example Number 1

A three span continuous beam, composed of three identical, symmetrical, parabolic, high arches, loaded and supported as shown in figure 6, will be analyzed by the slope deflection method.

The variation of the moment of inertia of the arch section is

$$I = I_o \sec \alpha .$$

The modulus of elasticity is constant for all arches and will be denoted by (E).

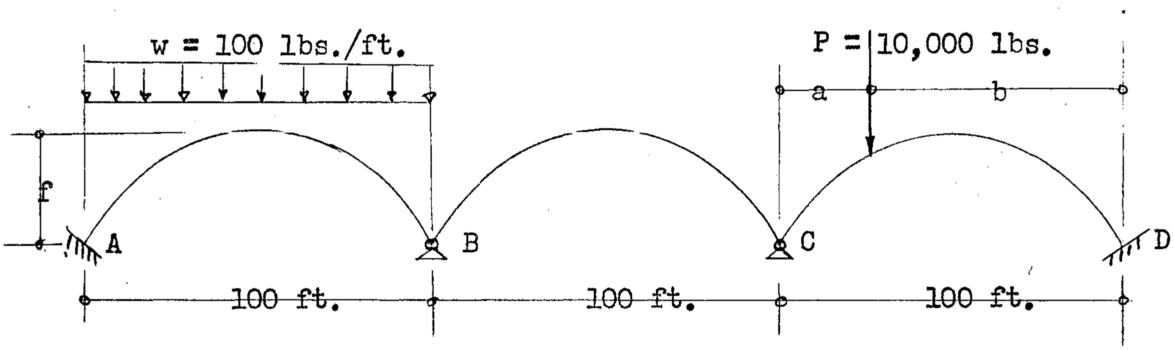


FIG. 6

Solution:

a. Stiffness factor $K_1 = K_2 = K_3 = \frac{I}{L} = \frac{1000}{100} = 10 \text{ in}^3.$

b. Fixed end moments

$$FM_{AB} = GM_{AB} + H_{AB}e = -\frac{wL_{AB}^2}{12} + \frac{wL_{AB}^2}{12} = 0 \ ,$$

$$FM_{BA} = GM_{BA} - H_{AB}e = +\frac{wL_{AB}^2}{12} - \frac{wL_{AB}^2}{12} = 0 \ ,$$

$$FM_{CD} = GM_{CD} + H_{CD}e = -\frac{Pab^2}{L^2} + \frac{5Pa^2b^2}{2L^3} = -203{,}000 \ ,$$

$$FM_{DC} = GM_{DC} - H_{CD}e = +\frac{Pa^2b}{L^2} - \frac{5Pa^2b^2}{2L^3} = +\ 18{,}300 \ .$$

c. Deformations

$$\theta_A = 0, \quad \theta_B = ?, \quad \theta_C = ?, \quad \theta_D = 0,$$

$$\Delta_{AX} = 0, \quad \Delta_{BX} = 0, \quad \Delta_{CX} = 0, \quad \Delta_{DX} = 0,$$

$$\Delta_{AY} = 0, \quad \Delta_{BY} = 0, \quad \Delta_{CY} = 0, \quad \Delta_{DY} = 0,$$

d. Deformation equations

$$M_{AB} = 2EK\theta_B - 5EK\theta_B = -\ 3EK\theta_B \ , \qquad (71)$$

$$M_{BA} = 4EK\theta_B + 5EK\theta_B = \ 9EK\theta_B \ , \qquad (72)$$

$$M_{BC} = 4EK\theta_B + 2EK\theta_C + 5EK\theta_B - 5EK\theta_C$$

$$= 9EK\theta_B - 3EK\theta_C \ , \qquad (73)$$

$$M_{CB} = 4EK\theta_C + 2EK\theta_B - 5EK\theta_B + 5EK\theta_C$$

$$= -3EK\theta_B + 9EK\theta_C \ , \qquad (74)$$

$$M_{CD} = 4EK\theta_C + 5EK\theta_C - 203{,}000$$

$$= 9EK\theta_C - 203{,}000 \ , \qquad (75)$$

$$M_{DC} = 2EK\theta_C - 5EK\theta_C + 18{,}300$$

$$= -\ 3EK\theta_C + 18{,}300 \ . \qquad (76)$$

e. Equations of equilibrium

$$\Sigma M_B = 0 \equiv M_{BA} + M_{BC} = 0 \quad ,$$

$$180\ E\theta_B - 30\ E\theta_C = 0 \quad , \tag{77}$$

$$\Sigma M_C = 0 \equiv M_{CB} + M_{CD} = 0 \quad ,$$

$$-30\ E\theta_B + 180 E\theta_C - 203{,}000 = 0 \quad . \tag{78}$$

f. Solving equations (77 & 78) simultaneously

$$\theta_B = \frac{193.5}{E}, \quad \text{and} \quad \theta_C = \frac{1{,}161}{E} \quad . \tag{79}$$

g. Deformation equations in terms of values of θ_B and θ_C

$$M_{AB} = -\ 5{,}800$$

$$M_{BA} = +\ 17{,}415$$

$$M_{BC} = -\ 17{,}415$$

$$M_{CB} = +\ 98{,}685$$

$$M_{CD} = -\ 98{,}510$$

$$M_{DC} = -\ 16{,}530$$

These values are the end moments at points A, B, C, and D. All moments are in feet pounds.

13. Illustrative Example Number 2

A frame with a parabolic top member and two side bents as shown in figure 7, will be analyzed. The parabolic top member is of the type

$$I = I_o \sec \alpha .$$

The modulus of elasticity (E) is constant for the whole structure. The slope deflection method will be used.

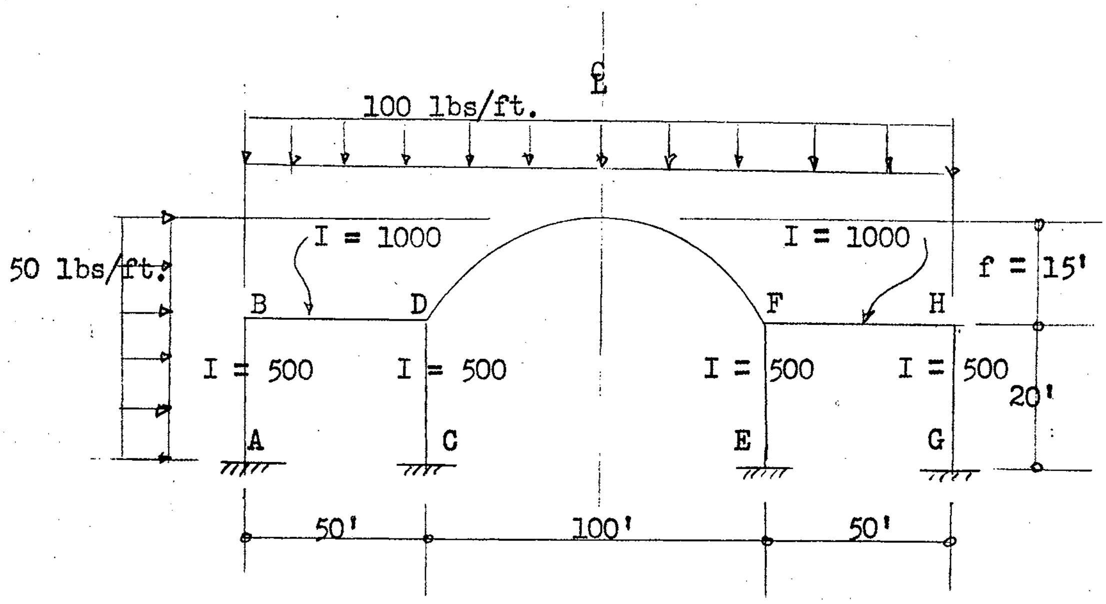


FIG. 7

This problem can be solved very conveniently by resolving the frame (Fig. 7) into (I) a symmetrical system, (Fig. 8) and (II) an antisymmetrical system (Fig. 9), and to analyze only half of the frame instead of the whole. Each system will be analyzed independently and the results will be superimposed.

I. Solution by symmetrical system

a. Stiffness factors

$$K_{AB} = \frac{500}{20} = 25 = K_{DC} = K_{FE} = K_{HG} ,$$

$$K_{BD} = \frac{1000}{50} = 20 = K_{FH} ,$$

$$K_{DF} = \frac{1000}{100} = 10.$$

b. Fixed end moments

$$FM_{AB} = \frac{-wL_{AB}^2}{12} = -\frac{25(20)^2}{12} = -\ 833 \quad \text{ft. lbs.} = -FM_{BA} ,$$

$$FM_{BD} = \frac{-wL_{BD}^2}{12} = -\frac{100(50)^2}{12} = -\ 20,833 \quad \text{ft. lbs.} = -FM_{DB} ,$$

$$FM_{DF} = \frac{-wL_{DF}^2}{12} + \frac{wL_{DF}^2}{12} + \frac{3w_1e^2}{5} - \frac{6w_1e^2}{7} = \frac{3(25)100}{5} - \frac{6(25)100}{7}$$

$$= -\ 643 \text{ ft. lbs.}$$

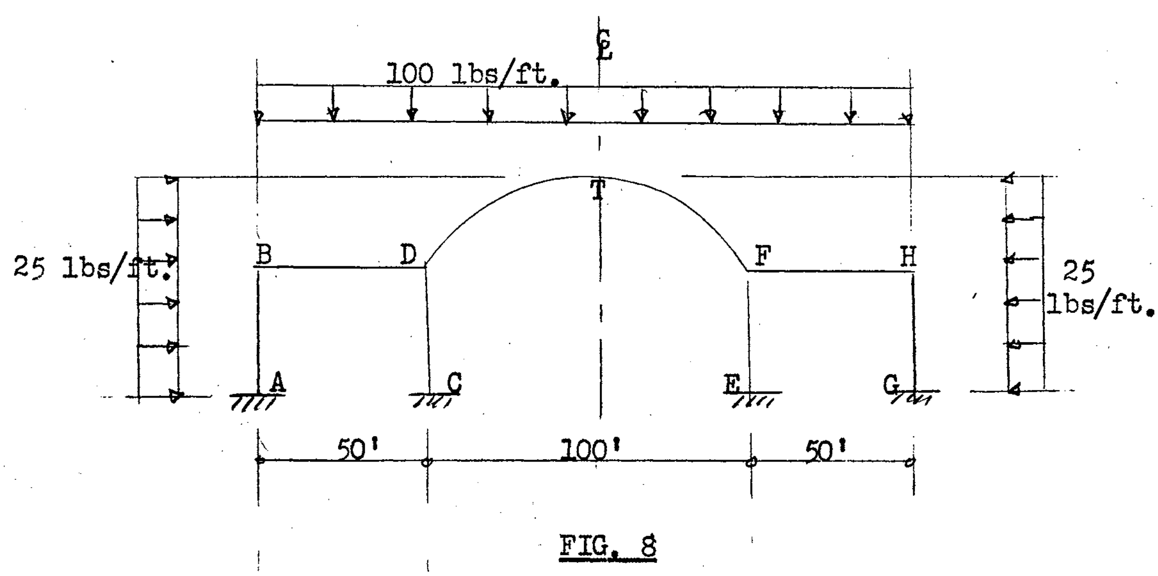


FIG. 8

c. Deformations

$$\theta_A = 0, \quad \theta_B = ?, \quad \theta_C = 0, \quad \theta_D = ?,$$
$$\Delta_{AX} = 0, \quad \Delta_{BX} = \Delta_1, \quad \Delta_{CX} = 0, \quad \Delta_{DX} = \Delta_1,$$
$$\Delta_{AY} = 0, \quad \Delta_{BY} = 0, \quad \Delta_{CY} = 0, \quad \Delta_{DY} = 0.$$

d. Deformation equations

$$M_{AB}^{(I)} = 2EK_{AB}(\theta_B - 3\psi_1) - FM_{AB}$$
$$= 50E\theta_B - 7.5E\Delta_1 - 833 \,, \tag{80}$$
$$M_{BA}^{(I)} = 2EK_{AB}(2\theta_B - 3\psi_1) + FM_{BA}$$
$$= 100E\theta_B - 7.5E\Delta_1 + 833 \,, \tag{81}$$
$$M_{BD}^{(I)} = 2EK_{BD}(2\theta_B + \theta_D) - FM_{BD}$$
$$= 80E\theta_B + 40E\theta_D - 20{,}833 \,, \tag{82}$$
$$M_{DB}^{(I)} = 2EK_{BD}(2\theta_D + \theta_B) + FM_{DB}$$
$$= 80E\theta_D + 40E\theta_B + 20{,}833 \,, \tag{83}$$
$$M_{DC}^{(I)} = 2EK_{DC}(2\theta_D - 3\psi_1)$$
$$= 100\,E\theta_D - 7.5E\Delta_1 \,, \tag{84}$$
$$M_{DF}^{(I)} = 2EK_{DF}(2\theta_D - \theta_D) + 10EK_{DF}\theta_D + \frac{100E\Delta_1}{e} - FM_{DF}$$
$$= 120E\theta_D + 10E\Delta_1 - 653 \,, \tag{85}$$
$$M_{CD}^{(I)} = 2EK_{DC}(\theta_D - 3\psi_1)$$
$$= 50E\theta_D - 7.5E\Delta_1 \,, \tag{86}$$

where

$$\psi_1 = \frac{\Delta_1}{20} \,. \tag{87}$$

e. Equations of equilibrium

$$\sum M_B = 0 \equiv M_{BA} + M_{BD} = 0, \tag{88}$$

$$180E\theta_B + 40E\theta_D - 7.5E\Delta_1 - 20,000 = 0,$$

$$\sum M_D = 0 \equiv M_{DB} + M_{DC} + M_{DF}, \tag{89}$$

$$40E\theta_B + 300E\theta_D + 2.5E\Delta_1 + 20,190 = 0.$$

f. Shear equation

$$R_{AX} + R_{CX} + H_{DF} + 875 = 0, \tag{90}$$

where

$$R_{AX} = \text{horizontal reaction at A,} = -\frac{M_{AB} + M_{BA}}{20} - \frac{w(20)^2}{2(20)}$$

$$= -\frac{150\ E\theta_B - 15E\Delta_1}{20} + 250, \tag{91}$$

$$R_{CX} = \text{horizontal reaction at C} = -\frac{M_{CD} + M_{DC}}{20}$$

$$= -\frac{150\ E\theta_D - 15E\Delta_1}{20}, \tag{92}$$

$$H_{DF} = \text{horizontal thrust of the parabolic arch DF at T}$$

$$= \frac{-wL_{DF}^2}{12e} - \frac{6wf}{7} - \frac{5EK_{DF}\theta_D + 5EK_{DF}\theta_D}{e} - \frac{5EK_{DF}(2\Delta_1)}{e^2}$$

$$= -10E\theta_D - E\Delta_1 - 8,654. \tag{93}$$

Substituting these values in equation (90) above, the final shear equation become

$$-7.5E\theta_B - 17.5E\theta_D - 2.5E\Delta_1 + 7,530 = 0. \tag{90}$$

g. Solving equations (88, 89, and 90) simultaneously

$$\theta_B = -\frac{3.12}{E}, \tag{94}$$

$$\theta_D = -\frac{42.1}{E}, \tag{95}$$

$$\Delta_1 = -\frac{2965}{E}. \tag{96}$$

h. Deformation equations in terms of θ_B , θ_D and Δ_1 .

$$M_{AB}^{(I)} = +19,890 = -M_{GH}^{(I)}$$
$$M_{BA}^{(I)} = +21,980 = -M_{HG}^{(I)}$$
$$M_{BD}^{(I)} = -22,180 = -M_{HF}^{(I)}$$
$$M_{DB}^{(I)} = +17,510 = -M_{FH}^{(I)}$$
$$M_{DC}^{(I)} = +15,720 = -M_{FE}^{(I)}$$
$$M_{DF}^{(I)} = -33,250 = -M_{FD}^{(I)}$$
$$M_{CD}^{(I)} = +18,010 = -M_{EF}^{(I)}$$

These values are the end moments due to the symmetrical system of loading. All moments in feet pounds.

II. Solution by antisymmetrical system

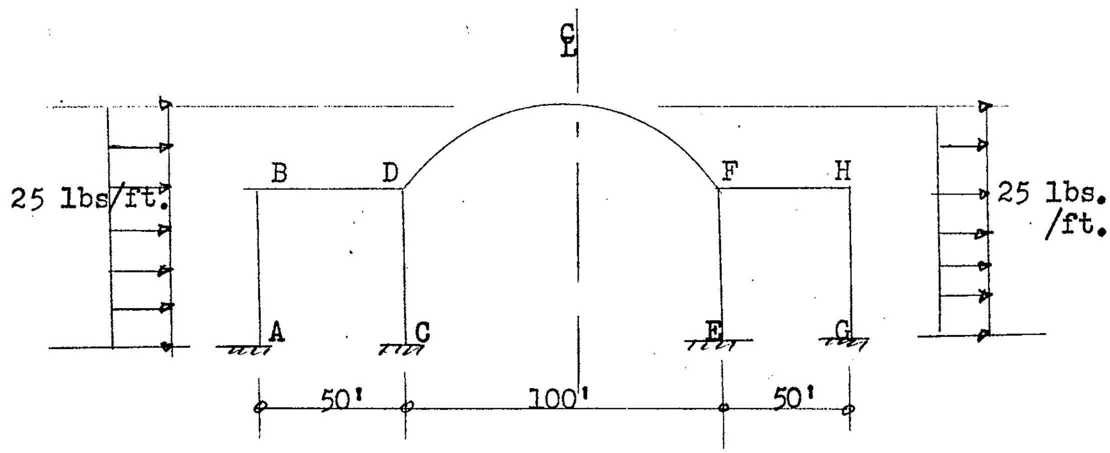


FIG. 9

a. Fixed end moments

$$FM_{AB} = -\frac{wL_{AB}^2}{12} = -\frac{25(20)^2}{12} = -833 = -FM_{BA} ,$$

$$FM_{DF} = \frac{3we^2}{5} - \frac{6we^2}{7} = 1,500 - 2,140 = -640 .$$

b. Deformations are

$$\theta_A = 0, \quad \theta_B = ?, \quad \theta_C = 0, \quad \theta_D = ?,$$

$$\Delta_{AX} = 0, \quad \Delta_{BX} = \Delta_2, \quad \Delta_{CX} = 0, \quad \Delta_{DX} = \Delta_2 ,$$

$$\Delta_{AY} = 0, \quad \Delta_{BY} = 0, \quad \Delta_{CY} = 0, \quad \Delta_{DY} = 0.$$

c. Deformation equations

$$M_{AB}^{II} = 50\ E\theta_B - 7.5\ E\Delta_2 - 833\ , \quad (97)$$

$$M_{BA}^{II} = 100E\theta_B - 7.5\ E\Delta_2 + 833\ , \quad (98)$$

$$M_{BD}^{II} = 80E\theta_B + 40E\theta_D\ , \quad (99)$$

$$M_{DB}^{II} = 80E\theta_D + 40E\theta_B\ , \quad (100)$$

$$M_{DC}^{II} = 100E\theta_D - 7.5E\Delta_2\ , \quad (101)$$

$$M_{DF}^{II} = 60E\theta_D - 640\ , \quad (102)$$

$$M_{CD}^{II} = 50E\theta_D - 7.5E\Delta_2\ . \quad (103)$$

d. Equations of equilibrium

$$\Sigma M_B = 0 \equiv M_{BA} + M_{BD} = 0\ , \quad (104)$$

$$180E\theta_B + 40E\theta_D - 7.5E\Delta_2 + 833 = 0\ ,$$

$$\Sigma M_D = 0 \equiv M_{DB} + M_{DC} + M_{DF}\ , \quad (105)$$

$$40E\theta_B + 240E\theta_D - 7.5E\Delta_2 - 640 = 0\ .$$

e. Shear equation

$$R_{AX} + R_{CX} = 875\ , \quad (106)$$

Knowing the values of R_{AX} and R_{CX} from equations (91 and 92) and substituting these values into equation (106) above, the final shear equation is obtained

$$-7.5E\theta_B - 7.5E\theta_D + 1.5E\Delta_2 = 625\ . \quad (106)$$

f. Solving equations (104, 105, and 106) simultaneously,

$$\theta_B = \frac{15.7}{E}, \qquad (107)$$

$$\theta_D = \frac{18.35}{E}, \qquad (108)$$

and

$$\Delta_2 = \frac{586.3}{E}. \qquad (109)$$

g. Deformation equations in terms of θ_B, θ_D, and Δ_2

$$M^{II}_{AB} = -4{,}000 = M^{II}_{GH} \qquad (110)$$

$$M^{II}_{BA} = -2{,}000 = M^{II}_{HG} \qquad (111)$$

$$M^{II}_{BD} = +2{,}000 = M^{II}_{HF} \qquad (112)$$

$$M^{II}_{DB} = +2{,}100 = M^{II}_{FH} \qquad (113)$$

$$M^{II}_{DC} = -2{,}600 = M^{II}_{FE} \qquad (114)$$

$$M^{II}_{DF} = +500 = M^{II}_{FD} \qquad (115)$$

$$M^{II}_{CD} = -3{,}500 = M^{II}_{EF} \qquad (116)$$

These values are the end moments due to the antisymmetrical system of loading. All are in feet pounds.

h. Superimposing the effects of the symmetrical and the antisymmetrical systems of loading, the final true end moments are obtained.

$$M = M^{I} + M^{II} . \tag{117}$$

The final end moments are

$M_{AB} = + 15,890,$

$M_{BA} = + 19,980,$

$M_{BD} = - 20,180,$

$M_{DB} = + 19,610,$

$M_{DC} = + 13,120,$

$M_{DF} = - 32,750,$

$M_{CD} = + 14,510,$

$M_{GH} = - 23,890,$

$M_{HG} = - 23,980,$

$M_{HF} = + 24,180,$

$M_{FH} = - 15,410,$

$M_{FE} = - 18,320,$

$M_{FD} = + 33,750,$

$M_{EF} = - 21,510.$

All moments in feet pounds.

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