76-15,798

CHEN, Tommy Fu-Tuan, 1937-
HEAT ABSORPTION VIA ELECTROMAGNETIC FIELDS IN DROSOPHILA MELANOGASTER.

The University of Oklahoma, D.Engr., 1975
Engineering, electronics and electrical

THE UNIVERSITY OF OKLAHOMA

GRADUATE COLLEGE

HEAT ABSORPTION VIA ELECTROMAGNETIC FIELDS IN DROSOPHILA MELANOGASTER

A DISSERTATION

SUBMITTED TO THE GRADUATE FACULTY

in partial fulfillment of the requirements for the

degree of

DOCTOR OF ELECTRICAL ENGINEERING

BY

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Norman, Oklahoma

1975

HEAT ABSORPTION VIA ELECTROMAGNETIC FIELDS

IN

DROSOPHILA MELANOGASTER

APPROVED BY

DISSERTATION COMMITTEE

ACKNOWLEDGMENT

I would like to express my appreciation to Dr. Leon W. Zelby for his assistance and guidance in the preparation of this work. His comments and suggestions were most helpful in performing this research. Finally I am indebted to the committee members, who have been very generous in providing detailed criticisms that significantly improved the manuscript.

ABSTRACT

We, human beings, are exposing ourselves to electromagnetic fields for the duration of our lifetimes, and we cannot help asking what an increasing intensity of electromagnetic radiation of all types does to us. It is important to know whether there is a biological state that can be affected by this invisible radiation which surrounds us. This problem calls for worldwide attention; as we know the maximum "safe" limits (as set by law) for electromagnetic exposure are $10 mW/cm^2$ for the U.S., $1 mW/cm^2$ for Sweden, and $0.01 mW/cm^2$ for the U.S.S.R.

This work is a step to determine the biological effects of electromagnetic radiation. Drosophila melanogaster have been used as analytical and experimental objects because their lifetimes are so short that they can be reproduced easily.

Drosophila melanogaster is assumed to be an ellipsoid of revolution (prolate spheroid) or revolution solid of a nephroidal shape with permittivity ε, conductivity σ, and permeability μ. The input electromagnetic field is assumed to be a plane wave. Two methods of determining the electromagnetic field of the prolate spheroid scatterer have been used: (1) solving the vector Helmholtz's equation for the spheroid coordinate and (2) solving the vector Helmholtz's equation for the spherical elementary vector and then using the boundary conditions to match both transmitted and scattered waves to find the unknown

coefficients of the transmitted and scattered fields. The second method used in solving for the prolate spheroid scatterer has also been used to solve the nephroidal scatterer.

The power absorbed by the prolate spheroid representing Drosophila melanogaster has been calculated at a frequency of 4 GHz. The following table lists several values of incident power and the corresponding absorbed power :

$P_{in}\left(\frac{W}{m^2}\right)$	$E_z\left(\frac{V}{m}\right)$	$Q_{ab}(W)$
1.	1.94×10	7.19×10^{-12}
10.	6.14×10	7.19×10^{-11}
100.	1.94×10^2	7.19×10^{-10}
1000.	6.14×10^2	7.19×10^{-9}

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CHAPTER 1

INTRODUCTION

In our modern environment, we are utilizing all types of electric and electronic devices at an increasing rate. Thus, the environment is becoming saturated with electromagnetic radiation of all types — very low frequency associated with power distributing systems; low and medium frequency associated with broadcasting; very high and ultrahigh frequency associated with television, radio, navigation, and communication; and microwave frequency associated with line-of-sight communication radar and cooking. The biological effects of this radiation depend in part on the frequency and on the intensity of the signal. Some of the biological effects of radiation, particularly from radiation in the high energy range and radiation associated with heating, have been investigated relatively extensively and are known [1,2,3,4].

The dielectric behavior of human body tissue in the microwave range led to the introduction of microwave diathermy in the medical practice at the beginning of the 1950's. At first, the use of microwaves in medical diagnosis and therapy was very controversial, but through the years, the behavior of body tissue in a microwave field has become better understood. In short, wave diathermy frequencies of 27 MHz (11 meters) and 40.68 MHz (7 meters) are employed. The heating is carried out in the capacitor or coil field depending upon the part of the body and on the disease to be treated. The heating of the body area depends largely on the distribution of the electromagnetic field on the complex dielectric constant and the volume of different adjoining tissues [5].

The energy density of the radiated wave must not exceed a certain value, in order to protect the body tissue against biological injury resulting from strong radiation. Because of the rising radiation capability of newly developed high power radar installations, the maximum radiation density permissible for human beings in the microwave range must be calculated [6].

The biological effects of electromagnetic radiation on the human body in the microwave range have been discussed by many authors [6-20]. They researched the behavior of various forms of human tissue under the influence of electromagnetic waves in the microwave range in order to discover the differences and the advantages and disadvantages of diathermy derived from the frequency dependence of conductivity, permittivity, and permeability of tissue. Therapy with electromagnetic radiation is based solely on the development of heat [7,8] and the tissue changes associated with it through the dilation of blood vessels. When the radiation energy absorbed and converted into heat exceeds a particular value, large surface irradiation has harmful effects because of the general rise or increase of the body temperature. In the irradiation of small tissue areas, the danger lies in local temperature increases in organs particularly sensitive to heat such as the lense of the eye, the brain of the head, the gall bladder, the bladder, and the testicles.

The eye is one of the organs most sensitive to microwave radiation. The eye has a weak vascular system, and the heat produced by radiation cannot be conducted away quickly enough. Gray cataract is a typical eye disease occurring through protracted exposure to radiation when the radiation density exceeds a specified value [9,10]. In this case, the protein in the lens coagulates into visible white flakes. At 2400 MHz, a radiation density of less than 80 mW/cm^2 was calculated by Carpenter [9,10] for rabbits whose eyes resemble the human eye in size and shape. A cataract develops above a critical frequency of about 200 MHz. Short radiation periods of high radiation density are considerably more harmful than prolonged radiation with a uniform medium radiation density; in other words, pulse-form overloading can lead to quicker cataract formation. For example, the eyes of rabbits were exposed for 20 minutes to a radiation density of 140 mW/cm^2 without showing any indication of cataract. When, on the other hand, the energy was supplied in pulse form, a cataract developed with the same radiation time and density [9,10,11,12,13,14].

The brain and the extension of the spinal cord are sensitive to changes of pressure and temperature. Therefore, abnormal temperature changes induced by irradiation of the head can have serious consequences. Cranial bones produce strong reflections, and estimating the energy absorbed by the cranial bones is very difficult. Indication of the radiation absorption can be obtained only in a dummy as done by Antharvedi Anne.* The temperature rises most rapidly in the brain when it is irradiated from above or when the thorax is irradiated, since it is through

*Antharvedi Anne, Scattering and Absorption of Microwaves by Dissipative Dielectric Objects: The Biological Significance and Hazards to Mankind (University of Pennsylvania, 1963).

the thorax that the heated blood flows directly into the head. The rectal temperature remains unaltered for a long period or only rises very little. In apes, irradiation of the head led to a condition of somnolence and subsequently to unconsciousness. With protracted radiation, convulsions occurred and then continuous paralysis [15].

The male sex organs are extraordinarily sensitive to heat and are therefore especially endangered by radiation. The injurious radiation density was given by Ely [16] at a maximum of 5 mW/cm^2, thus it is lowest in comparison with the densities of other radiation-sensitive organs. Slight radiation damage to the spermatic duct leads to sterility, which according to observations disappears again after a short time. If the prescribed radiation density is exceeded, injuries can occur which result in permanent sterility [17,18].

Some biological effects which are not due to heat were observed. McAfee [3,4] was able to observe that animals exposed to a radiation field are very restless even before their body temperature has risen. They try to excape from the beam, turn round in circles, and show convulsive spasms. It can be shown that these symptoms are caused by local temperature rises and are not traceable to irradiation of the brain.

In addition, phenomena were observed in microwave radiation which are not due to thermal effect [19]. Amoebae and protozoa in a high frequency field of a capacitor (1-100 MHz) move perpendicular or parallel to the electric field lines; the direction of movement and speed vary according to the frequency of the irradiating field and the species of amoeba [20]. It can happen that the paths of two amoeba cultures cross and a further type persists at this point. This persistence is used for segregating different cultures. Internal structural changes were also noticed in amoebae, leading to fission and destruction of the body. Mutations can be produced as well.

Red and white blood corpuscles (erithrocytes and leucocytes) in the high frequency field display behavior similar to that of the amoebae. They arrange themselves in the direction of propagation of radiation like iron filings are oriented in a magnetic field. Fat globules in milk and lymphocytes make bead-like chains which also run in the direction of radiation [20].

It is almost impossible to predict the amount of radiation energy absorbed by man at a particular area in the electromagnetic field and converted into heat, because to a great extent radiation is dependent on the prevailing electrical properties (the position, size, and structure of the muscular and fatty tissue and the incident direction of the wave), that is to say, on the input impedance of this complex structure. The direction of polarization of the incident wave in relation to the body axis also plays a considerable part here. In each individual case, the symptoms demand an exact study of the existing conditions. The actual amount of temperature rise in the body is determined by such environmental conditions as temperature, humidity, and the cooling mechanism of the body. A simple successful and precise method of measurement can be obtained by reproducing the human body in a life-size dummy of equal dielectric properties.

A. Anne used the solution of KCℓ-Dioxane-Water mixture in the spherical dummy to simulate the electrical properties of human tissue. The relative absorption cross section was calculated. In order to simulate the shape of mankind and its effect on relative absorption cross section, hollow plastic dolls of the type used by children were adapted as dummies which were filled with saline solution. The plastic dolls were believed to be relatively accurate models of babies or children of the same physical size. Hence the results could also be applied to an adult if the doll dummies were scaled to the correct size.

CHAPTER 2

EFFECTS OF ELECTROMAGNETIC FIELDS ON DROSOPHILA CAUSING SOME GENETIC PHENOMENA

Very little is known about the biological effects of an electromagnetic field which does not produce heating (i.e., low average power density) or of a low energy field* (e.g., in extra low frequency range). In order to consider some biological effects in this frequency range, the athermal influence of nonionizing electric and magnetic fields on several genetic phenomena in Drosophila melanogaster is being tested [21]. These biological effects include mutation, meiotic exchange, nondisjunction of chromosomes, and effects on developmental time and fecundity.

2.1 Electric Fields

Recent work at this institution on Drosophila melanogaster using static electric fields, both homogeneous and inhomogeneous, with field intensities up to 300 kV/m, involved both males and females kept in the fields for 11 days with no noticeable untoward effects on viability and fertility. This appeared somewhat inconsistent with the report of tests by Horlacher [22] for sex-linked recessive lethal mutations in Drosophila. In Horlacher's first experiment with 33 $\frac{kV}{m}$ at 60 Hz, treatments from 1 to 30 minutes had obvious immediate effects on the flies, with those not

*The distinction here is that the power density is proportional to the square of the field amplitude, whereas the energy is proportional to frequency.

immediately killed being seriously affected. Apparently only a small proportion of the flies surviving returned to normal. In his second experiment with 225 $\frac{kV}{m}$ at a frequency of 1.225 MHz, a one minute exposure was the longest practical time used, and this short treatment killed half the flies. Horlacher used alternating currents in his work which apparently generated heat internally by induction in the treated male flies. This heat was probably responsible for these striking effects.

2.2 Magnetic Fields

Chevais and Manigault [23] indicated in a preliminary report that an inhomogeneous magnetic field with intensities of several tenths of a Tesla produced some genetic effects in a D. melanogaster egg that had been in the fields for 24 hours and then removed to develop. The CIB method for sex-linked lethals was employed, and some visible wing mutants as well as some lethals were recovered.

A recent study by Close and Beischer, cited by Beischer [24], indicated a lack of effect in an extremely high intensity homogeneous magnetic field on all stages of Drosophila development. Base (Muller-5) females were mated with treated males. One possible mutation at the y-locus was observed, but only 258 treated gametes were tested.

Mulay and Mulay [25] reported the production of noninheritable abnormalities or deformities in flies that had been reared in magnetic fields of $3. \times 10^{-1}$ and 4.4×10^{-1} Tesla for two or three generations. Fields of lower intensities (1.5×10^{-1} Tesla and less) did not apparently produce any deformities. The authors claimed that possibilities for dominant, recessive, viable, and recessive lethal mutants (both autosomal and sex-linked) were checked and ruled out.

In a recently published study of the effects of magnetic fields in D. melanogaster, Tegenkamp [26], using two permanent U-shaped magnets, subjected flies to field intensities of 0 through 5.2×10^{-2} Tesla. He reported the effects on the sex ratio, the induction of autosomal mutants, and a sex-linked recessive lethal. Braver and Zelby consider the authenticity of these effects to be questionable. The reported alteration in sex ratio, in favor of male offspring, is interesting in that quite often in normal cultures, female progeny outnumber their brothers. In many of Tegenkamp's crosses with treated flies, the males were in excess. This also appeared to happen in the controls, although the alterations in sex ratios were not as great as in the experimentals. The fact that some matings occurred with a highly excessive male count, and one of these in the control series, may indicate that something other than the treatment was responsible for the excess of male progeny.

2.3 Work Conducted at the University of Oklahoma

At this institution, Braver and Zelby have been looking for genetic effects of static electric fields in Drosophila and have, to this date, confined their studies to possible electric field and magnetostatic field influence on chromosome movements. If the chromosomes were to respond to the fields, then their movements prior to and leading to pairing during gametogenesis could be affected. This could lead to altered frequencies of meiotic exchange in females or altered rates of nondisjunction in both sexes.

Research to date has centered on the degree to which microwave radiation can induce sex-linked recessive lethal mutations in males of D. melanogaster. The flies were exposed to microwave radiation by being placed about 5 cm from the mouth of a pyramidal horn. The frequency in use was 16.5 GHz produced by an SDF-342 Varian magnetron. Power delivered to the horn was measured with an HP432A power bridge, a P486 thermistor mount, and 30 dB couplers; the frequency was measured by a Polarad Spectrum Analyzer Model SA84 WA. The average power was 43 watts delivered to the horn at 9 ma filament current and 9.75 kV on the anode. Peak power measured at the center of the horn was 62 kW, with a duty cycle of 0.00072 produced by a pulse width of about 1.2 microsecond and PRF slightly below 600 Hz.

Attempts to determine the extent of heating of the flies were made with standard thermocouple and a YSI model 47 scanning telethermometer. Initially, the flies were contained in glass vials with the thermocouple measuring the temperature of the air in the vial. Several measurements indicated that the rise in temperature was due primarily to the heating of the vials themselves rather than the flies. Subsequent experiments were therefore conducted by enclosing the flies in specially made containers of balsa wood and bolting cloth. This prevented the heating of flies due to heating of the walls of the containers and also allowed for free air flow and ventilation during the experiments. Thermocouples were inserted into the containers, and the temperature was monitored throughout the treatment period and recorded on a Bausch & Lomb recorder. In the experiments with the latter containers, the temperature of the air in the containers was kept below 30°C and usually ranged from room temperature (23°C - 25°C) to 27°C. The control temperatures stayed at room temperature.

Several tests were made to determine whether the thermocouples absorbed microwaves and recorded the temperature as a result of absorption rather than as a result of changes in the ambient temperature. The thermocouples were first exposed to microwaves, and the temperature was monitored. With air flow provided by a special fan, thermocouples indicated the room temperature (23°C - 25°C). The thermocouples were then inserted into glass vials with cotton stoppers, with and without flies, and temperatures in excess of 41°C (the limit of the bridge) were recorded.

As a final determination of the validity of temperature measurement, a number of flies were packed very closely between two layers of bolting cloth. This produced a dense volume of flies with a microwave exposure area of about one square centimeter and thickness of about one millimeter. Two such sets were exposed to microwave radiation simultaneously, each monitored by a thermocouple, one on the side of the microwave source with one set of flies and the other away from the microwave source with the other set of flies. The two sets were moved laterally in the radiation field and were also rotated 180 degrees. Under the usual experimental conditions using forced air cooling, no heating was evident, even though the fly density was very high. In vials with slighly lower fly density, the flies were heated above 41°C within a few minutes and were all killed.

Continuous measurements of temperature in various locations in the radiation field, interchange of thermocouples, and constant monitoring indicated that the temperature measurements were reasonably reliable and that the flies were not being significantly heated by the microwaves.

Other temperature measurements of the flies were also attempted, one using cholesteric crystals and one using thermography. The former was not very productive because of the problem associated with painting the flies with crystals and then observing them in the electromagnetic field. Thermographic measurements using flies heated by an infrared lamp were very promising.

Thus far, some of the biological uses and effects of electromagnetic radiation have been mentioned. In the following chapter the energy absorbed by a prolate spheroid representing the Drosophila melanogaster has been calculated. The fly is represented as a prolate spheroid and also as a nephroid of revolution upon which a plane electromagnetic wave is incident. The calculation of energy absorbed is checked using the optical theorem.

CHAPTER 3

BACKGROUND

3.1 The Scattering of an Electromagnetic Wave by a Spheroid [27]

The problem of a perfectly conducting prolate spheroid illuminated by a plane wave at arbitrary incidence has been considered by T.B.A. Senior [27]. The known low-frequency solution is presented in a much simplified form; based on symmetries, the next terms in the expansion are predicted. T.B.A. Senior reviewed a classic paper published by Stevenson [28] on the solution of electromagnetic scattering problems as power series in the propagation constant k. He pointed out that Stevenson's theory provides the only systematic approach to the calculation of more than the leading term in a low-frequency expansion. The scope of the method is so great that any homogeneous body of finite dimensions can be treated, and in view of this generality, it is not surprising that the solution is not always obtained in a form convenient for specific applications. Even in a companion paper [29], where the analysis for an ellipsoid is presented, the results in such limiting cases as a metallic spheroid still demand considerable simplification before they can be put to practical use.

3.1.1 Stevenson's solution [29]

The problem considered here is that of a plane wave incident on a perfectly conducting prolate spheroid. In terms of the Cartesian coordinates the spheroid is defined by the equation

$$\frac{x^2 + y^2}{\xi^2 - 1} + \frac{z^2}{\xi^2} = d^2 \tag{3.1.1}$$

with $1 \leq \xi < \infty$. The semimajor and semiminor axes are ξd and $(\xi^2-1)^{1/2}d$ respectively. Following Stevenson [29], the incident field is assumed to be

$$E^i = (\ell_1,m_1,n_1)\ \exp[ik(\ell x + my + nz)], \tag{3.1.2a}$$

$$H^i = Y(\ell_2,m_2,n_2)\ \exp[ik(\ell x + my + nz)], \tag{3.1.2b}$$

where k is propagation constant, and (ℓ,m,n), (ℓ_1,m_1,n_1), and (ℓ_2,m_2,n_2) are three sets of direction cosines of direction of propagation, electric vector, and magnetic vector respectively, which satisfy the relations

$$(\ell_1,m_1,n_1) = (\ell_2,m_2,n_2) \times (\ell,m,n), \tag{3.1.3a}$$

$$(\ell_2,m_2,n_2) = (\ell,m,n) \times (\ell_1,m_1,n_1). \tag{3.1.3b}$$

Y is the intrinsic admittance of free space, and a time factor $e^{-i\omega t}$ has been suppressed. The expressions for far-zone electric field components provided by Stevenson are as follow:

$$E_\theta = \left[\frac{\partial P}{\partial\theta} + \frac{1}{\sin\theta}\frac{\partial\overline{P}}{\partial\phi}\right]\frac{e^{ikR}}{R}\ , \tag{3.1.4a}$$

$$E_\phi = \left[\frac{1}{\sin\theta}\frac{\partial P}{\partial\phi} - \frac{\partial\overline{P}}{\partial\theta}\right]\frac{e^{ikR}}{R}\ . \tag{3.1.4b}$$

Here, $(R, \theta, \phi,)$ are the spherical polar coordinates of the field point. P and $\overline{P}$ are functions of associated Legendre function [27].

3.1.2 Senior's alternative representation [27]

For any finite body, the components of the scattered electric vector in the far-zone can be written as:

$$E_\theta = \frac{e^{ikR}}{R} \sum_{r=0}^{\infty} \sum_{s=1}^{\infty} \left[\left\{ \alpha_{rs} \frac{\partial}{\partial\theta} P_s^r(\cos\theta) + r\bar{\beta}_{rs} \frac{P_s^r(\cos\theta)}{\sin\theta} \right\} \cos(r\phi) + \left\{ \beta_{rs} \frac{\partial}{\partial\theta} P_s^r(\cos\theta) - r\,\bar{\alpha}_{rs} \frac{P_s^r(\cos\theta)}{\sin\theta} \right\} \sin(r\phi) \right] ,$$

$$E_\phi = -\frac{e^{ikR}}{R} \sum_{r=0}^{\infty} \sum_{s=1}^{\infty} \left[\left\{ \bar{\alpha}_{rs} \frac{\partial}{\partial\theta} P_s^r(\cos\theta) - r\beta_{rs} \frac{P_s^r(\cos\theta)}{\sin\theta} \right\} \cos(r\phi) + \left\{ \bar{\beta}_{rs} \frac{\partial}{\partial\theta} P_s^r(\cos\theta) + r\alpha_{rs} \frac{P_s^r(\cos\theta)}{\sin\theta} \right\} \sin(r\phi) \right] ,$$

where the four sets of coefficients α_{rs}, β_{rs}, $\bar{\alpha}_{rs}$, and $\bar{\beta}_{rs}$ are, as yet, unrelated to one another and specified only by the boundary conditions at the surface of the body [27].

3.2 Scattering by Spherically Symmetrical Objects [30]

According to L. Shafai [30], scattering by spherically symmetrical objects can be expressed in terms of two auxiliary functions, related respectively to the phase and amplitude of the resulting field. It is shown that these auxiliary functions satisfy first-order differential equations of the radial coordinate, and the scattered field is described by the phase functions alone. Furthermore, the differential equations satisfied by the phase functions are found to be independent of the amplitude functions and are solved numerically by using the initial phase shifts obtained from the boundary conditions.

An electromagnetic field in a spherical coordinate system may be found from two radial vector potentials A and F , respectively, for the electric and magnetic vector potentials. The solutions of these functions in a radially stratified region and in terms of the spherical harmonics are readily known and given by [31]

$$A = \frac{\cos\phi}{\mu\omega} \sum_{n=1}^{\infty} i^{n-1} \frac{(2n+1)}{n(n+1)} P_n^1(\cos\theta)\, W_n(R) \quad , \tag{3.2.1a}$$

$$F = \frac{1}{K} \sin\phi \sum_{n=1}^{\infty} i^{n-1} \frac{(2n+1)}{n(n+1)} P_n^1(\cos\theta)\, G_n(R) \quad , \tag{3.2.1b}$$

where $R = kr$, the propagation constant of free space and the radial distance in the spherical coordinates are k and r , respectively. The functions $G_n(R)$ and $W_n(R)$ are two radial functions with $W_n(R)$ satisfying the following differential equation

$$\frac{d^2W_n}{dR^2} - \frac{1}{\varepsilon_r}\frac{d\varepsilon_r}{dR}\frac{dW_n}{dR} + \left[\mu_r\varepsilon_r - \frac{n(n+1)}{R^2}\right] W_n = 0 \tag{3.2.2}$$

where ϵ_r and μ_r are the relative permittivity and permeability of the medium. The function G_n satisfies a similar differential equation which may be obtained by interchanging ε_r and μ_r in the above equation. Equation (3.2.2), however, may be modified to

$$\frac{d^2W_n}{dR^2} + \left[1 - \frac{n(n+1)}{R^2}\right] W_n = L_w W_n \quad , \tag{3.2.3}$$

where L_w is an operator given by

$$L_w = 1 - \mu_r\varepsilon_r + \frac{1}{\varepsilon_r}\frac{d\varepsilon_r}{dR}\frac{d}{dR} \quad , \tag{3.2.4}$$

and a similar operator L_g for the function G_n may be found by interchanging ε_r and μ_r in equation (3.2.4). Equation (3.2.3) is a Sturm-Liouville equation and has a solution of the form

$$W_n(R) = C_n^1\hat{j}_n(R) + C_n^2\hat{y}_n(R) + \int^R L_w W_n(R')R'dR'\left[\hat{j}_n(R')\hat{y}_n(R) - \hat{j}_n(R)\hat{y}_n(R')\right] \tag{3.2.5}$$

where C_n^1 and C_n^2 are two constants yet to be determined, and $\hat{j}_n(R)$ and $\hat{y}_n(R)$ are Riccati Bessel functions of order n. Now, similar to the case of a conducting sphere, the solution of W_n and G_n may be expressed in terms of two auxiliary amplitude and phase functions of the form

$$W_n(R) = A_n(R)\left[\hat{j}_n(R) \cos \delta_n(R) + \hat{y}_n(R) \sin \delta_n(R)\right] \quad , \tag{3.2.6a}$$

$$G_n(R) = B_n(R)\left[\hat{j}_n(R) \cos \varepsilon_n(R) + \hat{y}_n(R) \sin \varepsilon_n(R)\right] \quad , \tag{3.2.6b}$$

where $A_n(R)$ and $\delta_n(R)$ are the amplitude and phase functions associated with W_n , and $B_n(R)$ and $\varepsilon_n(R)$ are the aplitude and phase functions associated with G_n . An introduction of these functions into the differential equations satisfied by W_n and G_n gives

$$\frac{d}{dR}A_n(R) = \frac{-(L_w W_n)}{A_n^2} W_n \quad , \tag{3.2.7a}$$

$$\frac{d}{dR} A_n(R) = -(L_w W_n)\left[\hat{j}_n(R) \sin \delta_n(R) - \hat{y}_n(R) \cos \delta_n(R)\right], \tag{3.2.7b}$$

and

$$\frac{d}{dR} \varepsilon_n(R) = \frac{-(L_g G_n)}{B_n^2} G_n \quad , \tag{3.2.7c}$$

$$\frac{d}{dR} B_n(R) = -(L_g G_n)\left[\hat{j}_n(R) \sin \varepsilon_r(R) - \hat{y}_n(R) \cos \varepsilon_r(R)\right]. \tag{3.2.7d}$$

Equations (3.2.7a) to (3.2.7d) provide a set of first-order differential equations which may be solved numerically to give the required scattered field.

3.3 The Inverse Problem of Scattering from a Perfect Conducting Prolate Spheroid [32]

F. H. Vandenberghe and W. M. Boerner [32] have discussed the inverse problem of electromagnetic scattering from a prolate spheroidal scatterer. The approach is based on the inverse scattering model theory as developed in Boerner, Vandenberghe, and Hamie [33] and Boerner and Vandenberghe [34] for the circular cylindrical and the spherical cases, respectively. In this model theory, the transverse scattered field is expressed in terms of a truncated series expansion of the associated wave functions. The unknown expansion coefficients are recovered from the bistatic scattered field data by employing a matrix inversion procedure. Based on the hypothesis that all the information pertaining to these simple shapes is implicitly contained in the expansion coefficients, closed form expressions for the electrical radii ka are derived in terms of a limited number of contiguous expansion coefficients. Instead of directly using an expansion in spheroidal wave functions, an alternative expansion of the scattered field by Senior [27] is employed. It is then shown that the characteristic parameters of the ellipse generating the prolate spheroid (the interfocal distance d and the eccentricity $\varepsilon = \frac{1}{\xi}$) can be directly recovered from Senior's expansion coefficients.

In order to formulate the scattered far field matrix, Senior's [27] solution is again reviewed. He considered the plane wave incidence given by equations (3.1.2a) and (3.1.2b). The perfectly conducting prolate spheroid is defined by equation (3.1.1). Fig. 3.3.1 illustrates the case of nose-on incidence on a perfectly conducting prolate spheroid.

Senior then showed that the transverse scattered far field components can be represented by equations (3.1.4a) and (3.1.4b), where (R, θ, ϕ) define the spherical coordinate parameters at the observation point. Retaining only the leading expression in a low-frequency expansion, P and $\overline{P}$ are given by

$$P = k^2[k_1 \cos\phi + k_2 \sin\phi)\sin\theta + k_3 \cos\theta] + O(k^4), \quad (3.3.1a)$$

$$\overline{P} = k^2[(\overline{k}_1 \cos\phi + \overline{k}_2 \sin\phi)\sin\theta + \overline{k}_3 \cos\theta] + O(k^4), \quad (3.3.1b)$$

where the coefficients k_j and $\overline{k}_j$ $(j = 1,2,3)$ are explicit functions of the direction cosines (ℓ_1, m_1, n_1) and (ℓ_2, m_2, n_2) and implicit functions of the geometrical parameters d and ξ of the prolate spheroid. Following Senior [27], the coefficients k_j and $\overline{k}_m$ can be expressed by

$$k_1 = -\frac{2}{3}d^3\ell_1 \frac{P_1^1(\xi)}{Q_1^1(\xi)}, \qquad \overline{k}_1 = -\frac{2}{3}d^3\ell_2 \frac{P_1^1(\xi)'}{Q_1^1(\xi)'}$$

$$k_2 = -\frac{2}{3}d^3 m_1 \frac{P_1^1(\xi)}{Q_1^1(\xi)}, \qquad \overline{k}_2 = -\frac{2}{3}d^3 m_2 \frac{P_1^1(\xi)'}{Q_1^1(\xi)'}$$

$$k_3 = \frac{1}{3}d^3 n_1 \frac{P_1^0(\xi)}{Q_1^0(\xi)}, \qquad \overline{k}_3 = \frac{1}{3}d^3 n_2 \frac{P_1^0(\xi)'}{Q_1^0(\xi)'}$$

where $P_\alpha^\beta(\xi)$, $Q_\alpha^\beta(\xi)$ are associated Legendre functions of order α and degree β of the first and second kind, respectively, and the primed expressions represent first-order partial derivatives with respect to ξ .

Neglecting terms of $O(k^4)$ and higher-order of equations (3.3.1a) and (3.3.1b), the transverse scattered far field components can be expressed in a matrix form. Extracting the radial components according to equations (3.1.4a), (3.1.4b), (3.3.1a), and (3.3.1b), the normalized field components are related unknown coefficients k_j and $\overline{k}_j$ (the coefficients k_j, $\overline{k}_j$ for j = 1, 2, 3 are retained in this low-frequency expansion) by

$$[E] = [S(\theta,\phi)]\ [K] \quad ,$$

where

$$[E] = \begin{bmatrix} E\theta_1 \\ E\theta_2 \\ E\theta_3 \\ E\phi_1 \\ E\phi_2 \\ E\phi_3 \end{bmatrix} , \quad [k] = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ \overline{k}_1 \\ \overline{k}_2 \\ \overline{k}_3 \end{bmatrix} ,$$

and

$$[S(\theta,\phi)] = \begin{bmatrix} \cos\theta_1\cos\phi_1 & \cos\theta_1\sin\phi_1 & -\sin\theta_1 & -\sin\phi_1 & \cos\phi_1 & 0 \\ \cos\theta_2\cos\phi_2 & \cos\theta_2\sin\phi_2 & -\sin\theta_2 & -\sin\phi_2 & \cos\phi_2 & 0 \\ \cos\theta_3\cos\phi_3 & \cos\theta_3\sin\phi_3 & -\sin\theta_3 & -\sin\phi_3 & \cos\phi_3 & 0 \\ -\sin\phi_1 & \cos\phi_1 & 0 & -\cos\theta_1\cos\phi_1 & -\cos\theta_1\sin\phi_1 & \sin\theta_1 \\ -\sin\phi_2 & \cos\phi_2 & 0 & -\cos\theta_2\cos\phi_2 & -\cos\theta_2\sin\phi_2 & \sin\theta_2 \\ -\sin\phi_3 & \cos\phi_3 & 0 & -\cos\theta_3\cos\phi_3 & -\cos\theta_3\sin\phi_3 & \sin\theta_3 \end{bmatrix}$$

3.4 Scattering Properties of Oblate Raindrops and Cross Polarization of Radio Waves Due to Rain: Calculations at 19.3 and 34.8 GHz [35]

T. Qguchi in his paper [35] explained that boundary value problems for scattering of radio wave by oblate raindrops may be solved by three different techniques: (a) point-matching technique, (b) spheroidal function expansions, and (c) perturbation method.

A raindrop is assumed to be an oblate spheroid, and the relation between deformation and drop size is approximated by a linear relation

$$S = 1 - \frac{0.41}{4.5} R \tag{3.4.1}$$

where S is the ratio of minor to major axis and R is the effective drop radius in mm, meaning the radius of the sphere with a volume equal to that of the oblate drop.

The geometry for calculation of the scattering property is illustrated in Fig. 3.4.1. If the electric field of a unit plane wave impinging on the oblate drop is denoted by

$$\vec{E}^i = \hat{e} \exp(-i k_0 r \; \hat{k}_1 \cdot \hat{k}_2) \quad , \tag{3.4.2}$$

the electric field of the scattered wave, in the far field region, is written as

$$\vec{E}^s = \vec{f} \; (\hat{k}_1,\hat{k}_2) r^{-1} \exp(-i k_0 r) \tag{3.4.3}$$

where $\hat{e}$ is a unit vector specifying the polarization state of the incident field, $\hat{k}_1$ is a unit vector in direction of propagation of the incident wave, $\hat{k}_2$ is a unit vector directed from the origin to the observation point, k_0 is the free-space propagation constant, r is the distance from the origin to the observation point, and $\vec{f}(\hat{k}_1,\hat{k}_2)$ is a function denoting vector-scattering amplitude. $\vec{E}^s$ in (3.4.3) is obtained by the solution of the boundary-value problem on the surface of a spheroid. The following three techniques are used for obtaining $\vec{E}^s$ [hence, obtaining $\vec{f}(\hat{k}_1,\hat{k}_2)$].

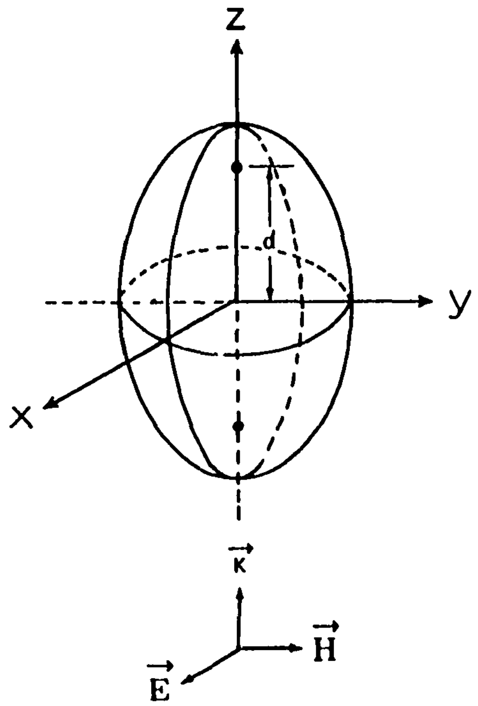

Fig. 3.3.1. Prolate spheroid scattering geometry for nose-on incidence.

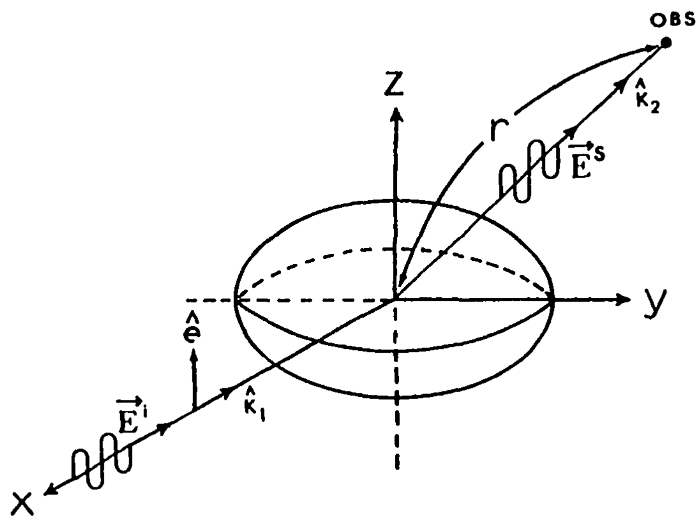

Fig. 3.4.1. Geometry for calculation of the scattering property of an oblate raindrop.

3.4.1 Point-matching technique

The incident field is expanded in terms of a spherical elementary vector solution with known expansion coefficients, and scattered and transmitted fields are expanded by spherical elementary vector solutions with unknown expansion coefficients. If the infinite modal summations in the expressions of fields are truncated at some modal index (M,N) and if the boundary conditions are satisfied for the representative points on the spheroid [whose number is appropriate to the index (M,N)], these conditions give simultaneous linear equations for the determination of unknown coefficients. Truncated modal summations with these coefficients give approximate fields $\vec{E}^s$ and $\vec{f}(\hat{k}_1,\hat{k}_2)$.

3.4.2 Spheroidal function expansions

Incident, scattered, and transmitted fields are expanded in terms of spheroidal elementary vector solutions with known and unknown expansion coefficients respectively. Application of the boundary conditions for the field gives simultaneous linear equations of infinite extent for the unknown expansion coefficients, because the spheroidal elementary vector solutions are not orthogonal. These equations are solved as in section 3.4.1 after they are truncated at some modal index (M,L) . These procedures are almost analogous to those used in the electromagnetic scattering by a conducting spheroid [36, 37, 38], except that the fields exist also in the lossy dielectric spheroid. The procedures of calculation are shown in appendix 2 of Oguchi's paper [35].

3.4.3 Perturbation method

If the eccentricity of a spheroid is small, electromagnetic scattering by a spheroid can be formulated as a perturbation of the corresponding solution for a sphere [39]. By setting $\nu = 1 - \frac{a^2}{b^2}$ (where a and b denote the minor and major semiaxes respectively), the scattered fields can be expressed as a power series in ν. Only the first order perturbation is considered in numerical computation in the paper [35].

This background information, though based on only four papers [27, 30, 32, 35] represents, I believe, the most important work in dealing with scattering from a spheroid object and is the most relevant information to the problem at hand. The main difference, aside from specific solution techniques, is that the foregoing work dealt with far-field scattering and asymptotic solutions; whereas in the problem at hand, the concern is the field distribution within the scatterer. Furthermore the other solutions represent approximations, in the following work the basic analytic solution is exact. Approximation were introduced subsequently by the elimination of terms whose contribution to energy absorption would be negligible because of the size of the spheroid and the frequency at which the calculation was made.

CHAPTER 4

MODELING

In order to determine the electromagnetic energy absorbed by a Drosophila melanogaster exposed to an electromagnetic field, it is necessary to calculate the field distribution inside that insect. To make this problem tractable, it is necessary to generate some simplifying assumptions. First, the Drosophila melanogaster is assumed to be an ellipsoid of revolution (prolate spheroid) or revolution solid of nephroidal shape with permittivity ε, permeability μ, and conductivity σ. Second, the incident field is assumed to be a plane wave. This last assumption is not limiting, as other wave fronts can be synthesized by superposition of plane waves.

Two methods of determing the electromagnetic field of the prolate spheroid scatterer have been used in this research: (1) solving the vector Helmholtz's equation for the spheroid coordinate and (2) solving the vector Hemholtz's equation for the spherical elementary vector and then applying the boundary conditions to match both transmitted and scattered waves to find the unknown coefficients of the transmitted and scattered fields. The second method used in solving for the prolate spheroid scatterer has also been used to solve the case of a nephroidal scatterer.

After the electromagnetic field distribution of the Drosophila melanogaster has been found, the power which is dissipated in Joule heat was calculated using $\int_V \sigma|\vec{E}|^2 dV$, with calculation confirmed by the use of the optical theorem.

CHAPTER 5

SOLUTION ANALYSIS

5.1 Prolate Speroid Scatterer

5.1.1 The scalar Helmholtz wave equation

The scalar Helmholtz wave equation is:

$$\nabla^2\psi + k^2\psi = 0 \ , \tag{5.1.1}$$

where

$$k^2 = \mu\varepsilon\omega^2 + i\mu\sigma\omega \tag{5.1.2}$$

with the Laplacian of the scalar, ψ

$$\nabla^2\psi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial}{\partial u_1}\psi\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial}{\partial u_2}\psi\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial}{\partial u_3}\psi\right)\right] . \tag{5.1.3}$$

If ϕ measures the angle of rotation from the x-axis in the z-plane and r the perpendicular distance of a point from the z-axis, then (see Figs. 5.1.1 and 5.1.2):

$$x = r\cos\phi \ , \ y = r\sin\phi \ . \tag{5.1.4}$$

In prolate spheroidal coordinates, the variables are

$$u_1 = \xi \ , \quad u_2 = \eta \ , \quad u_3 = \phi \ . \tag{5.1.5}$$

The equations of two confocal systems defining the coordinates are

$$\frac{z^2}{\xi^2} + \frac{r^2}{\xi^2 - 1} = d^2 \quad \text{and} \quad \frac{z^2}{\eta^2} - \frac{r^2}{1 - \eta^2} = d^2 \ , \tag{5.1.6}$$

from which we deduce

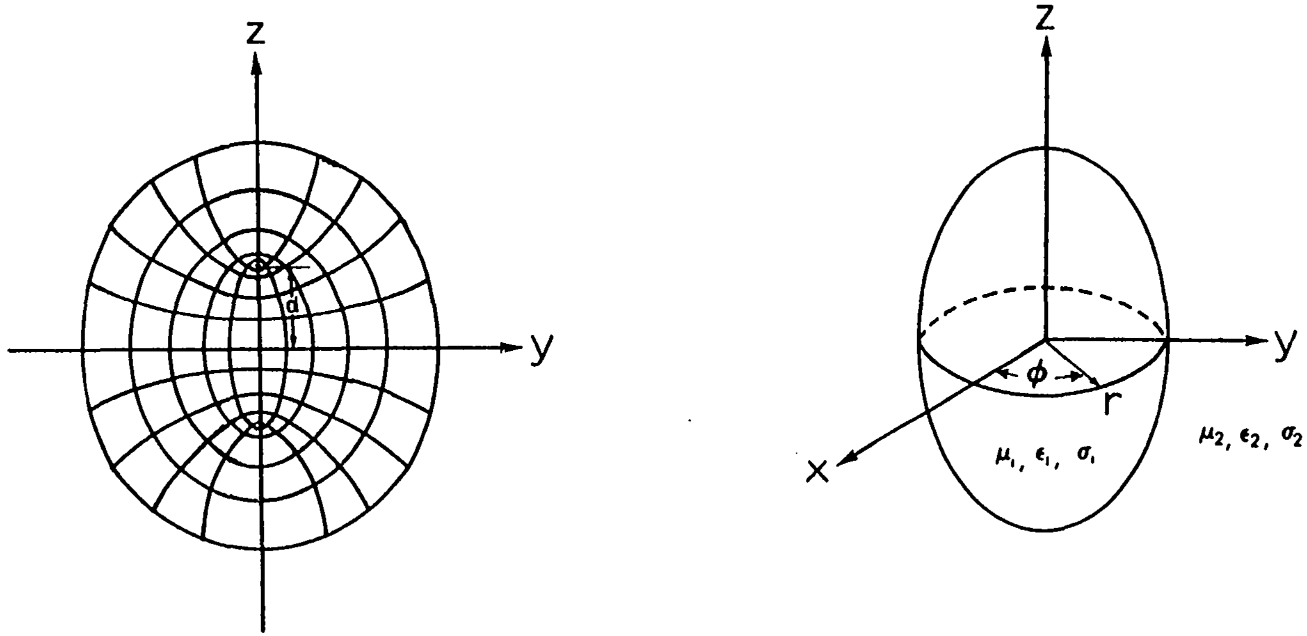

Figure 5.1.1. Two confocal systems given by

$$\frac{z^2}{\xi^2} + \frac{r^2}{\xi^2 - 1} = d^2 \text{ and } \frac{z^2}{\eta^2} - \frac{r^2}{1 - \eta^2} = d^2$$

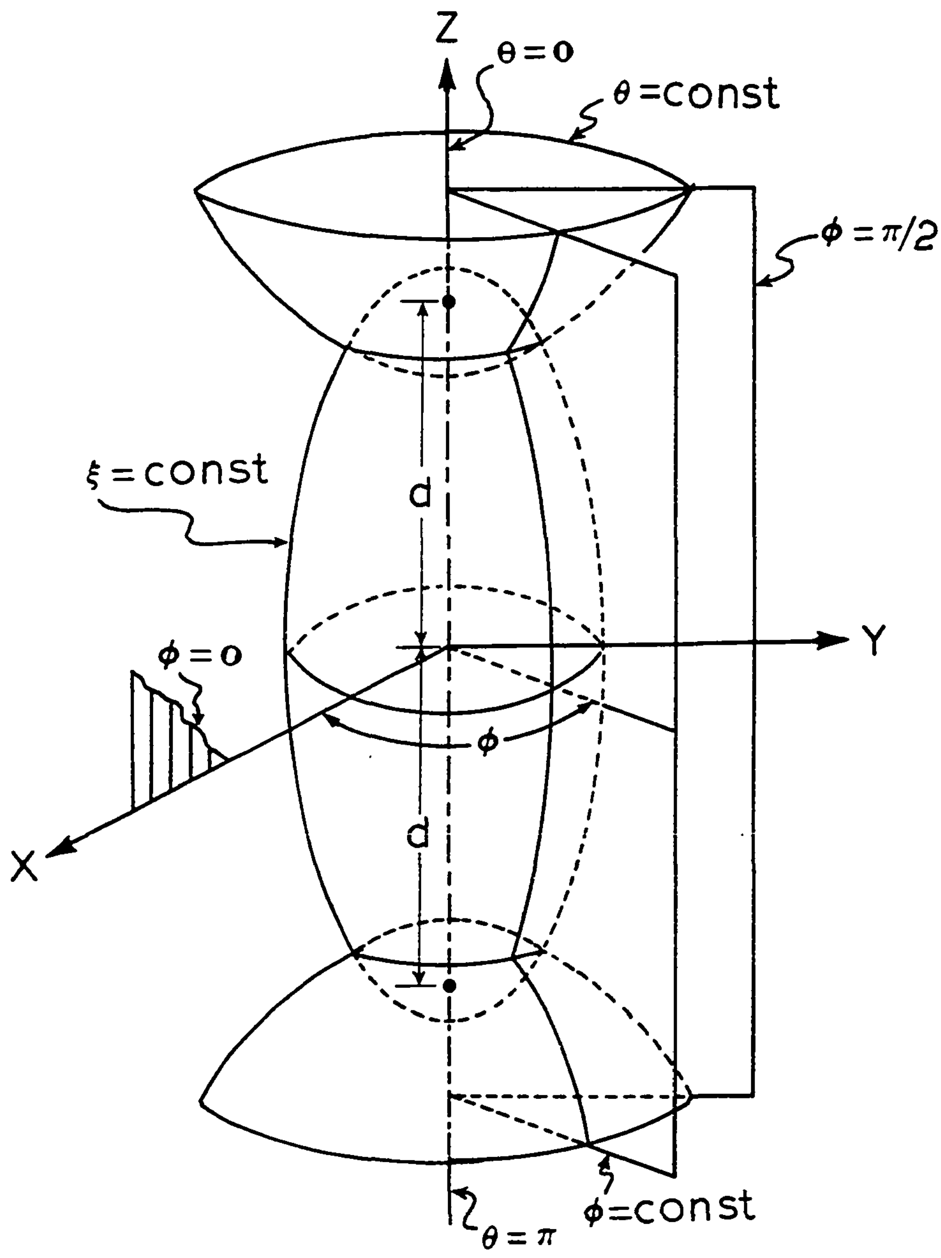

Figure 5.1.2 Prolate spheroidal coordinates (ξ,θ,ϕ). Coordinate surfaces are prolate spheroids (ξ = const) , hyperboloids of revolution (θ = const) , and half-plane (ϕ = const) .

$$x = d\left[(\xi^2 - 1)(1 - \eta^2)\right]^{1/2} \cos\phi ,$$

$$y = d\left[(\xi^2 - 1)(1 - \eta^2)\right]^{1/2} \sin\phi \text{ and } z = d\xi\eta , \qquad (5.1.7a)$$

where $\xi \geq 1$, $-1 \leq \eta \leq 1$, and $0 \leq \phi \leq 2\pi$.

With the metrical coefficients,

$$h_1 = d\left(\frac{\xi^2 - \eta^2}{\xi^2 - 1}\right)^{1/2} , \; h_2 = d\left(\frac{\xi^2 - \eta^2}{1 - \eta^2}\right)^{1/2} , \text{ and } h_3 = d\left[(\xi^2 - 1)(1 - \eta^2)\right]^{1/2} . \qquad (5.1.7b)$$

Substituting equations (5.1.3), (5.1.5), and (5.1.7b) into (5.1.1) leads to

$$\frac{\partial}{\partial\xi}\left[(\xi^2 - 1)\frac{\partial\psi}{\partial\xi}\right] + \frac{\partial}{\partial\eta}\left[(1 - \eta^2)\frac{\partial\psi}{\partial\eta}\right] + \left[\frac{1}{\xi^2 - 1} + \frac{1}{1 - \eta^2}\right]\frac{\partial^2\psi}{\partial\phi^2} + k^2d^2(\xi^2 - \eta^2)\psi = 0. \qquad (5.1.8)$$

Letting $\psi = \psi_1(\xi)\, \psi_2(\eta)\, \psi_3(\phi)$ [an $e^{-i\omega t}$ time convention is assumed and is suppressed], equation (5.1.8) separates in the following three equations:

$$\frac{d}{d\xi}\left[(\xi^2 - 1)\frac{d}{d\xi}\psi_1\right] - \left[A_{m\ell}(h) - h^2\xi^2 + \frac{m^2}{\xi^2 - 1}\right]\psi_1 = 0 , \qquad (5.1.9a)$$

$$\frac{d}{d\eta}\left[(1 - \eta^2)\frac{d}{d\eta}\psi_2\right] + \left[A_{m\ell}(h) - h^2\eta^2 - \frac{m^2}{1 - \eta^2}\right]\psi_2 = 0 , \qquad (5.1.9b)$$

$$\frac{d^2}{d\phi^2}\psi_3 + m^2\psi_3 = 0 , \qquad (5.1.9c)$$

where $h = kd = \frac{\omega d}{c} = \frac{2\pi}{\lambda} d$, and $A_m (h)$ is a separation constant.

The solutions of equations (5.1.9a to 5.1.9c) are (see appendix A):

$$\psi_1 = \begin{cases} je_{m\ell}(h,\xi) = \frac{(\ell - m)!}{(\ell + m)!}\left(\frac{\xi^2 - 1}{\xi^2}\right)^{m/2} \sum_n{}' i^{n+m-\ell}\frac{(n + 2M)!}{n!} d_n(h|m,\ell) j_{n+m}(h\xi) & (5.1.10a) \\ ne_{m\ell}(h,\xi) = \frac{(\ell - m)!}{(\ell + m)!}\left(\frac{\xi^2 - 1}{\xi^2}\right)^{m/2} \sum_n{}' i^{n+m-\ell}\frac{(n + 2m)!}{n!} d_n(h|m,\ell) n_{n+m}(h\xi) & (5.1.10b) \\ he_{m\ell}(h,\xi) = je_{m\ell}(h,\xi) + i\, ne_{m\ell}(h,\xi) \quad , & (5.1.10c) \end{cases}$$

$$\psi_2(\eta) = S_{m\ell}(h,\eta) = \sum_n d_n P^m_{m+n}(\eta) = (1 - \eta^2)^{m/2} \sum_n d_n\, T^m_n(\eta) \; , \tag{5.1.11}$$

$$\psi_3(\phi) = {\cos \atop \sin}(m\phi) \quad , \tag{5.1.12}$$

where $je_{m\ell}$, $ne_{m\ell}$, and $he_{m\ell}$ are the spheroidal radial functions of the first, second, and third kind respectively. $T^m_n(\eta)$ is the Gegenbauer polynomial*, $P^m_{m+n}(\eta)$ is the associated Legendre function, and $S_{m\ell}$ is the spheroidal angle function.

The complete solution of the scalar wave equation (5.1.8) is

$$\psi = \sum_{m,\ell} he_{m\ell}(h_1,\eta)\; S_{m\ell}(h_1,\eta)\; {\cos \atop \sin}(m\phi); \text{ for outside the prolate} \tag{5.1.13a}$$

$$= \sum_{m,\ell} je_{m\ell}(h_2,\xi)\; S_{m\ell}(h_2,\eta)\; {\cos \atop \sin}(m\phi); \text{ for inside the prolate} \tag{5.1.13b}$$

---

*
$$T^m_{\ell-m}(x) = \frac{d^m}{dX^m}\left[P_\ell(x)\right] = \frac{1}{2^\ell \ell!}\,\frac{d^{\ell+m}}{dX^{\ell+m}}\,(x^2 - 1)^\ell \; .$$

where

$$h_1 = k_1 d = d(\mu_1\varepsilon_1\omega^2 + i\,\mu_1\sigma_1\omega)^{1/2} , \qquad (5.1.14a)$$

$$h_2 = k_2 d = d(\mu_2\varepsilon_2\omega^2 + i\,\mu_2\sigma_2\omega)^{1/2} . \qquad (5.1.14b)$$

[The "$h_{1,2}$" variables used here should not be confused with the metrical coefficients h_1, h_2, h_3 of equation (5.1.7b).]

5.1.2 The plane wave expansion

The plane wave expansion in terms of the prolate spheroidal coordinate is [42]

$$e^{i\vec{k}\cdot\vec{r}} = 2\sum_{m,\ell} \frac{\varepsilon_m i^\ell}{\Lambda_{m\ell}(h)} S_{m\ell}(h,\cos\theta_0)\cos m(\phi - \phi_0) S_{m\ell}(h,\cos\theta) je_{m\ell}(h,\cosh u)$$

(see appendix A for an explanation of $\Lambda_{m\ell}$) (5.1.15)

where $\xi = \cosh u$, $\eta = \cos\theta$, and

$$\vec{k}\cdot\vec{r} = h[\cosh u \cos\theta \cos\theta_0 + \sinh u \sin\theta \sin\theta_0 \cos(\phi - \phi_0)] ;$$

θ_0 and ϕ_0 are the coordinates of the angular location of the plane wave source. The Neumann factor, ε_m , is 1 when $m = 0$ and is 2 when $m > 0$.

5.1.3 Solutions of the vector Helmholtz wave equation

The vector Helmholtz wave equation is

$$\nabla^2\vec{A} + k^2\vec{A} = 0 \qquad (5.1.16a)$$

or

$$\nabla(\nabla\cdot\vec{A}) - \nabla\times\nabla\times\vec{A} + k^2\vec{A} = 0 \qquad (5.1.16b)$$

Suppose there is a function f which is an arbitrary function of prolate spheroidal coordinates ξ, η, and ϕ . We also define a vector $\vec{M}$ as

$$\vec{M} = \nabla \times (\hat{\phi} f)$$

$$= \frac{1}{h_2 h_3} \frac{\partial}{\partial \eta} (h_3 f)\hat{\xi} - \frac{1}{h_3 h_1} \frac{\partial}{\partial \xi} (h_3 f)\hat{\eta} \ , \qquad (5.a.17)$$

[h_1, h_2, and h_3 are the metrical coefficients in equation (5.a.7b).] $\vec{M}$ is tangential to the surface ϕ = const . Since the divergence of $\vec{M}$ is zero, it could be one solution, if f is such a scalar field as to make M satisfy the equation (5.1.16a); that is

$$-\nabla^2\vec{M} = -\nabla(\nabla \cdot \vec{M}) + \nabla \times \nabla \times \vec{M} = \nabla \times \nabla \times \vec{M} = k^2\vec{M} \quad (\text{since } \nabla \cdot \vec{M} = 0)$$

We try to arrange the equation so that $\nabla \times \vec{M} = \nabla \times \nabla \times (\hat{\phi} f)$ is equal to $k^2\hat{\phi} f$ plus the gradient of some scalar, because the curl of this will just be $k^2\vec{M}$.

By taking the curl of both sides of equation (5.1.17), we have

$$\nabla \times \vec{M} = \nabla \times \nabla \times (\hat{\phi} f)$$

$$= - \frac{\hat{\phi}}{h_1 h_2} \left\{ \frac{\partial}{\partial \xi}\left[\frac{h_2}{h_3 h_1} \frac{\partial}{\partial \xi}(h_3 f)\right] + \frac{\partial}{\partial \eta}\left[\frac{h_1}{h_2 h_3} \frac{\partial}{\partial \eta}(h_3 f)\right]\right\}$$

$$+ \frac{\hat{\eta}}{h_3 h_1}\left\{\frac{\partial}{\partial \phi}\left[\frac{h_1}{h_3 h_2} \frac{\partial}{\partial \eta}(h_3 f)\right]\right\} + \frac{\hat{\xi}}{h_2 h_3}\left\{\frac{\partial}{\partial \phi}\left[\frac{h_2}{h_3 h_1} \frac{\partial}{\partial \xi}(h_3 f)\right]\right\}$$

After rearranging, the preceding

$$= \frac{1}{h_3^2} \nabla\left[\frac{\partial}{\partial \phi}(h_3 f)\right] - \hat{\phi}\left\{\frac{1}{h_3} \frac{\partial^2}{\partial \phi^2}(h_3 f) + \frac{1}{h_1 h_2} \frac{\partial}{\partial \eta}\left[\frac{h_1}{h_2 h_3} \frac{\partial}{\partial \eta}(h_3 f)\right]\right.$$

$$\left. + \frac{1}{h_1 h_2} \frac{\partial}{\partial \xi} \left[\frac{h_2}{h_1 h_3} \frac{\partial}{\partial \xi}(h_3 f)\right]\right\} .$$

In order to make the curl of $\frac{1}{h_3^2} \nabla\left[\frac{\partial}{\partial \phi}(h_3 f)\right]$ equal to zero, we choose a function that is independent of ϕ . (We make this choice because $h_3 = d(\xi^2 - 1)^{1/2} (1 - \eta^2)^{1/2}$ is already independent of ϕ .) If f

is the function of ξ and η only [i.e., $f = \Phi(\xi,\eta)$] , then by substituting this f , $\nabla \times \vec{M}$ can be rewritten as

$$\nabla \times \overline{M} = - \frac{\hat{\phi}}{h_1 h_2}\left\{ \frac{\partial}{\partial\eta}\left[\frac{h_1}{h_2 h_3}\frac{\partial}{\partial\eta}(h_3\Phi)\right] + \frac{\partial}{\partial\xi}\left[\frac{h_2}{h_1 h_3}\frac{\partial}{\partial\xi}(h_3\Phi)\right]\right\}$$

$$= - \frac{1}{(\xi^2 - \eta^2)}\left\{\frac{\partial}{\partial\xi}\left[(\xi^2 - 1)\frac{\partial}{\partial\xi}\Phi\right]\right.$$

$$\left. + \frac{\partial}{\partial\eta}\left[(1 - \eta^2)\frac{\partial}{\partial\eta}\Phi\right] - \frac{(\xi^2 - \nu^2)}{(\xi^2 - 1)(1 - \eta^2)}\Phi\right\}\hat{\phi} \quad . \qquad (5.1.18)$$

We now show that $\Phi(\xi,\eta)$ is equal to $\psi_1(\xi)\ \psi_2(\eta)$, which is the solution of the radial and angular parts of the scalar Helmholtz equation (5.1.8). Let

$$\psi = \Phi(\xi,\eta)\ {\cos \atop \sin}(m\phi) \qquad (5.1.19)$$

be the solution of scalar Helmholtz equation $\nabla^2\psi + k^2\psi = 0$, which is

$$\frac{\partial}{\partial\xi}\left[(\xi^2 - 1)\frac{\partial}{\partial\xi}\psi\right] + \frac{\partial}{\partial\eta}\left[(1 - \eta^2)\frac{\partial}{\partial\eta}\psi\right] + \frac{(\xi^2 - \eta^2)}{(\xi^2 - 1)(1 - \eta^2)}\frac{\partial^2}{\partial\phi^2}\psi$$

$$+ k^2(\xi^2 - \eta^2)\psi = 0$$

$$(5.1.20)$$

Substituting equation (5.1.19) into (5.1.20), we have

$$\left\{\frac{\partial}{\partial\xi}\left[(\xi^2 - 1)\frac{\partial}{\partial\xi}\Phi\right] + \frac{\partial}{\partial\eta}\left[(1 - \eta^2)\frac{\partial}{\partial\eta}\Phi\right] - \frac{(\xi^2 - \eta^2)}{(\xi^2 - 1)(1 - \eta^2)}m^2\Phi\right.$$

$$\left. + k^2(\xi^2 - \eta^2)\Phi\right\}{\cos \atop \sin}(m\phi) = 0$$

which simplifies to

$$\frac{1}{(\xi^2 - \eta^2)}\left\{\frac{\partial}{\partial\xi}\left[(\xi^2 - 1)\frac{\partial}{\partial\xi}\Phi\right] + \frac{\partial}{\partial\eta}\left[(1 - \eta^2)\frac{\partial}{\partial\eta}\Phi\right]\right.$$

$$\left. - \frac{(\xi^2 - \eta^2)}{(\xi^2 - 1)(1 - \eta^2)}m^2\Phi\right\} = - k^2\Phi \quad . \qquad (5.1.21)$$

From equations (5.1.18) and (5.1.21) we see that

$$\nabla \times \vec{M} = k^2 \Phi \hat{\phi} \ ,$$

provided $m = 1$. By taking the curl of both sides and applying simple algebra, we obtain

$$-\nabla \times \nabla \times \vec{M} + k^2 \overbrace{\nabla \times (\Phi\hat{\phi})}^{\vec{\mathbf{M}}} = 0 \ .$$

By substituting the definiton of $\vec{M}$ given previously

$$-\nabla \times \nabla \times \vec{M} + k^2 M = 0 \ .$$

From this and the vector identity $\nabla^2\vec{M} = \nabla(\nabla \cdot \vec{M}) - \nabla \times \nabla \times \vec{M}$, we concluded that $\vec{M}$ is one of the solutions of the vector Helmholtz equation $\nabla^2\vec{M} + k^2\vec{M} = 0$, provided that $\psi = \Phi(\xi,\eta)^{\cos}_{\sin}(m\phi)$ is a solution of the scalar Helmholtz equation.

Let $\quad \vec{N} = \dfrac{\nabla \times \vec{M}}{k} = \hat{\phi}(k\Phi) \qquad \therefore \quad \nabla \cdot \vec{N} = 0$

$\qquad \vec{M} = \dfrac{\nabla \times \vec{N}}{k} = \nabla \times (\hat{\phi}\Phi) \qquad \therefore \quad \nabla \cdot \vec{M} = 0$

and $\quad \vec{L} = \nabla\psi$.

The vector potential $\vec{A}$ is a combination of $\vec{N}$, $\vec{M}$, and $\vec{L}$;

$$\vec{A} = -\frac{i}{\omega}\left[\sum_{m,\ell} a_{m\ell}\vec{L}_{m\ell} + b_{m\ell}\vec{M}_{m\ell} + c_{m\ell}\vec{N}_{m\ell}\right]$$

$$= -\frac{i}{\omega}\sum_{\ell}\left[a_{1\ell}\vec{L}_{1\ell} + b_{1\ell}\vec{M}_{1\ell} + c_{1\ell}\vec{N}_{1\ell}\right] \qquad (5.1.22)$$

for $m = 1$ only [52,53]

5.1.4 Electromagnetic fields inside and outside the prolate spheroid

Having solved for the vector potential $\vec{A}$, the electromagnetic field can be found by

$$\vec{E} = -\nabla\Psi - \frac{\partial}{\partial t}\vec{A} \qquad (5.1.23)$$

where $\Psi = \sum_{\ell} a_{1\ell}\psi_{1\ell}$.

Substituting equation (5.1.22) into (5.1.23) yields

$$\vec{E} = (b_{1\ell}\vec{M}_{1\ell} + c_{1\ell}\vec{N}_{1\ell}) \ . \tag{5.1.24}$$

Recall that the vectors $\vec{M}$ and $\vec{N}$ are:

$$\vec{M} = \nabla \times \left[\hat{\phi}\Phi(\xi,\eta)\right] = \frac{1}{h_2h_3}\frac{\partial}{\partial\eta}(h_3\Phi)\hat{\xi} - \frac{1}{h_3h_1}\frac{\partial}{\partial\xi}(h_3\Phi)\hat{\eta}$$

$$= \frac{1}{d(\xi^2 - \eta^2)^{1/2}}\left\{\frac{\partial}{\partial\eta}\left[(1 - \eta^2)^{1/2}\Phi\right]\hat{\xi} - \frac{\partial}{\partial\xi}\left[(\xi^2 - 1)^{1/2}\Phi\right]\hat{\eta}\right\},$$

(h_1, h_2, and h_3 are metrical coefficients)

$$\vec{N} = \hat{\phi}k\Phi \ ,$$

where

$$\Phi_{scat}(\xi,\eta) = \psi_1(\xi)\psi_2(\eta) = he_{1,\ell}(h_2,\xi)\ S_{1,\ell}(h_2,\eta) \tag{5.1.25a}$$

$$\Phi_{in}(\xi,\eta) = \psi_1(\xi)\psi_2(\eta) = je_{1,\ell}(h_1,\xi)\ S_{1,\ell}(h_1,\eta) \tag{5.1.25b}$$

where Φ_{scat} and Φ_{in} represent, respectively, a scattering solution and a solution inside the spheroid.

The incident plane wave $\vec{E}_{inct}$ is chosen to be

$$\vec{E}_{inct} = \hat{k}\ E_z e^{i\vec{k}\cdot\vec{r}} \ .$$

Using equations (5.1.15) and (B.9b, appendix B),

$$\vec{E}_{inct} = \frac{E_z}{(\xi^2 - \eta^2)^{1/2}}\left[\eta(\xi^2 - 1)^{1/2}\hat{\xi} + \xi(1 - \eta^2)^{1/2}\hat{\eta}\right]$$

$$\times \sum_{m,\ell} \frac{2 i^{\ell}\epsilon_m}{\Lambda_{m\ell}(h_2)}\ S_{m\ell}(h_2,\cos\theta_0)\cos m(\phi - \phi_0)\ S_{m\ell}(h_2,\cos\theta)je_{m\ell}(h_2,\xi)$$

(5.1.26a)

where E_z is the amplitude of the incident plane wave with propagation constant $\vec{k}$ parallel to the x-axis. That is, $\theta_0 = \frac{\pi}{2}$, $\phi_0 = 0$, and $\vec{E}_{inct}$ are simplified to (see Fig. 5.1.3).

$$\vec{E}_{inct} = \frac{E_z}{(\xi^2 - \eta^2)^{1/2}}\left[\eta(\xi^2 - 1)^{1/2}\hat{\xi} + \xi(1 - \eta^2)^{1/2}\hat{\eta}\right]$$

$$\sum_{m,\ell}\frac{2 i^{\ell}\epsilon_m}{\Lambda_{m,\ell}(h_2)}S_{m,\ell}(h_2,0)\cos(m\phi)S_{m\ell}(h_2,\eta)je_{m\ell}(h_2,\xi) \qquad (\vec{\nabla}\cdot\vec{E}_{inct} = 0)$$

(5.a.26b)

We can write the E-field inside the prolate spheroid and the field scattered by the prolate spheroid as

$$\vec{E}_{in} = \sum_{\ell} b'_{1\ell}\frac{1}{d(\xi^2 - \eta^2)^{1/2}}\left\{\frac{\partial}{\partial\eta}\left[(1 - \eta^2)^{1/2}je_{1\ell}(h_1,\xi)S_{1\ell}(h_1,\eta)\right]\hat{\xi} - \frac{\partial}{\partial\xi}\left[(\xi^2 - 1)^{1/2}je_{1\ell}(h_1,\xi)S_{1\ell}(h_1,\eta)\right]\hat{\eta}\right\}$$

$$+ \sum_{\ell} c'_{1\ell}k_1 je_{1\ell}(h_1,\cancelto{0}{\xi)S_{1\ell}}(h_1,\eta)\hat{\phi} \tag{5.1.27a}$$

$$\vec{E}_{scat} = \sum_{\ell} b_{1\ell}\frac{1}{d(\xi^2 - \eta^2)^{1/2}}\left\{\frac{\partial}{\partial\eta}\left[(1 - \eta^2)^{1/2}he_{1\ell}(h_2,\xi)S_{1\ell}(h_2,\eta)\right]\hat{\xi} - \frac{\partial}{\partial\xi}\left[(\xi^2 - 1)^{1/2}he_{1\ell}(h_2,\xi)S_{1\ell}(h_2,\eta)\right]\hat{\eta}\right\}$$

$$+ \sum_{\ell} c_{1\ell}k_2 he_{1\ell}(h_2,\cancelto{0}{\xi)S_{1\ell}}(h_2,\eta)\hat{\phi} \tag{5.1.27b}$$

where $b_{1\ell}$, $b'_{1\ell}$, $c_{1\ell}$, and $c'_{1\ell}$ are constants to be determined by using boundary conditions. We can immediately set $c_{1\ell}$ and $c'_{1\ell}$ equal to zero, because the $\vec{E}_{in}$ and $\vec{E}_{scat}$ fields do not have $\hat{\phi}$ components when the incident field $\vec{E}_{inct}$ does not have $\hat{\phi}$ components.

5.1.5 Boundary conditions [Appendix C]

The normal component of $\vec{D}(=\epsilon\vec{E})$ is continuous at the boundary, or equivalently $\hat{n} \cdot (\vec{D}_2 - \vec{D}_1) = 0$; i.e.,

$$\epsilon_2 \vec{E}_{\xi \text{ outside}} = \epsilon_1' \vec{E}_{\xi \text{ inside}} \text{ at the boundary } \xi = \xi_b$$

where $E_{\xi \text{ outside}}$ is the ξ component of the E-field outside the prolate spheroid ($E_{\xi \text{ OUTSIDE}} = E_{\xi \text{ inct}} + E_{\xi \text{ scat}}$); i.e.,

$$\sum_\ell b_{1\ell} \epsilon_2 \frac{\partial}{\partial\eta}\left[(1 - \eta^2)^{1/2} he_{1\ell}(h_2,\xi_b) S_{1\ell}(h_2,\eta)\right]$$

$$- \sum_\ell b'_{1\ell} \frac{\partial}{\partial\eta}\left[(1 - \eta^2)^{1/2} je_{1\ell}(h_1,\xi_b) S_{1\ell}(h_1,\eta)\right]$$

$$= \sum_{\ell,m} \epsilon_2 d E_z \eta (\xi_b - 1)^{1/2} \frac{2 i^\ell \epsilon_m}{\Lambda_{m\ell}(h_2)} S_{m\ell}(h_2,0)\cos(m\phi) S_{m\ell}(h_2,\eta) j_{m\ell}(h_2,\xi_b)$$

(5.1.28a)

The tangential components of E are continuous at the boundary or equivalent to $\hat{n} \times (\vec{E}_2 - \vec{E}_1) = 0$; i.e.,

$$E_{\eta \text{ outside}} = E_{\eta \text{ inside}} \quad \text{at the boundary } \xi = \xi_b \text{ ; i.e.,}$$

$$\sum_\ell b_{1\ell} \frac{\partial}{\partial\xi}\left[(\xi^2 - 1)^{1/2} he_{1\ell}(h_2,\xi) S_{1\ell}(h_2,\eta)\right]_{\xi = \xi_b}$$

$$- \sum_\ell b'_{1\ell} \frac{\partial}{\partial\xi}\left[(\xi^2 - 1)^{1/2} je_{1\ell}(h_1,\xi) S_{1\ell}(h_1,\eta)\right]_{\xi = \xi_b}$$

$$= \sum_{\ell,m} E_z d \xi_b (1 - \eta^2)^{1/2} \frac{2 i^\ell \epsilon_m}{\Lambda_{m\ell}(h_2)} S_{m\ell}(h_2,0)\cos(m\phi) S_{m\ell}(h_2,\eta) je_{m\ell}(h_2,\xi_b)$$

(5.1.28b)

We have two equations, (5.1.28a) and (5.1.28b), and two unknowns, $b_{1\ell}$ and $b'_{1\ell}$. These two unknowns can be obtained by means of Cramer's rule. The solutions are

$$b_{1\ell} = \frac{\begin{vmatrix} -\varepsilon_2 \eta(\xi_b^2 - 1)^{1/2} & \varepsilon_1 \frac{\partial}{\partial\eta}\left[(1 - \eta^2)^{1/2} je_{1\ell}(h_1,\xi_b) S_{1\ell}(h_1,\eta)\right] \\ \xi_b(1 - \eta^2)^{1/2} & -\frac{\partial}{\partial\xi}\left[(\xi^2 - 1)^{1/2} je_{1\ell}(h_1,\xi) S_{1\ell}(h_1,\eta)\right]_{\xi = \xi_b} \end{vmatrix}}{\Delta}$$

$$\times\ E_z d \sum_m \frac{2 i^\ell \epsilon_m}{\Lambda_{m\ell}(h_2)} S_{m\ell}(h_2,0) \cos(m\phi) S_{m\ell}(h_2,\eta) je_{m\ell}(h_2,\xi_b)$$

(5.1.29)

$$b'_{1\ell} = \frac{\begin{vmatrix} \varepsilon_2 \frac{\partial}{\partial\eta}\left[(1 - \eta^2)^{1/2} he_{1\ell}(h_2,\xi_b) S_{1\ell}(h_2,\eta)\right] & -\varepsilon_2 \eta(\xi_b^2 - 1)^{1/2} \\ \frac{\partial}{\partial\xi}\left[(\xi^2 - 1)^{1/2} he_{1\ell}(h_2,\xi) S_{1\ell}(h_2,\eta)\right]_{\xi = \xi_b} & \xi_b(1 - \eta^2)^{1/2} \end{vmatrix}}{\Delta}$$

$$\times\ E_z d \sum_m \frac{2 i^\ell \epsilon_m}{\Lambda_{m\ell}(h_2)} S_{m\ell}(h_2,0) \cos(m\phi) S_{m\ell}(h_2,\eta) je_{m\ell}(h_2,\xi_b)$$

(5.a.30)

where

$$\Delta = \begin{vmatrix} \varepsilon_2 \frac{\partial}{\partial\eta}\left[(1-\eta^2)^{\frac{1}{2}} he_{1\ell}(h_2,\xi_b) S_{1\ell}(h_2,\eta)\right] & -\varepsilon_1 \frac{\partial}{\partial\eta}\left[(1-\eta^2)^{\frac{1}{2}} je_{1\ell}(h_1,\xi_b) S_{1\ell}(h_1,\eta)\right] \\ \frac{\partial}{\partial\xi}\left[(\xi^2-1)^{\frac{1}{2}} he_{1\ell}(h_2,\xi) S_{1\ell}(h_2,\eta)\right]_{\xi=\xi_b} & -\frac{\partial}{\partial\xi}\left[(\xi^2-1)^{\frac{1}{2}} je_{1\ell}(h_1,\xi) S_{1\ell}(h_1,\eta)\right]_{\xi=\xi_b} \end{vmatrix}$$

(5.1.31)

We are not interested in the particular set of ϵ_1, ϵ_2, k_1a, and k_2a which may cause Δ to be equal to zero, for that case represents the natural modes of the system, and is outside the scope of this work.

5.1.6 An alternate approach

Again we find the E-fields as before, this time in spherical coordinates. The solutions of the scalar Helmholtz equation in spherical coordinates are

$$\psi_{m\ell}\begin{Bmatrix}\text{even}\\ \text{odd}\end{Bmatrix} = Z_\ell(kr)\ P_\ell^m(\cos\theta)\begin{Bmatrix}\cos m\phi\\ \sin m\phi\end{Bmatrix} \tag{5.1.32}$$

where Z_ℓ is the spherical Bessel function [which may be specified as $j_\ell(kr)$ or $n_\ell(kr)$].

Again, we define two functions which we shall call $\vec{M}$ and $\vec{N}$ [43,44]:

$$\vec{M}_{m\ell} = \nabla \times (\vec{r}\ \psi_{m\ell}) \tag{5.1.33}$$

$$\vec{N}_{m\ell} = \frac{\nabla \times \vec{M}_{m\ell}}{k} \tag{5.1.34}$$

The reciprocal relation $\vec{M}_{m\ell} = \dfrac{\nabla \times \vec{N}_{m\ell}}{k}$ follows from the fact that both $\vec{M}$ and $\vec{N}$ satisfy the vector Helmholtz equation, in addition to being functions with vanishing divergence. In light of the above, in spherical component form, the vector solutions are

$$\vec{M}_{m\ell}\begin{Bmatrix}\text{even}\\ \text{odd}\end{Bmatrix} = \mp \frac{m}{\sin\theta} Z_\ell(kr)P_\ell^m(\cos\theta)\begin{Bmatrix}\sin m\phi\\ \cos m\phi\end{Bmatrix}\hat{\theta} - Z_\ell(kr)\frac{d}{d\theta}P_\ell^m\begin{Bmatrix}\cos m\phi\\ \sin m\phi\end{Bmatrix}\hat{\phi} \tag{5.1.35}$$

$$\vec{N}_{m\ell}\begin{Bmatrix}\text{even}\\ \text{odd}\end{Bmatrix} = \frac{\ell(\ell+1)}{kr} Z_\ell(kr)P_\ell^m(\cos\theta)\begin{Bmatrix}\cos m\phi\\ \sin m\phi\end{Bmatrix}\hat{r} + \frac{1}{kr}\frac{d}{dr}\, r Z_\ell(kr)\ \frac{d}{d\theta}P_\ell^m\begin{Bmatrix}\cos m\phi\\ \sin m\phi\end{Bmatrix}\hat{\theta}$$
$$\mp \frac{m}{kr\sin\theta}\frac{d}{dr}\, rZ_\ell(kr)\ P_\ell^m(\cos\theta)\begin{Bmatrix}\sin m\phi\\ \cos m\phi\end{Bmatrix}\hat{\phi} \tag{5.1.36}$$

The surface of the prolate spheroid is defined by

$$\frac{x^2+y^2}{a^2} + \frac{z^2}{b^2} = 1 \quad \text{or} \quad r = \frac{ab}{(a^2\cos^2\theta + b^2\sin^2\theta)^{\frac{1}{2}}} \tag{5.1.37}$$

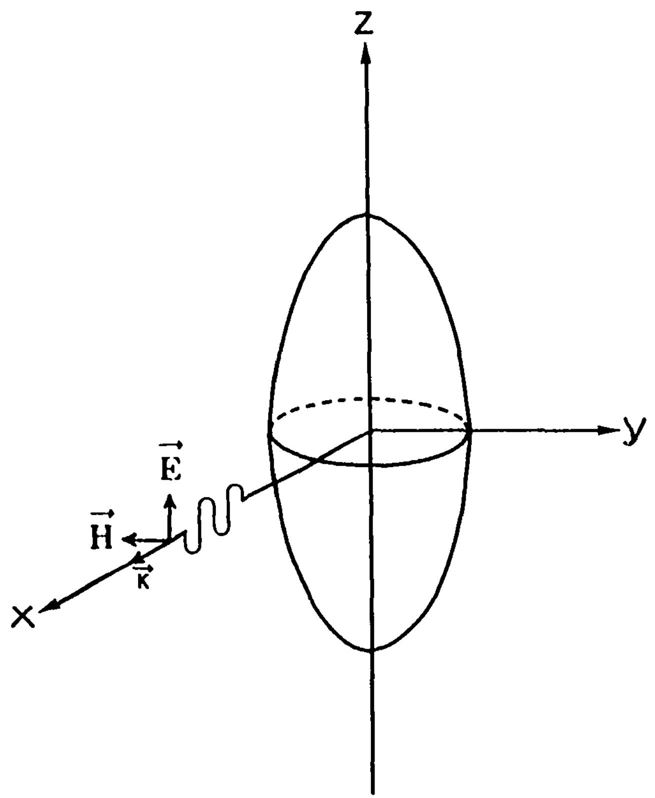

Figure 5.1.3. Geometry for the prolate spheroid and the incident wave.

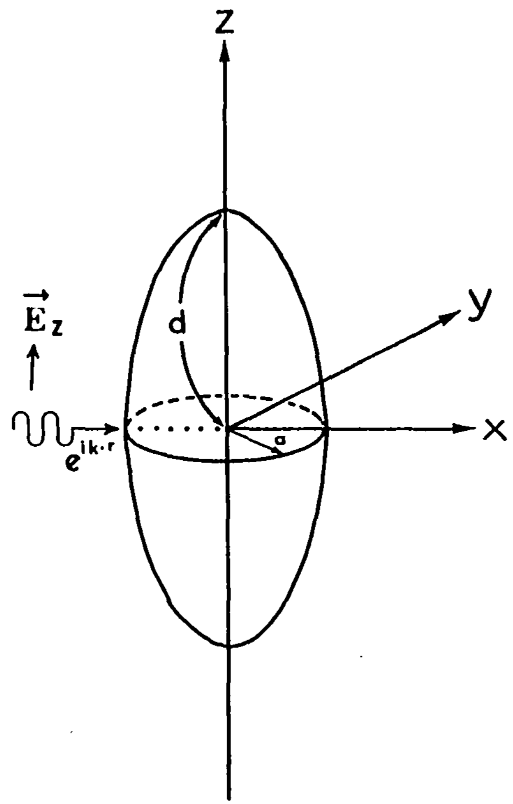

Figure 5.1.4. Geometry for prolate spheroid and the incident wave.

where $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, and $z = r \cos\theta$.

The plane wave expansion in terms of spherical coordinates is [45]

$$e^{i\vec{k}\cdot\vec{r}} = \sum_{\ell=0}^{\infty}(2\ell+1)i^{\ell}\sum_{m=0}^{\ell}\frac{\varepsilon_m(\ell-m)!}{(\ell+m)!}\cos m(\phi-\phi_0)P^m(\cos\theta_0)P_\ell^m(\cos\theta)j_\ell(kr) \tag{5.1.38}$$

If the plane wave is propagating in the x-direction; i.e., propagation constant $\vec{k} \,//\, \hat{i}$ or $\vec{k}$ has the spherical angles $\theta_0 = \frac{\pi}{2}$ and $\phi_0 = 0$ (see Fig. 5.1.4), the above may be reduced to

$$e^{i\vec{k}\cdot\vec{r}} = \sum_{\ell=0}^{\infty}(2\ell+1)i^{\ell}\sum_{m=0}^{\ell}\frac{\varepsilon_m(\ell-m)!}{(\ell+m)!}\cos m\phi\; P_\ell^m(0)P_\ell^m(\cos\theta)j_\ell(kr) \tag{5.1.39}$$

The incident plane wave, which is denoted by $\vec{E}_{inct}$, is assumed to be

$$\begin{aligned}\vec{E}_{inct} &= E_0\hat{k}e^{i\vec{k}\cdot\vec{r}}\\ &= E_0(\cos\theta\hat{r}-\sin\theta\hat{\theta})\sum_{\ell=0}^{\infty}(2\ell+1)i^{\ell}\sum_{m=0}^{\ell}\frac{\varepsilon_m(\ell-m)!}{(\ell+m)!}\cos(m\phi)P_\ell^m(0)P_\ell^m(\cos\theta)j_\ell(kr)\\ &= E_{r\ inct}\hat{r} + E_{\theta\ inct}\hat{\theta}\end{aligned} \tag{5.1.40}$$

where E_0 is an amplitude of the incident wave.

The scattering wave, which is denoted by $\vec{E}_{scat}$, is

$$\vec{E}_{scat} = \sum_{\ell,m}\left[A_{m\ell}\vec{M}^{(s)}_{m\ell} + B_{m\ell}\vec{N}^{(s)}_{m\ell}\right]$$

$$= \sum_{\ell,m}\left\{B_{m\ell}\frac{\ell(\ell+1)}{k_2 r}h_\ell(k_2 r)P^m_\ell(\cos\theta)\cos m\phi\hat{r}\right.$$

$$+ \left[A_{m\ell}\frac{m}{\sin\theta}h_\ell(k_2 r)P^m_\ell(\cos\theta) + \frac{B_{m\ell}}{k_2 r}\left[rh_\ell\right]\frac{d}{d\theta}P^m_\ell\right]\cos m\phi\ \hat{\theta}$$

$$\left. + \left[-A_{m\ell}h_\ell(P^m_\ell)' - B_{m\ell}\frac{m}{k_2 r\sin\theta}(rh_\ell)'\ P^m_\ell\right]\sin m\phi\ \hat{\phi}\right\} \qquad (5.1.41)$$

$$\left[\ (P^m_\ell)' \equiv \frac{d}{d\theta}\left[P^m_\ell(\cos\theta)\right],\quad (rh_\ell)' \equiv \frac{d}{dr}\left[rh_\ell(k_2 r)\right] \text{ and } (rj_\ell)' \equiv \frac{d}{dr}\left[rj_\ell(k_1 r)\right]\ \right]$$

The electric field inside the sphere will then be

$$\vec{E}_{in} = \sum_{\ell,m}\left[A'_{m\ell}\vec{M}^{(i)}_{m\ell} + B'_{m\ell}\vec{N}^{(i)}_{m\ell}\right]$$

$$= \sum_{\ell,m}\left\{B'_{m\ell}\frac{\ell(\ell+1)}{k_1 r}j_\ell(k_1 r)P^m_\ell(\cos\theta)\cos m\phi\hat{r}\right.$$

$$+\left[A'_{m\ell}\frac{m}{\sin\theta}j_\ell P^m_\ell + B'_{m\ell}\frac{1}{k_1 r}\left[rj_\ell\right]'(P^m_\ell)'\right]\cos m\phi\hat{\theta}$$

$$\left. + \left[-A'_{m\ell}j_\ell(P^m_\ell)' - B'_{m\ell}\frac{m}{k_1 r\sin\theta}\left[rj_\ell\right]'\ P^m_\ell\right]\sin m\phi\hat{\theta}\right\} \qquad (5.1.42)$$

where the odd function of $\vec{M}_{m\ell}$ and the even function of $\vec{N}_{m\ell}$ are used, because in the equation for $\vec{E}_{inct}$, $\cos m\phi$ is multiplied by the $\hat{r}$ and $\hat{\theta}$ components.

We do not expect $\vec{E}_{scat}$ and $\vec{E}_{in}$ to have a $\hat{\phi}$ component when the incident electric field $\vec{E}_{inct}$ has no $\hat{\phi}$ component. It follows that

$$A_{m\ell}h_\ell(P^m_\ell)' - B_{m\ell}\frac{m}{k_2 r\sin\theta}\left[rh_\ell\right]'\ P^m_\ell = 0 \qquad (5.1.43)$$

$$A'_{m\ell} j_\ell (P_\ell^m)' - B'_{m\ell} \frac{m}{k_1 r \sin\theta} \left[r j_\ell \right]' P_\ell^m = 0 \qquad (5.1.44)$$

The unit normal $\hat{n}$ to the surface of the prolate spheroid, in terms of a spherical coordinate, is

$$\hat{n} = \frac{\nabla f}{|\nabla f|} = \frac{1}{|\nabla f|} \left[\hat{r} + r^2 \left(\frac{1}{a^2} - \frac{1}{b^2} \right) \sin\theta \cos\theta \, \hat{\theta} \right] = n_r \hat{r} + n_\theta \hat{\theta}$$

(5.1.45)

where $f = r - \dfrac{ab}{(a^2\cos^2\theta + b^2\sin^2\theta)^{1/2}}$, and n_r and n_θ are the components of $\hat{n}$ in the $\hat{r}$ and $\hat{\theta}$ directions.

By using equations (5.1.43) and (5.1.44) to find $A_{\ell m}$ in terms of $B_{m\ell}$, and $A'_{m\ell}$ in terms of $B'_{m\ell}$, we can by substitution rewrite equations (5.1.41) and (5.1.42) as

$$\vec{E}_{scat} = \sum_{\ell,m} B_{m\ell} \left\{ \frac{\frac{-m}{k_2 r \sin\theta} (rh_\ell)' \; P_\ell^m}{h_\ell (P_\ell^m)'} \vec{M}_{m\ell}^{(s)} + \vec{N}_{m\ell}^{(s)} \right\}$$

$$= \sum_{\ell,m} B_{m\ell} \left\{ \frac{(\ell+1)\ell}{k_2 r} h_\ell \; P_\ell^m \cos m\phi \; \hat{r} \right.$$

$$\left. + \left[- \frac{\left[\frac{m}{\sin\theta}\right]^2 \left(P_\ell^m\right)^2}{k_2 r \left(P_\ell^m\right)'} + \frac{1}{k_2 r} \left(P_\ell^m\right)' \right] \left(rh_\ell\right)' \cos m\phi \; \hat{\theta} \right\}$$

$$= E_{r\ scat} \hat{r} + E_{\theta\ scat} \hat{\theta} \qquad (5.1.46)$$

$$\vec{E}_{in} = \sum_{\ell,m} B'_{m\ell} \left\{ \frac{\frac{-m}{k_1 r \sin\theta} \left(rj_\ell\right)' P_\ell^m}{j_\ell \left(P_\ell^m\right)'} \vec{M}_{m\ell}^{(i)} + \vec{N}_{m\ell}^{(i)} \right\}$$

$$= \sum_{\ell,m} B'_{m\ell} \left\{ \frac{\ell(\ell+1)}{k_1 r} j_\ell P_\ell^m \cos m\phi \hat{r} \right.$$

$$\left. + \left[- \frac{\left[\frac{m}{\sin\theta}\right]^2 \left(P_\ell^m\right)^2}{k_1 r \left(P_\ell^m\right)'} + \frac{1}{k_1 r}\left(P_\ell^m\right)' \right] \left(r j_\ell\right)' \cos m\phi \; \hat{\theta} \right\}$$

$$= E_{r\ in} \hat{r} + E_{\theta\ in} \hat{\theta} \qquad (5.1.47)$$

where $E_{r\ scat}$ and $E_{\theta\ scat}$ are the $\hat{r}$ and $\hat{\theta}$ components of $\vec{E}_{scat}$, and $E_{r\ in}$ and $E_{\theta\ in}$ are the $\hat{r}$ and $\hat{\theta}$ components of $\vec{E}_{in}$.

We are now ready to apply the boundary conditions to find the values of $B_{m\ell}$ and $B'_{m\ell}$.

The normal component of $\vec{D}$ is continuous $\hat{n}\cdot(\vec{D}_2 - \vec{D}_1) = 0$ at $r = r_0$; i.e.,

$$\overbrace{B_{m\ell}\varepsilon_2 \left\{ \frac{\ell(\ell+1)}{k_2 r} h_\ell P_\ell^m + \left[- \frac{\left(\frac{m}{\sin\theta} P_\ell^m\right)^2}{k_2 r_0 \left(P_\ell^m\right)'} + \frac{\left(P_\ell^m\right)'}{k_2 r} \right] \left[r h_\ell \right]'_{r=r_0} r_0^2 \left(\frac{1}{a^2} - \frac{1}{b^2}\right) \sin\theta \cos\theta \right\}}^{\alpha}$$

$$\overbrace{-B'_{m\ell}\varepsilon_1 \left\{ \frac{\ell(\ell+1)}{k_1 r_0} j_\ell P_\ell^m + \left[- \frac{\left(\frac{m}{\sin\theta} P_\ell^m\right)^2}{k_1 r_0 \left(P_\ell^m\right)'} + \frac{\left(P_\ell^m\right)'}{k_1 r_0} \right] \left[r j_\ell \right]'_{r=r_0} r_0^2 \left(\frac{1}{a^2} - \frac{1}{b^2}\right) \sin\theta \cos\theta \right\}}^{\beta}$$

$$= -E_0 \varepsilon_2 \left[1 - r_0^2 \left(\frac{1}{a^2} - \frac{1}{b^2}\right) \sin\theta^2 \right] \cos\theta \, (2\ell+1) i^\ell \frac{\varepsilon_m (\ell-m)!}{(\ell+m)!} P_\ell^m(0) P_\ell^m j_\ell(k_2 r_0)$$

$$(5.1.48)$$

where $r_0 = \dfrac{ab}{\left[a^2 \cos^2\theta + b^2 \sin^2\theta\right]^{1/2}}$.

The tangential components of $\vec{E}_{scat}$, $\vec{E}_{inct}$, and $\vec{E}_{in}$ are:

$$\hat{n} \times \vec{E}_{scat} = \hat{\phi}(n_r E_{\theta\ scat} - n_\theta E_{r\ scat})$$

$$= \frac{\sum_{\ell,m} B_{m\ell}}{|\nabla f|} \left\{ \left[\frac{-\left(\frac{m\, P_\ell^m}{\sin\theta}\right)^2}{k_2 r \left(P_\ell^m\right)'} + \frac{\left(P_\ell^m\right)'}{k_2 r} \right] \left(r h_\ell\right)' \right.$$

$$\left. - r^2\left(\frac{1}{a^2} - \frac{1}{b^2}\right)\sin\theta\ \cos\theta\ \frac{\ell(\ell+1)}{k_2 r} h_\ell P_\ell^m \right\} \cos m\phi\ \hat{\phi}$$

(5.a.49)

$$\hat{n} \times \vec{E}_{in} = \hat{\phi}(n_r E_{\theta inct} - n_\theta E_{r\ inct})$$

$$= \frac{\sum_{\ell,m}}{|\nabla f|}\left[-1 - r^2\left(\frac{1}{a^2} - \frac{1}{b^2}\right)\cos^2\theta\right]\sin\theta\ E_0(2\ell+1)\, i^\ell \frac{\epsilon_m (\ell-m)!}{(\ell+m)!}$$

$$\times \cos m\phi\ P_\ell^m(0) P_\ell^m j_\ell(k_2 r)\hat{\phi}$$

(5.1.50)

$$\hat{n} \times \vec{E}'_{in} = \hat{\phi}(n_r E_{\theta\ in} - n_\theta E_{r\ in})$$

$$= \frac{\sum_{\ell,m} B'_{m\ell}}{|\nabla f|} \left\{ \left[\frac{\left(\frac{m P_\ell^m}{\sin\theta}\right)^2}{k_1 r \left(P_\ell^m\right)'} + \frac{\left(P_\ell^m\right)'}{k_1 r} \right] \left[r j_\ell\right]' \right.$$

$$\left. - r^2\left(\frac{1}{a^2} - \frac{1}{b^2}\right)\sin\theta\ \cos\theta\ \frac{\ell(\ell+1)}{k_1 r} j_\ell P_\ell^m \right\} \cos m\phi\ \hat{\phi}$$

(5.1.51)

The tangential components of $\vec{E}$ are continuous or equivalently $\hat{n} \times (\vec{E}_2 - \vec{E}_1) = 0$ at $r=r_0$, that is

$$B_{m\ell}\left\{\overbrace{\left[\frac{\left(\frac{mP_\ell^m}{\sin\theta}\right)^2}{k_2r_0\left(P_\ell^m\right)'}+\frac{\left(P_\ell^m\right)'}{k_2r_0}\right]\left[rh_\ell\right]'_{r=r_0}-r_0^2\left(\frac{b^2-a^2}{a^2b^2}\right)\sin\theta\,\cos\theta\,\frac{\ell(\ell+1)}{k_2r_0}\,h_\ell P_\ell^m}^{\alpha'}\right\}$$

$$-B'_{m\ell}\left\{\overbrace{\left[\frac{\left(\frac{mP_\ell^m}{\sin\theta}\right)^2}{k_1r_0\left(P_\ell^m\right)'}+\frac{\left(P_\ell^m\right)'}{k_1r_0}\right]\left[rj_\ell\right]'_{r=r_0}-r_0^2\left(\frac{b^2-a^2}{a^2b^2}\right)\sin\theta\,\cos\theta\frac{\ell(\ell+1)}{k_2r_0}\,j_\ell P_\ell^m}^{\beta'}\right\}$$

$$=\left[1+r_0^2\left(\frac{b^2-a^2}{a^2b^2}\right)\cos^2\theta\right]\sin\theta\;E_0(2\ell+1)i^\ell\,\frac{\varepsilon_m(\ell-m)!}{(\ell+m)!}\,P_\ell^m(0)P_\ell^m j_\ell(k_2r_0) \qquad (5.1.52)$$

By applying Cramer's rule to equations (5.1.48) and (5.1.52), the coefficients of $B_{m\ell}$ and $B'_{m\ell}$ are found to be

$$B_{m\ell}=\frac{\begin{vmatrix}-\varepsilon_2\left[1-r_0^2\left(\frac{b^2-a^2}{a^2b^2}\right)\sin^2\theta\right]\cos\theta & -\beta\\ \left[1+r_0^2\left(\frac{b^2-a^2}{a^2b^2}\right)\cos^2\theta\right]\sin\theta & -\beta'\end{vmatrix}}{\begin{vmatrix}\alpha & -\beta\\ \alpha' & -\beta'\end{vmatrix}}\,E_0(2\ell+1)i^\ell\frac{\varepsilon_m(\ell-m)!}{(\ell+m)!}P_\ell^m(0)P_\ell^m j_\ell(k_2r_0) \qquad (5.1.53)$$

$$B'_{m\ell}=\frac{\begin{vmatrix}\alpha & -\varepsilon_2\left[1-r_0^2\left(\frac{b^2-a^2}{a^2b^2}\right)\sin^2\theta\right]\cos\theta\\ \alpha' & \left[1+r_0^2\left(\frac{b^2-a^2}{a^2b^2}\right)\cos^2\theta\right]\sin\theta\end{vmatrix}}{\begin{vmatrix}\alpha & -\beta\\ \alpha' & -\beta'\end{vmatrix}}\,E_0(2\ell+1)i^\ell\frac{\varepsilon_m(\ell-m)!}{(\ell+m)!}P_\ell^m(0)P_\ell^m j_\ell(k_2r_0) \qquad (5.1.54)$$

5.2 Nephroid of Revolution

This section describes a fly as a revolution solid of nephroidal shape, which is expressed by rotation of the function

$$r = r_1 + r_2 \cos^2\theta \tag{5.2.1}$$

where r_1 and r_2 are constants (see Fig. 5.2.1a).

We use the same method as we did in section 5.1.6. Equation (5.1.45) is replaced by

$$\hat{n} = \frac{\nabla f}{|\nabla f|} = \frac{1}{|\nabla f|}\left[\hat{r} + \frac{2r_2}{r}\cos\sin\theta\ \hat{\theta}\right] \tag{5.2.2}$$

and the equation (5.1.48) is replaced by

$$\underbrace{B_{m\ell}\varepsilon_2\left\{\frac{\ell(\ell+1)}{k_2r_0} h_\ell P_\ell^m + \left[\frac{\left(\frac{mP_\ell^m}{\sin}\right)^2}{k_2r\left(P_\ell^m\right)'} + \frac{\left(P_\ell^m\right)'}{k_2r_0}\right]\left[rh_\ell\right]'_{r=r_0} \frac{2r_2}{r_0}\cos\theta\sin\theta\right\}}_{\gamma}$$

$$\underbrace{-B'_{m\ell}\varepsilon_1\left\{\frac{\ell(\ell+1)}{k_1r_0} j_\ell P_\ell^m + \left[\frac{\left(\frac{mP_\ell^m}{\sin}\right)^2}{k_1r_0\left(P_\ell^m\right)'} + \frac{\left(P_\ell^m\right)'}{k_1r_0}\right]\left[rj_\ell\right]'_{r=r_0} \frac{2r_2}{r_0}\cos\theta\sin\theta\right\}}_{\delta}$$

$$= -E_0\varepsilon_2\left[1 - \frac{2r_2\sin^2\theta}{r_0}\right]\cos\ (2\ell+1)i^\ell\frac{\varepsilon_m(\ell-m)!}{(\ell+m)!} P_\ell^m(0)P_\ell^m j_\ell(k_2r_0) \tag{5.2.3}$$

where $f = r - r_1 - r_2\cos^2\theta$ and $r_0 = r_1 + r_2\cos^2\theta$. Equation (5.1.52) is replaced by

$$B_{m\ell}\underbrace{\left\{\left[\frac{\left(\dfrac{mP_\ell^m}{\sin\theta}\right)^2}{k_2r_0\left(P_\ell^m\right)'}+\frac{\left(P_\ell^m\right)'}{k_2r_0}\right]\left[rh_\ell\right]'_{r=r_0}-\frac{2r_2\cos\theta\,\sin\theta}{r_0}\,\frac{\ell(\ell+1)}{k_2r_0}\,h_\ell P_\ell^m\right\}}_{\gamma'}$$

$$-\,B'_{m\ell}\underbrace{\left\{\left[\frac{\left(\dfrac{mP_\ell^m}{\sin\theta}\right)^2}{k_1r_0\left(P_\ell^m\right)'}+\frac{\left(P_\ell^m\right)'}{k_2r_0}\right]\left[rj_\ell\right]'_{r=r_0}-\frac{2r_2\cos\theta\,\sin\theta}{r_0}\,\frac{\ell(\ell+1)}{k_1r_0}\,j_\ell P_\ell^m\right\}}_{\delta'}$$

$$=\left[1+\frac{2r_2\cos^2\theta}{r_0}\right]\sin\theta\;E_0(2\ell+1)i^\ell\,\frac{\varepsilon_m(\ell-m)!}{(\ell+m)!}\,P_\ell^m(0)P_\ell^m j_\ell(k_2r_0) \tag{5.2.4}$$

The coefficients $B_{m\ell}$ and $B'_{m\ell}$, in this case, are found to be

$$B_{m\ell}=\frac{\begin{vmatrix}-\varepsilon_2\left[1-\dfrac{2r_2\sin^2\theta}{r_0}\right]\cos\theta & -\delta\\ \left[1+\dfrac{2r_2\cos^2\theta}{r_0}\right]\sin\theta & -\delta'\end{vmatrix}}{\begin{vmatrix}\gamma & -\delta\\ \gamma' & -\delta'\end{vmatrix}}\,E_0(2\ell+1)i^\ell\,\frac{\varepsilon_m(\ell-m)!}{(\ell+m)!}\,P_\ell^m(0)P_\ell^m j_\ell(k_2r_0) \tag{5.2.5}$$

$$B'_{m\ell}=\frac{\begin{vmatrix}\gamma & -\varepsilon_2\left[1-\dfrac{2r_2\sin^2\theta}{r_0}\right]\cos\theta\\ \gamma' & \left[1+\dfrac{2r_2\cos^2\theta}{r_0}\right]\sin\theta\end{vmatrix}}{\begin{vmatrix}\gamma & -\delta\\ \gamma' & -\delta'\end{vmatrix}}\,E_0(2\ell+1)i^\ell\,\frac{\varepsilon_m(\ell-m)!}{(\ell+m)}\,P_\ell^m(0)P_\ell^m j_\ell(k_2r_0) \tag{5.2.6}$$

a) $r = r \ + r_2\cos^2\theta$ b) Drosophila Melanogaster c) prolate spheroid

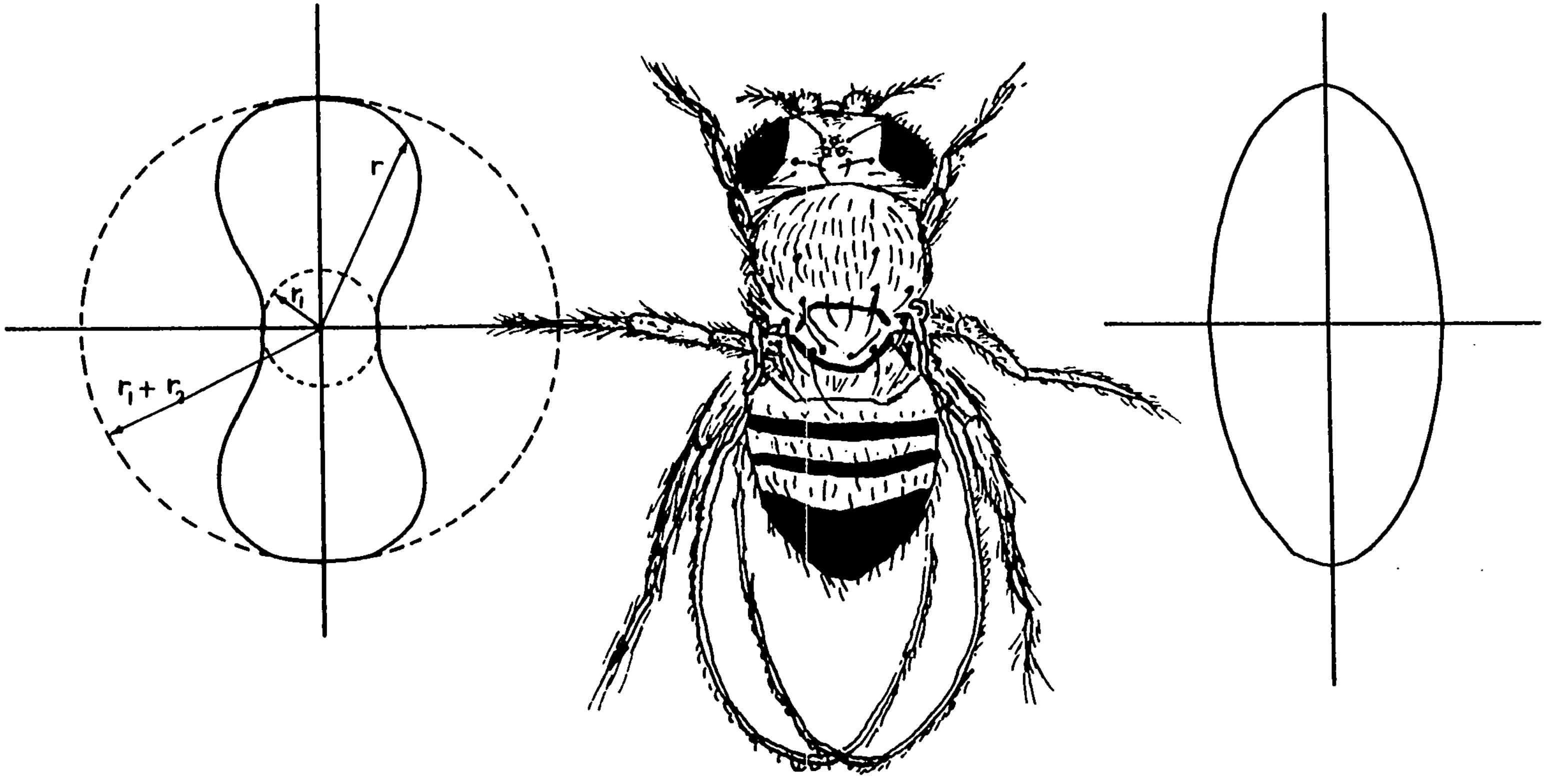

Fig. 5.2.1. Comparison of mathematical models with an actual specimen.

In order to check that the results of solutions (5.1.27a) and (5.1.47) are the same, the prolate spheroid has been reduced to a sphere. As we can see from appendix D, equations (5.1.27a) and (5.1.47) are equivalent in the limit of a sphere to the solution given in Stratton [51]. The absorption of a sphere scatterer has been calculated using the result on page 569 of Stratton; this solution has been proven to be the optical Theorem [appendix E], and agrees numerically with $\int \sigma E^2 dV$ [appendix F].

In the case of a perfect conductor, $k_1 = \infty$ as $\sigma = \infty$; thus $j_\ell(k_1 r) = 0$ as $k_1 r = \infty$. Therefore, from equation (5.1.27a), the electromagnetic field inside a perfector vanish.

Some identities, recursion relations, and behaviors of spherical Bessel functions and Legendre functions have been listed in appendix G.

CHAPTER 6

ENERGY ABSORBED BY A DROSOPHILA MELANOGASTER

The energy absorbed by a Drosophila melanogaster (fly) can be calculated by the equation

$$\int_V \vec{E}\cdot\vec{J}\, dv = \int_V \sigma|\vec{E}|^2 dv \qquad (6.1)$$

which represents the power dissipated in Joule heat - an irreversible transformation, where σ and v are the conductivity and volume of the fly respectively.

As we can see from equations (5.1.29-30, 5.1.52-53, 5.2.5-6), the calculation of equation (6.1) is so tedious that it is not practical. Since the first several terms of the equations (5.1.29-30, 5.1.53-54, 5.2.5-6) are the dominant ones when ξh is small (i.e., $\xi h < 1$), we attempt to find the values of equation (6.1) with $\ell = 1$ only.

6.1 Power Series Expansion for the d's and Other Constants for h Small

Power series expansions for the d's and other constants for h small may be obtained [46]. For instance, setting

$$S_{00} = d_0 P_0(\eta) + d_2 P_2(\eta) + d_4 P_4(\eta) + \cdots$$

and

$$A_{00} = a_2 h^2 + a_4 h^4 + \cdots$$

we have

$$\left[\frac{1}{3}h^2d_0 + \frac{2}{15}h^2d_2 - A_{00}d_0\right]P_0(\eta) + \left[6d_2 + \frac{2}{3}h^2d_0 + \frac{11}{21}h^2d_2 + \frac{4}{21}h^2d_4 - A_{00}d_2\right]P_2(\eta)$$

$$+ \left[20d_4 + \frac{12}{35}h^2d_2 + \frac{39}{77}h^2d_4 + \frac{30}{143}h^2d_6 - A_{00}d_4\right]P_4(\eta) + \cdots = 0$$

Setting $d_2 = [h^2\alpha_2 + h^4\alpha_4 + \cdots]d_0$ and $d_4 = [h^4\beta_4 + \cdots]d_0$ and equating the coefficients of $P_0(\eta), P_2(\eta), \cdots$ to zero separately, we obtain $a_2 = \frac{1}{3}$; $a_4 = \frac{2}{15}\alpha_2$ and $\alpha_2 = -\frac{1}{9}$; $-\alpha_2 a_2 + 6\alpha_4 + \frac{11}{21}\alpha_2 = 0$ or $a_4 = \frac{2}{135}$; $\alpha_4 = \frac{2}{567}$ and $20\beta_4 + \frac{12}{35}\alpha_2 = 0$ or $\beta_4 = \frac{1}{525}$.

Consequently

$$S_{00} \approx d_0P_0 - d_0\left[\frac{1}{9}h^2 - \frac{2}{567}h^4\right]P_2 + \frac{h^4}{525}d_0P_4 + \cdots$$

Since S_{00} is equal to unity at $\eta = 1$, by applying equation (A.3, appendix A) we must have

$$d_0\left[1 - \frac{1}{9}h^2 + \frac{11}{2025}h^4 \cdots\right] = 1$$

or

$$d_0 = 1 + \frac{1}{9}h^2 + \frac{14}{2025}h^4 + \cdots$$

By this means, approximate formulas can be built up for small values of h :

$$\left.\begin{aligned}
&S_{00} \approx \underbrace{(1 + \tfrac{1}{9}h^2 + \tfrac{14}{2025}h^4)}_{d_0(h|0,0)}P_0(\eta) - \underbrace{(\tfrac{1}{9}h^2 + \tfrac{5}{567}h^4)}_{d_2(h|0,0)}P_2(\eta) + \underbrace{\tfrac{h^4}{525}}_{d_4(h|0,0)}P_4(\eta) \\
&A_{00} \approx \tfrac{1}{3}h^2 - \tfrac{2}{135}h^4 \;; \qquad \Lambda_{00} \approx 2\left[1 + \tfrac{2}{9}h^2 + \tfrac{116}{2025}h^4\right] \\
&S_{01} \approx \underbrace{(1 + \tfrac{1}{25}h^2 + \tfrac{144}{55125}h^4)}_{d_1(h|0,1)}P_1(\eta) - \underbrace{(\tfrac{1}{25}h^2 + \tfrac{29}{6525}h^4)}_{d_3(h|0,1)}P_3(\eta) + \underbrace{\tfrac{h^4}{2205}}_{d_5(h|0,1)}P_5(\eta) \\
&A_{01} \approx 2 + \tfrac{3}{5}h^2 - \tfrac{6}{875}h^4 \;; \qquad \Lambda_{01} \approx \tfrac{2}{3} + \tfrac{4}{75}h^2 + \tfrac{92}{18375}h^4
\end{aligned}\right\}$$

$$\left. \begin{array}{l} S_{11} \approx \underbrace{(1 + \frac{2}{25}h^2 + \frac{1193}{275625}h^4)}_{d_0(h|1,1)} P_1^1(\eta) - \underbrace{(\frac{h^2}{75} + \frac{16}{16875}h^4)}_{d_2(h|1,1)} P_3^1(\eta) + \underbrace{\frac{h^4}{11025}}_{d_4(h|1,1)} P_5^1(\eta) \\ A_{11} \approx 2 + \frac{1}{5}h^2 - \frac{4}{875}h^4 \; ; \qquad \Lambda_{11} \approx \frac{4}{3} + \frac{16}{75}h^2 + \frac{17104}{826875}h^4 \end{array} \right) \tag{6.1.1}$$

6.2 The Value of h at the Long-Wavelength Limit ($he^u \ll 1$)

The values of d and ξ_b related to the size of Drosophila are assume to be $d \sim 1.23 \times 10^{-3}$m

$\xi_b \sim 1.05 \quad (\xi_b = \cosh\mu_b,\ \mu_b \approx 0.2 \times \frac{\pi}{2} \approx 0.31).$

We further assume that the values of μ_1, ε_1, and σ_1 at a frequency of 4 GHz to be (representing known value of tissue [47])

$$\mu_1 \approx \mu_0 = 4\pi \times 10^{-7} \text{ henry/m}$$

$$\varepsilon_1 \approx 40\varepsilon_0 = 40 \times \frac{1}{36\pi} \times 10^{-9} \text{ farad/m}$$

$$\sigma_1 \approx 2.5 \text{ mho/m}$$

$$\omega = 2\pi \times 4 \times 10^9 \text{ sec}^{-1}.$$

Recall that

$$h_1^2 = d^2k_1^2 = d^2(\mu_1\varepsilon_1\omega^2 + i\mu_1\sigma_1\omega)$$

or

$$|h_1| = d\left[(\mu_1\varepsilon_1\omega^2)^2 + (\mu_1\sigma_1\omega)^2\right]^{1/4}$$

$$\approx 0.664 \tag{6.2.1}$$

and

$$|h_2| = d(\mu_2\varepsilon_2\omega^2)^{1/2} = d(\mu_0\varepsilon_0\omega^2)^{1/2}$$

$$\approx 0.103. \qquad (6.2.2)$$

6.3 The Value of $b'_{1\ell}$ of Equation (5.1.30) with $\ell = 1$

In the case of long wavelengh approximation, the Bessel functions j_ℓ are dominated by j_1 ; the remaing terms ($\ell \geq 2$) are negligible in comparison to j_1 in the case of the fly.

The value of b'_{11} is found to be [see appendix H]:

$$b'_{11} \approx 3.08 \times 10^{-2} E_z d \cos (1-\eta^2)^{1/2} \, i$$

6.4 The Evaluation of $\int_V \sigma |\vec{E}|^2 dv$

The electric field inside the fly can be written as

$$\vec{E}_{in} = \frac{b'_{11}}{d(\xi^2 - \eta^2)^{1/2}} \left\{ \frac{\partial}{\partial\eta}\left[(1 - \eta^2)^{1/2} je_{11}(h_1,\xi) S_{11}(h_1,\eta)\right]\hat{\xi} - \frac{\partial}{\partial\xi}\left[(\xi^2 - 1)^{1/2} je_{11} S_{11}\right]\hat{\eta} \right\}$$

$$\approx 3.08 \times 10^{-2} E_z \cos\phi (1 - \eta^2)^{1/2} \frac{i}{(\xi^2 - \eta^2)^{1/2}} \left\{ - 2\eta je_{11}\hat{\xi} - \left[(1 + \frac{1}{\xi^2}) j_1 + (\xi - \frac{1}{\xi}) h_1 \frac{1}{3} j_0 d_0 (1 - \eta^2)^{1/2}\right]\hat{\eta} \right\}$$

$$|\vec{E}_{in}|^2 = 9.5 \times 10^4 E_z^2 \cos^2\phi \frac{d_0^2(1 - \eta^2)}{(\xi^2 - \eta^2)} \left\{ 4\eta^2 j_1^2(h_1\xi)\left(\frac{\xi^2 - 1}{\xi^2}\right) + (1 - \eta^2)\left[\left(1 + \frac{1}{\xi^2}\right)^2 j_1^2 + \frac{2h_1}{3}\left(\frac{\xi^2 + 1}{\xi^2}\right)\left(\frac{\xi^2 - 1}{\xi}\right) j_0 j_1 + \frac{h_1^2}{9}(\xi - \frac{1}{\xi})^2 j_0^2\right] \right\}$$

The volume element dV for prolate spheroidal coordinates is

$$dV = d^3(\sinh^2 u + \sin^2\theta)\sinh u \sin\theta \, d\mu d\theta d\phi$$
$$= -d^3(\xi^2 - \eta^2)d\xi d\eta d\phi$$

$$\int_V \sigma|\vec{E}|dV \approx -9.5 \times 10^{-4}E_z^2 d_0^2 d^3 \overbrace{\int_0^{2\pi} \cos^2\phi d\phi}^{\pi} \int_1^{1.05} d\xi \left\{ j_1^2(h_1\xi)\left(\frac{\xi^2 - 1}{\xi^2}\right) \int_{-1}^{1} 4\eta^2(1 - \eta^2)d\eta \right.$$

$$+ \left[\left(1 + \frac{1}{\xi^2}\right)^2 j_1^2 + \frac{2h_1}{3}\left(\frac{\xi^2 + 1}{\xi^2}\right)\left(\frac{\xi^2 - 1}{\xi^2}\right)j_0 j_1\right.$$

$$\left.\left. + \frac{h_1^2}{9}\left(\xi - \frac{1}{\xi}\right)^2 j_0^2\right] \int_{-1}^{1} (1 - \eta^2)^2 d\eta \right\} \approx 1.89$$

$$\times 10^{-12}E_z^2 \left\{\frac{16}{15}\int_1^{1.05}\left(\frac{\xi^2 - 1}{\xi^2}\right)j_1^2 + \frac{16}{15}\int_1^{1.05}\left(1 + \frac{1}{\xi^2}\right)^2 j_1^2\right.$$

$$\left. + \frac{32h_1}{45}\int_1^{1.05}\left(\frac{\xi^2 - 1}{\xi^2}\right)\left(\frac{\xi^2 - 1}{\xi}\right)j_0 j_1 + \frac{16h_1^2}{135}\int_1^{1.05}\left(\xi - \frac{1}{\xi}\right)^2 j_0^2\right\}$$

The above integrations of transcendental functions can be carried out by approximate integration. Using the Simpson's rule, some computer work is needed to accomplish the job. The numerical results are:

$$\int_1^{1.06}\left(\frac{\xi^2 - 1}{\xi^2}\right)j_1^2(h_1\xi)d\xi \simeq 0.1133 \times 10^{-3}$$

$$\int_1^{1.05}\left(1 + \frac{1}{\xi^2}\right)^2 j_1^2(h_1\xi)d \simeq 0.8932 \times 10^{-2}$$

$$\int_1^{1.05}\left(\frac{\xi^2 + 1}{\xi^2}\right)\left(\frac{\xi^2 - 1}{\xi}\right)j_0(h_1\xi)j_1(h_1\xi)d = 0.959 \times 10^{-3}$$

$$\int_{1}^{1.05}\left(\xi - \frac{1}{\xi}\right)^{2} j_0^2(h_1\xi)d\xi \simeq 0.1394 \times 10^{-3}$$

The result of the integration of $\int_V \sigma|\vec{E}|^2 dV$ is

$$\int_V \sigma|\vec{E}|^2 dV \simeq 1.91 \times 10^{-14} E_z^2 \qquad (6.4.1)$$

Using the optical theorem (see appendix E), the amount of absorbed energy is approximately $2.19 \times 10^{-14} E_0^2$, a value within 13 % of equation (6.4.1). In view of the approximation the aggrement can be considered quite good.

Suppose there is an incident plane wave with the input power $P_{in}\left(\frac{W}{m^2}\right)$, then the E_z of equation (6.4.1) can be carried out by the following:

$$\frac{|E_z|}{|H_y|} \approx 376.6\ \Omega \qquad (6.4.2)$$

and

$$|E_z \times H_y| = P_{in}\left(\frac{W}{m^2}\right) \qquad (6.4.3)$$

i.e.,

$$E_z^2 = 376.6 \times P_{in} \qquad (6.4.4)$$

The following table lists the power absorbed Q_{ab} by the prolate spheroid representing the fly given the electromagnetic properties of the spheroid assumed above, and the E_z of the incident plane wave when power per unit area of the incident wave P_{in} is known.

P_{in} $\frac{W}{m^2}$	E_z $\frac{V}{m}$	$Q_{ab}(W)$
1.	1.94×10	7.19×10^{-12}
10.	6.14×10	7.19×10^{-11}
100.	1.94×10^{2}	7.19×10^{-10}
1000.	6.14×10^{2}	7.19×10^{-9}

CHAPTER 7

DISCUSSION

The models of an ellipsoid of revolution and a nephroid of revolution have been used to describe a fly. As we can see from Fig. 5.2.1, both the ellipsoid of revolution and nephroid of revolution models, although close approximations, are not exactly the same as a fly. Because it is impossible to model a fly perfectly by a single mathematical function, some simplifications of the fly's shape have to be made. The relatively unimportant portions of the fly, such as legs, wings, and the slight departure from revolutional symmetry, were neglected.

The fly is about 2.6 mm long. This length corresponds to a frequency of 115 GHz. As the skin depth, which is $\delta = \sqrt{\frac{2}{\omega\mu\sigma}}$, approaches or exceeds the size of the fly, the shape of the fly becomes uncritical for that particular frequency ω , at which the wavelengh is much larger than the fly.

The skin of the insect is almost an insulator; its body is about 70 percent saline. No information of its a.c. conductivity σ , permeability μ , and permittivity ε are known. The values of a.c. (σ, μ, and ε) used in section 6.2 are estimates and, therefore, are not authoritative. In addition, these quantities vary depending on the specimen and biological state. Therefore, sections 6.2, 6.3, and 6.4

serve as an example of how to calculate the power absorbed by a fly for any given conductivity σ , permeability μ , and permittivity ε of the fly.

In section 6.1, we are dealing with a small h , which is near the low frequency (or long wavelength) limit. In the high frequency (or short wavelength) case, h is no longer small, but the ideas used in approaching the problem are still the same. Instead of expanding the functions of $A_{m\ell}(h)$ and $d_n(h|m,\ell)$ into power series of h , they can be represented by powers of $(h-k)$. For example, $A_{00}(h) = \sum_n a_n(h-k)^n$, $d_2(h|0,0) = \sum_n \alpha_n(h-k)^n$, $d_4(h|0,0) = \sum_n \beta_n(h-k)^n$; etc., where k is some constant chosen to insure that $|h-k|$ is small. In this case, the number of terms, ℓ , necessary to obtain an accurate numerical summation calculating the $\vec{E}$-field, must increase to include all the dominant terms. This ℓ is dependent on the magnitude of $h\xi_b$.

The power absorbed by the Drosophila melanogaster represented in terms of a spheroid has been calculated, the calculation was performed at a frequency of 4 GHz. The reason for this choice is the fact the electromagnetic properties of fly are not known. Consequently the values of ε, μ, and σ of human tissue measured at 4 GHz have been used as an approximation. Several given incident powers and the corresponding power absorbed by the spheroid have been listed in the following table.

$P_{in}\left(\frac{W}{m^2}\right)$	$E_z\left(\frac{V}{m}\right)$	$Q_{ab}(W)$
1.	1.94×10	7.19×10^{-12}
10.	6.14×10	7.19×10^{-11}
100.	1.94×10^2	7.19×10^{-10}
1000.	6.14×10^2	7.19×10^{-9}

We, human beings, are exposing ourselves to the nonionizing electromagnetic radiantion fields for the duration of our lifetimes, and we cannot help but ask what this increasing intensity of electromagnetic radiation of all types does to us. It is important to know whether there is a biological state that can be affected by this invisible radiation which surrounds us. This problem increasingly calls for worldwide attention; as we know the maximum "safe" limits (as set by law) for electormagnetic exposure are 10 mW/cm^2 for the U.S., 1 mW/cm^2 for Sweden, and 0.01 mW/cm^2 for the U.S.S.R.[54] .

This work represents another step in an attempt at quantitative determination of the biological effects. Drosophila melanogaster have been used in the experiment and were represents by a spheroid in the analysis.

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APPENDIX A

THE SOLUTIONS OF EQUATIONS (5.1.9a-c) [40]

The solutions of equations (5.1.9a-c) are:

$$\psi_3(\phi) = {\cos \atop \sin}(m\phi)$$

$$\psi_2(\eta) = S_{m\ell}(h,\eta) = \sum_n d_n P^m_{m+n}(\eta) = (1 - \eta^2)^{m/2} \sum_n d_n T^m_n(\eta) ,$$

where the recursion formula relating successive coefficients is

$$\frac{n(h-1)h^2}{(2n+2m-1)(2n+2m-3)} d_{n-2} + \frac{(n+2m+1)(n+2m+2)h^2}{(2n+2m+3)(2n+2m+5)} d_{n+2}$$

$$+ \left[h^2 \frac{2(n+m)(n+m+1) - m^2 - 1}{(2n+2m+3)(2n+2m-1)} + (n+m)(n+m+1) - A_{m\ell}\right] d_n = 0 . \tag{A.1}$$

There are two sets of finite solutions of angle for equation (5.1.9b), one for even values of n and the other for odd values. We arrange each in order of increasing value of $A_{m\ell}(\eta)$ to obtain the final solution:

$$S_{m\ell}(h,\eta) = \sum_n{}' d_n(h|m,\ell) P^m_{n+m}(\eta) = (1 - \eta^2)^{m/2} \sum{}' d_n T^m_n(\eta) \tag{A.2}$$

which is finite and continuous over the range $-1 \leq \eta \leq 1$ for different allowed values of the separation constant $A_{m\ell}(\eta)$ (ℓ=m, m+1, m+2, $\cdots$) , where $A_{m,\ell}(\eta) < A_{m,\ell+1}(\eta)$. The prime on the summation sign indicates that only even values of n are included if ℓ-m is even, and odd values of n are included if ℓ-m is odd. When $h \to 0$, the equation for $S_{m\ell}(\eta)$ reduces to that for a single spherical harmonic $P_\ell^m(\eta) = (1 - \eta^2)^{m/2} T_{\ell-m}^m(\eta)$ and $A_{m\ell} \to \ell(\ell + 1)$ [this can be obtained from equation (A.1)]. We can normalize our function $S_{m\ell}(\eta)$ so that its behavior near $\eta = 1$ is close to that of $P_\ell^m(\eta)$, no matter what value η has. In other words, since we have

$$T_{\ell-m}^m(1) = \frac{(\ell + m)!}{2^m m!(\ell - m)!}$$

(which can be obtained from the generating function for Gegenbauver polynomials; i.e.,

$$\frac{2^m \Gamma(m + 1/2)/\sqrt{\pi}}{(1 + t^2 - 2t\eta)^{m+1/2}} = \sum_{n=0}^{\infty} t^2(\eta) \;;\; (|t| < 1)$$, we require that

$$\sum_n{}' \frac{(n + 2m)!}{n!} d_n(h|m,\ell) = \frac{(\ell + m)!}{(\ell - m)!} \,. \qquad \text{(A.3)}$$

The functions $S_{m\ell}$ represent a set of orthogonal eigenfunctions. The normalizing constant is:

$$\int_{-1}^{1} |S_{m\ell}|^2 d\eta = \sum_n{}' [d_n(h|m,\ell)]^2 \left[\frac{2}{2n + 2m + 1}\right] \frac{(n + 2m)!}{n!} = \Lambda_{m\ell}(\eta) \qquad \text{(A.4)}$$

There are three kinds of solutions for the radial equation (5.1.9a). Solutions of the first kind, finite at $\xi = \pm 1$ are:

$$je_{m\ell}(h,\xi) = \frac{(\ell - m)!}{(\ell + m)!} \left[\frac{\xi^2 - 1}{\xi^2}\right]^{m/2} \sum_n{}' i^{n+m-\ell} \frac{(n + 2m)!}{n!} d_n(h|m,\ell) j_{n+m}(h\xi)$$

$$\to \frac{1}{h\xi} \cos[h\xi - \frac{\pi}{2}(\ell + 1)] \,;\; h\xi \to \infty$$

$$= \left[\frac{1}{\lambda_{m\ell}(\eta)}\right] S_{m\ell}(h,\xi) \qquad \text{(A.5)}$$

where d's are the same coefficients as for $S_{m\ell}(h,\eta)$,

$$\left.\begin{aligned}\lambda_{m\ell}(\eta) &= \frac{i}{d_0(h|m,\ell)}\,\frac{(2m+1)}{(2h)^m m!}\,\frac{(\ell+m)!}{(\ell-m)!}\,S_{m\ell}(h,0)\ ; \\ &\qquad \ell=m,\ m+2,\ m+4,\ \cdots \\ &= \frac{2i^{\ell-1}}{d_1(h|m,\ell)}\,\frac{(2m+2)(2m+3)}{(2h)^{m+1}\,m!}\,\frac{(\ell+m)!}{(\ell-m)!}\left[\frac{d}{d\eta}\,S_{m\ell}(h,\eta)\right]_{\eta=0}\ ; \\ &\qquad \ell=m+1,\ m+3,\ m+5,\ \cdots\end{aligned}\right\}\qquad (A.6)$$

and where $j_m(h\xi)$ is the spherical Bessel function of the first kind.

For $he^{\mu}<<1$, some particular approximate formulas [41] are:

$$je_{00}(h,\cosh u) \approx 1 - \frac{1}{18}h^2 - \frac{1}{6}h^2\sinh^2 u\ , \qquad (A.7a)$$

$$je_{0\ell}(h,\cosh u) \approx \left(\frac{1}{3}h + \frac{1}{75}h^3\right)\cosh u\ , \qquad (A.7b)$$

$$je_{1\ell}(h,\cosh u) \approx \left(\frac{1}{3}h + \frac{2}{75}h^3\right)\sinh u\ , \qquad (A.7c)$$

where $u = \cosh^{-1}\xi$ and $\theta = \cos^{-1}\eta$.

Solutions of the second kind [41] are:

$$ne_{m\ell}(h,\xi) = \frac{(\ell-m)!}{(\ell+m)!}\left[\frac{\xi^2-1}{\xi^2}\right]^{m/2}\sum_n{}'\, i^{n+m-\ell}\,\frac{(n+2m)!}{n!}\,d_n(h|m,\ell)\,n_{n+m}(h\xi)$$

$$\rightarrow \frac{1}{h\xi}\sin[h\xi - \frac{\pi}{2}(\ell+1)]\ ;\ h\xi\rightarrow\infty \qquad (A.8)$$

For $he^{u}<<1$

$$ne_{00}(h,\cosh u) \approx -\frac{2}{h}\left[\frac{1}{3}h^2\cosh u + \left(1 + \frac{11}{72}h^2 - \frac{1}{24}h^2\cosh 2u\right)\tanh^{-1}e^{-u}\right] \qquad (A.9a)$$

$$ne_{0\ell}(h,\cosh u) \approx -\frac{6}{h^2}\left\{\left[\left(1 + \frac{17}{200}h^2\right)\cosh u - \frac{1}{40}h^2\cos 3u\right]\tanh^{-1}e^{-u} - \frac{1}{2}\left[\left(1 + \frac{1}{12}h^2\right) - \frac{1}{20}h^2\cosh 2u\right]\right\} \qquad (A.9b)$$

$$ne_{11}(h,\cosh u) \simeq -\frac{3}{2h^2}\frac{1}{\sinh u}\left\{\left[\left(1 - \frac{19}{75}h^2\right)\cosh u - \frac{1}{75}h^2\cosh^2 u\right]\right.$$

$$\left. - 2\sinh^2 u\left[\left(1 + \frac{3}{50}h^2\right) + \frac{1}{10}h^2\cosh^2 u\right]\tanh^{-1} e^{-u}\right\} . \tag{A.9c}$$

Solutions of the third kind [41] are:

$$he_{m\ell}(h,\xi) = je_{m\ell}(h,\xi) + i\, ne_{m\ell}(h,\xi)$$

$$\longrightarrow \frac{i^{-\ell-1}}{h\xi} e^{ih\xi} \;;\quad h\xi \longrightarrow \infty . \tag{A.10}$$

For $he^u << 1$, in particular we have

$$he_{0\ell}(h,\cosh u) = i^{\ell}\left[j_0(\alpha)h_0(\beta) - \left(\frac{4}{5}\frac{d_2}{d_0} - 1\right)j_1(\alpha)h_1(\beta)\right.$$

$$\left. + \left(\frac{16}{21}\frac{d_4}{d_0} - \frac{4}{7}\frac{d_2}{d_0} + 1\right)j_2(\alpha)h_2(\beta) - \cdots\right] ;\ell=0,2,4,\cdots$$

$$= i^{\ell-1}\,\frac{12\cosh u}{h}\left[j_1(\alpha)h_1(\beta) - \left(\frac{12}{7}\frac{d_3}{d_1} + 1\right)j_2(\alpha)j_2(\beta)\right.$$

$$\left. + \left(\frac{80}{33}\frac{d_5}{d_1} \quad \frac{4}{3}\frac{d_3}{d_1} + 2\right)j_3(\alpha)h_3(\beta) - \cdots\right] ;\ell=1,3,5,\cdots \tag{A.11a}$$

$$he_{1\ell}(h,\cosh u) = i^{\ell-1}\,\frac{12\sinh u}{h}\left[j_1(\alpha)h_1(\beta) - \left(\frac{24}{5}\frac{d_2}{d_0} - 3\right)j_2(\alpha)h_2(\beta)\right.$$

$$\left. + \left(\frac{80}{11}\frac{d_4}{d_0} - 8\,\frac{d_2}{d_0} + 6\right)j_3(\alpha)h_3(\beta) - \cdots\right] ;\ \ell=1,3,5,\cdots \tag{A.11b}$$

where $\alpha = \frac{1}{2}he^{-u}$ and $\beta = \frac{1}{2}he^{u}$.

APPENDIX B

Unit Vectors in a Curvilinear System

The unit vectors $\hat{i}, \hat{j}, \hat{k}$ in the rectangular coordinates are worked in terms of the prolate spheroidal coordinates:

$$
\begin{aligned}
x &= d[(\xi^2 - 1)(1 - \eta^2)]^{1/2}\cos\phi = d \sinh u \sin\theta \cos\phi \\
y &= d[(\xi^2 - 1)(1 - \eta^2)]^{1/2}\sin\phi = d \sinh u \sin\theta \sin\phi \\
z &= d\xi\eta = d \cosh u \cos\theta
\end{aligned}
\tag{B.1}
$$

Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be the position vector. With the substitution of equation (B.1) into the position vector $\vec{r}$, we have

$$\vec{r} = d[\sinh u \sin\theta \cos\phi\ \hat{i} + \sinh u \sin\theta \sin\phi\ \hat{j} + \cosh u \cos\theta\ \hat{k}] \tag{B.2a}$$

$$= d[(\xi^2 - 1)^{1/2}(1 - \eta^2)^{1/2}\cos\phi\ \hat{i} + (\xi^2 - 1)^{1/2}(1 - \eta^2)^{1/2}\sin\phi\ \hat{j} + \xi\eta\hat{k}] \tag{B.2b}$$

The unit tangent vectors $\hat{u}, \hat{\theta}, \hat{\phi}(\hat{\xi},\hat{\eta},\phi)$ are:

$$\hat{u} = \frac{\frac{\partial\vec{r}}{\partial u}}{\left|\frac{\partial\vec{r}}{\partial u}\right|} = \frac{\cosh u \sin\theta \cos\phi\ \hat{i} + \cosh u \sin\theta \sin\phi\ \hat{j} + \sinh u \cos\theta\ \hat{k}}{(\cosh^2 u \sin^2\theta + \sinh^2 u \cos^2\theta)^{1/2}} \tag{B.3a}$$

$$\hat{\xi} = \frac{\frac{\partial\vec{r}}{\partial \xi}}{\left|\frac{\partial\vec{r}}{\partial \xi}\right|} = \frac{\xi(1 - \eta^2)^{1/2}\cos\phi\ \hat{i} + \xi(1 - \eta^2)^{1/2}\sin\phi\ \hat{j} + (\xi^2 - 1)^{1/2}\eta\hat{k}}{(\xi^2 - \eta^2)^{1/2}} \tag{B.3b}$$

$$\hat{\theta} = \frac{\frac{\partial \vec{r}}{\partial \theta}}{\left|\frac{\partial \vec{r}}{\partial \theta}\right|} = \frac{\sinh u \cos\theta \cos\phi \; \hat{i} + \sinh u \cos\theta \sin\phi \; \hat{j} - \cosh u \sin\theta \; \hat{k}}{(\cosh^2 u \sin^2\theta + \sinh^2 u \cos^2\theta)^{1/2}} \tag{B.4a}$$

$$\hat{\eta} = \frac{\frac{\partial \bar{r}}{\partial \eta}}{\left|\frac{\partial \bar{r}}{\partial \eta}\right|} = \frac{-\eta(\xi^2 - 1)^{1/2}\cos\phi \; \hat{i} - \eta(\xi^2 - 1)^{1/2}\sin\varepsilon \; \hat{j} + \xi(1 - \eta^2)^{1/2} \; \hat{k}}{(\xi^2 - \eta^2)^{1/2}} \tag{B.4b}$$

$$\hat{\phi} = \frac{\frac{\partial \bar{r}}{\partial \phi}}{\left|\frac{\partial \bar{r}}{\partial \phi}\right|} = \sin\phi \; \hat{i} + \cos\phi \; \hat{j} \tag{B.5}$$

We rewrite equations (B.3a-b, B.4a-b, B.5) in matrix form:

$$\begin{bmatrix} \hat{u} \\ \hat{\theta} \\ \hat{\phi} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{p}\cosh u \sin\theta \cos\phi & \frac{1}{p}\cosh u \sin\theta \sin\phi & \frac{1}{p}\sinh u \cos\theta \\ \frac{1}{p}\sinh u \cos\theta \cos\phi & \frac{1}{p}\sinh u \cos\theta \sin\phi & -\frac{1}{p}\cosh u \sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix}}_{A} \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix}$$

$$= \begin{bmatrix} \\ A \\ \\ \end{bmatrix} \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix} \tag{B.6a}$$

$$\begin{bmatrix} \hat{\xi} \\ \hat{\eta} \\ \hat{\phi} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{q}\xi(1 - \eta^2)^{1/2}\cos\phi & \frac{1}{q}\xi(1 - \eta^2)^{1/2}\sin\phi & \frac{1}{q}\eta(\xi^2 - 1)^{1/2} \\ -\frac{1}{q}\eta(\xi^2 - 1)^{1/2}\cos\phi & -\frac{1}{q}\eta(\xi^2 - 1)^{1/2}\sin & \frac{1}{q}\xi(1 - \eta^2)^{1/2} \\ -\sin\phi & \cos\phi & 0 \end{bmatrix}}_{B} \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix}$$

$$= \begin{bmatrix} & \\ B & \\ & \end{bmatrix} \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix} \tag{B.6b}$$

where $p = (\cosh^2 u \sin^2\theta + \sinh^2 u \cos^2\theta)^{1/2}$ and $q = (\xi^2 - \eta^2)^{1/2}$.

The determinants and the transposes of A and B (as defined above) are:

$$\det A = 1 \ , \qquad \det B = -1 \ ,$$

$$A^t = \begin{bmatrix} \frac{1}{p} \cosh u \sin\theta \cos\phi & \frac{1}{p} \sinh u \cos\theta \cos\phi & -\sin\phi \\ \frac{1}{p} \cosh u \sin\theta \sin\phi & \frac{1}{p} \sinh u \cos\theta \sin\phi & \cos\phi \\ \frac{1}{p} \sinh u \cos\theta & -\frac{1}{p} \cosh u \sin\theta & 0 \end{bmatrix},$$

$$B^t = \begin{bmatrix} \frac{1}{q}\xi(1 - \eta^2)^{1/2} \cos\phi & -\frac{1}{q}\eta(\xi^2 - 1)^{1/2}\cos\phi & -\sin\phi \\ \frac{1}{q}\xi(1 - \eta^2)^{1/2} \sin\phi & -\frac{1}{q}\eta(\xi^2 - 1)^{1/2}\sin\phi & \cos\phi \\ \frac{1}{q}\eta(\xi^2 - 1)^{1/2} & \frac{1}{q}\xi(1 - \eta^2)^{1/2} & 0 \end{bmatrix}.$$

The adjoint of A , written "adjA", is defined to be the transposed matrix of cofactors of A . Thus,

$$\mathrm{adj}A = \begin{bmatrix} \frac{1}{p} \cosh u \sin\theta \cos\phi & \frac{1}{p} \sinh u \cos\theta \cos\phi & -\sin\phi \\ \frac{1}{p} \cosh u \sin\theta \sin\phi & \frac{1}{p} \sinh u \cos\theta \sin\phi & \cos\phi \\ \frac{1}{p} \sinh u \cos & -\frac{1}{p} \cosh u \sin & 0 \end{bmatrix}. \tag{B.7a}$$

Similarly, we have

$$\mathrm{adj}B = \begin{bmatrix} -\frac{1}{q}\xi(1-\eta^2)^{1/2}\cos\phi & \frac{1}{q}\eta(\xi^2-1)^{1/2}\cos\phi & \sin\phi \\ -\frac{1}{q}\xi(1-\eta^2)^{1/2}\sin\phi & \frac{1}{q}\eta(\xi^2-1)^{1/2}\sin\phi & -\cos\phi \\ -\frac{1}{q}\eta(\xi^2-1)^{1/2} & -\frac{1}{q}\xi(1-\eta^2)^{1/2} & 0 \end{bmatrix} . \tag{B.7b}$$

We are now ready to write the inverse of A and B :

$$A^{-1} = \frac{\mathrm{adj}A}{\det B} = \mathrm{adj}A \tag{B.8a}$$

and

$$B^{-1} = \frac{\mathrm{adj}B}{\det B} = -\mathrm{adj}B \tag{B.8b}$$

Finally, we get

$$\begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix} = \begin{bmatrix} A^{-1} \end{bmatrix} \begin{bmatrix} \hat{u} \\ \hat{\theta} \\ \hat{\phi} \end{bmatrix} \tag{B.9a}$$

or

$$= \begin{bmatrix} B^{-1} \end{bmatrix} \begin{bmatrix} \hat{\xi} \\ \hat{\eta} \\ \hat{\phi} \end{bmatrix} \tag{B.9b}$$

APPENDIX C

BOUNDARY CONDITIONS [48,49]

At the interface of two media, the transition of the tangential component of $\vec{E}$ and the normal component of $\vec{D}$ is expressed by [48,49].

$$\hat{n} \times (\vec{E}_2 - \vec{E}_1) = 0 \ , \qquad \hat{n} \cdot (\vec{D}_2 - \vec{D}_1) = \delta \ ,$$

where n is the unit normal directed from the medium 1 into 2 and δ denotes the surface charge. The flow of charge across or to the boundary must also satisfy the equation of continuity in case either or both the conductivities are finite and not zero.

$$\hat{n} \cdot (\vec{J}_2 - \vec{J}_1) = - \frac{\partial \delta}{\partial t}$$

Suppose now that the time enters only as a common factor $\exp(-i\omega t)$ and that apart from the boundary the two media are homogeneous and isotropic. Then 1 and 2 together give

$$\varepsilon_2 E_{2n} - \varepsilon_1 E_{1n} = \delta$$

$$\sigma_2 E_{2n} - \sigma_1 E_{1n} = i\omega\delta$$

or

$$\varepsilon_2 E_{2n} - \varepsilon_1' E_{1n} = 0$$

where

$$\varepsilon_1' = \varepsilon_1 - \frac{\sigma_1}{i\omega} \ , \qquad \text{if} \quad \sigma_2 = 0$$

APPENDIX D

REDUCE THE SPHEROID PROLATE TO A SPHERE

The conditions for reducing a spheroid to a sphere are $\xi \to \infty$ and $d \to 0$ (and $h = kd \to 0$) ,

$$x = d\sqrt{(\xi^2 - 1)(1 - \eta^2)}\ \cos\phi \to d\xi(1 - \eta^2)^{\frac{1}{2}}\cos\phi$$

$$y = d\sqrt{(\xi^2 - 1)(1 - \eta^2)}\ \sin\phi \to d\xi(1 - \eta^2)^{\frac{1}{2}}\sin\phi$$

$$z = \xi\eta$$

From the above equations we have

$$x^2 + y^2 + z^2 = (d\xi)^2 = r^2$$

When $h \to 0$, the separation constant $A_{m\ell} \to \ell(\ell+1)$, and equation (5.1.9a) will give

$$\frac{d}{d\xi}\left[(\xi^2 - 1)\frac{d}{d\xi}\psi_1\right] - \left[A_{m\ell} - h^2\xi^2 + \frac{m^2}{\xi^2 - 1}\right]\psi_1 = 0$$

which can be reduced to

$$\rho^2\frac{d}{d\rho}\psi_1 + 2\rho\frac{d}{d\rho}\psi_1 + \left[\rho^2 - \ell(\ell+1)\right]\psi_1 = 0 \quad (\rho \equiv h\xi)$$

This equation has particular solutions of the spherical Bessel functions of the first kind $j_\ell(\rho)$, the second kind $n_\ell(\rho)$, and third kind $h_\ell(\rho)$.

The equation (5.1.9b)

$$\frac{d}{d\eta}\left[(1-\eta^2)\frac{d}{d\eta}\psi_2\right] + \left[A_{m\ell} - h^2\eta^2 - \frac{m^2}{1-\eta^2}\right]\psi_2 = 0$$

can be reduced to

$$\frac{d}{d\eta}\left[(1-\eta^2)\frac{d}{d\eta}\psi_2\right] + \left[\ell(\ell+1) - \frac{m^2}{1-\eta^2}\right]\psi_2 = 0$$

This equation has a solution of a single spherical harmonic $P_\ell^m(\eta)$. We can also directly reduce $je_{m\ell} \to j_\ell$ and $S_{m\ell} \to P_\ell^m$ when $h \to 0$.

Now, we reduce $\vec{E}_{in}$, which is in a spheroid coordinate, to a spherical case. The equation (5.1.27a) is

$$\vec{E}_{in} = \sum_\ell b_1' \frac{1}{d(\xi^2-\eta^2)^{1/2}}\left\{\frac{\partial}{\partial\eta}\left[(1-\eta^2)^{1/2} je_{1\ell}(h_1,\xi)S_{1\ell}(h_1,\eta)\right]\hat{\xi}\right.$$

$$\left. - \frac{\partial}{\partial\xi}\left[(\xi^2-1)^{1/2} je_{1\ell}(h_1,\xi)S_{1\ell}(h_1,\eta)\right]\hat{\eta}\right\}$$

Letting $h \to 0$, $\xi \to \infty$, and $d\xi \to r$, it follows

$$je_{m\ell} \to j_\ell \ , \quad S_{m\ell} \to P_\ell^m$$

$$\hat{\xi} \to \hat{r} \ , \quad \hat{\eta} \to -\hat{\theta}$$

We rewrite (5.1.27a) as

$$E_{in} = b_{11}'\frac{1}{r}\left\{-2\eta j_1(k_1 r)\hat{r} + \frac{\partial}{\partial r}\left[r j_1(k_1 r)\right](1-\eta^2)^{1/2}\,\hat{\theta}\right\}; \quad \ell=1$$

$$= \frac{\varepsilon_2\left\{2h_1(k_2 a) - \frac{\partial}{\partial r}\left[r h_1(k_2 r)\right]_{r=a}\right\}\frac{3i}{2}E_0 a j_1(k_2 a)}{\left\{\varepsilon_2 h_1(k_2 a)\frac{\partial}{\partial r}\left[r j_1(k_1 r)\right]_{r=a} - \varepsilon_1 j_1(k_1 a)\frac{\partial}{\partial r}\left[r h_1(k_2 r)\right]_{r=a}\right\}}$$

$$\times \frac{1}{r}\left\{z\cos\theta\, j_1(k_1 r)\hat{r} - \frac{\partial}{\partial r}\left[r j_1(k_1 r)\right]\sin\theta\,\hat{\theta}\right\}\sin\theta\cos\phi \qquad \text{(D.1)}$$

where b_{11}' is

$$b'_{11} = \frac{\begin{vmatrix} \varepsilon_2 \frac{\partial}{\partial \eta}\left[(1-\eta^2)^{\frac{1}{2}} he_{11} S_{11}\right] & -\varepsilon_2 \eta (\xi_b^2-1)^{\frac{1}{2}} \\ \frac{\partial}{\partial \xi}\left[(\xi^2-1)^{\frac{1}{4}} he_{11} S_{11}\right]_{\xi=\xi_b} & \xi_b (1-\eta^2)^{\frac{1}{2}} \end{vmatrix}}{\Delta}$$

$$\times E_0 d \frac{2i \cdot 2}{\Lambda_{11}(0)} S_{11}(0,0) \cos\phi \; S_{11}(0,\eta) \; je_{11}(0,\xi_b)$$

$$= \frac{\begin{vmatrix} \varepsilon_2(-2\eta) h_1(k_2 a) & -\varepsilon_2 \eta \xi_b \\ \frac{\partial}{\partial r}\left[r h_1(k_2 r)\right]_{r=a} (1-\eta^2)^{\frac{1}{2}} & \xi_b (1-\eta^2)^{\frac{1}{2}} \end{vmatrix}}{\Delta} E_0 d \frac{4i}{\frac{4}{3}} (1-\eta^2)^{\frac{1}{2}} \cos\phi \, j_1(k_2 a)$$

$$= \frac{-\varepsilon_2 \left[2h_1(k_2 a) - (r h_1)'_{r=a}\right]}{\left[\varepsilon_2 h_1 (r j_1)'_{r=a} - \varepsilon_1 j_1 (r h_1)'_{r=a}\right]} \frac{3i}{2} E_0 a \sin\theta \cos\phi \, j_1(k_2 a) \; ; \quad \xi d = a$$

where

$$\Delta = \begin{vmatrix} \varepsilon_2 \frac{\partial}{\partial \eta}\left[(1-\eta^2)^{\frac{1}{2}} he_{11}(h_2,\xi_b) S_{11}(h_2,\eta)\right] & -\varepsilon_1 \frac{\partial}{\partial \eta}\left[(1-\eta^2)^{\frac{1}{2}} je_{11}(h_1,\xi_b) S_{11}(h_1,\eta)\right] \\ \frac{\partial}{\partial \xi}\left[(\xi^2-1)^{\frac{1}{2}} he_{11}(h_2,\xi) S_{11}(h_2,\eta)\right]_{\xi=\xi_b} & -\frac{\partial}{\partial \xi}\left[(\xi^2-1)^{\frac{1}{2}} je_{11}(h_1,\xi) S_{11}(h_1,\eta)\right]_{\xi=\xi_b} \end{vmatrix}$$

$$= \begin{vmatrix} \varepsilon_2(-2\eta) h_1(k_2 a) & -\varepsilon_1(-2\eta) j_1(k_1 a) \\ \frac{\partial}{\partial r}\left[r h_1(k_2 r)\right](1-\eta^2)^{\frac{1}{2}} & -\frac{\partial}{\partial r}\left[r j_1(k_1 r)\right](1-\eta^2)^{\frac{1}{2}} \end{vmatrix}_{r=a}$$

$$= 2\eta(1-\eta^2)^{\frac{1}{2}} \left\{ \varepsilon_2 h_1(k_2 a) \frac{\partial}{\partial r}\left[r j_1(k_1 r)\right]_{r=a} - \varepsilon_1 j_1(k_1 a) \frac{\partial}{\partial r}\left[r h_1(k_2 r)\right]_{r=a} \right\}$$

The equation (5.1.47) is

$$\vec{E}_{in} = \sum_{\ell,m} B'_{m\ell} \frac{1}{k_1 r} \left\{ \ell(\ell+1) j_\ell P_\ell^m \hat{r} + \left[- \frac{\left[\frac{m}{\sin\theta} P_\ell^m\right]^2}{\left(P_\ell^m\right)'} + \left(P_\ell^m\right)' \right] \left[r j_\ell\right]' \hat{\theta} \right\} \cos m\phi$$

$$= B'_{11}\,\frac{1}{k_1 r}\left\{2j_1(k_1 r)\sin\theta\,\hat{r} - \frac{\sin^2\theta}{\cos\theta}\Big[rj_\ell(k_1 r)\Big]'\hat{\theta}\right\}\cos\phi\ ; \qquad \ell=1,\ m=1$$

$$= \frac{\varepsilon_2\left\{2h_1(k_2 a) - \Big[rh_1(k_2 a)\Big]'_{r=a}\right\}}{\left\{\varepsilon_2 h_1(k_2 a)\Big[rj_1(k_1 r)\Big]'_{r=a} - \varepsilon_1 j_1(k_1 a)\Big[rh_1(k_2 r)\Big]_{r=a}\right\}}\,\frac{3i}{2}\,E_0 a j_1(k_2 a)$$

$$\times\,\frac{1}{r}\left\{2j_1(k_1 r)\cos\theta\,\hat{r} - \frac{\partial}{\partial r}\Big[rj_1(k_1 r)\Big]\sin\theta\,\hat{\theta}\right\}\sin\theta\,\cos\phi \qquad (D.2)$$

where

$$B'_{11} = \frac{\begin{vmatrix} \alpha & -\varepsilon_2\left[1-r_0^2\left(\frac{b^2-a^2}{a^2b^2}\right)\sin^2\theta\right]\cos\theta \\ \alpha' & \left[1+r_0^2\left(\frac{b^2-a^2}{a^2b^2}\right)\cos^2\theta\right]\sin\theta \end{vmatrix}}{\Delta} \times E_0 3i\,\frac{2}{2}\,\sin\theta\ j_1(k_2 a)$$

$$= \frac{\begin{vmatrix} \frac{2\epsilon_2}{k_2 a}h_1\ \sin\theta & -\ \varepsilon_2\cos\theta \\ -\ \frac{\sin^2\theta}{k_2 a\cos\theta}\Big[rh_1\Big]'_{r=a} & \sin\theta \end{vmatrix}}{\frac{2\sin^3\theta}{k_1 k_2 a^2}\Big[\epsilon_2 h_1(rj_1)' - \epsilon_1 j_1(rh_1)'\Big]_{r=a}}\,E_0 3i\sin\theta j_1(k_2 a)\ ; \qquad a=b$$

$$= \frac{\varepsilon_2\Big[2h_1 - (rh_1)'\Big]_{r=a}k_1 a}{\frac{\sin\theta}{\cos\theta}\Big[\varepsilon_2 h_1(rj_1)' - \varepsilon_1 j_1(rh_1)'\Big]_{r=a}}\,\frac{3i}{2}\,E_0\ \sin\theta\ j_1(k_2 a)\ .$$

where

$$\Delta = \begin{vmatrix} \alpha & -\beta \\ \alpha' & \beta' \end{vmatrix}\ ;\quad \beta = \varepsilon_1\frac{2}{k_1 a}j_i(k_1 a)\sin\theta\ ,\quad \beta' = \frac{\sin^2\theta}{k_1 a\cos\theta}\Big[rj_1(k_1 r)\Big]'_{r=a}\ ,$$

$$\alpha = \varepsilon_2\frac{2}{2k_2 a}h_1(k_2 a)\ \sin\theta\ ,\quad \alpha' = -\ \frac{\sin^2\theta}{k_2 a\cos\theta}\Big[rh_1(k_2 r)\Big]'_{r=a}\ .$$

From equations (D.1) and (D.2), we concluded that equations 5.1.27a and 5.1.47 have the same result.

APPENDIX E

NUMERICAL CHECK OF OPTICAL THEOREM [51]

The scattered energy for a sphere of radius a is

$$W_s = \pi \frac{E_0^2}{k_2^2} \sqrt{\frac{\varepsilon_2}{\mu_2}} \sum_{\ell=1}^{\infty} (2\ell+1)\left(|a_\ell^r|^2 + |b_\ell^r|^2 \right) ;$$

the sum of absorbed and scattered energies are

$$W_t = \pi \frac{E_0^2}{k_2^2} \sqrt{\frac{\varepsilon_2}{\mu_2}} R_e \sum_{\ell=1}^{\infty} (2\ell+1)(a_\ell^r + b_\ell^r) ;$$

To find the absorbed energy

$$W_a = W_t - W_s$$

where

$$a_\ell^r = - \frac{\mu_1 j_\ell(N\rho)\left[\rho j_\ell(\rho)\right]' - \mu_2 j_\ell(\ell)\left[N\rho j_\ell(N\rho)\right]'}{\mu_1 j_\ell(N\rho)\left[\rho h_\ell(\rho)\right]' - \mu_2 h_\ell(\rho)\left[N\rho j_\ell(N\rho)\right]'}$$

$$b_\ell^r = - \frac{\mu_1 j_\ell(\rho)\left[N\rho j_\ell(N\rho)\right]' - \mu_2 N^2 j_\ell(N\rho)\left[\rho j_\ell(\rho)\right]'}{\mu_1 h_\ell(\rho)\left[N\rho j_\ell(N\rho)\right]' - \mu_2 N^2 j_\ell(N\rho)\left[\rho h_\ell(\rho)\right]'} ;$$

$$k_1 = Nk_2 , \quad \rho = k_2 a , \quad k_1 a = N\rho$$

Suppose we use the following parameters:

$\omega = 2\pi \times 10^9 \text{ Sec}^{-1}$

$\sigma = 2.5 \text{ mho}/_m$

$\varepsilon_1 = 40\ \varepsilon_2 = 40\ \varepsilon_0$

$\mu_1 = \mu_2 = \mu_0$

$k_1 \simeq 162.5$

$k_2 \simeq 20.94$

$N \simeq {}^{k1}/_{k2} = 7.77$

$\rho = k_2 a \simeq 0.02094$

$N\rho = k_1 a \simeq 0.1625$

$\varepsilon_0 = \dfrac{10^{-9}}{36\pi} \text{ farad}/_m$

$\mu_0 = 4\pi \times 10^{-7} \text{ henry}/_m$

$a = 10^{-3}\text{m}$

$j_1(N\rho) \simeq 0.05402$

$\left[N\rho j_1(N\rho)\right]' \simeq 0.1078$

$j_1(\rho) \simeq 0.00698$

$n_1(\rho) \simeq -2281.087$

$\left[\rho j_1(\rho)\right]' \simeq 0.01396$

$\left[\rho h_1(\varepsilon)\right]' \simeq 0.01396 + 2281.066i$

We find that

$$a_1^r \simeq -2.06 \times 10^{-17} + 4.54 \times 10^{-9} i$$

$$b_1^r \simeq -3.39 \times 10^{-11} + 5.82 \times 10^{-6} i$$

hence

$$W_a \simeq -3.8 \times 10^{-15}\ E_0^2 \tag{E.1}$$

To check absorbed energy, we calculate

$$W_a = \int_V \sigma |E_{in}|^2 dV = -(2.5)(1.0473 \times 10^{-6}) E_0^2 \frac{\int r^2 \sin^2\theta\ \cos^2\phi\ d(\cos\theta) d\phi dr}{\frac{4}{3}\frac{a^3}{3}\pi}$$

$$\simeq -3.7 \times 10^{-15}\ E_0^2 \tag{E.2}$$

where

$$\vec{E}_{in} = \frac{\varepsilon_2 \left\{ 2h_1(k_2a) - \frac{\partial}{\partial r}\left[rh_1(k_2r)\right]_{r=a} \right\} \frac{3i}{2} E_0 a j_1(k_2a)}{\left\{ \varepsilon_2 h_1(k_2a) \frac{\partial}{\partial r}\left[rj_1(k_1r)\right]_{r=a} - \varepsilon_1' j_1(k_1a) \frac{\partial}{\partial r}\left[rh_1(k_2r)\right]_{r=a} \right\}}$$

$$\times \frac{1}{r}\left\{ 2\cos\theta j_1(k_1r)\hat{r} - \frac{\partial}{\partial r}\left[rj_1(k_1r)\right]\sin\theta\ \hat{\phi} \right\} \sin\theta \cos\phi \ ; \qquad \ell=1$$

$$= \frac{\left[2(0.00698-2281.087i)-(0.01396+2281.066i)\right] iE_0(10^{-3})(0.00698)(162.5)}{\left[(0.00698-2281.087i)(0.1078)-(40+45i)(0.05402)(0.01396+2281.066i)\right]}$$

$$\times\ [\cos\theta\ \hat{r} - \sin\theta\ \hat{\theta}]\sin\theta\cos\phi \ ; \qquad \varepsilon_1' = \varepsilon_1 + i\frac{\sigma}{\omega} = (40 + 45i)\varepsilon_0 \ ,$$

$$j_1(k_1r) \approx \frac{k_1r}{3} \ ,$$

$$\frac{\partial}{\partial r}\left[rj_1(k_1r)\right] \approx \frac{2k_1r}{3} \ .$$

$$\approx \frac{-7.762\ E_0}{5.545 \times 10^3 - 5.175 \times 10^3 i} (\cos\theta\ \hat{r} - \sin\theta\ \hat{\phi})\sin\theta\cos\phi$$

$$|E_{in}|^2 \approx 1.0473 \times 10^{-6}\ E_0^2 \sin^2\theta \cos^2\phi \ ;$$

$$dV = -r^2\ d(\cos\theta)d\phi\ dr$$

As we can see from (E.1) and (E.2), the two methods give values $-3.8 \times 10^{-15}E_0^2$ and $-3.7 \times 10^{-15}E_0^2$, respectively; they agree within 3 percent.

In the case of the spheroid, the scattering E-fields [equation (5.1.27b)] is

$$E_{scat} = b_{11}\frac{1}{d(\xi^2-\eta^2)^{\frac{1}{2}}}\left\{\frac{\partial}{\partial\eta}\left[(1-\eta^2)^{\frac{1}{2}}he_{11}(h_2,\xi)S_{11}(h_2,\eta)\right]\hat{\xi}\right.$$

$$\left. - \frac{\partial}{\partial\xi}\left[(\xi^2-1)^{\frac{1}{2}}he_{11}(h_2,\xi)S_{11}(h_2,\eta)\right]\hat{\eta}\right\}$$

At large distances $he_{m\ell}$, $S_{m\ell}$, $d\xi$, and $\hat{\eta}$ can be reduced to

$$he_{m\ell} \longrightarrow h_\ell \longrightarrow \frac{i^{-1-\ell}}{kr}\, e^{ikr}$$

$$S_{m\ell} \longrightarrow P_\ell^m$$

$$d\xi \longrightarrow r$$

$$\hat{\eta} \longrightarrow -\hat{\theta}$$

Asymptotically the scattering field at large distances from the origin may be written [50]

$$\vec{E}_{scat} = \frac{E_0}{k_2 r}\,\vec{F}(\theta,\phi)\, e^{ik_2 r}$$

$$= -\frac{ib_{11}}{r}\, e^{ik_2 r}\,\sin\theta\ \hat{\theta}$$

$$\therefore\ \vec{F}(\theta,\phi) = \frac{-b_{11}}{E_0}\, k_2\, i\sin\theta\ \hat{\theta}$$

Suppose we use the following parameters:

$d \simeq 1.23 \times 10^{-3}$ m $\qquad\qquad \xi_b \simeq 1.05$

$\mu_1 = \mu_0 \simeq \mu_2 = 4\pi \times 10^{-7}$ h/m $\qquad \sigma_1 \simeq 2.5$ mho/m

$\varepsilon_1 \simeq 40\varepsilon_0 = 40 \times \frac{10^{-9}}{36\pi}$ f/m $\qquad \omega = 4 \times 2\pi \times 10^{-9} \text{sec}^{-1}$

$\varepsilon_1' = \varepsilon_1 + \frac{\sigma}{\omega} \simeq (40 + 11.25\, i)\varepsilon_0 \qquad h_1^2 = d^2 k_1^2 = d^2(\mu_1 \varepsilon_1 \omega^2 + i\mu_1 \sigma\omega)$

$k_1 = 5.4 \times 10^2 (0.99 + 0.138\, i) \qquad h_1 = dk_1 = 0.664(0.99 + 0.138\, i)$

$h_2 \approx 0.103 \qquad k_2 \approx 83.8$

$\Lambda_{11} \simeq 1.34 \qquad je_{11}(h_1, \xi_b) \simeq 0.07$

$je_{11}(h_2, \xi_b) \approx 0.01 \qquad he_{11}(h_2, \xi_b) \simeq 0.01 - 379\, i$

$$\frac{\partial}{\partial \xi}\left[(\xi^2 - 1)^{\frac{1}{2}} je_{11}(h_1, \xi)\right]_{\xi=\xi_b} \simeq 0.45$$

$$\frac{\partial}{\partial \xi}\left[(\xi^2 - 1)^{\frac{1}{2}} he_{11}(h_2, \xi_b)\right] \simeq 0.072 + 281\, i$$

By using above data $\vec{F}(\theta,\phi)$ is found to be

$$\vec{F}(\theta,\phi) \approx (5.87 \times 10^{-12} - 1.1 \times 10^{-10}\ )\, k_2 \sin^2\theta \cos\phi\ \hat{\vartheta}$$

The cross section is

$$\sigma_t = -\frac{1}{i k_2^2}\int_0^{2\pi} d\phi\ (\vec{F} - \vec{F}^*)\cdot \hat{x}\Big|_{\vartheta = \pi/2} = 8.23 \times 10^{-12}\ \ m^2$$

The scattering cross section is

$$\sigma_{scat} = \frac{1}{k_2^2}\int_0^{2\pi} d\phi \int_0^{\pi} |\vec{F}|^2 \sin\theta d\theta \simeq 4.07 \times 10^{-20}\ \ m^2$$

$$\sigma_{abs} = \sigma_t - \sigma_{scat} \approx 8.23 \times 10^{-12} \ m^2$$

The power W absorbed by the prolate spheroid becomes

$$W = \sigma_{abs} \times E_0^2 \sqrt{\frac{\varepsilon_0}{\mu_0}}$$

$$\approx 8.23 \times 10^{-12} \times 2.65 \times 10^{-3} \ E_0^2$$

$$\approx 2.19 \times 10^{-14} \ E_0^2$$

APPENDIX F

DERIVATION OF STRATTON'S SCATTERING CROSS-SECTION FORMULA FROM THE OPTICAL THEOREM [50,51]

On page 235 of Panofsky and Phillips [50], or on page 564 of Stratton [51], the electromagnetic field scattered by a sphere is given by

$$\vec{E}_s = E_0 \sum_{\ell=1}^{\infty} i^\ell \frac{2\ell+1}{\ell(\ell+1)}\left[a_\ell \vec{M}_{1\ell} - i b_\ell \vec{N}_\ell\right]$$

$$= E_0 \sum_{\ell=1}^{\infty} i^\ell \frac{2\ell+1}{\ell(\ell+1)}\Bigg\{a_\ell\left[\frac{1}{\sin\theta} h_\ell(k_2 r)P_\ell^{'}(\cos\theta)\cos\phi\ \hat{\theta} - h_\ell P_\ell^{1'} \sin\phi\ \hat{\phi}\right]$$

$$- i b_\ell\left[\frac{\ell(\ell+1)}{k_2 r} h_\ell P_\ell^1 \cos\phi \hat{r} + \frac{1}{k_2 r}\left[r h_\ell\right]' P_\ell^{1'} \cos\phi\ \hat{\theta}\right.$$

$$\left.- \frac{1}{k_2 r \sin\theta}\left[r h_\ell\right]' P_\ell^1 \sin\phi\ \hat{\phi}\right]\Bigg\}$$

$$= E_0 \sum_{\ell=1}^{\infty} i^\ell \frac{2\ell+1}{\ell(\ell+1)}\Bigg\{- i b_\ell \frac{\ell(\ell+1)}{k_2 r} h_\ell P_\ell^{'} \cos\phi\ \hat{r}$$

$$+ \left[a_\ell \frac{1}{\ell \sin\theta} h_\ell P_\ell^{'} - i b_\ell \frac{1}{\ell k_2 r}\left[r h_\ell\right]' P_\ell^{1'}\right]\cos\phi\ \hat{\theta}$$

$$- \left[a_\ell h_\ell P_\ell^{1'} - i \frac{b_\ell}{k_2 r \sin\theta}\left[r h_\ell\right]' P_\ell^{'}\right]\sin\phi\ \hat{\phi}\Bigg\} .$$

Asymptotically, the scattered field at large distances from the origin may be written as [50].

$$E_s = E_0 \frac{1}{k_2 r} \vec{F}(\theta,\phi) e^{ik_2 r}$$

where $\vec{F}(\theta,\phi)$ can be extracted from above, we get

$$\vec{F}(\theta,\phi) = \sum_{\ell=1}^{\infty} i^{\ell} \frac{2\ell+1}{\ell(\ell+1)} \left\{ \left[a_\ell i^{-\ell-1} \frac{P'_\ell}{\sin\theta} - i b_\ell i^{\ell} P_\ell^{1'} \right] \cos\phi \, \hat{\theta} \right.$$

$$\left. - \left[a_\ell i^{-\ell-1} P_\ell^{1'} - i b_\ell \frac{i^{-\ell}}{\sin\theta} P^1_\ell \right] \sin\phi \, \hat{\phi} \right\}$$

$$= \sum_{\ell=1}^{\infty} \frac{2\ell+1}{\ell(\ell+1)} i \left\{ - \left[a_\ell \frac{P'_\ell}{\sin\theta} + b_\ell P_\ell^{1'} \right] \cos\phi \, \hat{\theta} \right.$$

$$\left. + \left[a_\ell P_\ell^{1'} + b_\ell \frac{P'}{\sin\theta} \right] \sin\phi \, \hat{\phi} \right\}$$

$$P_\ell^m(x) = (1 - x^2)^{\frac{1}{2}} T_{\ell-m}^{m}(x) \quad ; \quad x = \cos\theta$$

$$T_{\ell-m}^{m}(1) = \frac{(\ell+m)!}{2^m m!(\ell-m)!} , \quad T'_{\ell-1} = \frac{\ell(\ell+1)}{2}$$

$$(1-x^2) \frac{d}{dx} P_\ell^m(x) = (\ell+1) \times P_\ell^m(x) - (\ell-m+1) P_{\ell+1}^{m}(x)$$

$$(1-x^2)^{\frac{1}{2}} \frac{d}{d\theta} P'_\ell = \ell P'_{\ell+1} - (\ell+1) \times P'_\ell$$

$$= \ell(1-x^2)^{\frac{1}{2}} T'_{\ell+1-1}(x) - (\ell+1) \times (1-x^2)^{\frac{1}{2}} T'_{\ell-1}(x)$$

i.e. $P_\ell^{1'}(1) = \frac{\ell(\ell+1)}{2}$

$$= \sum_{\ell=1}^{\infty} \frac{2\ell+1}{\ell(\ell+1)} i \left\{ - \frac{\ell(\ell+1)}{2} (a_\ell + b_\ell) \cos\phi \, \hat{\theta} + \frac{\ell(\ell+1)}{2} (a_\ell + b_\ell) \sin\phi \, \hat{\phi} \right\}_{\theta=0}$$

$$= \sum_{\ell=1}^{\infty} \frac{(2\ell+1)}{2} i \left\{ - (a_\ell + b_\ell) \cos\phi \, \hat{\theta} + (a_\ell + b_\ell) \sin\phi \, \hat{\phi} \right\}_{\theta=0} .$$

$$\frac{2\pi}{i k_2^2} \hat{x} \cdot (\vec{F} - \vec{F}^*) = \frac{2\pi}{k_2^2} \sum_{\ell=1}^{\infty} (2\ell+1) R_e (a_\ell + b_\ell)$$

$$E_s = E_0 \frac{1}{k_2 r} \vec{F}(\theta,\phi) e^{ik_2 r}$$

where $\vec{F}(\theta,\phi)$ can be extracted from above, we get

$$\vec{F}(\theta,\phi) = \sum_{\ell=1}^{\infty} i^{\ell} \frac{2\ell+1}{\ell(\ell+1)} \left\{ \left[a_\ell i^{-\ell-1} \frac{P_\ell^{1}}{\sin\theta} - i b_\ell i^{\ell} P_\ell^{1'} \right] \cos\phi \, \hat{\theta} \right.$$

$$\left. - \left[a_\ell i^{-\ell-1} P_\ell^{1'} - i b_\ell \frac{i^{-\ell}}{\sin\theta} P_\ell^{1} \right] \sin\phi \, \hat{\phi} \right\}$$

$$= \sum_{\ell=1}^{\infty} \frac{2\ell+1}{\ell(\ell+1)} i \left\{ -\left[a_\ell \frac{P_\ell^{1}}{\sin\theta} + b_\ell P_\ell^{1'} \right] \cos\phi \, \hat{\theta} \right.$$

$$\left. + \left[a_\ell P_\ell^{1'} + b_\ell \frac{P^{1}}{\sin\theta} \right] \sin\phi \, \hat{\phi} \right\}$$

$$P_\ell^m(x) = (1 - x^2)^{\frac{1}{2}} T_{\ell-m}^{m}(x) \; ; \quad x = \cos\theta$$

$$T_{\ell-m}^{m}(1) = \frac{(\ell+m)!}{2^m m!(\ell-m)!} \, , \quad T_{\ell-1}^{1} = \frac{\ell(\ell+1)}{2}$$

$$(1-x^2)\frac{d}{dx} P_\ell^m(x) = (\ell+1) \times P_\ell^m(x) - (\ell-m+1) P_{\ell+1}^m(x)$$

$$(1-x^2)^{\frac{1}{2}} \frac{d}{d\theta} P_\ell^{1} = \ell P_{\ell+1}^{1} - (\ell+1) \times P_\ell^{1}$$

$$= \ell (1-x^2)^{\frac{1}{2}} T_{\ell+1-1}^{1}(x) - (\ell+1) \times (1-x^2)^{\frac{1}{2}} T_{\ell-1}^{1}(x)$$

$$\text{i.e.} \quad P_\ell^{1'}(1) = \frac{\ell(\ell+1)}{2}$$

$$= \sum_{\ell=1}^{\infty} \frac{2\ell+1}{\ell(\ell+1)} i \left\{ - \frac{\ell(\ell+1)}{2} (a_\ell + b_\ell) \cos\phi \, \hat{\theta} + \frac{\ell(\ell+1)}{2} (a_\ell + b_\ell) \sin\phi \, \hat{\phi} \right\}_{\theta=0}$$

$$= \sum_{\ell=1}^{\infty} \frac{(2\ell+1)}{2} i \left\{ - (a_\ell + b_\ell) \cos\phi \, \hat{\theta} + (a_\ell + b_\ell) \sin\phi \, \hat{\phi} \right\}_{\theta=0} .$$

$$\frac{2\pi}{i k_2^2} \hat{x} \cdot (\vec{F} - \vec{F}^*) = \frac{2\pi}{k_2^2} \sum_{\ell=1}^{\infty} (2\ell+1) R_e (a_\ell + b_\ell)$$

From page 421 of Panofsky and Phillips [50], the sum of the cross section for scattering and absorption is

$$Q_{total} = Q_{scat} + Q_{abs} = \frac{2\pi}{i k_2^2} \hat{x} \cdot (\vec{F}-\vec{F}^*)_{\theta=0}$$

where

$$Q_{scat} = \frac{1}{k_2^2}\int_0^{2\pi} d\phi \int_0^{\pi} |\vec{F}|^2 \sin\theta \, d\theta \ ,$$

This integration can be evaluated with the help of

$$\int_0^{\pi}\left[\frac{d}{d\theta}P'_\ell \frac{d}{d\theta}P'_m + \frac{1}{\sin^2\theta}P'_\ell P'_m\right]\sin\theta d\theta = \begin{cases} 0 & \text{if } \ell \neq m \\ \dfrac{2\ell^2(\ell+1)^2}{(2\ell+1)} & \text{if } \ell = m \end{cases}$$

and the relation

$$\int_0^{\pi} \left[\frac{P'_m}{\sin\theta}\frac{d}{d\theta}P'_\ell + \frac{P'_\ell}{\sin\theta}\frac{d}{d\theta}P'_m\right]\sin\theta d\theta = 0 \quad .$$

i.e.,

$$\frac{1}{k_2^2}\int_0^{2\pi} d\phi \int_0^{\pi} |\vec{F}|^2 \sin\theta \, d\theta = \frac{2\pi}{k_2^2}\sum_{\ell=1}^{\infty} (2\ell+1)\left(|a_\ell|^2 + |b_\ell|^2\right)$$

which have been shown above to be equivalent to the expression on page 569 of Stratton [51].

$$Q_{total} = \frac{2\pi}{k_2^2}\sum_{\ell=1}^{\infty} (2\ell+1)R_e(a_\ell+b_\ell)$$

$$Q_{scat} = \frac{2\pi}{k_2^2}\sum_{\ell=1}^{\infty} (2\ell+1)(|a_\ell|^2 + |b_\ell|^2)$$

APPENDIX G

SPHERICAL BESSEL FUNCTIONS AND LEGENDRE FUNCTIONS

Spherical Bessel Functions

$$j_\ell(z) = \sqrt{\frac{\pi}{2z}}\, J_{\ell+\frac{1}{2}}(z) \xrightarrow[z\to 0]{} \frac{z^\ell}{1\cdot 3\cdot 5\cdot \ldots (2\ell+1)}$$

$$\xrightarrow[z\to\infty]{} \frac{1}{z}\cos\left[z - \frac{\pi}{2}(\ell+1)\right] \quad ; \qquad \ell \text{ an integer}$$

$$n_\ell(z) = \sqrt{\frac{\pi}{2z}}\, N_{\ell+\frac{1}{2}}(z) \xrightarrow[z\to 0]{} - \frac{1\cdot 1\cdot 3\cdot 5\cdot \ldots (2\ell+1)}{z^{\ell+1}}$$

$$\xrightarrow[z\to\infty]{} \frac{1}{z}\sin\left[z - \frac{\pi}{2}(\ell+1)\right] \quad ; \qquad n_\ell(z) = (-1)^{\ell+1} j_{-\ell-1}(z)$$

$$h_\ell(z) = j_\ell(z) + i\, n_\ell(z) \xrightarrow[z\to\infty]{} \frac{1}{z} i^{-\ell-1} e^{iz} \quad ; \qquad h_{-\ell}(z) = i(-1)^{\ell-1} h_{\ell-1}(z)$$

$$j_\ell(z) = (-1)^\ell z^\ell \left(\frac{d}{zdz}\right)^\ell \left(\frac{\sin z}{z}\right) \quad ; \qquad n_\ell(z) = -\,(-1)^\ell z^\ell \left(\frac{d}{zdz}\right)^\ell \left(\frac{\cos z}{z}\right) \; ;$$

$$h_\ell(z) = -i\,(-1)^\ell z^\ell \left(\frac{d}{zdz}\right)^\ell \left(\frac{e^{iz}}{z}\right)$$

$$j_0(z) = \frac{1}{z}\sin z \; ; \qquad n_0(z) = -\frac{1}{z}\cos z \; ; \quad h_0(z) = -\frac{i}{z} e^{iz}$$

$$j_1(z) = \frac{\sin z}{z^2} - \frac{\cos z}{z} \; ; \qquad n_1(z) = -\frac{\cos z}{z^2} - \frac{\sin z}{z} \; ; \; h_1(z) = -\left(\frac{z+i}{z^2}\right) e^{iz}$$

If $f_\ell(z)$ is a linear combination of $j_\ell(z)$ and $n_\ell(z)$, with coefficients independent of ℓ or z, we have

$$\frac{1}{z^2}\frac{d}{dz}\left(z^2\frac{d}{dz}f_\ell\right) + \left[1 - \frac{\ell(\ell+1)}{z^2}\right]f_\ell = 0$$

$$\left(\frac{2\ell+1}{z}\right)f_\ell(z) = f_{\ell-1}(z) + f_{\ell+1}(z)$$

$$(2\ell+1)\frac{d}{dz}f_\ell(z) = \ell f_{\ell-1}(z) - (\ell+1)f_{\ell+1}(z)$$

Legendre Functions

$$P_\ell^m(z) = (1-z^2)^{\frac{m}{2}}\frac{d^m}{dzm}P_\ell(z) = (1-z^2)^{\frac{m}{2}}T_{\ell-m}^m(z)$$

$$= \frac{(1-z^2)^{\frac{m}{2}}}{2^\ell \ell!}\frac{d^{\ell+m}}{dz\ell+m}(z^2-1)^\ell \; ; \quad m, \ell = 0,1,2,\ldots \; ; \quad \ell \geq m$$

$$(2\ell+1)(1-z^2)^{\frac{1}{2}} P_\ell^m(z) = P_{\ell+1}^{m+1}(z) - P_{\ell-1}^{m+1}(z)$$

$$= (\ell+m)(\ell+m-1)P_{\ell-1}^{m-1}(z) - (\ell-m+1)(\ell-m+2)P_{\ell+1}^{m-1}(z)$$

$$(2\ell+1)zP_\ell^m(z) = (\ell-m+1)P_{\ell+1}^m(z) + (\ell+m)P_{\ell-1}^m(z)$$

$$(1-z^2)\frac{d}{dz}P_\ell^m(z) = (\ell+1)zP_\ell^m(z) - (\ell-m+1)P_{\ell+1}^m(z)$$

$$\frac{d}{dz}\left[(1-z^2)^{\frac{m}{2}}P_\ell^m(z)\right] = -(\ell-m+1)(\ell+m)(1-z^2)^{\frac{m}{2}-\frac{1}{2}}P_\ell^{m-1}(z)$$

APPENDIX H

THE VALUE OF $b'_{1\ell}$ OF EQUATION (5.1.30) WITH $\ell = 1$

The values of equations (6.2.1) and (6.2.2) can be substituted into the d's of equation (6.1.1) to yield

$$d_0(h_1|1,1) \simeq 1 + \frac{2}{25}h_1^2 + \frac{1193}{275625}h_1^4 \qquad d_0(h_2|1,1) \simeq 1.001$$

$$\simeq 1.036$$

$$d_2(h_1|1,1) \simeq \frac{-1}{75}h_1^2 - \frac{16}{16875}h_1^4 \qquad d_2(h_2|1,1) \simeq 0.00014$$

$$\simeq -0,00588$$

$$d_4(h_1|1,1) \simeq \frac{h_1^4}{11025} \simeq 1.76 \times 10^{-5} \qquad d_4(h_2|1,1) \simeq 1.02 \times 10^{-8}$$

$$\simeq 0 \qquad\qquad \simeq 0$$

and

$$S_{11}(h_1,\eta) \simeq d_0P_1^1(\eta) + d_2P_3^1(\eta) + d_4P_5^1(\eta)$$

$$\simeq 1.036P_1^1(\eta) - 0.00588P_3^1(\eta) + 1.76 \times 10^{-5}P_5^1(\eta)$$

$$\simeq P_1^1(\eta) = (1 - \eta^2)^{1/2}$$

$$S_{11}(h_2,\eta) \simeq P_1^1(\eta) = (1 - \eta^2)^{1/2}$$

The value of $je_{11}(h_1,\xi_b)$ can be obtained either from equation (A.5) or (A.7c), which is

$$je_{11}(h_1,\xi_b) \simeq 0.073 ,$$

and the values of $|he_{11}(h_2,\xi_b)|$ can be obtained either from equation (A.9c) or (A.11b), which is

$$|he_{11}(h_2,\xi_b)| \xrightarrow[h\to 0]{} |ne_{11}(h_2,\xi_b)| \simeq 386 .$$

The values of $\frac{\partial}{\partial\eta}[(1-\eta^2)^{1/2}S_{11}(h_2,\eta)]$, $\frac{\partial}{\partial\eta}[(1-\eta^2)^{1/2}S_{11}(h_1,\eta)]$, $\frac{\partial}{\partial\xi}[(\xi^2-1)^{1/2}he_{11}(h_2\xi)]_{\xi=\xi_b}$, $\frac{\partial}{\partial\xi}[(\xi^2-1)^{1/2}je_{11}(h_1,\xi)]_{\xi=\xi_b}$, and $\Lambda_{11}(h_2)$ are:

$$\frac{\partial}{\partial\eta}[(1-\eta^2)^{1/2}S_{11}(h_2,\eta)] \simeq \frac{\partial}{\partial\eta}[(1-\eta^2)^{1/2}P_1^1(\eta)]$$

$$\simeq \frac{\partial}{\partial\eta}[(1-\eta^2)^{1/2}(1-\eta^2)^{1/2}] = -2\eta$$

$$\frac{\partial}{\partial\eta}[(1-\eta^2)^{1/2}S_{11}(h_1,\eta)] \simeq -2\eta$$

$$\frac{\partial}{\partial\xi}[(\xi^2-1)^{1/2}je_{11}(h_1,\xi)]_{\xi=\xi_b} = \frac{\partial}{\partial\xi}\left\{(\xi^2-1)^{1/2}\cdot\frac{1}{2}\left[\frac{\xi^2-1}{\xi^2}\right]^{1/2}[2d_0j_1(h_1\xi) - 12d_2j_3(h_1\xi) - 30d_4j_5(h_1\xi) + \cdots]\right\}_{\xi=\xi_b}$$

$$\simeq \frac{\partial}{\partial\xi}\left[\left(\xi-\frac{1}{\xi}\right)d_0j_1(h_1\xi)\right]^*_{\xi=\xi_b}$$

$$\simeq d_0\left\{\left(1+\frac{1}{\xi^2}\right)j_1(h_1\xi) + \left(\xi-\frac{1}{\xi}\right)h_1\,\frac{1}{3}\left(j_0(h_1\xi) - 2j_2(h_1\xi)\right)\right\}_{\xi=\xi_b}$$

$$\simeq 0.452$$

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*$(2n+1)\,\frac{d}{dz}f_n(z) = nf_{n-1}(z) - (n+1)f_{n+1}(z)$, where f_n is a linear combination of $j_n(z)$ and $n_n(z)$ with coefficients independent of n or z.

$$\frac{\partial}{\partial\xi}\left[(\xi^2 - 1)^{1/2}he_{11}(h_2,\xi)\right]_{\xi=\xi_b} \simeq \frac{\partial}{\partial\xi}[(\xi^2 - 1)^{1/2}ne_{11}(h_2,\xi)]_{\xi=\xi_b} \quad ; \; h_2<<1$$

(by using equation A.9c)

$$\simeq -\frac{\partial}{\partial\xi}\left\{\left[\left(1 - \frac{19}{75}h_2^2\right)\xi - \frac{h_2^2}{75}\xi^2\right]\right.$$

$$- 2(\xi^2 - 1)\left[\left(1 + \frac{3}{50}h_2^2\right)\right.$$

$$\left.\left. + \frac{h_2^2}{10}\xi^2\right]\tanh^{-1}[\xi - (\xi^2 - 1)^{1/2}]\right\}_{\xi=\xi_b}\frac{3}{2h_2^2}$$

$$\simeq 278$$

and

$$\Lambda_{11}(h_2) \simeq \left|d_0(h_2|1,1)\right|^2 \frac{2}{3} \cdot \frac{2}{1} \simeq 1.34 \text{ [using equation (A.4)]}.$$

The quantity of Δ in equation (4.a.31) is

$$\Delta \simeq \begin{vmatrix} -\varepsilon_0 \times 386 \times 2\eta & 40\varepsilon_0 \times (0.073) \times 2\eta \\ 278 \times (1 - \eta^2)^{1/2} & -(0.452) \times (1 - \eta^2)^{1/2} \end{vmatrix} \simeq -1275\varepsilon_0 \,(1 - \eta^2)^{1/2}$$

Finally

$$b_{11}' \simeq \frac{\begin{vmatrix} -\varepsilon_0\times386\times2\eta & -\varepsilon_0\eta(0.32) \\ 278\times(1-\eta^2)^{1/2} & (1.05)\times(1-\eta^2)^{1/2} \end{vmatrix}}{\Delta} E_z d \, \frac{0.073\,i}{\Lambda_{11}(h_2)} \underbrace{S_{11}(h_2,0)}_{\simeq 1}\cos\phi(1-\eta^2)^{1/2}$$

$$\simeq 3.08 \times 10^{-2} E_z d \cos\phi(1-\eta^2)^{1/2}\, i$$