ON LATTICE VALUED METRICS

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PREFACE

The present study originated as a result of the author's desire to become more familiar with the foundations of mathematics, particularly general topology and modern algebra. Greater familiarity with the system of real numbers, which constitutes one of the most fundamental structures of mathematics, was another goal. The study of metric spaces and their generalizations seemed an excellent way to accomplish these ends.

This thesis is a study of certain topological neighborhood spaces which the author defined in a manner not previously done in the mathematical literature.

Indebtedness is acknowledged to Dr. D. O. Ellis of the University of Florida for providing the author with reprints of his published articles on subjects related to this study, and to the mathematics faculty of the Oklahoma A. and M. College, particularly Dr. O. H. Hamilton, for guidance and constructive criticism during the preparation of this thesis.

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I. INTRODUCTION

The present paper is devoted to the consideration of topological spaces in which the topology is defined by means of a function of two variables in the space taking values in a lattice. The conditions on the function are analogous to the conditions on an ordinary metric taking values in the set of non-negative real numbers. Hence the name lattice-valued metric.

This system has not been treated in the mathematical literature. However, several related mathematical systems have been treated. References to these will be given in the following pages.

Many generalizations of the notion of metric space have been studied. The present study suggested itself to the author because a lattice seems to be the most general system in which the triangle axiom can be treated. That is, for the triangle axiom it is necessary to have some binary operation and some partial ordering relation. A lattice is an algebra which provides just these properties.

The lattice used in our discussion will be described completely in the next section. The topological space, defined by means of the lattice, is introduced and treated in Section III.

There follows a list of symbols, with their meanings, which will be used in this paper.

$a \varepsilon A$	the element a is a member of the set A
$a \notin A$	the element a is not a member of the set A
$a \neq b$	the elements a and b are different
$A \subset B$ . . .	the set A is a subset of the set B
$A \cap B$ . . .	the intersection of the sets A and B
$A \cup B$ . . .	the union of the sets A and B
$A-B$ . . .	the intersection of the sets A and the complement of B
$E(x: P(x))$.	the set of elements having property P
$a+b$	the lattice join of a and b
$a.b$	the lattice meet of a and b
$a<b$	the element a precedes the element b
$a \leq b$	the element a precedes or is equal to the element b.

II. THE LATTICE

Birkhoff (1) (see bibliography at the end of the paper) has defined a lattice as a partly ordered set in which each pair of elements has a join and meet. These will be defined presently. A partly ordered set is a set in which is defined a relation $\leq$ with the properties:

O1. $a \leq a$ (reflexive)

O2. $a \leq b$ and $b \leq a$ imply $a = b$ (anti-symetric)

O3. $a \leq b$ and $b \leq c$ imply $a \leq c$ (transitive)

Such a relation is called an order relation. We take $b \geq a$ to mean $a \leq b$. We write $a < b$ to mean $a \leq b$ but $a \neq b$.

If in addition the property:

O4. $a \leq b$ or $a = b$ or $a \geq b$ (trichotomy)

holds, the set is said to be simply or linearly ordered. Such a set is also called a chain. In the foregoing expressions, the letters represent arbitrary elements of the set: thus universal quantifiers are not written.

An upper bound of a subset X of a partly ordered set L is an element b such that $x \leq b$ for all $x \varepsilon X$. A least upper bound of a subset X of a partly ordered set L is an element b such that:

$x \leq b$ for all $x \varepsilon X$ and

$x \leq c$ for all $x \varepsilon X$ implies $b \leq c$.

A lower bound of a subset X of a partly ordered set L is an element b such that $b \leq x$ for all $x \varepsilon X$. A greatest lower bound of a subset X of L is an element b such that:

$b \leq x$ for all $x \varepsilon X$ and

$c \leq x$ for all $x \varepsilon X$ implies $c \leq b$.

The least upper bound of a set X is denoted by sup X; the greatest lower bound is written inf X. These are read supremum and infimum, respectively.

If X is a two element subset of L, say $X=(a,b)$, we denote sup X by $a+b$ (read join) and inf X by $a.b$ (read meet). Thus a lattice is a partly ordered set in which each pair of elements has a meet and join.

The introduction of the + and . notation leads to an algebraic treatment of lattices. The following properties hold:

L1. $x=x+x=x.x$ (idempotent)

L2. $x+y=y+x$, $x.y=y.x$ (commutative)

L3. $(x+y)+z=x+(y+z)$, $(x.y).z=x.(y.z)$ (assosciative)

L4. $x=x+(x.y)=x.(x+y)$ (absorptive)

These laws can be proved as follows:

Let us note first that the meet and join are unique. For if x and y are elements of L, and a and b are two meets of x and y, we have

$a \leq b$ because b is a meet of x and y, and

$b \leq a$ because a is a meet of x and y.

Hence $a=b$ by O2. A similar proof holds for joins.

To prove L1, we see that x is an upper bound of the pair x,x. But no smaller element can be an upper bound of a set containing x. Thus $x=x+x$. A similar argument proves $x=x.x$.

To see that L2 is true, we only need to notice that the definitions of meet and join are symetric in the two elements x and y.

For a proof of L3, we observe that $(x+y)+z$ and $x+(y+z)$ are both suprema for the set consisting of x, y, and z. Hence they are equal because of the uniqueness of suprema. The assosciative law for meet is proved similarly.

To prove L4, we have $x\geq x.y$ since $x.y$ is a lower bound of the set containing x. But $x\geq x$ by O1. Hence $x\geq x+x.y$. But $x+x.y\geq x$, since $x+x.y$ is an upper bound of the set containing x. Hence $x=x+xy$ by O2. The proof of the other half of L4 is analogous.

A complete lattice is one in which every subset has a sup and inf. Thus a complete lattice has universal bounds O and I; $0\leq a$ for all $a\varepsilon L$, $a\leq I$ for all $a\varepsilon L$. Here $0=\inf L$, and $I=\sup L$.

An element a of a lattice is said to be meet irreducible if it cannot be written as the meet of a pair of elements of the lattice each distinct from a. Thus we say that 0 is meet irreducible if $a.b=0$ implies $a=0$ or $b=0$. An atom is an element which covers 0, that is, which follows 0 but follows no other element. Note that if 0 is meet irreducible, the lattice is non-atomic. For if a and b are atoms, $a.b=0$, so that 0 is meet reducible.

Hereafter, the letter L will denote a complete lattice in which 0 is meet irreducible. The reason for this restriction will appear presently. Let us call this property L5;

L5. $0 \varepsilon L$ is meet irreducible.

Birkhoff (1) has discussed several topologies which may be considered in L itself. The interval topology of a lattice is defined by taking the closed intervals as a sub-base of closed sets. These terms will now be explained. A closed interval is the set $E(x: a \leq x \leq b)$ for any elements a and b of L. Here the notation $E(x: \ldots)$ means the set of all elements x having the indicated property.

A base of closed sets is a collection of sets such that every closed set is an intersection of sets in the base. A sub-base of closed sets is a collection whose finite unions form a base.

Our definition of the interval topology then means that a set is closed if and only if it is an intersection of finite unions of closed intervals.

Since a point is a closed interval, we note that a lattice is a T_1 space in its interval topology. A T_1

space is a space in which points are closed sets.

Birkhoff (1) has proved that a complete lattice is compact in its interval topology. The usual definition of compactness is that every open covering of a set contains a finite subcovering. This means that a set A is compact if $A \subset \bigcup G_\alpha$ implies $A \subset \bigcup G_i$, where the G_α is any collection of open sets, and G_i is a finite subcollection of the G_α.

An equivalent formulation of this concept can be given in terms of the finite intersection property. We say that a collection of sets has the finite intersection property if when the intersection of every finite subcollection is non-empty, the intersection of the collection is non-empty. A space is compact if and only if every subcollection of its closed sets has the finite intersection property.

To prove that a complete lattice is compact in its interval topology, it is sufficient to prove that its closed intervals have the finite intersection property.

Let (a_α, b_α) be a collection of closed intervals, such that any two of them have a non-empty intersection. Then $a_\alpha \leq b_{\alpha'}$ for each α, α'. Hence $a = \sup a_\alpha \leq \inf b_\alpha = b$. We see that the closed interval (a,b) is contained in each of the closed intervals of the collection, so that their intersection is not empty.

III. THE SPACE

Let S be a set and D a function mapping S><S into L. Here S><S is the Cartesian square of S, or the set of all ordered pairs of elements of S. Thus D is a function of two variables in S. Let the mapping D have the following properties:

D1. $D(a,b)=0$ if and only if $a=b$

D2. $D(a,b)=D(b,a)$

D3. $D(a,b)+D(b,c)\geq D(a,c)$.

Here $D(a,b)$ denotes the element of L which corresponds under D to the pair a,b of S. We shall use lower case letters near the beginning of the alphabet for the elements of S. Elements of L will, when necessary, be denoted by the letters near the middle of the alphabet. We shall abbreviate $D(a,b)$ by ab; thus ab denotes the element of L which corresponds to the pair a,b of S under the mapping D. The axioms above now read:

D1. $ab=0$ if and only if $a=b$

D2. $ab=ba$

D3. $ab+bc\geq ac$.

We say that S is an L-metrized space.

Some systems related to that under consideration have been treated recently in the literature. Blumenthal (2) defined a Boolean metric space as a system of the present type in which the lattice is a Boolean algebra. No topology is introduced. In a private communication to the author, Dr. Blumenthal has indicated that a paper to appear later treats the convergence topology of the Boolean algebra.

Ellis and Sprinkle (4) discussed the topology of such a B-metrized space for a sigma-complete Boolean algebra. A sigma-complete Boolean algebra is one in which every countable subset has a supremum and infimum. Kelly and Lapidus (6) discussed the geometry of an L-metrized space. In case S=L, L is said to be autometrized. Autometrized Boolean algebras were discussed in Ellis (3).

As in the case of the ordinary metric axioms, we could replace D2 and D3 by one condition D4:

D4. $ab+cb \geq ac$.

Theorem 1. Axioms D1 and D4 are equivalent to axioms D1, D2, and D3.

Proof:

Put a=b in D4. Thus $bc \leq bb+cb=cb$, since bb=0 by D1. Now permute the letters of D4; $cb \leq ca+ba$ and put a=c. Then $cb \leq cc+bc=bc$ by D1. Now O2 gives bc=cb and we have proved D2. D3 now follows immediately. The converse is obvious.

We shall define a toplogy on the set S by using the mapping D to define a closure operator on the subsets of S. We define $D(A,B)=\inf ab$, for $a\varepsilon A$, $b\varepsilon B$, as the distance between A and B, where A and B are any two subsets of S. Obviously, $D(A,B)=D(B,A)$. Now we define the closure A of a subset A of S:

$\overline{A}=E(x:\ \ D(x,A)=0)$.

The closure of a subset of S is thus another subset of S consisting of the elements at lattice distance 0 from S.

We shall show that the closure operator defined above satisfies the usual axioms for a closure operator.

The axioms for a closure operator are:

C1. $\overline{X}\supset X$ (isotone)

C2. $\overline{\overline{X}}=\overline{X}$ (idempotent)

C3. $\overline{X}\cup\overline{Y}=\overline{X\cup Y}$ (distributive)

C4. $\overline{\phi}=\phi$, where ϕ is the null set.

Theorem 2. The operator $\overline{A}$ defined on the subsets of S by the distance function D is a closure operator.

Proof:

Axioms C1 and C4 are obvious. To establish C2, we need only prove that $\overline{\overline{X}}\subset\overline{X}$, since the reverse inclusion is obvious. But $a\varepsilon\overline{\overline{X}}$ means that for any $m\varepsilon L$, $ay<m$ for some $y\varepsilon\overline{X}$. This means that for some $x\varepsilon X$, $yx<m$ also. Then $ax<ay+yx<m+m=m$. But m is arbitrary, so we have proved $a\varepsilon\overline{X}$. To prove C3, let $a\varepsilon\overline{X}\cup\overline{Y}$. If $a\varepsilon\overline{X}$, we have $D(a,X)=0$,

so that $D(a, X\cup Y)=0$ and $a\varepsilon\overline{X\cup Y}$. Similarly for $a\varepsilon\overline{Y}$. We have proved that $\overline{X}\cup\overline{Y}\subset\overline{X\cup Y}$. For the reverse inclusion, if $a\notin\overline{X}\cup\overline{Y}$, then $a\notin\overline{X}$ and $a\notin\overline{Y}$, and so for some m and n of L, and all x, y of X, Y, we have $ax>m$, $ay>n$ and hence $ax>m.n>0$ for $x\varepsilon X\cup Y$ by L5. This means that $D(a, X\cup Y)>0$, so that $a\notin\overline{X\cup Y}$. This completes the proof.

Theorem 3. S is T_1 space.

Proof:

A T_1 is a space in which points are closed. Our theorem follows directly from D1. Let $a\varepsilon S$. Then $\overline{a}=a$, for the set at zero distance from a is a itself, because if $ab=0$, we have $a=b$ by D1.

In order to define a neighborhood topology directly in S, let us define:

$N_m(a)=E(x\colon\ ax<m)$, $a\varepsilon A$, $m\varepsilon L$

as a neighborhood of a.

The neighborhood axioms are:

N1. $a\varepsilon N_m(a)$ and each point has a neighborhood.

N2. $b\varepsilon N_m(a)$ implies that some $N_n(b)\subset N_m(a)$.

N3. $a\varepsilon N_m(a)$, $a\varepsilon N_n(a)$ imply that some

$N_p(a)\subset N_m(a)\cap N_n(a)$.

In this last expression, $\cap$ is used to denote set intersection.

Theorem 4. S is a neighborhood space.

Proof:

N1 is obvious, since $N_I(a)$ is a neighborhood of a, and $aa=0<m$. To prove N2, let $b\varepsilon N_m(a)$ and consider $x\varepsilon N_m(b)$. Then $ax\leq ab+bx<m+m=m$, so that $x\varepsilon N_m(a)$. Thus $N_m(b)\subset N_m(a)$. To show that N3 is satisfied, observe that $N_{m.n}(a)$ is contained in $N_m(a)$ and $N_n(a)$, since $ax<m.n$ implies that $ax<m$ and $ax<n$.

We now define as usual an open set as a set which is a union of neighborhoods. A closed set is one whose complement is open.

Theorem 5. S is a T_1 space in its neighborhood topology.

Proof:

If $a\neq b$ are in S, we have $a\notin N_{ab}(b)$. We have shown that of any two points of S, each has a neighborhood not containing the other. To show that points are closed, let $a\varepsilon S$, and for each $b\neq a$, let N(b) be a neighborhood of b which does not contain a. Then $S-a=\bigcup_{b\varepsilon S} N(b)$, so that the complement of a is open. Thus a is a closed set.

Theorem 6. A set is closed if and only if it is equal to its closure.

Proof:

Let $A\subset S$ be closed and $b\varepsilon S-A$. Then for some m,

$b\varepsilon N_m(b) \subset S-A$. Thus $ab \geq m$ for all $a\varepsilon A$. Hence $D(b,A) \geq m > 0$, so that $b \not\varepsilon \overline{A}$. This proves that $\overline{A} \subset A$. Since we always have $A \subset \overline{A}$, we have proved that $A = \overline{A}$.

Now assume $A = \overline{A}$. We show that the complement of A is open. Let $b\varepsilon S-A$. Then $ab \geq m > 0$ for some $m\varepsilon L$ and all $a\varepsilon A$, so that $N_m(b) \subset S-A$. Thus S-A is open and so A is closed. This completes the proof.

The need for condition L5 is seen from the following:

Theorem 7. If S is a neighborhood space defined by an L-metric D, then condition L5 holds.

Proof:

Condition L5 means that $m.n=0$ implies that $m=0$ or $n=0$. Consider $N_m(a) \cap N_n(a)$. If this intersection is to contain a neighborhood of a, we must have some $p\varepsilon L$, $p>0$ for which $N_p(a) \subset N_m(a) \cap N_n(a)$. But $p \leq m.n$, since $p \leq m$ and $p \leq n$ follow from $a\varepsilon N_m(a)$, $a\varepsilon N_n(a)$, respectively. Thus p is a lower bound of the pair m, n so that $p \leq m.n$. This completes the proof.

A Hausdorff space is a space in which each of two distinct points have disjoint neighborhoods.

Theorem 8. S is a Hausdorff space.

Proof:

We show that two elements a, b of S have disjoint neighborhoods. Now $N_{ab}(a)$ and $N_{ab}(b)$ are disjoint, for if $x \varepsilon N_{ab}(a) \cap N_{ab}(b)$, we have $ax < ab$, and $bx < ab$, so that $ab \leq ax + xb < ab + ab = ab$. This is a contradiction.

Theorem 9. The lattice metric is a continous function of its variables.

Proof:

It will suffice to prove that for each $m \varepsilon L$, $x \varepsilon N_m(a)$ and $y \varepsilon N_m(b)$ imply that $xy < ab + m$ and $ab < xy + m$.

But if $ax < m$ and $by < m$, then

$ab \leq ax + xb \leq ax + xy + yb < m + xy + m = xy + m$.

Similarly, $xy < ab + m$.

A topological space is said to be normal if every pair of disjoint closed subsets of it are contained in disjoint open subsets of it. We can prove that the space S has this property.

Theorem 10. S is normal.

Proof:

Let A and B be disjoint closed subsets of S. Then $D(A,B) = m > 0$. For each $a \varepsilon A$ and $b \varepsilon B$, consider $N_m(a)$, $N_m(b)$; $A \subset \bigcup_{a \varepsilon A} N_m(a)$, $B \subset \bigcup_{b \varepsilon B} N_m(b)$. If these open sets are not disjoint, let c be a common point. This means that for some $a \varepsilon A$, and $b \varepsilon B$, we have $c \varepsilon N_m(a)$, $c \varepsilon N_m(b)$. Thus $ab \leq ac + bc < m + m = m$, contrary to $D(A,B) = m$.

A subset X of a topological space is said to be connected if it is not the union of two disjoint non-empty open sets. A set is connected if and only if it contains no proper subset which is both open and closed. For if $A \subset X$ is open and closed, then $X-A$ is open, so that $X=A \cup (X-A)$, contrary to connectedness of X. On the other hand, if $X=A \cup B$, where A and B are both open, then A is also closed, being the complement of an open set.

A space is called totally disconnected if it contains no connected subset of more than one point. We can show that the space S is totally disconnected.

Theorem 11. S is totally disconnected.

Proof:

We first show that a neighborhood is closed as well as open in S. Consider $N_m(a)$ and let $D(b,N_m(a))=0$. Then for some $x \varepsilon N_m(a)$, $bx<m$. But $ax<m$. Hence $ab \leq ax+bx<m+m=m$. Thus $b \varepsilon N_m(a)$, so that $N_m(a)$ is closed. Now suppose $A \subset S$ contains more than one point. Let a, $b \varepsilon A$. Then $A \cap N_{ab}(a)$ is a subset of A both open and closed in A, so that A is not connected. We have thus established that S does not contain a connected subset of more than one point.

In an ordinary metric space, a sequence $a_1, a_2, \dots$ is called a Cauchy sequence if for each positive number m, there is an integer k such that $a_i a_j<m$ if $i>k$ and $j>k$. Here $a_i a_j$ denotes the ordinary distance from a_i to a_j. We can generalize this notion to L-metrized spaces as follows. Let a_1, a_2, ... be a sequence of elements of S. We say that this is a Cauchy sequence if for each $m \varepsilon L$ there is an integer k such that $a_i a_j < m$ if $i > k$ and $j > k$. We can prove the following theorem about Cauchy sequences in S.

Theorem 12. A sequence is a Cauchy sequence if and only if for each $m \varepsilon L$ there is an integer k such that $a_i a_{i+1} < m$ if $i > k$.

Proof:

The necessity is obvious.

To prove the sufficiency, let a_1, a_2, ... be a sequence satisfying the conditions of the theorem.

Then:

$a_i a_{i+1} < m$ if $i > k$, where k is the integer given by the theorem. Suppose $j > i$. Now:

$$a_i a_j \leq a_i a_{i+1} + a_{i+1} a_{i+2} + \dots + a_{j-1} a_j \text{ by D3}$$

$$< m + m + \dots + m \text{ by O3}$$

$$= m \text{ by L1.}$$

This proves the sufficiency.

The following theorem gives a partial characterization of spaces having a lattice valued metric. A related result appears in Hausdorff (5), who proves that a compact space for which each pair of points have distanz zero is connected. The distanz ∂ between two points of a metric space is given by :

$\partial(a,b)=\inf \max x_i x_{i+1}$, where $x_i x_{i+1}=D(x_i, x_{i+1})$ is the metric distance between x_i and x_{i+1} and $a=x_1$, x_2, ..., $x_n=b$ is a finite sequence of points in the space.

An ultra-metric space is an ordinary metric space in which the triangle inequality is strengthened to read $ac\leq\max(ab, bc)$, where ab denotes the distance from a to b. The concept of ultra-metric space is explained and discussed further in examples 2 and 3 of this paper.

Theorem 13. A compact metric space S has an ultra-metric if and only if it is totally disconnected.

Proof:

It follows from theorem 11 that a space is totally disconnected if it has an ultra-metric.

To prove the converse for compact metric spaces, it is sufficient to show that in a totally disconnected compact metric space, the distanz is an ultra-metric equivalent to the original metric.

First we prove axiom D1 for ∂. If $a=b$, clearly $\partial(a,b)=0$. Let $\partial(a,b)=0$, and suppose $a\neq b$. Consider the set $A=E(x: \partial(a,x)=0)$. This set contains at least two elements by hypothesis. We show that it is connected, contrary to the hypothesis that S is totally disconnected. If A is not connected, let $A=P\cup Q$ be a separation into disjoint closed subsets. Then because of compactness we have two sequences $p_n \varepsilon P$ and $q_n \varepsilon Q$ such that $D(p,p_n) \to 0$ for some $p \varepsilon P$ and $D(q,q_n) \to 0$ for some $q \varepsilon Q$ and $D(p,q)=D(P,Q)$. But $p \neq q$, since P and Q are disjoint, so we have $D(P,Q)>0$. This shows that the distanz ∂ between a point of P and a point of Q is not zero, contrary to hypothesis.

The distanz clearly satisfies D2.

If D3 should fail for ∂, we would have some three points a,b, and c for which $\max(\partial(a,b), \partial(b,c)) < \partial(a,c)$. But then $\partial(a,c)$ would not be the infimum as defined, since the sequences of the form a, ..., b, ..., c are among those over which the infimum is taken.

To show the equivalence of D and ∂, let us first remark that $\partial(a,b) \leq D(a,b)$ for all a and b of S, since the pair a,b is a sequence from a to b. Now suppose $\partial(a,a_n) \to 0$. Then $D(a_{ni},p) \to 0$ for some $p \varepsilon S$ because of compactness. Here a_{ni} is a subsequence of a_n. But this implies that $\partial(a_{ni},p) \to 0$, since $\partial(a,b) \leq D(a,b)$. This gives a=p, since limits are unique under an ultra-metric. Thus we have proved $D(a_{ni},a) \to 0$. We could not have $D(a_{nj},q) \to 0$ for another point $q \varepsilon S$ and subsequence a_{nj} of a_n, since in that case we would have $\partial(a_{nj},q) \to 0$, contrary to the assumption that a is the sequential limit of a_n. This proves the equivalence of D and ∂.

IV. EXAMPLES

Example 1.

The following example is discussed in Blumenthal (2) and Ellis (3).

Let B be a Boolean algebra, and define $D(B><B) \subset B$ as $D(a,b)=ab=(a.b')+(a'.b)$. Here we take $S=L=B$. This is the Boolean metric, making B an autometrized Boolean space. A Boolean algebra is a lattice in which the distributive laws

$a.(b+c)=a.b+a.c$ and

$a+b.c=(a+b).(a+c)$

hold, and in which each element has a complement. The complement a' of a is an element having the properties

$a.a'=0$, $a+a'=I$, where 0 and I are the universal bounds.

We shall show that the conditions D1, D2, and D3 of a lattice metric are satisfied by the Boolean metric defined above.

D1. If $ab=0$, that is $a.b'+a'.b=0$, then $a.b'=0$ and $a'.b=0$, that is $a\leq b$ and $b\leq a$ so $a=b$.

If $a=b$, then $a.b'=0$, $a'.b=0$, so $ab=0+0=0$. This proves D1.

D2. To prove D2, we observe that ab is symetric by definition.

D3. We have, using the definitions

$$
\begin{aligned}
ab+bc &= a.b'+a'.b+b.c'+b'.c \\
&= a.b'+b.c'+a'.b+b'.c \\
&\geq a.c'.b'+b.a.c'+a'.c.b+b'.a'.c \\
&= a.c'.(b'+b)+a'.c.(b+b') \\
&= a.c'+a'.c \\
&= ac.
\end{aligned}
$$

This proves D3.

This example illustrates what is perhaps the most natural way in which an L-metric can arise. It is treated in both Blumenthal (2) and Ellis (3). The only proofs so far published are in Ellis (3). They deal with the geometry of the autometrized space, i.e., with the distance preserving transformations of the space.

Example 2.

Another example of L-metrized spaces is provided by the ultra-metric spaces which first appear in Hausdorff (5), page 158.

An ultra-metric space is an ordinary metric space in which the triangle inequality is strengthened to read $\partial(a,c) \leq \max(\partial(a,b),\ \partial(b,c))$, where ∂ denotes the ordinary metric.

Now let L be the set of real numbers in the unit interval. This set of numbers forms a complete lattice under the ordinary order relation $\leq$. Here sup X and inf X have the same meaning as usual. But a+b means the sup of the two element set containing a and b; thus a+b is the maximum of a and b. Similarly a.b means the minimum of a and b. Clearly 0 is meet irreducible.

We see that an ultra-metric space is an L-metrized space with the metric taking values in the lattice described above. Consequently, all of the theorems proved above for an L-metrized space are valid in any ultra-metric space.

Example 3.

The following example shows that an L-metrized space may be dense-in-itself. This means that each point of it may be a limit point. Hence the theorem on total disconnectedness cannot be strengthened to read that some of the points are isolated.

Let S be the Cantor set in the closed unit interval, and L the lattice of real numbers in the closed unit interval. The Cantor set is the set of numbers of the unit interval which can be written in the triadic system without using ones. We can think of it as the set remaining after removing the open middle third of the interval, the open middle third of the remaining intervals, etc. We shall define an ultra-metric on the Cantor set which is topologically equivalent to the usual metric.

We define the distance ab between a and b of C as the length of the longest complementary interval which lies between them on the closed unit interval. Then ab is clearly equivalent to $|a-b|$, the usual metric. But $\max(ab,bc) \geq ac$ for any three points of C. Thus ab is an ultra-metric.

The Cantor set is well known to be dense-in-itself. It is in fact perfect, that is, it is equal to the set of its limit points.

V. CONCLUSIONS

The L-metrized spaces have been proved to be both a generalization and a specialization of ordinary metric spaces. The lattice L is more general that the set of non-negative real numbers in that the ordering need not be linear nor must the lattice contain a countable dense subset. However, the operation of join is only a partial analogue of ordinary addition since it is defined by means of the order relation, whereas the addition of arithmetic is a field operation less simply related to the ordering of real numbers.

On the other hand, the triangle axiom on the L-metric is perhaps more properly an analogue of the stronger ultra-metric axiom (see example 2). We see in the theorems of Section III that the strength of the lattice metric axioms partially compensates for the greater generality of the lattice. Thus, the separation theorems on metric spaces are also true in L-metrized spaces. However, the proofs are obtained by different methods.

The similarity between the L-metrized spaces and the ordinary metric spaces ends with the separation theorems. The theorem that an L-metrized space is totally disconnected has no analogue in the theory of metric spaces, nor does the theorem on Cauchy sequences. The L-metrized space is more nearly similar to the ultra-metric space in these latter properties.

The theory of L-metrized spaces constitutes a chapter in the general theory of distance geometries as outlined in the work of Blumenthal and Ellis, listed in the bibliography. However, their work is concerned principally with the distance relations, while the interest in this paper has been centered on the topological properties.

VI. BIBLIOGRAPHY

1. Garrett Birkhoff, Lattice Theory, New Edition, American Mathematical Society Colloquium Publications, Vol. XXV, 1948.

2. L. M. Blumenthal, Boolean Geometry I, Abstract in Bulletin of the American Mathematical Society, Vol. 58, No. 4, July 1952.

3. D. O. Ellis, Autometrized Boolean Algebras, Canadian Journal of Mathematics, Vol. 3, 1951.

4. D. O. Ellis and H. O. Sprinkle, Topology in B-metrized Spaces, Abstract in Bulletin of the American Mathematical Society, Vol. 58, No. 5, Sept. 1952.

5. Hausdorff, Mengenlehre, Dover, New York.

6. L. M. Kelley and Leo Lapidus, The Geometry of L-metrized Spaces, Abstract in the Bulletin of the American Mathematical Society, Vol. 58, No. 6, Nov. 1952.

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