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GENERALIZED CONVEXITY STRUCTURES AND THEIR PRODUCTS

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DISSERTATION COMMITTEE

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CONVEXITY STRUCTURES AND THEIR PRODUCTS

INTRODUCTION

Various attempts have been made to place convexity in an axiomatic setting, but only scant attention has been devoted to the product of general convexity structures. In this paper, as in [11], we consider a set X, a family of subsets of X closed under intersection (which are called convex sets), and a closure operator conv on P(X) satisfying certain convexity properties. Relationships between the Carathéodory, Helly, and Radon numbers in such a setting are explored in [11] and we consider them here in connection with products.

Shirley [16] introduced a topology in this setting culminating in a proof of the Krein-Mil'man theorem. In Chapter One we will introduce the basic definitions and axioms needed for the remaining chapters and give proofs of a few classical convexity theory theorems as applied to the general setting.

Eckhoff [5] was the first to define the product of these generalized convexity structures and proved a Radon theorem in a general setting. Reay [14] proved a Carathéodory theorem when the factor spaces in the Eckhoff product are Euclidean. In Chapter Two we obtain results concerning the Eckhoff product and bounds concerning the Carathéodory and Helly numbers in a general setting.

A method of defining lines in the product of two generalized convexity spaces was recently given by Sandstrom [15] by considering a family of lines in both factor spaces and families of real-valued functionals. In Chapter Three we define a new type of product for generalized convexity structures and give results concerning the Carathéodory and Radon numbers in the product space.

Finally, in Chapter Four we explore another method of defining the product of two convexity structures when the underlying factor spaces are vector spaces. We prove that this product generates the usual convexity structure when each factor space has the usual convexity structure. We then obtain Carathéodory and Helly theorems for this product and derive results concerning affine mappings of the factor and product spaces.

CHAPTER I

AXIOMATIC CONVEXITY

There have been many recent papers dealing with the theory of generalized convexity structures in which the authors have defined the class of "convex" sets axiomatically and proceeded to develop a theory that yields analogues of the theorems in the classical setting. See, for example, Bryant and Webster [2], Kay [10], and Cantwell [4]. In this chapter we shall present the basic definitions needed for the remaining chapters where products of generalized convexity structures are considered, and we shall include the proofs of some analogues of classical convexity theorems which have not yet been published.

<u>1.1. DEFINITION</u>. A collection of subsets $\mathcal{C}$ of a set X will be called a <u>convexity structure</u> if and only if

i) X and $\emptyset$ belong to $\mathcal{C}$, and

ii) $\mathcal{C}$ is closed under intersections; that is, if $C_i \in \mathcal{C}$ for each $i \in I$ then $\bigcap_{i \in I} C_i \in \mathcal{C}$.

<u>1.2. DEFINITION</u>. If $\mathcal{C}$ is a convexity structure for a set X and if $E \subseteq X$ then the <u>convex hull</u> of E is denoted by

conv E and is defined by conv $E = \bigcap\{C \in \mathcal{C} \mid E \subseteq C\}$. A set E will be called <u>convex</u> if and only if E = conv E. A subfamily $\mathcal{B}$ of $\mathcal{C}$ is called a <u>basis</u> of $\mathcal{C}$ if and only if each member of $\mathcal{C}$ is obtainable as an intersection of members of $\mathcal{B}$.

In order to promote brevity and permit easier reading, a singleton set $\{x\}$ will be denoted by x, and the convex hull of a two element set, a <u>segment</u>, will be denoted by juxtaposition, i.e., conv $\{p,q\}$ = pq.

<u>1.3. THEOREM</u>. If $\mathcal{C}$ is a convexity structure and A and B are subsets of X, then,

i) $A \subseteq$ conv A,

ii) if $A \subseteq B$ then conv $A \subseteq$ conv B, and

iii) conv $\{$conv A$\}$ = conv A.

The next two definitions are commonly used as axioms to be imposed on a convexity structure; see [8] and [11]. The concept of regular segments was recently introduced in [10].

<u>1.4. DEFINITION</u>. A convexity structure $(X, \mathcal{C})$ has the property of <u>domain finiteness</u> if for each $S \subseteq X$, conv $S = \bigcup\{$conv $F \mid F$ is finite and $F \subseteq S\}$.

<u>1.5. DEFINITION</u>. A convexity structure $(X, \mathcal{C})$ is said to be <u>join-hull commutative</u> if for each $S \subseteq X$, conv $\{x \cup S\} = \bigcup\{xs \mid s \in$ conv $S\}$.

1.6. DEFINITION. The segment xy is said to be nondiscrete if and only if the open segment $(x,y) = xy \setminus \{x,y\} \neq 0$. The segment is decomposable if and only if for every $z \in xy$, $xz \cap zy = z$ and $xz \cup zy = xy$. The segment is extendible if and only if $xy \subsetneq xz \setminus z = [x,z)$ for some $z \neq y$. A segment is called regular if and only if it is nondiscrete, decomposable, and extendible.

1.7. DEFINITION. A convexity structure $(X, \mathcal{C})$ is an interval convexity structure when conv $A = A$ if and only if $xy \subseteq A$ whenever $x \in A$ and $y \in A$.

As is proved in [11], a convexity structure that is domain finite and join-hull commutative is necessarily an interval convexity structure. An interval convexity structure is domain finite but not necessarily join-hull commutative.

For the remainder of this chapter $\mathcal{C}$ will denote a convexity structure for a set X in which the following axioms hold:

i) $(X, \mathcal{C})$ is domain finite,

ii) $(X, \mathcal{C})$ is join-hull commutative,

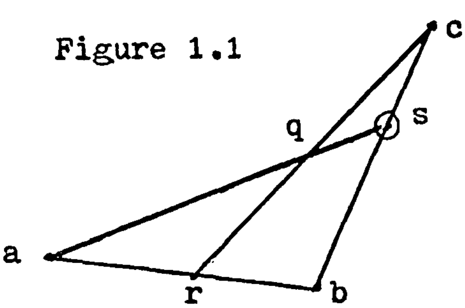
iii) Segments are regular.

That these axioms are independent can be seen from the following convexity structures.

1.8. EXAMPLE. a) Let $X = R^1$ and let $C \in \mathcal{C}$ if and only if C is a closed interval. Then $(X, \mathcal{C})$ will be a convexity structure satisfying ii) and iii) but not i). b) Let $X = R^3$ and let $C \in \mathcal{C}$ if and only if $C = X$ or C is a convex set with dimension less than or equal to 2. then $(X, \mathcal{C})$ is a convexity structure satisfying i) and iii) but not ii). c) Let X be any nonempty set and let $C \in \mathcal{C}$ if and only if $C \in 2^X$. Then $(X, \mathcal{C})$ will be a convexity structure satisfying i) and ii) but not iii).

1.9. LEMMA. In a convexity structure satisfying axioms i-iii, if $r \in ab$ and $q \in cr$ then there exists an $s \in cb$ such that $q \in as$.

Figure 1.1



Proof. Since $r \in ab$ and $q \in cr$, then $q \in \operatorname{conv}\{a,b,c\}$. By axiom ii) $q \in \operatorname{conv}\{a,b,c\} = \bigcup\{ad \mid d \in bc\}$. Hence, there exists an $s \in bc$ such that $q \in as$.

1.10. DEFINITION. Let $S \subseteq X$. The kernel K of S is the set of all points $z \in X$ such that $zx \subseteq S$ for all $x \in S$.

The following theorem was originally proved by Brunn in 1913 [1] for finite dimensional Euclidean spaces. It has been extended to linear spaces, e.g. see Valentine [21]. The following shows that it is also valid in our axiomatic setting.

1.11. THEOREM. The kernel of any set is convex.

Proof. Let $S \subseteq X$ and K be the kernel of S, and suppose $x, y \in K$, $x \neq y$, and $u \in xy$. Let z be an arbitrary point of

S. If $z = x$ or $z = y$ then $uz \subseteq S$. Let $v \in zu$. We know that $zy \subseteq S$ by the definition of K. Also, by Lemma 1.9, there exists a $w \in zy$ such that $v \in wx$. But then $w \in S$ and $wx \subseteq S$ by the definition of K; hence $v \in S$. Since v was an arbitrary point of zu, then $zu \subseteq S$. Hence $u \in K$ and $xy \subseteq K$ which says that K is convex.

1.12. DEFINITION. The kernel of a set S will be referred to as its convex kernel and we denote it by ck S.

1.13. DEFINITION. A set S is star-shaped if and only if ck S $\neq \emptyset$.

Valentine [21, Research Problem 9.3] posed the problem of characterizing the star-shaped sets S in a finite dimensional linear space in terms of the maximal convex subsets of S. The problem was solved by Guay [7], and extended by Smith [18] for linear spaces of arbitrary dimension. A characterization of star-shaped sets in our axiomatic setting is also possible, as the following development shows.

1.14. THEOREM. Let S be a nonempty subset of X and let $\{M_i\}_{i \in I}$ be the collection of maximal convex subsets of S, then $S = \cup M_i$.

Proof. Let $x \in S$ and consider the collection of all convex subsets of S which contain x. The collection is nonempty since x is convex, and this collection may be partially ordered by inclusion; that is $A \leq B$ if and only if $A \subseteq B$. Also, each linearly ordered subset has an upper bound, namely the union of the sets in the class, and so by Zorn's Lemma, there exists a maximal element, namely a maximal convex set containing X. Thus each element of S is contained in some maximal convex set, so $S = \cup M_i$.

1.15. THEOREM. A set S is star-shaped if and only if the intersection of all the maximal convex subsets of S is nonempty.

Proof. Let the collection of all maximal convex subsets of S be denoted by $\{M_i\}_{i \in I}$. If $z \in \cap M_i$, let $s \in S$. By Theorem 1.14 $s \in M_i$ for some $i \in I$. Hence $zs \subseteq M_i$ since M_i is convex, and since s was an arbitrary point of S, S is star-shaped with respect to z.

To finish the proof it will be sufficient to show that $\mathrm{ck}\ S \subseteq \cap M_i$. So suppose $z \in \mathrm{ck}\ S$ and suppose that there exists a maximal convex set M_i such that $z \notin M_i$. Then for all $m \in M_i$, $\cup\{zm \mid m \in M_i\} = \mathrm{conv}\ \{z, M_i\}$ is a convex set by axiom ii, and moreover it is a subset of S since $z \in \mathrm{ck}\ S$ and it properly contains M_i, contradicting the maximal nature of M_i. Hence $z \in M_i$ for each i and $\mathrm{ck}\ S \subseteq \cap M_i$.

Many of the important theorems of convexity are intimately related to an underlying topological structure and it is possible at this time to introduce a topology on the set X and to relate it to the convexity structure. Womble [22] introduced a topology into the convexity structure by considering the set X to be in E^n with the resulting topology and hyerplanes. Shirley [16] did it by considering the Hausdorff topology for P(X).

It might be natural to ask what additional axioms would be required to yield the usual convex structure when X is the Euclidean space E^n with the usual topology. Kay and Womble [11] have succeeded in listing a set of axioms for the convexity structure on E^n that will give the usual convex sets.

Recently in a paper by Mah, Naimpally, and Whitfield [12], a characterization of a linear topological space among all topological convexity structures was given.

CHAPTER II

THE ECKHOFF PRODUCT

The concept of defining the product of two mathematical systems is a logical outgrowth of the mathematical process in almost all areas of mathematics. The first work on the product of generalized convexity structures was done by J. E. Eckhoff in [5]. Recently Reay [14] considered Eckhoff's product in a restricted setting, and Sandstrom [15] considered products of generalized linear spaces.

The following definition follows the one given by Eckhoff.

<u>2.1. DEFINITION</u>. Given two convexity structures $(X, \mathcal{C}_x)$ and $(Y, \mathcal{C}_y)$, a set C in the Cartesian product space $X \times Y$ is convex if and only if C is the Cartesian product of a convex set in $\mathcal{C}_x$ with a convex set in $\mathcal{C}_y$. This product will be termed the <u>Eckhoff product</u> of X and Y, and denoted $(X \times Y, \mathcal{C}^E_{x \times y})$.

The following three definitions will be used in the remaining chapters. Relationships between the Carathéodory, Helly, and Radon numbers for generalized convexity structures were proved in [11].

2.2. DEFINITION. A convexity structure $(X,\mathcal{C})$ is said to have Carathéodory number c if and only if c is the least cardinal number for which it is true that for any $A \subseteq X$, $\operatorname{conv} A = \bigcup \{\operatorname{conv} B \mid B \subseteq A, \operatorname{card} B \leq c\}$.

2.3. DEFINITION. A convexity structure $(X,\mathcal{C})$ is said to have a Helly number h if and only if h is the least cardinal number for which it is true that each finite subfamily $\mathcal{F}$ of $\mathcal{C}$ having h+1 members has nonempty intersection if each h of them meet.

2.4. DEFINITION. A convexity structure $(X,\mathcal{C})$ is said to have Radon number r if and only if r is the least cardinal number for which it is true that each set $A \subseteq X$ having cardinality at least r possesses a partition (A,B) such that $(\operatorname{conv} A) \cap (\operatorname{conv} B) \neq \emptyset$.

We state here for later reference a result of Eckhoff on the Eckhoff product. The proof may be found in [5].

2.5. THEOREM. (Eckhoff) If $(X,\mathcal{C}_x)$ and $(Y,\mathcal{C}_y)$ are convexity structures with Radon numbers m and n respectively, then $(X \times Y, \mathcal{C}^E_{x \times y})$ has a Radon number r and, $\max(m,n) \leq r \leq m+n+1$.

It was recently proved by Sierksma and Boland [17] that the least upper bound for r in the above theorem is m+n. Also, Reay [14] had proved earlier that if the factor spaces are Euclidean spaces having the usual convexity structure then $\mathcal{C}^E_{x \times y}$ will have a Carathéodory number less than or equal to the sum of the dimensions of the factor spaces.

A few original results concerning the Eckhoff product will now be given; alternative definitions for products of generalized convexity structures will be considered later.

2.6. LEMMA. For any subsets $A \subseteq X$ and $B \subseteq Y$, $\operatorname{conv}(A \times B) = \operatorname{conv} A \times \operatorname{conv} B$.

Proof. Since $\operatorname{conv} A \times \operatorname{conv} B$ is a convex set in $X \times Y$ containing $A \times B$, we have $\operatorname{conv}(A \times B) \subseteq \operatorname{conv} A \times \operatorname{conv} B$. To reverse the inclusion, since $\operatorname{conv}(A \times B) \in \mathcal{C}^E_{x \times y}$, $\operatorname{conv}(A \times B) = C \times D$ for some $C \in \mathcal{C}_x$, $D \in \mathcal{C}_y$. But for any $x \in A$, $y \in B$ we have $(x,y) \in \operatorname{conv}(A \times B) \Rightarrow x \in C$ and $y \in D$, so C and D are convex sets containing A and B, and thus $\operatorname{conv} A \subseteq C$, $\operatorname{conv} B \subseteq D$. Hence $\operatorname{conv} A \times \operatorname{conv} B \subseteq C \times D = \operatorname{conv}(A \times B)$.

2.7. COROLLARY. If $p_1, p_2, \ldots, p_k \in X \times Y$ then $\operatorname{conv}(p_1, p_2, \ldots, p_k) = \operatorname{conv}(\pi_x(p_1), \pi_x(p_2), \ldots, \pi_x(p_k)) \times \operatorname{conv}(\pi_y(p_1), \pi_y(p_2), \ldots, \pi_y(p_k))$.

Therefore we know that if $A \subseteq X \times Y$ and $(x,y) \in \operatorname{conv} A$, where there exist m points $p_1, \ldots, p_m \in A$ with $x \in \operatorname{conv}\{\pi_x(p_1), \ldots, \pi_x(p_m)\}$ by X having Carathéodory number less than or equal to m, and n points $q_1, \ldots, q_n \in A$ with $y \in \operatorname{conv}\{\pi_y(q_1), \ldots, \pi_y(q_n)\}$ by Y having Carathéodory number less than or equal to n, then

$$\begin{aligned}(x,y) \in\ & \operatorname{conv}\{\pi_x(p_1), \ldots, \pi_x(p_m), \pi_x(q_1), \ldots, \pi_x(q_n)\} \times \\ & \operatorname{conv}\{\pi_y(p_1), \ldots, \pi_y(p_m), \pi_y(q_1), \ldots, \pi_y(q_n)\} \\ = \ & \operatorname{conv}\{p_1, \ldots, p_m, q_1, \ldots, q_n\}.\end{aligned}$$

This result yields the following two theorems.

2.8. THEOREM. If X is domain finite and Y is domain finite, then $X \times Y$ is domain finite.

2.9. THEOREM. If X has a Carathéodory number m and Y has Carathéodory number n, then $X \times Y$ has a Carathéodory number less than or equal to m+n.

2.10. REMARK. Although it is an open question as to whether the upper bound, m+n, for the Carathéodory number can be improved, it is possible to show that max (m,n) is less than or equal to the Carathéodory number c for $\mathcal{C}^E_{x \times y}$. Thus, $\max(m,n) \leq c \leq m+n$. Moreover, this lower bound is the best possible result as the following example shows.

2.11. EXAMPLE. Let $(X, \mathcal{C}_x)$ be defined as $X = R^1$ with $\mathcal{C}_x$ the usual convexity structure on R^1 and $(Y, \mathcal{C}_y)$ as $Y = R^1$, $\mathcal{C}_Y = \{y \mid |y| \leq 1\}$. Then $A = \text{conv}\,\{(1,1),\ (2,1)\}$ is the horizontal line segment joining the two points; hence $(3/2,1) \in A$ and $(3/2,1)$ is not in the convex hull of less than the two original points.

We shall now prove the assertion made in the above remark concerning the lower bound on the Carathéodory number in the product space.

2.12. THEOREM. Let $(X, \mathcal{C}_x)$ and $(Y, \mathcal{C}_y)$ be two convex structures having Carathéodory numbers m and n respectively, then the Carathéodory number c of the product structure $(X \times Y, \mathcal{C}^E_{X \times Y})$ satisfies the inequality $\max(m,n) \leq c$.

Proof. Assume $m = \max(m,n)$; then since there exist m points $x_1, \ldots, x_m$ and m is the Carathéodory number of $(X, \mathcal{C}_X)$, there exists a point p such that $p \in \mathrm{conv}\{x_1, \ldots, x_m\}$ and p does not belong to the convex hull of any m-1 of the points. Let $y \in Y$; we claim that $(p,y) \in \mathrm{conv}\{(x_1,y), \ldots, (x_m,y)\}$ but not in any proper subset. That (p,y) belongs to the convex hull is clear from the definition of the Eckhoff product. Now assume that (p,y) belongs to the convex hull of $\{(x_1,y), \ldots, (x_{i-1},y), (x_{i+1},y), \ldots, (x_m,y)\}$. But then $\{(x_1,y), \ldots, (x_{i-1},y), (x_{i+1},y), \ldots, (x_m,y)\} \subseteq \mathrm{conv}\{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m\} \times Y$, which is a convex set excluding (p,y) contrary to our assumption.

Having obtained results concerning the Radon and Carathéodory numbers one would naturally hope for a result concerning the Helly number for the Eckhoff product. This is contained in the following theorem.

2.13. THEOREM. Let $(X, \mathcal{C}_X)$ and $(Y, \mathcal{C}_Y)$ have Helly numbers of h_1 and h_2, respectively, and let $h = \max\{h_1, h_2\}$. Then $(X \times Y, \mathcal{C}^E_{X \times Y})$ has a finite Helly number which is less than or equal to h.

Proof. Suppose $\mathcal{F}$ is a finite subfamily of $\mathcal{C}$, having h+1 members such that for every h members of $\mathcal{F}$, their intersection is nonempty. Then consider the family $\pi_X \mathcal{F}$ defined to be $\{\pi_X C_i \mid C_i \in \mathcal{F}\}$. The intersection of every h_1 members is non-empty and hence the intersection of the whole family is non-empty. That is, there exists a p_1 such that for each $C_i \in \mathcal{F}$, there is y_i such that $(p_1, y_i) \in C_i$. Similarly there exists p_2 such that for each $C_i \in \mathcal{F}$, there is x_i such that $(x_i, p_2) \in C_i$. By the definition of the Eckhoff product (p_1, p_2) belongs to each C_i, hence $(X \times Y, \mathcal{C}^E_{x \times y})$ has a Helly number less than or equal to h.

It is natural to ask if the result in Theorem 2.13 is the best possible. The following example answers in the affirmative.

<u>2.14. EXAMPLE</u>. Let $\mathcal{C}_x = \{C \mid |C| \leq 4\}$, $\mathcal{C}_y = \{C \mid |C| \leq 2\}$, then h_1 for $\mathcal{C}_x = 5$, h_2 for $\mathcal{C}_y = 3$; also let

$a = (1,2)$ $\quad b = (2,2)$ $\quad c = (3,2)$ $\quad d = (4,2)$ $\quad e = (5,2)$

$\alpha = (1,1)$ $\quad \beta = (2,1)$ $\quad \gamma = (3,1)$ $\quad \delta = (4,1)$ $\quad \epsilon = (5,1)$

and $C_1 = \{a,b,c,d,\alpha,\beta,\gamma,\delta\}$ $\quad C_2 = \{a,b,c,e,\alpha,\beta,\gamma,\epsilon\}$

$C_3 = \{a,b,d,e,\alpha,\beta,\delta,\epsilon\}$ $\quad C_4 = \{a,c,d,e,\alpha,\gamma,\delta,\epsilon\}$

$C_5 = \{b,c,d,e,\beta,\gamma,\delta,\epsilon\}$

It is routine to show that the intersection of any four of the above sets is not empty but $\bigcap_{i=1}^{5} C_i = \emptyset$. A similar example could be constructed for any two numbers h_1 and h_2, showing that $h = \max\{h_1, h_2\}$ is the best possible result.

A final remark concerning Eckhoff's product is in order. A basic property concerning convexity structures is how they are related to line segments and lines. The Eckhoff product of two interval convexity structures may contain two distinct lines having more than one point of intersection, where the line containing a and b, L (a,b) is defined as $ab \cup \{c \mid a \in bc\} \cup \{d \mid b \in da\}$.

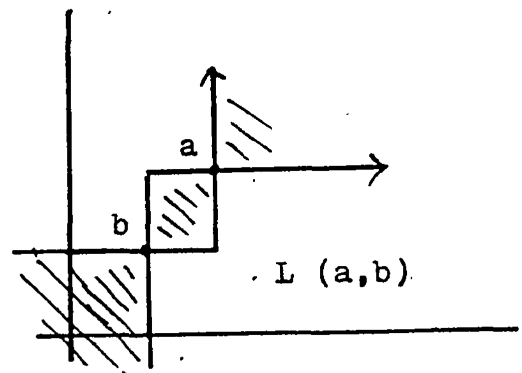


Figure 2.1.

If X = Y = the ordinary convexity space R^1, the line containing points a and b is as illustrated in the shaded region in Figure 2.1.

Also, if X = Y = R^1 and $\mathcal{C}_x$ and $\mathcal{C}_y$ are the convexity structures consisting of <u>closed</u> convex sets, then $C \in \mathcal{C}^E_{x \times y}$ if and only if C is a closed rectangle in R^2. Let S be the open rectangle with vertices at (1,1), (1,2), (3,1), (3,2). Then the convex hull of S is the closed rectangle having the same vertices but yet $xy \subseteq S$ for each $x,y \in S$; i.e. the convexity structure is <u>not</u> generated by its line segments, and thus, is not an interval convexity structure.

The Eckhoff product is easy to define but the convex sets in the product space are noticably restricted. Even for the simple case when X and Y are one-dimensional Euclidean spaces with the usual convexity structures, the class of convex sets in the product space are the rectangles with sides parallel to the coordinate axes. It would seem desirable to have a product concept which, when restricted to the classical setting, yields the classical convex sets in the product space.

CHAPTER III

THE COMPLEMENT PRODUCT

In this chapter we shall define a product of two convexity structures which contains more convex sets than the Eckhoff product. Our definition is motivated by the Tychonoff Product in general topology.

As before, it suffices to define the product of two convexity structures, as the method of defining the product of any finite number of structures will then be obvious. For convenience let $co_x A$ denote the relative complement of A in X; that is, for arbitrary sets $A \subset X$, $co_x A = \{a \in X \mid a \notin A\}$.

3.1. DEFINITION. Let $(X, \mathcal{C}_x)$ and $(Y, \mathcal{C}_y)$ be two convexity structures. In the Cartesian product $X \times Y$, define the complement convexity structure $\mathcal{C}^C_{x \times y}$ by taking as a basis sets of the form $co_{x \times y}(co_x C \times co_y D)$ where $C \in \mathcal{C}_x$ and $D \in \mathcal{C}_y$. Then $C \in \mathcal{C}^C_{x \times y}$ if and only if there exists an indexing set I and a family of sets $C_i \in \mathcal{C}_x$ and $D_i \in \mathcal{C}_y$ for $i \in I$ such that $C = \bigcap_{i \in I} co_{x \times y}(co_x C_i \times co_y D_i)$.

3.2. REMARK. Since there is little chance of confusion, we shall drop the subscript notation and simply use "co" for

$co_{x \times y}$, co_x, and co_y. It is then a routine exercise to show that $\emptyset$, $X \times Y \in \mathcal{C}^C_{x \times y}$ and $\mathcal{C}^C_{x \times y}$ is closed under intersection and hence $\mathcal{C}^C_{x \times y}$ is a convexity structure on $X \times Y$.

The following lemma is presented to shorten some of the later theorems. The proofs are omitted since they are elementary set theoretic exercises.

3.3. LEMMA. For all $A \subseteq X$, $B \subseteq Y$,

a) $\text{co}\,(A \times B) = ((\text{co}\,A) \times B) \cup (A \times (\text{co}\,B)) \cup ((\text{co}A) \times (\text{co}B))$.

b) $\text{co}\,((\text{co}A) \times (\text{co}B)) = ((\text{co}A) \times B) \cup (A \times (\text{co}B)) \cup (A \times B)$.

c) $\text{conv}\,(A \times B) \subseteq \text{conv}\,A \times \text{conv}\,B$.

3.4. EXAMPLE. If $(X, \mathcal{J}_x)$ and $(Y, \mathcal{J}_y)$ are two topological spaces and $(X, \mathcal{C}_x)$ and $(Y, \mathcal{C}_y)$ are the convexity structures obtained by defining a set to be convex if and only if it is a closed set in the topology. Then the convexity structure $\mathcal{C}^C_{x \times y}$ is precisely the closed sets in the product topology for $X \times Y$. For if C and D are closed sets in X and Y respectively, then $\text{co}\,C \times \text{co}\,D$ is the product of open sets and hence open in the topology for $X \times Y$, and $\text{co}\,(\text{co}\,C \times \text{co}\,D)$ is closed in the topology for $X \times Y$, so that if F is in $\mathcal{C}^C_{x \times y}$ then F is closed. Finally if F is a closed set in the topology for $X \times Y$ and $p \notin F$ then by the definition of the product topology there are open sets A and B in X and Y respectively such that $p \in A \times B$ and $(A \times B) \cap F = \emptyset$. But then $\text{co}\,(A \times B)$ is a convex set containing F but excluding p, showing that if F is closed in the product topology then it is in $\mathcal{C}^C_{x \times y}$.

3.5. EXAMPLE. Let $X = Y = R^1$ and suppose that $\mathcal{C}_x$ and $\mathcal{C}_y$ are the closed convex sets in X and Y. Then a basis for $\mathcal{C}^c_{x\times y}$ consists of any three closed quadrants in E^2 and their translations. The shaded portions of Figures 3.1 and 3.2 below illustrate two convex sets in the basis and Figures 3.3 and 3.4 illustrate two convex sets in $\mathcal{C}^c_{x\times y}$.

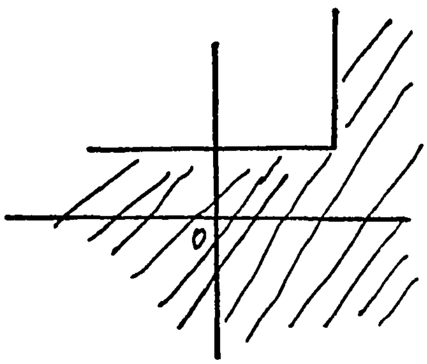


Figure 3.1

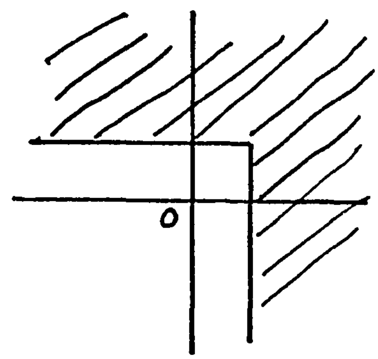


Figure 3.2

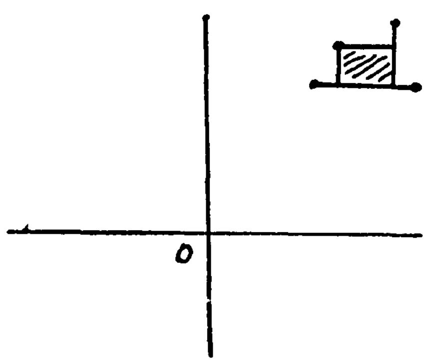


Figure 3.3

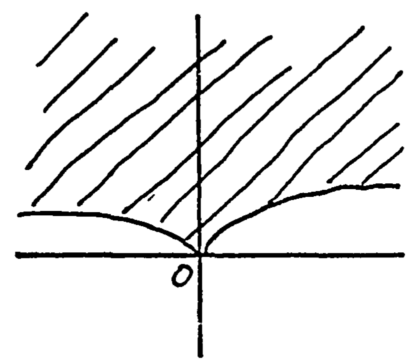


Figure 3.4

3.6. THEOREM. If $X = R^m$ and $Y = R^n$, and $\mathcal{C}_x$ and $\mathcal{C}_y$ are the usual convexity structures for X and Y, and S is a convex set in the usual sense in the linear space $X \times Y = R^{m+n}$ then S belongs to the product convexity structure of Definition 3.1.

__Proof.__ It is sufficient to show that for each $z \in X \times Y \setminus S$ there is a set in $\mathcal{C}^C_{x\times y}$ which contains S but not z. Let $z = (x,y)$ where $x \in X$ and $y \in Y$. We know that $\pi_x^{-1}(x)$ and $A = \pi_x^{-1}(x) \cap S$ are ordinary convex sets in $X \times Y \equiv R^{m+n}$. Note that if $A = \emptyset$ then $\mathrm{co}(\mathrm{co}\,\pi_x(S) \times \mathrm{co}\,\emptyset)$ is a convex set containing S but not z, so we may assume $A \neq \emptyset$. Also $\pi_y(A)$ is a convex set in Y which excludes y. Hence there exists a maximal convex set M_y in Y containing $\pi_y(A)$ but excluding y. (See figure 3.5.)

Similarly, $B = \pi_y^{-1}(y) \cap S$ is a convex set in the usual sense and $\pi_x(B)$ is a convex set in X which excludes x. Hence there is a maximal convex set M_x containing $\pi_x(B)$ but excluding x.

We then claim that $\mathrm{co}(\mathrm{co}\,M_x \times \mathrm{co}\,M_y)$ is a convex set in $\mathcal{C}^C_{x\times y}$ which contains S but not z. It is obvious that $z \notin \mathrm{co}(\mathrm{co}\,M_x \times \mathrm{co}\,M_y)$, since $x \in \mathrm{co}\,M_x$ and $y \in \mathrm{co}\,M_y$. To show that $S \subseteq \mathrm{co}(\mathrm{co}\,M_x \times \mathrm{co}\,M_y)$ let $c = (a,b) \in S$ where $a \in X$, $b \in Y$. It is then sufficient to show that if $a \notin M_x$ then $b \in M_y$.

Assume $b \notin M_y$. Again the maximal property of M_y implies $y \in \mathrm{conv}\,(\{b\} \cup \pi_y(A))$, for if not, then letting M_y' be a maximal convex set containing $\mathrm{conv}\,(\{b\} \cup \pi_y(A))$ but not y, then M_y' contains $\pi_y(A)$ and not y or $M_y' \subseteq M_y$ implying the contradiction $b \in M_y$. Hence there exists a $d \in \pi_y(A)$ such that y lies on the segment bd, and we may suppose $p \in A$ such that $\pi_y(p) = d$. Since π_y maps segments onto segments, π_y takes cp to bd; then $y \in \pi_y(cp)$ or cp meets $\pi_y^{-1}(y)$. That is, since $cp \subseteq S$ and $B = \pi_y^{-1}(y) \cap S$, cp meets B at a point q.

Now consider $\pi_x(B)$ and M_x. Since π_x maps the segment cp onto segment xa, then $e \equiv \pi_x(q) \in xa$. But $e \in M_x$ and under the assumption that $a \notin M_x$ then $xa \cap M_x = \emptyset$, a contradiction.

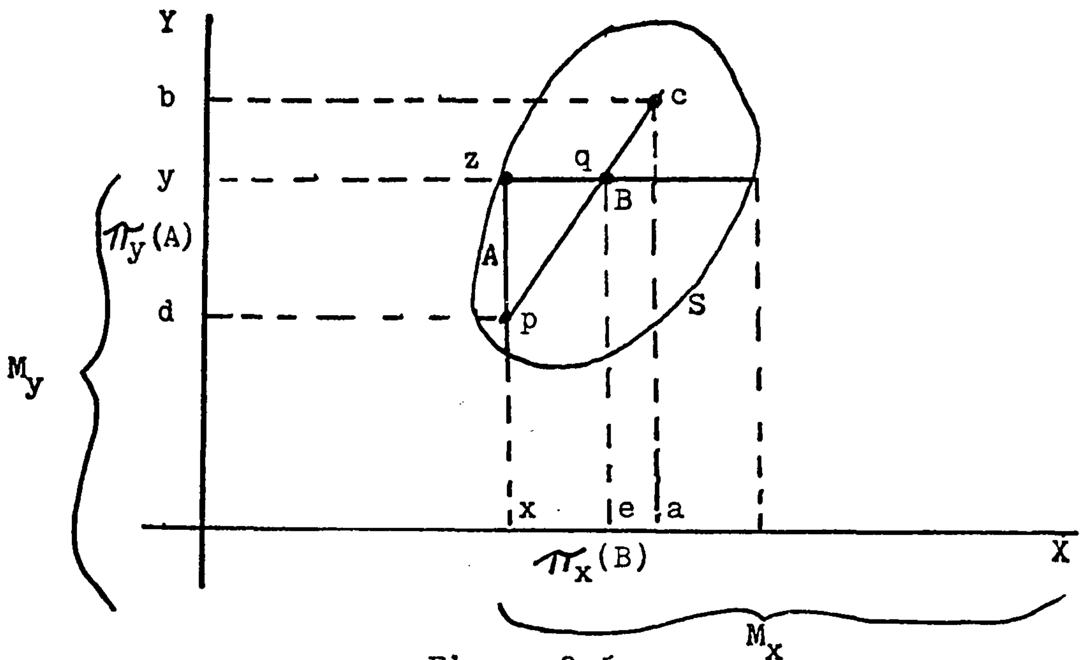


Figure 3.5

3.7. REMARKS. Using Definition 3.1 for the product of two ordinary convexity structures, the projection of a convex set may not be connected. Also the connected components of the projection of a convex set need not be convex in the factor space. To see this one need only study the example illustrated in Figure 3.6.

Although the projections of convex sets to the factor spaces do not behave as well as in the Eckhoff product (where the projection of a convex set is always convex), the next theorem shows a result that can be obtained and will serve as a comparison for the definition introduced in the next chapter.

a convex set in the complement product of E^1 and E^2-- it is the hull of the 10 indicated points.

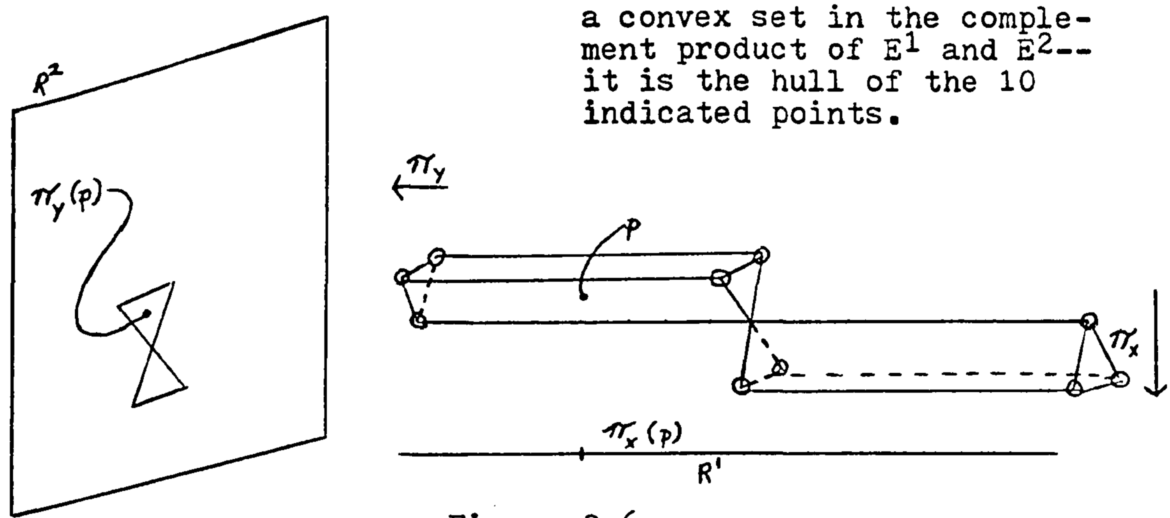


Figure 3.6

3.8. THEOREM. If $E \subseteq X \times Y$, then $\pi_x \operatorname{conv}_{x \times y} E \subseteq \operatorname{conv}_x \pi_x(E)$, but equality need not occur.

Proof. The example shown in Figure 3.6 shows that the inclusion may be proper. Assume $x \notin \operatorname{conv}_x \pi_x(E)$ and let $A = \operatorname{conv}_x \pi_x(E)$. Then $\operatorname{co}(\operatorname{co} A \times \operatorname{co} \emptyset)$ is a convex set that contains E but not (x,y) for any $y \in Y$. Hence, for each $y \in Y$, $(x,y) \notin \operatorname{conv}_{x \times y}(E)$ and then $x \notin \pi_x \operatorname{conv}_{x \times y} E$.

We now show that the definition of the complement product of two convexity structures is an associative operation which will indicate a method for defining the product of a finite number of convexity structures.

3.9. THEOREM. For any three convexity structures $(X, \mathcal{C}_x)$, $(Y, \mathcal{C}_y)$, and $(Z, \mathcal{C}_z)$ and under identification of the points in $(X \times Y) \times Z$ and $X \times (Y \times Z)$ then $\mathcal{C}^c_{x \times (y \times z)} = \mathcal{C}^c_{(x \times y) \times z}$.

Proof. Let $C \in \mathcal{C}^C_{(X\times Y)\times Z}$. Then $C = \bigcap_{i\in I} co(coD_i \times coE_i)$ where $D_i \in \mathcal{C}^C_{X\times Y}$ and $E_i \in \mathcal{C}_Z$. Also, for each $i \in I$ there exists an index set $J(i)$ such that $D_i = \bigcap_{j\in J(i)} co(coA_{ij} \times coB_{ij})$ where for each $i \in I$ and $j \in J(i)$, $A_{ij} \in \mathcal{C}_X$ and $B_{ij} \in \mathcal{C}_Y$. We define $F_{ij} = co(coB_{ij} \times coE_i) \in \mathcal{C}^C_{Y\times Z}$ for each $i \in I$, $j \in J(i)$. We shall show that $C = C'$ where $C' = \bigcap_{i\in I} [\bigcap_{j\in J(i)} co(co\, A_{ij} \times co\, F_{ij})] \in \mathcal{C}^C_{X\times(Y\times Z)}$. Let $(x,y,z) \in C$. Then $((x,y),z) \notin co\, D_i \times co\, E_i$ for all $i \in I$. Hence either $(x,y) \notin co\, D_i$ or $z \notin co\, E_i$. If $z \notin co\, E_i$ then it follows that $(y,z) \notin co\, F_{ij}$ and $(x,y,z) = (x,(y,z)) \notin co\, A_{ij} \times co\, F_{ij}$ or $(x,y,z) \in co(co\, A_{ij} \times co\, F_{ij})$ for all $j \in J(i)$. Hence $(x,y,z) \in C'$.

If $(x,y) \notin co\, D_i$ then $(x,y) \notin co\, A_{ij} \times co\, B_{ij}$ for all $j \in J(i)$. Again, it follows easily that $(x,y,z) \in C'$, showing $C \subseteq C'$. If $(x,y,z) \in C'$, then $(x,(y,z)) \notin coA_i \times coB_i$ for all i and j. We now fix $i \in I$ and consider the two cases $z \in co\, E_i$ or $z \notin co\, E_i$. If $z \notin co\, E_i$ then, obviously, $(x,y,z) \in co(coD_i \times co\, E_i)$ for each $i \in I$. If $z \in co\, E_i$ then either $x \notin co\, A_{ij}$ or $(y,z) \notin co\, F_{ij} = co\, B_{ij} \times co\, E_i$; then $x \notin co\, A_{ij}$ or $y \notin co\, B_{ij}$ for all $j \in J(i)$. That is $(x,y) \in \bigcap_{j\in J(i)} co(co\, A_{ij} \times co\, B_{ij}) = D_i$ and again $(x,y,z) \in co(co\, D_i \times co\, E_i)$. Since i was arbitrary, $(x,y,z) \in C$. Thus $\mathcal{C}^C_{(X\times Y)\times Z} \subseteq \mathcal{C}^C_{X\times(Y\times Z)}$. A similar argument reverses the inclusion, proving $\mathcal{C}^C_{(X\times Y)\times Z} = \mathcal{C}^C_{X\times(Y\times Z)}$.

The next results will compare and contrast the definition of convexity product as given in this chapter with the Eckhoff product in the previous chapter and the definition to be given in the next chapter.

3.10. THEOREM. Let $X = R^1$ with the usual convexity structure and a Carathéodory number of two, and let $(Y, \mathcal{C}_y)$ be any convexity structure with a Carathéodory number of n. Then $(X \times Y, \mathcal{C}^C_{x \times y})$ has Carathéodory number 2n.

Proof. Let $S \subseteq X \times Y$ and let $p = (x,y) \in \text{conv } S$. Note that $y \in \text{conv}_y \pi_y S$; for if not, then $\text{co}[\text{co } \emptyset \times \text{co}(\text{conv}_y \pi_y S)]$ is a convex set containing S and not p, which contradicts the hypothesis that $p \in \text{conv } S$.

We claim that there exist n or fewer points $s_1, \ldots, s_n$ such that $y \in \text{conv}\{\pi_y(s_1), \ldots, \pi_y(s_n)\}$ and such that $\pi_x(s_1), \ldots, \pi_x(s_n) \subseteq (-\infty, x] = A$. If not, let $S_a = \{s \in S \mid \pi_x(s) \in A\}$. If $y \notin \text{conv } \pi_y S_a$ then $\text{co}[\text{co}(\text{co } A) \times \text{co}(\text{conv } \pi_y S_a)]$ is a convex set which contains S but not p, which is a contradiction. Similarly, there exists a second set of n or fewer points $\{t_1, \ldots, t_n\} \subseteq S$ such that $y \in \text{conv}_y\{\pi_y(t_1), \ldots, \pi_y(t_n)\}$ and $\{\pi_x(t_1), \ldots, \pi_x(t_n)\} \in [x, \infty)$. We now claim that $p \in \text{conv}\{s_1, \ldots, s_n, t_1, \ldots, t_n\}$. To that end, let $\text{conv}(s_1, \ldots, s_n, t_1, \ldots, t_n) = \bigcap_{i \in I} \text{co}(\text{co } A_i \times \text{co } B_i)$, where A_i, B_i are convex in $\mathcal{C}_x$ and $\mathcal{C}_y$ respectively. Now for each i, $\{s_1, \ldots, s_n, t_1, \ldots, t_n\} \subseteq \text{co}(\text{co } A_i \times \text{co } B_i)$. Let $s_j = (x_j, y_j)$ and $t_j = (\bar{x}_j, \bar{y}_j)$ for $j = 1, \ldots, n$, and we shall suppose $p \notin \text{co}(\text{co } A_i \times \text{co } B_i) \Rightarrow x \notin A_i$ and $y \notin B_i$. If for

every j we have $x_j \notin A_i$ and $\bar{x}_j \notin A_i$ then $x_j, \bar{x}_j \in \text{co } A_i$ from which it follows that $y_j, \bar{y}_j \notin \text{co } B_i$ (since $s_j, t_j \in \text{co}(\text{co } A_i \times \text{co } B_i)$ or $y_j, \bar{y}_j \in B_i$ for every j). But this means that
$y \in \text{conv}\{\pi_y(s_1), \ldots, \pi_y(s_n)\} = \text{conv}\{y_1, \ldots, y_n\} \subseteq B_i$, a contradiction. Hence for some j, x_j or $\bar{x}_j \in A_i$. The argument being symmetric, suppose $x_j \in A_i$. Then for every j and $y \in \text{conv}\{\pi_y(t_1), \ldots, \pi_y(t_n)\} = \text{conv}\{\bar{y}_1, \ldots, \bar{y}_n\} \subseteq B_i$, a contradiction. Hence $p \in \text{co}(\text{co } A_i \times \text{co } B_i)$ or $p \in \text{conv}\{s_1, \ldots, s_n, t_1, \ldots, t_n\}$ as asserted.

Elementary examples may easily be constructed to show that the upper bound for the Carathéodory number in the preceding theorem is the best possible result.

Although we were able to obtain a result concerning the Carathéodory number for the product space when one of the factor spaces is E^1, in general there does not exist a finite Carathéodory number. We present an example below to show that the product of E^2 having the usual convexity structure with itself is infinite.

It then follows as an immediate corollary that in general, the convexity structure defined by the complement product of two spaces need not be domain finite, even if the factor spaces have this property.

Also, it should be mentioned that since any subset of the line y = x is a convex set in the convexity structure obtained by using E^1 with the usual convexity structure for both of the factor spaces, no finite Radon number exists.

We shall now end this chapter with an example showing that the Carathéodory number of $E^2 \times E^2$, where the usual convexity structure is used for each factor space, is greater than or equal to 32. Similar examples can be constructed for any power of two, showing that the Carathéodory number is infinite.

3.11. EXAMPLE. Let $p = ((0,0),(0,0))$ and let

$p_1 = ((\cos 0, \sin 0),(\cos 0, \sin 0))$,

$p_2 = ((\cos(1/32)\cdot 2\pi, \sin(1/32)\cdot 2\pi,$

$(\cos(1/16)\cdot 2\pi, \sin(1/16)\cdot 2\pi))\ldots$

$p_{32} = ((\cos(31/32)\cdot 2\pi, \sin(31/32)\cdot 2\pi),$

$(\cos(15/16)\cdot 2\pi, \sin(15/16)\cdot 2\pi))$.

Figure 3.7 shows the method by which all 32 points are defined, where the points $\pi_x(p_i)$ and $\pi_y(p_i)$ have been labeled as the point i for convenience.

It is then routine to show that the point p (the origin in E^4) is in the convex hull of the 32 points. That is, it is impossible to find a convex set C in $E^2 = X$ that does not contain the origin, and a convex set D in $E^2 = Y$ that does not contain the origin but such that $\pi_x(p_i) \in C$ or $\pi_y(p_i) \in D$ for each i. (Again, if one tries to exclude the origin from convex sets in both X and Y, then one also excludes both projections of some point p_i.)

It is now an elementary exercise to show that p does not belong to the convex hull of any proper subset of the original 32 points. That is, it is possible to find a convex set C in $E^2 = X$ that excludes the origin in X and a convex set D in $E^2 = Y$ that excludes the origin in Y and such that $\pi_x(p_i) \in C$ or $\pi_y(p_i) \in D$ for all i except one.

Figure 3.8 shows the convex sets needed in each factor space to construct a convex set in the product space that will contain $(\cup\{p_i\})\setminus\{p_2\}$ but not the origin p.

Figure 3.9 shows the convex sets needed in each factor space to construct a convex set in the product space that will contain $(\cup\{p_i\})\setminus\{p_{13}\}$ but not the origin p.

Figure 3.10 shows the convex sets needed in each factor space to construct a convex set in the product space that will contain $(\cup\{p_i\})\setminus\{p_{14}\}$ but not the origin p.

Figure 3.11 shows the convex sets needed in each factor space to construct a convex set in the product space that will contain $(\cup\{p_i\})\setminus\{p_{29}\}$ but not the origin p.

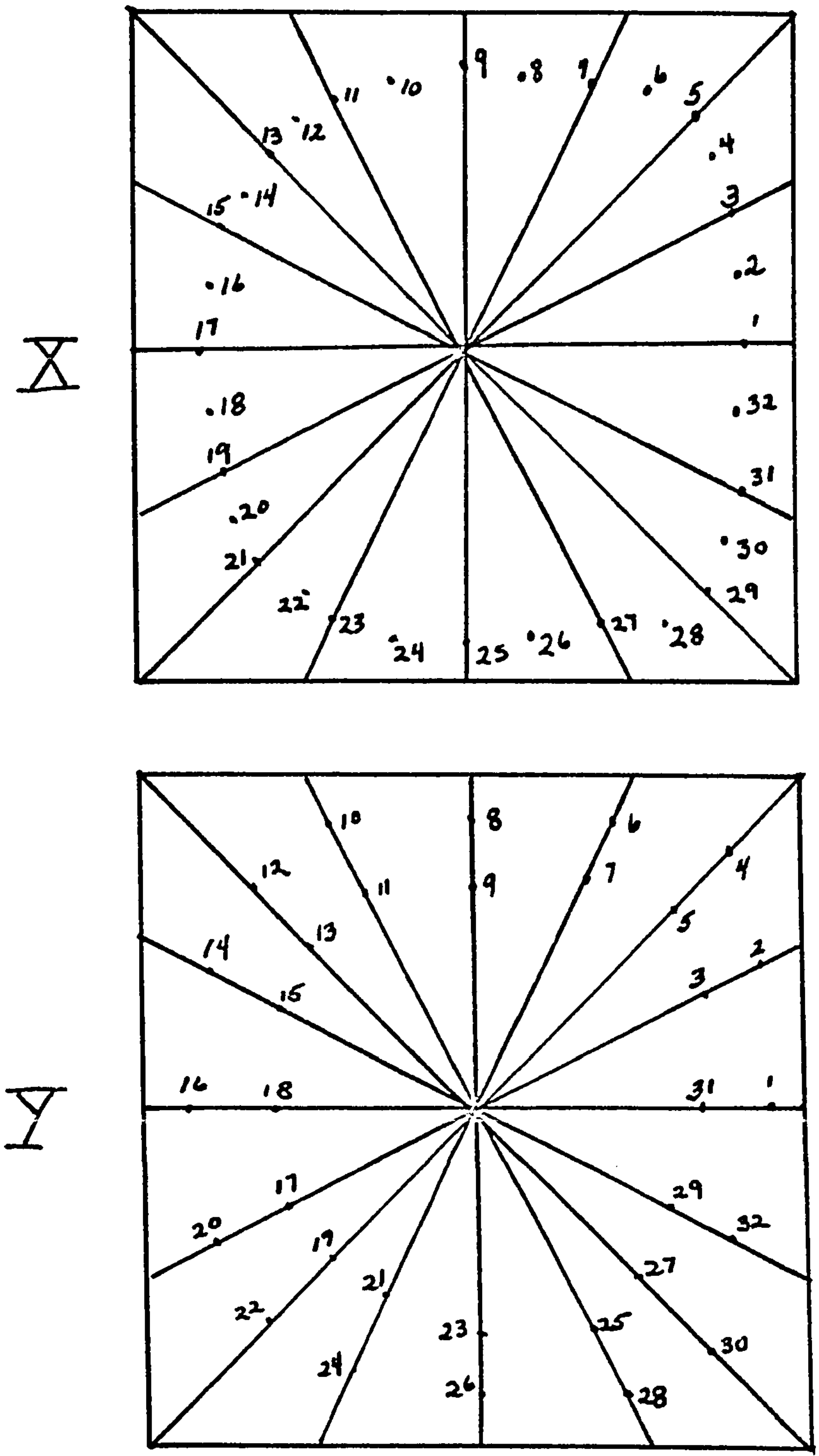

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Figure 3.7

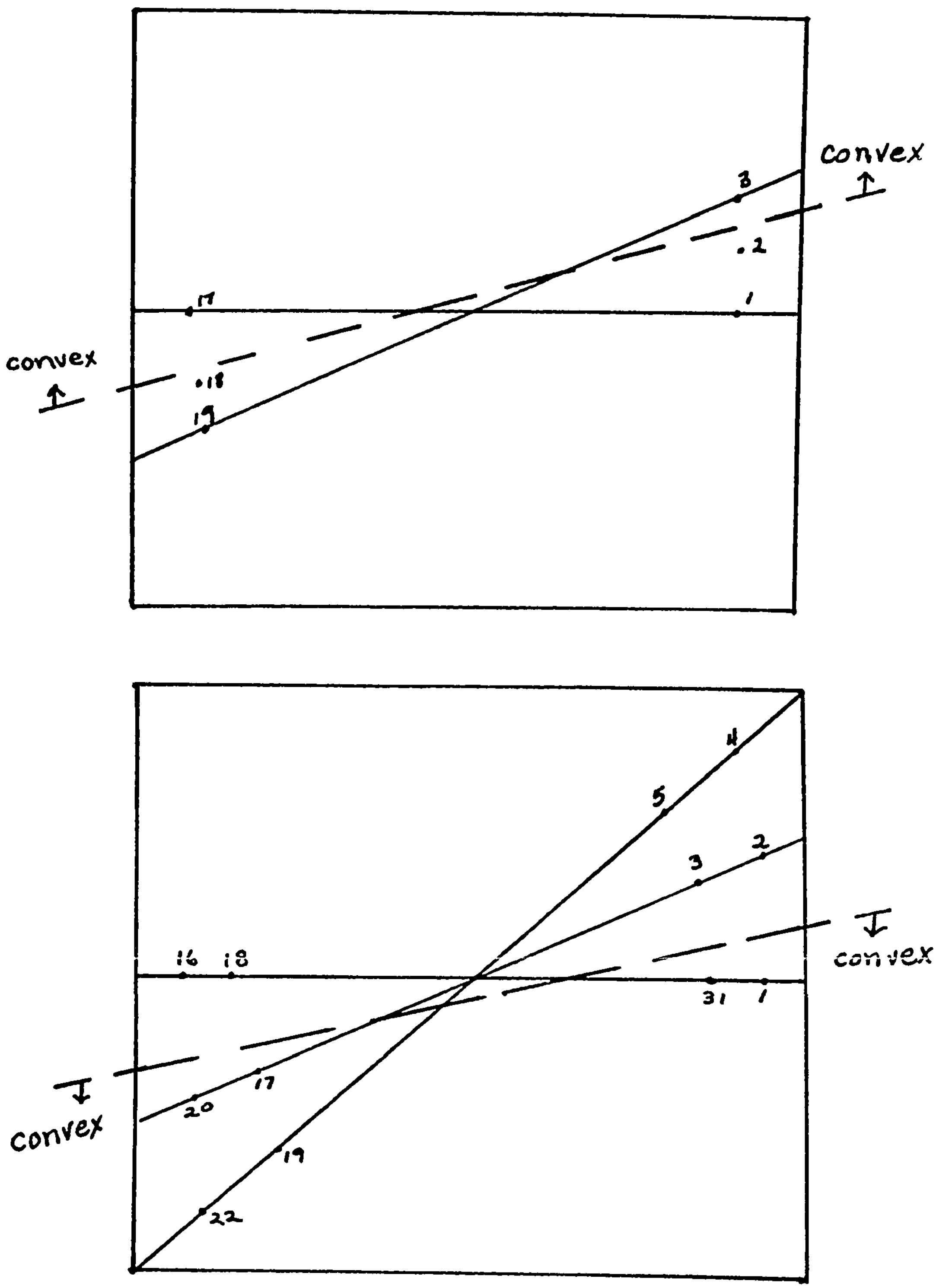
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Figure 3.8

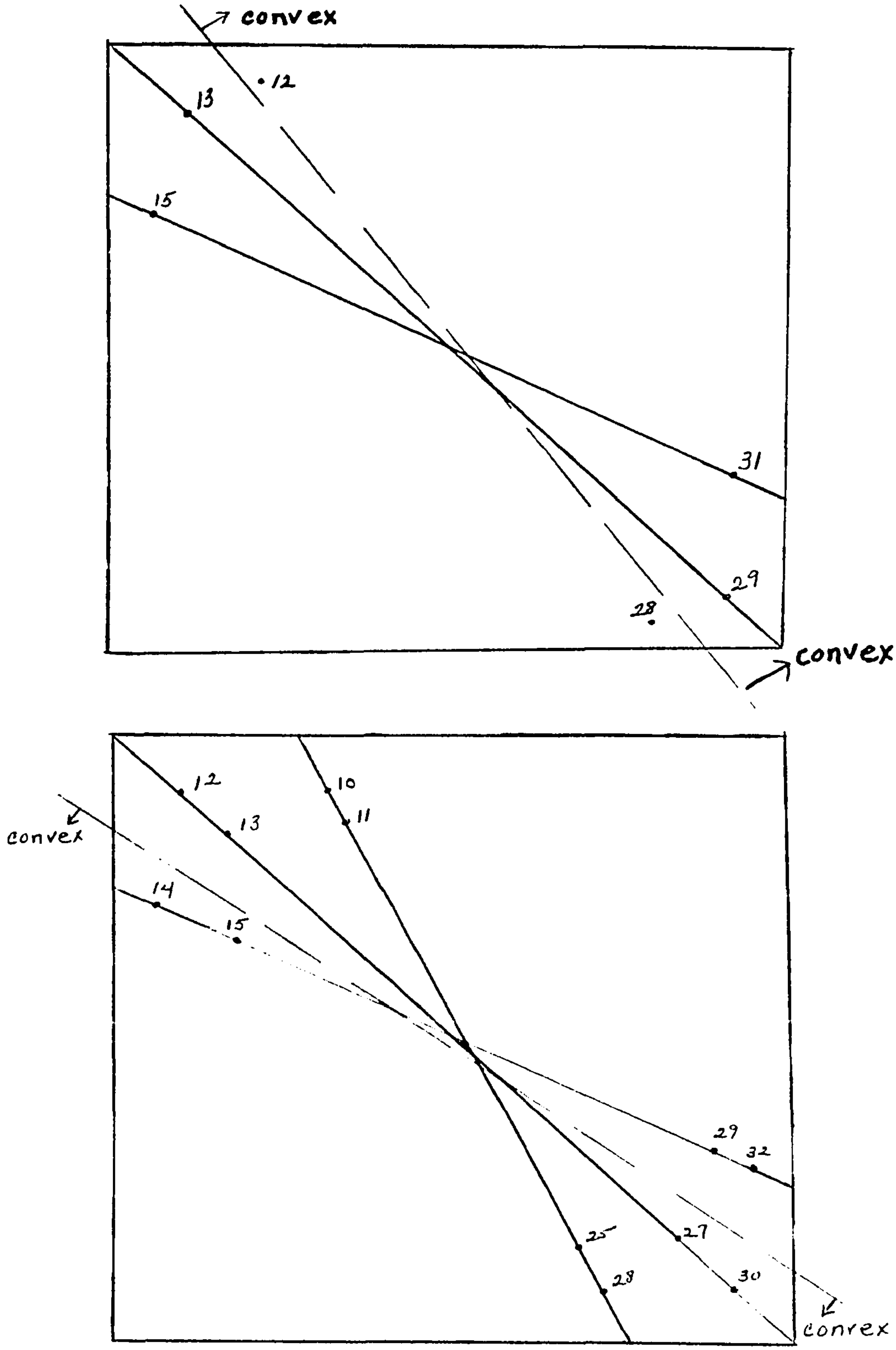
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Figure 3.9

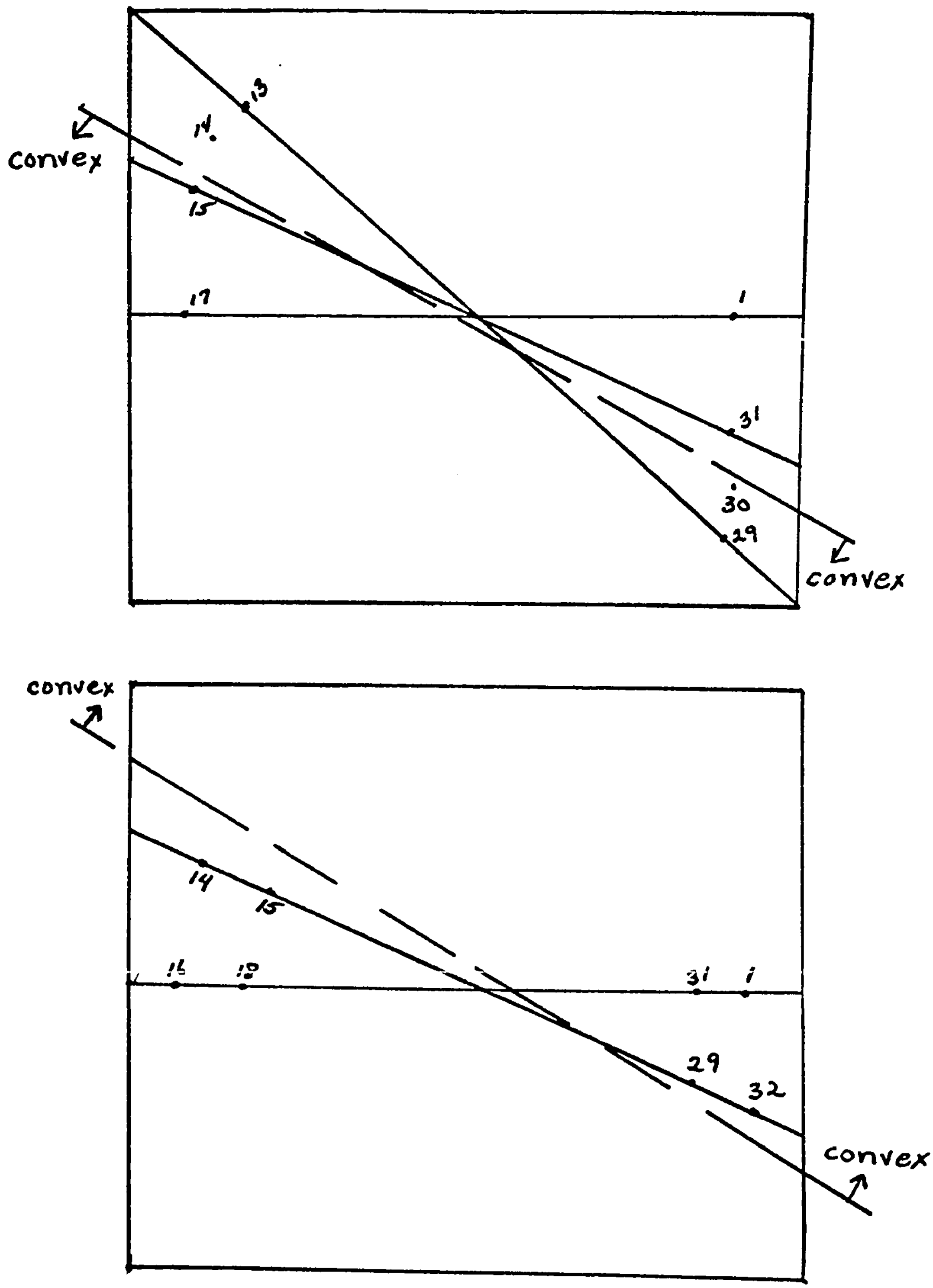
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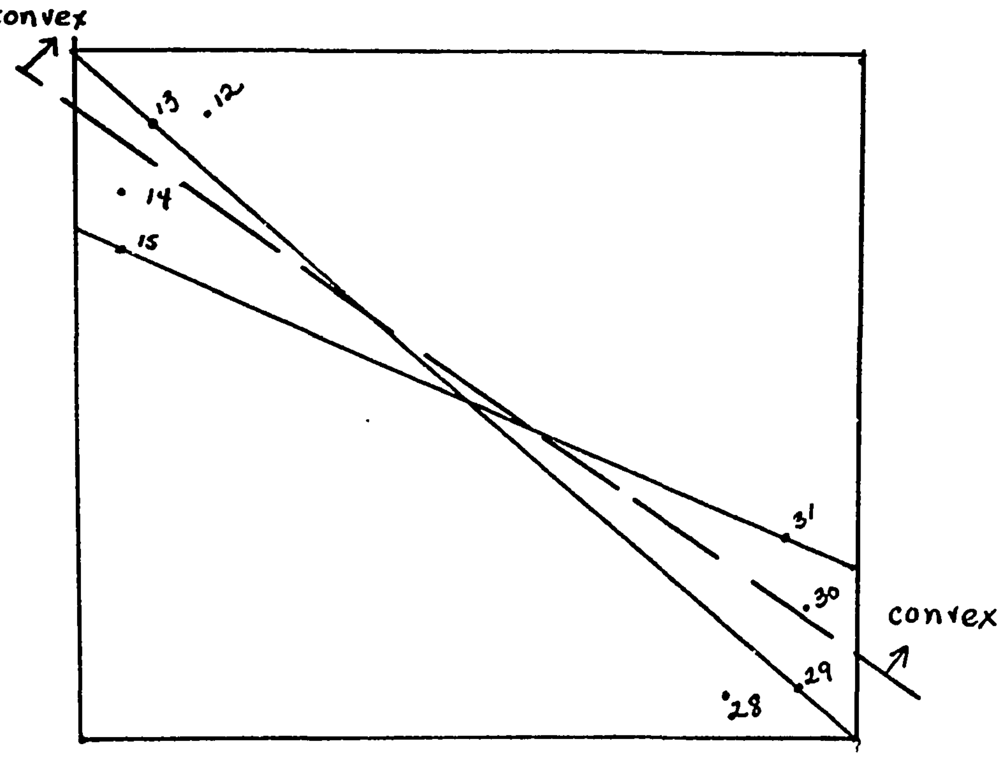
Figure 3.10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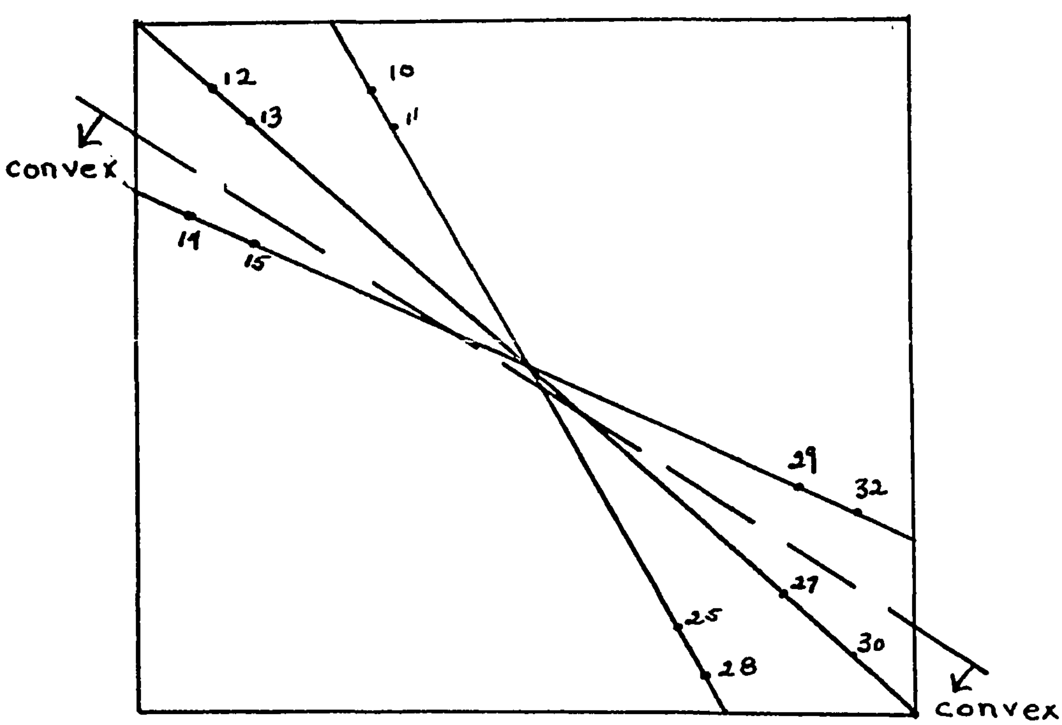
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Figure 3.11

CHAPTER IV

THE PROJECTIVE PRODUCT

In this chapter, in contrast to the product concepts considered in the preceding two chapters, we define a true generalization of the ordinary product of the usual convexity structures in real vector spaces. We shall restrict our attention to only those factor spaces that are point-convex convexity structures and to only the case $X=R^m$ and $Y=R^n$. Throughout this chapter π_x and π_y will denote the orthogonal or Cartesian projections from R^{m+n} to R^m and R^n respectively, i.e. $\pi_x(a,b)=a$ and $\pi_y(a,b)=b$. Also, p_x and p_y will denote projections (not necessarily orthogonal) from R^{m+n} onto $R^m\times\{0\}$ and $\{0\}\times R^n$ respectively. That is, p_x is a linear operator with domain R^{m+n} and range $R^m\times\{0\}$ such that $p_x^2 = p_x$. We let $\mathcal{P}_x$ and $\mathcal{P}_y$ denote the respective classes of all such projections. Since there is little chance of confusion we can identify $R^m\times\{0\}$ with R^m and say p_x is a projection onto R^m.

<u>4.1. DEFINITION</u>. Let $(R^m, \mathcal{C}_x)$ and $(R^n, \mathcal{C}_y)$ be point-convex convexity structures, and define the <u>projective product</u>

<u>convexity structure</u> $\mathcal{C}^{P}_{x\times y}$ on R^{m+n} as the collection of all sets C such that $\pi_y(p_x^{-1}(a)\cap C)\in\mathcal{C}_y$ and $\pi_x(p_y^{-1}(b)\cap C)\in\mathcal{C}_x$ for each projection $p_x\in\mathcal{P}_x$ and $p_y\in\mathcal{P}_y$ and for each $a\in R^m$ and $b\in R^n$.

The set $\text{conv}_{x\times y}C$ will denote the convex hull of a set C relative to $\mathcal{C}^{P}_{x\times y}$ and $\mathcal{C}^{u}_{x}$ will denote the usual convexity structure for $X = R^m$ and u-conv (A) for the usual convex hull of A.

<u>4.2.1. EXAMPLE</u>. If $(X, \mathcal{C}_x)$ is the convexity structure where $X=R^1$ and $A\in\mathcal{C}_x$ if and only if $A=X$ or $|A|\leq 1$, and $(Y, \mathcal{C}_y)$ is the convexity structure $Y=R^1$, with $B\in\mathcal{C}_y$ if and only if $|B|\leq 1$ or $B=Y$, then $C\in\mathcal{C}^{P}_{x\times y}$ if and only if $|C|\leq 1$ or C is a line in R^2.

<u>4.2.2. EXAMPLE</u>. If $X=Y=R^1$ and if $\mathcal{C}_x=\mathcal{C}^u_x$ and $B\in\mathcal{C}_y$ if and only if $|B|\leq 1$ or $B=Y$, then $C\in\mathcal{C}^{P}_{x\times y}$ if and only if C is any convex subset of a horizontal line in $X\times Y$.

<u>4.2.3. EXAMPLE</u>. If $X=Y=R^1$ and if $A\in\mathcal{C}_x=\mathcal{C}_y$ if and only if $|A|\leq 2$ or $A=R^1$ then $C\in\mathcal{C}^{P}_{x\times y}$ if and only if either $C=R^2$, C does not contain three collinear points, C is a line or pair of lines in R^2, or C is a line together with 2 points from a line parallel to it.

<u>4.2.4. EXAMPLE</u>. If $X=Y=R^1$ and if $A\in\mathcal{C}_x=\mathcal{C}_y$ if and only if $A=R^1$ or the diameter of A is less than or equal to one, then $C\in\mathcal{C}^{P}_{x\times y}$ if and only if either $C=R^2$, C is any line

in R^2, or C is any subset of a unit square with sides parallel to the coordinate axes.

4.2.5. EXAMPLE. If $X=R^1$ and $A\in\mathcal{C}_x$ if and only if $A=R^1$ or $|A|\leq 1$; $Y=R^2$ and $B\in\mathcal{C}_y$ if and only if $B=R^2$, $|B|\leq 1$, or B is a vertical line in R^2 then $C\in\mathcal{C}^P_{x\times y}$ if and only if either $C=R^3$, $|C|\leq 1$, or C is a plane, or any line in that plane, of the form $\pi_y^{-1}(\ell)$ where ℓ is a vertical line in Y.

4.3. LEMMA. If z_1 and z_2 belong to $X\times Y$ and $\pi_y(z_1)\neq\pi_y(z_2)$ then there exists a projection from $X\times Y$ onto X such that the images of z_1 and z_2 coincide.

Proof. The identity map on X is a projection from X onto X and our problem is to find an extension of this linear map that will satisfy the conditions of the theorem.

Let M be the space spanned by X and $\pi_y(z_2-z_1)$. Then each element of M can be written uniquely as an element of X and a multiple of $\pi_y(z_2-z_1)$ in the form $x+\alpha\cdot(\pi_y(z_2-z_1))$ where α is a real scalar.

In particular, $z_2-z_1=\pi_x(z_2-z_1)+1\cdot(\pi_y(z_2-z_1))$. We now define an operator F_o from M onto X by setting $F_o(x+\alpha\cdot\pi_y(z_2-z_1))=x+\alpha(-\pi_x(z_2-z_1))$. Then F_o is linear, a projection, an extension of the identity on X, and $F_o(z_2-z_1)=F_o(\pi_x(z_2-z_1)+1\cdot\pi_y(z_2-z_1))=\pi_x(z_2-z_1)+1\cdot(-\pi_x(z_2-z_1))=0$. Hence $F_o(z_1)=F_o(z_2)$.

It is now a standard result to extend F_o to all of $X \times Y$. See, for example, Taylor [19], p.40.

4.4. THEOREM. Using the definition 4.1. given above, $(X \times Y, \mathcal{C}^P_{x \times y})$ is a convexity structure.

Proof. It is easily seen that $X \times Y = R^{m+n}$ and $\emptyset$ belong to $\mathcal{C}^P_{x \times y}$, thus we must show that $\mathcal{C}^P_{x \times y}$ is closed under intersections. Let $C_i \in \mathcal{C}^P_{x \times y}$ for every $i \in I$; then $\pi_y(p_x^{-1}(a) \cap (\bigcap_{i \in I} C_i)) = \pi_y(\bigcap_{i \in I} (p_x^{-1}(a) \cap C_i))$, and we shall now show that $\pi_y(\bigcap_{i \in I} p_x^{-1}(a) \cap C_i) = \bigcap_{i \in I} \pi_y(p_x^{-1}(a) \cap C_i)$. This will complete the proof, since the argument is symmetric in X and Y.

If $d \in \pi_y(\bigcap_{i \in I} p_x^{-1}(a) \cap C_i)$ then there exists a $c \in X$ such that $(c,d) \in (\bigcap_{i \in I} p_x^{-1}(a) \cap C_i)$, and therefore $(c,d) \in p_x^{-1}(a) \cap C_i)$ for every $i \in I$. Therefore $d \in \pi_y(p_x^{-1}(a) \cap C_i)$ for every $i \in I$ and $d \in \bigcap_{i \in I} \pi_y(p_x^{-1}(a) \cap C_i)$, showing $\pi_y(\bigcap_{i \in I} p_x^{-1}(a) \cap C_i) \subseteq \bigcap_{i \in I} \pi_y(p_x^{-1}(a) \cap C_i)$.

If $d \in \bigcap_{i \in I} \pi_y(p_x^{-1}(a) \cap C_i)$ then $d \in \pi_y(p_x^{-1}(a) \cap C_i)$ for every $i \in I$, and therefore there exists a c_i for every $i \in I$ such that $(c_i, d) \in p_x^{-1}(a) \cap C_i$ for every $i \in I$. Since p_x is a linear projection, $c_i = c_j$, for every $i, j \in I$, and therefore there exists a c such that $(c,d) \in p_x^{-1}(a) \cap C_i$ for every $i \in I$.

Therefore $(c,d) \in \bigcap_{i \in I} (p_x^{-1}(a) \cap C_i)$ and thus $d \in \pi_y(\bigcap_{i \in I} p_x^{-1}(a) \cap C_i)$.

The next theorem shows that our main objective for $\mathcal{C}^P_{x \times y}$ has been achieved.

<u>4.5. THEOREM</u>. If $\mathcal{C}_x$ and $\mathcal{C}_y$ are the usual convexity structures for R^m and R^n respectively, then $\mathcal{C}^P_{x\times y}$ is precisely the usual convexity structure for R^{m+n}.

<u>Proof</u>. Let $C\in\mathcal{C}^P_{x\times y}$, and let $r=(a,b)$ and $s=(c,d)$ belong to C, where a and c belong to R^m and b and d belong to R^n. We must show that the usual convex hull of r and s belongs to C to show that $\mathcal{C}^P_{x\times y}$ is a subset of the class of usual convex sets. Assume that $t\in$ u-conv$\{r,s\}$ and that $t\notin C$. If $b\neq d$ then by Lemma 4.3 there exists a projection p_x from R^{m+n} onto X such that $p_x(r)=p_x(s)=p_x(t)$. Since C is convex in the product space, $\pi_y(p_x^{-1}(p_x(r))\cap C)\in\mathcal{C}_y$. Since this set contains b and d and since it is a usual convex set in Y, it also contains $\pi_y(t)$. We know p_x is a linear projection onto X; thus $t\in C$, which is contrary to our assumption.

If b=d, then the orthogonal projection π_y is such that $\pi_y(r)=\pi_y(s)=\pi_y(t)$. Then since C is convex in $\mathcal{C}^P_{x\times y}$, $\pi_x(\pi_y^{-1}(\pi_y(t))\cap C)\in\mathcal{C}^u_x$ and hence $\pi_x(t)\in\pi_x(\pi_y^{-1}(\pi_y(t))\cap C)$, and $t\in C$; again a contradiction to the assumption. It then follows that $\mathcal{C}^P_{x\times y}$ is a subset of the class of usual convex sets.

Now let $D\in\mathcal{C}^u_{x\times y}$ and let $a\in R^m$, p_x and p_y be arbitrary linear projections of R^{m+n} onto R^m and R^n respectively. We must show that $\pi_y(p_x^{-1}(a)\cap D)\in\mathcal{C}^u_y$. Since p_x is a linear projection of R^{m+n} onto R^m, $p_x^{-1}(a)$ is a usual convex set in R^{m+n}; hence $p_x^{-1}(a)\cap D\in\mathcal{C}^u_{x\times y}$ and since usual convex sets are preserved by linear projections, $\pi_y(p_x^{-1}(a) \cap D) \in \mathcal{C}_y^u$. Similarly, $\pi_x(p_y^{-1}(b) \cap D) \in \mathcal{C}_x^u$ for any $b \in R^n$, showing that $\mathcal{C}_{x \times y}^P$ contains the usual convex sets.

The next result follows immediately from the definition of the projective product and is stated without proof.

4.6. THEOREM. The product $(X \times Y, \mathcal{C}_{x \times y}^P)$ is point convex if and only if both $(X, \mathcal{C}_x)$ and $(Y, \mathcal{C}_y)$ are point convex.

Some questions about the existence and uniqueness of factor spaces which generate a given product space will now be considered. First, if we start with a convexity structure $(X, \mathcal{C}_x)$ and a structure $\mathcal{C}_{x \times y}$ for $X \times Y$, it may be that there are no structures or many structures for $(Y, \mathcal{C}_y)$ such that $\mathcal{C}_{x \times y}^P = \mathcal{C}_{x \times y}$.

4.7. EXAMPLE. If $(X, \mathcal{C}_x)$ is the usual convexity structure on R^1 and if $C \in \mathcal{C}_{x \times y}$ if and only if $C = X \times Y$ or $|C| \leq 2$, then there does not exist a $\mathcal{C}_y$ for R^1 such that $\mathcal{C}_{x \times y}$ is the projective product of $\mathcal{C}_x$ and $\mathcal{C}_y$.

4.8. EXAMPLE. If $\mathcal{C}_x$ is the usual convexity structure on R^1, then if $Y = R^1$ and $C \in \mathcal{C}_y$ if and only if C is empty or C contains all rational numbers, then the product structure $\mathcal{C}_{x \times y}^P$ is the trivial product structure $\{\emptyset, R^2\}$. Likewise the structure $\mathcal{C}_y'$ defined by $C \in \mathcal{C}_y'$ if and only if C is empty or C is a superset of the irrationals, also yields the trivial product structure.

We shall now consider a concept introduced by Calder [3].

4.9. DEFINITION. A convexity structure $(X, \mathcal{C}_x)$ is an interval convexity structure when conv A = A if and only if $xy \subseteq A$ whenever $x \in A$ and $y \in A$.

4.10. THEOREM. If $(X, \mathcal{C}_x)$ and $(Y, \mathcal{C}_y)$ are both interval convexity structures, then $(X \times Y, \mathcal{C}^P_{x \times y})$ is an interval convexity structure.

Proof. If A is convex in $\mathcal{C}^P_{x \times y}$ then $z_1 z_2 \subseteq A$ whenever $z_k \in A$, k=1,2 by the definition of the convex hull operator.

Assume $z_1 z_2 \subseteq A$ whenever $z_k \in A$, k=1,2. We shall show that $A \in \mathcal{C}^P_{x \times y}$ by showing for all $p_x \in \mathcal{P}_x$, $p_y \in \mathcal{Q}_y$, $a \in X$, $b \in Y$ that $\pi_y(p_x^{-1}(a) \cap A) \in \mathcal{C}_y$ and $\pi_x(p_y^{-1}(b) \cap A) \in \mathcal{C}_x$. Since the arguments for X and Y are symmetric, it suffices to prove the first of these. Let $y_1, y_2 \in \pi_y(p_x^{-1}(a) \cap A)$. Thus there exist $z_k \in p_x^{-1}(a) \cap A$ such that $\pi_y(z_k) = y_k$, k=1,2. By hypothesis $z_1 z_2 \subseteq A$. Therefore, since $\pi_y(p_x^{-1}(a) \cap z_1 z_2) \in \mathcal{C}_y$ by the product convexity of $z_1 z_2$, $y_1 y_2 \subseteq \text{conv}_y[\pi_y(p_x^{-1}(a) \cap z_1 z_2)] = \pi_y(p_x^{-1}(a) \cap z_1 z_2) \subseteq \pi_y(p_x^{-1}(a) \cap A)$. Since $\mathcal{C}_y$ is an interval convexity structure this proves $\pi_y(p_x^{-1}(a) \cap A) \in \mathcal{C}_y$, as desired.

There are two conditions on pairs of convexity structures which will be useful later. It is easy to construct convexity structures which are so inherently dissimilar that the product structure contains only some of the translates of convex sets in the factor spaces. For example, if $X=Y=R^1$ and $C\in\mathcal{C}_X$ if and only if $C=X$ or $|C|\leq 1$ or $C=[n,\infty)$ for some integer n, and $D\in\mathcal{C}_Y$ if and only if $D=Y$ or $|D|\leq 1$, then $\mathcal{C}^P_{X\times Y}$ contains only points and sets of points of the form $[n,\infty)\times r$ where n is an integer and r is a real number.

At times it will be desirable to avoid such behavior for products.

4.11. DEFINITION. If $p_X^{-1}(C)\in\mathcal{C}^P_{X\times Y}$ for every $C\in\mathcal{C}_X$, $p_X\in\mathcal{P}_X$, [respectively, $\pi_X^{-1}(C)\in\mathcal{C}^P_{X\times Y}$, for $C\in\mathcal{C}_X$], then $\mathcal{C}_X$ is said to be compatible in the product, [respectively, orthogonally compatible in the product].

We state a preliminary result concerning this concept.

4.12. LEMMA. If $\mathcal{C}_X$ is compatible [respectively, orthogonally compatible] in the product then for $A\subseteq X\times Y$ and $p_X\in\mathcal{P}_X$, $p_X(\mathrm{conv}_{X\times Y}A)\subseteq \mathrm{conv}_X p_X(A)$, [respectively, $\pi_X(\mathrm{conv}_{X\times Y}A)\subseteq \mathrm{conv}_X \pi_X(A)$].

Proof. It suffices to prove $\mathrm{conv}_{X\times Y}A\subseteq p_X^{-1}(\mathrm{conv}_X p_X(A))$. Since $\mathrm{conv}_X p_X(A)$ is convex in X, by compatibility, $p_X^{-1}(\mathrm{conv}_X p_X(A))\in\mathcal{C}^P_{X\times Y}$. Thus $\mathrm{conv}_{X\times Y}A\subseteq \mathrm{conv}_{X\times Y}(p_X^{-1}(p_X(A)))\subseteq \mathrm{conv}_{X\times Y}(p_X^{-1}(\mathrm{conv}_X p_X(A))) = p_X^{-1}(\mathrm{conv}_X p_X(A))$, proving the contention. The statement regarding π_X may be proved similarly.

A condition somewhat analagous to that of compatibility in the product, in that it also deals with the interplay between the product and each of the factor spaces, is introduced next.

4.13. DEFINITION. A convexity structure will be said to be projective in the product [respectively orthogonally projective in the product] if $p_x(C)\in\mathcal{C}_x$ whenever $C\in\mathcal{C}^P_{x\times y}$ and $p_x\in\mathcal{P}_x$ [respectively, if $\pi_x(C)\in\mathcal{C}_x$ for $C\in\mathcal{C}^P_{x\times y}$].

4.14. LEMMA. If $\mathcal{C}_x$ is projective [respectively, orthogonally projective] in $\mathcal{C}^P_{x\times y}$ then for $A\subseteq X\times Y$ and $p_x\in\mathcal{P}_x$, $\mathrm{conv}_x p_x(A)\subseteq p_x(\mathrm{conv}_{x\times y}A)$, [respectively, $\mathrm{conv}_x\pi_x(A)\subseteq\pi_x(\mathrm{conv}_{x\times y}A)$].

Proof. Since $\mathcal{C}_x$ is projective in $\mathcal{C}^P_{x\times y}$, $p_x(\mathrm{conv}_{x\times y}A)$ is convex in X. Hence,

$\mathrm{conv}_x p_x(A)\subseteq\mathrm{conv}_x p_x(\mathrm{conv}_{x\times y}A) = p_x(\mathrm{conv}_{x\times y}A)$.

The statement regarding π_x may be proved in an analagous manner.

4.15. COROLLARY. If $\mathcal{C}_x$ is compatible and projective in the product then for any set $A\subseteq X\times Y$ and projection $p_x\in\mathcal{P}_x$, $p_x(\mathrm{conv}_{x\times y}A) = \mathrm{conv}_x(p_x(A))$.

4.16. EXAMPLE. If $X=Y=R^1$ and if $\mathcal{C}_x = \{C \mid |C| \le 2\}$, $\mathcal{C}_y=\mathcal{C}^u_y$ then neither $\mathcal{C}_x$ nor $\mathcal{C}_y$ is compatible or projective in the product. In Example 4.2.5 both $\mathcal{C}_x$ and $\mathcal{C}_y$ are projective but not compatible in the product, and in

Example 4.2.3 both $\mathcal{C}_x$ and $\mathcal{C}_y$ are compatible but not projective in the product.

It is easily seen that sets in $\mathcal{C}^E_{x\times y}$ are always orthogonally compatible and orthogonally projective in the product, and the complement product is orthogonally compatible but not generally projective in the product.

<u>4.17. THEOREM</u>. If $\mathcal{C}_x$ and $\mathcal{C}_y$ are compatible and orthogonally projective in the product, and are domain finite, then the product $\mathcal{C}^P_{x\times y}$ is domain finite.

<u>Proof</u>. Let $A \subseteq X \times Y$ and define the set $C = \bigcup\{\mathrm{conv}_{x\times y}B \mid B \subseteq A,\ |B| < \infty\}$. Since $A \subseteq C \subseteq \mathrm{conv}_{x\times y}A$ it suffices to prove that C is convex, for it would then follow that $\mathrm{conv}_{x\times y}A = \bigcup\{\mathrm{conv}_{x\times y}B \mid B \subseteq A,\ |B| < \infty\}$, thus proving $\mathcal{C}^P_{x\times y}$ is domain finite. Consider $b \in Y$ and $p_y \in \mathcal{P}_y$. We show that $\pi_x(p_y^{-1}(b) \cap C) \in \mathcal{C}_x$.

Let $x \in \mathrm{conv}_x \pi_x(p_y^{-1}(b) \cap C)$. Then there exist $x_1, \ldots, x_k \in \pi_x(p_y^{-1}(b) \cap C)$ and thus $z_i = (x_i, y_i) \in X \times Y$ such that $x \in \mathrm{conv}_x\{x_1, \ldots, x_k\}$ and $z_i \in p_y^{-1}(b) \cap C$, $1 \le i \le k$. Hence $p_y(z_i) = b$ and $z_i \in C$ for all i. But by definition of C there must exist finite subsets $B_i \subseteq A$ such that $z_i \in \mathrm{conv}_{x\times y}B_i$, $1 \le i \le k$. Define the finite subset of A, $B = \bigcup_{i=1}^{k} B_i$. Then $\mathrm{conv}_{x\times y}B \subseteq C$ so that $\mathrm{conv}_{x\times y}\{z_1, \ldots, z_n\} \subseteq C$. Therefore, $\pi_x[p_y^{-1}(b) \cap \mathrm{conv}_{x\times y}\{z_1, \ldots, z_k\}] \subseteq \pi_x(p_y^{-1}(b) \cap C)$. By compatibility and point-convexity of $\mathcal{C}_x$ and $\mathcal{C}_y$,

$p_y^{-1}(b) \in \mathcal{C}^P_{X\times Y}$. Hence $p_y^{-1}(b) \cap \mathrm{conv}_{X\times Y}\{z_1,\ldots,z_k\}$ is a convex set in $X \times Y$ containing $z_1,\ldots,z_k$. It follows that $\mathrm{conv}_{X\times Y}\{z_1,\ldots,z_k\} \subseteq p_y^{-1}(b) \cap \mathrm{conv}_{X\times Y}\{z_1,\ldots,z_k\}$. Therefore, using the above and Lemma 4.14,

$x \in \mathrm{conv}_X\{x_1,\ldots,x_k\} = \mathrm{conv}_X\{\pi_X(z_1),\ldots,\pi_X(z_k)\}$
$\subseteq \pi_X(\mathrm{conv}_{X\times Y}\{z_1,\ldots,z_k\}) \subseteq \pi_X(p_y^{-1}(b) \cap C)$. That is, $\mathrm{conv}_X \pi_X(p_y^{-1}(b) \cap C) = \pi_X(p_y^{-1}(b) \cap C)$ proving that $\pi_X(p_y^{-1}(b) \cap C) \in \mathcal{C}_X$. Similarly, it may be shown that for $a \in X$, and $p_x \in \mathcal{P}_X$, $_Y(p_x^{-1}(a) \cap C) \in \mathcal{C}_Y$. Therefore, $C \in \mathcal{C}^P_{X\times Y}$ as was to have been shown.

The next two examples show that join-hull commutativity and regularity of segments are not, in general, productive properties.

<u>4.18. EXAMPLE</u>. Let $X=R^1$ and $C \in \mathcal{C}_X$ if and only if $C=X$ or $|C| \leq 1$. Let $Y=R^2$ and $\mathcal{C}_Y = \mathcal{C}^u_Y$. Then $C \in \mathcal{C}^P_{X\times Y}$ if and only if either $C=X\times Y$, C is a usual convex set in a plane orthogonal to X, or C is a line or plane in $X \times Y$.

Then if z_1, z_2, z_3 are three non-collinear points in a plane parallel to X and if z_2 and z_3 are in a plane orthogonal to X, then $\mathrm{conv}_{X\times Y}\{z_1,z_2,z_3\}$ is the plane containing the three points but $\bigcup\{z,q\}$, where $q \in z_2z_3$, is a proper subset of that plane, showing that join-hull commutativity is not productive.

<u>4.19. EXAMPLE</u>. Let $X=R^2$ and let $T\colon R^2 \to R^2$ be the homeomorphism of R^2 defined by $T(x,y)=(x,y^3)$. Define

$C \in \mathcal{C}_x$ if and only if $C = T(D)$ where $D \in \mathcal{C}_x^u$. Let $Y = R^1$ and $\mathcal{C}_y = \mathcal{C}_y^u$. Then $\mathcal{C}_x$ and $\mathcal{C}_y$ have the property that segments are regular, but $\mathcal{C}^P_{x \times y}$ does not have that property.

Cantwell's Axiom A [4], which says that each line is order isomorphic to the reals, is not productive as can be shown from Example 4.19. Thus the projective product of two Cantwell spaces need not be a Cantwell space.

<u>4.20. LEMMA</u>. Given an r-flat S in $X \times Y = Z$ with $r \leq \dim X$, there exists a projection $p_x \in \mathcal{P}_x$ such that p_x is one-to-one on S.

<u>Proof</u>. Without loss of generality, we may assume that S is a subspace of Z. If $S \subseteq X$ then $p_x = \pi_x$ is the desired projection. If $S \nsubseteq X$ then $\ell = \dim(S \cap X) < \dim X = m$. Hence, $m \geq \ell + 1$, and the following bases for the various subspaces of Z may be assumed: $S \cap X = \mathrm{span}\{z_1, \ldots, z_\ell\}$,
$X = \mathrm{span}\{z_1, \ldots, z_\ell, z_{\ell+1}, \ldots, z_m\}$,
$S = \mathrm{span}\{z_1, \ldots, z_\ell, z_{m+1}, \ldots, z_t\}$, $t = m + r - \ell$. (Since $S \nsubseteq X$, $\dim S > \ell$ and z_{m+1} exists.) Since $\{z_1, \ldots, z_t\}$ is linearly independent, it may be extended to a basis for Z:
$Z = \mathrm{span}\{z_1, \ldots, z_t, \ldots, z_k\}$ where $k = \dim Z$. Now define $f : \mathcal{B}_z \to \mathcal{B}_x \cup \{0\}$, where $\mathcal{B}_z$ and $\mathcal{B}_x$ are the bases for Z and X respectively, by setting

$$f(z_i) = z_i, \quad 1 \leq i \leq m$$
$$f(z_i) = z_{i-m+\ell}, \quad m+1 \leq i \leq t$$
$$f(z_i) = 0, \quad t+1 \leq i \leq k.$$

Since $t-m+\ell=r\leq m$, $f(\mathcal{B}_z)\subseteq \mathcal{B}_x\cup\{0\}$. Define the linear extension p_x of f to all of Z: $p_x(\sum_{i=1}^{k}\lambda_i z_i)=\sum_{i=1}^{k}\lambda_i f(z_i)$.

It is then clear that p_x is linear, onto X, and is the identity map on X, so $p_x\in\mathcal{P}_x$. Finally, if $p_x(s)=0$ for some $s\in S$ then

$$p_x(s)=p_x\Big(\sum_{\substack{1\leq i\leq \ell\\ m+1\leq i\leq t}}\lambda_i z_i\Big)=\sum_{\substack{1\leq i\leq \ell\\ m+1\leq i\leq t}}\lambda_i f(z_i)=0,$$

or $\lambda_1 z_1+\lambda_2 z_2+\dots+\lambda_\ell z_\ell+\lambda_{m+1}z_{\ell+1}+\lambda_{m+2}z_{\ell+2}+\dots+\lambda_t z_r=0$. By the linear independence of $\mathcal{B}_x$,

$\lambda_1=\lambda_2=\dots=\lambda_\ell=0=\lambda_{m+1}=\dots=\lambda_t$ and $s=0$. Hence the kernel of p_x is zero and p_x is injective on S.

<u>4.21. DEFINITION</u>. In a convexity structure $(X,\mathcal{C}_x)$, if whenever $x_5\in x_1x_2$ and $x_4\in x_1x_3$ there exists an $x_6\in x_3x_5\cap x_2x_4$ then the convexity structure is said to satisfy the <u>Ellis property</u>.

<u>4.22. THEOREM</u>. If $(X,\mathcal{C}_x)$ and $(Y,\mathcal{C}_y)$ have regular segments and are compatible and projective in the product, then the Ellis property is productive.

<u>Proof</u>. Let $\dim X=\dim Y=1$, $Z=X\times Y$ and in Z let $z_5\in z_1z_2$, $z_4\in z_1z_3$. Without loss of generality we may assume there is a $p_x\in\mathcal{P}_x$ such that $p_x(z_2)=p_x(z_4)=x_4$. Let $p_x(z_1)=x_1, p_x(z_5)=x_5$ and $p_x(z_3)=x_3$. Since $z_5\in z_1z_2$, by compatibility $x_5\in x_1x_4$ and since $z_4\in z_1z_3$, $x_4\in x_1x_3$. Then $x_5\in x_1x_3$ and $x_4\in x_1x_3$ so by regularity, $x_4\in x_1x_3=x_1x_5\cup x_5x_3$. Since segments are regular $x_4\notin x_1x_5$ so $x_4\in x_5x_3$. But then

by projectivity, there exists a $z_6 \in z_5z_3$ such that $p_x(z_6) = x_4$.

Now let p_y be such that $p_y(z_5)=p_y(z_3)=y_3$. Let $p_y(z_1)=y_1$, $p_y(z_4)=y_4$ and $p_y(z_2)=y_2$. Since $z_6 \in z_5z_3$ then by compatibility, $p_y(z_6)=y_3$. Now $z_4 \in z_1z_3$ hence by compatibility $y_4 \in y_1y_3$ and by regularity $y_3 \in y_1y_4 \cup y_4y_2$, and by decomposability, $y_3 \in y_4y_2$. Hence $z_6 \in z_4z_2$ as we wished to show.

If dim $X \geq 2$ then the points z_1, z_2, z_3, and $z_5 \in z_1z_2$ and $z_4 \in z_1z_3$ all lie in some 2-flat F, by Lemma 4.20 there is a projection of F onto X that is one-to-one. It is then clear that the points in the Ellis property can be mapped into F, proving it for $X \times Y$.

<u>4.23. COROLLARY</u>. If $(X, \mathcal{C}_x)$ and $(Y, \mathcal{C}_y)$ are compatible and projective in the product then regularity of segments is a productive property.

<u>4.24. DEFINITION</u>. Two non-empty convex sets $C, D \in \mathcal{C}^P_{x \times y}$ are said to be <u>complementary</u> if $C \cup D = X \times Y$ and $C \cap D = \emptyset$.

The following theorem is a basic separation theorem for convex sets. The theorem for linear spaces is proven in Kakutani [9], and Tukey [20], and was extended by Ellis [6]. The following proof is an adaptation of the one used by Valentine [21] for linear spaces to our more general setting.

<u>4.25. THEOREM</u>. Suppose $(X, \mathcal{C}_x)$ and $(Y, \mathcal{C}_y)$ have regular segments, are compatible and projective in the product, and suppose $\mathcal{C}^P_{x \times y}$ is join-hull commutative. If A and B are non-empty disjoint sets in $X \times Y$ then there exist complementary convex sets C and D in $\mathcal{C}^P_{x \times y}$ such that $A \subseteq C$ and $B \subseteq D$.

<u>Proof</u>. Let P be the class of all ordered pairs (A_i, B_i) of convex sets in $\mathcal{C}^P_{x \times y}$ such that $A_i \cap B_i = \emptyset$, $A_i \supseteq A$, $B_i \supseteq B$. First we note that P is non-empty since $(A,B) \subseteq P$. Partially order P by defining $(A_i, B_i) < (A_j, B_j)$ if and only if $A_i \subseteq A_j$ and $B_i \subseteq B_j$. The union of every linearly ordered subset of elements in P belongs to P and then by Zorn's Lemma, there is a maximal element (C,D) in P. To show that $C \cup D = X \times Y$, suppose $p \in X \times Y \setminus \{C \cup D\}$. Since $\text{conv}_{x \times y}(C \cup \{p\}) = \bigcup \{ pc \mid c \in C \}$ and $\text{conv}_{x \times y}(D \cup \{p\}) = \bigcup \{ pd \mid d \in D \}$ and since (C,D) is maximal, there exists $d_1 \in D \cap \text{conv}_{x \times y}\{C \cup \{p\}\}$ and there exists $c_1 \in C \cap \text{conv}_{x \times y}\{D \cup \{p\}\}$. Since $d_1 \notin C$, $c_1 \notin D$ then there exist points $c \in C$ and $d \in D$ such that $d_1 \in \text{intv}(cp)$ and $c_1 \in \text{intv}(dp)$. By Theorem 4.22, $dd_1 \cap cc_1 \neq \emptyset$, which is a contradiction.

The following associative law is presented to show how to define the projective product of a finite number of convexity structures.

4.26. THEOREM. Let $(X, \mathcal{C}_x)$ and $(Y, \mathcal{C}_y)$ and $(Z, \mathcal{C}_z)$ be convexity structures on R^ℓ, R^m, and R^n respectively. Then $\mathcal{C}^P_{x\times(y\times z)} = \mathcal{C}^P_{(x\times y)\times z}$.

Proof. Let $C \in \mathcal{C}^P_{(x\times y)\times z}$, to show that $C \in \mathcal{C}^P_{x\times(y\times z)}$ we must show that given $(0,\bar{y},\bar{z})$ and onto projections $p_{y\times z}$: $X\times(Y\times Z)\to Y\times Z$ and π_x: $X\times(Y\times Z)\to X$ that,

a) $\pi_x(p^{-1}_{y\times z}(0,\bar{y},\bar{z})\cap C) \in \mathcal{C}_x$,

and given $\bar{x} \in X$ and onto projections p_x and $\pi_{y\times z}$ that

b) $\pi_{y\times z}(\bar{x},0,0)\cap C) \in \mathcal{C}^P_{y\times z}$.

To establish a) we note that $p_{y\times z}$ and π_x have the following block matrix representations,

$$p_{y\times z} = \begin{pmatrix} 0 & 0 & 0 \\ A & I_m & 0 \\ B & 0 & I_n \end{pmatrix}, \qquad \pi_x = \begin{pmatrix} I_\ell & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where A is an $m\times\ell$ matrix, B is an $n\times\ell$ matrix and I_ℓ, I_m, and I_n are identity matrices of order ℓ, m, and n respectively. We define

$$\pi_{x\times y} = \begin{pmatrix} I_\ell & 0 & 0 \\ 0 & I_m & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad p_z = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ B & 0 & I_n \end{pmatrix}, \quad p_y = \begin{pmatrix} 0 & 0 & 0 \\ A & I_m & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

then it is routine to show that

$$\pi_x(p^{-1}_{y\times z}(0,\bar{y},\bar{z})\cap C) = \pi_x(p_y^{-1}(0,\bar{y},0)\cap(\pi_{x\times y}(p_z^{-1}(0,0,\bar{z})\cap C))),$$

where the right side of the above equation belongs to $\mathcal{C}_x$ since $C \in \mathcal{C}^P_{(x\times y)\times z}$, establishing a).

To establish b) there are two similar arguments. We will show that given $(0,\bar{y},0)$ and onto projections

p_y: $Y \times Z \longrightarrow Y$ and π_z: $Y \times Z \longrightarrow Z$ that $\pi_z(p_y^{-1}(0,\bar{y},0) \cap T) \in \mathcal{C}_z$, where $T = \pi_{y \times z}(p_x^{-1}(\bar{x},0,0) \cap C)$.

We note that p_x: $X \times (Y \times Z) \longrightarrow X$ and p_y: $0 \times Y \times Z \longrightarrow Y$ have matrix representations

$$p_x = \begin{pmatrix} I_\ell & D & E \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad p_y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & I_m & F \\ 0 & 0 & 0 \end{pmatrix},$$

where D,E, and F are of order $\ell \times m$, $\ell \times n$, and $m \times n$ respectively. If we define

$$p_{x \times y} = \begin{pmatrix} I_\ell & 0 & E-DF \\ 0 & I_m & F \\ 0 & 0 & 0 \end{pmatrix}$$

and let $q = (\bar{x} - D\bar{y}, \bar{y}, 0) \in X \times Y$, then it is routine to show that $\pi_z(p_y^{-1}(0,\bar{y},0) \cap T) = \pi_z(p_{x \times y}^{-1}(q) \cap C)$, where the right side of the above equation belongs to $\mathcal{C}_z$ since $C \in \mathcal{C}^P_{(x \times y) \times z}$, establishing b), and completing the proof.

We now present some results leading to a Carathéodory theorem for the projective product.

<u>4.27. LEMMA</u>. If $\mathcal{C}_x$ and $\mathcal{C}_y$ are compatible in the product then the usual lines of $Z = X \times Y$ are members of $\mathcal{C}^P_{x \times y}$.

<u>Proof</u>. Let F be any line in Z; then by choosing $q \in F$ the set $S = F - q$ is a one-dimensional subspace of Z. It is clear that either $S \cap X = 0$ or $S \cap Y = 0$, say $S \cap X = 0$. Let $S = \operatorname{span}\{s\}$. First, if $s \in Y$ then assume the following bases for X and Y, $X = \operatorname{span}\{z_1, \ldots, z_m\}$, $Y = \operatorname{span}\{z_{m+1}, \ldots, z_{m+n}\}$ where $z_{m+1} = s$. Define the projections $\pi_x(\sum_{i=1}^{m+n} \lambda_i z_i) = \sum_{i=1}^{m} \lambda_i z_i$,

and $p_{x_r}(\sum_{i=1}^{m+n}\lambda_i z_i)=(\sum_{i=1}^{m}\lambda_i z_i)-\lambda_{m+r}z_1$, $2\le r\le n$. We note that $p_{x_r}\in\mathcal{P}_x$ and $p_{x_r}(s)=0$ for every $r=2,\ldots,n$. We now claim that $S=\pi_x^{-1}(0)\cap(\bigcap_{r=2}^{n}p_{x_r}^{-1}(0))$; for if $z=\sum_{i=1}^{m+n}\lambda_i z_i\in\ker\pi_x\cap\ker p_{r_2}\cap\ldots\cap\ker p_{r_n}$, then $\lambda_1=\ldots=\lambda_m=0$ and $\lambda_{m+2}=\lambda_{m+3}=\ldots=\lambda_{m+n}=\lambda_1$ or $z=\lambda z_{m+1}=\lambda s$ as desired.

If $s\notin Y$ define the following bases for X and Y,

$X=\operatorname{span}\{z_1,\ldots,z_m\}$ where $z_1=\pi_x(s)$ and

$Y=\operatorname{span}\{z_{m+1},\ldots,z_{m+n}\}$ where $z_{m+1}=\pi_y(s)$. Define

$p_x(\sum_{i=1}^{m+n}\lambda_i z_i)=(\sum_{i=1}^{m}\lambda_i z_i)-\lambda_{m+1}z_1$,

$p_y(\sum_{i=1}^{m+n}\lambda_i z_i)=(\sum_{i=m+1}^{m+n}\lambda_i z_i)-\lambda_1 z_{m+1}$. Note that $p_x\in\mathcal{P}_x$, $p_y\in\mathcal{P}_y$ and $s\in\ker p_x\cap\ker p_y$. We claim that $S=p_x^{-1}(0)\cap p_y^{-1}(0)$ for if $z=\sum_{i=1}^{m+n}\lambda_i z_i\in\ker p_x\cap\ker p_y$ then $\lambda_1=\lambda_{m+1}$, $\lambda_2=\ldots=\lambda_m=0$ and $\lambda_1=\lambda_{m+1}$, $\lambda_{m+2}=\ldots=\lambda_{m+n}=0$ and so $z=\lambda z_1+\lambda z_{m+1}=\lambda(\pi_x(s)+\pi_y(s))=\lambda s$ as desired.

<u>4.28. LEMMA</u>. If $\mathcal{C}_x$ and $\mathcal{C}_y$ are compatible and projective in the product, then for any one-dimensional member C of $\mathcal{C}_x$ and affine map T: $X\to X$, $T(C)\in\mathcal{C}_x$, and similarly for $\mathcal{C}_y$. That is, the subfamilies of $\mathcal{C}_x$ and $\mathcal{C}_y$ consisting of their one-dimensional members are closed under affine maps.

<u>Proof</u>. Suppose $C\subseteq F$ where F is a 1-flat in X. Since the problem is trivial if C is singleton, assume without loss that $0\in C$, $x_0\in C$ with $x_0\ne 0$, and $T(0)=0$. Let $x_0'=T(x_0)$ and consider a point $z_0\in\pi_x^{-1}(x_0)\setminus X$. The line $L(0,z_0)$ belongs to $\mathcal{C}^P_{x*y}$ Lemma 4.27. Also there exists

$p_x \in \mathcal{P}_x$ such that $p_x(z_o)=x_o'$. Both p_x and π_x are one-to-one on $L(0,z_o)$ and hence $T(C)=p_x[\pi_x^{-1}(C)\cap L(0,z_o)] \in \mathcal{C}_x$.

<u>4.29. LEMMA</u>. Let $\mathcal{C}_x$, $\mathcal{C}_y$ be compatible and projective in the product and suppose that segments in $\mathcal{C}_x$ are closed topologically, relative to $X=R^m$. Then each segment pq in $\mathcal{C}_x$ is either $\{p,q\}$, the usual segment joining p and q, or the line determined by p and q.

<u>Proof</u>. Let $x_o,x_1 \in X$ and suppose $x_2 \in x_ox_1$ where $x_2 \neq x_i$, i=1,2. By Lemma 4.27 $x_ox_1 \subseteq L(x_o,x_1)$; then without loss of generality assume $x_2 \in$ u-conv $x_o x_1$. It is clear that a sequence of affine transformations $T_2,\ldots,T_i,\ldots$ and a sequence of points $x_3,\ldots,x_{i+1},\ldots$ $i \geq 2$ may be constructed such that if $Q_i=\{x_j \mid 0 \leq j \leq i\}$ then

1) $T_i(x_2)=x_{i+1}$,
2) $T_i(Q_1)\subseteq Q_i$, and
3) $Q=\bigcup_{i\geq 1}Q_i$ is topologically dense on u-conv $x_o x_1$.

Each T_i may be assumed injective so that if S is any one-dimensional subset of X then $T_i(\text{conv}_x S)=\text{conv}_x T_i(s)$ by Lemma 4.28. To show $Q\subseteq x_ox_1$, it suffices to show $Q_i\subseteq x_ox_1$ for $i\geq 1$. Obviously, $Q_1\subseteq x_0x_1$ so assume $Q_i\subseteq x_ox_1$ and consider Q_{i+1}; we have $x_{i+1}=T_i(x_2)\in T_i(x_ox_1)=T_i(\text{conv}_x Q_1)= \text{conv}_x T_i(Q_1)\subseteq \text{conv}_x Q_i\subseteq x_ox_1$. Hence $Q_{i+1}=x_{i+1}\cup Q_i\subseteq x_ox_1$. Hence $Q\subseteq x_ox_1$ and then x_ox_1 contains the closure of Q or u-conv x_ox_1. So if $x_ox_1\neq \{x_o,x_1\}$ then u-conv $x_ox_1\subseteq x_ox_1$. If there is an $x_2\in x_ox_1\setminus$u-conv x_ox_1 then by Lemma 4.28 $x_ox_1=L(x_o,x_1)$. Otherwise $x_ox_1=$u-conv x_ox_1.

4.30. REMARK. The segments of $\mathcal{C}_x$ can be the ordinary lines of X so it does not necessarily follow that the betweeness relations in R^m coincide with those of $\mathcal{C}_x$ under the hypothesis of Lemma 4.29.

4.31. REMARK. Inductive proofs of the classical Carathéodory theorem are rare; only one is known to the author, namely that advanced by B. Peterson in [13]. If we specialize to the case $\mathcal{C}_x = \mathcal{C}_x^u$ and $\mathcal{C}_y = \mathcal{C}_y^u$ then the following theorem becomes an inductive proof of the classical Carathéodory theorem in R^{m+n}, different from that given by Peterson.

4.32. THEOREM. Suppose $(X, \mathcal{C}_x)$ and $(Y, \mathcal{C}_y)$ are convexity structures that are compatible and projective in the product having topologically closed segments, and such that $\mathcal{C}_{x \times y}^P$ is join-hull commutative. Then if $\mathcal{C}_x$ has Carathéodory number c and $Y=R^1$, then $\mathcal{C}_{x\times y}^P$ has Carathéodory number less than or equal to c+1.

Proof. Let $z \in \mathrm{conv}_{x\times y} S$ for $S \subseteq X \times Y$ and let $\pi_y(z)=y$. First, if $S \subseteq \pi_y^{-1}(y)$ then $x=\pi_x(z) \in \pi_x(\mathrm{conv}_{x\times y} S)=\mathrm{conv}_x \pi_x(S)$, so there exist c points $x_1, \ldots, x_c$ in $\pi_x(S)$ such that $x \in \mathrm{conv}_x\{x_1, \ldots, x_c\}$ and then there exist points $s_i \in S$ such that $\pi_x(s_i)=x_i$, $i=1, \ldots, c$. Since π_x is bijective on $\pi_y^{-1}(y)$, $z \in \mathrm{conv}_{x\times y}\{s_1, \ldots, s_c\}$ the convex hull of c or fewer points of S.

Now suppose $S \not\subseteq \pi_y^{-1}(y)$ and let $s^* \in S$ be such that $\pi_y(s^*) \neq y$. By join-hull commutativity $z \in \mathrm{conv}_{x \times y}[s^* \cup (S \setminus s^*)] = s^* \mathrm{jn}\ \mathrm{conv}_{x \times y}(S \setminus s^*)$. Since $\mathcal{C}^P_{x \times y}$ is domain finite by Theorem 4.17 $S \setminus s^*$ may be assumed to be finite so there exists a minimal subset $S^* \subseteq S \setminus s^*$ such that $z \in s^* \mathrm{jn}\ \mathrm{conv}_{x \times y} S^*$.

<u>Case 1.</u> If $z \notin \mathrm{conv}_{x \times y} S^*$ there exists a $q \in \mathrm{conv}_{x \times y} S^*$ such that $z \in qs^*$. Since $z \notin \mathrm{conv}_{x \times y} S^*$, $z \neq q$. Also, $z \neq s^*$ since $\pi_y(s^*) \neq \pi_y(z)$. By Lemma 4.29 $qs^* \supsetneq$ u-conv qs^*. Since $s^* \notin \pi^{-1}(y)$ let $p_x \in \mathcal{P}_x$ such that $p_x(q) = p_x(s^*)$. Then $q' = p_x(q) \in p_x(\mathrm{conv}_{x \times y} S^*) = \mathrm{conv}\ p_x(S^*)$ so there exist points $s_i \in S^*$, $i = 1, \ldots, c$ such that $q' \in \mathrm{conv}_x\{p_x(s_1), \ldots, p_x(s_c)\} = p_x(\mathrm{conv}_{x \times y}\{s_1, \ldots, s_c\})$. Thus there exists an $s \in \mathrm{conv}_{x \times y}\{s_1, \ldots, s_c\}$ so that $q' = p_x(s)$.

There are now a number of cases corresponding to the possible locations of s on the usual line containing q and s^*. If $s^* \in$ u-conv zs then $z \in$ u-conv qs. Now either $qs = \{q, s\}$ or $qs \supsetneq$ u-conv qs, but since the one-dimensional members of $\mathcal{C}^P_{x \times y}$ are affinely related and $qs^* \neq \{q, s^*\}$ then $qs \neq \{q, s\}$. Thus $qs \supsetneq$ u-conv qs and $z \in qs \subseteq \mathrm{conv}_{x \times y} S^*$, a contradiction. If $s \in$ u-conv zs^* then $z \in$ u-conv qs and again $z \in$ u-conv $qs \subseteq qs \subseteq \mathrm{conv}_{x \times y} S^*$, a contradiction. Finally, if $s \in$ u-conv qz or $q \in$ u-conv sz then $z \in$ u-conv $ss^* \subseteq ss^*$, and then $z \in \mathrm{conv}_{x \times y}\{s_1, \ldots, s_c, s^*\}$--the convex hull of c+1 points of S.

<u>Case 2.</u> If $z \in \mathrm{conv}_{X \times Y} S^*$, we repeat the procedure and consider the two cases $S^* \subseteq \pi_y^{-1}(y)$ and $S^* \nsubseteq \pi_y^{-1}(y)$. If $S^* \subseteq \pi_y^{-1}(y)$, then as before it follows that z lies in the convex hull of c or fewer points of $S^* \subseteq S$. If $S^* \nsubseteq \pi_y^{-1}(y)$ then we choose $\bar{s}^* \in S^* \setminus \pi_y^{-1}(y)$ as before, let $\bar{S}^* \subseteq S^* \setminus \bar{s}^*$ be minimal such that $z \in \bar{s}^* jn\ \mathrm{conv}_{X \times Y} \bar{S}^*$, and then $\bar{s}^*$ and $\bar{S}^*$ play the roles of s^* and S^* in the preceding argument. But in this case since S^* is minimal such that $z \in s^* \mathrm{conv}_{X \times Y} S^*$ we cannot have $z \in \mathrm{conv}_{X \times Y} \bar{S}^*$. Hence $z \notin \mathrm{conv}_{X \times Y} \bar{S}^*$ and it ultimately follows that z lies in the convex hull of c+1 or fewer points $\bar{s}_1, \bar{s}_2, \ldots, \bar{s}_c, \bar{s}^*$ of $S^* \subseteq S$, which completes the proof.

We shall now prove a Helly theorem for the projective product. The lemma following the next two definitions is an adaptation of a theorem in Valentine [21].

<u>4.33. DEFINITION.</u> A <u>polyhedron</u> in a convexity structure $(X, \mathcal{C}_X)$ is the convex hull of a finite set.

<u>4.34. DEFINITION.</u> A convexity structure $(X \times Y, \mathcal{C}^P_{X \times Y})$ is said to have the <u>polyhedron separation property</u> if and only if given a polyhedron P and a convex set C there exists a projection $p_z \in \mathcal{P}_X \cup \mathcal{P}_Y$ and a point $a \in X \cup Y$ such that $p_z^{-1}(a) \cap (P \cup C) = \emptyset$, and if $z_1 \in P$, $z_2 \in C$ then $p_z^{-1}(a) \cap \mathrm{conv}\ z_1 z_2 \neq \emptyset$.

<u>4.35. LEMMA</u>. Let C_i, $i=1,\dots,h+1$ be h+1 convex sets in a convexity structure such that the intersection of each h of them is nonempty. Then there exist h+1 polyhedrons P_i, $i=1,\dots,h+1$ such that $P_i \subseteq C_i$ and the intersection of each h of them is nonempty.

<u>Proof</u>. Since every h of the h+1 sets C_i has nonempty intersection, there exists $p_i \in \bigcap_{\substack{1\le j\le h+1\\ j\ne i}} C_j$, $i=1,\dots,h+1$.

Define $P_i=\operatorname{conv}\Big(\bigcup_{\substack{1\le j\le h+1\\ j\ne i}} \{p_j\}\Big)$, then $P_i \subseteq C_i$, $i=1,\dots,h+1$,

and $P_1 \cap \Big(\bigcap_{\substack{2\le j\le h+1\\ j\ne i}} C_j\Big) \ne \emptyset$, $2 \le i \le h+1$; hence P_1 can replace C_1. It then follows by finite induction that each set C_i can be replaced by a polyhedron P_i such that the intersection of every h of the polyhedrons is nonempty.

<u>4.36. THEOREM</u>. If $(X, \mathcal{C}_X)$ and $(Y, \mathcal{C}_Y)$ have finite Helly numbers h_1 and h_2 respectively, and if $\mathcal{C}^P_{X\times Y}$ has the polyhedron separation property, then $\mathcal{C}^P_{X\times Y}$ has a finite Helly number $h \le \max\{h_1, h_2\} + 1$.

<u>Proof</u>. By the above lemma, it is sufficient to prove that all members of a family $\mathcal{F}$ of h+1 polyhedrons have a point in common if every h have nonempty intersection.

Suppose that this is false, that there exist h+1 polyhedrons P_i, $i=1,\dots,h+1$ such that each h of them have a point in common but $\bigcap_1^{h+1} P_i=\emptyset$. Since $P_1 \cap \dots \cap P_h$ and P_{h+1} are disjoint, by the polyhedron separation property there exists, without loss of generality, a projection p_x and a point $a \in X$ such that $p_x^{-1}(a) \cap [(P_1 \cap \ldots \cap P_h) \cup P_{h+1}] = \emptyset$, and if $z_1 \in P_1 \cap \ldots \cap P_h$ and $z_2 \in P_{h+1}$ then $p_x^{-1}(a) \cap \operatorname{conv}\{z_1, z_2\} \neq \emptyset$.

Since P_{h+1} and every combination of h-1 members of $P_1, \ldots, P_h$ have nonempty intersection, then the intersection of every h-1 members of $P_1, \ldots, P_h$ must have a nonempty intersection with $p_x^{-1}(a)$. Since each P_i is convex in $\mathcal{C}^P_{x \times y}$, $\pi_y(p_x^{-1}(a) \cap P_i)$ is convex in Y for $1 \leq i \leq h$ and each h-1 of them have nonempty intersection. But $h-1 \geq h_1$; therefore all h sets $\pi_y(p_x^{-1}(a) \cap P_i)$ have nonempty intersection. That is, there exists a $y \in Y$ such that $y \in \bigcap_{i=1}^{h} \pi_y(p_x^{-1}(a) \cap P_i)$, and then there exists a point $q \in p_x^{-1}(a) \cap \{P_1 \cap \ldots \cap P_h\}$, a contradiction.

We shall end this chapter with some results relating the projective product and affine mappings.

4.37. THEOREM. If $(X, \mathcal{C}_x)$ and $(Y, \mathcal{C}_y)$ are convexity structures that are compatible in the product, then $\mathcal{C}_x$ and $\mathcal{C}_y$ are closed under translations.

Proof. Let $T: X \to X$ be a translation; i.e. $T(x) = x + a$ for some $a \in X$. Let π_x and π_y be the orthogonal projections from $X \times Y$ onto X and Y respectively, and let $0 \neq y \in Y$. Let p_x be a projection onto X such that $p_x(x+a, y) = x$. It is then routine to show that $T(C) = C + a = \pi_x(\pi_y^{-1}(y) \cap p_x^{-1}(C))$, and since $\mathcal{C}_x$ and $\mathcal{C}_y$ are compatible in the product, $p_x^{-1}(C)$ is a convex set in $\mathcal{C}^P_{x \times y}$ and hence $\pi_x(\pi_y^{-1}(y) \cap p_x^{-1}(C)) = T(C) \in \mathcal{C}_x$, which completes the proof.

If $(X, \mathcal{C}_x)$ is the usual convexity structure on R^1 and if $(Y, \mathcal{C}_y)$ is the structure defined on R^1 by $C \in \mathcal{C}_y$ if and only if card $C \leq 2$ or $C=Y$, then both $\mathcal{C}_x$ and $\mathcal{C}_y$ are closed under translations, but $(X, \mathcal{C}_x)$ and $(Y, \mathcal{C}_y)$ are not compatible in the product, showing that the converse to the above theorem is not necessarily true.

4.38. THEOREM. If $(X, \mathcal{C}_x)$ and $(Y, \mathcal{C}_y)$ are convexity structures that are compatible in the product and if $\mathcal{C}^P_{x \times y}$ is closed under affine mappings, then $\mathcal{C}_x$ and $\mathcal{C}_y$ are both closed under affine mappings.

Proof. Let $T: X \to X$ be an affine map on X. Then $T'(x,y)=(T(x),y)$ defines an affine mapping T' on $X \times Y$, and hence if $C \in \mathcal{C}_x$, $T(C)=\pi_x(\pi_y^{-1}(0) \cap T'(\pi_x^{-1}C))$ belongs to $\mathcal{C}_x$.

As a counterexample to the converse of the above theorem, let $X=R^1$, $C \in \mathcal{C}_x$ if and only if $|C| \leq 2$ or $C=X$, and $Y=R^2$, $\mathcal{C}_y = \{C \mid C=Y$ or C is a pair of parallel lines$\}$ $\cup \{C \mid C$ consists of the vertices of a non-degenerate parallelogram$\}$ $\cup \{C \mid C$ is a set of three non-collinear points$\}$ $\cup \{C \mid |C| \leq 2\}$. Then $\mathcal{C}_x$ and $\mathcal{C}_y$ are compatible in the product and both are closed under affine mappings since a set of cardinality five, no three of which are collinear and lying in a plane containing X, will belong to $\mathcal{C}^P_{x \times y}$ but the image of this set under a rotation need not belong to $\mathcal{C}^P_{x \times y}$.

The following example shows that even if $(X, \mathcal{C}_x)$ and $(Y, \mathcal{C}_y)$ are compatible in the product, $\mathcal{C}_x$ and $\mathcal{C}_y$ need not be closed under affine mappings.

4.39. EXAMPLE. Let $X=R^2$ and let $C \in \mathcal{C}_x$ if and only if $C=X$ or $C \in \{C \mid C$ is a line or a pair of parallel lines$\} \cup$ $\{C \mid |C| \leq 4$, no three of which are collinear$\} \cup$ $\{C \mid C$ is a translate of $y=x^2\}$.
Let $Y=R^1$ and let $C \in \mathcal{C}_y$ if and only if $C=Y$ or $|C| \leq 2$. Then it is tedious but routine to show that $(X, \mathcal{C}_x)$ and $(Y, \mathcal{C}_y)$ are compatible in the product but $\mathcal{C}_x$ is not closed under affine mappings.

The following theorem presents a positive result relating compatibility and affine mappings when $X=Y=R^1$.

4.40. THEOREM. If $(X, \mathcal{C}_x)$ and $(Y, \mathcal{C}_y)$ are convexity structures on $X=Y=R^1$ such that $\mathcal{C}^P_{x \times y}$ is closed under affine mappings, then $(X, \mathcal{C}_x)$ and $(Y, \mathcal{C}_y)$ are compatible in the product.

Proof. It is routine to show that the mappings $F(\bar{x})=\pi_x(p_y^{-1}(y) \cap p_x^{-1}(\bar{x}))$ and $G(\bar{x})=\pi_y((p_x')^{-1}(x) \cap p_x^{-1}(\bar{x}))$ are either trivial or affine mappings from $X \times \{0\}$ to X and $X \times \{0\}$ to Y, respectively.

4.41. EXAMPLE. Let $X=Y=R^2$, $\mathcal{C}_x=\mathcal{C}_y=\{C \mid |C| \leq 1$ or $C=R^2\}$. Then $\mathcal{C}^P_{x \times y}=\{C \mid |C| \leq 1$ or $C=R^4\}$, and $\mathcal{C}_x$, $\mathcal{C}_y$ and $\mathcal{C}^P_{x \times y}$ are all closed under affine mappings but $(X, \mathcal{C}_x)$ and $(Y, \mathcal{C}_y)$

are not compatible in the product since $\mathcal{C}_x$ and $\mathcal{C}_y$ do not contain lines, forcing $\mathcal{C}^P_{x\times y}$ to not contain planes, e.g. $\pi_x^{-1}(0)$.

Finally, there appears to be no connection at all between the closure of $\mathcal{C}_x$ and $\mathcal{C}_y$ under affine maps and the property of $\mathcal{C}^P_{x\times y}$ being projective as the examples below show.

If $X=Y=R^1$ and if $\mathcal{C}_x=\mathcal{C}_y=\{C \mid C=R^1 \text{ or } |C| \leq 2\}$, then $\mathcal{C}_x$ and $\mathcal{C}_y$ are closed under affine maps but not projective in the product: Three non-collinear points will belong to $\mathcal{C}^P_{x\times y}$ but their projection will not necessarily belong to $\mathcal{C}_x$ or $\mathcal{C}_y$.

If $X=R^2$ and $\mathcal{C}_x=\{C \mid |C| \leq 1$, C is a line not parallel to the x-axis or $C=R^2\}$ and $Y=R^1$, $\mathcal{C}_y=\{C \mid |C| \leq 1 \text{ or } C=R^1\}$, then $(X,\mathcal{C}_x)$ and $(Y,\mathcal{C}_y)$ will be projective in the product but not closed under affine mappings.

We remark that we have been only partly successful in creating a product concept for $\mathcal{C}_{x\times y}$ that is both natural and useful. We have been unable to remove the requirement that X and Y are themselves vector spaces so that the product concept could be defined for more general convexity structures. Also, we have been unable to obtain a Radon theorem or general Carathéodory and Helly theorems for this product.

APPENDIX

SYMBOLS USED IN THE TEXT

Symbol	Meaning	Page Defined
$\emptyset$	Empty set	
dim X	Dimension of X	
span$\{x_i\}$	Linear space spanned by $\{x_i\}$	
$\lvert C \rvert$	Cardinality of a set C	
$\mathcal{C}$	Convexity structure	3
conv S	Convex hull of a set S	4
pq	Convex hull of $\{p,q\}$	4
ckS	Convex kernel of a set S	7
$\mathcal{C}^E_{x \times y}$	Eckhoff product of X and Y	10
$co_x A$	Complement of A in X	17
$\mathcal{C}^C_{x \times y}$	Complement product of X and Y	17
π	Orthogonal projection	33
p_x	Projection onto X	33
$\mathcal{P}_x$	Class of projections onto X	33
$\mathcal{C}^P_{x \times y}$	Projective product of X and Y	33
$\mathcal{C}^u_x$	Usual convexity structure on X	34
u-conv	Usual convex hull operator	34

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