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DOMINANT EIGENVALUES ASSIGNMENT METHODS
FOR STATE FEEDBACK SYSTEMS

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ABSTRACT

A new approach for eigenvalue assignment in a linear time-invariant, deterministic feedback system is presented. The problem of pole assignment with complete or incomplete state feedback is investigated. A sufficient condition for placing an r number of poles using an r number of measurable states is derived. Algorithms for complete and for incomplete state feedback are developed separately.

The application of dominant eigenvalue assignment for model simplification is developed.

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CHAPTER 1

INTRODUCTION

This dissertation presents a new eigenvalue placement procedure by application of state feedback and system decomposition principle. The design of feedback controllers eigenvalue placement, for linear time-invariant multivariable systems has received considerable attention [1-17]. Anderson and Luenberger [1] have developed an effective method which transforms the system matrix into companion form. The state variable feedback may then be easily computed from the difference between coefficients of the closed loop characteristic polynomial and the open loop characteristic polynomial. In practice, however, not all the state variables are measurable, hence an observer is required to provide complete state feedback [2]. Wonham's method [3] requires conversion of a multi-input controllable system into a single-input controllable one. Heyman [4], and Chen [5] have proposed an improved multi-input controllable to single-input controllable conversion algorithm separately. Ding and Pearson [6] have shown that it is possible to obtain eigenvalue assignment by using a dynamic compensator which requires repeated differentiation of the input. This method was extended by Brash and Pearson [7] to show that arbitrary pole assignment may be obtained by adding a compensator of the order $P = \min[(V_c-1)(V_o-1)]$, where V_c and V_o are respectively the controllability and observability indices of the plant. Similar results are obtained by Hse and Chen [9]. These techniques lead to the design of an augmented system with guarranteed stability. Retallack and MacFarlane [10] have derived a state feedback pole-shifting algorithm using HSU-CHEN theorem [8]. Fallside and Seraji [11], and Widodo [12] used the concept of the return-difference matrix to find the state feedback matrix. Park and Seborg [13] have shown that it is possible to assign the eigenvalues of the augmented system, while eliminating steady state error arising from sustained disturbance by using proportional-integral state feedback. A new eigenvalue assignment method via singular perturbation has been shown by Porter and Shenton [14].

However, all the poles of the system do not have to be preassigned for stabilization purposes. (Jameson [15]) Therefore, a pole assignment method using incomplete state feedback to assign some of the closed loop poles was investigated by Davison [16], Sridhar and Lindorff [17].

Davison [16], Sridhar and Lindorff [17] have shown that, for a completely controllable and completely observable system, the maximum number of poles that could be assigned using output feedback is equal to Max (Rank[B], Rank[C])· For this case, an observer is not required.

However, the remaining unassigned eigenvalues will assume any values. Therefore, even if the open loop system is stable, by applying output feedback to achieve Max (Rank[B], Rank[C]) pole placement, the closed loop system might become unstable.

Dominant eigenvalue placement has been investigated for the purpose of system reduction. Davison [18] has shown how to use a simplified model, consisting of the dominant eigenvalues of the original system, to approximate the original system. A more accurate result is given by Chidambara [19]. Chen's [20] simplification method requires continued fraction expansion and turncation of the transfer function matrix. The suboptimal control based on the simplified model has been developed by Rao and Lamba [21,22]. Suboptimal control has been applied to the control of power systems using a simplified model by Elangovan and Kuppuajulu [23].

This dissertation presents a new eigenvalue placement method which can apply complete or incomplete state feedback to assign real or complex eigenvalues.

Chapter 2 develops eigenvalue assignment in single variable systems. An algorithm for placing n eigenvalues with complete state feedback is developed. The algorithm is then attend to allow placement of r number of poles using an r number of measurable states. Sufficient conditions for eigenvalue assignment of incomplete state feedback is determined.

Chapter 3 is concerned with the multivariable systems. In parallel with the structure of Chapter 2, algorithms are derived to assign all or part of the n eigenvalues by using complete or incomplete state feedback. The sufficient conditions for eigenvalue assignment of incomplete state feedback is also determined.

In Chapter 4, the eigenvalue assignment method is applied to model reduction with results compared with those of Childambara's simplification method [19].

Some numerical examples are given in Appendix to demonstrate application of the new method.

Related areas of further investigation along with comparative observation of current effort are presented in the conclusion.

CHAPTER 2

SINGLE VARIABLE SYSTEMS

2.0 INTRODUCTION:

It was shown by Anderson and Luenberger [1], and by Wonham [2] that if a linear time-invariant system is controllable, then all the closed loop poles can be placed at the desired locations by using state feedback. However, their methods require complete state feedback. In practice, not all the states are available for feedback. In this Chapter, a new pole assignment method, from which incomplete state feedback can be derived, is developed. This is achieved by introducing a non-singular matrix T to the system equation. A state feedback matrix K is then found such that $TA + TBK = A_c$, where $A_c = \left[\begin{array}{c|c} A_{c1} & 0 \\ \hline 0 & A_{c2} \end{array}\right]$. A_{c1} and A_{c2} are companion matrices with desired eigenvalues $\lambda_1, \lambda_2 \ldots, \lambda_r$, and $\lambda_{r+1}, \ldots, \lambda_n$, respectively. Unlike the incomplete state feedback method proposed by Davison [16], Sridhar and Lindorff [17], which uses output vector to assign Max(Rank[B], Rank[C]) poles but let the remaining poles blindly assuming any values. The new incomplete state feedback method will set a bound to the remaining poles. The sufficient conditions for incomplete state feedback is found. A stability criterion is derived from that sufficient conditions.

2.1 COMPLETE STATE FEEDBACK SYSTEMS

Consider a linear time-invariant controllable system described by

$$\dot{X} = AX + bu \tag{2.1}$$

where X is an n x 1 state vector.

A is an n x n system matrix.

b is an n x 1 input matrix.

u is a scalar input.

The dynamic equation (2.1) is controllable by assumption; hence the set of n x 1 column vectors b, Ab, ... $A^{n-1}b$ is linear independent. Consequently, the following set of n x 1 vectors:

$$\begin{aligned}
q^n &\triangleq b \\
q^{n-1} &\triangleq Aq^n + \partial_1 q^n = Ab + \partial_1 b \\
q^{n-2} &\triangleq Aq^{n-1} + \partial_2 q^n = A^2 b + \partial_1 Ab + \partial_2 b \\
&\vdots \\
q^1 &\triangleq Aq^2 + \partial_{n-1} q^n = A^{n-1} b + \partial_1 A^{n-2} b + \ldots + \partial_{n-1} b
\end{aligned} \tag{2.2}$$

is linearly independent and qualify as a basis of the state space of (2.1). Observe that

$$\left.\begin{aligned}
Aq^1 &= (A^n + \partial_1 A^{n-1} + \ldots + \partial_{n-1} A + \partial_n I)b - \partial_n b \\
&= -\partial_n b = -\partial_n q^n = \begin{bmatrix} q^1 & q^2 & \ldots & q^n \end{bmatrix} \begin{bmatrix} 0 \\ \cdot \\ 0 \\ \vdots \\ -\partial_n \end{bmatrix} \\
Aq^2 &= q^1 - \partial_{n-1} q^n = \begin{bmatrix} q^1 & q^2 & \ldots & q^n \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ -\partial_{n-1} \end{bmatrix} \\
&\vdots \\
Aq^n &= q^{n-1} - \partial_1 q^n = \begin{bmatrix} q^1 & q^2 & \ldots & q^n \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ -\partial_1 \end{bmatrix}
\end{aligned}\right\} \tag{2.3}$$

(2.3) can be rewritten as

$$A\begin{bmatrix} q^1 & q^2 & \dots & q^n \end{bmatrix} = \begin{bmatrix} q^1 & q^2 & \dots & q^n \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ -\partial_n & -\partial_{n-1} & & \dots & -\partial_1 \end{bmatrix} \tag{2.4}$$

and

$$b \triangleq q^n = \begin{bmatrix} q^1 & \dots & q^n \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}$$

Let $p^{-1} \triangleq \begin{bmatrix} q^1 & q^2 & \dots & q^n \end{bmatrix}$, and let $\bar{x} = PX$, then the dynamical equation (2.1) can be transformed into

$$\left.\begin{aligned} \dot{\bar{x}} &= PAP^{-1}\bar{x} + Pbu \\ &= \bar{A}\bar{x} + \bar{b}u \end{aligned}\right\} \tag{2.5}$$

where

$$\left.\begin{aligned} \bar{A} &= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & & \\ -\partial_n & & \dots & & -\partial_1 \end{bmatrix}, \\ \bar{b} &= \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix} \end{aligned}\right. \tag{2.6}$$

Now introducing state feedback

$$U = V + Kx = \nu + Kp^{-1}\bar{x} = \nu + \bar{K}\bar{x} \tag{2.7}$$

where

$$\bar{K} = Kp^{-1} \tag{2.8}$$

Since the eigenvalues are invariant under equivalence transformation, the set of eigenvalues of $(A + bK)$ is equal to the set of the eigenvalues of $(\bar{A} + \bar{b}\bar{K})$. Let the characteristic polynomial of the matrix

$(\bar{A} + \bar{b}\bar{K})$ with desired eigenvalues be

$$S^n + a_1S^{n-1} + \ldots + a_n.$$

If $\bar{K}$ is chosen as

$$\bar{K} = (\partial_n - a_n, \partial_{n-1} - a_{n-1}, \ldots, \partial_1 - a_1) \qquad (2.9)$$

then the state feedback dynamical equation becomes

$$\dot{x} = \begin{bmatrix} 0 & 1 \,.. & 0 \\ 0 & 01.. & 0 \\ \vdots & & \\ -a_n & \ldots & -a_1 \end{bmatrix} \bar{x} + \begin{bmatrix} 0 \\ . \\ \vdots \\ 1 \end{bmatrix} \nu \qquad (2.10)$$

Since the characteristic polynomial of the A matrix in (2.10) is $S^n + a_1S^{n-1} + \ldots + a_n$, it follows that the state feedback equation (2.10) has the desired eigenvalues.

Note that if any of the states are not available for feedback, an observer is needed. Therefore, a new pole placement method, which can apply both complete or incomplete state feedback will be derived. This is achieved by introducing a non-singular matrix T to the closed-loop system equation

$$\dot{x} = (\bar{A} + \bar{b}\bar{K})\bar{x} + \bar{b}V \qquad (2.11)$$

yielding

$$\dot{x} = T^{-1}(T\bar{A} + T\bar{b}\bar{K})\bar{x} + \bar{b}V \qquad (2.12)$$

A complete state feedback matrix $\bar{K}$ can be found so that

$$T\bar{A} + T\bar{b}\bar{K} = A_c \qquad (2.13)$$

where

$$A_c = \left(\begin{array}{c|c} A_{c1} & 0 \\ \hline 0 & A_{c2} \end{array}\right), \qquad (2.14)$$

A_{c1} and A_{c2} are companion matrices with desired eigenvalues λ_1, λ_2, ..., λ_r, and λ_{r+1}, ..., λ_n, respectively. Note that A_c is chosen as (2.14) because that form lends itself to treating an incomplete state feedback. Most important, the form lends itself to treating the placement of the selected sub-set of eigenvalues as in the case of dominant eigenvalue selection.

2.2 THEORY DEVELOPMENT

Let T_c be a linear operator, T_c: V → V over the complex field C^n. Consider V be decomposed into 2 subspaces V_1 and V_2 such that $V = V_1 \oplus V_2$. Now let T_c be the direct sum of operator T_{c1} and T_{c2} such that V_1 and V_2 are invariant under T_c. That is $T_{c1}V_1 \rightarrow V_1$, $T_{c2}V_2 \rightarrow V_2$. From linear algebra theory ([35], p. 159), it may be readily shown that the matrix analogue to T_c is the block diagonal matrix A_c, where

$$A_c = \left[\begin{array}{c|c} A_{c1} & 0 \\ \hline 0 & A_{c2} \end{array}\right],$$

and where A_{c1} and A_{c2} are r x r and (n-r) x (n-r) matrices respectively. Since T_c is the direct sum of T_{c1} and T_{c2}. Then the characteristic polynomial of T_c is the product of characteristic polynomial of T_{c1} and T_{c2} [35], or $\det(SI - A_{c1}) \cdot \det(SI - A_{c2}) = \det(SI - A_c)$
where

$$\det(SI - A_{c1}) = S^r + \partial_1 S^{r-1} + \ldots + \partial_r, \tag{2.15}$$

$$\det(SI - A_{c2}) = S^{n-r} + \beta_1 S^{n-r-1} + \ldots + \beta_{n-r}, \tag{2.16}$$

and

$$\det(SI - A_c) = S^n + a_1 S^{n-1} + \ldots + a_n. \tag{2.17}$$

If A_c has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then the decomposition yields a subset of the λ_i's to A_{c1} and the remainder to A_{c2}. Note that each set must retain pair-wise complex conjugate eigenvalues. For convenience, A_{c1} and A_{c2} may be put into companion form. Note that the assigned set of eigenvalues which determined the characteristic polynomial of A_{c1} and A_{c2} directly relate to the elements of A_{c1} and A_{c2}, if A_{c1} and A_{c2} are represented in companion form, or

$$A_{c1} = \begin{bmatrix} 0 & 1 & 0 \ldots & 0 \\ 0 & 0 & 1 \ldots & 0 \\ \vdots & & & \\ -\partial_r & & \ldots\ldots & -\partial_1 \end{bmatrix} \text{ and }$$

$$A_{c2} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & & & & \\ -\beta_{n-r} & & & \ldots & -\beta_1 \end{bmatrix}. \tag{2.18}$$

Now let the matrix A_f be an n x n matrix over field C^n such that A_f is in companion form with characteristic polynomial identical to A_c,

or

$$A_f = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & & & & \\ -a_n & -a_{n-1} & & \ldots & -a_1 \end{bmatrix} \tag{2.20}$$

Since both matrices A_f and A_c have the same characteristic polynomial then they are similar. ([35], p. 165).

Theorem 2.1

If A_c is a quasi-diagonal matrix over the field C^n, and A_f is a companion form matrix over the field C^n, and $\det(SI - A_c) = \det(SI - A_f)$, then there exists a non-singular T over C^n such that $A_c = TA_f$, where all the eigenvalues of A_c (or A_f) must be non-zero.

proof:

If T exists, then $T = A_c A_f^{-1}$, which requires that A_f be non-singular. Thus it is necessary that A_f must possess non-zero eigenvalues.

Since A_c and A_f are similar, then $\det(A_c) = \det(A_f)$. Therefore, $\det(T) = \det(A_c A_f^{-1}) = \det(A_c) \det(A_c^{-1}) = 1$.

Hence T exists. T is non-singular.

2.3 TRANSFORM ALGORITHM:

Since $T = A_c A_f^{-1}$, where A_c and A_f have the form as shown in (2.14) and (2.20) respectively. By [39], A_f^{-1} can be easily obtained by inspection as shown below:

$$A_f^{-1} = \begin{bmatrix} \frac{-a_{n-1}}{a_n} & \frac{-a_{n-2}}{a_n} & \cdots & \frac{-a_1}{a_n} & \frac{-1}{a_n} \\ 1 & 0 & & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & & 1 & 0 \end{bmatrix} \qquad (2.21)$$

Therefore,

$$T = \left[\begin{array}{ccccc|cccc} 0 & 1 & 0 & \cdots & 0 & & & & \\ 0 & 0 & 1 & \cdots & 0 & & & & \\ \cdot & \cdot & \cdot & \cdots & & & & & \\ -\partial_r & -\partial_{r-1} & & \cdots & -\partial_1 & & & & \\ \hline & & & & & 0 & 1 & & \cdots\ 0 \\ & & & & & 0 & 0 & 1 & \cdots\ 0 \\ & & & & & \cdot & & & \\ & & & & & \cdot & & & \\ & & & & & \cdot & & & \\ & & & & & -\beta_{n-r} & & & \cdots\ \beta_1 \end{array}\right] \left[\begin{array}{cccc} \frac{-a_{n-1}}{a_n} & \frac{-a_{n-1}}{a_n} & \cdots \frac{-a_1}{a_n} & \frac{-1}{a_n} \\ 1 & 0 & \cdots\ 0 & 0 \\ 0 & 1 & \cdots\ 0 & 0 \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ 0 & & \cdots\ 1 & 0 \end{array}\right]$$

(2.22)

$$\begin{array}{c|ccc|c|c|ccccc|} & & & & \begin{array}{c}r\text{-}1\text{th}\\ \text{column}\end{array} & \begin{array}{c}r\text{th}\\ \text{column}\end{array} & & & & & \\ = & 1 & 0 & 0 & & & & & & & 0 \\ & 0 & 1 & 0 & & & & & & & 0 \\ & \cdot & & & & & & & & & \\ & \cdot & & & & & & & & & \\ & 0 & 0 & & 1 & 0 & & & & & 0 \\ \hline \begin{array}{c}r\text{th}\\ \text{row}\end{array} & \frac{a_{n-1}}{a_n}\partial_r-\alpha_{r-1} & \frac{a_{n-2}}{a_n}\partial_r-\alpha_{r-2} & \cdots\cdot & \frac{a_{n-r-1}}{a_n}\partial_r-\alpha_1 & \frac{a_{n-r}}{a_n}\partial_r & & \cdots\cdot & & \frac{a_1}{a_n}\partial_r & \frac{\alpha_r}{a_n} \\ \hline & & & & & 0 & 1 & & \cdots & 0 & 0 \\ & & & & & 0 & 0 & 1 & & 0 & 0 \\ & & & & & \cdot & & & & & \\ & & & & & \cdot & & & & & \\ & & & & & \cdot & & & & & \\ & & & & & \cdot & & & & & \\ & & & & & \cdot & 0 & & .. & 1 & 0 \\ & & & & & -\beta_{n-r} & -\beta_{n-r-1} & & \cdot & -\beta_1 & 0 \end{array}$$

The determinant of the matrix T is

$$\det(T) = (-1)^{n+r}\left(\frac{\partial_r}{a_n}\right) \begin{array}{c} \begin{array}{ccccccccc} & & & & n-1 & & & & \end{array} \\ \left(\begin{array}{ccccccccc} 1 & 0 & \cdot & \cdot & \cdot & \cdot & \cdots & & 0 \\ 0 & 1 & \cdot & \cdot & \cdot & \cdot & & & 0 \\ \cdot & \cdot & & & & & & & \\ \cdot & \cdot & & & & & & & \\ 0 & \cdots & 1 & 0 & \cdot & \cdot & \cdots & & 0 \\ 0 & \cdots & 0 & 0 & 1 & \cdot & \cdots & & 0 \\ \cdot & & \cdot & 0 & 0 & 1 & \cdots & & 0 \\ \cdot & & \cdot & \cdot & \cdot & \cdot & & & \cdot \\ \cdot & & \cdot & \cdot & \cdot & \cdot & & & \cdot \\ \cdot & & \cdot & \cdot & \cdot & \cdot & & & \cdot \\ 0 & & 0 & -\beta_{n-r} & -\beta_{n-r-1} & & \cdots & -\beta_1 & 0 \end{array}\right) \end{array}$$

$$= (-1)^{(n+r)}(-1)^{(n+r-1)}\left(\frac{\partial_r}{a_n}\right)(-1)(\beta_{n-r}) \left(\begin{array}{cccccc} 1 & 0 & \cdots & & & 0 \\ 0 & 1 & \cdots & & & 0 \\ 0 & 0 & 1 & \cdots & & 0 \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ 0 & \cdot & & \cdots & & 1 \end{array}\right)$$

$$= (-1)^{2(n+r)}\frac{(\partial_r)(\beta_{n-r})}{a_n}$$

$$= 1. \qquad (2.22a)$$

Thus, it has been verified that the determinant of T is equal to 1.

Let $\partial_0 = \beta_0 = 1$ and $a_0 = 1$. The T matrix can be expressed as:

$$
Tij = \begin{cases}
\delta ij, & (i = r,\ 1 \le i \le n-1,\ 1 \le j \le n). \\
\frac{a_{n-j}}{a_n}\partial_r - \partial_{r-j}, & (i = r,\ j = 1,\ \ldots,\ r). \\
\frac{a_{n-j}}{a_n}\partial_r, & (i = r,\ j = r+1,\ \ldots,\ n). \\
0 & (i = n,\ j = 1,\ \ldots,\ r-1) \\
-\beta_{n-j}, & (i = n,\ j = r,\ \ldots,\ n).
\end{cases} \tag{2.23}
$$

The inverse of the matrix T is:

$$
T^{-1} = (A_c A_f^{-1})^{-1} = A_f A_c^{-1}
$$

$$
= \begin{pmatrix}
0 & 1 & \ldots & 0 \\
0 & 0 & 1 \ldots & 0 \\
\cdot & & & \\
\cdot & & & \\
\cdot & & & \\
-a_n & & \ldots & -a_1
\end{pmatrix}
\left(\begin{array}{cccc:cccc}
\frac{-\partial_{r-1}}{\partial_r} & \frac{-\partial_{r-1}}{\partial_r} & \ldots \frac{-\partial_1}{\partial_r} & \frac{-1}{\partial_r} & & & & \\
1 & 0 & 0 & 0 & & & & \\
0 & 1 & 0 & 0 & & & & \\
\cdot & & & & & & & \\
\cdot & & & & & & & \\
0 & 0 & 1 & 0 & & & & \\
\hdashline
 & & & & \frac{-\beta_{n-r-1}}{\beta_{n-r}} & \ldots & \frac{-\beta_1}{\beta_{n-r}} & \frac{-1}{\beta_{n-r}} \\
 & & & & 1 & 0 & \ldots & 0 \\
 & & & & 0 & 1 & \ldots & \cdot \\
 & & & & \cdot & \cdot & & \\
 & & & & 0 & \ldots & 1 & 0
\end{array}\right) \tag{2.24}
$$

$$
= \begin{array}{r|cccccc:c:cccc|}
 & & & & & & & \text{rth column} & & & & \\
 & 1 & . & 0 & \ldots & 0 & & & & & & \\
 & 0 & . & 1 & \ldots & 0 & & & & & & \\
 & \vdots & & & & & & & & & & \\
 & 0 & . & . & \ldots & 1 & & & & & & \\
\text{rth row} & 0 & . & 0. & \ldots & 0, & 0 & \frac{-\beta_{n-r-1}}{\beta_{n-r}} & \frac{-\beta_{n-r-2}}{\beta_{n-r}} & \ldots & \frac{-\beta_1}{\beta_{n-r}} & \frac{-1}{\beta_{n-r}} \\
\hdashline
 & & & & & & & 1 & 0\ . & \ldots & 0\ . & 0 \\
 & & & & & & & 0 & 1\ . & \ldots & 0\ . & 0 \\
 & & & & & & & 0 & 0 & 1 \ldots & 0\ . & 0 \\
\text{nth row} & \multicolumn{6}{c:}{a_n\frac{\partial_{r-1}}{\partial_r} - a_{n-1} \ \ldots + \frac{a_n\partial_1}{\partial_r} - a_{n-r+1}} & \frac{a_n}{\partial_r} & \frac{a_{n-r}\beta_{n-r-1}}{\beta_{n-r}} - a_{n-r-1} & \ldots & \frac{a_{n-r}\beta_1}{\beta_{n-r}} - a_1 & \frac{a_{n-r}}{\beta_{n-r}}
\end{array}
$$

Let $\beta_0 = 1$. The T^{-1} can be written as:

$$T_{ij}^{-1} = \begin{cases} \delta ij, & (i \neq r,\ i = 1, \dots n-1,\ j = 1, \dots n). \\ 0, & (i = r,\ j = r). \\ \dfrac{-\beta_{n-j}}{\beta_{n-r}}, & (i = r,\ j = r+1, \dots, n). \\ \dfrac{a_n \partial_{r-j}}{\partial_r} - a_{n-j}, & (i = n,\ j = 1, \dots, r-1). \\ \dfrac{a_n}{\partial_r}, & (i = n,\ j = r). \\ \dfrac{a_{n-r}}{\beta_{n-r}} \beta_{n-j} - a_{n-j}, & (i = n,\ j = r+1, \dots, n-1). \\ \dfrac{a_n}{\beta_{n-r}}, & (i = n,\ j = n). \end{cases} \qquad (2.25)$$

2.4 FEEDBACK MATRIX DETERMINATION:

Consider now the system equation with state feedback

$$\dot{x} = (A + bK)x + bV \qquad (2.26)$$

where A, b have the form

$$A = \begin{pmatrix} 0 & 1 & 0 & .. & 0 \\ 0 & 0 & 1 & .. & 0 \\ . & & & & . \\ . & & & & . \\ 0 & . & . & . & 1 \\ -q_n & -q_{n-1} & & \cdots & -q_1 \end{pmatrix},$$

$$b = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \qquad (2.27)$$

Without changing the system, T may be introduced so that

$$\dot{x} = T^{-1}(TA + TbK)x + bV \tag{2.28}$$

The problem now is to find the K matrix so that

$$TA + TbK = A_c$$

where A_c has $\lambda_1, \lambda_2, \ldots \lambda_n$ as its eigenvalues, let

$$\begin{aligned} \det(SI - A_{c1}) &= (S - \lambda_1) \ldots (S - \lambda_r) \\ &= S^r + \partial_1 S^{r-1} + \ldots + \partial_r, \end{aligned} \tag{2.29}$$

Let the remaining eigenvalues be denoted $\lambda_{r+1}, \ldots \lambda_n$.

Let

$$\begin{aligned} \det(SI - A_{c2}) &= (S - \lambda_{r+1}) \ldots (S - \lambda_n) \\ &= S^{n-r} + \beta_1 S^{n-r-1} + \ldots + \beta_{n-r}, \end{aligned} \tag{2.30}$$

Let

$$\begin{aligned} \det(SI - A_f) &= \det(SI - A_{c1}) \cdot \det(SI - A_{c2}) \\ &= S^n + a_1 S^{n-1} + \ldots + a_n. \end{aligned} \tag{2.31}$$

With the ∂_i's, β_i's and a_i's known, the matrices A_f and A_c can be found as in (2.18) and (2.20). Hence the T matrix can be generated as in (2.22). Substitute this T matrix into (2.28) and solving for K, then

$$\left(\begin{array}{ccccc:c:cccc}
1 & . & 0 & . & 0 & . & & & & \\
0 & . & 1 & . & 0 & . & & & & \\
. & . & . & . & . & & & & & \\ \hdashline
\mathrm{Tr.1} & \mathrm{Tr.2} & \ldots & & & \mathrm{Tr.r} & \ldots\ldots & \mathrm{Tr.n{-}1} & & \frac{\partial_r}{a_n} \\ \hdashline
& & & & & & 1 & 0 & \ldots & 0 \\
& & & & & & 0 & 1 & \ldots & 0 \\
& & & & & & \vdots & & & \\
& & & & & -\beta_{n-r} & -\beta_{n-r-1} & \cdot\cdot & -\beta_1 & 0
\end{array}\right)
\left(\begin{array}{cccc}
0 & 1 & 0 & \cdot\cdot 0 \\
0 & 0 & 1 & \ldots \\
. & & & \\
. & & & \\
. & & & \\
. & & & \\
-q_n & & \ldots\ldots & -q_1
\end{array}\right)$$

$$+\left(\begin{array}{ccc:c:ccc}
1 & 0 & & 0 & & & \\
0 & 1 & & 0 & & & \\
& & & & & & \\ \hdashline
\mathrm{Tr.1} & \mathrm{Tr.2} & \ldots & \mathrm{Tr.r} & & \mathrm{Tr.n{-}1} & \frac{\partial_r}{a_n} \\ \hdashline
& & & & & & \\
& & & & & & \\
& & & -\beta_{n-r} & -\beta_{n-r-1} & -\beta_1 & 0
\end{array}\right)
\left(\begin{array}{c}
0 \\ . \\ . \\ . \\ . \\ 0 \\ 1
\end{array}\right)
[k_1 \;\ldots\ldots\; k_n]$$

$$= \left(\begin{array}{cccc:cccc}
0 & 1 & & \ldots 0 & & & & \\
0 & 0 & 1 & \cdot\cdot 0 & & & & \\
\vdots & & & & & & & \\
-\partial_r & -\partial_{r-1} & \cdot\cdot & -\partial_1 & & & & \\ \hdashline
& & & & 0 & 1 & 0 & \cdot\cdot\ 0 \\
& & & & 0 & 0 & 1 & \cdot\cdot\ 0 \\
& & & & \vdots & & & \\
& & & & -\beta_{n-r} & -\beta_{n-r-1} & \cdot\cdot & -\beta_1
\end{array}\right) \qquad (2.32)$$

After multiplication, the left hand side of (2.32) yields

$$\left[\begin{array}{ccccccc}
0 \;.\; 1 \ldots\ldots 0 & & & & & & \\
0 \;.\; 0 \;\; 1 \ldots 0 & & & & & & \\
\vdots \qquad\quad 0 & & & & & & \\
\cdot \qquad\quad \cdot & & & & & & \\
\cdot \qquad\quad \cdot & & & & & & \\
\qquad\qquad 1 & & & & & & \\
\hdashline
Wr.1 & \ldots & Wr.r & \ldots., & Wr.n-1 & & \frac{-\partial_r q_1}{a_n} \\
\hdashline
 & & 0 & 0 \;.\; 1 & \ldots & & 0 \\
 & & . & 0 \quad 0 \quad 1 & \ldots & & \\
 & & . & & & & \\
 & & \cdot & -\beta_{n-r} & \cdots & & -\beta_1
\end{array}\right]$$

$$+ \left[\begin{array}{ccccc}
 & & & & \\
\hdashline
\frac{\partial_r}{a_n}k_1 & \frac{\partial_r}{a_n}k_2 & \cdot \quad \cdot \quad \cdot & & \frac{\partial_r}{a_n}k_n \\
\hdashline
 & & & &
\end{array}\right] \tag{2.33}$$

Where

$$Wr.j = Tr.j-1 - \frac{\partial_r}{a_n} q_{n-j+1} \quad (Tr.0 = 0,\ j = 1,\ \ldots,\ n) \tag{2.34}$$

Comparing the rth row of both sides of (2.32) yields

$$Wr.j + \frac{\partial_r}{a_n}k_j = \begin{cases} -\partial_{r+1-j} & (j = 1, \ldots r) \\ 0 & (j = r+1, \ldots n) \end{cases} \tag{2.35}$$

Next solve (2.35) for k_j.

$$k_j = \begin{cases} (-\partial_{r+1-j} - Wr.j)\ \frac{a_n}{\partial_r}, & (j = 1, \ldots, r) \\ -Wr.j\ \frac{a_n}{\partial_r}, & (j = r+1, \ldots n) \end{cases} \tag{2.36}$$

With the k_j chosen as (2.36), the desired relation $TA + TbK = A_c$ is met. Now (2.27) can be transformed to its original form as

$$\dot{x} = T^{-1}A_c x + bV = A_f x + bV \tag{2.37}$$

Since A_f is similar to A_c, then the desired closed loop eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ are still maintained.

The above procedure of finding the feedback matrix K can be summarized as follows:

(1) Use (2.29), (2.30) and (2.31) to define the matrices A_c and A_f.

(2) Use (2.22) to generate the matrix T.

(3) Use (2.34) to find each Wr.j.

(4) Find every k_j as shown in (2.36).

(5) $k = [k_1, \ldots, k_n]$ is the desired feedback matrix.

2.5 INCOMPLETE STATE FEEDBACK:

In the above section, a complete state feedback is applied to place n poles at the desired locations. However, in practice, it is not necessary to preassign all the poles. In fact, only some of the n poles need to be placed at the desired locations. In doing so, not all the states are needed in feedback. Generally speaking, to place an r number of poles only a r number of available states is needed in feedback, provided certain conditions hold. In this section, an incomplete state feedback matrix k and the conditions under which incomplete state feedback can be applied will be found.

Consider a linear time-invariant controllable system described by

$$\dot{x} = Ax + bU \tag{2.38}$$

and an r number of poles to be assigned, $\lambda_1, \ldots, \lambda_r$. It is desired to find an incomplete state feedback matrix k such that with feedback from an r number of measurable states (i.e. U = Kx + V, where only first r elements out of n of the K vector have non-zero values) so that the closed loop system

$$\dot{x} = (A + bK)x + bV \tag{2.39}$$

has at least $\lambda_1, \lambda_2, \ldots, \lambda_r$ as its eigenvalues.

The following assumptions are needed before the solution procedure can be found:

<u>Assumption 1</u>: The number of measurable states is greater than or equal to the number r.

<u>Assumption 2</u>: All the states associated with A_{c1} of (2.14) are measurable.

From the known eigenvalues $\lambda_1, \ldots, \lambda_r$, form the characteristic polynomial as in (2.15):

$$\begin{aligned} \det(SI - A_{c1}) &= (S - \lambda_1) \ldots (S - \lambda_r) \\ &= S^r + \partial_1 S^{r-1} + \ldots + \partial_r, \end{aligned} \tag{2.40}$$

and let

$$\det(SI - A_{c2}) = S^{n-r} + \beta_1 S^{n-r-1} + \ldots + \beta_{n-r}, \tag{2.41}$$

denote the characteristic polynomial of A_{c2}, where β_i's are unknown constants. Next let $S^n + a_1 S^{n-1} + \ldots + a_n$ denote the characteristic polynomial of A_f. Then one obtains the following relation:

$$\begin{aligned} &S^n + a_1 S^{n-1} + \ldots \\ &+ a_n = (S^r + \partial_1 S^{r-1} + \ldots + \partial_r) \times (S^{n-r} + \beta_1 S^{n-r-1} + \beta_{n-r}) \end{aligned} \tag{2.42}$$

Compare the coefficients of the same power of S yields

$$\begin{aligned} \partial_1 + \beta_1 &= a_1 \\ \beta_2 + \partial_1 \beta_1 + \partial_2 &= a_2 \end{aligned} \tag{2.43}$$

(2.43) can be rewritten as:

$$\left(\begin{array}{c} 1 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_r \\ a_{r+1} \\ \vdots \\ \\ a_{n-r} \\ \hline a_{n-r+1} \\ \vdots \\ \\ \\ a_{n-1} \\ a_n \end{array}\right) = \left(\begin{array}{ccccccccc|ccccc} 1 & 0 & 0 & \cdots & & & & & 0 & 0 & \cdots & 0 & & \\ \partial_1 & 1 & 0 & \cdots & & & & & & & & & & \\ \partial_2 & \partial_1 & 1 & 0 & \cdots & & & & & & & & & \\ \partial_3 & \partial_2 & \partial_1 & 1 & 0 & \cdots & & & & & & & & \\ \vdots & & & & & & & & & & & & & \\ \partial_r & \partial_{r-1} & \cdots & & 1 & 0 & & & & & & & & \\ 0 & \partial_r & \cdots & & & 1 & 0 & & & & & & & \\ \vdots & & & & & & & & 0 & 0 & & & & \\ 0 & \cdots & & & 0 & \partial_r & \partial_{r-1} & \partial_1 & 1 & 0 & & & & \\ \hline 0 & \cdots & & & & 0 & \partial_r & \cdots & \partial_1 & 1 & 0 & 0 & \cdots & 0 \\ \vdots & & & & & & & & & & 1 & 0 & \cdots & 0 \\ & & & & & & & & & & & 1 & \cdots & 0 \\ \vdots & & & & & & & & & & & & & \\ 0 & \cdots & & & & & & \partial_r & \partial_{r-1} & \cdots & & & 1 & \\ 0 & \cdots & & & & 0 & 0 & & \partial_r & \cdots & & & & 1 \end{array}\right) \left(\begin{array}{c} 1 \\ \beta_1 \\ \beta_2 \\ \vdots \\ \\ \\ \\ \\ \\ \beta_{n-r} \\ \hline 0 \\ \vdots \\ \\ \\ \\ 0 \end{array}\right) \tag{2.44}$$

Following the same procedure as shown in section 2.4, the K matrix can be found as in (2.36). However, because only the first r states are available in feedback, it follows that

$$K_{r+1} = K_{r+2} = \ldots = K_n = 0 \tag{2.45}$$

By (2.36), that is the same as

$$Wr.j = 0. \quad (j = r+1, \ldots, n) \tag{2.46}$$

the above n-r conditions serve as the sufficient conditions that the placement of r poles using r states for feedback is possible. Therefore, one has the following theorem:

Theorem 2.2:

For a completely controllable system described by (2.38) together with r eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_r$, the sufficient conditions for placement of r poles using r states for feedback is

$$a_i = q_i. \quad (i = 1, \ldots, n-r) \tag{2.47}$$

proof:

It is shown above that the sufficient condition for placement of r poles using r states for feedback is

$$Wr.j = 0; \quad (j = r + 1, \ldots, n)$$

But from (2.34) and (2.25)

$$Wr.j = Tr.j-1 - \frac{\partial_r}{a_n} q_{n-j+1}$$

$$Tr.j = \frac{a_{n-j}}{a_n} \partial_r \tag{2.48}$$

Hence $Wr.j = 0$ implies $Tr.j-1 = \frac{\partial_r}{a_n} q_{n-j+1}$ for $j = r + 1, \ldots, n$. Which is the same as

$$\frac{a_{n-(j-1)}}{a_n} \partial_r = \frac{\partial_r}{a_n} q_{n-j+1} \qquad (j = r + 1, \ldots, n)$$

Therefore

$$a_{n-j+1} = q_{n-j+1} \qquad (j = r + 1, \ldots, n)$$

$$a_i = q_i \qquad (i = 1, \ldots, n-r) \quad \text{Q.E.D.}$$

Equation (2.44) can be rewritten as:

$$\begin{bmatrix} a^1 \\ \hline a^2 \end{bmatrix} \quad \left[\begin{array}{c|c} C_{11} & 0 \\ \hline C_{21} & C_{22} \end{array}\right] \quad \begin{bmatrix} \beta \\ \hline 0 \end{bmatrix} \tag{2.49}$$

Where

$$a^1 = (1, a_1, \ldots, a_{n-r})^T,$$

$$a^2 = (a_{n-r+1}, \ldots, a_n)^T,$$

$$\beta = (1, \beta_1, \ldots, \beta_{n-r})^T. \qquad (2.50)$$

$$C_{11} = \begin{bmatrix} 1 & 0 & . & . & . & . & . & . & 0 \\ \partial_1 & 1 & 0 & . & . & . & . & . & 0 \\ \partial_2 & \partial_1 & 1 & . & 0 & . & . & . & 0 \\ . & & & & & & & & \\ . & & & & & & & & \\ . & & & & & & & & \\ 0 & . & . & . & \partial_r & . & . & . & 1 \end{bmatrix} \qquad (2.51)$$

C_{11} is an (n-r+1) x (n-r+1) matrix,

C_{22} is an r x r identity matrix, and

C_{21} is an r x (n-r+1) matrix with forms

$$C_{21} = \begin{bmatrix} 0 & . & . & . & . & 0 & \partial_r & & & & \partial_1 \\ 0 & . & . & . & . & 0 & 0 & \partial_r & . & . & \partial_2 \\ . & & & & & & & & . & & \\ . & & & & & & & & . & & \\ 0 & . & . & . & . & & & 0 & \partial_r & & \partial_{r-1} \\ 0 & . & . & . & . & & & 0 & 0 & & \partial_r \end{bmatrix} \qquad (2.52)$$

Equation (2.49) can be rewritten as

$$a^1 = C_{11}\beta \qquad (2.53)$$

$$a^2 = C_{21}\beta \qquad (2.54)$$

Using the following relation

$$\det\begin{pmatrix} M & 0 \\ Q & N \end{pmatrix} = \det(M) \cdot \det(N) \qquad (2.55)$$

repeatedly, it may be shown that

$$\det(C_{11}) = 1 \qquad (2.56)$$

Therefore, C_{11}^{-1} exists.

From eq. (2.53) and eq. (2.56)

$$\beta = C_{11}^{-1} a^{1} \tag{2.57}$$

(2.57) shows how to find β's coefficients from the known values of ∂'s and a_i's ($i = 1, \ldots, n-r$).

The remaining r coefficients of a's can be found as follows:

$$a^{2} = C_{21} C_{11}^{-1} \beta \tag{2.58}$$

The above procedure of finding an incomplete state feedback matrix K is summarized in the following:

(1) Use (2.29), (2.30) and (2.31) to define the matrices A_c and A_f.

(2) Use (2.57) to solve for β_i, ($i = 1, \ldots, n-r$) with the aid of (2.47).

(3) Use (2.22) to generate the matrix T.

(4) Find each k_j, ($j = 1, \ldots, r$), with the aid of (2.36)

(5) $K = [k_1, \ldots, k_r, 0, 0, \ldots 0]$ is the state feedback vector.

It is possible that, using an r number of measurable states to assign an r number of poles, the closed loop system might become unstable because the other n-r poles have to meet the n-r conditions defined by (2.47). Therefore, a criterion to determine the closed loop stability has to be developed.

The characteristic polynomial of the matrix A_f in (2.20) is $D(S) = S^n + a_1 S^{n-1} + \ldots + a_n$. Form the so-called Hurwitz matrix

$$H \triangleq \begin{bmatrix} a_1 & a_3 & a_5 \cdots & a_n & 0 \cdots\cdots 0 \\ 1 & a_2 & a_4 \cdots & a_{n-1} & 0 \cdots\cdots 0 \\ 0 & a_1 & a_3 \cdots & \cdot \quad a_n & 0 \cdots 0 \\ 0 & 1 & a_2 \quad a_4 & \cdots\cdots & 0 \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ 0 & \cdots\cdots & a_1 & & a_3 \cdots\cdots a_n \end{bmatrix} \tag{2.59}$$

Note that the elements of the diagonal of H are a_1, a_2, ... a_n. By ([36], Vol. 2, p. 221; [34] p. 327), it follows that the polynomial D(S) is a Hurwitz polynomial if the following leading minors of H

$$\begin{aligned} \Delta_1 &= a_1 \\ \Delta_2 &= \det\begin{bmatrix} a_1 & a_3 \\ 1 & a_2 \end{bmatrix} \\ \Delta_3 &= \det\begin{bmatrix} a_1 & a_3 & a_5 \\ 1 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{bmatrix} \\ &\;\;\vdots \\ \Delta_n &= \det H \end{aligned} \tag{2.60}$$

are all positive.

Now, define

$$Q(S) \triangleq S^{n-r} + q_1 S^{n-r-1} + \ldots + q_{n-r} \tag{2.61}$$

$$D_1(S) \triangleq S^{n-r} + a_1 S^{n-r-1} + \ldots + a_{n-r} \tag{2.62}$$

Similarly, $D_1(S)$ is a Hurwitz polynomial if the following determinants

$$\Delta_1 = a_1$$

$$\Delta_2 = \det\begin{bmatrix} a_1 & a_3 \\ 1 & a_2 \end{bmatrix}$$

$$\cdot$$

$$\Delta_{n-r} = \det\begin{bmatrix} a_1 & a_3 & \cdots & a_{n-r} & 0 \ldots 0 \\ 1 & a_2 & \cdots & a_{n-r-1} & 0 \ldots 0 \\ 0 & & & & \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ 0 & \cdots & & a_1 & \cdots \; a_{n-r} \end{bmatrix} \qquad (2.63)$$

are all positive.

Therefore, if $D_1(S)$, or which is identical to $Q(S)$, is a Hurwitz polynomial, then the first n-r determinants of H defined as (2.63) are all positive. However, the remaining r coefficients of a_i's (i.e. $a_{n-r+1}, \ldots, a_n$) are determined by the r eigenvalues to be assigned. But those r poles can be chosen at any value. Therefore, all the r poles can be appropriately chosen to have negative real parts such that the $a_{n-r+1}, \ldots, a_n$ together with $a_1, \ldots, a_{n-r}$ forms a Hurwitz polynomial as in (2.20), then the closed loop system is stable.

If the assumption 2 does not hold, one needs the following assumption:

Assumption 2a: The number of pairs of complex conjugate poles to be assigned is less than or equal to the number of pairs of adjoining measurable states.

If the above assumption holds, one can still handle the case where some of the states in A_{c1} are not measurable and some of the states in A_{c2} are measurable.

Consider a linear time-invariant controllable system described by

$$\dot{x} = Ax + bU \tag{2.64}$$

where $x = (x_1, \ldots, xm_1, xm_{1+1}, \ldots, xm_2, xm_{2+1}, \ldots, xm_3)^T$, $m_1 + m_2 + m_3 = n$. $m_1 < r$ and $m_1 + m_3 \geq r$. Suppose the first m_1 states are measurable, the next m_2 states are not measurable, and the last m_3 states are again measurable. A state feedback matrix K will be found so that with state feedback from the first m_1 states and the last m_3 states, (i.e. $U = KX + V$, where $K = [k_1, \ldots, km_1, 0, \ldots 0, km_{2+1}, \ldots, km_3]$) the closed loop system

$$\dot{x} = (A + bK)x + bV \tag{2.65}$$

has $\lambda_1, \lambda_2, \ldots, \lambda_r$ as its eigenvalues.

First select an m_1 number of eigenvalues out of the r desired eigenvalues, denoted $\lambda_1, \lambda_2, \ldots, \lambda_{m_1}$. Form the matrices A_{c1}, A_{c2} and A_f as in (2.18) and (2.20) respectively. Then the T matrix can be found as in (2.23) such that

$$TA_f = A_c = \left[\begin{array}{c|c} A_{c1} & 0 \\ \hline 0 & A_{c2} \end{array}\right]$$

where A_{c1} is an $m_1 \times m_1$ matrix and all the states in A_{c1} are measurable. Another T_1 matrix is found so that

$$T_1 A_{c2} = \left[\begin{array}{c|c} A_{c21} & 0 \\ \hline 0 & A_{c22} \end{array}\right] \tag{2.66}$$

where A_{c22} is an $(r-m_1) \times (r-m_1)$ matrix. A_{c21} is an $(n-r) \times (n-r)$ matrix, and all the states in A_{c22} are measurable. Now define a n x n matrix P

$$P \triangleq \left[\begin{array}{c|c} I & 0 \\ \hline 0 & T_1 \end{array}\right],$$

where I is an m_1 x m_1 identity matrix, and T_1 is an $(n-m_1)$ x $(n-m_1)$ matrix defined in (2.66), then

$$P^{-1} = \left[\begin{array}{c|c} I & 0 \\ \hline 0 & T_1^{-1} \end{array}\right].$$

Introduce matrices P and T into (2.65) yielding

$$\begin{aligned} \dot{x} &= T^{-1}(TA + TbK)x + bV \\ &= T^{-1}P^{-1}(PTA + PTbK)x + bV \end{aligned} \tag{2.67}$$

Therefore, the problem becomes to find a k matrix such that

$$PTA + PTbK = \left[\begin{array}{c|c|c} A_{c1} & 0 & 0 \\ \hline 0 & A_{c21} & 0 \\ \hline 0 & 0 & A_{c22} \end{array}\right] \tag{2.68}$$

where A_{c1} and A_{c22} has $(\lambda_1, \ldots \lambda m_1)$ and $(\lambda m_1+1, \ldots, \lambda_r)$ respectively as its eigenvalues.

The matrix K can be found by a procedure similar to that in section 2.5.

CHAPTER 3

MULTIVARIABLE SYSTEMS

3.0 INTRODUCTION

Instead of considering single input system, this chapter is devoted to the pole-assignment problem of linear multivariable systems. Anderson and Luenberger [1] were first to treat this problem. However, their method sometimes fails to assign pairs of complex conjugate poles. Wonham [2] suggests a way which introduces state feedback so that the resulting dynamical equation is controllable by a single component of input vector V, and then applies the result established for the single variable systems. However, Wonham's method requires complete state feedback. When any of the states are not available for feedback, an observer needs to be constructed.

Therefore, a new pole-assignment method which can apply complete or incomplete state feedback to assign pairs of complex conjugate eigenvalues needs to be considered.

In parallel with the structure of Chapter 2, algorithms are derived to assign all or some of the n poles of the closed loop systems by applying complete or incomplete state feedback. The sufficient condition for incomplete state feedback is found. A stability criterion is established from that condition.

3.1 COMPLETE STATE FEEDBACK

Consider a linear time-invariant system described by

$$\dot{x} = Ax + BU \tag{3.1}$$

where x is an n x 1 state vector.

U is an m x 1 control input.

A is an n x n system matrix.

B is an n x m input matrix

Since the system is assumed controllable, one could use the tranformation method in section 2.1 so that A and B have the forms:

$$A = \left[\begin{array}{cccccc|cccc|ccc|cccc}
0 & 1 & 0 & \cdot & \cdot & 0 \\
0 & 0 & 1 & \cdot & \cdot & 0 \\
\cdot \\
\cdot \\
\cdot \\
-e_r & \cdot & \cdot & \cdot & \cdot & -e_1 \\ \hline
-m_1 & & & & & 0 & 1 & 0 & & 0 \\
-m_2 & & & & & 0 & 0 & 1 & & 0 \\
\cdot & & & & & \cdot \\
\cdot & & & & & \cdot \\
\cdot & & & & & \cdot \\
-m_q & & & & & & -f_q & & & -f_1 \\ \hline
 & & & & & & & & & & \cdot \\
 & & & & & & & & & & & \cdot \\
 & & & & & & & & & & & & \cdot \\ \hline
-n_1 & & & & & & & & & & & & & 0 & 1 & & 0 \\
\cdot & & & & & & & & & & & & & 0 & 0 & 1 & 0 \\
\cdot & & & & & & & & & & & & & \cdot \\
\cdot & & & & & & & & & & & & & \cdot \\
\cdot & & & & & & & & & & & & & \cdot \\
-n_e & & & & & & & & & & & & & -h_e & \cdot\cdot\cdot & & -h_1
\end{array}\right]$$

$$B = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0\cdot\cdot & \vdots \\
\cdot & \cdot & 1 \\
\cdot & \cdot & \cdot \\
 & 1 & \cdot \\
\cdot & 0 & \cdot \\
0 & \cdot & \cdot \\
1 & 0 & 0
\end{bmatrix} \qquad (3.2)$$

For simplicity, the system described by (3.1) is assumed to have 2 inputs only. Under this assumption, A and B become

$$A = \left[\begin{array}{cccccc:cccccc} 0 & .. & 1 & . & . . . & 0 & & & & & & \\ 0 & .. & 0 & . & 1 & . . & & & & & & \\ . & & & & & & & & & & & \\ . & & & & & & & & & & & \\ -e_r & & . & . & . . & e_1 & & & & & & \\ \hdashline -m_1 & & & & & & 0 & 1 & . & . & . . & 0 \\ -m_2 & & & & & & 0 & 0 & 1 & . & . & \\ . & & & & & & . & & & & & \\ . & & & & & & . & & & & & \\ -m_q & & & & & & -f_q & . & . & . & . & -f_1 \end{array}\right] \quad \text{and}$$

(3.3)

$$B = \left[\begin{array}{cc} 0 & 0 \\ . & . \\ . & . \\ . & 1 \\ & . \\ 0 & . \\ 1 & 0 \end{array}\right] \leftarrow \text{rth row}$$

Consider a linear time-invariant controllable system

$$\dot{x} = Ax + BU \tag{3.4}$$

where A and B have the forms as in (3.3) and an n number of eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. The problem in this section is to find a state feedback matrix K so that with $U = Kx + V$, the closed loop system

$$\dot{x} = (A + BK)x + BV \tag{3.5}$$

has $\lambda_1, \ldots, \lambda_n$ as its eigenvalues.

3.2 THEORY DEVELOPMENT

Let T_c be a linear operator, $T_c: V \to V$ over the complex field of C^n. Consider V be decomposed into 2 subspace V_1 and V_2 such that $V = V_1 \oplus V_2$. Now let $T_c = T_{c1} \oplus T_{c2}$ such that $T_{c1}V_1 \to V_1$, $T_{c2}V_2 \to V_2$.

From linear algebra theory [35] it may be readily shown that the matrix analogue to T_c is the block diagonal matrix A_c, where

$$A_c = \left[\begin{array}{c|c} A_{c1} & 0 \\ \hline 0 & A_{c2} \end{array}\right],$$

and where A_{c1} and A_{c2} are r x r and (n-r) x (n-r) matrices respectively. Since $T_c = T_{c1} \oplus T_{c2}$, then the characteristic polynomial of T_c is the product of the characteristic polynomial of T_{c1} and T_{c2}.[35]. or $\det(SI - A_{c1}) \cdot \det(SI - A_{c2}) = \det(SI - A_c)$, where

$$\det(SI - A_{c1}) = S^s + r_1 S^{s-1} + \ldots + r_s \tag{3.6}$$

$$\det(SI - A_{c2}) = S^p + a_1 S^{p-1} + \ldots + a_p \tag{3.7}$$

If A_c has eigenvalues $\lambda_1, \ldots, \lambda_n$, then the decomposition yields a subset of the λ_i's to A_{c1} and the remainder to A_{c2}. Note that each set must retain pair-wise complex conjugate eigenvalues. For convenience, A_{c1} and A_{c2} may be put into companion form. Note that the assigned set of eigenvalues determined the characteristic polynomial of A_{c1} and A_{c2} directly relate to the elements of A_{c1} and A_{c2} are represented in companion form or

$$A_{c1} = \begin{bmatrix} 0 & 1 & \cdot\;\cdot\;\cdot & & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ -r_s & & \cdot\;\cdot\;\cdot\;\cdot & & -r_1 \end{bmatrix},$$

$$A_{c2} = \begin{bmatrix} 0 & 1 & & 0 \\ 0 & 0 & 1 & 0 \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ -a_p & & & -a_1 \end{bmatrix} \tag{3.8}$$

Let the matrix A_f be a quasi-diagonal matrix over the field C^n such that

$$A_f = \left[\begin{array}{c|c} A_{f1} & 0 \\ \hline M & A_{f2} \end{array}\right]$$

when A_{f1} and A_{f2} are (r x r) and (n - r) x (n - r) matrices with forms

$$A_{f1} = \begin{bmatrix} 0 & 1 & \cdots & \cdot & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ -\partial_r & & \cdots & \cdot & -\partial_1 \end{bmatrix},$$

$$A_{f2} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ -g_q & & \cdots & \cdot & -g_1 \end{bmatrix} \qquad (3.9)$$

and

$$M = \begin{bmatrix} -m_1 \cdot\cdot & 0 & \cdots & 0 \\ -m_2 & 0 & \cdots & 0 \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ -m_q & 0 & \cdots & 0 \end{bmatrix}$$

is an (n - r) x r matrix, where $\det(SI - A_f) = \det(SI - A_c)$, that is $\det(SI - A_{f1}) \det(SI - A_{f2}) = \det(SI - A_{c1}) \cdot \det(SI - A_{c2})$. Since both A_f and A_c have the same characteristic polynomial, then they are similar. [35]

Theorem 3.1

If A_c and A_f are quasi-diagonal matrices over the field C^n, A_c and A_f have the form as (3.8) and (3.9) respectively, and $\det(SI - A_c) = \det(SI - A_f)$, then there exists a non-singular matrix T over C^n such that $TA_f = A_c$ where all the eigenvalues of A_c (or A_f) must be non-zero.

proof:

If T exists, then $T = A_cA_f^{-1}$, which requires that A_f be non-singular. Thus it is necessary that A_f must possess non-zero eigenvalues.

Since matrices A_c and A_f are similar, then $\det(A_c) = \det(A_f)$. Therefore, $\det T = \det(A_cA_f^{-1}) = \det(A_c)\ \det(A_f^{-1}) = \det(A_c)\ \det(A_c^{-1}) = 1$. Hence T exists. T is non-singular. Q.E.D.

Since

$$T = A_cA_f^{-1}, \tag{3.10}$$

and

$$A_f^{-1} = \left[\begin{array}{c|c} A_{f1} & 0 \\ \hline M & A_{f2} \end{array}\right] = \left[\begin{array}{c|c} A_{f1}^{-1} & 0 \\ \hline -A_{f2}^{-1}MA_{f1}^{-1} & A_{f2}^{-1} \end{array}\right] \tag{3.11}$$

$$A_{f1}^{-1} = \left[\begin{array}{ccccc} \frac{-g_{q-1}}{g_q} & \frac{-g_{q-2}}{g_q} & \cdots & \frac{-g_1}{g_q} & \frac{-1}{g_q} \\ 1 & 0 & \cdots & & 0 \\ 0 & 1 & \cdots & & 0 \\ \cdot & & & & \\ \cdot & & & & \\ 0 & \cdots 0 & \cdots & 1 & 0 \end{array}\right] \tag{3.12}$$

$$A_{f2}^{-1} = \begin{pmatrix} \frac{-\partial_{r-1}}{\partial_r} & \frac{-\partial_{r-2}}{\partial_r} & \cdots & \frac{-\partial_1}{\partial_r} & \frac{-1}{\partial_r} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdot & & & & \\ \cdot & & & & \\ 0 & \cdots & & 1 & 0 \end{pmatrix} \qquad (3.13)$$

Substituting (3.13) (3.12) into (3.11) yields

$$-A_{f2}^{-1} M A_{f1}^{-1} = \begin{pmatrix} \frac{-\partial_{r-1}}{\partial_r}\left(\sum_{i=1}^{q-1} \frac{M_{q-i}}{g_q} + \frac{M_q}{g_q}\right) & \frac{-\partial_{r-2}}{\partial_r}\left(\sum_{i=1}^{q-1} \frac{M_{g-i}}{g_q} + \frac{M_g}{g_q}\right) & \cdots & -\frac{1}{\partial_r}\left(\quad + \quad \right) \\ m_1 \frac{\partial_{r-1}}{\partial_r} & m_1 \frac{\partial_{r-2}}{\partial_r} & \cdots & \frac{m_1}{\partial_r} \\ m_2 \frac{\partial_{r-1}}{\partial_r} & m_2 \frac{\partial_{r-2}}{\partial_r} & \cdots & \frac{m_2}{\partial_r} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ m_q \frac{\partial_{r-1}}{\partial_r} & \cdots & & \frac{m_{q-1}}{\partial_r} \end{pmatrix}$$

(3.14)

Substituting (3.12) (3.13) and (3.14) into (3.11) and multiplying out the right hand side of (3.10). Let $\beta_0 = 0$, $\partial_0 = g_0 = 1$, and $S-r = e$. Equation (3.10) yields

$$
Ti.j = \begin{cases}
\delta i.j & (i = 1, \ldots r-1,\ j = 1, \ldots, n) \\
Tr.r{+}d = \dfrac{-g_{q-d}}{g_q} & (i = r,\ d = 1, \ldots q) \\
Tr.r = \dfrac{-1}{\partial_r} \displaystyle\sum_{d=1}^{q} \dfrac{M_d g_{q-d}}{g_q} & (i = j = r) \\
Tr.r\ \partial_{r-j} & (i = r,\ j = 1, \ldots, r-1) \\
Tr{+}d.j = \dfrac{-M_d}{\partial_r}\ \partial_{r-j} & (d = 1, \ldots, e-1,\ j = 1, \ldots, r) \\
\delta i.j & (i = r+1, \ldots, s-1,\ j = r+1, \ldots, n) \\
Ts.n = \dfrac{r_e}{g_q} & (i = s,\ j = n) \\
Ts.r{+}d = -r_{e-d} + \dfrac{g_{q-d}}{g_q}\ r_e & (i = s,\ d = 1, \ldots, e-1) \\
Ts.s{+}d = g_{p-d} \cdot \dfrac{r_e}{g_q} & (i = s,\ d = 0, \ldots, p-1) \\
Ts.j = -r_{s-j} + Ts.r \cdot \partial_{r-j} & (i = s,\ j = 1, \ldots, r-1) \\
Ts{+}d.j = \dfrac{-1}{\partial_r} M_{e+d} \cdot \partial_{r-j} & (d = 1, \ldots, n-s-1,\ j = 1, \ldots, r) \\
\delta i.j & (i = s+1, \ldots, n-1,\ j = r+1, \ldots, n) \\
Tn.s{+}d = -a_{p-d} & (i = n,\ d = 1, \ldots, p) \\
Tn.r = \dfrac{-1}{\partial_r} \displaystyle\sum_{d=1}^{p-1} M_{d+e} a_{p-d} & (i = n,\ j = r) \\
Tn.r \cdot \partial_{r-j} & (i = n,\ j = 1, \ldots, r-1) \\
0 & (i = n,\ j = r+1, \ldots, s-1)
\end{cases}
\tag{3.15}
$$

3.3 FEEDBACK MATRIX DETERMINATION:

Consider now the system equation with state feedback

$$\dot{x} = (A + BK)x + BV \qquad (3.16)$$

where A and B have the following forms:

$$A = \left(\begin{array}{cccccc} 0 & 1 & \ldots 0 & & & \\ 0 & 0 & 1..0 & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ -e_r & & -e_1 & & & \\ \hdashline -m_1 & & & 0 & 1 & \ldots\ 0 \\ \cdot & & & 0 & 0 & 1..\ 0 \\ \cdot & & & \vdots & & \\ -m_g & & & -f_g & & -f_1 \end{array}\right) \quad \text{and,}$$

$$B = \left(\begin{array}{cc} 0 & 0 \\ \cdot & \vdots \\ \cdot & 0 \\ \cdot & 1 \\ \cdot & 0 \\ \cdot & \cdot \\ 0 & \cdot \\ 1 & 0 \end{array}\right) \qquad (3.17)$$

Without changing the system, T may be introduced so that

$$\dot{x} = T^{-1}\ (TA + TBK)x + BV \qquad (3.18)$$

The problem is to find the K matrix so that

$$TA + TBK = A_c \qquad (3.19)$$

where A_c has $\lambda_1, \ldots, \lambda_n$ as desired eigenvalues for the closed loop system.

Select an s number of eigenvalues from n desired eigenvalues, denoted $\lambda_1, \ldots, \lambda_s$.

Let
$$\begin{aligned} \det(SI - A_{c1}) &= (S - \lambda_1) \ldots (S - \lambda_s) \\ &= S^s + r_1 S^{s-1} + \ldots + r_s \end{aligned} \qquad (3.20)$$

Let

$$\begin{aligned}\det(SI - A_{c2}) &= (S - \lambda_{s+1}) \cdots (S - \lambda_n) \\ &= S^p + a_1 S^{p-1} + \ldots + a_p \qquad (3.21)\end{aligned}$$

Also, select an r number of eigenvalues from n desired eigenvalues, denoted $\lambda_1, \ldots, \lambda_r$.

Let

$$\begin{aligned}\det(SI - A_{f1}) &= (S - \lambda_1) \cdots (S - \lambda_r) \\ &= S^r + \partial_1 S^{\,r-1} + \ldots + \partial_r \qquad (3.22)\end{aligned}$$

$$\begin{aligned}\det(SI - A_{f2}) &= (S - \lambda_{r+1}) \cdots (S - \lambda_n) \\ &= S^q + g_1 S^{q-1} + \ldots + g_q \qquad (3.23)\end{aligned}$$

With r_i's, a_i's, g_i's and λ_i's known, the T matrix can be found as (3.15). Substituting this T matrix into (3.18) yields

$$\left(\begin{array}{cccc|cccccc|ccc}
1 & 0 & \ldots & 0 & & & & & & & & & \\
0 & 1 & \ldots & 0 & & & & & & & & & \\
 & & & & & & & & & & & & \\
0 & \ldots & 1 & 0 & & & & & & & & & \\
\hline
Tr.1 & \ldots & Tr.r & & \ldots\ldots & & Tr.s & & & & & & Tr.n \\
\hline
T_{r+1} & 1 \ldots & T_{r+1} & r & 1 & 0 & \ldots\ldots & 0 & & & & & \\
\cdot & & & & 0 & 1 & \ldots\ldots & 0 & & & & & \\
\cdot & & & & 0 & \ldots\ldots & 1 & 0 & & & & & \\
\hline
Ts.1 & \ldots & Ts.r & & & \ldots\ldots & Ts.s & & \ldots\ldots & & & & Ts.n \\
\hline
\cdot & & & & & & & & 1 & 0 & \ldots\ldots & & 0 \\
\cdot & & & & & & & & 0 & 1 & \ldots\ldots & & \\
\vdots & & & & & & & & \ldots\ldots & & & & \\
\hline
Tn.1 & \ldots & Tn.r & & & & Tn.s & & \ldots\ldots & & Tn.n-1 & 0 & \\
\hline
\end{array}\right)
\left(\begin{array}{cccc|cccc}
0 & 1 & \ldots & 0 & & & & \\
0 & 0 & 1 \ldots & 0 & & & & \\
\cdot & \cdot & \cdot & \cdot & & & & \\
-e_r & & -e_1 & & & & & \\
\hline
-m_1 & & & & 0 & 1 & \ldots & 0 \\
\cdot & & & & 0 & 0 & 1 & \ldots \\
\cdot & & & & \cdot & & & \\
\cdot & & & & \cdot & & & \\
\cdot & & & & \cdot & & & \\
 & & & & \cdot & & & \\
-m_q & & & & -f_q & \ldots & & -f_1
\end{array}\right)$$

$$\begin{pmatrix} T_{11} & \cdots & T_{1n} \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ Tn.1 & \ldots & Tn.n \end{pmatrix}
\begin{pmatrix} 0 & 0 \\ & \vdots \\ \vdots & 1 \\ & \vdots \\ 1 & 0 \end{pmatrix}
\begin{pmatrix} k_{11} \ldots k_{1n} \\ \\ \\ k_{21} \ldots k_{2n} \end{pmatrix}
=
\left(\begin{array}{cccc|ccccc}
0 & 1 & \ldots & 0 & & & & & \\
0 & 0 & 1 \ldots & 0 & & & & & \\
\vdots & & & & & & & & \\
-r_s & & \ldots & -r_1 & & & & & \\
\hline
-m_1 & & & & 0 & 1 & 0 & \ldots & 0 \\
\vdots & & & & 0 & 0 & 1 & \ldots & 0 \\
 & & & & \vdots & & & & \\
-m_q & & & & -a_p & & & & -a_1
\end{array}\right)$$

(3.24)

After multiplication, (3.26) can be rewritten as

$$
\left(\begin{array}{c|c|c}
\begin{matrix}0 & 1 & \dots & 0\\ 0 & 0 & 1.. & 0\\ \cdot & & & \\ \cdot & & & \end{matrix} & & \\
\hline
\mathrm{Wr.1}\ \dots\ \mathrm{Wr.r} & \mathrm{Wr.r{+}1}\ \dots\ \mathrm{Wr.s} & \mathrm{Wr.n}\\
\hline
 & \begin{matrix}0 & 1 & 0\\ 0 & 0 & 1\end{matrix} & \\
\hline
\mathrm{Ws.1}\quad \mathrm{Ws.r} & \mathrm{Ws.s} & \mathrm{Ws.n}\\
\hline
 & & \begin{matrix}0 & 1 & \dots & 0\\ 0 & 0 & 1.. & 0\\ \mathrm{Wn.s{+}1} & \dots & & \mathrm{Wn.n}\end{matrix}
\end{array}\right)
$$

$$
+\left(\begin{array}{cc}
0 & 0\\
\hline
\mathrm{Tr.n} & \mathrm{Tr.r}\\
\hline
0 & 0\\
\hline
\mathrm{Ts.n} & \mathrm{Ts.r}\\
\hline
0 & \mathrm{Tn.r}
\end{array}\right)
\begin{bmatrix} k_{11} & \dots & k_{1n}\\ k_{21} & \dots & k_{2n}\end{bmatrix}
$$

$$
=\left(\begin{array}{c|c}
\begin{matrix}0 & 1 & 0 \dots & 0\\ 0 & 0 & 1 \dots & 0\\ \dots & & & \\ -r_s & \dots & & -r_1\end{matrix} & \\
\hline
\begin{matrix}-m_1\\ \cdot\\ \cdot\\ -m_q\end{matrix} & \begin{matrix}0 & 1 \dots & 0\\ 0 & 0\ 1 \dots & \\ -a_p & \dots & -a_1\end{matrix}
\end{array}\right) \qquad (3.25)
$$

Where

$$
Wi.j = \begin{cases}
-\sum_{d=1}^{q} Tr.r{+}d\ M_d - Tr.r\ e_r & (i = r,\ j = 1) \\
Tr.j{-}1 - Tr.r\ e_{r-j+1} & (i = r,\ j = 2,\ \ldots,\ r{-}1) \\
Wr.r{+}d = Tr.n\ f_{q-d+1} + Tr.r{+}d{-}1 & (i = r,\ d = 1,\ \ldots,\ q) \\
-Ts.r\ e_r - \sum_{d=1}^{q} Ts.s{+}d\ M_d & (i = s,\ j = 1) \\
Ts.j{-}1 - Tr.r\ e_{r+1-j} & (i = s,\ j = 2,\ \ldots,\ r) \\
Ws.r{+}d{+}1 = -Ts.r{+}d - Ts.n\ f_{q-d} & (i = s,\ d = 1,\ \ldots,\ q{-}1) \\
-Ts.nf_q & (i = s,\ j = r{+}1) \\
-Tn.r - \sum_{d=0}^{e-1} Tn.s{+}d\ M_{q-e+d} & (i = n,\ j = 1) \\
0 & (i = n,\ j = 2,\ \ldots,\ s) \\
Tn.s{+}d{-}1 & (i = n,\ d = s{+}1,\ \ldots,\ n)
\end{cases}
\tag{3.26}
$$

By comparing the elements on both sides, one obtains:

$$
Wr.j + Tr.nk_{1.j} + Tr.rk_{2.j} = \begin{cases} 0 & j \neq r + 1 \\ 1 & j = r + 1 \end{cases} \tag{3.27}
$$

$$
Ws.j + Ts.nk_{1.j} + Ts.rk_{2.j} = \begin{cases} -r_{s+r-j} & j = 1,\ \ldots,\ s \\ 0 & j = s{+}1, \ldots, n \end{cases} \tag{3.28}
$$

and

$$
Wn.s{+}d + Tn.rk_{21s+d} = -a_{p+d-1} \qquad d = 1,\ \ldots,\ p \tag{3.29}
$$

Solving (3.27), (3.28) and (3.29) for k yields

$$k_{1.j} = \begin{cases} \dfrac{r_{s-j+1}\, Tr.r + Tr.rWs.j - Ts.rWr.j}{Ts.rTr.n - Tr.rTs.n}, & j = 1, \ldots, s-1 \\[2ex] \dfrac{-Tr.r}{Tr.n}\left(\dfrac{-Wn.s}{Tn.r}\right) - \dfrac{Wr.s}{Tr.n} & j = s \\[2ex] k_{1.s+d} = \dfrac{Tr.r(a_{p-d+1} + W_{n.s+d})}{Tn.r\ Tr.n} - \dfrac{Wr.s+d}{Tr.n} & (d = 1, \ldots, p) \end{cases} \tag{3.30}$$

and

$$k_{2j} = \begin{cases} \dfrac{r_{s-j+1} Tr.n + Tr.nWs.j - Ts.nWr.j}{Tr.rTs.n - Tr.nTs.r} & j = 1, \ldots, s-1 \\[2ex] \dfrac{-Wn.s}{Tn.r} & j = s \\[2ex] k2.s+d = \dfrac{-a_{p+1-d} - Wn.s+d}{Tn.r\ \ Tr.n} & d = 1, \ldots, p \end{cases} \tag{3.31}$$

Therefore, with the k chosen as (3.30) and (3.31), the desired relationship $TA + TBK = A_c$ is met, so the closed loop system has $\lambda_1, \ldots, \lambda_n$ as its eigenvalues.

The above procedure of finding the matrix K can be summarized as follows:

1. Define matrices A_c and A_f as in (3.8) and (3.9).
2. Find the matrix T as in (3.15).
3. Find each Wi.j as in (3.26).
4. Use (3.30) and (3.31) to find $k_{1.j}$ and $k_{2.j}$ respectively.
5. $k = \begin{bmatrix} k_{11} \ldots k_{1n} \\ k_{21} \ldots k_{2n} \end{bmatrix}$ is the state feedback matrix.

3.4 INCOMPLETE STATE FEEDBACK

In the previous section, a complete state feedback matrix is applied to placed n poles at the desired locations. However, in practice, only some of the poles need to be placed at the desired locations. Therefore, an incomplete state feedback matrix K will be found so that an s number of poles could be preassigned by using feedback from an s number of states.

Consider a linear time-invariant controllable system described by

$$\dot{x} = Ax + BU \tag{3.32}$$

and an s number of eigenvalues $\lambda_1, \ldots, \lambda_s$. The problem in this section is to find a state feedback matrix K such that with feedback from an s number of measurable states (i.e. $U = [K_1, 0]x + V$, where K_1 is an $2 \times S$ matrix), the closed loop system

$$\dot{x} = (A + BK)x + BV \tag{3.33}$$

has $\lambda_1, \ldots, \lambda_s$ as its eigenvalues.

The following assumptions are needed before the solution procedure can be found.

Assumption 3-1: The number of measurable states is greater than or equal to the number s. $(s > r)$

Assumption 3-2: All the states in A_{cl} of (3.7) are measurable.

From the known eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_s$, form the characteristic polynomial

$$\begin{aligned} \det(SI - A_{cl}) &= (S - \lambda_1) \ldots (S - \lambda_s) \\ &= S^s + r_1 S^{r-1} + \ldots + r_s \end{aligned} \tag{3.34}$$

and let

$$\det(SI - A_{c2}) = S^p + a_1 S^{p-1} + \ldots + a_p \tag{3.35}$$

denote the characteristic polynomial of A_{c2}, where $a_1, \ldots, a_p$ are unknown constants.

Choose an r number of the s eigenvalues and let it be denoted $\lambda_1, \ldots, \lambda_r$. Form the characteristic polynomial of A_{f1} as in (3.9)

$$\begin{aligned}\det(SI - A_{f1}) &= (S - \lambda_1) \ldots (S - \lambda_r) \\ &= S^r + \partial_1 S^{r-1} + \ldots + \partial_r\end{aligned} \tag{3.36}$$

Also let

$$\det(SI - A_{f2}) = S^q + g_1 S^{q-1} + \ldots + g_q \tag{3.37}$$

denote the characteristic polynomial of A_{f2}, where $g_1, \ldots, g_q$ are unknown constants that must satisfy

$$\det(SI - A_{c1}) \cdot \det(SI - A_{c2}) = \det(SI - A_{f1}) \cdot \det(SI - A_{f2}) \tag{3.38}$$

Since A_f is similar to A_c, then the desired closed loop eigenvalues $\lambda_1, \ldots, \lambda_s$ are still maintained. Following the same procedure as in Section 3.1, the state feedback matrix K can be found as in (3.30). However, because only the first s states are available for feedback, it is necessary that

$$K_{i.s+d} = 0 \qquad (i = 1, 2;\ d = 1, \ldots, n-s) \tag{3.39}$$

This implies

$$\frac{Tr.r(a_{p-d+1} + Wn.s+d)}{Tn.r\ Tr.n} - \frac{Wr.s+d}{Tr.n} = 0 \quad (d = 1, \ldots p) \tag{3.40}$$

(3.39) and (3.40) can be further reduced to

$$Wr.s+d = 0 \qquad (d = 1, \ldots, p) \tag{3.41}$$

$$a_{p-d+1} + Wn.s{+}d = 0 \qquad (d = 1, \ldots, p) \qquad (3.42)$$

By (3.15) and (3.26), it follows that

$$a_{p-d+1} + Wn.s{+}d = 0 \qquad (d = 1, \ldots, p) \qquad (3.43)$$

Hence (3.42) is an identity equation. Therefore (3.41) and (3.42) reduced to (3.41) only. That is

$$Wr.s{+}d = 0 \qquad (d = 1, \ldots, p) \qquad (3.44)$$

The above p = n-s number of conditions serve as the sufficient conditions that the placement of s poles using s states for feedback is possible. Therefore, one obtains the following:

Theorem 3.2

For a completely controllable system described by (3.32), together with s eigenvalues $\lambda_1, \ldots, \lambda_s$, the sufficient condition for the placement of s poles using s states for feedback is

$$f_i = g_i \qquad (i = 1, \ldots, q-e) \qquad (3.45)$$

Note that theorem (3.2) is a generalization of theorem (2.2). Following the same procedure as in section 2.4, an equation may be obtained so that the remaining e coefficients of g_i's can be determined from the known values of f_i's $(i = 1, \ldots, q-e)$.

The above procedure of using an s number of states to assign an s number of poles is summarized as follows:

1. Define matrix A_c and A_f as in (3.8) and (3.9).
2. Solve (3.45) for g_i $(i = 1, \ldots, q-e)$. The remaining e coefficients can be found by using (2.58).

3. Find the matrix T as in (3.15).

4. Use (3.30) and (3.31) to find $k_{1.j}$ and $k_{2.j}$ for $j = 1, \ldots, s$.

5. $k = \begin{bmatrix} k_{11} & \ldots & k_{1s}, & 0, & \ldots 0 \\ k_{21}, & \ldots, & k_{2s}, & 0, & \ldots 0 \end{bmatrix}$ is the state feedback matrix.

It is possible that, using an s number of states to assign an s number of poles, the closed loop system might become unstable because the remaining n-s poles have to meet n-s conditions shown in (3.48). Therefore, a criterion to determine the closed loop system's stability has to be considered.

Define $$Q(S) \triangleq S^{n-s} + f_1 S^{n-s-1} + \ldots + f_{n-s} \qquad (3.46)$$

$$D(S) \triangleq S^{n-s} + g_1 S^{n-s-1} + \ldots + g_{n-s} \qquad (3.47)$$

Following the same procedure as in section 2.4, one may show that the closed loop system can be stabilized if Q(S) or D(S) is a Hurwitz polynomial.

If the assumption 3-2 does not hold, the following assumption is needed:

Assumption 3-2-a: The number of pairs of complex conjugate poles to be assigned is less than or equal to the number of pairs of adjoining measurable states.

With the above assumption holds, one can still handle the case where some of the states in A_{c1} are not measurable and some of the states in A_{c2} are measurable.

Consider a linear time-invariant controllable system

$$\dot{x} = Ax + BU \qquad (3.48)$$

where $x = (x_1, \ldots, x_m, x_{m_1+1}, \ldots, x_2, x_{m_2+1}, \ldots, x_{m_3})^T,$

$$m_1 + m_2 + m_3 = n, \quad m_1 < s \quad \text{and} \quad m_1 + m_3 \geq s.$$

For simplicity, suppose the first m_1 states are measurable, the next m_2 states are unmeasurable, and the last m_3 states are again measurable. One can use the procedure similar to that in section 2.4 to find a state feedback K such that with state feedback from the first m_1 states and the last m_3 states,

i.e. $U = Kx + V,$

$$k = \begin{bmatrix} k_{11}, \ldots k_{1m_1}, 0 \ldots 0, k_{1m_2+1}, \ldots k_{1\,m_3} \\ k_{21}, \ldots k_{2m_1}, 0 \ldots 0, k_{2m_2+1}, \ldots k_{2\,m_3} \end{bmatrix}$$

the closed loop system

$$\dot{x} = (A + BK)x + BV \tag{3.49}$$

has $\lambda_1, \ldots, \lambda_s$ as its eigenvalues.

CHAPTER 4

APPLICATION

4.0 INTRODUCTION

In this Chapter, the eigenvalue assignment method is applied to dynamic model reduction. The results are compared with those of Chidambara's simplification method [19]. It is found that both methods yield the same system matrix. However, the input matrices are slightly different. A review of Chidambara's simplification method is shown in section 4.1 and in the same section, Rao's [21] notations are used for all equations. A simplification procedure based on the decomposition approach is derived in section 4.2. The application of the decomposition method to the suboptimal control is presented, with examples, in section 4.3 to provide a comparison to Chidambara's approach.

4.1 REVIEW OF CHIDAMBARA'S SIMPLIFICATION TECHNIQUE

Consider a linear time-invariant system described by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} U \tag{4.1}$$

which may be written as

$$\dot{x} = Ax + BU \tag{4.2}$$

where x is an n vector and U is an m vector. The matrices A and B are of order (n x n) and (n x m), respectively. The vector x_1 contains r elements of the state vector that are to be retained in the simplified model.

Let $$x = Mz \qquad (4.3)$$

where M is the modal matrix of A. Then

$$\dot{z} = Hz + LU \qquad (4.4)$$

where

$$H = M^{-1}AM \qquad (4.5)$$

and

$$L = M^{-1}B \qquad 4.6)$$

If A has distinct eigenvalues,

$$H = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & \\ \vdots & & & \\ 0 & & & \lambda_n \end{bmatrix} \qquad (4.7)$$

one may arrange the eigenvalues such that

$$|\lambda_1| < |\lambda_2| < \dots |\lambda_n| \qquad (4.8)$$

Since (4.4) can be rewritten as

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} U \qquad (4.9)$$

$\dot{z}_1$ and $\dot{z}_2$ can be put as

$$\dot{z}_1 = H_1 z_1 + L_1 U \qquad (4.10)$$

$$\dot{z}_2 = H_2 z_2 + L_2 U \qquad (4.11)$$

Where z_1 is an r x 1 vector, and z_2 is an (n-r) x 1 vector.

$$H_1 = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_r \end{bmatrix}$$

$$H_2 = \begin{bmatrix} \lambda_{r+1} & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ & & & \lambda_n \end{bmatrix}$$

L_1 is the top r x m submatrix of $m^{-1}B$.

L_2 is the bottom (n - r) x m submatrix of $m^{-1}B$.

By taking the Laplace transform of both sides of (4.11) with an initial condition of zero, one obtains:

$$z_2\ (s) = (SI - H_2)^{-1}\ L_2U(s) \tag{4.12}$$

$$\approx -\ H_2^{-1}L_2U(s) \tag{4.13}$$

Hence

$$z_2(t) = -H_2^{-1}L_2U(t) \tag{4.14}$$

Eq.(4.3) can be rewritten as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \tag{4.15}$$

that is

$$x_1 = M_{11}z_1 + M_{12}z_2 \tag{4.16}$$

$$x_2 = M_{21}z_1 + M_{22}z_2 \tag{4.17}$$

Solving (4.16) for z_1, one obtains

$$z_1 = M_{11}^{-1}x_1 - M_{11}^{-1}M_{12}z_2 \tag{4.18}$$

By substituting (4.18) and (4.14) into (4.17)

$$x_2 = M_{21}M_{11}^{-1}x_1 - (M_{22} - M_{21}M_{11}^{-1}M_{12})H_2^{-1}L_2U \tag{4.19}$$

Substitution of (4.19) into (4.1) yields a simplified model represented by:

$$\dot{x}_1 = Fx_1 + GU \tag{4.21}$$

where

$$F = A_{11} + A_{12}M_{21}M_{11}^{-1} \tag{4.22}$$

and

$$G = B_1 - [A_{12}M_{22} - A_{12}M_{21}M_{11}^{-1}M_{12}]H_2^{-1}L_2 \tag{4.23}$$

4.2 SIMPLIFICATION BASED ON DECOMPOSITION METHOD

Consider a linear time-invariant controllable system described by

$$\dot{x} = Ax + BU \tag{4.24}$$

It is noted that, without loss of generality, one can assume A, B to have the form as (3.4) and (3.5).

Suppose $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A and they are selected in such a way that

$$|\lambda_1| < |\lambda_2| < \ldots < |\lambda_n|$$

Let

$$\begin{aligned} \det(SI - A_{c1}) &= (S - \lambda_1) \ldots (S - \lambda_r) \\ &= S^r + \partial_1 S^{r-1} + \ldots + \partial_r \end{aligned} \tag{4.25}$$

$$\begin{aligned} \det(SI - A_{c2}) &= (S - \lambda_{r+1}) \ldots (S - \lambda_n) \\ &= S^{n-r} + \beta_1 S^{n-r-1} + \ldots + \beta_{n-r} \end{aligned} \tag{4.26}$$

Also assume $|\lambda_i| << |\lambda_j|$, $i = 1, \ldots, r$; $j = r+1, \ldots, n$. By theorem (3.1), the T matrix can be found such that

$$TA = \begin{pmatrix} A_{c1} & 0 \\ 0 & A_{c2} \end{pmatrix} \tag{4.27}$$

where

$$A_{c1} = \begin{pmatrix} 0 & 1 & 0 \ldots 0 \\ 0 & 0 & 1 \ldots 0 \\ \vdots & & \\ -\partial_r & \ldots & -\partial_1 \end{pmatrix} \text{ and } A_{c2} = \begin{pmatrix} 0 & 1 & 0 \ldots 0 \\ 0 & 0 & 1 \ldots 0 \\ \vdots & & \\ -\beta_{n-r} & \ldots & -\beta_1 \end{pmatrix} \tag{4.28}$$

Using the T matrix, eq. (4.24) can be put

$$\dot{x} = T^{-1}(TAx + TBU) \tag{4.29}$$

Let

$$TB = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \tag{4.30}$$

where B_1 and B_2 are of order r x m and (n -r)x m, respectively.

Then eq. (4.29) becomes

$$\begin{vmatrix} \dot{x}_1 \\ \dot{x}_2 \end{vmatrix} = T^{-1} \left[\begin{pmatrix} A_{c1} & 0 \\ 0 & A_{c2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} U \right] \tag{4.31}$$

By virtue of the dominant eigenvalue associate with A_{c1} and with respect to the system response, it may be readily shown that the dominant dynamics of the system may be approximated by

$$\dot{x}_1 = A_{c1}x_1 + B_1U \tag{4.32}$$

where vector x_1 is defined on an r dimensional space.

Consider a numerical example given by Rao [21], where

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.5 & -5.6 & -6.1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} U \qquad (4.33)^*$$

The objective is to find a second-order simplified model.

Using the Chidambara's simplification method, one obtains the following:

$$\frac{dx_1}{dt} = \begin{bmatrix} 0 & 1 \\ -0.1 & -1.1 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 0.204 \end{bmatrix} U \tag{4.34}$$

---

* See Appendix

However, the proposed decomposition method yields

$$\frac{dx_1}{dt} = \begin{bmatrix} 0 & 1 \\ -0.1 & -1.1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0.2 \end{bmatrix} U \qquad (4.35)$$

It can be seen that both methods yield the same system matrix but different input matrices. This occurs because Chidambara's method makes use of the approximation as shown in eq. (4.13), whereas the decomposition method does not.

4.3 MODEL REDUCTION APPLIED TO SUBOPTIMAL CONTROL

Consider the standard linear regulator problem for a system

$$\dot{x} = Ax + BU \qquad (4.36)$$

with a performance inde[illegible]

$$J = \text{[illegible]} \qquad (4.37)$$

where Q is a pos[illegible]atrix and R is a positive definite symmet[illegible]

The opt[illegible]he simplified model

$$\dot{x}_1 = \text{[illegible]} \qquad (4.38)$$

where A_{c1} and B_1 are [illegible]and (4.30), respectively.

Since the matrix A in (4.36) is assumed in companion form, by the theorem of Wonham and Johnston [33], A could be written as

$$A = NHN^{-1} \qquad (4.39)$$

where

$$H = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \qquad (4.40)$$

However, the proposed decomposition method yields

$$\frac{dx_1}{dt} = \begin{bmatrix} 0 & 1 \\ -0.1 & -1.1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0.2 \end{bmatrix} U \qquad (4.35)$$

It can be seen that both methods yield the same system matrix but different input matrices. This occurs because Chidambara's method makes use of the approximation as shown in eq. (4.13), whereas the decomposition method does not.

4.3 MODEL REDUCTION APPLIED TO SUBOPTIMAL CONTROL

Consider the standard linear regulator problem for a system

$$\dot{x} = Ax + BU \qquad (4.36)$$

with a performance indes

$$J = \frac{1}{2} \int_0^\infty (x^T Qx + U^T RU)dt \qquad (4.37)$$

where Q is a positive semidefinite symmetric matrix and R is a positive definite symmetric matrix.

The optimization is carried out on the simplified model

$$\dot{x}_1 = A_{c1}x_1 + B_1U \qquad (4.38)$$

where A_{c1} and B_1 are defined in (4.28) and (4.30), respectively.

Since the matrix A in (4.36) is assumed in companion form, by the theorem of Wonham and Johnston [33], A could be written as

$$A = NHN^{-1} \qquad (4.39)$$

where

$$H = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \cdot & \\ & & & \cdot \\ & & & & \lambda_n \end{bmatrix} \qquad (4.40)$$

and

$$N = \begin{bmatrix} 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & & \lambda_n \\ \lambda_1^{\,2} & \cdot & & \cdot \\ \vdots & \cdot & & \cdot \\ \lambda_1^{\,n-1} & \dot{\lambda}_2^{\,n-1} & & \dot{\lambda}_n^{\,n-1} \end{bmatrix} \tag{4.41}$$

The matrix N is known as a Vandermonde matrix [36].

Using the following transformation

$$x = NW \tag{4.42}$$

eq. (4.36) can be put

$$\dot{W} = N^{-1}ANW + N^{-1}BU \tag{4.43}$$

$$= HW + \bar{B}U \tag{4.44}$$

or

$$\begin{bmatrix} \dot{W}_1 \\ \dot{W}_2 \end{bmatrix} = \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} + \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix} U \tag{4.45}$$

where

$$H_1 = \begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \lambda_r \end{bmatrix} \tag{4.46}$$

$$H_2 = \begin{bmatrix} \lambda_{r+1} & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \ \lambda_n \end{bmatrix} \tag{4.47}$$

and λ_i's are arranged in a way such that

$$|\lambda_1| < |\lambda_2| < \dots < |\lambda_n| \tag{4.48}$$

Solving (4.45) for $W_2(t)$ yields

$$W_2(t) = e^{H_2 t} W_2(0) + \int_o^t e^{H_2(t-G)} \bar{B}_2 U(G)\, dG \tag{4.49}$$

If every eigenvalue of H_2 is negative and very large, one can assume that

$$\lim_{t \to \infty} e^{H_2 t} = 0 \tag{4.50}$$

Therefore, one can further assume that

$$W_2(t) \approx 0 \tag{4.51}$$

Eq. (4.42) can be written as

$$x_1 = N_{11} W_1 + N_{12} W_2 \tag{4.52a}$$

$$x_2 = N_{21} W_1 + N_{22} W_2 \tag{4.52b}$$

when $W_2 = 0$ the above equation becomes

$$x_1 = N_{11} W_1 \tag{4.53}$$

$$x_2 = N_{21} W_1$$

Thus x_2 can be found as

$$x_2 = N_{21} N_{11}^{-1} x_1 \tag{4.54}$$

Since eq. (4.37) can be written as

$$J = \frac{1}{2} \int_0^\infty \left\{ [x_1^T \; x_2^T] \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12} & Q_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + U^T RU \right\} dt$$

$$= \frac{1}{2} \int_0^\infty \left\{ x_1^T Q_{11} x_1 + 2x_1^T Q_{12} x_2 + x_2^T Q_{22} x_2 + U^T RU \right\} dt \qquad (4.55)$$

the new performance index for the simplified system (4.38) can be obtained by replacing x_2 by x_1 in (4.55) with the relation defined in (4.54). The performance index is then

$$J_1 = \frac{1}{2} \int_0^\infty (x_1^T Q_1 x_1 + U^T RU) dt \qquad (4.56)$$

where Q_1 is obtained as

$$Q_1 = \begin{bmatrix} I & (N_{21} N_{11}^{-1})^T \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12} & Q_{22} \end{bmatrix} \begin{bmatrix} I \\ N_{21} \; N_{11}^{-1} \end{bmatrix} \qquad (4.57)$$

The optimal control of a simplified system described by (4.38) and (4.56) is given by Anderson and Moore [31] as

$$U = -Kx_1 \qquad (4.58)$$

where

$$K = R^{-1} B_1 P \qquad (4.59)$$

and P is the solution of the matrix Ricatti equation

$$PA_{c1} + A_{c1}^T P - PB_1 R^{-1} B_1^T P - Q_1 = 0 \qquad (4.60)$$

Then the equation (4.58) can be used as the suboptimal control policy of the original system given by equation (4.36), i.e.

$$Usub = [-K \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad (4.61)$$

Consider the following numerical example which was presented in example 4 of Appendix .

For the given system equation

$$\dot{x} = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.5 & -5.6 & -6.1 \end{vmatrix} x + \begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix} U \qquad (4.62)$$

and the performance index

$$J = \frac{1}{2} \int_0^\infty \left(x^T \begin{bmatrix} 5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} x + U^T U \right) dt \qquad (4.63)$$

Using the computer program developed by Melson and Jones [40], the optimal solution of the problem presented by (4.62) and (4.61) was obtained where

$$U = -K_{opt} x = - [1.79 \quad 2.08 \quad 0.41] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad (4.64)$$

and the optimal control of Chidambara's simplified model for the same system was found as

$$\begin{aligned} U &= -K_c x_1 \\ &= -[1.82 \quad 2.41] x_1 \end{aligned} \qquad (4.65)$$

The proposed decomposition method gives the optimal control of the simplified model as given below. That is

$$\begin{aligned} U &= -K_d x_1 \\ &= -[1.812 \quad 2.39] x_1 \end{aligned} \qquad (4.66)$$

Noted that the values of K_d are very close to that of K_c, which was to be expected in that the simplified models were similar.

Figures 1 and 2 show the simulation results of Chidambara's simplified model, the original system, and the decomposition simplified model, when the optimal control is applied.

Figure 1 shows how the x_1 state variable varies with time. It has shown that both simplified models provide good approximation to the original system. The x_1 value of Chidambara's simplified model seems more close to that of the original system with this particular example.

Figure 2 shows how the x_2 state variable varies with time. It is noted that the difference between the values of x_2 of the proposed simplified model and that of the original system become sharply increased when time is longer than 8 seconds.

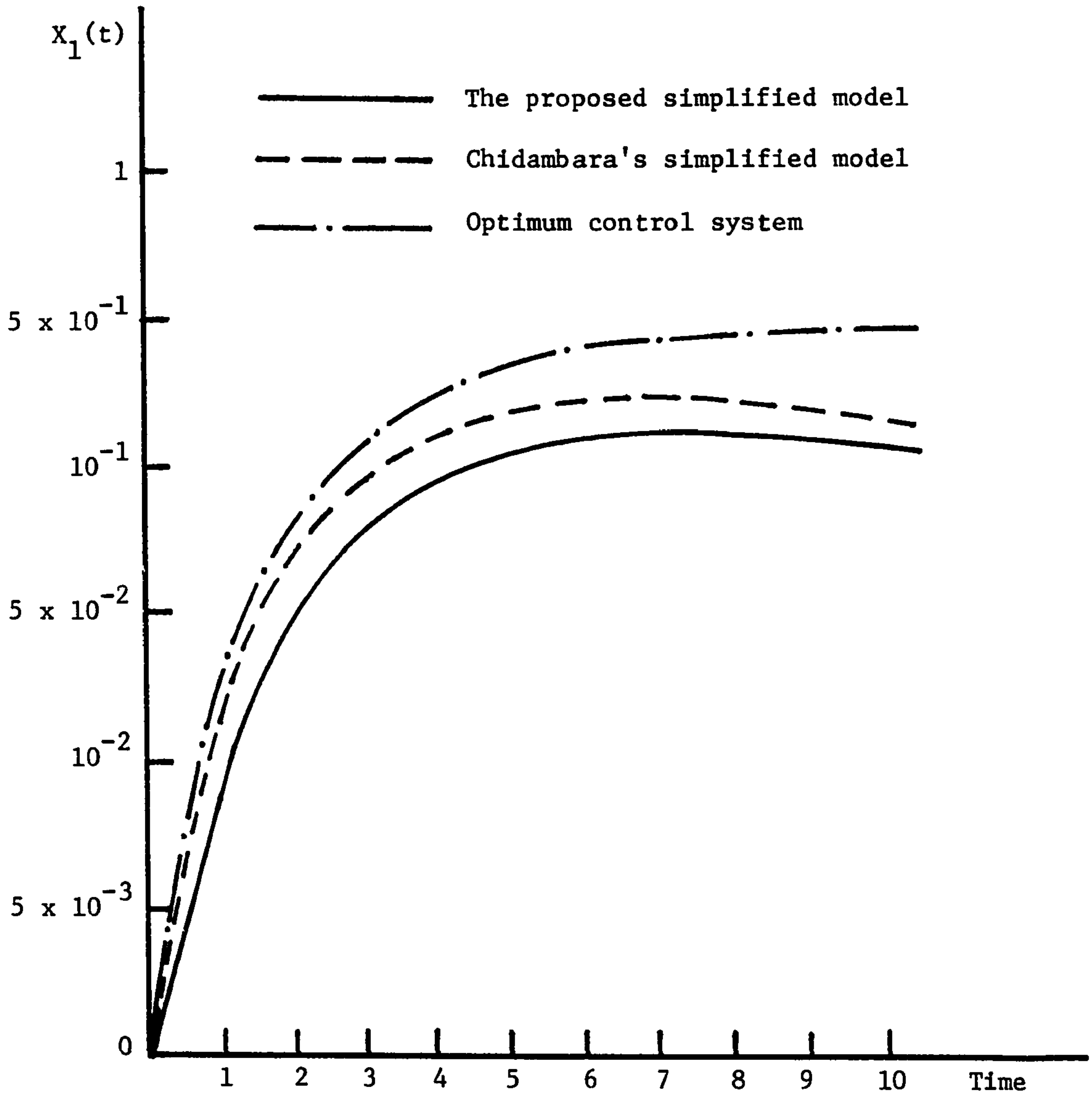

Figure 1. $X_1(t)$ versus time.

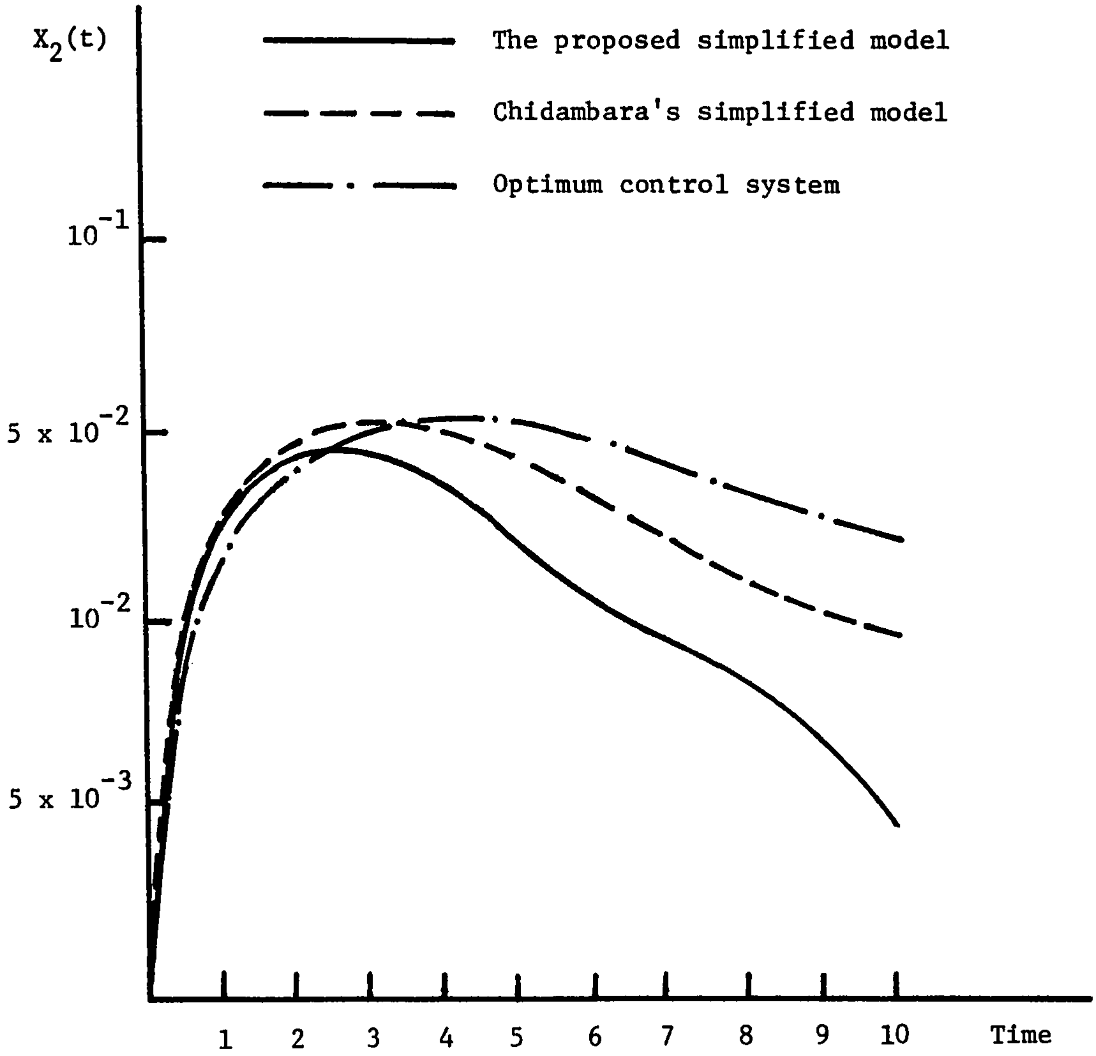

Figure 2. $X_2(t)$ versus time.

CHAPTER 5

CONCLUSION

In this dissertation, an alternate method is presented for assigning eigenvalues to a linear time-invariant system using state feedback. This method relies on the simple decomposition of the system matrix (in companion form) in order to define subdivided subsets of the system eigenvalues. This decomposition is accomplished by an operator matrix T whose coefficients may be determined by a relatively simple algorithm. It has also been shown how the coefficients of T inverse may be easily computed.

It is noted that if all the states are available for feedback, one can arbitrarily assign all the closed loop poles. If only an r number of states are available for feedback, then only an r number of poles can be arbitrarily preassigned, provided some sufficient conditions hold. Treatment of the incomplete state feedback in Chapters 2 and 3 show how to relate n - r coefficients of the closed loop companion form system matrix directly to those of the open loop companion form system matrix in order to provide a new and easy method for determining the stability of a system that uses incomplete state feedback to arbitrarily assign an r number of its closed loop poles.

Unlike prior methods, which apply incomplete state feedback by using an output vector to assign r poles, but let the remaining n - r poles blindly assume any value, the new method presented by this paper sets a bound on these n - r poles as determined by theorem 2.2 and 3.2.

Although Chapter 3 was concerned with control systems driven by 2 inputs, the result may be generalized to a control system with more than 2 inputs by the repeated use of theorem 2.1 and 3.1 under the assumptions provided. An example demonstrating the eigenvalue assignment procedure to a multivariable system is presented in Appendix.

An appropriate application of the decomposition technique has been applied to model reduction. The method is developed which assigns r dominant poles on the upper r rows of the companion form system matrix. It has been found that once the T matrix defined in (2.22) is constructed, the simplified model can be easily obtained. Results of the decomposition method have been shown to be similar to that composed by Chidambara's method [21]. The noted difference is that the decomposition approach generates a more accurate input matrix than Chidambara's. An example applying the model reduction technique to a suboptimal control is presented in Appendix. Sufficiency of the technique with respect to Chidambara's method is demonstrated.

While the procedures developed are complete by themselves, two directions for further research should be mentioned. One direction is the theoretical treatment of placing bounds other than stability on those (n - r) non-specified eigenvalues resulting from incomplete state feedback. Perhaps the model reduction approach can be applied relating specified eigenvalues to desired dominant eigenvalues. Another research direction might be in the application of the procedure developed to adaptive control. Fruitful application is a possibility due to the computational simplicity of determining the feedback matrix for eigenvalue placement. This feature is important when considering that the feedback matrix coefficients must be continually updated as the control system coefficients vary slowly with time.

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APPENDIX

NUMERICAL EXAMPLES

EXAMPLE 1 (COMPLETE STATE FEEDBACK)

Consider a linear time-invariant system described by

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -12 & -16 & -7 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} U \qquad (1)$$

The problem is to find a state feedback matrix K where $U = Kx + V$; the closed loop system has -2, $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$ as its poles.

Solution

Step 1. Define matrices A_c and A_f

Let $\det(SI - A_{c1}) = (S + \frac{1}{2} + \frac{\sqrt{3}}{2}i)(S + \frac{1}{2} - \frac{\sqrt{3}}{2}i)$

$$= S^2 + S + 1 \qquad (2)$$

Therefore, $\partial_1 = 1$, and $\partial_2 = 1$. (3)

Let $\det(SI - A_{c2}) = (S + 2)$,

then $\beta_1 = 2$. (4)

$$\det(SI - A_f) = \det(SI - A_{c1}) \cdot \det(SI - A_{c2})$$

$$= S^3 + 3S^2 + 3S + 2 \qquad (5)$$

Hence $a_1 = 3$, $a_2 = 3$, and $a_3 = 2$. (6)

Now define the matrices A_c and A_f as follows:

$$A_f = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -3 & -3 \end{bmatrix} \tag{7}$$

and

$$A_c = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \tag{8}$$

Step 2. Generate the T matrix.

The T matrix can be found as

$$T = A_c A_f^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{3}{2} & \frac{1}{2} \\ 0 & -2 & 0 \end{bmatrix} \tag{9}$$

The inverse of T matrix is

$$T^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \\ -1 & 2 & \frac{3}{2} \end{bmatrix} \tag{10}$$

Step 3. Find $W_2.j$ for j = 1, 2, 3.

$$W_{2\cdot 1} = -6 \quad W_{2.2} = -7.5 \quad \text{and} \quad W_{2.3} = -2. \tag{11}$$

Step 4. Find K_j for j = 1, 2, 3.

$$K_1 = 10 \quad K_2 = 13 \quad \text{and} \quad K_3 = 4 \tag{12}$$

Step 5. K = [10, 13, 4] is the desired state feedback matrix.

EXAMPLE 2 (INCOMPLETE STATE FEEDBACK)

Consider the following system

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -3 & -3 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} U \tag{13}$$

Assume that the x_1 state is the only state that is available for feedback. The problem is to find an incomplete state feedback matrix K that will observe only the x_1 state, and use this single state to give the closed loop system a pole at -1.

Solution

Step 1. Define the matrices A_c and A_f.

$$\text{Let } \det(SI - A_{c1}) = (S + 1), \text{ then } \partial_1 = 1. \tag{14}$$

$$\text{Let } \det(SI - A_{c2}) = S^2 + \beta_1 S + \beta_2 \tag{15}$$

$$\det(SI - A_f) = S^3 + a_1 S^2 + a_2 S + a_3 \tag{16}$$

Step 2. Solve β_1, β_2 and a_3.

By theorem (2.2), the sufficient condition for incomplete state feedback is

$$a_1 = q_1 \text{ and } a_2 = q_2$$

$$\text{where} \qquad q_1 = 3 \text{ and } q_2 = 3. \tag{17}$$

From eq. (2.69), one obtains

$$\left(\begin{array}{c} 1 \\ a_1 \\ a_2 \\ \hline a_3 \end{array}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \partial_1 & 1 & 0 & 0 \\ 0 & \partial_1 & 1 & 0 \\ 0 & 0 & \partial_1 & 1 \end{pmatrix} \left(\begin{array}{c} 1 \\ \beta_1 \\ \beta_2 \\ \hline 0 \end{array}\right) \tag{18}$$

or

$$\begin{bmatrix} a^1 \\ \hline a^2 \end{bmatrix} = \begin{bmatrix} C_{11} & 0 \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} \beta \\ \hline 0 \end{bmatrix} \tag{19}$$

Hence, $\beta = C_{11}^{-1}a^1$. β_1 and β_2 can be found as

$$\begin{bmatrix} 1 \\ \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \tag{20}$$

And a^2 can be found as

$$a^2 = a_3 = C_{21}C_{11}^{-1}\beta = [0 \quad 0 \quad 1] \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = 1 \tag{21}$$

Step 3. Generate the T matrix

$$T = \begin{pmatrix} 3 & 3 & 1 \\ 0 & 1 & 0 \\ -1 & -2 & 0 \end{pmatrix} \tag{22}$$

Step 4. Find $W_{1.1}$.

$$W_{1.1} = -2 \tag{23}$$

Step 5. Find k_1.

$$k_1 = 1 \tag{24}$$

Step 6. $K = [k_1, 0, 0] = [1, 0, 0]$ is the desired state feedback matrix.

EXAMPLE 3 (INCOMPLETE STATE FEEDBACK)

Consider a linear time-invariant system represented by

$$\dot{x} = \left[\begin{array}{ccc|ccc|cc} 0 & 1 & 0 & & & & & \\ 0 & 0 & 1 & & 0 & & & 0 \\ -4 & -5 & -4 & & & & & \\ \hline -1 & & & 0 & 1 & 0 & & 0 \\ -2 & 0 & & 0 & 0 & 1 & & \\ -1 & & & -1 & -3 & -2 & & \\ \hline -1 & & & & & & 0 & 1 \\ -2 & & & & 0 & & -1 & -2 \end{array}\right] x + \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 0 \end{array}\right] \qquad (25)$$

or

$$\begin{bmatrix} \dot{x}^1 \\ \dot{x}^2 \\ \dot{x}^3 \end{bmatrix} = \begin{bmatrix} A_1 & 0 & 0 \\ M_1 & A_2 & 0 \\ M_2 & 0 & A_3 \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \\ x^3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & b_1 \\ 0 & b_2 & 0 \\ b_3 & 0 & 0 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} \qquad (26)$$

Assume that the states that are available for feedback are x_1 and x_7 states. The problem is to find an incomplete state feedback matrix K that will observe only two states x_1 and x_7, and use these states to give the closed loop system 2 poles at -2 and -3 respectively.

Solution

This problem can be treated as to place one pole at -2 of the following system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = A_1 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + b_1 U_3 \qquad (27)$$

by using state feedback from x_1 state only, and to place another pole at -3 of the following system

$$\begin{bmatrix} \dot{x}_7 \\ \dot{x}_8 \end{bmatrix} = A_3 \begin{bmatrix} x_7 \\ x_8 \end{bmatrix} + b_3 U_1 \qquad (28)$$

by using state feedback from x_7 state.

Consider the case of placing one pole at -2 of the following system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ -4 & -5 \end{bmatrix} \begin{bmatrix} 0 & x_1 \\ 1 & x_2 \\ -4 & x_3 \end{bmatrix} + b_1 U_3 \qquad (29)$$

Following the same procedure as in example 2, one can find the T_1 matrix such that $T_1 = A_{c1} A_{f11}{}^{-1}$

where

$$A_{c1} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -2 \end{bmatrix} \qquad (30)$$

and

$$A_{f1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -5 & -4 \end{bmatrix} \qquad (31)$$

Therefore,

$$T_1 = \begin{bmatrix} 5 & 4 & 1 \\ 0 & 1 & 0 \\ -1 & -2 & 0 \end{bmatrix} \qquad (32)$$

and

$$T_1{}^{-1} = \begin{bmatrix} 0 & -2 & -1 \\ 0 & 1 & 0 \\ 1 & 6 & 5 \end{bmatrix} \qquad (33)$$

Let

$$U_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ k_{31} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + V_3 \qquad (34)$$

The value of k_{31} can be found by the same procedure as in example 2.

$$k_{31} = 2 \tag{35}$$

Consider the case of placing one pole at -3 of the following system

$$\begin{pmatrix} \dot{x}_7 \\ \dot{x}_8 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} x_7 \\ x_8 \end{pmatrix} + b_3 U_1 \tag{36}$$

The matrix T_3 is found such that $T_3 = A_{c3} A_{f3}^{-1}$ where

$$A_{c3} = \begin{pmatrix} -3 & 0 \\ 0 & -2 \end{pmatrix} \tag{37}$$

and

$$A_{f3} = \begin{pmatrix} 0 & 1 \\ 3 & -2 \end{pmatrix} \tag{38}$$

Therefore,

$$T_3 = \begin{pmatrix} -2 & -1 \\ 1 & 0 \end{pmatrix} \tag{39}$$

$$T^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} \tag{40}$$

Let

$$U_1 = \begin{pmatrix} k_{17} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{41}$$

Similarly, k_{17} can be found as

$$k_{17} = 4 \tag{42}$$

Therefore, the state feedback matrix is

$$K = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} + V \qquad (43)$$

EXAMPLE 4 (MODEL SIMPLIFICATION AND SUBOPTIMAL CONTROL)

Consider a linear regulator problem given by Rao [21] as

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.5 & -5.6 & -6.1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} U \qquad (44)$$

and a performance index

$$J = \frac{1}{2} \int_0^\infty (s^T Qx + U^T RU)dt$$

where

$$Q = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (45)$$

and $R = 1$

The object is to find a suboptimal control via a second-order simplified model.

Solution

Method 1 (Chidambara's method)

Let the second-order simplified model be represented by

$$\frac{dx^1}{dt} = Fx^1 + GU \qquad (47)$$

where x^1 is assumed to contain first two state variables of the system governed by eq. (44). i.e.

$$x^1 \triangleq \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{48}$$

The eigenvalues of system matrix A in eq. (44) are -0.1, -1, and -5, respectively. The eigenvector associated with those eigenvalues are

$$V^1 = \begin{bmatrix} 1 \\ -0.1 \\ 0.01 \end{bmatrix} \tag{49}$$

$$V^2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \tag{50}$$

and

$$V^3 = \begin{bmatrix} 1 \\ -5 \\ 25 \end{bmatrix} \tag{51}$$

Therefore, the model matrix is given by

$$M = \begin{bmatrix} 1 & 1 & 1 \\ -0.1 & -1 & -5 \\ 0.01 & 1 & 25 \end{bmatrix} \tag{52}$$

$$= \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

From eq. (4.22)

$$F = A_{11} + A_{12}M_{21}M_{11}^{-1}$$

$$= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [0.01 \quad 1] \begin{bmatrix} 1 & 1 \\ -0.1 & -1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 0 & 1 \\ -0.1 & -1.1 \end{bmatrix} \tag{53}$$

From eq. (4.14)

$$z_2(t) = -H_2^{-1}L_2U(t)$$

$$= -0.0102U(t) \qquad (54)$$

From eq. (4123)

$$G = B_1 - [A_{12}M_{22} - A_{12}M_{21}M_{11}^{-1}M_{12}]H_2^{-1}L_2$$

$$= \begin{bmatrix} 0 \\ 0.204 \end{bmatrix} \qquad (55)$$

Therefore, the simplified model equation is

$$\dot{x}^1 = \begin{bmatrix} 0 & 1 \\ -0.1 & -1.1 \end{bmatrix} x^1 + \begin{bmatrix} 0 \\ 0.204 \end{bmatrix} U \qquad (56)$$

From eq. (4.57) one obtains the equivalent performance index

$$Q_1 = \begin{bmatrix} 5.1 & 1 \\ 1 & 14 \end{bmatrix} \qquad (57)$$

Using the computer program written by Melsa and Jones [40], one obtains the optimal control of the above simplified model as

$$U = -K_c x^1$$

$$= - [1.82 \quad 2.41] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad (58)$$

Method 2 (The proposed decomposition method)

The eigenvalues of the system matrix A in eq. (44) are known as $\lambda_1 = -0.1$, $\lambda_2 = -1$, and $\lambda_3 = -5$.

Let

$$\det(SI - A_{c1}) = (S - \lambda_1)(S - \lambda_2)$$

$$= S^2 + 1.1S + 0.1 \qquad (59)$$

$$\det(SI - A_{c2}) = S + 5 \tag{60}$$

From eq. (2.18)

$$A_c = \begin{bmatrix} 0 & 1 & 0 \\ -0.1 & -1.1 & 0 \\ 0 & 0 & -5 \end{bmatrix} \tag{61}$$

the T matrix can be found as

$$\begin{aligned} T &= \begin{bmatrix} 0 & 1 & 0 \\ -0.1 & -1.1 & 0 \\ 0 & 0 & -5 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.5 & -5.6 & -6.1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0.02 & 1.22 & 0.2 \\ 0 & -5 & 0 \end{bmatrix} \end{aligned} \tag{62}$$

From eq. (4.30)

$$\begin{aligned} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = TB &= \begin{bmatrix} 1 & 0 & 0 \\ 0.02 & 1.22 & 0.2 \\ 0 & -5 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0.2 \\ 0 \end{bmatrix} \end{aligned} \tag{63}$$

So

$$\beta_1 = \begin{bmatrix} 0 \\ 0.2 \end{bmatrix} \tag{64}$$

From eq. (4.32), the simplified model equation is

$$\dot{x}_1 = \begin{bmatrix} 0 & 1 \\ -0.1 & -1.1 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 0.2 \end{bmatrix} U \tag{65}$$

From eq. (4.41), the Vandermonde matrix is given by

$$N = \begin{bmatrix} 1 & 1 & 1 \\ -0.1 & -1 & -5 \\ 0.01 & 1 & 25 \end{bmatrix}$$

$$= \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \quad (66)$$

Therefore,

$$N_{11} = \begin{bmatrix} 1 & 1 \\ -0.1 & -1 \end{bmatrix} \quad (67)$$

$$N_{21} = [0.01 \quad 1] \quad (68)$$

The equivalent performance index for the simplified model can be obtained by using eq. (4.57) as

$$Q_1 = [T \quad (N_{21} N_{11}{}^{-1})^T] \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} I \\ (N_{21} N_{11}{}^{-1})^T \end{bmatrix}$$

$$= \begin{bmatrix} 5.1 & 1 \\ 1 & 14 \end{bmatrix} \quad (69)$$

The optimal control of the proposed simplified model for the same system is found as

$$U = -K_d x_1 = -[1.812 \quad 2.39] x_1 \quad (70)$$

while the optimal control of the original system is

$$U = -K_{opt}\, x = -[1.79 \quad 2.08 \quad 0.41] \begin{bmatrix} 1 \\ x_2 \\ x_3 \end{bmatrix} \quad (71)$$