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ement of Problem: The synthetic study of Euclidean spaces of dimension higher than 4 is not extensive. This is due, to some extent, to the fact that it is impossible to draw figures in higher spaces, and therefore it is hard to visualize the properties of these spaces. To overcome this difficulty, we introduce a transformation which maps the Euclidean n-space $E_n$ into a space $P_n$ lying in a plane. Then it is possible to study properties of spaces $E_n$ by studying the corresponding properties of space $P_n$.

lod of Procedure: Consider a set of n parallel lines lying on a plane E. A one to one correspondence between the points of space $E_n$ and sets of n ordered points lying on the n parallel lines is established. This correspondence, with certain additional restrictions, furnishes the desired transformation $T$.

lings and Conclusions: The space $P_n$, the immage of space $E_n$ under the transformation $T$, satisfies all the non-metric postulates of Euclidean n-space. Some of the problems of Euclidean n-space, particularly problems involving points and lines, become simpler in the space $P_n$. The space $P_n$ also contains certain properties which arise from the nature of the transformation $T$ and therefore do not neccessarily hold for space $E_n$. Some of these properties are introduced and studied here.

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A MAPPING OF EUCLIDEAN

N-SPACE ON THE PLANE

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A MAPPING OF EUCLIDEAN

N-SPACE ON THE PLANE

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# PREFACE

This is a synthetic approach to the study of real
Euclidean n-space. This has been done, especially for
spaces of dimensions one to four. But for spaces of
dimension higher than four, the study is not so extensive.
To some extent this is due to the fact that it is impossible
to draw figures for spaces of higher dimension, and conse-
quently it is hard to visualize all the properties of these
spaces.

To overcome this difficulty, we introduce a mapping
which will map the Euclidean n-space $E_n$ into another space
$P_n$ which lies in a plane. Then we prove that the non-metric
axioms of $E_n$ are true of $P_n$. Consequently it is possible to
study $E_n$ by studying $P_n$.

The space $P_n$ has some extra properties which do not hold
in $E_n$. These properties arise from this special kind of
mapping, and will be studied along with the rest of the
properties of $P_n$.

In the final part of the paper, a metric will be
introduced and this will lead to certain new results and in
particular to a graphical solution of a system of four linear
homogeneous equations in four unknowns.

My thanks are due to Drs. L. Wayne Johnson and E. F. Allen
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O. H. Hamilton for reading the manuscript and for his valuable
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## TABLE OF CONTENTS

# LIST OF ILLUSTRATIONS

PART I. PRELIMINARY NOTIONS, POINTS, LINES AND PLANES IN $E_n$

The following relations concerning points, lines and planes in $E_n$, will be used in this paper.

A real Euclidean n-space, is defined to be the set of all ordered n-tuples $(x_1, \ldots x_n)$ of real numbers, called points of the space, metrized by the function:

$$d(x, y) = \left[ \sum_{i}^{n} (x_i - y_i)^2 \right]^{\frac{1}{2}} \qquad .^{[1]}$$

A point $(x_1^1, x_2^1, \ldots x_n^1)$ is collinear with two distinct points $(x_1^2, x_2^2, \ldots x_n^2)$ and $(x_1^3, x_2^3, \ldots x_n^3)$ if the following equations hold:

$$\frac{x_j^3 - x_j^2}{x_j^2 - x_j^1} = \frac{x_k^3 - x_k^2}{x_k^2 - x_k^1} \qquad j, k = 1, 2, \ldots n$$

The set of all points collinear with two given distinct points constitutes a line.

Two distinct lines intersect if they have a point in common.

A plane is the set of all lines intersecting two intersecting lines in two distinct points. Any point belonging to a line of a plane is said to belong to the plane.

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[1] Witold Hurewicz and Henry Wallman, _Dimension Theory_, (Princeton, 1948), p. 158.

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A plane is also called a 2-plane.

An i-plane is defined by induction:

Consider two distinct $(i-1)$ - planes, $2 \leqslant i < n$, having an $(i-2)$ - plane in common. The set of all $(i-1)$ - planes having 2 distinct $(i-2)$ - planes in common with the two given $(i-1)$ - planes is called an i-plane.

An i-plane and a j-plane $(j \leqslant i < n)$ intersect if they have a $(j-1)$ - plane in common.

A point $(x_1, x_2, \ldots x_n)$ is said to be a point at infinity if at least one of the real numbers $x_i$ is infinite. A point at infinity is not uniquely determined unless it belongs to a given definite line.

Two lines are parallel if they intersect and their point of intersection is at infinity.

An i-plane is said to be at infinity if all its points are at infinity.

An i-plane is parallel to a j-plane $(j \leqslant i < n)$ if they intersect and their common $(j-1)$-plane is at infinity. This definition of parallelism is equivalent to what many authors call complete parallelism.[2]

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[2]Frederick S. Woods, Higher Geometry, (New York, 1922), p. 371.

PART II. $P_n$ SPACE. DEFINITION OF POINTS, LINES, AND PLANES IN

$P_n$ SPACE

INTRODUCTION OF THE TRANSFORMATION: Consider a plane E
with n parallel lines $x_1$, $x_2$, . . . $x_n$ in it. Since it is
possible to establish a one to one correspondence between
the points on a line and the set of all real numbers, it is
possible to establish a one to one correspondence between the
points of $E_n$ and the ordered set of n points $A_1$, $A_2$, . . . . $A_n$
lying on the lines $x_1$, $x_2$ . . . $x_n$ respectively. The inverse
of this transformation is also one to one.[1]

Suppose the point $(x_1, . . . x_n)$ of $E_n$ corresponds to the
set $A_1$, . . . $A_n$ and the point $(y_1 . . . y_n)$ corresponds to the
set $B_1$, . . . $B_n$. In plane E, the lines $A_i A_j$ and $B_i B_j$
intersect at the points $X_{ij}$ for i, j = 1, 2, . . . n and i $\neq$ j.
Since there exists $\frac{n(n-1)}{2}$ lines $A_i A_j$, there exist $\frac{n(n-1)}{2}$
points of intersection.

Now the following restriction will be imposed on the
above correspondence:

If the point $(z_1, . . . z_n)$ of $E_n$ which transforms into
the set $C_1$, . . . $C_n$ is collinear with the two points
$(x_1 . . . x_n)$ and $(y_1, . . . y_n)$, the lines $C_i C_j$ must pass
through the points $X_{ij}$ respectively.

The above correspondence with this restriction
is the transformation which is under consideration in this

---

[1]One such correspondence will be introduced and studied in
part V of this paper.

paper. It will be denoted by $\mathcal{T}$.

The space into which $E_n$ is transformed by $\mathcal{T}$ is called a parallel space and is denoted by $P_n$. The lines $x_1, \ldots x_n$ are called the n axes of the space $P_n$. The set of n points $A_1, A_2 \ldots A_n$ lying on the n axes respectively, determines a point in space $P_n$ which is denoted by $P(A_n)$ and $A_i$ is called the $i^{th}$ coordinate of this point.

It follows from this definition and the previous discussion that $\mathcal{T}$ transforms the space $E_n$ into $P_n$ and the inverse transformation $\mathcal{T}^{-1}$ transforms $P_n$ into $E_n$.

IMAGE OF A LINE: A line $l$ of space $E_n$ is determined by two of its points A and B. The image of these two points, under the transformation $\mathcal{T}$, will be the two points $P(A_n)$ and $P(B_n)$ of $P_n$.

Any other point of $l$ will transform into a point $P(C_n)$ of $P_n$ such that the lines $C_i\,C_j$ will pass through the points $X_{ij}$. Since $\mathcal{T}$ and $\mathcal{T}^{-1}$ are both one to one, the inverse image of any point $P(D_n)$, such that $D_iD_j$ passes through $X_{ij}$, will belong to the line $l$. Hence it is possible to consider the set of $\frac{n(n-1)}{2}$ points $X_{ij}$ as the image of the line $l$ of $E_n$. This image is denoted by $L_n(x)$.

It should be noticed that this set of $\frac{n(n-1)}{2}$ points are not all independent.

Given $n-1$ points $X_{ij}$ such that the indices i and j of these points include all the numbers from 1 to n and no index

being repeated more than twice, it is possible to determine the rest of the points uniquely:

Suppose m and n are the two indices that occur only once. Consider two points $A_m$ and $B_m$ on $x_m$ and connect them to $X_{mk}$, $k \neq n$. The lines $A_m X_{mk}$ and $B_m X_{mk}$ will intersect $x_k$ in two points $A_k$ and $B_k$. Connect $A_k$ and $B_k$ to $X_{k\ell}$, $\ell \neq m \neq n$, and extend them to find $A_\ell$ and $B_\ell$ on $x_\ell$. Continue this procedure until all the points $A_1$ to $A_n$ and $B_1$ to $B_n$ are found. These two sets of points will determine $\frac{n(n-1)}{2}$ points $X_{ij}$ which include the original n-1 points.

From this discussion follows that instead of the set of $\frac{n(n-1)}{2}$ points $X_{ij}$, this given set of n-1 points could be considered as the image of the line $\ell$.

The n-1 points having the above properties are called the n-1 components of the line $L_n(x)$. The set of n-1 points $X_{i,i+1}$, where i = 1, . . . . n-1, is one such set of components.

Throughout this paper, the set of n-1 points $X_{i,\,i+1}$ will be used to denote the image of a line $\ell$ in $E_n$. The points $X_{i,\,i+1}$ are simply called the n-1 components of this image.

PART III.  SOME PROPERTIES OF SPACE $P_n$

I-DEFINITIONS:

DIRECTIONS OF A LINE:  Consider a line $L_n(x)$ and a point
$P(A_n)$ of this line such that 3 of its successive coordinates
$A_i$, $A_{i+1}$, $A_{i+2}$ are collinear in the Euclidean plane E.

Since it is always possible to connect two points of a
plane, there always exist on every line of $P_n$    n-2 points having
the above property.

These points are called the n-2 directions of the line
and are indicated by the first direction, the second direction,
.........the n-2$^{nd}$ direction of the line $L_n(x)$.  The n-2$^{nd}$
direction is also called the last direction of the line.

It should be noticed that in space $P_1$ and $P_2$ lines
have no directions, and in $P_3$ they have only one, provided
that $X_{12}$ and $X_{23}$ do not coincide.  If they do, the line will
have infinitely many directions.

Since directions of a line are points of that line; a
line is determined uniquely by two of its directions or a
direction and one of its points.

POINT AT INFINITY:  The point $P(A_n)$ is said to be a point at
infinity if at least one of its coordinates is at infinity,
in the Euclidean plane E and in the direction $x_i$.

If more than one of the coordinates of a point are
infinite, the point is not determined uniquely.  To determine

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such a point uniquely, it has to be given on which line the point lies. This is equivalent to considering a line and determining the ideal point of that line, which is of course a unique point.

INTERSECTION OF TWO LINES: Two lines are said to intersect if they have one and only one point in common.

It should be noticed that given any two lines in $P_n$ they do not necessarily intersect except in case n = 2. In that case they always have a point in common, because each line has only one component.

PARALLEL LINES: Two lines are said to be parallel if they intersect and the point of intersection is at infinity.

This means that in the Euclidean plane E, the lines joining the respective components of the two lines should be parallel to the axes.

2-THEOREM I:

Given a line $L_n(x)$ and a point $P(A_n)$ not lying on the line, there exists one and only one line passing through the given point and parallel to the given line.

PROOF: The proof follows from the fact that in Euclidean plane E, it is possible to draw one and only one line through a given point and parallel to a given line, and two lines either intersect or they are parallel.

3-THEOREM 2:

Given two lines in $P_n$ such that all the n-1 components of each line coincide, they always have a point in common.

PROOF: Consider P and P' to represent the coinciding components

of the two lines. In Euclidean plane E, it is always possible
to connect these two points and the line PP' intersects the n
axes in n-collinear points which determine a point $P(A_n)$ in
$P_n$. This point belongs to both lines and hence it is their
point of intersection.

A-PLANE IN $P_n$:

The definition of plane in $E_n$, given on page 1 is also
valid for the plane in $P_n$.

It follows from the definition that any two distinct
intersecting lines always determine a plane.

THEOREM 3: Any two lines of a plane always intersect.

PROOF: Consider 2 lines $L(\overline{x}_n)$ and $L(\overline{\overline{x}}_n)$ which have a point
$P(A_n)$ in common. (fig. 1).

Consider a line $L(x^1_n)$ in this plane. This means consider
2 points $P(\overline{A}^1_n)$ and $P(\overline{\overline{A}}^1_n)$ belonging to $L(\overline{x}_n)$ and $L(\overline{\overline{x}}_n)$ respect-
ively. They determine a line $L(x^1_n)$ in the plane of $L(\overline{x}_n)$

and $L(\overline{\overline{x}}_n)$. Similarly consider 2 points $P(\overline{A}^2_n)$ and $P(\overline{\overline{A}}^2_n)$. The
line $L(x^2_n)$ determined by these 2 lines also belongs to the plane.
We prove that these 2 lines always have a point in common.
This means that in the Euclidean plane E (fig. 1), the lines
$x^2_{12}$ $x^1_{12}$ and $x^2_{23}$ $x^1_{23}$ must intersect at some point of $x_2$, the

lines $x^2_{23}$ $x^1_{23}$ and $x^2_{34}$ $x^1_{34}$ at some point of $x_3$ and in general

the lines $x^2_{i,i+1}$ $x^1_{i,i+1}$ and $x^2_{i+1,\,i+2}$ $x^1_{i+1,i+2}$ must intersect

$A_n$

$x$

$A_{n-1}$

$x$
$n$

$X^1_{34}$

$X^2_{34}$

$Q$

$\overline{A}^2_4$ $\overline{\overline{A}}^2_4$ $D_4$ $A_4$

$\overline{A}^1_4$

$X^2_{23}$

$\overline{\overline{X}}_{34}$

$\overline{X}_{34}$ $X^1_{12}$

$A_3$ $\overline{A}^2_3$ $\overline{\overline{A}}^1_3$ $D_3$

$\overline{A}^1_3$ $\overline{\overline{A}}^2_3$

$O$ $O'$

$\overline{X}_{23}$ $\overline{\overline{X}}_{23}$

$\overline{A}^1_2$ $\overline{A}^2_2$ $\overline{\overline{A}}^1_2$ $\overline{\overline{A}}^2_2$ $A_2$ $D_2$

$\overline{\overline{X}}_{12}$

$\overline{X}_{12}$

$A_1$ $\overline{A}^2_1$ $\overline{A}^1_1$ $\overline{\overline{A}}^2_1$ $\overline{\overline{A}}^1_1$ $D_1$
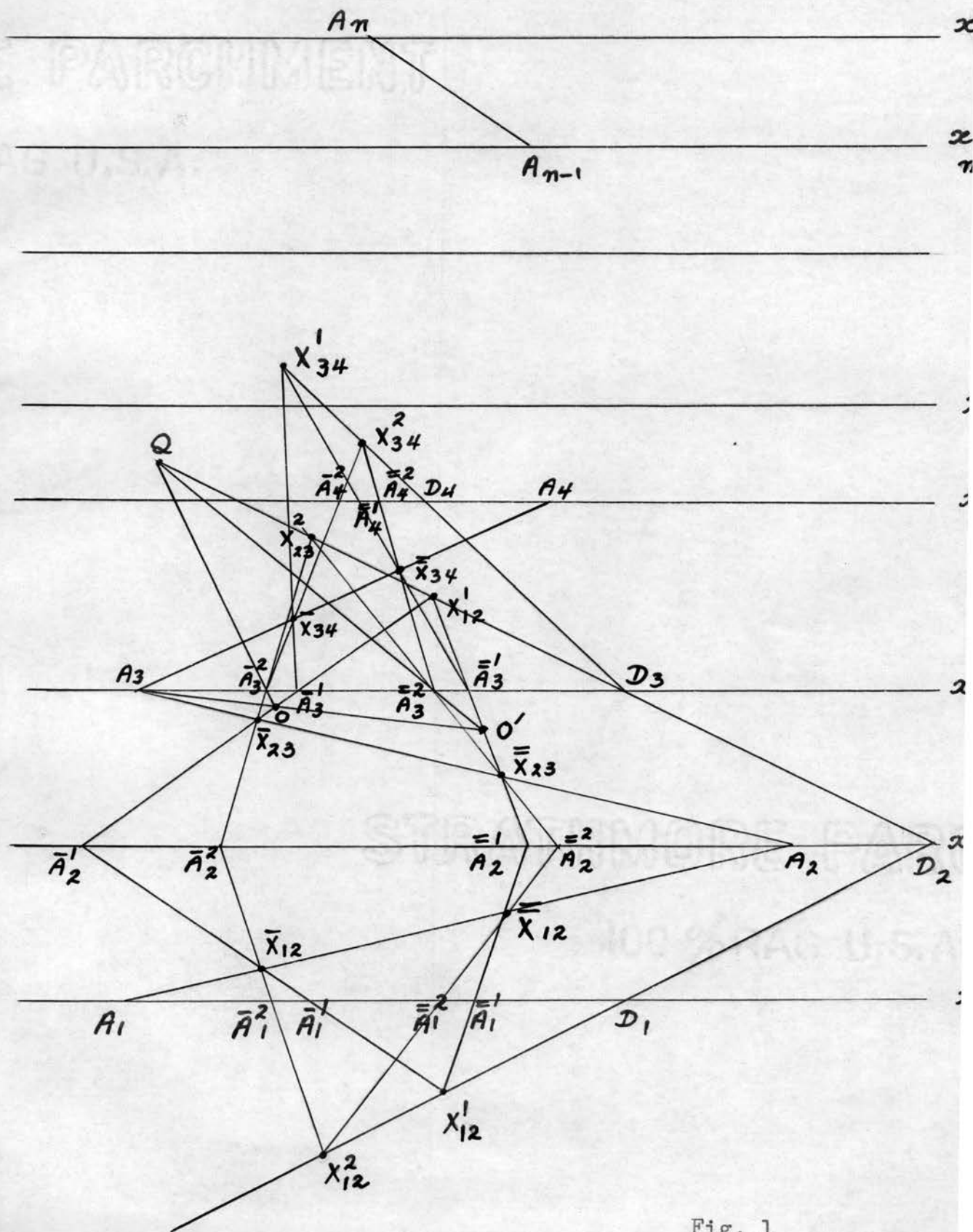
$X^1_{12}$

$X^2_{12}$

Fig. 1

at some point of $x_{i+1}$. The point $P(D_n)$, where $D_i$s are the above mentioned intersection points, is the point of intersection of the two lines.

To prove this, first consider the 3 axes $x_1$, $x_2$ and $x_3$. From the properties of ordinary homology it follows that $X_{12}^2 X_{12}^1$ and $X_{23}^2 X_{23}^1$ intersect at the point $D_2$ which lies on the axis $x_2$.

To complete the proof we must show that the 3 points $D_3$, $X_{34}^2$ and $X_{34}^1$ (in plane E) are collinear. Then we continue this process for $D_4$, $D_5$, . . . . . . . until we prove that

$D_i$, $X_{i,\ i+1}^2$ and $X_{i,\ i+1}^1$ are collinear. This will complete the proof.

To prove this collinearity we shall use Desargue's theorem as follows: Consider the points $O$ and $O'$ on $\bar{A}_3^1 \bar{X}_{23}$ and $\bar{\bar{A}}_3^1 \bar{\bar{X}}_{23}$ respectively, such that $OO'$ passes through $A_3$ in plane E. The two triangles $\bar{X}_{23} O\bar{A}_3^2$ and $\bar{\bar{X}}_{23} O' \bar{\bar{A}}_3^2$ have the property that the line joining the corresponding verteces passes through the point $A_3$ and therefore the points of intersection of the corresponding sides are collinear. Also consider the two triangles $O\bar{A}_3^2 \bar{X}_{34}$ and $O'\bar{\bar{A}}_3^2 \bar{\bar{X}}_{34}$. The same thing is true for these triangles. Now when $O \to \bar{A}_3^1$ and $O' \to \bar{\bar{A}}_3^1$ these two triangles approach $\bar{A}_3^2 \bar{A}_3^1 \bar{X}_{34}$ and $\bar{\bar{A}}_3^2 \bar{\bar{A}}_3^1 \bar{\bar{X}}_{34}$. From this follows that in this situation $O$ approaches the intersection of $X_{23}^2 X_{23}^1$ and $X_{34}^1$. (See fig. 1). It also should lie on $x_3$ (in this limit situation) therefore the 3 lines meet at a point which we denote by $D_3$. This

completes the proof of the theorem.

5-AXES OF A PLANE:

Consider a plane determined by the lines $L_n(x)$ and $L_n(\bar{X})$ which have the point $P(A_n)$ in common.

Consider the first directions of these two lines. They intersect at n-2 points $P_j^1$. This set of points ($p_1^1$, $p_2^1$, . . . $p_{n-2}^1$) actually determines a line which belongs to the given plane since it has a point in common with either line, and is called the first axes of the plane. Similarly we can consider any $m^{th}$ direction of the two lines and define the $m^{th}$ axes of the plane. The n-2 points $p_1^m$, $p_2^m$, . . . $p_{n-2}^m$ are called the n-2 components of the $m^{th}$ axis of the plane.

Since these n-2 axis of the plane are lines in that plane any two of them have a point in common and any two of them determine the plane. Since any other line in the plane must intersect all these axes of the plane, the following theorem follows:

THEOREM 4: The $j^{th}$ direction of any line in the plane must pass through the $j^{th}$ axes of the plane.

It is worth noting that in 3-space where every line has only one direction, the plane will have only one axis which consists of a single point in plane E and hence the direction of all the lines in a plane of this 3-space will pass through this unique point.

It is clear that for each of the n-2 axes of the plane there exist two components which coincide. (The general line has n-1 components but these special lines have only n-2).

If we denote the components of the $j^{th}$ axes of the plane by a superscrip $j$, where $j = 1 \ldots \ldots, n-2$, it follows that the component $p_j^j$ is the double component of the $j^{th}$ axis of the plane. For example, for the first axis of the plane we have the following components: $P_1^1, P_2^1, P_3^1 \ldots \ldots P_{n-2}^1$ and as we know $P_1^1$ is the double component of this axis.

THEOREM 5: Consider the set of all components of all the axes of a plane. The four components $P_j^j, P_{j+1}^j, P_j^{j+1}, P_{j+1}^{j+1}$ are always collinear for all $j = 1 \ldots \ldots, n-3$.

PROOF: From Desargue's theorem and the definition of components of the axes of the plane follows that

$P_j^j, P_j^{j+1}$ and $P_{j+1}^{j+1}$ are collinear. Similarly $P_{j+1}^{j+1}, P_{j+1}^j$ and $P_j^j$ are collinear. Therefore the four points

$P_j^j, P_j^{j+1}, P_{j+1}^j$ and $P_{j+1}^{j+1}$ are collinear.

As an illustration we can consider the space $P^4$. It has only two axes and each axis has two components and these four points are collinear.

6-INTERSECTION OF TWO PLANES:

Two planes intersect if they have one and only one line in common.

THEOREM 6: Two planes in n-space do not necessarily intersect nor have a point in common, except in $P^3$ and $P^4$ in which they have a line or a point in common respectively.

PROOF: To prove this theorem we begin with 3-space and extend

the result to the higher spaces. Consider two planes A and B in $P^3$, determined by their two axes $P_1^1$ and $\overline{P}_1^{-1}$ and two points $P(A_3)$, and $P(\overline{\overline{A}}_3)$, one belonging to each plane. (fig. 2). From the fact that $P_1^1$, and $\overline{P}_1^{-1}$ are 2 lines belonging to the two planes respectively it follows that the two planes have at least one point in common, namely $P(\overline{\overline{A}}_3)$.

Now in Euclidean plane E consider the lines $A_1A_2$, $A_2A_3$, $B_1B_2$ and $B_2B_3$ and the points $p_1^1$ and $\overline{p}_1^{-1}$. Inscribe a trapezoid $X_{23} X_{12} \overline{X}_{23} \overline{X}_{12}$ in the quadrilateral $HA_2 G\overline{\overline{A}}_2$ determined by the lines $A_1 A_2$, $B_1 B_2$, $A_2 A_3$ and $B_2 B_3$, such that the parallel sides $X_{12} \overline{X}_{12}$ and $\overline{X}_{23} X_{23}$ be parallel to the axis $x_2$ and sides $X_{12} X_{23}$ and $\overline{X}_{12} \overline{X}_{23}$ pass through $P_1^1$ and $\overline{P}_1^{-1}$ respectively.

To construct such a trapezoid consider the quadrilateral A B C D and the points P and Q (fig. 3). Draw the diagonals AC and BD and consider AC to be the diagonal which has to be parallel to the bases of the trapezoid. Draw any line PE through P and intersecting C B in a point E. Draw a line through E and parallel to AC to intersect AB in F. Connect F to R, the point of intersection of PE and BD. Draw a line through P and parallel to AC to intersect FR in P'. Connect P' to Q to intersect BD at R' and AD and AB at G and H. Connect R' to P to intersect DC and CB in G' and H'. G H G' H' is the inscribed trapezoid.
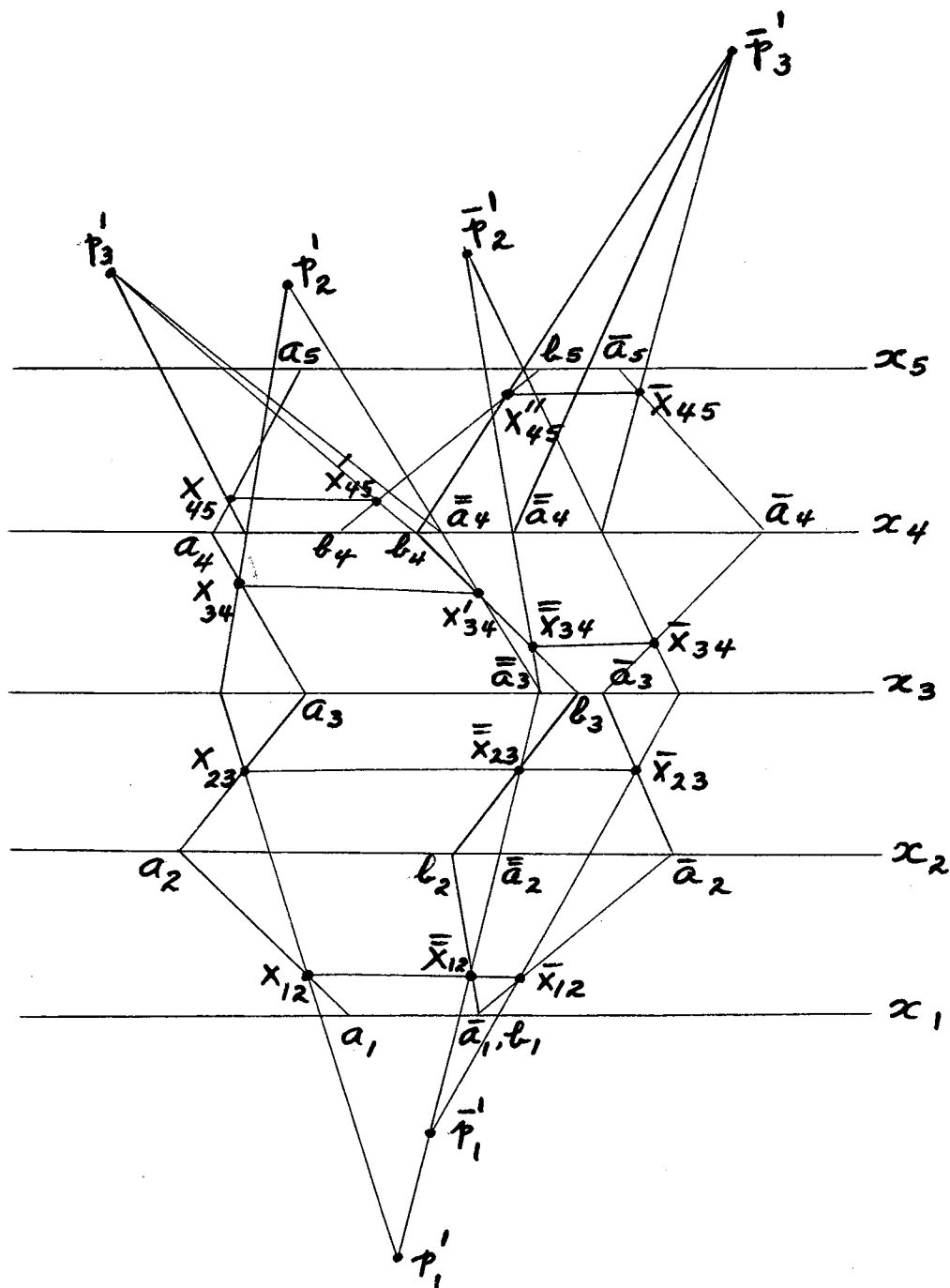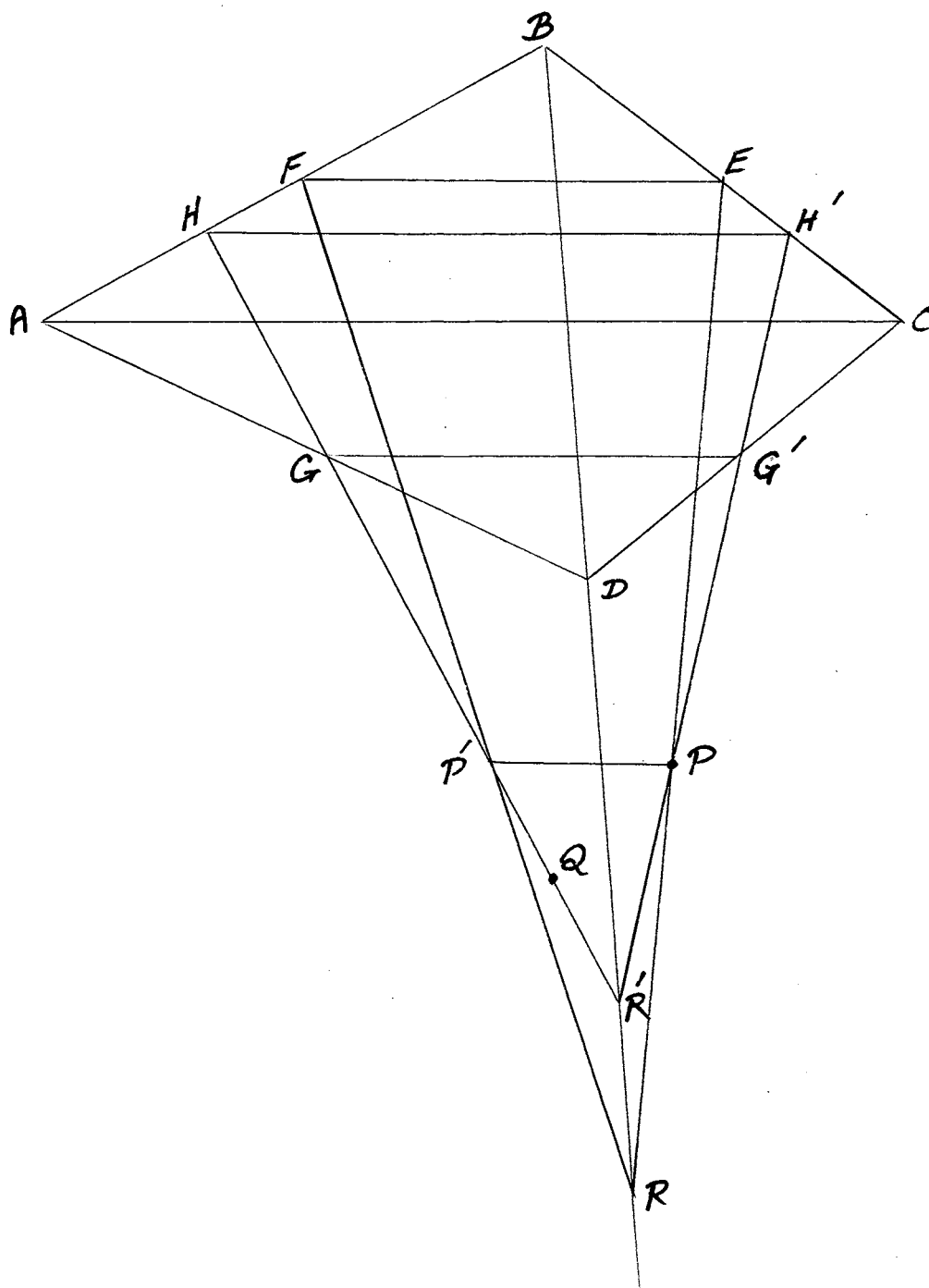
Fig. 2

Fig. 3

The proof follows from the fact that B and P determine a homology with the axis BD and center at infinity in the direction of AC. It follows that R H and R H', P and P', R P and R P', B G and A B, D G and D A are all corresponding pairs of this homology.

Therefore H and H' and G and G' are also corresponding pairs and hence H H' and G G' pass through the center of homology and therefore are parallel to A B.

Going back to the proof of theorem 6, the points $X_{12}$, $X_{23}$ and $\bar{X}_{12}$, $\bar{X}_{23}$ determine two parallel lines lying in the planes A and B of $P_3$. Draw a line $L_3(\bar{\bar{x}})$ through the point $P(\bar{\bar{A}}_3)$ and parallel to these two lines. $\bar{\bar{X}}_{12}$, $\bar{\bar{X}}_{23}$ are the components of this line and since $P(\bar{\bar{A}}_3)$ belongs to both planes A and B, the line $L_3(\bar{\bar{x}})$ belongs to both planes.

Since this construction of the above trapezoid is always possible and is unique, the two planes A and B of $P_3$ always have one and only one line in common, and consequently they always intersect.

Now consider $P_4$, that is add an axis $x_4$ to fig. 2. The two lines will become $L_4(z)$ and $L_4(\bar{\bar{z}})$ and the points $P(A_4)$ and $P(\bar{\bar{A}}_4)$. The two axes of the plane will be $p_1^1$, $p_2^1$ and $\bar{p}_1^1$, $\bar{p}_2^1$.

Since we can always choose $\bar{p}_2^1$ and $p_2^1$ such that in S the three points $p_2^1$, $\bar{p}_2^1$ and $\bar{\bar{A}}_3$ will not be collinear, and since the direction of any line of plane passes through the

axis of that plane, it follows that in $P^4$ two planes do not necessarily intersect.

However we can prove that they always have a point in common. To prove this consider the points $\bar{\bar{X}}_{12}$, $\bar{\bar{X}}_{23}$, $\bar{X}''_{34}$ and $\bar{\bar{X}}_{12}$, $\bar{\bar{X}}_{23}$, $\bar{X}'_{34}$ in $\Sigma$. They determine two lines in $P_4$, each line lying in one of the planes. These 2 lines always have a point $P(B_4)$ in common and hence the two planes always have a point in common.

There are some special cases however, in which the two planes in $P^4$ have a line in common.

If in fig. 2, $X'_{34}$ and $X''_{34}$ coincide the two planes intersect. This happens if in $\Sigma$ the line $\bar{p}_2^1$ $p_2^1$ passes through $\bar{\bar{A}}_3$ and also the line $X'_{34}$ $\bar{\bar{X}}_{34}$ is parallel to the axes of the space $\Sigma$.

In case of n-space, if all the respective components of the 2-axes of the two planes, $p_k^1$ and $\bar{p}_k^1$, pass through $\bar{\bar{A}}_{k+1}$ (in $\Sigma$) and $X_{k+1,\ k+2}$ $\bar{X}_{k+1,\ k+2}$ are parallel to $x_{k+1}$ axis (in $\Sigma$), then the two planes have a line in common.

Now we will prove that in $P^5$ two planes do not necessarily have a point in common.

Suppose the two planes do have a point in common. Let the first four coordinates of this point be $B_1$, $B_2$, $B_3$ and $B_4$ and the fifth one will be some $B_5$.

Now we can draw two lines, both passing through $P(B_5)$ and parallel to $L_5(\bar{x})$ and $L_5(x)$ respectively. These two lines

have $\bar{X}_{12}$, $\bar{X}_{23}$, $X'_{34}$ and $\bar{X}_{12}$, $\bar{X}_{23}$, $X''_{34}$ respectively for their first 3 components. The fourth components will be $X''_{45}$ and $X'_{45}$. $X'_{45}$ and $X''_{45}$ are determined by the two given planes and the coordinates $B_1$, $B_2$, $B_3$, $B_4$ of $P(B_5)$.

Now we have already assumed these two lines have the point $P(B_5)$ in common and since they are uniquely determined, they determine $B_4$ and $B_5$ uniquely. But by choosing $p^1_3$ and $\bar{p}^1_3$, we can always find $X'_{45}$ and $X''_{45}$ such that the component $B_4$ determined by them will be different from the $B_4$ that we had before.

Therefore the assumption of the two planes having a point in common leads to contradiction.

From what we have seen it follows that in $P^6$, two planes do not necessarily have a line or a point in common. Therefore in any space $P^n$ where $n \geqslant 5$, the two planes do not necessarily have a point or a line in common. This proves the theorem.

## 7- 3-PLANE IN N-SPACE:

A 3-plane is determined by two intersecting planes.

Consider two intersecting planes as given in figure 4. One plane is determined by its axis $\bar{p}^1_j$ and a point $P(A_n)$ belonging to the intersection of the two planes. The other plane is determined by its axis $p^1_j$ and the same point $P(A_n)$. According to the results obtained in the last section the line $l_n(x)$ of figure 4 is the intersection of the two planes.

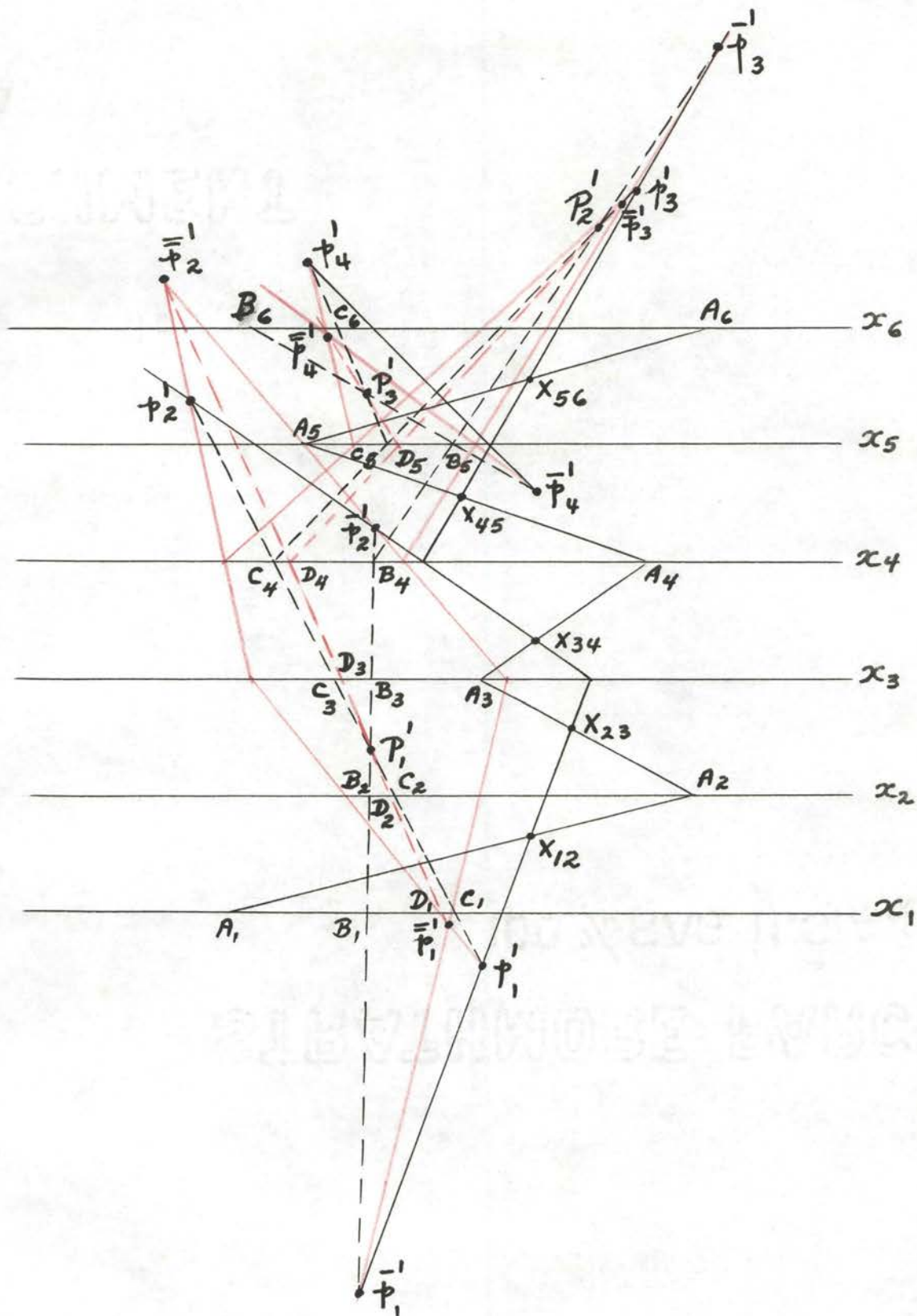Now consider the point $P(B_n)$ on the line $\bar{p}^1_j$ and $P(C_n)$

Fig. 4

on the line $p_j^1$. These two points belong to the 3-plane and hence the line they determine belongs to the 3-plane. Since each of the two points has 4 collinear coordinates, the line they determine will have only n-3 distinct components. (The first 3 components coincide). This line is called the axis of the 3-plane and its components are designated by

$$p_1^1, \ p_2^1 \ . \ . \ . \ . \ p_{n-3}^1.$$

Since a 3-plane is also determined by two non-intersecting lines, it can be determined by one of its axes and one of its lines which does not intersect the axis. In figure 4 the 3-plane is determined by $L_n(x)$ and the axis

$$p_1^1, \ p_2^1 \ . \ . \ . \ . \ p_{n-3}^1$$

It should be noticed that the axes of planes and n-planes are lines, and therefore have directions like any other line.

THEOREM 7: Given a 3-plane, the direction of the axes of any plane belonging to this 3-plane passes through the axis of the 3-plane.

PROOF: Consider the 3-plane of figure 4. Any plane of this 3-plane is determined by two intersecting lines lying in the two planes which determine the 3-plane, one in each plane. The direction of these lines passes through the axes of the two original planes, respectively.

In figure 4 the direction of these two lines are designated by red color. The axis of this new plane is denoted by

$$\bar{P}_1^1, \ \bar{P}_2^1, \ \ldots \ \bar{P}_{n-2}^1.$$

It follows from Desargue's theorem that in the plane $E$, the line $\bar{P}_1^1 \ \bar{P}_2^1$, passes through $P_1^1$. By a method of proof similar to that of theorem 4, it is possible to prove that the line $p_2^1 \ \bar{p}_3^1$ passes through $D_4$ and in general $p_{n-3}^1 \ \bar{p}_{n-2}^1$ passes through $D_{n-1}$. This completes the proof of the theorem.

As a particular example, consider $P_4$. A 3-plane in this space is determined by two intersecting planes. One plane is determined by its first axis $p_1^1$ and $p_2^1$ and a point $P(A_4)$. The second plane is determined by the same point and its first axis $\bar{p}_1^1, \ \bar{p}_2^1$. (fig. 5). The two planes intersect at the line having the components $X_{12}, \ X_{23}, \ X_{34}$.

Now if $p_1^1$ is connected to $p_2^1$ and $\bar{p}_1^1$ to $\bar{p}_2^1$, the two lines meet at a point P which is the axis of the 3-plane. Now if we consider any other plane in this 3-plane and connect the two distinct components of its axis, the connecting lines will pass through the point P.
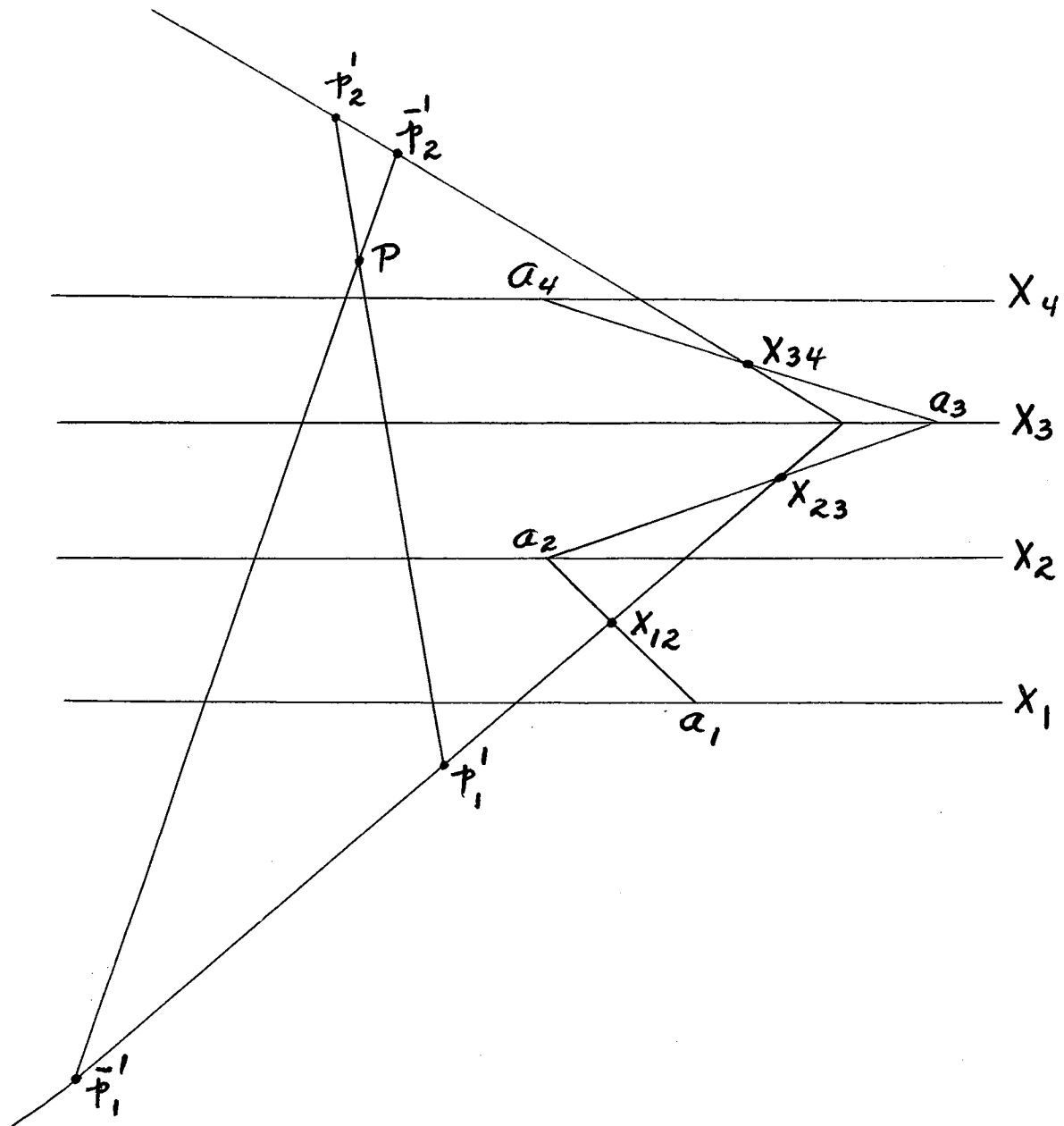
Fig. 5

PART IV—THE PROOF OF NON METRIC POSTULATES OF EUCLIDEAN N-SPACE

In this section we shall prove that space $P_n$ satisfies the non-metric postulates of the Euclidean space $E_n$.

I— THE POSTULATE SYSTEM OF HILBERT:[1]

For spaces of dimension 1 to 3, we shall use the set of postulates formulated by Hilbert, all of which except postulate I-7 hold for spaces of dimension higher than 3.

I— The Postulates of Connection:

1 and 2  Two distinct points determine one and only one straight
         line.

Proof:  It follows from the definition of the line in $P_n$.

3         There are at least two points on every line, and there
        are at least three points on every plane which do not lie
        on the same straight line.

Proof:  The proof of the first part follows from the fact that
        in Euclidean plane E, there exist at least two points
        on every straight line $x_1$. The second part follows
        from the definition of the plane in $P_n$.

4 and 5 Three points which do not lie on the same straight line
        determine one and only one plane.

Proof:  Follows from the definition of the plane and postulate 1.

6         If two points of a line lie on a plane, then all the
        points of the line lie on the plane.

---

[1] Harold E. Wolf, Introduction to non-Euclidean Geometry, (New York, 1948) pp. 12-16.

Proof: Follows from the definition of a point of a plane in $P_n$.

7    If two planes have one point in common they have at least one other point in common. (This postulate holds only for space $P_3$).

Proof: Consider two planes A and B in $P_n$, plane A being determined by its axis $p_1^1$ and a point $P(A_3)$ and plane B being determined by its axis $\bar{p}_1^1$ and the same point $P(A_3)$. If (in Euclidean plane E) the points $p_1^1$ and $\bar{p}_1^1$ are connected, the line $p_1^1 \bar{p}_1^1$ will intersect the lines $A_1 A_2$ and $A_2 A_3$ in two points $X_{12}$ and $X_{23}$. These two points determine a line in $P_3$ which belongs to both planes and therefore the two planes have at least one other point besides $P(A_3)$ in common.

8    There exist at least 4 points which do not lie on the same plane.

Proof: Consider 3 points which determine a plane and find the axis of this plane. It is always possible to find a line in $P_n$ such that its direction does not pass through the axis of this plane and hence does not belong to the plane. Any point of this line together with the three given points determines a set of four points which do not lie on the same plane.

II THE POSTULATES OF ORDER:

Definition: A point $P(C_n)$ is said to lie between two given points $P(A_n)$ and $P(B_n)$ if it is collinear with these two points, and in Euclidean plane E, one of its coordinates $C_i$ lies between the

coordinates $A_n$ and $B_n$ of the two points.

1.  If $P(A_n)$, $P(B_n)$ and $P(C_n)$ are points of a straight line and $P(B_n)$ is between $P(A_n)$ and $P(C_n)$, then $P(B_n)$ is also between $P(C_n)$ and $P(A_n)$.

Proof:  Proof follows from the definition.

2.  If $P(A_n)$ and $P(C_n)$ are two points of a straight line, there exists at least one other point of the line which lies between them.

Proof:  The proof follows from the above definition.

3.  Of any 3 points of a straight line, one and only one lies between the other two.

Proof:  Follows from the definition.

III THE POSTULATE OF PARALLELS

1.  Given a line $L_n(x)$ and a point $P(A_n)$ not lying on the line $L_n(x)$, then there exist one and only one line parallel to the given line and passing through the given point.

Proof:  See theorem I, page 7.

2-III CHARACTERISTIC POSTULATE OF $P_4$

The above postulates hold for all the space $P_n$, where n is a positive finite integer, except postulate I-7 which is true only for space $P_3$.

For space $P_4$, we must replace postulate I-7 by the following postulate.[2]

The planes in $P_4$ may have only one point in common.

_____

2. H. Baker, *Principles of Geometry*, Vol. IV, (Cambridge, 1925), pp. 33, 34.

Proof:  See theorem 6, on page 11.

Since there does not exist any uniform set of postulates for Euclidean spaces of dimension 4 or higher, we will not attempt to prove that all the different postulates proposed by different authors hold for $P_n$.

The above postulates are the ones that are usually contained in every set of postulates and have been used in this paper.

## PART V   INTRODUCTION OF A METRIC

Consider a space $P_n$ determined by a plane $E$ and $n$ parallel lines $x_i$, the distance between $x_i$ and $x_{i+1}$, being the same for all $i = 1 \ldots n-1$.

Also consider a line $O$, in the plane $E$ and perpendicular to the lines $x_i$, intersecting these lines at the points $O_i$ respectively.  Consider a unit of length on $x_i$ (the same unit for all $i$), and take the point $O_i$ as the origin, and the direction to the right of $O_i$ as the positive direction.

Now consider a point $M(a_1, \ldots a_i, \ldots a_n)$ in $E_n$. Determine a point $X_i$ on $x_i$ such that $O_i X_i = a_i$, for all $i$'s.  Thus for every point $M$ in $E_n$ one set of points $X_i$ is obtained, and conversely every set of $n$ points $X_i$ on the $n$ axes $x_i$ represents one and only one point $M(a_1, \ldots a_n)$ in $E_n$.  Thus a 1-1 correspondence is established between the points of the space $E_n$ and the sets of $n$ points $X_i$ on the lines $x_i$ of plane $E$.  It follows immediately that this correspondence satisfies all the conditions imposed on the transformation $\mathcal{T}$ introduced in part II.  The following is a study of this space.

I (n-1) - PLANE IN $P_n$:

An equation $\displaystyle\sum_1^n a^i x_i = c$ is the equation of an (n-1)-plane in the space $E_n$.

27

Any point of this (n-1)-plane could be mapped into a set of n points $X_i$ on the n lines $x_i$ of $E$, that is into a point of space $P_n$. In particular consider the following two points:

a) The point determined by the following equations:

$$\sum_1^n a^i x_i = c$$
$$x_1 = x_2$$
$$x_2 = x_3$$
$$\text{-----------------------}$$
$$x_{n-1} = x_n$$

This is a system of n equations in n unknowns and hence has a unique solution, namely $x_i = a$ for all i's.

b) The point determined by the following equations:

$$\sum_1^n a^i x_i = c$$
$$x_2 = x_1 + d$$
$$x_3 = x_2 + d = x_1 + 2d$$
$$\text{-----------------------}$$
$$x_n = x_{n-1} + d = x_1 + (n-1)\,d$$

For every given d this set has a unique solution.

Consider the image of the points a) and b) on $P_n$. (fig. 6). These two points determine a line. The image $\Omega$ of this line in $P_n$ has the property that all of its components coincide and is called the axis of the (n-1)-plane.

For any other value of d, the point determined by set b) will have the property that the line connecting its coordinates will also pass through $\Omega$.
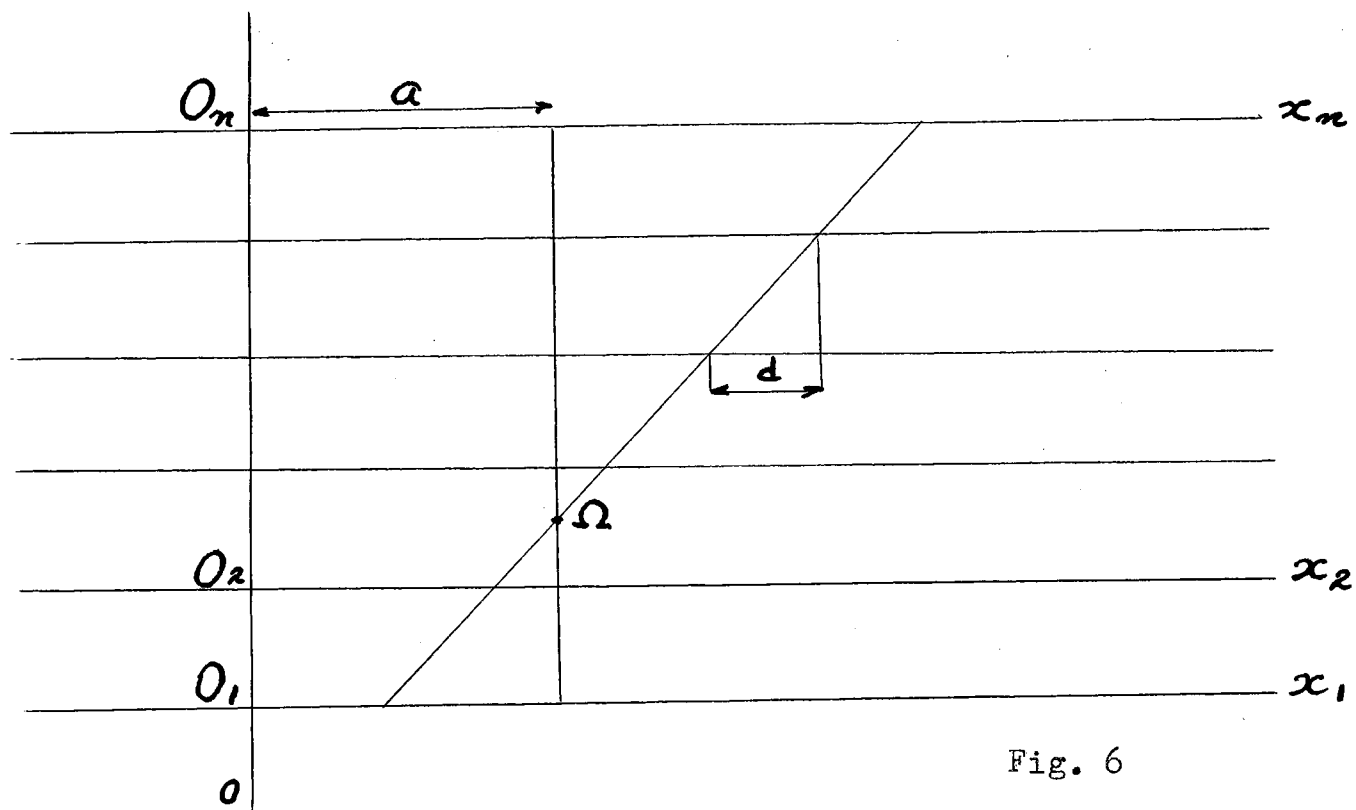
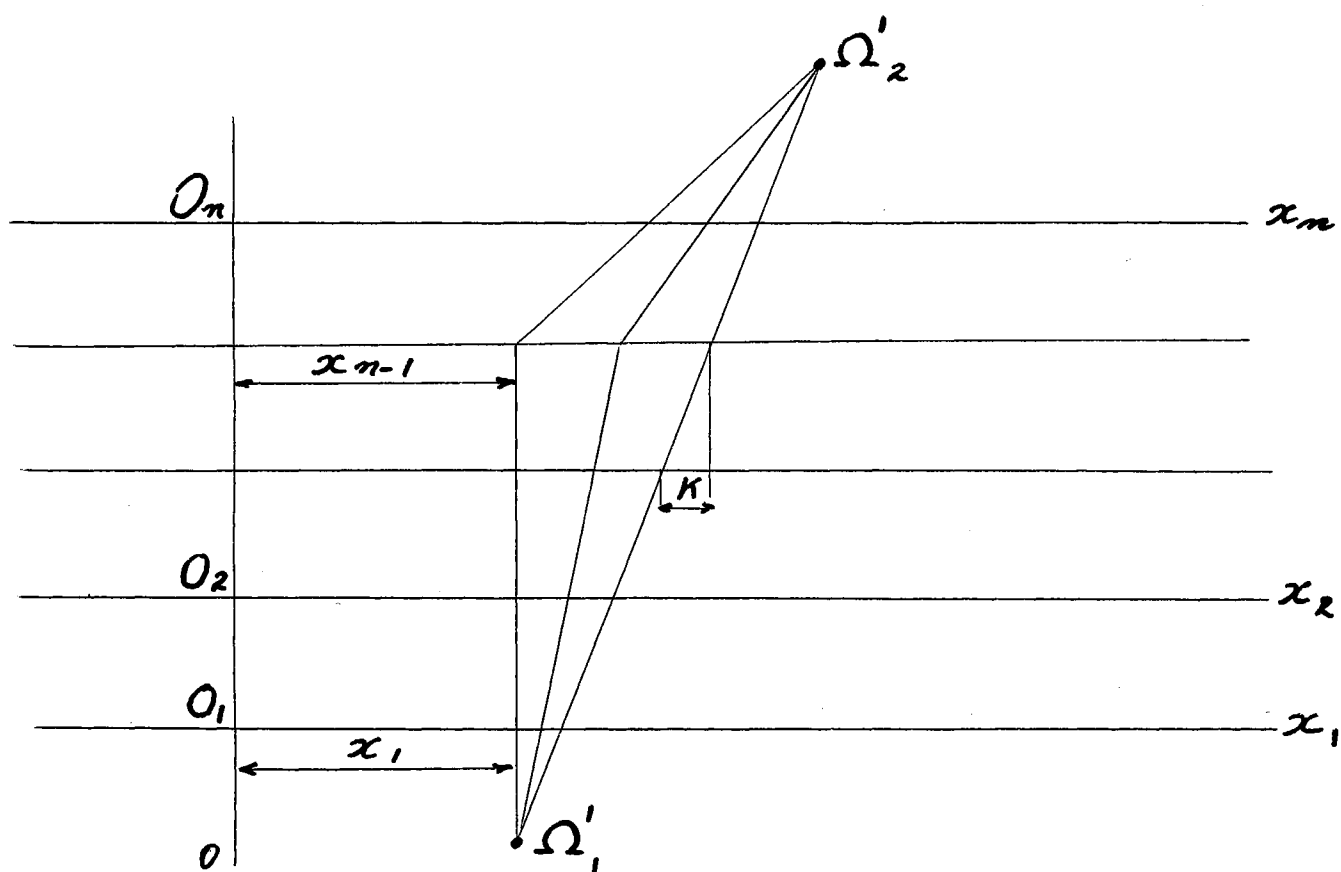To prove this consider $x_1$ as z-axis and 0 as y-axis

Fig. 6



Fig. 7

of a cartesian coordinate system. Find the coordinates of a point $\Omega$ in this system and show that for every other value of d in set b), the corresponding line will pass through $\Omega$.

It should be noticed that if the given (n-1)-plane has two distinct axes $\Omega_1$ and $\Omega_2$, then it contains infinitely many axes, all of which lie in a plane.

The above results could be incorporated in the following theorem.

THEOREM 8:

In every (n-1)-plane lying in an n-space, there always exists one line such that all its components coincide; and if there exist two such lines, then there exists a plane (2-plane) such that every line of it has this property.

We have already studied some special cases of this theorem in 3 and 4 spaces.

2 (n-2)-PLANE IN $P_n$:

Consider an (n-2)-plane in $E_n$, that is the set of two equations:

$$\Sigma a^i x_i = c$$

$$\Sigma \bar{a}^i x_i = \bar{c}$$

Consider the following two points:

a)
$$\Sigma a^i x_i = c$$

$$\Sigma \bar{a}^i x_i = \bar{c}$$

$$x_2 = x_1 + k$$

$$x_3 = x_2 + k = x_1 + 2k$$

$$x_n = x_{n-1} + k = x_1 + (n-1)\,k$$

b)

$$\sum a^i x_i = c$$

$$\sum a^i x_i = \bar{c}$$

$$x_2 = x_1$$

$$x_3 = x_2$$

$$x_{n-1} = x_{n-2}$$

Now consider the image of these two points in $P_n$ which is given in figure 7.

Consider the first n equations of the set a). For every given k this set represents a point in $P_n$ and the lines connecting the coordinates of the image of this point in plane $S$ will pass through $\Omega\,^1_1$ and $\Omega\,^1_2$.

It should be noticed that $\Omega\,^1_1$ and $\Omega\,^1_2$ determine a line in $P_n$ whose first n-2 components coincide and form $\Omega\,^1_1$ and whose last component is $\Omega\,^1_2$. It can be proved, in a similar way, that there exists another line $\Omega\,^2_1$ and $\Omega\,^2_2$ in the given (n-2)-plane such that its last n-2 component coincide and form $\Omega\,^2_2$ and whose first component is $\Omega\,^2_1$. The lines $\Omega\,^1_1, \Omega\,^1_2$ and $\Omega\,^1_2, \Omega\,^2_2$ are called the first and last axes of the (n-2)-plane.

From the method of construction of these two lines follows that $\Omega\,^1_1, \Omega\,^1_2, \Omega\,^2_1$ and $\Omega\,^2_2$ always exist and are collinear.

These results could be incorporated in the following theorem.

THEOREM 9:

In every (n-2)-plane lying in an n-space there always exist two lines $\Omega_1^1, \Omega_2^1$ and $\Omega_1^2, \Omega_2^2$ such that their first and last n-2 components coincide, respectively, and in plane E the four points $\Omega_1^1, \Omega_2^1, \Omega_1^2$ and $\Omega_2^2$ are collinear. Moreover, if there exist two distinct lines of one kind, then there exist infinitely many such lines.

The proof of the second part of theorem 9 is based on the fact that if two such lines exist at least one of the sets of equations a) and b) of page 30 has to have two distinct solutions and hence infinitely many solutions.

From the method of construction of the lines $\Omega_1^1, \Omega_2^1$ and $\Omega_1^2, \Omega_2^2$ it follows that:

THEOREM 10:

For all the (n-2)-planes in a given (n-1)-plane, the lines $\Omega_1 \Omega_2$ connecting $\Omega_1$ to $\Omega_2$ in E pass through the point $\Omega$ in E, where $\Omega$ is the axis of the given (n-1) plane.

This theorem was proved for cases of n = 3 and n = 4 in part III.

Now consider the following set of n equations:

$$\Sigma a_i^i x_i = c$$

$$\Sigma \bar{a}^i x_i = \bar{c}$$

$$x_2 = x_1 + d$$

$$x_3 = x_2 + d$$

. . . . . . . . . . . . .

$$x_i = x_{i-1} + d$$

$$x_{i+2} = x_{i+1} + d$$

. . . . . . . . . . . . .

. . . . . . . . . . . . .

$$x_n = x_{n-1} + d$$

For two different values of d two different points are obtained. The image of these points in $P_3$ is given in figure 6.

These two points determine a line with only three distinct components $\Omega_1$, $\Omega_2$ and $\Omega_3$. These 2 points are collinear in plane 3 and the line joining them also passes through $\Omega$ in 4, if the (n-2)-plane is lying on (n-1)-plane having $\Omega$ for its axis.

This is true for all values of i form 2 to n, and hence there exist n-1 such special lines, the first and last axes of the (n-2)-plane being two of them for which $\Omega_1$ coincides with $\Omega_2$ or $\Omega_2$ coincides with $\Omega_3$.

As an example consider space $P^4$. A plane in this space has three of these special lines namely the first and last ones which have been previously defined and a third line which has three distinct components.
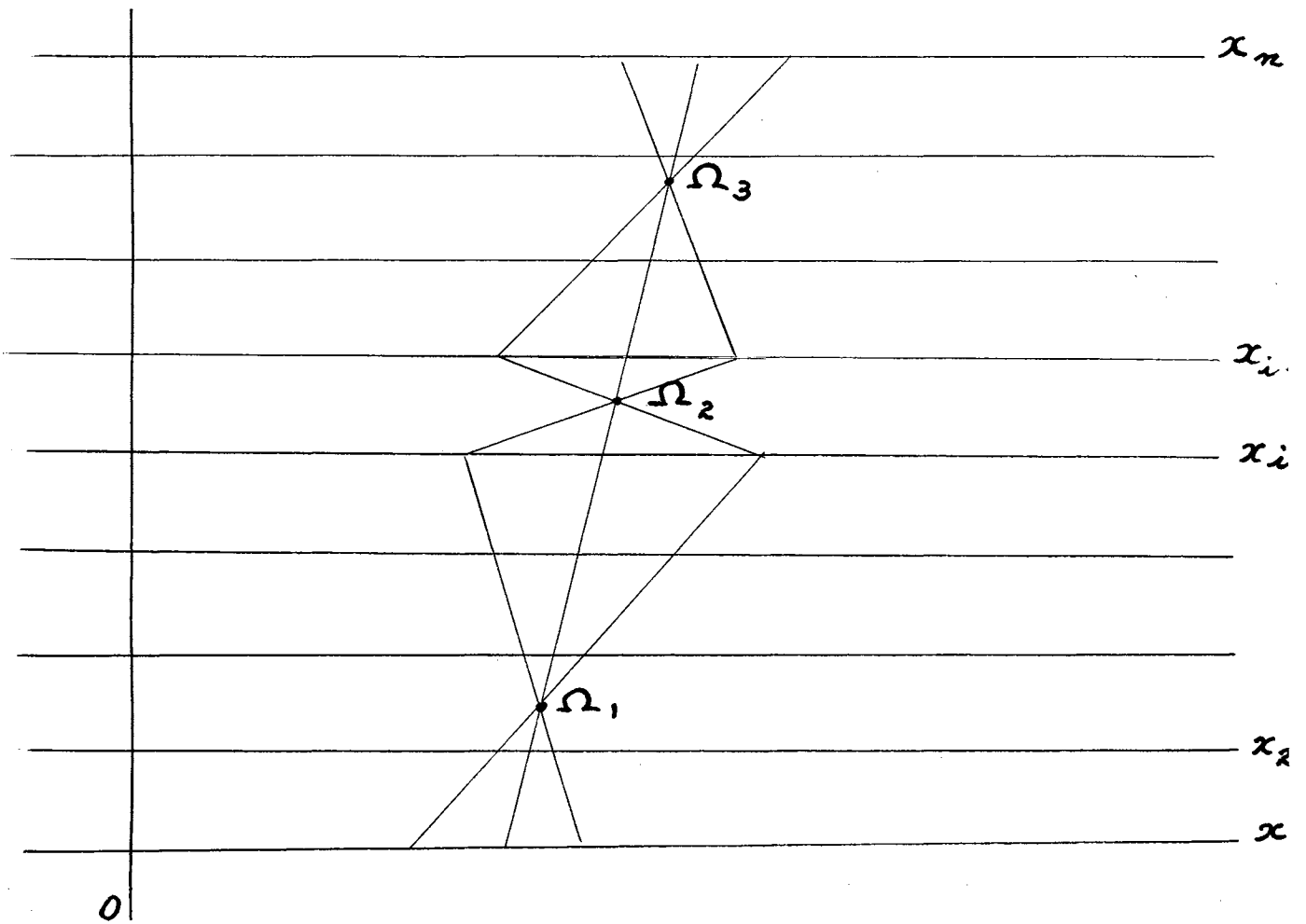
Fig. 8

## 3-The Graphical Solution of a System of Four Linear Equations in Four Unknowns

This is an application of the results obtained in parts III and V of this paper.

Consider the following four linear equations:

1) $ax + by + cz + du = e$
2) $a'x + b'y + c'z + d'u = e'$
3) $a''x + b''y + c''z + d''u = e''$
4) $a'''x + b'''y + c'''z + d'''u = e'''$

The first and last pairs of the above equations determine two planes in $E_4$. The intersection of these two planes is the solution of the system. This intersection is generally a point unless the two planes lie in the same 3-space. In this case the intersection is either a line and the system has infinitely many solutions, or the two planes are parallel and the system has no solution.

To find this intersection graphically we take the following two steps:

a) Map each of the two planes into two planes in space $P_4$

b) Find the intersection of the two planes.

The first step could be accomplished by choosing three points in each plane and mapping them on $P_4$. Since a point in a plane has two degrees of freedom we could always assign two of the coordinates of the point arbitrarily.

In particular we can choose two of the coordinates of two of the points such that they determine the axis of the plane. This could be done in a way similar to what we have

done in pages 31 and 32.

The step b) of the solution of the problem has already been given on pages 14-17.

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# VITA

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Thesis: A MAPPING OF EUCLIDEAN N-SPACE ON THE PLANE

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THESIS TITLE:    A MAPPING OF EUCLIDEAN N-SPACE ON THE PLANE

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